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Hecke Algebras and Combinatorics of Partitions

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Introduction

Symmetry, in addition to its natural influence on human mental processes, represents one of the key concepts in mathematics. The concept of a *group* originated with the intention of describing the set of symmetries of an object standing in particular relationship with each other. One example - one that the author favors - is the *symmetric group* S_n of all permutations of the first n natural numbers. It was during an epistolary conversation between Dedekind and Frobenius at the end of the 19th century that a new perspective on group theory emerged: *representation theory*. The insight, whose usefulness was soon questioned by prominent mathematicians like Burnside, lies in translating the often rather mysterious group structure into a language that, by the end of the 1800s, was far more comfortable - linear algebra. Roughly speaking, given a group G and a field $\mathbb{F}$, a representation of G over $\mathbb{F}$ is a vector space V together with an action of the group G. Thus, one associates to every element of G an endomorphism of V that, under specific conditions, can be translated in matrix language. In fact, when we talk about the representation of a group, we refer to a richer structure known as the *group algebra*. This object, denoted by $\mathbb{F}G$, contains all formal linear combinations of the elements of G with coefficients in $\mathbb{F}$. Initially, the focus was on the field of complex numbers; in this case, we talk about *ordinary* representation theory. It was especially with Richard Brauer in the 1930s that the perspective expanded to finite fields of positive characteristic, giving rise to *modular* representation theory. In 1963, Brauer himself proposed a list of problems [Br], many of which remain unsolved and occupy a large number of researchers.

It was in the second half of the 20th century that the utility and power of the new approach became evident. One example? The proof of Fermat's Last Theorem by Andrew Wiles, a problem that remained open for over 300 years, makes consistent use of Brauer's modular theory. Thus, representation theory started permeating various and transversal fields of mathematics, as well as physics and chemistry.

More generally, one can consider the representation theory of every $\mathbb{F}$-algebra A following the same idea as above, i.e. considering vector spaces on which A acts. In this document we will prefer using the more common term *A-module* to refer to a representation of A. One of the problems that arose with the advent of the theory and remains relevant today is the description of all possible representations of an algebra, particularly those we will call *simple*: once this collection is described, every representation is derived as a linear combination of simple representations. Another central problem is the study of the decomposition matrix, which we will return to shortly.

The starting point of our narrative is about 80 years from Frobenius's early works. In 1978, Gordon James published his book *"Representation Theory of Symmetric Groups"*, a work destined to influence generations of mathematicians up to the present day. The insight is as follows: the conjugacy classes of S_n are naturally in bijection with the partitions of the integer n. It is a general fact that the number of conjugacy classes of a finite group gives the

number of ordinary simple representations. With this in mind, for every partition of n, James defined a $\mathbb{F}S_n$-module, called *Specht*, and the set of Specht modules turns out to be crucial in the representation theory of S_n both in the ordinary and modular case. In characteristic zero, the Specht modules comprise a complete set of pairwise non-isomorphic simple modules; in positive characteristic an analogous set is found considering the (unique) head of the Specht modules labelled by the so-called *regular partitions*. The decomposition number problem for S_n then asks for the multiplicities of the simple modules as composition factors of the Specht ones. In Chapter 1 we will provide some elements of the theory recalling the *block theory* of S_n, i.e. the study of the decomposition of the algebra $\mathbb{F}S_n$ into minimal bilateral ideals. Since $\mathbb{F}S_n$ is cellular (in the sense of [GL]) with the Specht modules being the cell modules, in order to detect the blocks it is sufficient to see whether two Specht modules are modules for the same block. There are several characterization of this condition; some of these are mentioned in Section 1.4.

The representation theory of the symmetric group is one of the brightest examples of algebraic combinatorics, the interaction and mutual description of algebraic and combinatorial objects. Later, new perspectives and generalizations emerged, particularly with the study of the representations of the *Hecke algebras*, i.e., deformations of group algebras of Coxeter groups (of which S_n is a basic example). We refer to the *Iwahori–Hecke algebra of type A* for the Hecke algebra of S_n.

This thesis aims to analyze two types of algebras from a combinatorial perspective. Part I concerns the *centraliser algebra of symmetric groups*. Given non-negative integers $m > l$, we can see the symmetric group S_l as a subgroup of S_m, and then we can consider the set $C_{l,m}^{\mathbb{F}}$ as the subalgebra of $\mathbb{F}S_m$ whose elements commute with S_l. We first outline a strategy that involves Specht modules. Having two integers at hand, a suitable labelling set turns out to be the set of *skew-partitions* (Section 2.1 below). The aim is to characterize the blocks of $C_{l,m}^{\mathbb{F}}$. In this direction, at the end of Chapter 2, we give a conjecture (Conjecture 2.4.1) inspired by the Nakayama conjecture (Theorem 1.4.1) that characterizes the blocks of S_n in term of cores. With our *conjectural blocks* in mind, we develop a combinatorial strategy, following [EM2], to show that the conjecture holds in some special cases. This is the content of Chapter 3.

Part II focuses on *Ariki–Koike algebras*. These algebras can be seen as deformations of Coxeter groups of the form $C_r \wr S_n$ ([AK]). In this more general setting, the Iwahori–Hecke algebra of type A is just a basic case of Ariki–Koike algebra. For such algebras, there is still a strong combinatorial background. This concerns integer *multipartitions* that again serve as a labelling set for a family of Specht modules. In this context, a characterization of blocks is already known (Proposition 4.4.1). Starting from this characterization, Fayers [F1] defined the *weight* of a block as an invariant of the block intended to quantify its complexity. Our focus is on the level 2 Ariki–Koike algebra, known as *Iwahori–Hecke algebra of type B*. Denoting it by $\mathcal{H}_n$, our question deals with finding the decomposition numbers, i.e. as before, the multiplicities of the simple $\mathcal{H}_n$-modules as composition factors of the Specht modules. We ask which blocks of $\mathcal{H}_n$ have decomposition numbers equal to 0 or 1. In the symmetric group algebra as well as in type A, the blocks of weight at most 3 all share this property. A similar result was proved in type B by Fayers for blocks of weight at most 2. Here we deal with blocks of weight 3. In Chapter 5 we describe all *bipartitions* (multipartitions in two components which label the Specht modules in type B) in a weight 3 block of $\mathcal{H}_n$; Chapters 6 to 8 contain the proof that all decomposition numbers for these blocks are 0 or 1.

Chapter 1

Background and notation

The first chapter of the present thesis is introductory and serves to recall some basic facts about the representation theory of the symmetric group and of the underpinning combinatorics. This is also the occasion to establish the notation we will use throughout the document. In this thesis $\mathbb{N}$ denotes the set of positive integers. Fixed an integer $e \geqslant 2$, for any integer a, we write $\bar{a} = a + e\mathbb{Z} = \{a + em \mid m \in \mathbb{Z}\}$, and $\mathbb{Z}/e\mathbb{Z} = \{\bar{a} \mid a \in \mathbb{Z}\}$. The set $\mathbb{Z}/e\mathbb{Z}$ admits an additive action of $\mathbb{Z}$ via $a + \bar{b} = \overline{a+b}$. In general, given any set $\{x_i \mid i \in \mathbb{Z}/e\mathbb{Z}\}$ of objects indexed by $\mathbb{Z}/e\mathbb{Z}$, and given an integer a, we may write x_a to mean $x_{\bar{a}}$. Similarly, given any terminology involving an element of $\mathbb{Z}/e\mathbb{Z}$ (such as 'i-node'), we may substitute an integer a in place of $\bar{a}$ (so we may define an a-node to mean an $\bar{a}$-node).

1.1 Partitions

A *partition* is a weakly-decreasing sequence $\nu = (\nu_1, \nu_2, \dots)$ of non-negative integers such that $\nu_N = 0$ for $N \gg 0$. The integer ν_k is called the k-th *part* of ν and the number of non-zero parts is referred as the *length* of ν and denoted by $h(\nu)$. The (finite) sum $\nu_1 + \cdots + \nu_{h(\nu)}$ is referred as the *size* of ν and denoted by $|\nu|$. If $n \geqslant 0$ and ν is a partition of size n, we say that ν *is a partition of* n and we write $\nu \vdash n$. We denote by $\varnothing$ the unique partition of 0. When writing partitions we omit the trailing zeroes and group together equal parts with a superscript, e.g. $(3, 2^3, 1)$ stands for the length 5 partition $(3, 2, 2, 2, 1, 0, \dots) \vdash 10$. If ν is a partition, the *conjugate* partition ν' is defined by

$$(\nu')_r = |\{c \in \mathbb{N} \mid \nu_c \geqslant r\}|,$$

for every $r \geqslant 0$. For example, the conjugate of $(3, 2^3, 1)$ is the partition $(5, 4, 1)$.

We now briefly recall some facts about the representation theory of the symmetric group. A fundamental reference for the subject is [J]. Let $\mathcal{P}$ denote the set of all partitions and $\mathcal{P}_n$ the set of partitions of a fixed non-negative integer n. The set of partitions $\mathcal{P}_n$ is the labelling set for a relevant class of modules for the symmetric group algebra known in the literature as *Specht modules*. Let $\mathbb{F}$ be a field. The set of Specht modules $\{\mathrm{S}^\nu \mid \nu \in \mathcal{P}_n\}$ is a crucial object in the study of the representation theory of the symmetric group S_n because it provides a complete set of pairwise non-isomorphic simple $\mathbb{F}S_n$-modules whenever $\mathbb{F}$ has characteristic zero or bigger than n. When $\mathbb{F}$ has positive characteristic p, a complete family of pairwise non-isomorphic simple $\mathbb{F}S_n$-modules can be constructed from the Specht modules: if μ is a *p-regular* partition (meaning that it does not have p equal positive parts), then the Specht module S^μ has a unique simple head D^μ, and the simple modules obtained in this way give

a complete set of pairwise non-isomorphic simple $\mathbb{F}S_n$-modules. If $\mathcal{P}_n^{\text{reg}}$ denotes the set of p-regular partitions of n, the *decomposition matrix of* S_n is the matrix

$$(d_{\lambda\mu})_{\lambda\in\mathcal{P}_n,\mu\in\mathcal{P}_n^{\text{reg}}}$$

where $d_{\lambda\mu}$ denotes the multiplicity of the simple module D^μ as a composition factor of the Specht module S^λ. The computation of the *decomposition numbers* is a widely studied topic in algebraic combinatorics and several results, some of which are mentioned along the treatment, have been obtained in this direction.

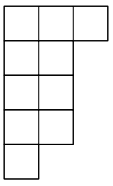
Analogue modules and matrices turn up when studying the *Iwahori–Hecke algebra of type A*. If q denotes an invertible element of the ground field, this is a *q-deformation* of the group algebra of the symmetric group - that, in fact, arises when specializing q to 1. These two algebras, from a representation theoretical point of view, as well as from a combinatorial one, behave really similarly. A reference for this subject and more generally for *cellular algebras* is [M]. Please note that, in Mathas' book, the Specht modules are the *dual* of James' ones. In this setting, the simple modules are labelled by *restricted* partitions we will define in Part II. The decomposition numbers problem for the symmetric group algebra and the type A algebra, in particular, have a strong connection: it goes back to Ariki [A2] that the above $d_{\lambda\mu}$ is the specialization at $q = 1$ of the decomposition number $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$ for the type A algebra over a field of characterisitc 0. Over fields of arbitrary characteristic, the decomposition matrices of the Hecke algebras are connected to the characteristic 0 case via *adjustment matrices* (see [M, Chapter 6] for further details). In Part II we will come back to this topic putting our attention on *level 2* Iwahori–Hecke algebras.

1.2 Young diagrams and standard tableaux

We now recall a good method for visualising partitions. The *Young diagram* of a partition ν is the set

$$[\nu] = \{(r,c) \in \mathbb{N}^2 \mid c \leqslant \nu_r\}.$$

The elements of $[\nu]$, and more generally of $\mathbb{N}^2$, are called *nodes*. We draw a Young diagram as an array of boxes in the plane such that the horizontal axis is oriented left-to-right and the vertical axis top-to-bottom (commonly referred as the *English* notation). For example, $[(3,2^3,1)]$ appears as follows.



Motivated by the drawing, we say that the node (r,c) is *above* the node (s,d), or that (s,d) is *below* (r,c), if $r < s$. Analogously, we say that (r,c) lies *to the left* of (s,d), or that (s,d) lies *to the right* of (r,c), whenever $c < d$.

Fix e is a positive integer. The *content* of a node (r,c) is the integer $c - r$, and the *e-residue* of (r,c) is $\overline{c-r}$. We just say residue if e is clear from the context. Note that nodes lying in the same NW to SE diagonal of $\mathbb{N}^2$ have the same content and therefore the same e-residue for all e. If $\nu \in \mathcal{P}$, we define the *e-content* and we write $\text{cont}_e(\nu)$, for the multiset of e-residues (modulo e) of the nodes in $[\nu]$. For example,

$$\text{cont}_3(3,2^3,1) = \{\overline{-4},\overline{-3},\overline{-2},\overline{-2},\overline{-1},\overline{-1},\bar{0},\bar{0},\bar{1},\bar{2}\} = \{\bar{0},\bar{0},\bar{0},\bar{1},\bar{1},\bar{1},\bar{2},\bar{2},\bar{2},\bar{2}\}.$$

The *rim* of a partition ν is the set of nodes $(r,c) \in [\nu]$ such that $(r+1,c+1) \notin [\nu]$. We regard this set as ordered from from SW to NE saying that a node (r,c) comes after (s,d) if (r,c) lies above or to the right of (s,d). Note that this coincides with ordering the nodes in the rim in increasing order of content. For $h \geqslant 1$, we define a *removable h-hook*, sometimes shorten as *rim h-hook* or simply as *rim hook*, of ν to be a set H of h consecutive nodes $(r_1,c_1),\ldots,(r_h,c_h)$ in the rim of ν such that $(r_1,c_1+1) \notin [\nu]$ and $(r_h+1,c_h) \notin [\nu]$. By definition the nodes of a removable h-hook can be removed from $[\nu]$ to leave the Young diagram of a partition of $|\nu|-h$. In particular, if $h=1$, we call a removable 1-hook simply a *removable node*.

Dually, we define the *neighbours* of $[\nu]$ to be the nodes $(s,d) \notin [\nu]$ such that either $s=1$ or $d=1$, or $(s-1,d-1) \in [\nu]$. The neighbours of ν are naturally ordered in increasing order of content. For $h \geqslant 1$, a set H of h consecutive neighbours $(s_1,d_1),\ldots,(s_h,d_h)$ ν is called an *addable h-hook* if $s_1=1$ or $(s_1-1,d_1) \in [\nu]$, and $s_h=1$ or $(s_h,d_h-1) \in [\nu]$. Again we can see that, picturally, an addable h-hook is a series of nodes which can be added to $[\nu]$ to obtain the Young diagram of a partition of $|\nu|+h$. An addable 1-hook is called an *addable node*. If H is either a removable or addable hook, the *foot node* $\mathrm{foot}(H)$ and the *hand node* $\mathrm{hand}(H)$ are the first and last nodes of H, respectively, and the *leg length* $\mathrm{ll}(H)$ is the difference in height between the highest and lowest nodes in H.

We give the last definition of this section. Fix $n \geqslant 0$ and $\nu \in \mathcal{P}_n$. A *ν-tableau* is a bijection $T : [\nu] \to \{1,\ldots,n\}$ represented by filling the nodes of the Young diagram $[\nu]$ with their images under T. A ν-tableau is *standard* if its entries increase along rows and down columns.

Example. Let $\nu = (5,3,2,1) \vdash 11$. The following are two ν-tableaux, the one on the left being standard.

1	3	7	8	10
2	5	9		
4	6			
11				

4	9	2	8	10
11	7	3		
1	6			
5				

The class of standard ν-tableaux plays a central role in the representetion theory of symmetric groups. These tableaux are in bijection with paths from the empty partition $\varnothing$ to ν in the branching graph of symmetric groups. Given a standard ν-tableau T, the corresponding path is inductively found starting from (the Young diagram of) $\varnothing$ and adding the node $T^{-1}(k)$ at the k-th step, for $k=1,\ldots,n$. As a consequence we have that the number of standard ν-tableaux gives the dimension of the Specht module S^ν over any commutative ring; see [OV] for further details and another approach to the representation theory of symmetric groups. This number can be computed directly from the Young diagram $[\nu]$ using the outstanding *Hook-Length Formula* by Frame, Robinson and Thrall. Different proofs of the formula have been given in the 70 years of its existence, see for example [B] for an elementary one.

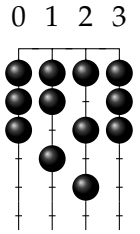
From now on we will abuse notation by omitting square brackets and not distinguishing anymore between partitions and Young diagrams.

1.3 Beta-sets and the abacus

In this section we introduce an idea of James and Kerber [JK] that gives another combinatorial perspective to the objects mentioned in Section 1.1. Fix e a positive integer. We consider an abacus display with e vertical runners labelled from left to right by the symbols $0,\ldots,e-1$. We index the *positions* in the i-th runner by the set $i+e\mathbb{Z} = \{i+ez \mid z \in \mathbb{Z}\}$ such

that position $i + ez$ is immediately above position $i + e(z+1)$. Then we align runners so that position x is immediately to the right of position $x-1$ whenever $e \nmid x$. Each position in the abacus is either a *bead* ● or a *space* ┆.

Now given a partition ν and an integer $C \geqslant h(\nu)$, we define the *beta-set for* λ with charge C to be the set $\{\nu_r + C - r \mid 1 \leqslant r \leqslant C\}$. The *abacus configuration* of ν with charge C is constructed by placing a bead on the abacus at positions marked by the integers in the just defined beta-set. We denote this object by $\mathrm{Ab}_e(\nu)$, or simply $\mathrm{Ab}(\nu)$ if e is clear from the context. We remark that the previous definition does not depend on the choice of C, provided it is big enough: changing the charge means simultaneously rescaling all the positions in the abacus display. Note that, by construction, every runner of $\mathrm{Ab}(\nu)$ has infinitely many consecutive spaces downwards. When we draw an abacus configuration, we will use the convention that all positions below those shown are spaces. The picture below exhibits the abacus configuration of $(7,3,2^2,1^2) \vdash 16$ with charge $C = 12$ (and hence with 12 beads) in an abacus display with $e = 4$ runners.



Given the abacus configuration $\mathrm{Ab}(\nu)$, we can read off the j-th part of ν counting the number of spaces which precede the j-th largest bead position in $\mathrm{Ab}(\nu)$. Note that, by definition, there is a correspondence between the rim of ν and the set of positions $-h(\nu) + C + 1 \leqslant x \leqslant \nu_1 + C - 1$ in $\mathrm{Ab}(\nu)$, where a node (r,c) corresponds to the position marked by $c - r + C$; this position is a bead if (r,c) lies below the next node in the rim of ν, or (r,c) is the last node of the rim, i.e. $r = 1$ and $c = \lambda_1$. Otherwise, this position is a space.

We remark that for $h \geqslant 1$, a removable h-hook arises from a position x such that x is a bead and $x - h$ is a space in $\mathrm{Ab}(\nu)$. The removal of this h-hook from ν corresponds to moving the bead at x into the space at $x - h$. Similarly, adding an addable h-hook corresponds to moving a bead at position x to position $x + h$, for some x. We will say that a removable or addable hook is *at runner* i if the movable bead is at some position $x \equiv i \pmod{e}$. In the abacus configuration above, for example, $(7,3,2^2,1^2)$ has a removable 3-hook at runner 2 since position 18 is a bead and position 15 is a space. Switching these two positions gives the abacus configuration of the partition $(4,3,2^2,1^2) \vdash 13$ obtained by removing nodes $(1,5)$,$(1,6)$ and $(1,7)$ from $(7,3,2^2,1^2)$.

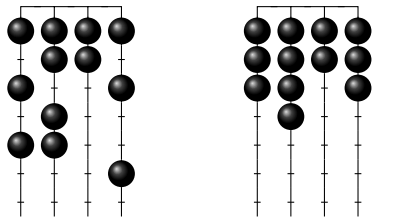
1.4 Cores and blocks

In this section we recall the combinatorics underlying the block theory of the symmetric groups.

