PERIODIC PERTURBATIONS OF A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-020-09928-2

Periodic Perturbations of a Class of Functional Differential Equations

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Received: 14 July 2020 / Accepted: 20 December 2020

Abstract
We study the existence of a connected "branch" of periodic solutions of T-periodic perturbations of a particular class of functional differential equations on differentiable manifolds. Our result is obtained by a combination of degree-theoretic methods and a technique that allows to associate the bounded solutions of the functional equation to bounded solutions of a suitable ordinary differential equation.

Keywords Functional differential equations · Branches of periodic solutions · Degree of a tangent vector field

Mathematics Subject Classification 34C25 · 34C40

1 Introduction

Consider, on a differentiable manifold $M \subseteq \mathbb{R}^k$, the class of constrained functional differential equations of the following form:

$$\dot{x}(t) = g\left(x(t), \int_{-\infty}^{t} \gamma_a^b(t-s)x(s)\,\mathrm{d}s\right), \tag{1}$$

where $g\colon M \times \mathbb{R}^k \to \mathbb{R}^k$ is a continuous map with the properties that $g(x, y)$ belongs to the tangent space T_xM to M at x for all $(x, y) \in M \times \mathbb{R}^k$. The integral kernel γ_a^b, for $a > 0$ and $b \in \mathbb{N} \setminus \{0\}$ is the gamma probability distribution

$$\gamma_a^b(s) = \frac{a^b s^{b-1} \mathrm{e}^{-as}}{(b-1)!} \text{ for } s \geq 0, \qquad \gamma_a^b(s) = 0 \text{ for } s < 0,$$

with mean b/a and variance b/a^2. The product $\gamma_a^b(t-s)x(s)$ inside the integral must be understood componentwise.

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Published online: 25 January 2021

Such equations (usually, with M an open subset of $\mathbb{R}^k$) arise naturally in many fields especially in the study of biological systems see, e.g., [4,5,18,19] and the many references therein; we mention in particular [6,7] where a model for HIV virus replication is proposed. See also [9] for a range of applications to the physical sciences.

In this paper we are interested in the periodic response of (1) subject to periodic perturbations. Namely, we consider the following perturbation of (1):

$$\dot{x}(t) = g\left(x(t), \int_{-\infty}^{t} \gamma_a^b(t-s)x(s)\,\mathrm{d}s\right) + \lambda f\big(t, x(t)\big), \tag{2}$$

where $f(t,x)$ belongs to the tangent space T_xM to M at x for all $x \in M$ and $t \in \mathbb{R}$ and, assuming that f is $T > 0$ periodic in t, we investigate the set of T-periodic solutions of (2). Recall that, by a T-periodic solution (on $\mathbb{R}$) of (2) we mean a C^1 function $x\colon \mathbb{R} \to M$ of period T that satisfies (2) identically.

Speaking loosely, our main result establishes a topological sufficient condition for the existence of a connected set of nontrivial pairs (λ, x), with x a T-periodic solution of (2) corresponding to λ, branching out of the set of zeros of the vector field $p \mapsto g(p,p)$ and whose closure is not compact. See Theorem 4.6 for a precise formulation. The techniques used and the results obtained can be seen as a generalization of [3] which is devoted to the special system arising from a model of HIV virus replication under periodic forcing. Unlike that paper, though, we do not deal with the local (near the zeros of $p \mapsto g(p,p)$) structure of the set of periodic solutions.

We will develop our analysis using methods inspired to [10,13]. However the techniques developed in those papers do not apply directly to perturbations of general functional differential equations but, for the particular form of g considered in Eq. (2) we observe that the study of periodic solutions can be reduced to the study of the periodic solution of an ordinary differential equation on a higher dimensional manifold (see Theorems 3.2 and 3.1 below). Indeed, the main contribution of this paper follows by specializing the techniques of [10,13] to such ordinary differential equation. This application is made possible by a formula, Theorem 2.2, for the computation of the degree (or rotation or characteristic) of the associated vector field (for $\lambda = 0$).

The idea behind the construction of an ordinary differential equation associated to (2), apart from some technical difficulties introduced by the manifold setting, is not new; see, e.g., [4,8,9,19]. Namely, it will be shown (see Remark 3.4) that the T-periodic solutions of (2) and of the following ordinary differential equation on the manifold $M \times \mathbb{R}^{kb} \subseteq \mathbb{R}^{k(b+1)}$ correspond in some sense:

$$\dot{\xi} = G(\xi) + \lambda F(t, \xi), \tag{3}$$

where $\xi = (x_0, x_1, \dots, x_b) \in M \times \mathbb{R}^{kb}$,

$$G(x_0, x_1, \dots, x_p) = (g(x_0, x_b), a(x_0 - x_1), \dots, a(x_{b-1} - x_b))$$

and

$$F(t, x_0, x_1, \dots, x_b) = \Big(f(t, x_0), \underbrace{0, \dots, 0}_{b\text{times}}\Big).$$

Clearly, by a T-periodic solution (on $\mathbb{R}$) of (3) we mean a C^1 function $x\colon \mathbb{R} \to N$ of period T that satisfies (3) identically.

The methods of [13] are applicable to (3) since, as we will show, the computation of the degree of G boils down to the degree of the tangent vector field $x \mapsto g(x,x)$. We show the validity of the last statement using the properties of the degree of a tangent vector field in combination with approximation results on manifolds. This somewhat technical proof is set apart in a dedicated section at the end of the paper in order not to divert the reader's attention from the main theme.

Our exposition is complemented by a special section, inspired by [2], devoted to techniques that can provide some visual representation of the set of T-periodic solutions of (2).

We point out that, at the price of some more minor technicalities, the same "reduction" strategy adopted in this paper could be applied to the more general case when the periodic perturbation f depends also on a retarded term, with delay that might be fixed or functional (possibly infinite). Roughly speaking, in these situations one can replace Theorem 4.2 with existing results appropriate for the type of perturbation considered [12,14]; although it should be noted that the case of infinite delay perturbation, unlike the finite-delay or no-delay perturbation case, requires more regularity assumptions on the perturbing term. Also notice that some further effort, following [20], would allow a slight generalization of the unperturbed equations. However, we feel that the these extensions introduce complications cluttering the simple construction underlying our analysis. For this reason, we will defer a brief discussion of these ideas to a later section on "Perspectives and further developments".

2 Basic Notions

In this section we recall a few notions about tangent vector fields on manifolds and on the notion of degree of an admissible tangent vector field which play a key role throughout this paper.

Consider a manifold $\mathcal{M} \subseteq \mathbb{R}^s$ and let w be a tangent vector field on $\mathcal{M}$, that is, a continuous map $w\colon \mathcal{M} \to \mathbb{R}^s$ such that $w(\xi) \in T_\xi\mathcal{M}$ for any $\xi \in \mathcal{M}$. It is known (see e.g. [17]) that if w is (Fréchet) differentiable at $\xi \in \mathcal{M}$ and $w(\xi) = 0$, then the its $dw_\xi : T_\xi\mathcal{M} \to \mathbb{R}^s$ at ξ maps $T_\xi\mathcal{M}$ into itself. Thus, the determinant $\det dw_\xi$ of dw_ξ is defined. If, in addition, $dw_\xi : T_\xi\mathcal{M} \to \mathbb{R}^s$ is injective (thus, dw_ξ being a linear map of the finite dimensional vector space $T_\xi\mathcal{M}$ into itself, is also surjective i.e. ξ is a nondegenerate zero) then ξ is an isolated zero and we get $\det dw_\xi \neq 0$.

Let $W \subseteq \mathcal{M}$ be open and assume w *admissible* (for the degree); namely we suppose that the set $w^{-1}(0) \cap W$ is compact. Then, one can associate an integer $\deg(w, W)$ to the pair (w, W), called the *degree (or rotation or characteristic) of the vector field w in W*, which, roughly speaking, counts algebraically the zeros of w in W (see e.g. [10,16,17] and references therein). In fact, when the zeros of w are all nondegenerate, then the set $w^{-1}(0) \cap W$ is finite and we have

$$\deg(w, W) = \sum_{\xi \in w^{-1}(0) \cap W} \operatorname{sign} \det dw_\xi . \tag{4}$$

In the general case, the degree is defined by approximation. That is, for an admissible vector field w in W, the degree is defined by constructing a suitable approximation ν of w in W as in formula (4) (see, e.g., Proposition 7.2) and by setting $\deg(w, W) = \deg(\nu, W)$.