Fix an integer $e \geqslant 2$, we say that a partition is an *e-core* if it has no removable e-hooks. Given $\nu \in \mathcal{P}$, we define the *e-core of* ν as the partition obtained by successively deleting removable e-hooks from ν until an e-core is reached. We denote this partition by $\mathrm{core}_e(\nu)$. In terms of the abacus, computing $\mathrm{core}_e(\nu)$ coincides with sliding all beads up as far as possible in the runners of $\mathrm{Ab}_e(\nu)$. Since this process clearly produces a unique abacus configuration, we deduce that the e-core of a partition is well-defined. We define the *e-weight* of ν, writ-

ten weight$_e(\nu)$, to be the number of e-hooks removed in working out core$_e(\nu)$. Note that $|\nu| = |\text{core}_e(\nu)| + e\,\text{weight}_e(\nu)$. For $i = 0, \dots, e-1$, we write $\nu^{(i)}$ for the partition which corresponds to the i-th runner of $\text{Ab}_e(\nu)$ considered as an abacus configuration with one single runner. The e-tuple of partitions $(\nu^{(0)}, \dots, \nu^{(e-1)})$ is called the *e-quotient* of ν. From the abacus interpretation of the weight, we immediately deduce that $\text{weight}_e(\nu) = |\nu^{(0)}| + \cdots + |\nu^{(e-1)}|$.

Example. Let $\nu = (12, 7^2, 5, 4, 2, 1^2) \vdash 39$ and $e = 4$. Then $\text{core}_4(\nu) = (2,1) \vdash 3$, as we see from the following abacus configurations.



We find that $\text{weight}_4(\nu) = 9$, and the 4-quotient of ν is $((2,1), (1^2), \varnothing, (3,1))$.

Now let n be a positive integer and $\mathbb{F}$ a field of prime characteristic p. We consider the *p-blocks* of S_n; these are the indecomposable summands of the algebra $\mathbb{F}S_n$ when written as the direct sum of minimal bilateral ideals. It is a standard fact that each Specht module is a module for a single p-block. Conversely, every p-block contains at least one Specht module (because every simple module is a composition factor of a Specht module). Then a first step in the direction of describing the p-block structure of S_n is to understand whenever two Specht modules lie in the same p-block. The following famous result, known as the *Nakayama conjecture*, gives a combinatorial characterization to understand whenever this happens.

Theorem 1.4.1. *Let $\alpha, \beta \in \mathcal{P}_n$. The Specht modules S^α, S^β belong to the same p-block of S_n if and only if* $\text{core}_p(\alpha) = \text{core}_p(\beta)$.

Theorem 1.4.1 was stated by Nakayama in 1940 and proved independently by Brauer and Robinson. Several proofs are known, see for example [JK, Section 6.1]. Before going on, we dwell a moment on the algebraic impact of Theorem 1.4.1. By Maschke's Theorem [I, Theorem 1.9], the algebra $\mathbb{F}S_n$ is semisimple if and only if $p > n$. If this is the case, Wedderburn's Theorem [I, Theorem 1.15] tells that the p-blocks of S_n are simple algebras, i.e. algebras with no proper non-trivial bilateral ideals, and are in 1:1 correspondence with the set of Specht modules. This means that non-isomorphic Specht modules belong to different p-blocks. This fact can be deduced also from Theorem 1.4.1 whenever $p > n$ noting that every partition of n is a p-core. If $p \leqslant n$, the algebra $\mathbb{F}S_n$ is not semisimple and it comes about that some Specht modules belong to the same p-block. The Nakayama conjecture shows how to quickly recognise whenever this happens. Later on we may abuse notation regarding a p-block B of S_n as the subset of partitions of n which labels the Specht $\mathbb{F}S_n$-modules belonging to B.

Clearly if two partitions of n have the same p-core then they have the same p-weight. Given a p-block B of S_n, we denote by γ_B and w_B, respectively, the p-core and the p-weight of *any* partition in B. It is clear that (the abacus configuration of) all partitions in B can be recovered starting from γ_B and then sliding w_B times beads down in $\text{Ab}_p(\gamma_B)$ in all possible ways.

We now give some more conditions equivalent to those in Theorem 1.4.1. Note that nothing of what follows requires p to be a prime number. We need all the abacus configurations to contain the same number of beads. Since the length of a partition of n is at most n, to ensure this is enough to fix the (common) charge C of the abacus configurations to be at least n. For every $\nu \in \mathcal{P}_n$ and every $i = 0, \dots, p-1$, let $b_i(\nu)$ denote the number of beads in the i-th runner of the abacus configuration $\mathrm{Ab}_p(\nu)$. It turns out that the p-tuple $(b_0(\nu), \dots, b_{p-1}(\nu))$ is another invariant of the p-block to which ν belongs. Actually, by definition, the p-core of a partition is obtained by sliding beads up in its abacus configuration. It follows that the number of beads in every runner remains unchanged at the end of the process. In particular, if α and β are in the same p-block B, then $b_i(\alpha) = b_i(\gamma_B) = b_i(\beta)$ for all $i = 0, \dots, p-1$. (Note that the numbers b_i depend on the choice of the charge; in fact C can be recovered as $b_0 + \cdots + b_{p-1}$. If C is increased by p, then each b_i is increased by 1.)

We end this section by mentioning a further interpretation of the combinatorial condition for two Specht modules to be in the same p-block. The following characterization also generalizes to *higher levels*, as we shall see in Part II. Recall that the p-content of a partition λ is the multiset of p-residues of the nodes of λ. The following result goes back to Littlewood [L].

Theorem 1.4.2. *Suppose $\lambda, \mu \in \mathcal{P}_n$. Then λ and μ have the same p-core if and only if they have the same p-content.*

Part I

Centraliser algebras of symmetric groups

Chapter 2

Centraliser algebras and their combinatorics

Let R be a commutative ring, G be a finite group and $H \leqslant G$. The *centraliser algebra* RG^H is defined by

$$RG^H = \{a \in RG \mid ah = ha \text{ for every } h \in H\}.$$

This algebra appears in a series of papers by Ellers ([E1, E2, E3, E4]) mainly motivated by the attempt of extending some *local-global* conjectures and theorems. Such kinds of statements allow one to detect certain invariants of the group working locally, i.e. considering normalizers of Sylow subgroups. Ellers' focus is especially on the Alperin Weight Conjecture [A]: his aim is to see that the Alperin bijection is in fact a special case of a more general framework. More details and motivations may be found in Ellers' papers. We remark that RG^H is isomorphic in the notation of [CR, Section 11D], to the Hecke algebra $\mathcal{H}(H \times G,\ H \times H,\ \mathbb{1}_{H\times H})$ over R, where $\mathbb{1}_{H\times H}$ denotes the trivial $R(H \times H)$-module. By this isomorphism and the theory in [CR] it follows that if both algebras RG and RH are semisimple, then so is RG^H.

In this thesis we analyse the case where $G = S_m$ and $H = S_l$ with $m > l$. The main goal in this setting is to find a suitable labelling for simple modules of the centraliser algebra and a complete description of its blocks. A strong result would be to give answers to these questions in terms of the combinatorics of integer partitions somehow extending the symmetric group situation we recalled in Chapter 1. At the end of the present chapter we give a conjecture which is intended to characterize the blocks in terms of cores; in Chapter 3 we show the validity of such conjecture for a special family of (conjectural) blocks.

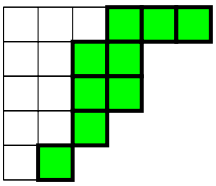
2.1 Skew-partitions

We start this chapter by presenting its combinatorial protagonist. Since the algebra of interest – as we shall see in Section 2.3 – relies on two natural numbers, a starting point is to consider a relation of *containment* between pairs of paritions.

Let $\lambda, \mu \in \mathcal{P}$. We say that λ *lies inside* μ if $[\lambda] \subseteq [\mu]$. In this case we say that $\mu \setminus \lambda$ is a *skew-partition*. We immediately see that, if $\mu \setminus \lambda$ is a skew-partition, then $|\mu| \geqslant |\lambda|$ with equality holding if and only if $\mu = \lambda$. We call the non-negative integer $|\mu| - |\lambda|$ the *size* of $\mu \setminus \lambda$. Given two non-negative integers $l \leqslant m$, we set $\mathcal{P}_{l,m} = \{\mu \setminus \lambda \mid \lambda \in \mathcal{P}_l,\ \mu \in \mathcal{P}_m\}$.

It is useful to visualise skew-partitions as (subsets of) Young diagrams. For a skew-partition $\mu \setminus \lambda$, we define the *skew-diagram* $[\mu \setminus \lambda]$ as the set subset of $\mathbb{N}^2$ comprising nodes

$[\mu] \setminus [\lambda]$. For example, if $\mu = (6, 4^2, 3, 2) \vdash 19$ and $\lambda = (3, 2^3, 1) \vdash 10$, the skew-diagram $[\mu \setminus \lambda]$ comprises the green nodes in the picture below.



Note that two different skew-partitions can have the same Young diagram; for example, $[(3,2) \setminus (3,1)] = [(2^2,1) \setminus (2,1^2)]$.

We remark that if λ lies inside μ, the Young diagram $[\lambda]$ can be recovered by repeatedly deleting removable hooks from $[\mu]$. The union of these hooks comprises the skew-diagram $[\mu \setminus \lambda]$. Note that this series of removals is not unique.

Similarly to the case of partitions, for e a positive integer, we define the *e-content* of a skew-partition $\mu \setminus \lambda$, and we write $\mathrm{cont}_e(\mu \setminus \lambda)$, as the multiset of e-residues of the nodes in $[\mu \setminus \lambda]$, e.g.

$$\mathrm{cont}_4((6,4^2,3,2) \setminus (3,2^3,1)) = \{\overline{-3}, \overline{-1}, \bar{0}, \bar{1}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = \{\bar{0}, \bar{0}, \bar{1}, \bar{1}, \bar{1}, \bar{1}, \bar{2}, \bar{3}, \bar{3}\}.$$

In conclusion to this section, we generalise the ideas of tableau and standard tableau to the case of skew-partitions. If $\mu \setminus \lambda$ is a skew-partition of size n, a *$\mu \setminus \lambda$-tableau* is a bijection $T : [\mu \setminus \lambda] \to \{1, \dots, n\}$ represented by filling the nodes of $[\mu \setminus \lambda]$ with their images under T. A tableau is *standard* if its entries increase along rows and down columns. On the algebraic *fil rouge* of our narration, we mention that also standard *skew-tableaux* appear in the representation of symmetric groups. Actually, if $\mu \setminus \lambda \in \mathcal{P}_{l,m}$, the set of standard $\mu \setminus \lambda$-tableaux is in bijection with the paths between the Specht modules S^λ and S^μ in the branching graph of symmetric groups. The branching rule itself also ensures that if R is a commutative ring, then the R-module

$$\mathrm{S}^{\mu\setminus\lambda} := \mathrm{Hom}_{RS_l}(\mathrm{S}^\lambda, \mathrm{S}^\mu \downarrow_{S_l})$$

is non-trivial if and only if $\mu \setminus \lambda$ is a skew-partition. In this case the dimension of $\mathrm{S}^{\mu\setminus\lambda}$ equals the number of standard $\mu \setminus \lambda$-tableaux; unlike in the case of symmetric groups, no closed formula for this number is known. We remark that the label of a Specht module depends on the partitions, not the skew-diagram.

From now on we will abuse notation by omitting square brackets also between skew-partitions and skew-diagrams.

2.2 The degenerate affine Hecke algebra

The next two sections are devoted to introducing the main algebraic objects of this part of the thesis. Before presenting the algebra which is at the heart, we introduce an auxiliary structure and some interesting related concepts. Let $\mathbb{F}$ be a field and n be a positive integer.

Definition. The *degenerate affine Hecke algebra of degree n over* $\mathbb{F}$, denoted by $\mathcal{H}_n^{\mathbb{F}}$, is the unital, associative $\mathbb{F}$-algebra with generators $z_1, \dots, z_n, s_1, \dots, s_{n-1}$ subject to the following relations:

1. $z_i z_j = z_j z_i$ for all i, j;
2. $(s_i s_j)^{m_{ij}} = 1$, where $m_{ii} = 1$, $m_{i(i+1)} = 3$ and $m_{ij} = 2$ for $|i - j| > 1$;

3. $s_i z_j = z_j s_i$, for $j \neq i, i+1$;

4. $s_i z_i = z_{i+1} s_i - 1$.

Looking at the relation (1), we see that the *polynomial* generators $z_1, \ldots, z_n$ generate a subalgebra $\mathbb{F}[z_1, \ldots, z_n]$ of $\mathcal{H}_n^{\mathbb{F}}$ isomorphic to the polynomial algebra in n indeterminates over $\mathbb{F}$. Moreover, this is a maximal commutative subalgebra of $\mathcal{H}_n^{\mathbb{F}}$. Considering relations in (2), we also observe that the *Coxeter* generators $s_1, \ldots, s_{n-1}$ generate a subalgebra of $\mathcal{H}_n^{\mathbb{F}}$ isomorphic to $\mathbb{F}S_n$. In fact, as $\mathbb{F}$-modules, $\mathcal{H}_n^{\mathbb{F}} \cong \mathbb{F}[z_1, \ldots, z_n] \otimes_{\mathbb{F}} \mathbb{F}S_n$. We remark that the centre $Z(\mathcal{H}_n^{\mathbb{F}})$ of $\mathcal{H}_n^{\mathbb{F}}$ is given by the ring of symmetric polynomials in the indeterminates $z_1, \ldots, z_n$ [K, Theorem 3.3.1].

We now focus on the polynomial subalgebra in the case where $\mathbb{F}$ is algebraically closed. Since $\mathbb{F}[z_1, \ldots, z_n]$ is commutative, every irreducible $\mathbb{F}[z_1, \ldots, z_n]$-module M has dimension one and corresponds (via a 1:1 correspondence) to the n-tuple $(a_1, \ldots, a_n) \in \mathbb{F}^n$ such that z_i acts as the scalar a_i on M. We define the *formal character* of a $\mathbb{F}[z_1, \ldots, z_n]$-module as the formal $\mathbb{Z}$-linear combination of the elements of $\mathbb{F}^n$ corresponding to its composition factors. The *formal character* of a finite dimensional $\mathcal{H}_n^{\mathbb{F}}$-module M is then defined as the formal character of its restriction $M \downarrow_{\mathbb{F}[z_1, \ldots, z_n]}$. More formally, following [K], one identifies $\mathbb{F}[z_1, \ldots, z_n]$ with the *parabolic subalgebra*

$$\mathcal{H}_{(1,\ldots,1)}^{\mathbb{F}} \cong \mathcal{H}_1^{\mathbb{F}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{H}_1^{\mathbb{F}}$$

of $\mathcal{H}_n^{\mathbb{F}}$, and defines the formal character $\mathrm{ch}(M)$ of M as the image of the class $[M]$ in the Groethendieck group $K(\mathcal{H}_n^{\mathbb{F}}\text{-mod})$, under the homomorphism

$$\mathrm{ch} : K(\mathcal{H}_n^{\mathbb{F}}\text{-mod}) \longrightarrow K(\mathcal{H}_{(1,\ldots,1)}^{\mathbb{F}}\text{-mod})$$

induced by the restriction from $\mathcal{H}_n^{\mathbb{F}}$ to $\mathcal{H}_{(1,\ldots,1)}^{\mathbb{F}}$. By [K, Theorem 5.3.1], ch is injective, therefore the composition factors of an $\mathcal{H}_n^{\mathbb{F}}$-module are determined by its formal character.

Before going on we associate another important object to any indecomposable $\mathcal{H}_n^{\mathbb{F}}$-module M. Let $(a_1, \ldots, a_n)$ be a summand in the formal character of M. The *central character* of M is defined as the map $Z(\mathcal{H}_n^{\mathbb{F}}) \to \mathbb{F}$ such that $f \mapsto f(a_1, \ldots, a_n)$ for all $f \in Z(\mathcal{H}_n^{\mathbb{F}})$. Since M is indecomposable, all summands of $\mathrm{ch}(M)$ lie in the same S_n-orbit (by [K, Lemma 4.2.2]), the previous definition does not depend on the choice of the summand $(a_1, \ldots, a_n)$. So we say that $\{a_1, \ldots, a_n\}$ is the central character of M and we write $\mathcal{C}(M)$ to denote it. Then two indecomposable $\mathcal{H}_n^{\mathbb{F}}$-modules lie in the same block if and only if they have the same central character.

2.3 The centraliser algebra and the Specht modules

We are ready to introduce the algebraic protagonist of Part I. Most of the material in this section is taken from [EM2]. Let R be a commutative ring and take m, l non-negative integers with $l \leqslant m$ and set $n := m - l$. Then RS_l is an R-subalgebra of the group ring RS_m. We define the *centraliser algebra* $\mathcal{C}_{l,m}^R$ to be the centraliser of RS_l in RS_m; that is, the algebra

$$\mathcal{C}_{l,m}^R := \{a \in RS_m \mid ab = ba \text{ for every } b \in RS_l\}.$$

This algebra has been studied extensively by Ellers and Murray [EM1, EM2], following earlier more general work on centraliser algebras for subgroups of finite groups by Ellers. However, $\mathcal{C}_{l,m}^R$ remains poorly understood, and in particular its blocks are not yet known. This

is the focus of the present part. the very effective description in [K, Section 2.1] provides a generating set for $\mathcal{C}^R_{l,m}$. This is the union of three subsets:

- ⋄ the centre $Z(RS_m)$;
- ⋄ the symmetric group (isomorphic to S_n) on the symbols $\{l+1,\dots,m\}$;
- ⋄ the set of *Jucys–Murphy elements* $L_j = (1\ j) + (2\ j) + \cdots + (j-1\ j)$ for $j = l+1,\dots,m$.

Now we introduce a family of $\mathcal{C}^R_{l,m}$-modules which will play a fundametal role in our treatment. Suppose that $\lambda \vdash l$ and $\mu \vdash m$. Then the R-module $\mathrm{S}^{\mu\setminus\lambda} = \mathrm{Hom}_{RS_l}(\mathrm{S}^\lambda, \mathrm{S}^\mu \downarrow_{S_l})$ introduced at the end of Section 2.1 is naturally a module for $\mathcal{C}^R_{l,m}$, via the action

$$(c\phi)(v) = c\phi(v) \qquad \text{for } c \in \mathcal{C}^R_{l,m},\ \phi \in \mathrm{S}^{\mu\setminus\lambda},\ v \in \mathrm{S}^\lambda\,.$$

As already remarked, $\mathrm{S}^{\mu\setminus\lambda}$ is non-trivial if and only if $\mu \setminus \lambda \in \mathcal{P}_{l,m}$. To emphasise the similarity with the symmetric group case, we refer to these $\mathcal{C}^R_{l,m}$-modules as *Specht modules*.

Let p be a prime number and let $\mathfrak{p}$ be a prime ideal of our commutative ring R. We consider a p-modular system $(R, \mathbb{F}, k)$, where $\mathbb{F}$ is the field of fractions of R and k is the residue field $R/\mathfrak{p}$ of (positive) characteristic p. Since $\mathbb{F}$ has characteristic zero both $\mathbb{F}S_m$ and $\mathbb{F}S_l$ are semisimple algebras by Maschke's Theorem. The general theory of centraliser algebras outlined above then implies that $\mathcal{C}^{\mathbb{F}}_{l,m}$ is semisimple as well, with a complete irredundant set of simple modules given by the set

$$\{\mathrm{S}^{\mu\setminus\lambda} \mid \mu \setminus \lambda \in \mathcal{P}_{l,m}\}$$

of Specht modules [K, Lemma 1.0.1].

Things change if the fraction field $\mathbb{F}$ is replaced by the residue field k: the centraliser algebra $\mathcal{C}^k_{l,m}$ is in general not semisimple and its Specht modules usually fail to be simple. The decomposition number problem for $\mathcal{C}^k_{l,m}$ asks for the composition factors of the Specht modules; unfortunately, at present we do not have a good labelling of simple module for $\mathcal{C}^k_{l,m}$. In this document we are concerned with determing the indecomposable summands of $\mathcal{C}^k_{l,m}$, the *p-blocks*. The Specht module $\mathrm{S}^{\mu\setminus\lambda}$ over k is a p-modular reduction of the corresponding (simple) Specht module over $\mathbb{F}$. So (analogously to the case of symmetric groups) describing the p-block structure of $\mathcal{C}^k_{l,m}$ reduces to finding the appropriate partition of the set of Specht $\mathcal{C}^k_{l,m}$-modules. For this, we use an approach pioneered by Ellers and Murray, which we describe in the next section.