Observe that when $\mathcal{M} = \mathbb{R}^s$, $\deg(w, W)$ is the classical Brouwer degree with respect to zero, $\deg_B(w, V, 0)$, where V is any bounded open neighborhood of $w^{-1}(0) \cap W$ whose closure is contained in W. Indeed, all the standard properties of the Brouwer degree for continuous maps on open subsets of Euclidean spaces, such as homotopy invariance, excision, additivity, existence, still hold in this more general context (see e.g. [10]). One could actually show that a few of these properties uniquely determine the notion of degree of a tangent vector field (see [11]).

Let $M \subseteq \mathbb{R}^k$ be a differentiable manifold, and let g and f be as in Eq. (2). We will be concerned with the tangent vector field on the manifold $\mathcal{M} = M \times \mathbb{R}^{kb}$ given by

$$G(\xi_0, \dots, \xi_b) = \big(g(\xi_0, \xi_b), a(\xi_0 - \xi_1), \dots, a(\xi_{b-1} - \xi_b)\big), \tag{5}$$

which is associated to Eq. (3).

Our first task will be to derive a formula for the computation of its degree in terms of the degree of the vector field $\hat{g}\colon M \to \mathbb{R}^k$, tangent on M, given by

$$\hat{g}(\xi) = g(\xi, \xi). \tag{6}$$

Remark 2.1 Observe that if $(x_0, \dots, x_b) \in G^{-1}(0)$ then $\hat{g}(x_0) = 0$ and $x_0 = x_1 = \dots = x_b$. Conversely, for any $x_0 \in \hat{g}^{-1}(0)$, then $G(x_0, \dots, x_0) = 0$.

Given an open set $U \subseteq M$, define the open subset $U^* = U \times \mathbb{R}^{kb}$ of $M \times \mathbb{R}^{kb}$. We have:

Theorem 2.2 *Suppose the vector field $\hat{g}$ is admissible for the degree in U. Then so is G, defined in (5), in U^* and we have* $\deg(G, U^*) = (-1)^{kb} \deg(\hat{g}, U)$.

This result will be crucial in one of the key step (Corollary 4.3) towards our main result. However, its proof is based on homotopy arguments and approximation techniques that would take us too far from the main topic of the paper. For this reason, the proof of Theorem 2.2 is deferred to a special section at the end of the paper.

3 Linear Chain Trick

In this section we show how bounded solutions of Eq. (2) correspond to bounded solutions of a particular system of ODEs. This fact applies, in particular, to periodic solutions (Remark 3.4). This correspondence is constructed using a well-known device sometimes called the "linear chain trick". In this section we follow essentially the formulation of [19].

Theorem 3.1 *Suppose x_0 is a bounded solution of* (2)*, and let*

$$x_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)x_0(s)\,\mathrm{d}s, \quad i = 1, \dots, b,$$

then $(x_0, \dots, x_b)$ is a bounded solution of (3).

Proof For $i = 1, \dots, b$ the functions x_i are obviously bounded; moreover,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x_i(t) &= \gamma_a^i(0)x_0(t) + \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\gamma_a^i(t-s)\,x_0(s)\,\mathrm{d}s \\
&= \int_{-\infty}^{t} a\left(\gamma_a^{i-1}(t-s) - \gamma_a^i(t-s)\right)\,\mathrm{d}s \\
&= a\left(\int_{-\infty}^{t} \gamma_a^{i-1}(t-s)\,\mathrm{d}s - \int_{-\infty}^{t} \gamma_a^i(t-s)\,\mathrm{d}s\right) \\
&= a\left(x_{i-1}(t) - x_i(t)\right).
\end{aligned}$$

So that the following relations hold:

$$\dot{x}_i(t) = a\big(x_{i-1}(t) - x_i(t)\big), \qquad i = 1, \dots, b.$$

Since x_0 is a solution of (2),

$$\begin{aligned}\dot{x}_0(t) &= g\left(x_0(t), \int_{-\infty}^{t} \gamma_a^i(t-s)x_0(s)\,\mathrm{d}s\right) + \lambda f\big(t, x_0(t)\big)\\ &= g\big(x_0(t), x_b(t)\big) + \lambda f\big(t, x_0(t)\big), \quad \lambda \geq 0,\end{aligned}$$

hence,

$$\begin{cases}\dot{x}_0(t) = g\big(x_0(t), x_b(t)\big) + \lambda f\big(t, x_0(t)\big), & \lambda \geq 0,\\ \dot{x}_i(t) = a\big(x_{i-1}(t) - x_i(t)\big), & i = 1, \ldots, b.\end{cases}$$

Whence the assertion. □

Conversely, we have the following:

Theorem 3.2 *Suppose* $(x_0, x_1, \ldots, x_b)$ *a bounded solution of* (3)*, then* x_0 *is a bounded solution of* (2).

The proof of Theorem 3.2 is based on the following slight generalization of [19, Prop. 7.3] concerning a class of nonhomogeneous linear differential equations. We provide a short proof for the sake of completeness.

Lemma 3.3 *Given a bounded function* $y_0 \colon \mathbb{R} \to \mathbb{R}^k$ *and a positive number a, there exists a unique bounded solution of the system in* $\mathbb{R}^{kb}$ *given by the following b equations in* $\mathbb{R}^k$*:*

$$\dot{y}_i(t) = a\big(y_{i-1}(t) - y_i(t)\big), \text{ for } i = 1, \ldots, b. \tag{7}$$

This solution is given by

$$y_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)y_0(s)\,\mathrm{d}s, \quad i = 1, \ldots, b. \tag{8}$$

Proof The homogeneous part of system (7) is the following equation

$$\dot{\eta} = A\eta \tag{9}$$

A being the following $kb \times kb$ matrix that we can write in block-matrix form as:

$$A = \begin{pmatrix} -I & 0 & \ldots\ldots & & 0\\ I & -I & 0 & \ldots & 0\\ 0 & I & -I & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \ldots & 0 & I & -I \end{pmatrix}$$

Where I is the $k \times k$ identity matrix. Clearly A has only the eigenvalue -1 (with multiplicity kb) thus the unique bounded solution of (9) is the identically zero one. The uniqueness of the bounded solution (on $\mathbb{R}$) of (7) follows by observing that if there are two bounded solution of (7), the their difference must be a bounded solution of (9), thus it is zero.

The last assertion follows by verifying directly, as in the proof of Theorem 3.1, that y_i defined as in (8) for $i = 1, \ldots, b$, satisfies (7). □

Proof of Theorem 3.2 Let $y_0(t) = x_0(t)$ for $t \in \mathbb{R}$. Thus, by Lemma 3.3,

$$y_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)y_0(s)\,\mathrm{d}s, \quad i = 1, \ldots, b,$$

is the unique bounded solution of (7). Hence, we necessarily have $x_i = y_i$ for $i = 0, \dots, n$. In particular, we have

$$x_b(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)x_0(s)\,\mathrm{d}s.$$

Thus, from (3),

$$\begin{aligned}\dot{x}_0(t) &= g\big(x_0(t), x_b(t)\big) + \lambda f\big(t, x_0(t)\big)\\ &= g\left(x_0(t), \int_{-\infty}^{t} \gamma_a^i(t-s)x_0(s)\,\mathrm{d}s\right) + \lambda f\big(t, x_0(t)\big), \quad \lambda \geq 0,\end{aligned}$$

for all $t \in \mathbb{R}$, whence the assertion.

By inspection of Theorems 3.2 and 3.1 we see that a bounded solution x_0 of (2), for a particular value of λ, corresponds to the (unique bounded) solution $t \mapsto \xi(t) = \big(\xi_0(t), \xi_1(t), \dots, \xi_b(t)\big)$ of (3), for the same value of λ, with initial conditions at time $t = 0$ given by

$$\xi_0(0) = x_0(0), \quad \text{and} \;\; \xi_i(0) = \int_{-\infty}^{0} \gamma_a^i(-s)x_0(s)\,\mathrm{d}s, \;\; \text{for } i = 1, \dots, b.$$

Observe that, normally, in order to determine a solution, say after time $t = 0$, of a retarded functional equation as (2) it is necessary to specify its whole history up to time $t = 0$ by giving an "initial value function" (see e.g. [1]). Loosely speaking, we may say that the possible initial values belong to an infinite dimensional space. However, for Eq. (2), if we confine ourselves to bounded solutions as done above, we have that our "interesting" initial conditions lie on a finite dimensional manifold. This fact will be important when it comes to graphical representation in Sect. 5.