2.4 Combinatorial blocks

Let l, m, n be as in Section 2.3. We have pointed out that in order to find the decomposition of $\mathcal{C}^k_{l,m}$ into p-blocks, we need to look for a way to partition the set of Specht $\mathcal{C}^k_{l,m}$-modules. We know by Theorem 1.4.1 how the Specht modules for the symmetric group split into p-blocks, this is a characterisation based on the concept of p-core. Our main conjecture (which is implicit in the work of Ellers and Murray [EM2]) predicts that the case of the centraliser algebra works in the same way.

Conjecture 2.4.1. *Let $\mu_1 \setminus \lambda_1, \mu_2 \setminus \lambda_2 \in \mathcal{P}_{l,m}$. The Specht $\mathcal{C}^k_{l,m}$-modules $S^{\mu_1\setminus\lambda_1}, S^{\mu_2\setminus\lambda_2}$ belong to the same p-block of $\mathcal{C}^k_{l,m}$ if and only if* $\mathrm{core}_p(\mu_1) = \mathrm{core}_p(\mu_2)$ *and* $\mathrm{core}_p(\lambda_1) = \mathrm{core}_p(\lambda_2)$.

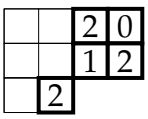
We can consider Conjecture 2.4.1 in terms of idempotents. We say that a non-zero central idempotent of $\mathcal{C}^k_{l,m}$ is a *block idempotent* if it is primitive, i.e. it cannot be written as the sum of two non-zero central idempotents. The p-blocks of $\mathcal{C}^k_{l,m}$ are then precisely the algebras $e\mathcal{C}^k_{l,m}$ for e a primitive central idempotent.

In the case of the symmetric group S_n, Theorem 1.4.1 tells that there is a bijection between the set of block idempotents of kS_n and the set of p-core partitions which are the p-core of at least one partition in $\mathcal{P}_n$. Given p-blocks B and B' of kS_m and kS_l respectively, let e_B and $e_{B'}$ be the corresponding block idempotents. Then it is straightforward to see that $e_B\, e_{B'}$ is a central idempotent of $\mathcal{C}^k_{l,m}$, so it is a sum of block idempotents, and $e_B\, e_{B'}\mathcal{C}^k_{l,m}$ is a (possibly zero) sum of p-blocks of $\mathcal{C}^k_{l,m}$. Moreover, $e_B\, e_{B'}$ acts as the identity on $S^{\mu\setminus\lambda}$ if S^μ lies in B and S^λ lies in B' (that is, if $\mathrm{core}_p(\mu) = \gamma_B$ and $\mathrm{core}_p(\lambda) = \gamma_{B'}$), and otherwise $e_B\, e_{B'} S^{\mu\setminus\lambda} = 0$. If $e_B\, e_{B'} \neq 0$, then we refer to $e_B\, e_{B'}\mathcal{C}^k_{l,m}$ as a *combinatorial block* of $\mathcal{C}^k_{l,m}$. Conjecture 2.4.1 then predicts that every combinatorial block is a p-block. Since the sum of the elements $e_B\, e_{B'}$ over all choices of B and B' is 1, this would then mean that every p-block of $\mathcal{C}^k_{l,m}$ is a combinatorial block. We will abuse notation by identifying a combinatorial block with the set of Specht modules in that combinatorial block, or with the set of labelling skew-partitions.

In [EM2], Ellers and Murray prove Conjecture 2.4.1 in the case $n \leqslant 3$. Our aim is to prove Conjecture 2.4.1 for a large family of combinatorial blocks of $\mathcal{C}^k_{l,m}$. We follow the idea of Ellers and Murray [EM2, Section 1.2] to exploit the relationship between the centraliser algebra $\mathcal{C}^k_{l,m}$ and the degenerate affine Hecke algebra $\mathcal{H}^k_n$. Their work shows that every $\mathcal{C}^k_{l,m}$-module naturally admits the structure of an $\mathcal{H}^k_n$-module: the symmetric group on $\{l+1,\dots,m\}$ and the Jucys–Murphy elements $L_{l+1},\dots,L_m$ generate a subalgebra of $\mathcal{C}^k_{l,m}$ which is a quotient of $\mathcal{H}^k_n$; restricting to this subalgebra and then inflating yields an $\mathcal{H}^k_n$-module structure. With respect to this, every Specht module is equipped with a formal character and a central character and it is an easy task to combinatorially compute these. Crucially, two $\mathcal{C}^k_{l,m}$-modules in the same combinatorial block have the same composition factors if and only if the corresponding $\mathcal{H}^k_n$-modules do. This enables us to work with formal characters of Specht modules to determine the p-block structure.

Let $\mu \setminus \lambda \in \mathcal{P}_{l,m}$ and consider a standard $\mu \setminus \lambda$-tableau T. We associate to T the permutation $a_T = (a_1,\dots,a_n)$ of $\mathrm{cont}_p(\mu \setminus \lambda)$ such that a_j is the p-residue of the node containing with number j in T, for $j = 1,\dots,n$. The formal character of the Specht module $S^{\mu\setminus\lambda}$, denoted by $\mathrm{ch}(\mu \setminus \lambda)$, is the formal sum of the n-tuples a_T with T varying in the set of standard $\mu \setminus \lambda$-tableaux.

Example. Let $p = 3$ and $\mu \setminus \lambda = (4^2,2) \setminus (2^2,1) \in \mathcal{P}_{5,10}$. The formal character of the Specht $\mathcal{C}^k_{5,10}$-module $S^{\mu\setminus\lambda}$ is as follows (we draw the skew-diagram $\mu \setminus \lambda$ filling every node with its 3-residue, omitting bars):



$$2\cdot(2,2,0,1,2) + 2\cdot(2,2,1,0,2) + 2\cdot(2,0,1,2,2) + 2\cdot(2,1,0,2,2) + (2,0,2,1,2) + (2,1,2,0,2).$$

By construction, every term of the formal character $\mathrm{ch}(\mu \setminus \lambda)$ is a permutation of the p-content of $\mu \setminus \lambda$. The central character of the Specht module $S^{\mu\setminus\lambda}$, as an $\mathcal{H}_n^k$-module, then coincides with the multiset $\mathrm{cont}_p(\mu \setminus \lambda)$. We observe the following fact.

Proposition 2.4.2. *Let $\mu_1 \setminus \lambda_1$ and $\mu_2 \setminus \lambda_2$ be skew-partitions in the same combinatorial block of $\mathcal{C}_{l,m}^k$. Then $\mathrm{cont}_p(\mu_1 \setminus \lambda_1) = \mathrm{cont}_p(\mu_2 \setminus \lambda_2)$.*

Proof. This follows from Theorem 1.4.2. □

Motivated by the above result, we say that a combinatorial block B has central character $\mathcal{C}(B)$ equal to the p-content of *any* skew-partition in B.

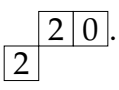
One of the main tools will concern the *decomposition matrix*. Let B be a combinatorial block and let D_B be the matrix whose rows are indexed by the elements of B and columns by those simple $\mathcal{H}_n^k$-modules appearing as composition factors of some Specht module $S^{\mu\setminus\lambda}$ with $\mu \setminus \lambda \in B$. To detect the entries of the row of D_B labelled by the skew-partition $\mu \setminus \lambda$, write $\mathrm{ch}(\mu \setminus \lambda)$ as a $\mathbb{Z}$-linear combination of formal characters of simple $\mathcal{H}_n^k$-modules. The coefficients in this expression give the desired decomposition numbers. B is then a p-block of $\mathcal{C}_{l,m}^k$ if and only if the decomposition matrix D_B is a connected matrix (i.e. if it cannot be put in block-diagonal form by permuting rows and columns).

We end this section and this chapter with a crucial remark.

Remark. Given a skew-partition $\mu \setminus \lambda$, fill the boxes of the skew-diagram $\mu \setminus \lambda$ with their p-residues, and define the *p-shape* of $\mu \setminus \lambda$ to be the multiset of connected components of the resulting diagram. It is easy to see that if two skew-partitions in B have the same p-shape, then the corresponding Specht modules have the same formal character, and therefore the same composition factors (with multiplicity) as $\mathcal{H}_n^k$-modules. This means that for our purposes, we can largely shorten the number of rows of D_B relabelling them with the elements of the set $\mathcal{B}$ obtained quotienting B by the equivalence relation that identifies skew-partitions with the same p-shape. We say that $\mathcal{B}$ is the *set of p-shapes* of B. We regard the elements of $\mathcal{B}$ as $\mathcal{H}_n^k$-modules and when talking about the formal character and the composition factors of a p-shape X we mean those of *any* Specht module in B whose p-shape is X. Also, saying that a p-shape belongs to a combinatorial block we mean that the combinatorial block contains a skew-partition with that p-shape. We observe that two Specht modules with the same p-shape are not necessarily isomorphic, even if they lie in the same combinatorial block, e.g. $S^{(3,1)\setminus(2)}$ and $S^{(2,1^2)\setminus(1^2)}$ in characteristic 2.

When drawing p-shapes, we follow [EM2] by drawing the connected components in a diagonal line in arbitrary order.

For example, if $p = 3$, the skew-partitions $(4,1) \setminus (2)$ and $(3,2) \setminus (2)$ have the same 3-shape



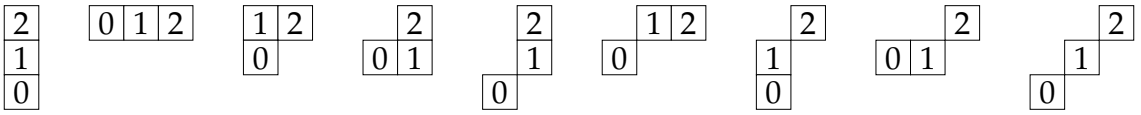
The corresponding Specht modules both have formal character $(2,0,2) + 2 \cdot (2,2,0)$.

Chapter 3

Ribbon blocks and belt blocks

This chapter is devoted to estabilishing Conjecture 2.4.1 for a special family of combinatorial blocks of the centraliser algebra. Again let m, l be non-negative integers with $m \geqslant l$ and let $n = m - l$. Recall that $(R, \mathbb{F}, k)$ is a p-modular system where p denotes the (prime) characteristic of the residue field k. Let B be a combinatorial block of $\mathcal{C}^k_{l,m}$. The central character of B can be viewed as a multiset of elements of $\mathbb{Z}/p\mathbb{Z}$. We will be concerned with the case where there are no repeated entries in the central character, so that the central character is a subset of $\mathbb{Z}/p\mathbb{Z}$. In this case clearly $n \leqslant p$, and we say that B is a *generalised ribbon block* if $n < p$ (so that the central character is a proper subset of $\mathbb{Z}/p\mathbb{Z}$), or a *belt block* if $n = p$. We will assume for the rest of Chapter 3 that B is either a generalised ribbon block or a belt block.

In the last section we explained that when working modulo p we can identify skew-partitions with the same formal character (since the rows of the decomposition matrix D_B labelled by them are identical), so from now on we will be concerned with just the set $\mathcal{B}$ of p-shapes of B. Firstly it is a good idea to give a visualisation of the elements of $\mathcal{B}$. If $\mu \setminus \lambda$ is a skew-partition in B, then its skew-diagram does not contain a 2×2-subdiagram (because otherwise it would have boxes with equal content, contrary to our assumption on B) and hence it is a disjoint union of rim hooks of μ. If X is the p-shape of $\mu \setminus \lambda$, we refer to a *connected component* of X for any of these rim hooks and we say that X is a *ribbon* if $\mu \setminus \lambda$ is connected, i.e. it is a single rim hook of μ. For example, if $p > n = 3$, and B has central character $\{\bar{0}, \bar{1}, \bar{2}\}$, then the p-shapes in $\mathcal{B}$ are among the nine drawings below (the first four on the left being the possible ribbons).



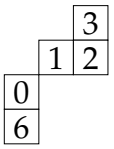
We close this long preface noting that, since there are no repetitions in the central character $\mathcal{C}(B)$, if X is a p-shape in $\mathcal{B}$, there is no risk of confusion writing $\boxed{i}$ for the unique node labelled by i in X, for every $i \in \mathcal{C}(B)$.

3.1 Generalised ribbon blocks

In this section we fix a generalised ribbon block B, with $\mathcal{C} = \mathcal{C}(B) \subset \mathbb{Z}/p\mathbb{Z}$ its central character. Our aim is to prove Conjecture 2.4.1 for B.

Before starting we give another combinatorial perspective to the p-shapes in a generalised ribbon block. The reason why we do this will be clarified in the next section. We define an *arrow graph* to be a directed graph with vertex set $\mathcal{C}$ such that every arrow has the form $i \to i+1$ or $i+1 \to i$. Given a p-shape X, we define an associated arrow graph Γ_X by draw an arrow $i \to i+1$ [resp. $i+1 \to i$] whenever $\boxed{i}$ lies on the left of [resp. below] $\boxed{i+1}$ in X, or no arrow between i and $i+1$ if these nodes lie in different connected components of X. It is clear that the set of all possible p-shapes is in bijection with the set of all possible arrow graphs. We define a p-shape to be *maximal* if its arrow graph is maximal, i.e. contains an arrow between i and $i+1$ whenever $i, i+1 \in \mathcal{C}$. Note that starting from an arrow graph we can recover the corresponding p-shape by mirroring the above procedure. Moreover, we can evaluate its formal character by taking the formal sum of all permutations $(a_1, \dots, a_n)$ of $\mathcal{C}$ with the property that i appears before j for every arrow $i \to j$.

Example. Suppose $p = 7$ and $\mathcal{C} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{6}\}$. We give an example of a p-shape and the corresponding arrow graph (omitting overlines).

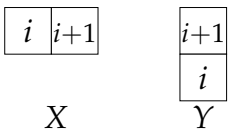


$$6 \longleftarrow 0 \qquad 1 \longrightarrow 2 \longleftarrow 3$$

The corresponding formal character is then (omitting bars)

$$\begin{aligned}&(0,1,3,2,6) + (0,1,3,6,2) + (0,1,6,3,2) + (0,3,1,2,6) + (0,3,1,6,2)\\ +&(0,3,6,1,2) + (0,6,1,3,2) + (0,6,3,1,2) + (1,0,3,2,6) + (1,0,3,6,2)\\ +&(1,0,6,3,2) + (1,3,0,2,6) + (1,3,0,6,2) + (1,3,2,0,6) + (3,0,1,2,6)\\ +&(3,0,1,6,2) + (3,0,6,1,2) + (3,1,0,2,6) + (3,1,0,6,2) + (3,1,2,0,6).\end{aligned}$$

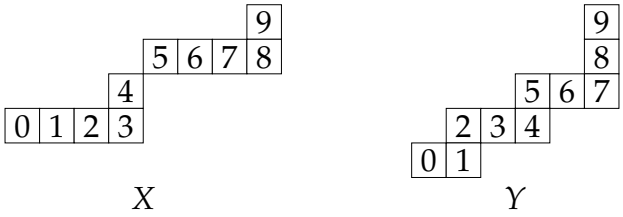
Now we introduce a fundamental tool in our treatment. Let X, Y be two p-shapes and let $\mathrm{D}(X,Y)$ be the subset of $\mathcal{C}$ containing those integers i such that $i+1 \in \mathcal{C}$ and $\boxed{i}$ lies on the left of $\boxed{i+1}$ in X and below $\boxed{i+1}$ in Y (see the picture below), or vice versa.



Define the *distance* $\mathrm{d}(X,Y)$ *between* X *and* Y as the cardinality of $\mathrm{D}(X,Y)$. This concept can be rephrased also in terms of arrow graphs: take Γ_X and Γ_Y and denote by $\Gamma_X \cup \Gamma_Y$ the picture obtained by overlapping these two arrow graphs. Then $\mathrm{d}(X,Y)$ coincides with the number of edges where two differently oriented arrows overlap in $\Gamma_X \cup \Gamma_Y$.

Note that d is *not* a metric: $\mathrm{d}(X,Y)$ can be zero even when $X \neq Y$.

Example. Suppose $p \geqslant 11$ and $\mathcal{C} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\}$ and consider the two following p-shapes:



Then $\mathrm{D}(X,Y) = \{\bar{1},\bar{3},\bar{7}\}$ and $\mathrm{d}(X,Y) = 3$. The union of the two arrow graphs $\Gamma_X \cup \Gamma_Y$ is given as follows, with Γ_X in red and Γ_Y in blue.

We start our analysis of generalised ribbon blocks by observing that the formal characters of two different maximal p-shapes have no summands in common.

Lemma 3.1.1. *Consider* $\mathbf{a} = (a_1,\ldots,a_n)$ *where* $a_1,\ldots,a_n$ *are the elements of* $\mathcal{C}$ *in some order. Then* $\mathbf{a}$ *appears in the formal character of exactly one maximal* p*-shape.*

Proof. We construct the unique maximal p-shape X such that $\mathbf{a}$ is a term of the formal character $\mathrm{ch}(X)$. Start drawing $\boxed{i}$, for some $i \in \mathcal{C}$ such that $i-1 \notin \mathcal{C}$. Then consider the order of i and $i+1$ in $\mathbf{a}$: if $i+1$ occurs before [resp. after] i, then draw $\boxed{i+1}$ just above [resp. to the right of] the node $\boxed{i}$. Now do the same for $i+1$ and $i+2$, and continue until an element $k \in \mathcal{C}$ is reached for which $k+1 \notin \mathcal{C}$. This determines one component of X. Now repeat with the other elements $i \in \mathcal{C}$ for which $i - i \notin \mathcal{C}$ to obtain the other components.

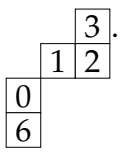
By construction, $\mathbf{a}$ is a summand of $\mathrm{ch}(X)$. We now see that X is unique. If Y is different maximal p-shape, then the set $\mathrm{D}(X,Y)$ is non-empty, because Γ_X and Γ_Y both have arrows joining i and $i+1$ whenever $i, i+1 \in \mathcal{C}$. Take $i \in \mathrm{D}(X,Y)$, and suppose that i appears before $i+1$ in $\mathbf{a}$ (the other case being similar). This means that we have an arrow $i \to i+1$ in Γ_X, and therefore an arrow $i \leftarrow i+1$ in Γ_Y, so that $\mathbf{a}$ does not appear in $\mathrm{ch}(Y)$. □

Corollary 3.1.2. *Different maximal* p*-shapes have no composition factors in common.*

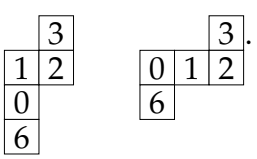
The next three results will finally give us a labelling set for the columns of the decomposition matrix of a ribbon block. In Lemma 3.1.3, if X is a p-shape, a standard X-tableau indicates a standard tableau of *any* skew-partition with p-shape X. We begin with a definition.

Definition. Let X be a p-shape. A *refinement* of X is a maximal p-shape Y that can be obtained joining connected components of X together.

Example. Suppose $p = 7$, and let X be the p-shape from an earlier example:



Then X has exactly two refinements, as follows.



Lemma 3.1.3. *Let* X *be a* p*-shape. The formal character of* X *is the sum of the formal characters of the refinements of* X*.*

Proof. This follows from a correspondence between standard tableaux: given a refinement Y of X, each standard Y-tableau gives rise to a standard X-tableau (just by breaking into individual connected components). Conversely, each standard X-tableau arises in this way for a unique refinement Y: for each i such that the nodes $\boxed{i}$ and $\boxed{i+1}$ lie in different connected components of X, the relative order of the entries $i, i+1$ tells us how they need to be joined together if the resulting tableau is to be standard. □

Corollary 3.1.4. *If X is a p-shape, then the composition factors of X are the composition factors of the refinements of X.*

Proof. This follows from the fact that the formal characters of simple modules of the degenerate affine Hecke algebra are linearly independent [K, Theorem 5.3.1]. □

Theorem 3.1.5. *Let X and Y be two p-shapes. The following are equivalent:*

1. *X and Y have a composition factor in common;*
2. *The formal characters of X and Y have a term in common;*
3. *$\mathrm{d}(X,Y) = 0$;*
4. *There is a maximal p-shape which is a refinement of both X and Y.*

Proof.

(1⇒ 2) This follows from the fact that the formal character of a $\mathcal{H}_n^k$-module is the sum of the formal characters of its composition factors.

(2 ⇒ 3) If there exists an integer $i \in D(X,Y)$, then $i+1$ occurs before i in every summand of $\mathrm{ch}(X)$ and after i in every summand of $\mathrm{ch}(Y)$, or vice versa. So $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common.

(3 ⇒ 4) Consider the overlapping $\Gamma_X \cup \Gamma_Y$. Since $\mathrm{d}(X,Y) = 0$, this gives rise to an arrow graph. Then every arrow graph obtained by filling the vacant edges in $\Gamma_X \cup \Gamma_Y$ corresponds to a refinement of both X and Y.

(4 ⇒ 1) This follows from Corollary 3.1.4. □

Theorem 3.1.5 tells us that the set of maximal p-shapes serves as a labelling set for an approximation to the decomposition matrix of B: by Corollary 3.1.4 any simple module S occurs as a composition factor of a unique maximal p-shape Y (say with multiplicity c). So by Corollary 3.1.4, if X is any p-shape then S occurs as a composition factor of X with multiplicity c if Y is a refinement of X, and 0 otherwise.

So we can define a matrix D_B with rows indexed by the set of p-shapes, and columns indexed by the set of maximal p-shapes, with entry $d_{XY} = 1$ if Y is a refinement of X and 0 otherwise. Then the actual decomposition matrix of B is obtained from D_B by possibly duplicating rows (because a given p-shape may correspond to more than one Specht module) and possibly duplicating and rescaling columns (because a Specht module labelled by a maximal p-shape might be reducible). As a consequence, to show that B is a single block of $\mathcal{C}_{l,m}^k$, we just need to show that D_B is a connected matrix, and our problem becomes purely combinatorial.

(In fact we suspect that every Specht module labelled by a maximal p-shape is irreducible, but we could not find a proof of this statement, and we do not need it.)

Example. Suppose $p = 5$, $l = 5$ and $m = 8$, and let B be the block consisting of Specht modules $S^{\mu\setminus\lambda}$ where $\mathrm{core}_5(\mu) = (2,1)$ and $\mathrm{core}_5(\lambda) = \varnothing$. Then $\mathcal{C} = \{\bar{0}, \bar{1}, \bar{4}\}$, and the p-shapes in B are as follows:

$$X_1 = \begin{array}{|c|c|c|}\hline 4&0&1\\\hline\end{array}, \quad X_2 = \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\!\!\begin{array}{|c|}\hline 4\\\hline\end{array}, \quad X_3 = \begin{array}{|c|c|}\hline 0&1\\\hline\end{array}\!\!\begin{array}{|c|}\hline 4\\\hline\end{array}, \quad X_4 = \begin{array}{|c|}\hline 1\\\hline\end{array}\!\!\begin{array}{|c|}\hline 0\\\hline 4\\\hline\end{array}, \quad X_5 = \begin{array}{|c|}\hline 1\\\hline 0\\\hline 4\\\hline\end{array}.$$

(For example, the p-shape X_2 arises from the skew-partitions $(7,1) \setminus (5)$ and $(5,3) \setminus (4,1)$.)

The matrix D_B is then given as follows.

	X_1	X_3	X_5
X_1	1	0	0
X_2	1	1	0
X_3	0	1	0
X_4	0	1	1
X_5	0	0	1

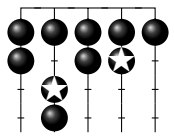
Definition. Let X, Y be two p-shapes in $\mathcal{B}$. We say that X and Y are *p-linked* if there exists a series of p-shapes $X = X_1, X_2, \ldots, X_{t-1}, X_t = Y$ in $\mathcal{B}$ such that $\mathrm{d}(X_j, X_{j+1}) = 0$ for every $1 \leqslant j \leqslant t-1$.

Our aim in this section is to show that any two p-shapes in B are p-linked. To do this, we will mainly work on abacus configurations. Given a skew-partition $\mu \setminus \lambda$, we consider the abacus configurations $\mathrm{Ab}(\mu)$ and $\mathrm{Ab}(\lambda)$ (with the same, big enough, charge) and we draw them one above the other. For convenience, for the rest of this chapter, we implicitly assume that the charge of all the abacus configurations is divisible by p. As remarked in Section 2.1, a skew-partition $\mu \setminus \lambda$ in B (that contains no 2×2-subdiagrams) can be thought as the union of a family of rim hooks of μ whose removal gives λ (recall that each of these rim hooks underlines a connected component of the p-shape of $\mu \setminus \lambda$). If H is one of them and $h, f \in \mathbb{Z}$ are the positions of $\mathrm{Ab}(\mu)$ corresponding to $\mathrm{hand}(H)$ and $\mathrm{foot}(H)$, respectively, then h is a bead and $f - 1$ is a space (and vice versa h is a space and $f - 1$ is a bead in $\mathrm{Ab}(\lambda)$). We represent the bead at position f in $\mathrm{Ab}(\mu)$ by a new symbol ✪ called a *star bead*. (So the star beads are precisely the beads occurring in the abacus configuration of μ but not of λ.)

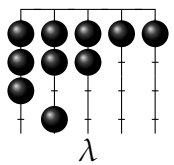
Example. Take $p = 5$ and $\mu \setminus \lambda = (7,3,1^2) \setminus (7,2)$, giving the following p-shape.

$$\begin{array}{|c|}\hline 1\\\hline\end{array}\\ \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}$$

This p-shape has two connected components, so that $\mu \setminus \lambda$ comprises two rim hooks. We draw abacus configurations of μ and λ as follows, with star beads in μ.



μ



λ

We need a powerful tool for proving our next result. This comes from the second characterization of the p-blocks of S_n mentioned in Section 1.4 and exploits the interplay between partitions belonging to the same p-block of S_n.

Proposition 3.1.6. *Let B be a p-block of S_n and let $\alpha, \beta \in B$. Suppose that for some $h \geqslant 1$ and some $i \in \{0, \dots, p-1\}$, α has a removable h-hook at runner i. Then at least one of the following occurs:*

- *α has an addable h-hook at runner $j \equiv i - h \pmod{p}$;*
- *β has a removable h-hook at runner i.*

The proof of this result is given in [EM2, Proposition 6] for $h = 1$. Our slight generalization follows with the same arguments.

Now we can give our main result.

Theorem 3.1.7. *Let $X, Y \in \mathcal{B}$ such that $\mathrm{d}(X, Y) > 0$. Then there is $Z \in \mathcal{B}$ such that $\mathrm{d}(X, Z) < \mathrm{d}(X, Y)$ and $\mathrm{d}(Y, Z) < \mathrm{d}(X, Y)$.*

Proof. For every $j \in \mathcal{C}$, consider the node $\boxed{j}$ in $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ and denote by x_j and y_j the positions corresponding to these nodes in the abacus display, respectively. (Note that by the choice of the charge, positions x_j and y_j are both in the j-th runner.) Fix $i \in \mathrm{D}(X, Y)$, and assume without loss of generality that the node $\boxed{i}$ lies on the left of $\boxed{i+1}$ in X and below it in Y. Then x_i is a space in $\mathrm{Ab}(\mu_X)$ and y_i is a bead (and *not* a star bead) in $\mathrm{Ab}(\mu_Y)$. We let r be maximal such that $i, i+1, \dots, i+r \in \mathcal{C}$ (where we add modulo p); note that there must be such an r because by assumption $\mathcal{C} \neq \mathbb{Z}/p\mathbb{Z}$. We prove that we can find the required p-shape Z by contradiction, via the following claim.

> **Claim 1.** Suppose that the required p-shape Z does not exist. Then for every $k \in \{i+1, \dots, i+r\}$ with $k \notin \mathrm{D}(X, Y)$, the following statements hold.
>
> - If x_k is a bead or a star bead in $\mathrm{Ab}(\mu_X)$, then
> - μ_X has no addable $(k-i)$-hooks at runner i, and
> - λ_Y has a removable $(k-i)$-hook at runner k.
> - If y_k is a space or a star bead in $\mathrm{Ab}(\mu_Y)$, then
> - μ_X has an addable $(k-i)$-hook at runner i, and
> - λ_Y has no removable $(k-i)$-hooks at runner k.

Note that the condition that x_k is a bead or star bead in $\mathrm{Ab}(\mu_X)$ is the same as saying that $\boxed{k+1}$ does not lie immediately to the right of $\boxed{k}$ in X. Similarly, the condition that y_k is a space or star bead in $\mathrm{Ab}(\mu_Y)$ means that $\boxed{k+1}$ does not lie immediately above $\boxed{k}$ in Y.

Claim 1 then gives the desired contradiction to complete the proof: if we let $k = i + r$, then x_k and y_k are star beads in $\mathrm{Ab}(\mu_X)$ and $\mathrm{Ab}(\mu_Y)$ respectively because $i + r + 1 \notin \mathcal{C}$. So Claim 1 cannot be true in the case $k = i + r$.

We prove Claim 1 by induction on k. So take $k \in \{i+1, \dots, i+m\} \setminus \mathrm{D}(X, Y)$, and assume that Claim 1 holds for all smaller values of k. The following statement is crucial.

> **Claim 2.** Suppose k is as in Claim 1, and that the inductive hypothesis holds. Then position $x_k + i - k$ is a space in $\mathrm{Ab}(\mu_X)$.

Proof. Consider the node $\boxed{k}$ in X. We have that $x_k = x_i - i + k + tp$, for some $t \in \mathbb{Z}$. If $t = 0$, then the claim is trivially true since we are supposing that x_i is a space. If $t \neq 0$, then the nodes $\boxed{i}$ and $\boxed{k}$ lie in different connected components of X. Let $j \in \{i+2, \ldots, k\}$ be such that $\boxed{j}$ is the foot node of the connected component of X containing $\boxed{k}$. It follows that $\boxed{j-1}$ is the hand node of a connected component of X, so that x_{j-1} is a star bead in $\mathrm{Ab}(\mu_X)$ and $j - 1 \notin \mathrm{D}(X, Y)$. By the inductive hypothesis, μ_X has no addable $(j-1-i)$-hooks at runner i, so position $x_j + i - j$ must be a space since $x_j - 1$ is a space in $\mathrm{Ab}(\mu_X)$. Finally, since $\boxed{j}$ and $\boxed{k}$ lie in the same connected component of X, we see that $x_j = x_k + j - k$. It follows that $x_j + i - j = x_k + i - k$, proving what we have claimed.

With a perfectly dual argument it can be shown that the inductive hypothesis forces the position $y_k + i - k$ to be a bead in $\mathrm{Ab}(\lambda_Y)$.

With Claim 2 established, we deal with the inductive step of Claim 1. We consider only the case when x_k is a bead or a star bead in $\mathrm{Ab}(\mu_X)$, the other case being similar. By Claim 2, $x_k + i - k$ is a space in $\mathrm{Ab}(\mu_X)$, and so μ_X has a removable $(k-i)$-hook at position x_k. Assume that there exists $a_1 \equiv i \pmod{p}$ such that μ_X has an addable $(k-i)$-hook at a_1. Let $1 \leqslant k_1 \leqslant k - i$ be the least integer such that $k - k_1 \notin \mathrm{D}(X, Y)$, position x_{k-k_1} is a space and $a_1 + k - i - k_1$ is a bead in $\mathrm{Ab}(\mu_X)$. The assumption that $k - k_1 \notin \mathrm{D}(X, Y)$ and x_{k-k_1} is a space means that y_{k-k_1} is a space or a star bead in $\mathrm{Ab}(\mu_Y)$. So if $k_1 < k - i$, then the inductive hypothesis says that there exists an integer $a_2 \equiv i$ such that μ_X has an addable $(k - i - k_1)$-hook at position a_2. Then we repeat the above process, letting $1 \leqslant k_2 \leqslant k - i - k_1$ be minimal such that $k - k_1 - k_2 \notin \mathrm{D}(X, Y)$, $x_{k-k_1-k_2}$ is a space and $a_2 + k - i - k_1 - k_2$ is a bead in $\mathrm{Ab}(\mu_X)$.

It is clear that this procedure always terminates at a step s such that $i = k - \sum_{1 \leqslant u \leqslant s} k_u$. Summarizing, we have found integers $a_1, \ldots, a_s \equiv i \pmod{p}$ and $k_1, \ldots, k_s$ such that, for every $1 \leqslant v \leqslant s$, in $\mathrm{Ab}(\mu_X)$, there is a bead at position $c_v := a_v + k - i - \sum_{1 \leqslant u \leqslant v} k_u$ and a space at position $d_v := a_v + k - i - \sum_{1 \leqslant u \leqslant v-1} k_u$. Let $i + 1 \leqslant j_1 < \cdots < j_t \leqslant k$ be those residues such that $\boxed{j_1}, \ldots, \boxed{j_t}$ are hand nodes of connected components of X.

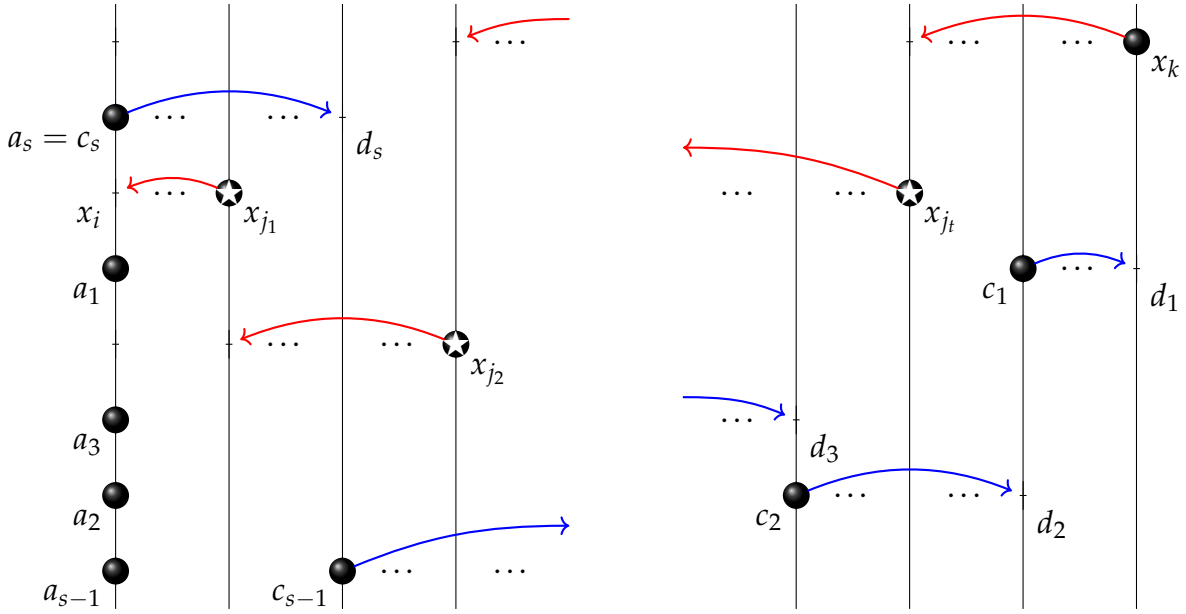
We now construct a new abacus configuration. We start from $\mathrm{Ab}(\mu_X)$ and make the following moves:

- ◇ move the bead (or star bead) at x_k to the space at $x_{j_t+1} - 1$,
- ◇ for every $2 \leqslant v \leqslant t$, move the star bead at x_{j_v} to the space at $x_{j_{v-1}+1} - 1$,
- ◇ move the star bead at x_{j_1} to the space at x_i,
- ◇ for every $1 \leqslant v \leqslant s$, move the bead at c_v to the space at d_v.

Note that in terms of the Young diagram, the first three types of moves simply entail removing all the nodes $\boxed{i+1}, \ldots, \boxed{k}$ from μ_X, and the moves of the last type entail putting these nodes back in new positions. So if we let μ_Z be the partition defined by this new abacus configuration, then $\lambda_X \subset \mu_Z$, and we have a skew-partition $\mu_Z \setminus \lambda_X$ in B. We let Z be the p-shape of this skew-partition. We will show that in most cases, Z is the p-shape desired in the theorem.

The diagram below illustrates the changes made in the abacus. Note that the rows of the diagram in which the given positions appear are illustrative only; these rows could easily

appear in a different order in practice.



For every $j \in \mathcal{C}$, let z_j be the position of the node $\boxed{j}$ in $\mathrm{Ab}(\mu_Z)$. Now take $j \in \{i+1, \ldots, k-1\}$ with $j \notin \mathrm{D}(X,Y)$. Then we claim that $j \notin \mathrm{D}(X,Z)$ and $j \notin \mathrm{D}(Y,Z)$. If x_j is a bead or a star bead then by the inductive hypothesis μ_X has no addable $(j-i)$-hooks at runner i, so that z_j is a bead in $\mathrm{Ab}(\mu_Z)$. If x_j is a space, then the choice of $k_1, \ldots, k_s$ means that z_j is a space or a star bead. This gives $j \notin \mathrm{D}(X,Z)$. We deduce similarly that $j \notin \mathrm{D}(Y,Z)$ (note that the inductive hypothesis means that it cannot be the case that x_j is a star bead and y_j is a space).

So our claim is true. Additionally, it is easy to see that $i \notin \mathrm{D}(X,Z)$ and $i \notin \mathrm{D}(Y,Z)$. Furthermore, $k \notin \mathrm{D}(X,Z)$ because x_k and z_k are both beads. It follows that $\mathrm{d}(X,Z) < \mathrm{d}(X,Y)$, and that $\mathrm{d}(Y,Z) < \mathrm{d}(X,Y)$ unless y_k is a space in $\mathrm{Ab}(\lambda_Y)$ and, in $\mathrm{Ab}(\mu_X)$, x_k is a star bead and $a_1 = x_{k+1} + k - i - 1$ is the only addable $(k-i)$-hook at runner i. If this is not the case, then Z is the desired p-shape and we conclude. Hence assume the contrary from now on.

We are under the assumption that y_k is a space in $\mathrm{Ab}(\lambda_Y)$. Then λ_Y has an addable $(k-i)$-hook at runner i because position $y_k + i - k$ is a bead by the dual version of Claim 2. The fact that μ_X has a removable $(k-i)$-hook at runner k and exactly an addable $(k-i)$-hook at runner i, together with Proposition 3.1.6, implies that μ_Y has a removable $(k-i)$-hook at runner k. Say that this removable hook occurs at position $r_1 \equiv k \pmod{p}$. Note that $r_1 \neq y_k$ because y_k is a space, so r_1 is a removable $(k-i)$-hook at runner k of λ_Y as well. Let $1 \leqslant k_1 \leqslant k-i$ with $k - k_1 \notin \mathrm{D}(X,Y)$ be the least integer such that y_{k-k_1} is a bead and $r_1 - k_1$ is a space in $\mathrm{Ab}(\lambda_Y)$. If $k_1 < k-i$, the inductive hypothesis ensures that λ_Y has a removable $(k-i-k_1)$-hook at some position $r_2 \equiv r_1 - k_1 \pmod{p}$. Then we define $1 \leqslant k_2 \leqslant k-i-k_1$ with $k - k_1 - k_2 \notin \mathrm{D}(X,Y)$ to be minimal such that $y_{k-k_1-k_2}$ is a bead and $r_2 - k_2$ is a space in $\mathrm{Ab}(\lambda_Y)$. We continue in this way; the process terminates at a step s such that $i = k - \sum_{1 \leqslant u \leqslant s} k_u$. Summarising we have collected integers $k_1, \ldots, k_s$ and $r_1, \ldots, r_s$ with $r_v \equiv k - \sum_{1 \leqslant u \leqslant v-1} k_u \pmod{p}$, for $1 \leqslant v \leqslant s$, such that, in $\mathrm{Ab}(\lambda_Y)$, there is a bead at position r_v and a space at position $r_v - k_v$ for every $1 \leqslant v \leqslant s$. Let $i+1 \leqslant j_1 < \cdots < j_t \leqslant k$ be the residues of nodes that are hand nodes of connected components of Y.

We modify $\mathrm{Ab}(\lambda_Y)$ as follows:

- move the bead at y_i to the space at y_{j_1},

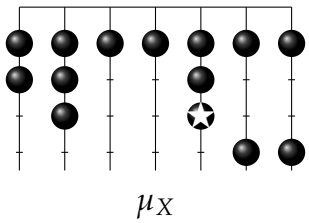
- ◇ for every $1 \leqslant v \leqslant t-1$, move the star bead at $y_{j_v+1} - 1$ to the space at $y_{j_{v+1}} - 1$,
- ◇ move the star bead at $y_{j_t+1} - 1$ to the space at x_k,
- ◇ for every $1 \leqslant v \leqslant s$, move the bead at r_v to the space at $r_v - k_v$.