Remark 3.4 Theorems 3.2 and 3.1are valid, in particular, for periodic solutions. Thus, we have that if $(x_0, x_1, \dots, x_b)$ a T-periodic solution of (3), then x_0 is a T-periodic solution of (2) and, conversely, if x_0 is a T-periodic solution of (2), setting

$$x_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)x_0(s)\,\mathrm{d}s, \quad i = 1, \dots, b,$$

we get the solution of (3) given by $(x_0, \dots, x_b)$. To see that this solution is T-periodic observe that with the change of variable $\sigma = s - T$

$$\begin{aligned}x_i(t+T) &= \int_{-\infty}^{t+T} \gamma_a^i(T+t-s)x_0(s)\,\mathrm{d}s = \int_{-\infty}^{t} \gamma_a^i(t-\sigma)x_0(\sigma+T)\,\mathrm{d}\sigma\\ &= \int_{-\infty}^{t} \gamma_a^i(t-\sigma)x_0(\sigma)\,\mathrm{d}\sigma = x_i(t),\end{aligned}$$

for $i = 1, \dots, b$.

4 Branches of *T*-Pairs

Let us introduce some notation. Given a subset D of $\mathbb{R}^s$, we will denote by $C_T(D)$, the metric subspace of the Banach space $(C_T(\mathbb{R}^s), \|\cdot\|)$ of all the T-periodic contiunous maps $x\colon \mathbb{R} \to D$ with the usual C^0 norm (i.e. the supremum norm, or sup-norm for short). Observe that $C_T(D)$ is not complete unless D is complete (i.e. closed in $\mathbb{R}^s$). Nevertheless, if D is locally compact, then $C_T(D)$ is locally complete.

Let $\phi\colon \mathbb{R}\times\mathcal{M}\to\mathbb{R}^s$ and $\psi\colon\mathcal{M}\to\mathbb{R}^s$ be continuous tangent vector fields defined on a (boundaryless) differentiable manifold $\mathcal{M}\subseteq\mathbb{R}^s$, with ϕ assumed T-periodic in the first variable. Consider the following parameterized ODE on $\mathcal{M}$:

$$\dot{x}(t)=\psi\big(x(t)\big)+\lambda\phi\big(t,x(t)\big),\quad \lambda\geq 0. \tag{10}$$

We say that a pair $(\lambda,x)\in[0,\infty)\times C_T(\mathcal{M})$ is a T*-pair* (for (10)) if x is a T-periodic solution of (10) corresponding to λ. If $\lambda=0$ and x is constant then the T-pair is said *trivial*. It is not hard to see that trivial T-periodic pairs of (10) correspond to zeros of ψ.

To make the last statement precise, for any $p\in\mathcal{M}$, denote by $\overline{p}$ the function $\overline{p}\colon\mathbb{R}\to\mathcal{M}$ constantly equal to p. We have that, given any $p\in\psi^{-1}(0)$, the pair $(0,\overline{p})\in[0,\infty)\times C_T(\mathcal{M})$ is a trivial T-pair of (10). Conversely, for a trivial T-periodic pair $(0,x)\in[0,\infty)\times C_T(\mathcal{M})$, from local uniqueness of the solutions of Cauchy problem it follows $x(t)\equiv p$ for some $p\in\psi^{-1}(0)$. Notice however that (10), depending on the map ψ, may have nonconstant T-periodic solutions for $\lambda=0$; consider, for example, $\mathcal{M}=\mathbb{R}^2$, $T=2\pi$ and $\psi(x_1,x_2)=(-x_2,x_1)$. Bear also in mind that all T-periodic pairs (λ,x) with $\lambda>0$ are nontrivial even when x is constant as, for instance, when $\mathcal{M}=\mathbb{R}$, $T=2\pi$, $\psi(x)=x$ and $\phi(t,x)=x\sin(t)$.

Remark 4.1 The injective map that to $p\in\mathcal{M}$ associates the pair $(0,\overline{p})\in[0,\infty)\times C_T(\mathcal{M})$ is an embedding that allowes us to regard $\mathcal{M}$ as a subset of $[0,\infty)\times C_T(\mathcal{M})$. Indeed, by the above discussion, it follows that this embedding actually identifies the zeros of ψ with the trivial T-pairs of (10).

We have the following fact concerning the T-pairs of (10):

Theorem 4.2 ([13]) *Let $\mathcal{N}$, $\phi\colon\mathbb{R}\times\mathcal{N}\to\mathbb{R}^s$ and $\psi\colon\mathcal{N}\to\mathbb{R}^s$ be as in* (10)*. Let $\mathcal{O}$ be an open in $[0,\infty)\times C_T(\mathcal{N})$, and assume that $\deg(\psi,\mathcal{O}\cap\mathcal{N})$ is well defined and nonzero. Then there exists a connected set Γ of nontrivial T-pairs in $[0,\infty)\times C_T(\mathcal{N})$ whose closure in $\mathcal{O}$ is not compact and meets the set of trivial T-pairs contained in $\mathcal{O}$, namely the set: $\big\{(0,\overline{p})\in\mathcal{O}:\psi(p)=0\big\}$.*

Let us look at (3) and let $\mathcal{M}=M\times\mathbb{R}^{kb}$, $\psi=G$ and $\phi=F$. By the above definition it makes sense to consider the set Y of T-pairs of (3). Theorems 4.2 and 2.2 yield the following result about Y:

Corollary 4.3 *Let $\Omega\subseteq[0,\infty)\times C_T(M)$ be open and let $\hat{g}$ and G be as in* (6) *and* (5)*, respectively, and put $\Omega_M:=\big\{q\in M:(0,\bar{q})\in\Omega\big\}$. Suppose that $\deg(\hat{g},\Omega_M)$ is well-defined and nonzero and let $\mathcal{O}=\Omega\times C_T(\mathbb{R}^{kb})$. Then, Y admits a connected subset of nontrivial T-pairs for* (3) *whose closure in $\mathcal{O}$ is not compact and intersects the set*

$$\big\{(0;\overline{q},\overline{q},\ldots,\overline{q})\in\mathcal{O}:\hat{g}(q)=0\big\}. \tag{11}$$

Proof Let G be as in (3). By Theorem 2.2 we have

$$\deg(G,\Omega_M\times\mathbb{R}^{kb})=\deg(\hat{g},\Omega_M)\neq 0,$$

Thus, by Theorem 4.2, there is connected set in Y consisting of nontrivial T-pairs for (3) whose closure in $\mathcal{O}$ is not compact and intersects the set

$$\big\{(0;\overline{q}_0,\overline{q}_1,\ldots,\overline{q}_b)\in\mathcal{O}:G(q_0,\ldots,q_b)=0\big\}. \tag{12}$$

By the particular form of the vector field G, see Eq. (5), we have that $G(q_0, \dots, q_b) = 0$ implies $q_0 = \dots = q_b$. Thus the set in (12) coincides with

$$\{(0; \overline{q}, \overline{q}, \dots, \overline{q}) \in \mathcal{O} : G(q, \dots, q) = 0\} = \{(0; \overline{q}, \overline{q}, \dots, \overline{q}) \in \mathcal{O} : \hat{g}(q) = 0\}.$$

Whence the assertion. □

Analogously to the notion of T-pair for (3) we can define T-pairs for Eq. (2): We say that a pair $(\lambda, x) \in [0, \infty) \times C_T(M)$ is a *T-pair* for (2) if x is a T-periodic solution of (2) corresponding to λ. If $\lambda = 0$ and x is constant then the T-pair is said *trivial*.