Let λ_W be the resulting partition. Then $\mu_Y \setminus \lambda_W \in B$. We let W be the p-shape of $\mu_Y \setminus \lambda_W$. Then we can deduce that $\mathrm{d}(X,W) < d$ and $\mathrm{d}(Y,W) < d$; this is done as for Z above, but now note that k cannot belong to $\mathrm{D}(X,W)$, e.g. because we are supposing that x_k is a star bead in $\mathrm{Ab}(\mu_X)$.) The p-shape W is finally the desired one, contradicting the hypothesis of Claim 1.

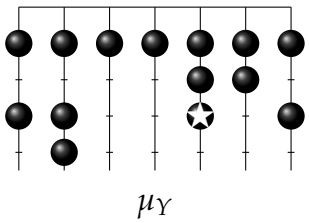
We can therefore assume that μ_X has no addable $(k-i)$-hooks at runner i and, hence, by Proposition 3.1.6, that μ_Y has a removable $(k-i)$-hook at runner k. Finally we note that this removable hook does not occur at position y_k since by the previous remark position $y_k + i - k$ is a bead in $\mathrm{Ab}(\mu_Y)$. So we conclude that λ_Y has a removable $(k-i)$-hook at runner i ending both the inductive step and the proof.

□

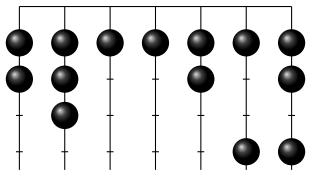
Example. Let $m = 42$, $l = 37$ and $p = 7$. The skew-partitions $\mu_X \setminus \lambda_X = (14^2,7,5,2) \setminus (14^2,4,3,2)$ and $\mu_Y \setminus \lambda_Y = (9,8,7,5^2,4^2) \setminus (9,8,4^5)$ have size $n = m - l = 5$ and belong to the same combinatorial block of $\mathcal{C}^{\mathbb{F}_7}_{37,42}$ having 7-content $\mathcal{C} = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Indeed, $\mathrm{core}_7(\mu_X) = \mathrm{core}_7(\mu_Y) = (5,3,2^3)$ and $\mathrm{core}_7(\lambda_X) = \mathrm{core}_7(\lambda_Y) = (7,3,2^3)$. Let X and Y be the p-shapes of $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$, respectively. The Young diagrams and the abacus configurations (with charge 14) are shown below.



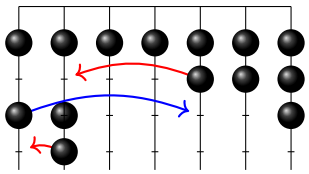
μ_X



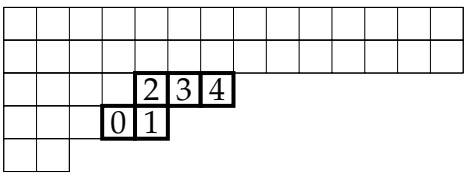
μ_Y



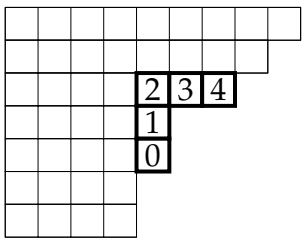
λ_X



λ_Y



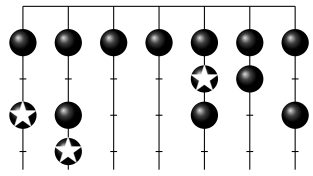
$\mu_X \setminus \lambda_X$



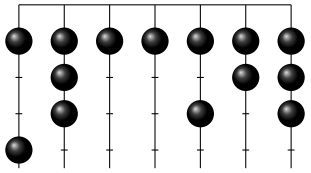
$\mu_Y \setminus \lambda_Y$

We see that $\mathrm{D}(X,Y) = \{\bar{0}\}$, so that $\mathrm{d}(X,Y) = 1$. We use the procedure explained in the proof of Theorem 3.1.7, faithfully retracing the notation, to construct a 7-shape that has distance 0 from both X and Y. Let $i = \bar{0}$: position $x_0 = y_0 = 14$ is a space in $\mathrm{Ab}(\mu_X)$ and a bead in $\mathrm{Ab}(\mu_Y)$. In this case the inductive procedure stops at $k = 4$ because $y_4 = 18$ is an addable

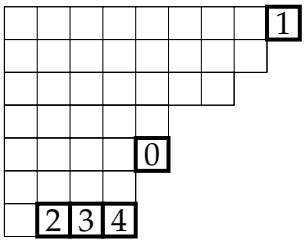
4-hook at runner 0 of $\mathrm{Ab}(\lambda_Y)$ and $r_1 = 11$ is a removable 4-hook at runner 4. It follows that $k_1 = 3$ since $r_1 - k_1 = 8$ is a space and $y_1 = 15$ is a bead in $\mathrm{Ab}(\mu_Y)$. The procedure ensures that λ_Y has a removable node at runner 1. Actually, this is at position $r_2 = 22$. Here we stop at the first step because $r_2 - 1 \equiv 0 \pmod 7$. This gives $k_2 = 1$ and $s = 2$ ending the 'removing' part. For the 'adding' one we have that $t = 1$ and $j_1 = \bar{4}$ is the only hand node of a connected component of X – that is actually a ribbon. We then move the nodes as explained in the proof of Corollary 3.1.8 and shown in figure. We find a partition $\lambda_Z = (8^2, 7, 5, 4^2, 1) \vdash 37$ such that $\mu_Y \setminus \lambda_Z$ has 7-shape Z such that $\mathrm{d}(X, Z) = \mathrm{d}(Y, Z) = 0$ as it can be easily deduced from the drawings.



λ_Z



μ_Z



$\mu_Z \setminus \lambda_Z$

We immediately deduce the following corollary, using induction on $\mathrm{d}(X, Y)$.

Corollary 3.1.8. *Suppose $X, Y \in \mathcal{B}$. Then X and Y are p-linked.*

We deduce as a corollary the main result of this section.

Corollary 3.1.9. *Conjecture 2.4.1 holds for generalised ribbon blocks.*

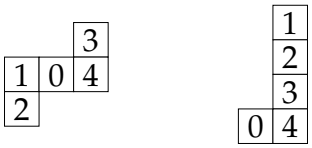
Proof. Let B be a generalised ribbon block as above. It follows from Corollary 3.1.8 and the implication $(3 \Rightarrow 4)$ in Theorem 3.1.5 that the matrix D_B is connected. As explained in the discussion following Theorem 3.1.5, this implies that the decomposition matrix of B is connected, so that B is a single p-block of $\mathcal{C}^k_{l,m}$. □

3.2 Belt blocks

For the rest of this section suppose that $p = n$. Our aim is to establish Conjecture 2.4.1 for a *belt block* B, i.e. a combinatorial block of $\mathcal{C}^k_{l,m}$ whose central character is an interval of $\mathbb{Z}/p\mathbb{Z}$, and hence coincides with $\mathbb{Z}/p\mathbb{Z} = \{\bar{0}, \ldots, \overline{p-1}\}$.

As in Section 3.1, we first replace B (again regarded as a subset of $\mathcal{P}_{l,m}$) with its set $\mathcal{B}$ of p-shapes. We keep going with the same notation established in the preface of the chapter talking about the connected components of a p-shape. Now the maximal p-shapes are ribbons. Unfortunately in this situation we cannot mimic at all the arguments we have used for ribbon blocks. Indeed, even if it is still true that the formal character of a disconnected p-shape is the sum of the formal characters of its refinements, a basic step of the reasoning in Section 3.1, namely the statement of Lemma 3.1.1, is no longer true in this setting. This means that, in general, different ribbons may share composition factors.

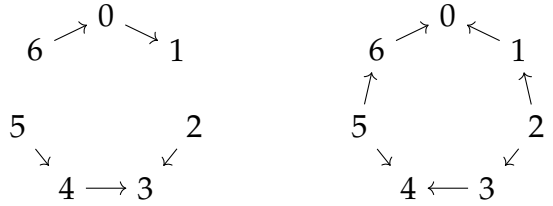
Example. Let $p = 5$. The formal characters of the following two ribbons share two summands: $(1,2,0,3,4)$ and $(1,2,3,0,4)$.



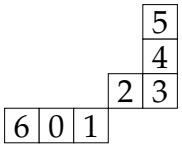
To deal with this inconvenience, we need to introduce a functional 'refinement' of the set of ribbons. To do this, we begin with the introduction of a graph analogue to that in Section 3.1. We define a (circular) arrow graph to be a directed graph on the set $\mathbb{Z}/p\mathbb{Z}$ such that every arrow has the form $i \to i+1$ or $i+1 \to i$ for some i and the graph is not a directed cycle. An arrow graph is a *belt* if for every i there is either an arrow $i \to i+1$ or an arrow $i+1 \to i$.

We will draw arrow graphs with $\bar{0}, \bar{1}, \ldots, \overline{p-1}$ in clockwise order. So we may say that the arrows of an arrow graph are all oriented clockwise to mean that they all have the form $i \to i+1$.

Example. Let $p = 7$. The following are two arrow graphs, the one on the right being a belt.



We observe that every p-shape X in a belt block is uniquely associated to an arrow graph Γ_X through the same procedure explained in the Section 3.1. For example, the arrow graph on the left of the above example corresponds to the p-shape below.



Note that in this case the procedure does not give rise to a bijection between all possible p-shapes and all possible arrow graphs. In particular, no p-shape corresponds to a belt. If Γ is an arrow graph, we define its formal character $\mathrm{ch}(\Gamma)$ to be the formal sum of all permutations $\mathbf{b} = (b_0, \ldots, b_{p-1})$ of $\mathbb{Z}/p\mathbb{Z}$ with the property that i appears before [resp. after] $i+1$ for every arrow $i \to i+1$ [$i+1 \to i$]. Note that if X is a p-shape, then $\mathrm{ch}(\Gamma_X) = \mathrm{ch}(X)$.

Our next task is to build for every belt Γ a $\mathcal{H}_p^k$-module whose formal character is $\mathrm{ch}(\Gamma)$. We take a vector space M_Γ with basis $\{e_{\mathbf{b}}\}$ labelled by the different summands of $\mathrm{ch}(\Gamma)$. If $\mathbf{b} = (b_0, \ldots, b_{p-1})$ is a summand of $\mathrm{ch}(\Gamma)$, for each $k \in \{1, \ldots, p-1\}$ we define $\sigma_k \mathbf{b} = (b_0, \ldots, b_{k+1}, b_k, \ldots, b_{p-1})$. We define an action of $\mathcal{H}_p^k$ on M_Γ as follows. Fix a summand $\mathbf{a}$ of $\mathrm{ch}(\Gamma)$. We set

$$z_k \, \mathrm{e}_{\mathbf{b}} = b_k \, \mathrm{e}_{\mathbf{b}}$$

and

$$s_k \, \mathrm{e}_{\mathbf{b}} = \begin{cases} \frac{1}{b_{k+1}-b_k} \, \mathrm{e}_{\mathbf{b}} + \mathrm{e}_{\sigma_k \mathbf{b}} & \text{if } b_k \text{ appears } \textit{after } b_{k+1} \text{ in } \mathbf{a}, \\ \frac{1}{b_{k+1}-b_k} \, \mathrm{e}_{\mathbf{b}} + \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} & \text{if } b_k \text{ appears } \textit{before } b_{k+1} \text{ in } \mathbf{a}. \end{cases}$$

Theorem 3.2.1. *With the above rule M_Γ is a $\mathcal{H}_p^k$-module with formal character* $\mathrm{ch}(\Gamma)$.

Proof. We just need to prove the first statement; the second part is then obvious from the definition of the formal character of a $\mathcal{H}_p^k$-module.

We prove that M_Γ is an $\mathcal{H}_p^k$-module showing that the above defined rules respect the defining relations of $\mathcal{H}_p^k$ shown in Section 2.2. Relations (1) and (3) are trivially satisfied and need no further investigations. Therefore we only consider the *braid* relations in (2) and the relations between the polynomial and the Coxeter generators in (4).

We start from the braid relations. These divide into three types depending on the integers $m_{i,j}$ such that $(s_i s_j)^{m_{i,j}} = 1$. For each relation several cases arise depending on the positions of the involved entries of $\mathbf{b}$ in the fixed summand $\mathbf{a}$ of $\mathrm{ch}(\Gamma)$. We only analyse one case for each relation, the others being similar.

- $s_k^2 = 1$. Suppose that b_k appears before b_{k+1} in $\mathbf{a}$. We have that

$$\begin{aligned} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k} \mathrm{e}_{\mathbf{b}} + \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} \quad \Rightarrow \\ s_k^2 \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)^2} \mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+1}-b_k} \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} \\ &\quad - \frac{1}{b_{k+1}-b_k} \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\sigma_k \mathbf{b}} + \left(1 - \frac{1}{(b_{k+1}-b_k)^2}\right) \mathrm{e}_{\mathbf{b}} \\ &= \mathrm{e}_{\mathbf{b}}. \end{aligned}$$

- $s_j s_k = s_k s_j$ with $|k-j| > 1$. Suppose that b_k appears after b_{k+1} and that b_j appears before b_{j+1} in $\mathbf{a}$. For the left-hand side we have

$$\begin{aligned} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k} \mathrm{e}_{\mathbf{b}} + \mathrm{e}_{\sigma_k \mathbf{b}} \quad \Rightarrow \\ s_j s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)(b_{j+1}-b_j)} \mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+1}-b_k} \left(1 - \frac{1}{(b_{j+1}-b_j)^2}\right) \mathrm{e}_{\sigma_j \mathbf{b}} \\ &\quad + \frac{1}{b_{j+1}-b_j} \mathrm{e}_{\sigma_k \mathbf{b}} + \left(1 - \frac{1}{(b_{j+1}-b_j)^2}\right) \mathrm{e}_{\sigma_j \sigma_k \mathbf{b}}, \end{aligned}$$

while for the right-hand side we have

$$
\begin{aligned}
s_j \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{j+1}-b_j}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_j\mathbf{b}} \quad \Rightarrow \\
s_k s_j \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)(b_{j+1}-b_j)}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{j+1}-b_j}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_j\mathbf{b}} + \left(1-\frac{1}{(b_{j+1}-b_j)^2}\right)\mathrm{e}_{\sigma_k\sigma_j\mathbf{b}}.
\end{aligned}
$$

Comparing the coefficients of the different basis elements we easily gain the proof.

◇ $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$. Suppose that b_{k+1} precedes b_{k+2} which precedes b_k in $\mathbf{a}$. The left-hand and right-hand sides are as follows

$$
\begin{aligned}
s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + \mathrm{e}_{\sigma_k\mathbf{b}} \quad \Rightarrow \\
s_{k+1} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\mathrm{e}_{\sigma_k\mathbf{b}} + \mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} \quad \Rightarrow \\
s_k s_{k+1} s_k \mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+1}-b_k)^2(b_{k+2}-b_{k+1})}\mathrm{e}_{\mathbf{b}} + \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+1}-b_k)(b_{k+2}-b_k)}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} \\
&\quad - \frac{1}{(b_{k+2}-b_k)(b_{k+1}-b_k)}\mathrm{e}_{\sigma_k\mathbf{b}} + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\sigma_k\mathbf{b}}
\end{aligned}
$$

and

$$
\begin{aligned}
s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \quad \Rightarrow \\
s_k s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+2}-b_{k+1})(b_{k+1}-b_k)}\mathrm{e}_{\mathbf{b}} + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_k\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} \quad \Rightarrow \\
s_{k+1}s_k s_{k+1}\mathrm{e}_{\mathbf{b}} &= \frac{1}{(b_{k+2}-b_{k+1})^2(b_{k+1}-b_k)}\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+2}-b_{k+1})(b_{k+1}-b_k)}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{(b_{k+2}-b_k)(b_{k+2}-b_{k+1})}\mathrm{e}_{\sigma_k\mathbf{b}} + \frac{1}{b_{k+2}-b_{k+1}}\mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}} \\
&\quad - \frac{1}{(b_{k+2}-b_k)(b_{k+2}-b_{k+1})}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\mathbf{b}} \\
&\quad + \frac{1}{b_{k+2}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\mathbf{b}} \\
&\quad + \frac{1}{b_{k+1}-b_k}\left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}} \\
&\quad + \left(1-\frac{1}{(b_{k+2}-b_{k+1})^2}\right)\mathrm{e}_{\sigma_{k+1}\sigma_k\sigma_{k+1}\mathbf{b}}.
\end{aligned}
$$

The coefficients of $\mathrm{e}_{\sigma_k\sigma_{k+1}\mathbf{b}}, \mathrm{e}_{\sigma_{k+1}\sigma_k\mathbf{b}}$ and $\mathrm{e}_{\sigma_k\sigma_{k+1}\sigma_k\mathbf{b}} = \mathrm{e}_{\sigma_{k+1}\sigma_k\sigma_{k+1}\mathbf{b}}$ are the same on both sides. The coefficients of $\mathrm{e}_{\mathbf{b}}, \mathrm{e}_{\sigma_k\mathbf{b}}$ and $\mathrm{e}_{\sigma_{k+1}\mathbf{b}}$ turn out to be equal from easy calculations comparing the two above formulas.

We finally focus on the relation (4): $s_k z_k = z_{k+1}s_k - 1$. Suppose that b_k precedes b_{k+1} in $\mathbf{a}$. We evaluate the action of both sides of (4) on $\mathrm{e}_{\mathbf{b}}$. On the left-hand we have

$$
\begin{aligned}
z_k\mathrm{e}_{\mathbf{b}} &= b_k\mathrm{e}_{\mathbf{b}} \quad \Rightarrow \\
s_k z_k \mathrm{e}_b &= \frac{b_k}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + b_k\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}},
\end{aligned}
$$

while on the right we have

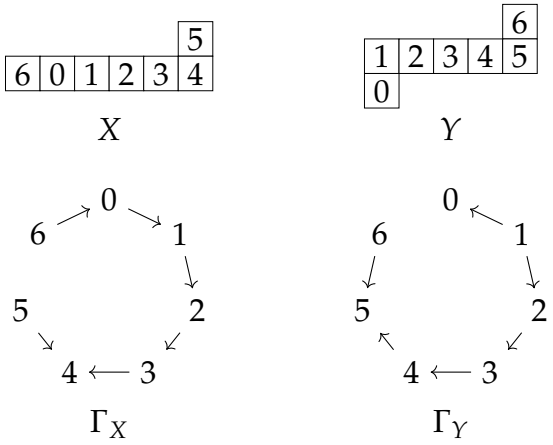
$$
\begin{aligned}
s_k\mathrm{e}_{\mathbf{b}} &= \frac{1}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + \left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}} \quad \Rightarrow \\
(z_{k+1}s_k - 1)\mathrm{e}_{\mathbf{b}} &= \frac{b_{k+1}}{b_{k+1}-b_k}\mathrm{e}_{\mathbf{b}} + b_k\left(1-\frac{1}{(b_{k+1}-b_k)^2}\right)\mathrm{e}_{\sigma_k\mathbf{b}} - \mathrm{e}_{\mathbf{b}}.
\end{aligned}
$$

It is now straightforward to check that the coefficients of $\mathrm{e}_{\mathbf{b}}$ and $\mathrm{e}_{\sigma_k\mathbf{b}}$ are equal on both sides. This ends the proof. □

Theorem 3.2.1 strengthens the above discussion and shows that *every* arrow graph corresponds to a $\mathcal{H}_p^k$-module. In particular, we can talk about the composition factors of a p-shape, meaning the composition factors of the corresponding $\mathcal{H}_p^k$-module. Later on we may look at the set of p-shapes $\mathcal{B}$ as a subset of the set of all possible arrow graphs.

As in Section 3.1 we rely on the notion of distance between p-shapes. Given p-shapes X and Y, we define $\mathrm{D}(X,Y)$ and $\mathrm{d}(X,Y)$ as in Section 3.1.

Example. Let $p = 7$, and let X and Y be the p-shapes below. From the corresponding arrow graphs, we see that $\mathrm{D}(X,Y) = \{\bar{0}, \bar{4}\}$, and therefore $\mathrm{d}(X,Y) = 2$.



We can now state a variant of Lemma 3.1.1.

Lemma 3.2.2. *Consider* $\mathbf{a} = (a_0, \ldots, a_{p-1})$ *where* $a_0, \ldots, a_{p-1}$ *equal* $\bar{0}, \ldots, \overline{p-1}$ *in some order. Then* $\mathbf{a}$ *appears in the formal character of exactly one belt.*

Proof. This result follows similarly to Lemma 3.1.1. Take an empty arrow graph and fill its edges according to the relative position between the entries of $\mathbf{a}$ until a belt X is reached. By construction $\mathbf{a}$ is a term of the $\mathrm{ch}(X)$. If a different belt Y is considered, then $\mathrm{D}(X,Y) \neq \emptyset$ and repeating the last argument in the proof of Lemma 3.1.1, we see that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common. □

Corollary 3.2.3. *Different belts have no composition factors in common.*

At this point we can mimic the arguments in Section 3.1 using belts in place of maximal p-shapes.

Definition. Let X be a p-shape. A belt Y is a *refinement* of X if Y can be obtained by adding edges to X.

Corollary 3.2.4. *Let X be a p-shape. The formal character of X is the sum of the formal characters of the refinements of X.*

Proof. Given Y a refinement of X, every term of $\mathrm{ch}(Y)$ is also a term of $\mathrm{ch}(X)$ (just because X is obtained forgetting some arrows of Y). Conversely, each term of $\mathrm{ch}(X)$ is a term of the formal character of a unique refinement of X: for every j and $j+1 \pmod p$ that are not linked by an arrow in X, the relative order of these entries in the chosen term of $\mathrm{ch}(X)$ indicate how they need to be linked to find the desired belt. □

As in Corollary 3.1.4, [K, Theorem 5.3.1] ensures the following.

Corollary 3.2.5. *If X is a p-shape, then the composition factors of X are the composition factors of its refinements.*

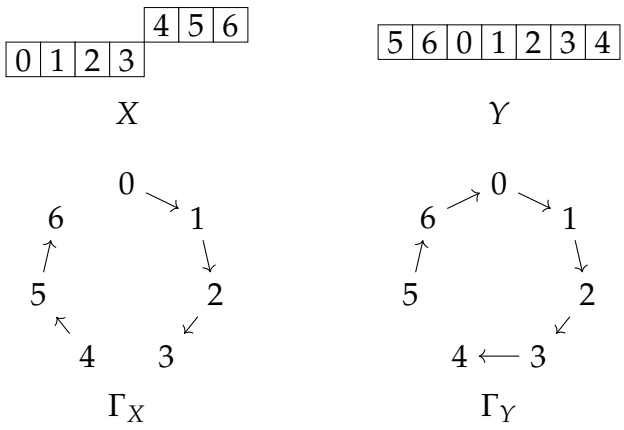
We are know ready to establish a result which gives us a way to recognise when two Specht $\mathcal{C}^k_{l,m}$-modules share a composition factor.

Theorem 3.2.6. *Let X and Y be p-shapes. The following are equivalent:*

1. *X and Y have a composition factor in common;*
2. *The formal characters of X and Y have a term in common;*
3. *$\mathrm{d}(X,Y) = 0$ and $\Gamma_X \cup \Gamma_Y$ is not a directed cycle;*
4. *There is a belt which is a refinement of both X and Y.*

Remark. Before proving this theorem we dwell a moment on the meaning of condition (3). Since $\mathrm{d}(X,Y) = 0$, the overlapping $\Gamma_X \cup \Gamma_Y$ is an arrow graph. We have that $\Gamma_X \cup \Gamma_Y$ is a directed cycle if and only if all arrows in Γ_X and Γ_Y are anticlockwise (or all clockwise) and there is no $k \in \mathbb{Z}/p\mathbb{Z}$ such that $\boxed{k}$ is simultaneously the foot node of a connected component of X and a connected component of Y.

Example. Let $p = 7$, and let X and Y be the p-shapes given below. All arrows in Γ_X and Γ_Y are oriented clockwise (because neither X nor Y has a node immediately above another). Moreover, X and Y do not have foot nodes of the same residue. So $\Gamma_X \cup \Gamma_Y$ is a directed cycle. As a consequence, the formal characters of X and Y have no term in common, so X and Y do not share a composition factor.



Proof of Theorem 3.2.6.

(1⇒ 2) This is trivial because the formal character of a $\mathcal{H}^k_p$-module is the sum of the formal characters of its composition factors.

(2⇒ 3) Suppose that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have a term in common. Then it immediately follows that $\mathrm{d}(X,Y) = 0$. Suppose for a contradiction that $\Gamma_X \cup \Gamma_Y$ is a clockwise directed cycle and set

$$F_X := \{k \in \mathcal{C}(B) \mid \boxed{k} \text{ is the foot node of a connected component of } X\}$$

and

$$F_Y := \{k \in \mathcal{C}(B) \mid \boxed{k} \text{ is the foot node of a connected component of } Y\}.$$

(For example, for the 7-shapes in the example above, $F_X = \{\bar{0}, \bar{4}\}$ and $F_Y = \{\bar{5}\}$.) The remark above shows that $F_X \cap F_Y = \emptyset$. Now since every term of $\mathrm{ch}(X)$ has an element of F_X as first component and every term of $\mathrm{ch}(Y)$ has an element of F_Y as first component, it follows that $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ have no terms in common and this contradicts the hypothesis. (If the arrows in $\Gamma_X \cup \Gamma_Y$ are all anticlockwise, the proof follows similarly replacing "foot" with "hand" in the definitions of F_X and F_Y above.)

(3$\Rightarrow$ 4) By the hypothesis the union $\Gamma_X \cup \Gamma_Y$ is an arrow graph. This can be refined to a belt by adding in arrows if necessary (ensuring that these arrows are not all oriented the same way as all the existing arrows). This belt is a refinement of both X and Y.

(4$\Rightarrow$ 1) This follows from Corollary 3.2.5. □

It seems natural that the set of belts gives a family of simple $\mathcal{H}_p^k$-modules. Despite this, we will not focus on the question in the present document. We write our suspect as a conjecture such that it can inspire further research.

Conjecture 3.2.7. *Let Γ be a belt. Then the $\mathcal{H}_p^k$-module M_Γ is simple.*

Now we construct an approximation to the decomposition matrix as in Section 3.1. We define the matrix D_B having rows indexed by the set of p-shapes $\mathcal{B}$ and colums indexed by belts, with entry $d_{XY} = 1$ if Y is a refinement of X, and 0 otherwise. As in Section 3.1, showing that D_B is a connected matrix is sufficient to show that B is a single block of $\mathcal{C}_{l,m}^k$.

Definition. Let X, Y be p-shapes in $\mathcal{B}$. We say that X and Y are *p-linked* if there exists a series of p-shapes $X = X_1, X_2, \ldots, X_{t-1}, X_t = Y$ in $\mathcal{B}$ such that X_j and X_{j+1} satisfy *any* of the equivalent conditions in Theorem 3.2.6, for every $1 \leqslant j \leqslant t-1$.

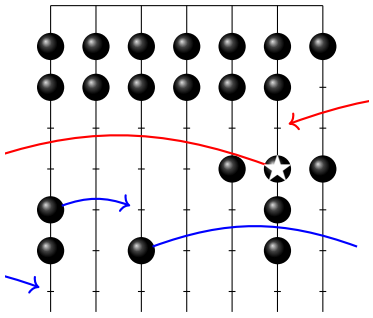
We now move to the heart of this section. First of all we show that we can reduce to the case where two p-shapes in a belt block have distance zero.

Proposition 3.2.8. *Let X and Y be p-shapes in $\mathcal{B}$ with $\mathrm{d}(X,Y) > 0$. There exists a p-shape $Z \in \mathcal{B}$ such that $\mathrm{d}(X,Z) < \mathrm{d}(X,Y)$ and $\mathrm{d}(Y,Z) < \mathrm{d}(X,Y)$.*

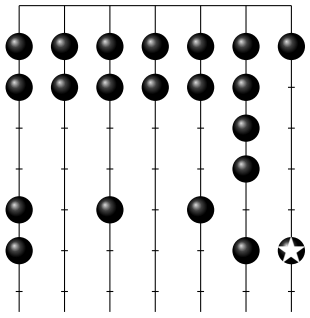
Proof. Let $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ be skew-partitions in B having p-shapes X and Y, respectively, and let $d = \mathrm{d}(X,Y)$. We want to show that there is a p-shape $Z \in \mathcal{B}$ that satisfies the conditions in the statement. The proof of this result is very similar to that of Theorem 3.1.7. Let $i \in \mathrm{D}(X,Y)$. Here we can state and prove an intermediate claim, i.e. a perfect analogue of Claim 1 in Theorem 3.1.7, running for $k \in \{i+1, \ldots, i+p+1\}$ with $k \notin \mathrm{D}(X,Y)$. If for any of these steps, a modifications of μ_X or λ_Y is allowed, then we are done performing the appropriate modification exploited in Theorem 3.1.7. Contrarily, if for all k no modifications have been allowed, we first find integers $k_1, \ldots, k_s$ with $\sum_{1 \leqslant u \leqslant s} k_s = p$, and positions $c_1, \ldots, c_s, d_1, \ldots, d_s$ such that $c_1, \ldots, c_s$ are beads and $d_1, \ldots, d_s$ are spaces in $\mathrm{Ab}(\mu_X)$. The proof then follows observing that μ_X has an addable p-hook at runner i (corresponding to the biggest bead position on runner i of $\mathrm{Ab}(\mu_X)$): the desired partition $\mu_Z \vdash m$ is obtained from μ_X deleting *all* the nodes in $\mu_X \setminus \lambda_X$ and adding them in different positions (moving the bead at c_v to the space at d_v for all v). The choice of $k_1, \ldots, k_s$ guarantees that $\mu_Z \setminus \lambda_X$ has the wanted p-shape Z. □

Example. Let $m = 119$, $l = 112$ and $p = n = 7$. Consider the skew-partitions $\mu_X \setminus \lambda_X = (20, 18, 17, 16, 12^4) \setminus (20, 18, 17, 16, 12^2, 11, 6)$ and $\mu_Y \setminus \lambda_Y = (21^2, 17, 15, 14, 13, 12, 6) \setminus$

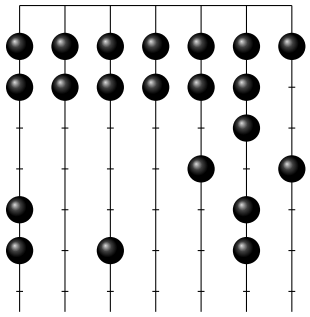
$(20, 16^2, 15, 14, 13, 12, 6)$ in $\mathcal{P}_{112,119}$ having 7-shapes X and Y, respectively. These belong to the same combinatorial block of $\mathcal{C}^k_{112,119}$ as can be easily checked (all the involved partitions have 7-core $(13, 7, 3, 2^2, 1) \vdash 28$). Below we show the abacus configurations with charge 21 and the arrow graphs of X and Y.



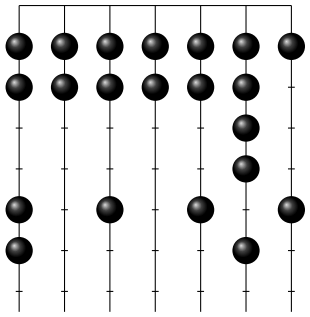
μ_X



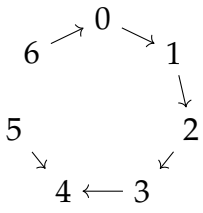
μ_Y



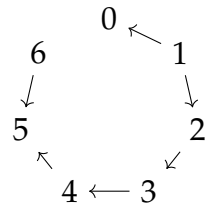
λ_X



λ_Y

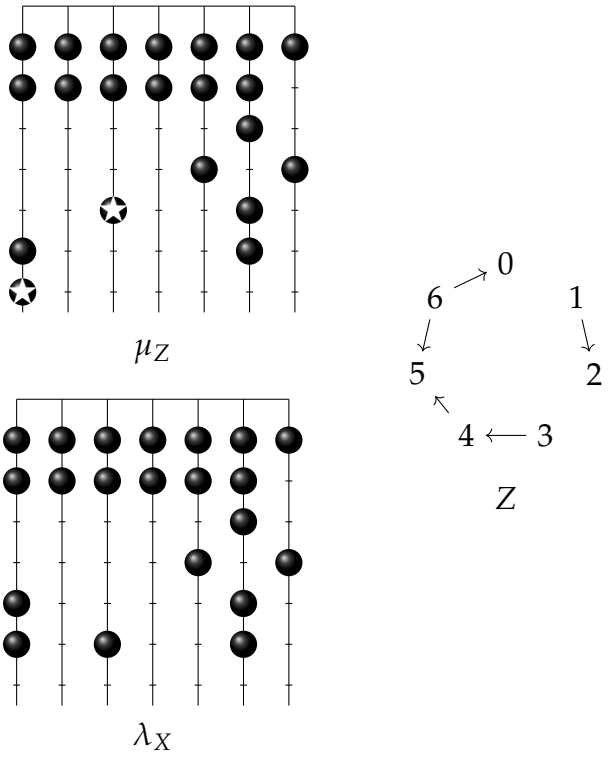


X



Y

We see that $\mathrm{D}(X, Y) = \{\bar{0}, \bar{4}\}$ and hence $\mathrm{d}(X, Y) = 2$. We follow the notation of Theorem 3.1.7. Let $i = \bar{0}$: position $x_0 = 21$ is a space in $\mathrm{Ab}(\mu_X)$ and position $y_0 = 35$ is a bead in $\mathrm{Ab}(\mu_Y)$. Here, for every $1 \leqslant k \leqslant 6$, no modifications of μ_X or λ_Y are allowed by the procedure. Consider the addable 7-hook at runner 0 of $\mathrm{Ab}(\mu_X)$ at position $a_1 = 35$. This gives $d_1 = 42$ from which follows that $k_1 = 5$ and $c_1 = 37$. The partition μ_X has an addable 2-hook at runner 0, this is at position $a_2 = 28$. Hence, $d_2 = 30$ and consequently $k_2 = 2$ and $c_2 = a_2 = 28$. Then $s = 2$ and we stop. For the 'removing' part, since X is a ribbon, $t = 1$ and $j_1 = \bar{5}$. We modify $\mathrm{Ab}(\mu_X)$ according to the rule given in Theorem 3.1.7 obtaining the partition $\mu_Z = (22, 21, 17, 16, 14, 12, 11, 6) \vdash 119$. Denoted by Z the 7-shape of $\mu_Z \setminus \lambda_X$, we see that $\mathrm{d}(X, Z) = 1$ and $\mathrm{d}(Y, Z) = 0$.



Now we use Proposition 3.2.8 to show that the combinatorial block B is actually a p-block of $\mathcal{C}^k_{l,m}$. This is slightly more complicated than in Section 3.1, because it is no longer the case that two p-shapes at distance 0 are p-linked. Take two p-shapes $X, Y \in \mathcal{B}$ such that $\mathrm{d}(X,Y) = 0$. By Theorem 3.2.6, X and Y are p-linked if $\Gamma_X \cup \Gamma_Y$ is not a directed cycle. Hence from now on we suppose that $\Gamma_X \cup \Gamma_Y$ is a directed cycle and we assume without loss of generality that all its arrows are clockwise. The aim of the rest of this section is to prove that X and Y are p-linked.

First of all we establish some notation. This is allowed by the following application of Littlewood's characterization of the p-blocks of S_n.

Proposition 3.2.9. *Let m, l as before. If $\mu \setminus \lambda \in \mathcal{P}_{l,m}$ and $\mathrm{cont}_p(\mu \setminus \lambda) = \{\bar{0}, \ldots, \overline{p-1}\}$, then $\mathrm{core}_p(\lambda) = \mathrm{core}_p(\mu)$ and $\mathrm{weight}_p(\lambda) = \mathrm{weight}_p(\mu) - 1$.*

Proof. Let $\nu = \mathrm{core}_p(\lambda)$, let B be the block of S_l containing λ, and let C be the block of S_m with p-core ν. Observe that the nodes of a removable p-hook have contents $a, a+1, \ldots, a+p-1$ for some a, and therefore have residues $\bar{0}, \bar{1}, \ldots, \overline{p-1}$ in some order. Therefore the p-content of C is obtained from the p-content of B by adding one copy of each residue $\bar{0}, \bar{1}, \ldots, \overline{p-1}$. From Theorem 1.4.2, this means that μ lies in C. □

Every skew-partition $\mu \setminus \lambda$ in B satisfies the hypothesis of Proposition 3.2.9 and hence $\mathrm{core}_p(\mu) = \mathrm{core}_p(\lambda)$. So there is a p-core γ and an integer w such that

$$\mathrm{core}_p(\mu) = \mathrm{core}_p(\lambda) = \gamma, \qquad \mathrm{weight}_p(\lambda) = w, \quad \mathrm{weight}_p(\mu) = w + 1$$

for every skew-partition $\mu \setminus \lambda$ in B. Let $c \in \mathbb{Z}/p\mathbb{Z}$ be the residue of the right-most node in the first part of γ, i.e. $c = \overline{\gamma_1 - 1}$. If γ is the empty partition, we set $c := \overline{p-1}$.

Let L be the p-block of S_l whose partitions all have p-core γ and p-weight w. For each $i \in \mathbb{Z}/p\mathbb{Z}$, let

$$L_i = \{\lambda \in L \mid \overline{\lambda_1 - 1} = i\}.$$

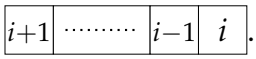
We remark that some L_i can be empty and that

$$L = \bigsqcup_{i \in \mathbb{Z}/p\mathbb{Z}} L_i.$$

For every $\lambda \in L$, let O_λ be the set of partitions $\mu \vdash m$ such that $\mu \setminus \lambda \in B$ and all the arrows of the p-shape of $\mu \setminus \lambda$ are clockwise (i.e. $\mu \setminus \lambda$ does not contain two nodes in the same column).

Remark. Suppose $\lambda_X, \lambda_Y \in L$, $\mu_X \in O_{\lambda_X}$ and $\mu_Y \in O_{\lambda_Y}$, and let X and Y be the p-shapes of $\mu_X \setminus \lambda_X$ and $\mu_Y \setminus \lambda_Y$ respectively. Then $\mathrm{d}(X, Y) = 0$, because all edges are oriented clockwise. Moreover, every pair X, Y such that $\Gamma_X \cup \Gamma_Y$ is a clockwise directed cycle arises in this way.

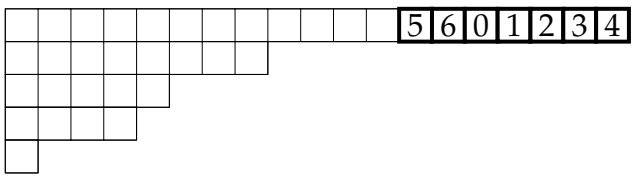
For every $i \in \mathbb{Z}/p\mathbb{Z}$, let X_i be the (ribbon) p-shape



For example, if $p = 7$ and $i = 4$, X_4 is the p-shape Y in the example in Theorem 3.2.6.

Let $\lambda \in L_i$ and let $\mu_\lambda = (\lambda_1 + p, \lambda_2, \ldots, \lambda_{l(\lambda)})$. Then the abacus configuration for μ_λ is obtained from the abacus configuration for λ by moving the last bead down one position. It is easy to see that $\mu_\lambda \in O_\lambda$ and $\mu_\lambda \setminus \lambda$ has p-shape X_i.

Example. Let $m = 37$, $l = 30$ and $p = n = 7$. Consider the partition $\lambda = (12, 8, 5, 4, 1) \vdash 30$. Then $\mu_\lambda = (19, 8, 5, 4, 1)$ and $\mu_\lambda \setminus \lambda$ has 7-shape X_4.



$\mu_\lambda \setminus \lambda$

Now take $\mu \in O_\lambda$ with $\mu \neq \mu_\lambda$, and call X the p-shape of $\mu \setminus \lambda$.

Proposition 3.2.10. *X and X_i are p-linked.*

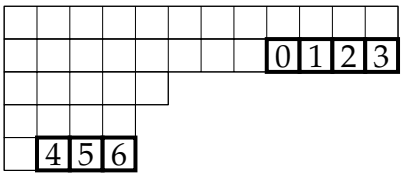
Proof. If a connected component of X has hand node $\boxed{i}$ then we are done because $\mathrm{ch}(X_i)$ and $\mathrm{ch}(X)$ share the term $(i+1, \ldots, i)$. Suppose that this is not the case, and let $j \in \mathbb{Z}/p\mathbb{Z}$ be the residue of the hand node of the connected component of X containing i. We firstly observe that $\mu_1 = \lambda_1$ because otherwise the nodes $(1, \lambda_1 + 1), \ldots, (1, \mu_1)$ of μ would form a connected component of X with $\boxed{i+1}$ as foot node and this would imply that X has a connected component whose hand node is $\boxed{i}$.