Remark 4.4 Observe that if $(0, \overline{p})$, with $\overline{p}(t) \equiv p \in M \subseteq \mathbb{R}^k$, is a trivial T-pair for (2), then so is $(0; \overline{p}, \dots, \overline{p})$ for Eq. (3) on $\mathcal{M} = M \times \mathbb{R}^{kp}$. Thus, by Remark 2.1, $p \in \hat{g}^{-1}(0)$. Conversely, again by Remark 2.1, for any $p \in \hat{g}^{-1}(0)$ we have $G(p, \dots, p) = 0$ so that $(0; \overline{p}, \dots, \overline{p})$ is a trivial T-pair for (3).

Denote by X the set of T-pairs of (2). Corollary 4.3 provides important information concerning X by means of the following straightforward lemma:

Lemma 4.5 *Let X and Y be the set of T-pairs of (2) and (3), respectively. Let $\mathfrak{h}\colon Y \to X$ be defined by $\mathfrak{h}(\lambda, x_0, \dots, x_b) = (\lambda, x_0)$. Then $\mathfrak{h}$ is bijective, continuous and has continuous inverse. Furthermore, $\mathfrak{h}$ establishes a bijective correspondence between trivial T-pairs of X and Y.*

Proof Bijectivity follows from Theorems 3.2 and 3.1, and from Remark 3.4. Continuity is obvious as $\mathfrak{h}$ is a projection.

To see the continuity of the inverse of $\mathfrak{h}$, recall that (see the proof of Theorem 3.2):

$$x_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) x_0(s)\, \mathrm{d}s, \quad i = 1, \dots, b.$$

The last part of the assertion now follows directly from Remark 2.1. □

We are now in a position to state and prove our main result concerning the set of T-pairs of (2).

Theorem 4.6 *Let $\Omega \subseteq [0, \infty) \times C_T(M)$ be open and let $\hat{g}$ be as in (6). Suppose that the open set $\Omega_M := \{p \in M : (0, \bar{p}) \in \Omega\}$ is such that $\deg(\hat{g}, \Omega_M)$ is well-defined and nonzero. Then, in $X \cap \Omega$ there is a connected subset Γ of nontrivial T-pairs whose closure relative to Ω intersects $\{(0, \overline{p}) \in \Omega : p \in \hat{g}^{-1}(0) \cap \Omega_M\}$ and is not compact.*

Proof By Corollary 4.3 there exists a connected subset Σ of nontrivial T-pairs for (3) whose closure in $\mathcal{O} = \Omega \times C_T(\mathbb{R}^{kb})$ is not compact and intersects the set

$$\{(0, \overline{p}, \dots, \overline{p}) \in \mathcal{O} : p \in \hat{g}^{-1}(0)\}.$$

Let $\Gamma := \mathfrak{h}^{-1}(\Sigma)$ with $\mathfrak{h}$ as in Lemma 4.5. Observe that the image under $\mathfrak{h}$ of the set (11) is

$$K := \{(0, \overline{p}) \in \Omega : p \in \hat{g}^{-1}(0) \cap \Omega_M\}.$$

Hence, by Lemma 4.5, Γ is a connected set that consists of nontrivial T-pairs of (2), intersects K and its closure in Ω is not compact, as in the assertion. □

Recalling Remark 4.1 and Lemma 4.5, it makes sense to say that the set Γ of the above theorem branches out from $\Omega_M \cap \hat{g}^{-1}(0)$.

5 Graphical Representation and Examples

In this section we illustrate Theorem 4.6 by the means of three simple examples for which, in a sense, we represent graphically (a portion of) the branch Γ. This set lives, so to speak, in an infinite dimensional space. So, what we do, as in [2], is to produce an image of a homeomorphic set which is finite dimensional or, in some cases more simply, to show a graph of some function of the elements of Γ. For simplicity, we assume some regularity on our equation and confine ourselves to the case when M is an open subset of $\mathbb{R}^k$.

Let us explain how this graphing technique works in our setting. Observe first that if g and f in (2) are locally Lipschitz, then so are G and F in (3). Let Y be the set of T-pairs of (3) and consider the set $S \subseteq [0,\infty) \times M \times \mathbb{R}^{kb}$ constituted by the points $(\lambda, p_0, \dots, p_b)$ such that $(p_0, \dots, p_b)$ is an initial condition (say at time $t = 0$) for T-periodic solutions of (3). Such points are called *starting points* for (3). Thus, by uniqueness and continuous dependence on initial data the map $\mathfrak{p}\colon Y \to S$ that to each $(\lambda, x_0, \dots .x_b) \in Y$ associates $\big(\lambda, x_0(0), \dots .x_b(0)\big) \in S$ is a homeomorphism.

Remark 5.1 It is not difficult to see that when f and g are smooth enough a Sard's Lemma argument implies that, "generically", the set of the starting points in S is the intersection of a boundaryless 1-dimensional manifold in $\mathbb{R}^{1+k(b+1)}$ with the closed half-space defined by $\lambda \geq 0$.

Let now X be the set of T-pairs of (2) and $\mathfrak{h}$ be the map of Lemma 4.5. Hence the composition $\mathfrak{p} \circ \mathfrak{h}^{-1}\colon X \to S$ is a homeomorphism. Therefore, to represent the set Γ of Theorem 4.6 one may sketch the set S. We warn the reader that, although it is tempting, a graph against λ of merely the initial value of the T-periodic solutions of (2) is obviously not sufficient to represent the set Γ.

Clearly, as S is a subset of $\mathbb{R}^{1+k(b+1)}$, it not possible to plot S directly if $1+k(b+1)$ is greater than 3. The idea is to compute numerically S in a selected box $B \subseteq [0,\infty) \times \mathbb{R}^{k(b+1)}$ and then either draw some simple functions (e.g., projections) of $S \cap B$, or use these initial values of T-periodic solutions of (3) to evaluate and draw a graph of some other relevant quantity as, for instance, the sup-norm or the diameter of the orbit (that represents the amplitude of the oscillation) of the corresponding elements of Y or, perhaps more interestingly, of X.

Suppose, for example, that we wish to obtain a graph of the sup-norm of the T-periodic solution of (2) for different values of $\lambda \geq 0$, assuming that the assumptions of Theorem 4.6 hold. We select a box B of the form $[0, \lambda_0] \times U$ with $U \subseteq \mathbb{R}^{k(b+1)}$ bounded and such that $\hat{g}$ is admissible with nonzero degree in U and compute numerically $S \cap B$. Having done so, it is possible to compute numerically the set of the corresponding T-pairs of (3). Now, to obtain the desired graph, it is enough to observe that for $(\lambda, x_0, \dots .x_b) \in Y$, $\mathfrak{h}(\lambda, x_0, \dots .x_b) = (\lambda, x_0)$ is the corresponding element of X.

Remark 5.2 The procedure just described is easy to apply because of the form of Eq. (2) which imply that (3) is an ordinary differential equation. More specifically, if the perturbing term f in (2) were dependent on the history of the system, then the only appropriate replacement for the finite dimensional set S would be a set of (numerically hard to determine) "initial functions" belonging to a suitable, infinite dimensional, manifold.

We first consider a very straightforward example.

Example 5.3 Let $M = \mathbb{R}$, $T = 1$ and consider the equation

$$\dot{x}(t) = x(t) - \frac{\int_{-\infty}^{t} \gamma_1^1(t-s)x(s)\,\mathrm{d}s}{1 + x(t)^2} - \lambda\big(1 + \sin(2\pi t)\big). \tag{13}$$

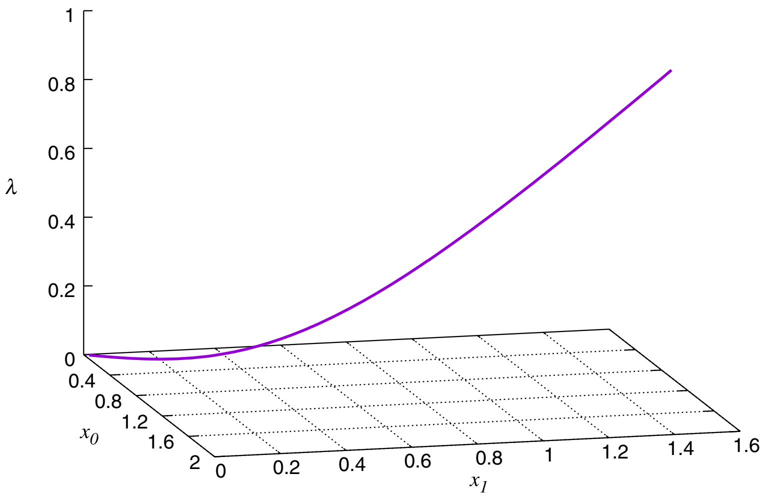
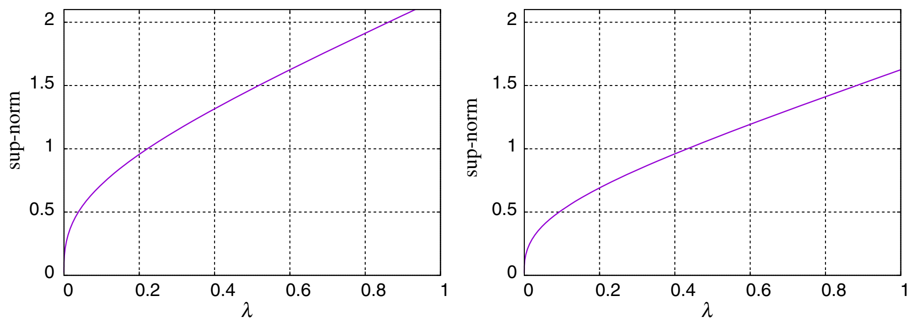


Fig. 1 Representation of the set S of Example 5.3 for $0 \leq \lambda \leq 1$



(a) sup-norm of elements of the set of T-pairs of (14)

(b) sup-norm of the T-periodic solutions of Equation (13)

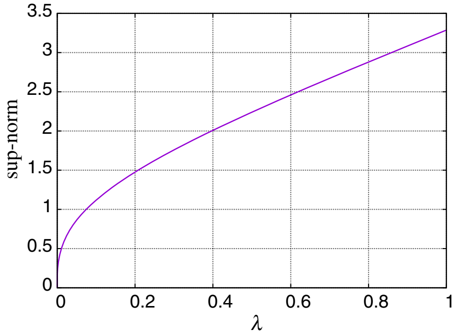
Fig. 2 Comparison between the graphs of the sup-norm of elements of the set of T-pairs of (14) and of (13) in Example 5.3

In the notation of Eq. (2), we have $a = b = 1$, $g(x, y) = 2x - y/(1 + x^2)$ and $f(t, x) = -1 - \sin(2\pi t)$. Then $\hat{g}(x) = x$, that has clearly degree 1 in M.

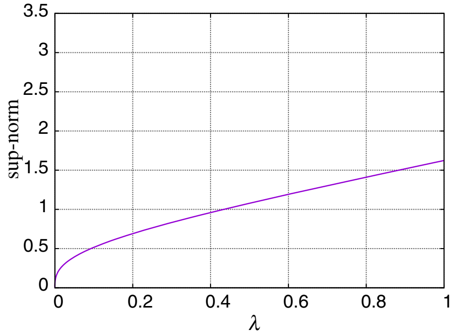
Setting $G(x_0, x_1) = (x_0 - x_1/(1 + x_0^2), x_0 - x_1)$ and $F(t, x_0, x_1) = \big(1 + \sin(2\pi t), 0\big)$, Eq. (3) becomes the following system:

$$\begin{cases} \dot{x}_0(t) = x_0(t) - \frac{x_1(t)}{1+x_0(t)^2} - \lambda\big(1 + \sin(2\pi t)\big), \\ \dot{x}_1(t) = x_0(t) - x_1(t) \end{cases} \tag{14}$$

The set of the starting points of (14), S, in a neighborhood of $(0, 0, 0) \subseteq [0, \infty) \times \mathbb{R}^2$ is represented in Fig. 1. As discussed above this can be viewed as a representation of the set of X of T-pairs of (13). Further information can be gathered from Fig. 2 which is a graph, against λ, of the sup-norm of elements of the set Y of T-pairs of (14) (i.e., of the T-periodic solutions corresponding to the elements of S) and of the T-periodic solution of (13).



(a) sup-norm of elements of the set of T-pairs of (16)



(b) sup-norm of the T-periodic solutions of Equation (15)

Fig. 3 Comparison between the graphs of the sup-norm of elements of the set of T-pairs of (16) and of (15) in Example 5.4

For comparison, consider an equation analogous to the one in the example above but with $b = 4$ (mantaining $a = 1$).

Example 5.4 Let $M = \mathbb{R}$, $T = 1$ and consider the equation

$$\dot{x}(t) = x(t) - \frac{\int_{-\infty}^{t} \gamma_1^4(t-s)x(s)\,\mathrm{d}s}{1 + x(t)^2} - \lambda\big(1 + \sin(2\pi t)\big). \tag{15}$$

In the notation of Eq. (2), as in Example 5.3, we have $g(x, y) = 2x - y/(1 + x^2)$ and $f(t, x) = -1 - \sin(2\pi t)$. Also, Eq. (3) becomes the following system:

$$\begin{cases} \dot{x}_0(t) = x_0(t) - \frac{x_4(t)}{1+x_0(t)^2} - \lambda\big(1 + \sin(2\pi t)\big), \\ \dot{x}_i(t) = x_{i-1}(t) - x_i(t) \quad i = 1, \ldots, 4 \end{cases} \tag{16}$$

Figure 3 shows graphs, against λ, of the sup-norm of elements of the set Y of T-pairs of (16) and of the T-periodic solution of (15).

A slightly more complex example is as follows:

Example 5.5 Let $M = \mathbb{R}^2$, $T = 1$ and consider the two-dimensional system

$$\begin{cases} \dot{x}(t) = y(t) + \int_{-\infty}^{t} \gamma_1^1(t-s)x(s)\,\mathrm{d}s, \\ \dot{y}(t) = x(t) + \lambda\big(3 + 5\sin(2\pi t)x(t)\big). \end{cases} \tag{17}$$

To write this equation in the notation of (2) we set

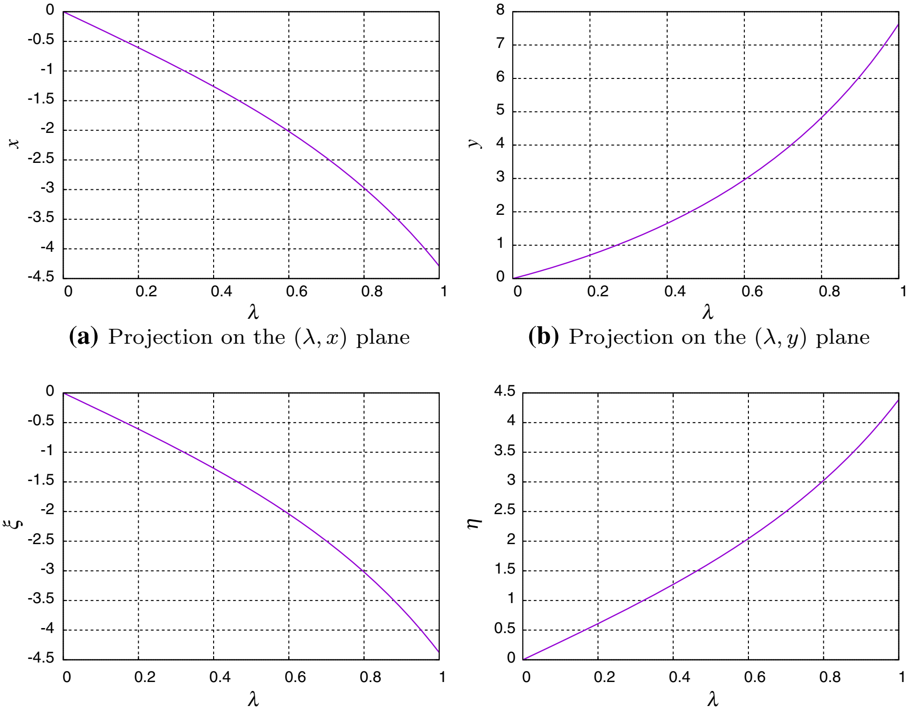
$$g\big((x, y), (\xi, \eta)\big) = \big(y + \xi, x\big) \quad \text{and} \quad f(t, x, y) = \big(0, 3 + 5\sin(2\pi t)x\big).$$

Clearly $\hat{g}(x, y) = g\big((x, y), (x, y)\big) = (x + y, x)$ that has degree 1 in M.