Now we can modify μ by moving the nodes of residue $i+1, i+2, \ldots, j$ in $\mu \setminus \lambda$ to the end of the first row. Let $\tilde{\mu}$ be the resulting partition. Then $\tilde{\mu} \setminus \lambda$ is in B and its p-shape $\tilde{X}$ is p-linked with both X_i and X: $\mathrm{ch}(X)$ and $\mathrm{ch}(\tilde{X})$ share the term $(i+1, i+2, \ldots, i)$ while $\mathrm{ch}(X_i)$ and $\mathrm{ch}(\tilde{X})$ share $(j+1, j+2, \ldots, j)$. So X and X_i are p-linked.

□

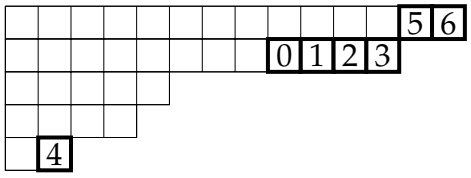
Example. Let m, l, p, n and λ be as in the previous example. Let $\mu = (12^2, 5, 4^2) \in O_\lambda$. The skew-partition $\mu \setminus \lambda$ belongs to the combinatorial block B of $\mathcal{C}^k_{30,37}$ characterised by the 7-core

$\gamma := \mathrm{core}_7(B) = (1^2) \vdash 2$. Call X the 7-shape of $\mu \setminus \lambda$.



$\mu \setminus \lambda$

Retracing the notation of Proposition 3.2.10, we have that $i = \bar{4}$ (because $\lambda_1 - 1 = 11 \equiv 4 \pmod 7$ and then $\lambda \in L_4$) and $j = \bar{6}$ because the node $\boxed{4}$ belongs to the connected component of X having hand node $\boxed{6}$. So we find $\tilde{\mu}$ by moving the nodes $\boxed{5}\boxed{6}$ to the end of the first row, giving $\tilde{\mu} = (14, 12, 5, 4, 2) \vdash 37$.



$\tilde{\mu} \setminus \lambda$

Now the 7-shape $\tilde{X}$ of $\tilde{\mu} \setminus \lambda$, is 7-linked with both X and X_4.

Remark. If $\lambda_1, \lambda_2 \in L_i$, then $\mu_{\lambda_1} \setminus \lambda_1$ and $\mu_{\lambda_2} \setminus \lambda_2$ have the same p-shape X_i. By the previous proposition we deduce that, for every $\mu_1 \in O_{\lambda_1}$ and $\mu_2 \in O_{\lambda_2}$, the p-shapes of $\mu_1 \setminus \lambda_1$ and $\mu_2 \setminus \lambda_2$ are p-linked.

We are now very close to the end of this section. The next result will link p-shapes X_i, X_j for all pairs of $i, j \in \mathbb{Z}/p\mathbb{Z}$. Before this we note that the set L_c is non-empty: for example, it contains the partition $\overline{\gamma} = (\gamma_1 + wp, \gamma_2, \gamma_3, \dots)$. In particular this shows that B contains at least a skew-partition with p-shape X_c.

Proposition 3.2.11. *Let $i \in \mathbb{Z}/p\mathbb{Z}$ such that $L_i \neq \emptyset$. Then X_i and X_c are p-linked.*

Proof. Let $\lambda \in L_i$ and consider the skew-partitions $\mu_\lambda \setminus \lambda$ and $\mu_{\overline{\gamma}} \setminus \overline{\gamma}$ in B having p-shapes X_i and X_c, respectively. For convenience we suppose that $c = \bar{0}$, though the proof can easily be modified for other values of c.

If $i = \bar{0}$, the result follows by the previous remark.

Suppose that $i \neq \bar{0}$. First of all we observe that in the abacus configuration of γ, the leftmost runner, i.e. runner 0, has the highest number of beads among the runners of $\mathrm{Ab}(\gamma)$. In symbols, following the notation of Section 1.4, we have that for every $j \in \mathbb{Z}/p\mathbb{Z} \setminus \{\bar{0}\}$,

$$b_0(\gamma) > b_j(\gamma). \tag{3.1}$$

The abacus configuration of λ is obtained by lowering down beads in $\mathrm{Ab}(\gamma)$ a total amount of w times. Now recall the p-quotient $(\lambda^{(0)}, \dots, \lambda^{(p-1)})$ of λ. The fact that $\lambda \in L_i$ implies that

$$\lambda^{(i)}{}_1 \geqslant \lambda^{(0)}{}_1 + b_0(\lambda) - b_i(\lambda), \tag{3.2}$$

where $\lambda^{(i)}{}_1$ and $\lambda^{(0)}{}_1$ denote the number of spaces above the lowest beads in the i-th and 0-th runners of $\mathrm{Ab}(\lambda)$, respectively. Recalling that $b_j(\lambda) = b_j(\gamma)$ for all $j \in \mathbb{Z}/p\mathbb{Z}$, we combine (1) and (2) getting

$$\lambda^{(i)}{}_1 > \lambda^{(0)}{}_1. \tag{3.3}$$

By construction, if $(\mu_\lambda^{(0)},\dots,\mu_\lambda^{(p-1)})$ denotes the p-quotient of μ_λ, then $(\mu_\lambda^{(j)})_1 = \lambda^{(j)}{}_1$ for every $j \in \mathbb{Z}/p\mathbb{Z} \setminus \{i\}$ and

$$(\mu_\lambda^{(i)})_1 = \lambda^{(i)}{}_1 + 1. \tag{3.4}$$

Comparing (3) and (4) we find that

$$(\mu_\lambda^{(i)})_1 > (\mu_\lambda^{(0)})_1 + 1.$$

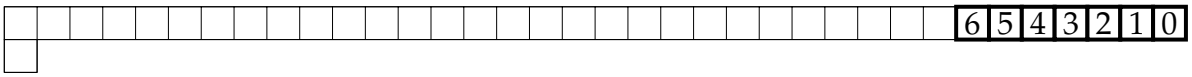
This inequality implies that μ_λ has at least two addable i-hooks at runner 0. We consider one of them, say at position $a_1 \equiv 0 \pmod p$, avoiding the one that affects the space immediately above the lowest bead in the i-runner of $\mathrm{Ab}(\mu_\lambda)$, if necessary. Now let $1 \leqslant k_1 \leqslant i$ be the least integer such that $a_1 + i - k_1$ is a bead. If $k_1 < i$, by (1) and the fact that $b_j(\mu_\lambda) = b_j(\gamma)$ for all $j \in \mathbb{Z}/p\mathbb{Z}$, μ_λ has an addable $(i-k_1)$-hook at runner 0, say at position a_2. Then we start again the process from position a_2: let k_2 be minimal such that position $a_2 + i - k_1 - k_2$ is a bead. It is clear that this procedure terminates at a step s such that $i = \sum_{1\leqslant u\leqslant s} k_u$. Then for $1 \leqslant v \leqslant s$ we have positions $c_v := a_v + i - \sum_{1\leqslant u\leqslant v} k_u$ and $d_v := a_v + i - \sum_{1\leqslant u\leqslant v-1} k_u$ such that c_v is a bead and $c_v + 1,\dots,d_v$ are all spaces in $\mathrm{Ab}(\mu_\lambda)$. Then we modify $\mathrm{Ab}(\mu_\lambda)$ as follows (remind that C denotes the charge of the abacus displays):

- ◇ move the star bead at $\lambda_1 + C + p - 1$ to the space at $\lambda_1 + C + p - i - 1$,
- ◇ for every $1 \leqslant v \leqslant s$, move the bead at c_v to the space d_v.

The resulting partition $\hat{\mu} \vdash m$ is such that $\hat{\mu} \setminus \lambda \in B$ and has p-shape $\hat{X}$ that is p-linked with both X_i and X_c: the formal characters $\mathrm{ch}(X_i)$ and $\mathrm{ch}(\hat{X})$ share the term $(i+1,\dots,i)$ while $\mathrm{ch}(X_c)$ and $\mathrm{ch}(\hat{X})$ share the term $(\bar{1},\bar{2},\dots,\overline{p-1},\bar{0})$. This ends the proof.

□

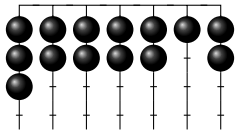
Example. Let B the combinatorial block of $\mathcal{C}^k_{30,37}$ from the previous examples. As already pointed out, following the notation, $\gamma = (1^2)$ and then $c = \gamma_1 - 1 \equiv 0 \pmod 7$. We see that $\bar{\gamma} = (29,1)$ and $\mu_{\bar{\gamma}} = (36,1)$.



$\mu_{\bar{\gamma}} \setminus \bar{\gamma}$

We show how to modify the skew-partition μ_λ from the previous example, to find a 7-shape that is 7-linked with both X_4 and X_0.

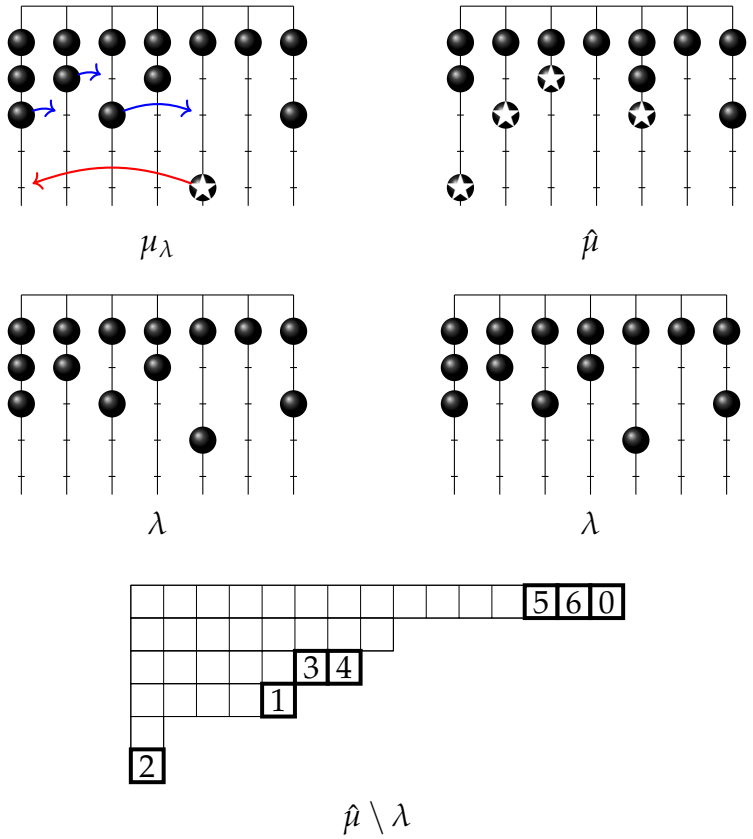
We first draw the abacus configuration of γ (all abacus configurations in the example have charge $C = 14$).



γ

Note that as remarked in the proof of Proposition 3.2.11, the number of beads in runner 0 of γ, $b_0(\gamma) = 3$, is bigger than the number of beads in any other runner of $\mathrm{Ab}(\gamma)$. Our proof points out that μ_λ has at least an addable 4-hook at runner 0 (in fact it has two). Take $a_1 = 14$ and $d_1 = 18$. We have that $k_1 = 2$ and so $c_1 = 16$. Now we consider the addable 2-hook of μ_λ at position $a_2 = 7$. This gives $d_2 = 9$ and consequently $k_2 = 1$ and $c_2 = 8$. Finally we

take the addable node at runner 0 of μ_λ at position $a_3 = 14$. We have that $d_3 = 15$, giving $k_3 = 1$ and $c_3 = 14$ (then $s = 3$). We modify $\mathrm{Ab}(\mu_\lambda)$ following the rule given in the proof of Proposition 3.2.11 (shown in figure). The resulting partition $\hat{\mu} = (15, 8, 7, 5, 1^2) \vdash 37$ is such that $\hat{\mu} \setminus \lambda$ has the desired 7-shape $\hat{X}$.



We have finally gained the key result of this section.

Corollary 3.2.12. *Suppose $X, Y \in \mathcal{B}$. Then X and Y are p-linked.*

Proof. By Proposition 3.2.8, we can reduce to the case when $\mathrm{d}(X, Y) = 0$. Moreover, Theorem 3.2.6 tells that we are done if $\Gamma_X \cup \Gamma_Y$ is not an oriented cycle. In the opposite case, the description of the remaining cases together with Propositions 3.2.10 and 3.2.11 ends the proof. □

Now, Corollary 3.2.12 and Theorem 3.2.6 imply that the decomposition matrix D_B of the belt block B is connected. As remarked along the treatment, this is enough to state our last result.

Corollary 3.2.13. *Conjecture 2.4.1 holds for belt blocks.*

Part II

Iwahori–Hecke algebras of type B

Chapter 4

Iwahori–Hecke algebras and their combinatorics

The second part of the thesis is focused on *Iwahori–Hecke algebras of type B*. These algebras, explicitly defined in Section 4.3 and denoted by $\mathcal{H}_n$, were first systematically studied by Dipper, James and Murphy [DJM], who introduced appropriate definitions of Specht modules labelled by bipartitions of n. This naturally leads to a combinatorial approach to the corresponding decomposition number problem, underpinned by the combinatorics of bipartitions.

Fayers introduced a combinatorial definition of the weight of a bipartition, and hence the weight of a block [F1], in the attempt of generalizing the classic notion of weight of a partitions we have seen in Section 1.4. The main advantage of having a weight is that it allows one to approach the decomposition number problem systematically by addressing blocks of a fixed weight. As in type A, the blocks of weight 0 are precisely the simple blocks, and blocks of weight 1 admit a very simple description (analogous to symmetric group blocks with cyclic defect groups). In [F2], Fayers gave a combinatorial formula for decomposition number for blocks of weight 2, analogous to Richards's formula [R] for type A.

In this part of the thesis we address blocks of weight 3. Our main result is as follows.

Theorem 4.0.1. *Suppose $\mathbb{F}$ is a field, $q \in \mathbb{F}$ is non-zero, and $k_1, k_2 \in \mathbb{Z}$. Let $\mathcal{H}_n$ denote the Iwahori–Hecke algebra of type B over $\mathbb{F}$ with parameters q, q^{k_1}, q^{k_2}, and let B be a block of $\mathcal{H}_n$ of weight 3. Then the decomposition numbers for B are all either 0 or 1.*

Note in particular that (unlike in type A) Theorem 4.0.1 applies even when $\mathbb{F}$ has characteristic 2 or 3.

Throughout this part, we fix an integer $e \geqslant 2$ and a pair $\kappa = (\kappa_1, \kappa_2) \in (\mathbb{Z}/e\mathbb{Z})^2$. A pair of integers k_1, k_2 such that $\kappa_1 = \overline{k_1}$ and $\kappa_2 = \overline{k_2}$ is called a *bicharge* for κ.

4.1 Bipartitions

A *bipartition* is an ordered pair $\lambda = \left(\lambda^{(1)} \,\middle|\, \lambda^{(2)}\right)$ of partitions, which we call the *components* of λ. We write $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}|$, and say that λ is a bipartition of $|\lambda|$.

Since we have two partitions at a time, we reset the notation for Young diagrams. The Young diagram of a bipartition λ is the set

$$[\lambda] = \left\{ (r, c, a) \in \mathbb{N}^2 \times \{1, 2\} \;\middle|\; c \leqslant \lambda_r^{(a)} \right\},$$

whose elements we call again the *nodes* of λ. In general, as before, a node means an element of $\mathbb{N}^2 \times \{1,2\}$. A node (r,c,a) of λ is *removable* if it can be removed from $[\lambda]$ to leave the Young diagram of a smaller bipartition (that is, if $c = \lambda_r^{(a)} > \lambda_{r+1}^{(a)}$), while a node not in λ is an addable node of λ if it can be added to $[\lambda]$ to leave a Young diagram.

We depict $[\lambda]$ by drawing the Young diagram of $\lambda^{(1)}$ above the Young diagram of $\lambda^{(2)}$ (using the English convention). Accordingly, we will say that a node (r,c,a) is *above* or *higher than* a node (s,d,b) if either $a < b$ or $a = b$ and $r < s$.

If λ and μ are two bipartitions of n, then we say that λ *dominates* μ (and write $\lambda \trianglerighteq \mu$) if

$$\lambda_1^{(1)} + \cdots + \lambda_r^{(1)} \geqslant \mu_1^{(1)} + \cdots + \mu_r^{(1)}$$

and

$$|\lambda^{(1)}| + \lambda_1^{(2)} + \cdots + \lambda_r^{(2)} \geqslant |\mu^{(1)}| + \mu_1^{(2)} + \cdots + \mu_r^{(2)}$$

for every $r \geqslant 1$. Another way of saying this is that $[\mu]$ can be obtained from $[\lambda]$ by moving one or more nodes to lower positions.

If λ is a bipartition, the conjugate bipartition is $\lambda' = \left(\lambda^{(2)'} \mid \lambda^{(1)'}\right)$. In other words, λ' is obtained by interchanging the two components, and then replacing each component with the conjugate partition. Note then that if λ, μ are two bipartitions, then $\lambda \trianglerighteq \mu$ if and only if $\mu' \trianglerighteq \lambda'$.

Now with our fixed pair $\kappa = (\kappa_1, \kappa_2) \in (\mathbb{Z}/e\mathbb{Z})^2$ at hand, we define the *residue* of a node (r,c,a), denoted $\mathrm{res}(r,c,a)$, as $c - r + \kappa_a \in \mathbb{Z}/e\mathbb{Z}$. If $i \in \mathbb{Z}/e\mathbb{Z}$, then an *i-node* means a node of residue i. If λ is a bipartition, then the *e-content* of λ is the multiset of the residues of the nodes of λ. For any $i \in \mathbb{Z}/e\mathbb{Z}$, we write $\mathrm{rem}_i(\lambda)$ for the number of removable i-nodes of λ.

We will also need to consider rim hooks. The *rim* of a bipartition λ is the set of nodes $(r,c,a) \in [\lambda]$ such that $(r+1, c+1, a) \notin [\lambda]$. Practically, this is the union of the rims of its components. A *rim hook* of λ is a connected subset of the rim which can be removed to leave a smaller bipartition (so it is a rim hook of a single component of λ).

Given a bipartition $\lambda = \left(\lambda^{(1)} \mid \lambda^{(2)}\right)$ and a bicharge $(k_1, k_2) \in \mathbb{N}^2$ with $k_a \geqslant h(\lambda^{(a)})$ for $a = 1,2$, the abacus configuration for λ with bicharge (k_1, k_2) is obtained by placing the abacus configuration for $\lambda^{(1)}$ with charge k_1 above the abacus configuration for $\lambda^{(2)}$ with charge k_2. All the abacus configurations in this part will have e runners. We often write $\mathrm{Ab}(\lambda)$ for this object (so in what follows this symbol will denote a pair of abacus configurations).

4.2 Restricted bipartitions

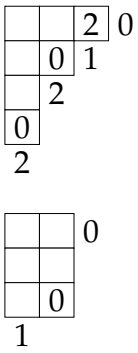
In [AM], Ariki and Mathas introduced a set of multipartitions to provide a combinatorial labelling for simple modules of Ariki–Koike algebras. They called these *Kleshchev multipartitions*, but here we prefer the term *restricted multipartitions* introduced by Brundan–Kleshchev [BK]. We now recall the definition here, specialising to the case of bipartitions. We will use the original recursive definition of Ariki–Mathas, though we note that a more efficient definition was found by Ariki, Kreiman and Tsuchioka [AKT].

The definition of restricted bipartitions depends on our fixed $\kappa \in (\mathbb{Z}/e\mathbb{Z})^2$. Take a bipartition λ and a residue $i \in \mathbb{Z}/e\mathbb{Z}$. Define the *i-signature* of λ to be the sequence of signs obtained by reading the Young diagram of λ from top to bottom, writing $+$ for each addable i-node and $-$ for each removable i-node. The *reduced i-signature* is the subsequence obtained by repeatedly deleting adjacent pairs $-+$. The removable nodes corresponding to the $-$

signs are called the *normal* i-nodes of λ, and the highest of these (if there are any) is the *good* i-node. The addable nodes corresponding to the $+$ signs in the reduced i-signature are the *conormal* i-nodes of λ, and the lowest of these (if there are any) is the *cogood* i-node.