Equation (3) becomes, for this example, the following four-dimensional system:

$$\begin{cases} \dot{x}(t) = y(t) + \xi(t), \\ \dot{y}(t) = x(t) + \lambda\big(3 + 5\sin(2\pi t)x(t)\big), \\ \dot{\xi}(t) = x(t) - \xi(t), \\ \dot{\eta}(t) = y(y) - \eta(t). \end{cases} \tag{18}$$

The set S of the starting points of (18) is a five-dimensional object. In Fig. 4 we show projections of S on the planes (λ, x), (λ, y), (λ, ξ), (λ, η). Figure 5 compares the set Y and



(a) Projection on the (λ, x) plane

(b) Projection on the (λ, y) plane

(c) Projection on the (λ, ξ) plane

(d) Projection on the (λ, η) plane

Fig. 4 Projections of a portion of S of Example 5.5

X, of T-pairs respectively of (18) and (17) by showing how the sup-norm and the orbit's diameter of the elements vary with λ for the two equations.

6 Perspectives and Further Developments

In this section we summarize some possible extensions, in different directions, of the results of the present paper.

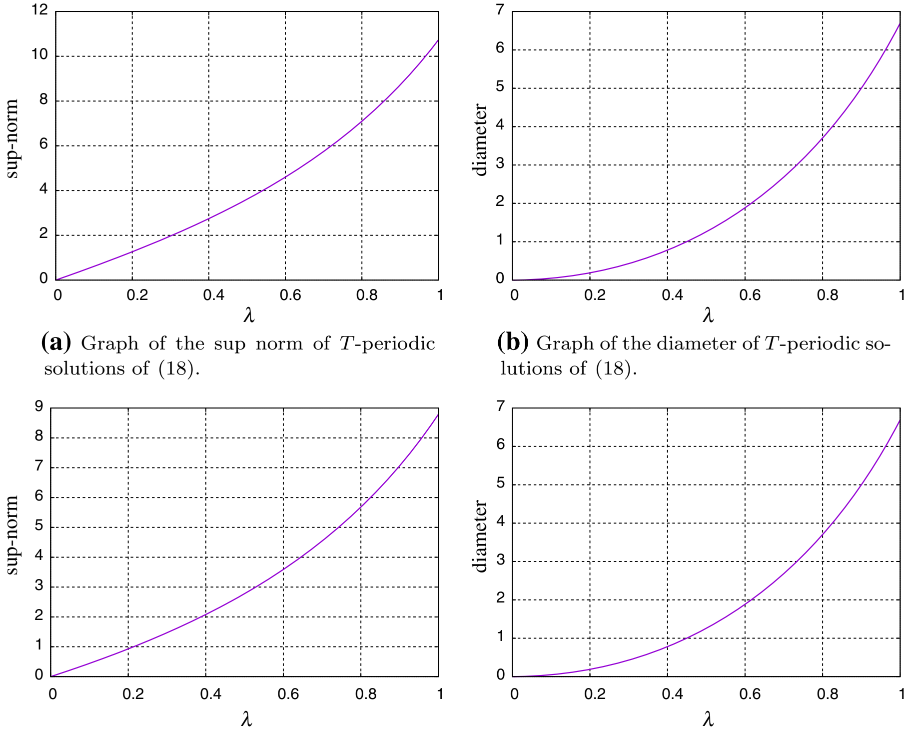
Let us now consider Eq. (1) subject to a perturbation that may depend on a fixed delay $r > 0$. Namely we consider the following differential equation on M:

$$\dot{x}(t) = g\left(x(t), \int_{-\infty}^{t} \gamma_a^b(t-s)x(s)\,\mathrm{d}s\right) + \lambda f\big(t, x(t), x(t-r)\big), \tag{19}$$

where g is as in (1) and $f\colon \mathbb{R} \times M \times M \to \mathbb{R}^k$ is such that $f(t, p, q) \in T_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$ and T-periodic in t. A T- periodic solution of (19) is an M-valued C^1 function on $\mathbb{R}$ that satisfies (19).

Since the "linear chain trick" discussed in Sect. 3 is applied only to the unperturbed part, we can proceed as in the previous paragraphs ending up with the following equation, similar to (3), on $M \times \mathbb{R}^{kb}$:

$$\dot{\xi}(t) = G\big(\xi(t)\big) + \lambda F\big(t, \xi(t), \xi(t-r)\big), \tag{20}$$



(a) Graph of the sup norm of T-periodic solutions of (18).

(b) Graph of the diameter of T-periodic solutions of (18).

(c) Graph of the sup norm of T-periodic solutions of (17).

(d) Graph of the diameter of T-periodic solutions of (17).

Fig. 5 Comparison between the sets of T-periodic solutions of (18) and of (17). The behavior of the sup-norm and of the diameter of the orbit is similar, but the values are quite different

where $F\colon \mathbb{R} \times M \times M \times \mathbb{R}^{kb} \to \mathbb{R}^{k(b+1)}$ is given by

$$F(t, x_0, y, x_1, \dots, x_b) = \big(f(t, x_0, y), 0, \dots, 0\big).$$

In order to prove a version of Theorem 4.6 for Eq. (19) we need to replace Theorem 4.2 (that above was applied to (3)) with the application of Theorem 5.1 of [14] to Eq. (20).

A similar strategy applies if one allowes functional retarded perturbations of (1). Namely, if we consider equations of the following type:

$$\dot{x}(t) = g\left(x(t), \int_{-\infty}^{t} \gamma_a^b(t-s)x(s)\,\mathrm{d}s\right) + \lambda f\big(t, x_t\big), \tag{21}$$

where $f\colon \mathbb{R}\times BU\big((-\infty, 0], M\big) \to \mathbb{R}^k$ is locally Lipschitz in the second variable, T-periodic in the first variable and such that $f(t, \varphi) \in T_{\varphi(0)}M$ for all $(t, \varphi) \in \mathbb{R} \times BU\big((-\infty, 0], M\big)$. Here $BU\big((-\infty, 0], M\big)$ denotes the metric subspace consisting of the M-valued functions, of the space of uniformly continuous bounded functions from $(-\infty, 0]$ into $\mathbb{R}^k$, with the supremum norm. As usual in the retarded functional differential equations context, given $t \in \mathbb{R}$, $x_t \in BU\big((-\infty, 0], M\big)$ is the function $\theta \mapsto x(t+\theta)$. Suppose also that g is locally Lipschitz. Notice that here we assume some regularity on f and g in contrast to the fixed-delay and no-delay case where f and g can be taken merely continuous. As above, a T-periodic solution of (21) is an M-valued C^1 function on $\mathbb{R}$ that satisfies (21).

With the linear chain trick, (21) can be written as

$$\dot{\xi}(t) = G\big(\xi(t)\big) + \lambda F\big(t, \xi_t\big). \tag{22}$$

As above, in order to prove a version of Theorem 4.6 for Eq. (21) one needs to replace Theorem 4.2 above (this is where the extra regularity assumptions are needed) with Theorem 4.1 of [12].

Let us now describe a different potential generalization that, this time, involves the unperturbed term. Consider the case when the manifold M is actually the Cartesian product of two manifolds: $M = M_1 \times M_2$ with $M_1 \subseteq \mathbb{R}^{k_1}$ and $M_2 \subseteq \mathbb{R}^{k_2}$, $k_1 + k_2 = k$, and assume that is "degenerate" in the sense that the first component of g on $T_{(x,y)}M$ vanishes identically, i.e., for all (x, ξ) and (y, η) in $M_1 \times M_2$ we have

$$g\big((x, \xi), (y, \eta)\big) = \Big(0, g_2\big((x, \xi), (y, \eta)\big)\Big) \in T_{(x,\xi)}M = T_x M_1 \times T_\xi M_2.$$

Of course, unless the manifold M_1 is compact, Theorem 4.6 cannot be applied if g is of the above form because $\hat{g}$ cannot be admissible. To remedy this undesirable fact one can apply the construction of [20] to the vector field G obtained by the linear chain trick for such a g. Doing so, one gets a version of Theorem 4.6 that involves the degree of a vector field that include some averaging of the forcing term.

We will not deal further on the last generalization. We only say that the main reason to contemplate this kind of extension is to bridge a gap existing between the kind of "branching" results from zeros of admissible vector fields, like those considered in the present paper, and existing "bifurcation" results that consider perturbations of the zero vector field; see [20] for a discussion.

We conclude this section with an observations that can add a new perspective to the results of this paper. Given $r > 0$, it can be shown (see e.g. [19]) that for any bounded continuous function y we have that for $n \to \infty$

$$\int_{-\infty}^{t} \gamma_{n/r}^{n}(t - s) y(s)\, \mathrm{d}s = \int_{0}^{+\infty} \gamma_{n/r}^{n}(s) y(t - s)\, \mathrm{d}s \longrightarrow y(t - r).$$

Thus, for $n \in \mathbb{N}$, Eq. (2) for $a_n = n/r$ and $b_n = n$ tends to

$$\dot{x}(t) = g\big(x(t), x(t - r)\big) + \lambda f\big(t, x(t)\big)$$

as $n \to \infty$, hinting at the possibility of using the methods of [13] and [14] along with the results of this paper to treat the case of periodic perturbations of an equation with delay. A feat which is not possible with merely the techniques of those two papers.

7 Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2 and to the necessary notions of differential topology.

We begin with the notion of transversality. Let $\mathcal{M}, \mathcal{N}$ be a differentiable manifolds and let $\mathcal{Z}$ be a boundaryless submanifold of $\mathcal{N}$. We say that a differentiable map $\phi\colon \mathcal{M} \to \mathcal{N}$ is *transversal* to $\mathcal{Z}$ if, for each $p \in \phi^{-1}(\mathcal{Z})$, we have

$$d\phi_p\big(T_p\mathcal{M}\big) + T_{\phi(p)}\mathcal{Z} = T_{\phi(p)}\mathcal{N}.$$

In this case we write $\phi \pitchfork \mathcal{Z}$. A well-known differential topology Theorem (see e.g. [15, Ch. 2, §4]) implies that if ϕ is transversal to $\mathcal{Z}$ the $\phi^{-1}(\mathcal{Z})$ is a differentiable submanifold of $\mathcal{M}$ with the same codimension (in $\mathcal{M}$) of $\mathcal{Z}$ in $\mathcal{N}$.

A tangent vector field $\omega\colon \mathcal{M} \to \mathbb{R}^s$ on the manifold $\mathcal{M} \subseteq \mathbb{R}^s$ can be regarded as the map $\tilde{\omega}\colon \mathcal{M} \to T\mathcal{M}$ on the tangent bundle

$$T\mathcal{M} = \big\{(p, v) \in \mathbb{R}^{2s} : p \in \mathcal{M} \text{ and } v \in T_p\mathcal{M}\big\} \subseteq \mathbb{R}^{2s},$$

given by $\tilde{\omega}(p) = \big(p, \omega(p)\big)$. The manifold $\mathcal{M}$ itself can be identified with the so called *zero-section of* $T\mathcal{M}$, that is with the submanifold $\mathcal{M}_0$ of $T\mathcal{M}$ given by $\mathcal{M}_0 = \big\{(p, 0) \in T\mathcal{M}\big\}$. Clearly $T_{(p,0)}\mathcal{M}_0 = T_p\mathcal{M} \times \{0\}$, and one can prove that at any point $(p, 0) \in \mathcal{M}_0$, we have $T_{(p,0)}T\mathcal{M} = T_p\mathcal{M} \times T_p\mathcal{M} \subseteq \mathbb{R}^{2s}$.

Take any $p \in \omega^{-1}(0)$ or, equivalently $p \in \tilde{\omega}^{-1}(\mathcal{M}_0)$. Since (see e.g. [17]) $d\omega_p$ maps $T_p\mathcal{M}$ into itself, we have $d\tilde{\omega}_p(v) = \big(v, d\omega_p(v)\big)$ for all $v \in T_p\mathcal{M}$.

With the above notation, let $\mathcal{Z} = \mathcal{M}_0$ and $\mathcal{N} = T\mathcal{M}$. Suppose that $\omega \pitchfork \mathcal{M}_0$, that is $\omega \pitchfork \mathcal{Z}$. Then, $\tilde{\omega}^{-1}(\mathcal{Z}) = \tilde{\omega}^{-1}(\mathcal{M}_0) = \omega^{-1}(0)$ is a submanifold of $\mathcal{M}$ whose codimension is is equal to the codimension of $\mathcal{Z} = \mathcal{M}_0$ in $\mathcal{N} = T\mathcal{M}$ which is equal to the dimension of $\mathcal{M}$. Hence, $\omega^{-1}(0)$ is a zero-dimensional manifold, in other words any zero p of ω is isolated and,

$$d\tilde{\omega}_p\big(T_p\mathcal{M}\big) + T_{(p,0)}\mathcal{M}_0 = T_{(p,0)}T\mathcal{M},$$

that is

$$T_p\mathcal{M} \times d\omega_p\big(T_p\mathcal{M}\big) + T_p\mathcal{M} \times \{0\} = T_p\mathcal{M} \times T_p\mathcal{M}.$$

Thus, if $\omega \pitchfork \mathcal{M}_0$, then $d\omega_p$ is surjective. The above relation shows that also the converse is true. Recalling that an isolated zero $p_0 \in \mathcal{M}$ is said to be *nondegenerate* if the differential $d_{p_0}\omega$ of ω at p_0 is surjective. we have that $\omega \pitchfork \mathcal{M}_0$ is equivalent to the statement that all zeros of ω are nondegenerate.

The so-called Transversality Theorem (see, e.g., [15,16]) shows the genericity of the tangent vector fields transversal to $\mathcal{M}_0$. One can easily deduce the following fact:

Lemma 7.1 *Let $\omega\colon \mathcal{M} \to \mathbb{R}^s$ be a tangent vector field on the manifold $\mathcal{M} \subseteq \mathbb{R}^s$, and let $W \subseteq \mathcal{M}$ be open and relatively compact. Assume $\omega^{-1}(0) \cap \overline{W} \subseteq W$. Then, given any $\varepsilon > 0$, there exists a tangent vector field $\omega_\varepsilon\colon \mathcal{M} \to \mathbb{R}^s$ with the following properties:*

(1) All of its zeros in W are nondegenerate;
(2) $\max_{p\in\overline{W}} \big\|\omega_\varepsilon(p) - \omega(x)\big\| < \varepsilon$.

We point out that, in the case when $\mathcal{M}$ is an open subset of $\mathbb{R}^s$ Lemma 7.1 can be quickly deduced from Sard's Lemma.

The following proposition shows that an admissible tangent vector field can always be replaced with one whose zeros are all nondegenerate without altering its degree.

Proposition 7.2 *Suppose $\omega\colon \mathcal{M} \to \mathbb{R}^s$ is a tangent vector field on the manifold $\mathcal{M} \subseteq \mathbb{R}^s$, admissible for the degree on an open subset $V \subseteq \mathcal{M}$. Then there exists a vector field $\nu\colon \mathcal{M} \to \mathbb{R}^s$ with the property that all of its zeros in V are nondegenerate and such that* $\deg(\omega, V) = \deg(\nu, V)$.

Sketch of the proof. Let $W_0 \subseteq V$ be a relatively compact open neighborhood of $\omega^{-1}(0) \cap V$. By the Excision property,

$$\deg(\omega, V) = \deg(\omega, W_0). \tag{23}$$

Let W_1 a "larger" relatively compact open neighborhood of $\omega^{-1}(0) \cap V$ in the sense that $\omega^{-1}(0) \cap V \subseteq W_0 \subseteq \overline{W}_0 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V$, and let

$$\delta := \min_{p \in \overline{W}_1 \setminus W_0} \|\omega(p)\|.$$

By a standard partition of unity argument one can show that there exists a continuous function $\sigma \colon \mathcal{M} \to [0, 1]$ such that $\sigma(p) = 1$ for all $p \in \overline{W}_0$ while it evaluates identically to 0 in $\mathcal{M} \setminus W_1$ (this fact is sometimes called the *Urysohn Lemma*). In Lemma 7.1 take $\varepsilon = \delta/2$ and consider the tangent vector field $\nu \colon \mathcal{M} \to \mathbb{R}^s$ given by

$$\nu(p) = \sigma(p)\omega_\varepsilon(p) + \big(1 - \sigma(p)\big)\omega(p), \quad p \in \mathcal{M},$$

where ω_ε is the vector field given by Lemma 7.1. Observe that ν coincides with ω in $\mathcal{M} \setminus W_1$ and is equal to ω_ε in $\overline{W}_0$. Also, we have

$$\min_{p \in \overline{W}_1 \setminus W_0} \|\nu(p)\| \geq \Big| \|\omega(p)\| - \sigma(p)\|\omega(p) - \omega_\varepsilon(p)\| \Big| \geq \frac{\delta}{2} > 0. \tag{24}$$

Thus $\nu^{-1}(0) \cap V \subseteq W_0$ so that, by excision,

$$\deg(\nu, W_0) = \deg(\nu, V). \tag{25}$$

Observe also that inequality (24) implies that the map

$$H(\lambda, p) := \lambda\omega(p) + (1 - \lambda)\nu(p)$$

establishes an admissible homotopy in W_0 between ω and ν. Hence

$$\deg(\omega, W_0) = \deg(\nu, W_0). \tag{26}$$

The assertion follows combining the identities (23), (25) and (26). □

Now that the preliminary facts have been established, we concentrate on the proof of Theorem 2.2. Let $M \subseteq \mathbb{R}^k$ be as in (2). Our first step is to simplify the computation of the degree of the tangent vector vector field G on the manifold $\mathcal{M} = M \times \mathbb{R}^{kb}$ defined in (5) by introducing the following homotopic tangent vector field on $\mathcal{M}$:

$$\widehat{G}(x_0, x_1, \ldots, x_b) := \big(\hat{g}(x_0), a\,(x_0 - x_1), \ldots, a\,(x_{b-1} - x_b)\big), \tag{27}$$

for all $(x_0, x_1, \ldots, x_b) \in M \times \mathbb{R}^{kb}$.

Lemma 7.3 *Suppose $\hat{g}$ defined in (6) is admissible for the degree in an open set $U \subseteq M$, then G and $\widehat{G}$ are admissibly homotopic in $U \times \mathbb{R}^{kb}$.*

Proof For $(\lambda, x_0, \ldots, x_b) \in [0, 1] \times U \times \mathbb{R}^{kb}$, consider the map

$$\mathcal{H}(\lambda, x_0, \ldots, x_b) = \Big(g\big(x_0, \lambda x_0 + (1 - \lambda)x_b\big), a\,(x_0 - x_1), \ldots, a\,(x_{b-1} - x_b)\Big),$$

where g is as in (2). As in Remark 2.1, we have that $\mathcal{H}(\lambda, x_0, \ldots, x_b) = 0$ if and only if $x_0 = x_1 = \ldots = x_b$ and $g(x_0, x_0) = \hat{g}(x_0) = 0$. Thus, $\mathcal{H}(\lambda, \cdot, \ldots, \cdot)$ is equal to the compact set $\big\{(x_0, \ldots, x_0) \in M \times \mathbb{R}^{kb} : x_0 \in \hat{g}^{-1}(0)\big\}$ for $\lambda \in [0, 1]$. Hence $\mathcal{H}$ is an admissible homotopy. □

We now show that whenever the zeros of $\hat{g}$ are nondegenerate, so are those of $\widehat{G}$.

Lemma 7.4 *If all zeros of $\hat{g}$ are nondegenerate, then all zeros of $\widehat{G}$ are nondegenerate as well.*

Proof We need to prove that for any $(x_0, \dots, x_b) \in \widehat{G}^{-1}(0)$ the differential

$$d\widehat{G}_{(x_0,\dots,x_b)} \colon T_{(x_0,\dots,x_b)}(M \times \mathbb{R}^{kb}) \to T_{\widehat{G}(x_0,\dots,x_b)}(M \times \mathbb{R}^{kb}).$$

is surjective. As in the proof of Lemma 7.3 we have that $(x_0, \dots, x_b) \in \widehat{G}^{-1}(0)$ if and only if $x_0 = x_1 = \dots = x_b$ and $\hat{g}(x_0) = 0$. Hence, what we actually have to prove is the surjectivity of

$$d\widehat{G}_{(x_0,\dots,x_0)} \colon T_{(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb}) \to T_{\widehat{G}(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb}).$$

In other words, given $w = (w_0, \dots, w_b) \in T_{\widehat{G}(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb})$ we need to exhibit a vector $\bar{v} = (\bar{v}_0, \dots, \bar{v}_b) \in T_{(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb})$ such that

$$d\widehat{G}_{(x_0,\dots,x_0)}(\bar{v}) = w. \tag{28}$$

Observe that for any $(v_0, v_1, \dots, v_b) \in T_{(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb})$ we have

$$d\widehat{G}_{(x_0,\dots,x_0)}(v_0, v_1, \dots, v_b) = \big(d\hat{g}_{x_0}(v_0), a\,(v_0 - v_1), \dots, a\,(v_{b-1} - v_b)\big).$$

Take any $w = (w_0, \dots, w_b) \in T_{\widehat{G}(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb})$. Since x_0 is a nondegenerate zero of $\hat{g}$, $d\hat{g}_{x_0}$ is surjective. Let $\bar{v}_0 \in T_{x_0}M$ be such that $d\hat{g}_{x_0}\bar{v}_0 = w_0$, and set, for $i = 1, \dots b$, $\bar{v}_i = \bar{v}_{i-1} - w_i \in \mathbb{R}^k$. Then, $\bar{v} = (\bar{v}_0, \dots, \bar{v}_b) \in T_{(x_0,\dots,x_0)}(M \times \mathbb{R}^{kb})$ satisfies (28). □

Let us now compute the degree of the tangent vector field $\widehat{G}$ in terms of that of $\hat{g}$ when all zeros of the latter field are nondegenerate.

Lemma 7.5 *Assume $\hat{g}$ is admissible on an open set $U \subseteq M$ and all its zeros are nondegenerate, then $\widehat{G}$ is admissible in $U^* = U \times \mathbb{R}^{kb}$, and*

$$\deg(\widehat{G}, U^*) = (-1)^{kb} \deg(\hat{g}, U).$$

Proof As in the proofs of Lemmas 7.4 and 7.3 we have that all zeros of $\widehat{G}$ are of the form $(x_0, \dots, x_0)$ with $\hat{g}(x_0) = 0$. Also, by Lemma 7.4, they are all nondegenerate. In particular, if $\hat{g}$ is admissible in U then $\hat{g}^{-1}(0) \cap U$ is compact and so is $\widehat{G}^{-1}(0) \cap U^*$, whence the admissibility of $\widehat{G}$ in U^*.

Let $x_0 \in M$ be such that $\hat{g}(x_0) = 0$ and let $V \subset M$ be an isolating neighborhood. Then $V^* = V \times \mathbb{R}^{kb}$ is an isolating neighborhood of $(x_0, \dots, x_0) \in \widehat{G}^{-1}(0) \subseteq M \times \mathbb{R}^{kb}$. We have, in block-matrix form

$$\det\big(d\widehat{G}_{(x_0,\dots,x_0)}\big) = \det \begin{pmatrix} d\hat{g}_{x_0} & 0 & \dots\dots & & 0 \\ I & -I & 0 & \dots & 0 \\ 0 & I & -I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I & -I \end{pmatrix} = (-1)^{kb} \det\big(d\hat{g}_{x_0}\big),$$

where I denotes the identity matrix in $\mathbb{R}^{k\times k}$. Thus, by formula (4),

$$\begin{aligned} \deg(\widehat{G}, U^*) &= \sum_{x_0 \in \hat{g}^{-1}(0)\cap U} \operatorname{sign}\det\big(d\widehat{G}_{(x_0,\dots,x_0)}\big) \\ &= \sum_{x_0 \in \hat{g}^{-1}(0)\cap U} (-1)^{kb} \operatorname{sign}\det\big(d\hat{g}_{x_0}\big) = (-1)^{kb} \deg(\hat{g}, U), \end{aligned}$$

whence the assertion. □

The proof of Theorem 2.2 now follows readily:

Proof of Theorem 2.2 By Proposition 7.2 we can assume that all zeros of $\hat{g}$ are nondegenerate. The assertion follows from Lemma 7.5.

Funding Open Access funding provided by Universitá degli Studi di Firenze

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