For any bipartition μ and any $i \in \mathbb{Z}/e\mathbb{Z}$, we write $\mathrm{nor}_i(\mu)$ for the number of normal i-nodes of μ.

Example. Suppose $e = 3$, $\kappa = (\bar{0}, \bar{1})$ and $\lambda = \left(3, 2, 1^2 \mid 2^3\right)$. The Young diagram of λ is shown below, with the residues of the addable and removable nodes indicated.



If we let $i = \bar{0}$, then we see that the i-signature of λ is $+ - - + -$. So the reduced i-signature is $+ - -$. So there are two normal i-nodes $(2,2,1)$ and $(3,2,2)$, with $(2,2,1)$ being the good i-node. There is a unique conormal (and therefore cogood) i-node $(1,4,1)$.

Now given two bipartitions λ and μ, write $\mu \xrightarrow{i} \lambda$ if μ is obtained from λ by removing the good i-node for $i \in \mathbb{Z}/e\mathbb{Z}$, or equivalently if λ is obtained from μ by adding the cogood i-node. We write $\mu \longrightarrow \lambda$ to mean that $\mu \xrightarrow{i} \lambda$ for some i, and let $\longleftrightarrow$ be the equivalence relation generated by $\longrightarrow$. Now define the set of restricted bipartitions (with respect to κ) to be the bipartitions in the $\longleftrightarrow$-class containing $\left(\varnothing \mid \varnothing\right)$.

The definition of restricted bipartitions derives from the theory of crystals for highest-weight representations of quantum groups. It follows from this theory (in particular, the uniqueness of highest-weight vectors) that the only restricted bipartition with no normal nodes of any residue is $\left(\varnothing \mid \varnothing\right)$. So to test whether a bipartition λ is restricted, one can repeatedly remove arbitrarily-chosen good nodes until there are no good nodes remaining; then λ is restricted if and only if the resulting bipartition is $\left(\varnothing \mid \varnothing\right)$.

We also make an observation which will be helpful later on. Say that a partition λ is *e-restricted* if $\lambda_r - \lambda_{r+1} < e$ for all r. Now the following is a well-known result; it reflects that restricted bipartitions label the vertices of a level-2 highest-weight crystal obtained as a component of the tensor product of two level-1 crystals, which themselves have vertices labelled by e-restricted partitions.

Lemma 4.2.1. *Suppose λ is a restricted bipartition. Then both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted partitions.*

Proof. It is an easy exercise to check that if λ, μ are bipartitions with $\mu \longrightarrow \lambda$, and $a \in \{1,2\}$, then $\lambda^{(a)}$ is e-restricted if and only if $\mu^{(a)}$ is. Since the components of the empty bipartition $\left(\varnothing \mid \varnothing\right)$ are both e-restricted, the result follows. □

We will also need to consider *regular* bipartitions. These are described in [F2, 1.3.3], where they are called *conjugate Kleshchev* bipartitions. We say that a bipartition λ is *regular*

if λ' is restricted. In fact we can modify the above definition of restricted bipartitions to give an alternative description of regular bipartitions, which will be useful later. To do this, define the *i*-signature of a bipartition as above, and define the *antireduced i*-signature by successively deleting pairs $+-$. The removable nodes corresponding to $-$ signs in the antireduced *i*-signature are called the *antinormal i*-nodes of λ, and the lowest of these (if there are any) is the *antigood i*-node. Similarly, the addable nodes corresponding to $+$ signs are the *anticonormal i*-nodes of λ, and the highest of these (if there are any) is the *anticogood i*-node. Given two bipartitions λ and μ, we write $\mu \stackrel{i}{\Longrightarrow} \lambda$ if μ is obtained from λ by removing the antigood *i*-node. Equivalently, if we write $i' = \kappa_1 + \kappa_2 - i$, then $\mu \stackrel{i}{\Longrightarrow} \lambda$ if and only if $\mu' \stackrel{i'}{\longrightarrow} \lambda'$. Now let $\Longrightarrow$ be the union of the relations $\stackrel{i}{\Longrightarrow}$, and let $\Longleftrightarrow$ be the equivalence relation generated by $\Longrightarrow$. The bipartitions in the $\Longleftrightarrow$-class containing $(\varnothing \mid \varnothing)$ are the regular bipartitions.

4.3 Iwahori–Hecke algebras

We fix a field $\mathbb{F}$ with q, Q_1, Q_2 non-zero elements of $\mathbb{F}$, and let $\mathcal{H}_n$ denote the *Iwahori–Hecke algebra of type B* with parameters q, Q_1, Q_2. This is the unital associative $\mathbb{F}$-algebra with generators $T_0, \ldots, T_{n-1}$, subject to relations

$$\begin{aligned}
(T_0 - Q_1)(T_0 - Q_2) &= 0, \\
(T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leqslant i < n, \\
T_i T_j &= T_j T_i && \text{if } j - i > 1, \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i < n-1.
\end{aligned}$$

The definition of $\mathcal{H}_n$ is unaffected if we interchange Q_1 and Q_2, but it will be important for us to consider (Q_1, Q_2) as an ordered pair; in particular, the construction of Specht modules depends on this ordered pair. (In Section 4.5 below, we consider in more detail the effect of interchanging Q_1 and Q_2.)

As explained in [F2, 1.3.2], using a Morita equivalence result of Dipper–James [DJ], we can restrict attention to the case where Q_1 and Q_2 are powers of q. We also assume that q is a primitive e-th root of unity, where $e \geqslant 2$; the case where q is not a root of unity works in the same way as the case where $e > n$ (and in fact most of the particular cases of blocks that we consider only arise if q is an e-th root of unity for $e \leqslant n$).

So with our fixed pair $\kappa = (\kappa_1, \kappa_2) \in (\mathbb{Z}/e\mathbb{Z})^2$, we will assume henceforth that $Q_1 = q^{k_1}$ and $Q_2 = q^{k_2}$, where (k_1, k_2) is any bicharge for κ.

As it happens for the symmetric group and the type *A* algebras, the representation theory of $\mathcal{H}_n$ is based on the theory of *Specht modules*. For each bipartition λ of n, we write S^λ for the corresponding Specht module, as defined in Mathas's survey article [M2]. The Specht modules are precisely the cell modules with respect to a particular cellular basis of $\mathcal{H}_n$, and a central problem in the representation theory of $\mathcal{H}_n$ is to determine their composition factors. To do this, we need a classification of the simple $\mathcal{H}_n$-modules. If λ is a restricted bipartition, then S^λ has a unique simple quotient D^λ, and the simple modules D^λ arising in this way give a complete irredundant set of simple $\mathcal{H}_n$-modules. Thus the problem of determining the composition factors of the Specht modules amounts to determining the *decomposition*

numbers $d_{\lambda\mu}$, where λ, μ are bipartitions of n with μ restricted. To make the notation clearer, thoughout this part, we will write $[S^\lambda : D^\mu]$ instead of $d_{\lambda\mu}$.

A fundamental result on decomposition numbers is the following.

Proposition 4.3.1 [F2, Theorem 1.2]. *If λ and μ are bipartitions of n with μ restricted, then $[S^\mu : D^\mu] = 1$, while if $[S^\lambda : D^\mu] > 0$ then $\lambda \trianglerighteq \mu$.*

4.4 Blocks and weight

An important tool for studying the decomposition number problem is the classification of blocks, since $[S^\lambda : D^\mu] = 0$ unless S^λ and D^μ lie in the same block. It is a consequence of the general theory of cellular algebras that a Specht module always belongs to a single block, and we will abuse notation by saying that λ lies in a block B to mean that S^λ lies in B. Of course, if λ is restricted then D^λ and S^λ lie in this block. Hence in order to classify the blocks of $\mathcal{H}_n$ it is sufficient to determine when two Specht modules lie in the same block. The block classification for $\mathcal{H}_n$ is a special case of the result for Ariki–Koike algebras proved by Lyle and Mathas.

Proposition 4.4.1 [LM, Theorem 2.11]. *Suppose λ and μ are bipartitions of n. Then S^λ and S^μ lie in the same block of $\mathcal{H}_n$ if and only if λ and μ have the same e-content.*

As explained in [F1, 3.2], this result can also be formulated in terms of the abacus: if we choose a bicharge (k_1, k_2) for κ with k_1, k_2 sufficiently large, and construct the abacus configurations for λ and μ with this bicharge, then λ and μ have the same content if and only if for each i the number of beads on runner i (across both components) is the same in each abacus configuration. This leads to another useful way to determine whether two bipartitions belong to the same block. If λ is a bipartition, then for each $i \in \mathbb{Z}/e\mathbb{Z}$ we define $\delta_i(\lambda)$ to be the number of removable i-nodes of λ minus the number of addable i-nodes of λ. Now we have the following result.

Proposition 4.4.2 [F1, Proposition 3.2]. *Suppose λ and μ are bipartitions of n. Then λ and μ lie in the same block if and only if $\delta_i(\lambda) = \delta_i(\mu)$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.*

As a consequence we can define $\delta_i(B)$ for a block B as $\delta_i(B) := \delta_i(\lambda)$ for any bipartition λ in B. If $a \in \mathbb{Z}$, then we may write $\delta_a(B)$ to mean $\delta_{a+e\mathbb{Z}}(B)$.

In [F1], Fayers introduced the notion of *weight* of a bipartition in order to generalize the notion of weight of a partition mentioned in Section 1.4. Consider the e-content of λ and for $i \in \mathbb{Z}/e\mathbb{Z}$ write $c_i(\lambda)$ for the number of i's in it, i.e. the number of i-nodes of λ. The *weight* of λ is defined as the non-negative integer

$$\mathrm{wt}(\lambda) = c_{k_1}(\lambda) + c_{k_2}(\lambda) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_i(\lambda) - c_{i+1}(\lambda))^2.$$

Example. Let $e = 3$ and $k = (\bar{1}, \bar{2})$ and $\lambda = \left(6^2, 1 \;\middle|\; 3, 2^2, 1\right)$. We draw the Young diagram of λ

filling every node with its residue.

1	2	0	1	2	0
0	1	2	0	1	2
2					

2	0	1
1	2	
0	1	
2		

We see that $c_0(\lambda) = 6$, $c_1(\lambda) = 7$ and $c_2(\lambda) = 8$ and hence the above formula gives $\mathrm{wt}(\lambda) = 12$.

[F2, 2.2.2] shows the non immediate fact that when considering a partition λ as a multipartition (λ) with a single component, then $\mathrm{wt}(\lambda) = \mathrm{weight}_e(\lambda)$.

Since the weight depends only on the content, it is the same for all bipartitions in a given block, and therefore one may define the weight of a block B to be the weight of any bipartition in B. The weight of B measures the complexity the block, and a fruitful approach to the representation theory of $\mathcal{H}_n$ is to consider blocks of a given small weight. We remark that blocks of weight zero exactly coincide with simple blocks.

We will use the following lemma several times.

Lemma 4.4.3 [F1, Lemma 3.6]. *Suppose λ is a bipartition, and that μ is a bipartition obtained by removing u removable i-nodes from λ. Then*

$$\mathrm{wt}(\mu) = \mathrm{wt}(\lambda) + u(\delta_i(\lambda) - u).$$

The work contained in Part II describes the decomposition matrix of a weight 3 block of $\mathcal{H}_n$. In type A we have that blocks of weight at most 3 - apart from a few exceptions for small characteristics - have all decomposition numbers equal to 0 or 1 (see [R, F4]). Our work is motivated by the following result.

Theorem 4.4.4. *Suppose B is a block of $\mathcal{H}_n$ of weight 0, 1 or 2, and λ, μ are bipartitions in B with μ restricted. Then $[S^\lambda : D^\mu] \leqslant 1$.*

Proof. The result for blocks of weight 0 follows from [F1, Theorem 4.1], and the result for weight 1 is [F1, Theorem 4.2]. For blocks of weight 2, the result is given in [F2, Theorems 3.18 & 4.13]. □

The aim is to prove the same result for blocks of weight 3. In fact for blocks of weight at most 2 explicit formulæ for the decomposition numbers are known; for weight 3 we just show that the decomposition numbers are at most 1 (which is enough to enable the decomposition numbers to be computed quickly using the Jantzen–Schaper formula of Section 4.7). This is analogous to the situation for Hecke algebras of type A [F4].

4.5 Conjugation and component-switching

The set of bipartitions of n has very natural symmetries, given by conjugation or by simply interchanging components. These symmetries provide useful tools for studying decomposition numbers, which we describe here.

We begin with conjugation, which relates to automorphisms. Let θ be the automorphism of $\mathcal{H}_n$ defined defined by mapping $T_0 \mapsto q^{k_1+k_2}T_0^{-1}$ and $T_i \mapsto -qT_i^{-1}$ for $i \geqslant 1$. Note that the inverse of the generators of $\mathcal{H}_n$ are defined; simple calculations show that

$$T_0^{-1} = -q^{-(k_1+k_2)}T_0 + q^{-(k_1+k_2)}(q^{k_1} + q^{k_2})$$

and

$$T_i^{-1} = q^{-1}T_i + (1 - q^{-1})$$

for $1 \leqslant i \leqslant n-1$. For any $\mathcal{H}_n$-module M, define the module M^θ by twisting the $\mathcal{H}_n$-action by θ. Obviously if M is simple then so is M^θ, and so we can define a bijection $\mu \mapsto \mu^\diamond$ from the set of restricted bipartitions to the set of regular bipartitions by $(\mathrm{D}^\mu)^\theta \cong \mathrm{D}^{(\mu^\diamond)'}$. Understanding what the automorphism θ does to Specht modules gives us more information on decomposition numbers, as follows.

Proposition 4.5.1. *Suppose λ and μ are bipartitions of n with μ restricted. Then $[\mathrm{S}^\lambda : \mathrm{D}^\mu] = [\mathrm{S}^{\lambda'} : \mathrm{D}^{(\mu^\diamond)'}]$.*

Proof. Our argument is based on [M1, Sections 4 and 5], where Mathas recalls the *dual Specht module* $\mathrm{S}'(\lambda)$ for each λ. The dual Specht modules are related to the usual Specht modules in two ways. On the one hand, one can check that $\mathrm{S}'(\lambda)$ is isomorphic to $(\mathrm{S}^\lambda)^\theta$. To see this, we recall how S^λ is constructed: a certain element $m_\lambda \in \mathcal{H}_n$ is defined, and S^λ is defined to be the right ideal generated by m_λ, modulo its intersection with the two-sided ideal generated by the elements m_ν for $\nu \rhd \lambda$. The dual Specht module $\mathrm{S}'(\lambda)$ is defined similarly using a different element n_λ. But one can check that $\theta(m_\lambda)$ equals n_λ times an invertible element of $\mathcal{H}_n$, so that the (right or two-sided) ideal generated by $\theta(m_\lambda)$ coincides with the (right or two-sided) ideal generated by n_λ. As a consequence, we obtain $[\mathrm{S}^\lambda : \mathrm{D}^\mu] = [\mathrm{S}'(\lambda) : (\mathrm{D}^\mu)^\theta]$ for all λ, μ.

On the other hand, Mathas considers contragredient duality $M \mapsto M^\circledast$ induced by the antiautomorphism of $\mathcal{H}_n$ that fixes each T_i. In [M1, Corollary 5.7] Mathas shows that $\mathrm{S}'(\lambda) \cong (\mathrm{S}^{\lambda'})^\circledast$. Since contragredient duality preserves composition multiplicities, and the simple modules are self-dual (which follows from the cellularity of $\mathcal{H}_n$), we find that $[\mathrm{S}'(\lambda) : \mathrm{D}^\mu] = [\mathrm{S}^{\lambda'} : \mathrm{D}^\mu]$ for every λ, μ. The result follows. □

Corollary 4.5.2. *Suppose λ and μ are bipartitions of n, with μ restricted. Then $[\mathrm{S}^{\mu^\diamond} : \mathrm{D}^\mu] = 1$, while if $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 0$ then $\mu^\diamond \trianglerighteq \lambda$.*

Proof. This follows from Propositions 4.3.1 and 4.5.1 using the fact that conjugation reverses the dominance order on bipartitions. □

Later we will need to be able to calculate the bijection $^\diamond$ combinatorially. Recall the relations $\xrightarrow{i}$ and $\overset{i}{\Longrightarrow}$ from Section 4.2.

Proposition 4.5.3. *Suppose λ and μ are restricted bipartitions. Then $\mu \xrightarrow{i} \lambda$ if and only if $\mu^\diamond \overset{i}{\Longrightarrow} \lambda^\diamond$.*

Proof. The Brundan–Kleshchev modular branching rules show that the relation $\xrightarrow{i}$ can be described algebraically via

$$\mu \xrightarrow{i} \lambda \qquad \text{if and only if} \qquad \mathrm{Hom}_{\mathcal{H}_{n-1}}(\mathrm{D}^\mu, \mathrm{e}_i \mathrm{D}^\lambda) \neq 0,$$

where e_i is the i-th restriction functor (see Section 4.6 below). The definition of e_i and the automorphism θ shows that $(e_i M)^\theta = e_{i'}(M^\theta)$ for any module M, where as above $i' = \kappa_1 + \kappa_2 - i$. Hence

$$\mathrm{Hom}_{\mathcal{H}_{n-1}}(D^\mu, e_i D^\lambda) \neq 0 \qquad \text{if and only if} \qquad \mathrm{Hom}_{\mathcal{H}_{n-1}}(D^{(\mu^\diamond)'}, e_{i'} D^{(\lambda^\diamond)'}) \neq 0,$$

and therefore

$$\mu \xrightarrow{i} \lambda \qquad \text{if and only if} \qquad (\mu^\diamond)' \xrightarrow{i'} (\lambda^\diamond)',$$

and from Section 4.2 this is equivalent to $\mu^\diamond \xRightarrow{i} \lambda^\diamond$. □

Now we come to component-switching, where we don't need to say as much. If λ is a bipartition, let $\lambda^\leftrightarrow$ denote the bipartition $\left(\lambda^{(2)} \mid \lambda^{(1)}\right)$ obtained by switching the two components of λ. Note that although the isomorphism type of $\mathcal{H}_n$ only requires the unordered pair (κ_1, κ_2), the definition of the Specht module (and the associated combinatorics of residues, blocks, restricted bipartitions etc) depends on the ordered pair (κ_1, κ_2). We let $S^\leftrightarrow_\lambda$ denote the Specht module constructed using the bipartition $\lambda^\leftrightarrow$ and the ordered pair (κ_2, κ_1).

Lemma 4.5.4. *Suppose λ is a bipartition of n. Then S^λ and $S^\leftrightarrow_\lambda$ have the same composition factors (with multiplicity).*

Proof. This uses a specialisation argument. Let $\hat{q}, \hat{Q}_1, \hat{Q}_2$ be algebraically independent indeterminates over $\mathbb{F}$, and consider the Iwahori–Hecke algebra $\hat{\mathcal{H}}_n$ over $\mathbb{F}(\hat{q}, \hat{Q}_1, \hat{Q}_2)$ with parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$. Then $\hat{\mathcal{H}}_n$ is semisimple by [A1, Main Theorem], with the Specht modules being the simple modules. If we define the Specht module $\hat{S}_\lambda$ for $\hat{\mathcal{H}}_n$ using the bipartition λ and the parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$, and we define the Specht module $\hat{S}^\leftrightarrow_\lambda$ using the bipartition $\lambda^\leftrightarrow$ and the parameters $\hat{q}, \hat{Q}_2, \hat{Q}_1$, then a consideration of characters shows that $\hat{S}_\lambda$ and $\hat{S}^\leftrightarrow_\lambda$ are isomorphic. The $\mathcal{H}_n$-modules S^λ and $S^\leftrightarrow_\lambda$ are obtained by specialising $\hat{q}$ to q, $\hat{Q}_1$ to q^{k_1} and $\hat{Q}_2$ to q^{k_2}, where $\kappa_1 = \overline{k_1}$, $\kappa_2 = \overline{k_2}$. It is a standard result on specialisation that specialisations of isomorphic modules have the same composition factors. □

We remark that since the simple modules are labelled by a different set of partitions in each case, S^λ and $S^\leftrightarrow_\lambda$ have the same composition factors but the labels of these may not agree.

Lemma 4.5.4 allows us to reduce the work we do in considering different types of blocks: given a block B of $\mathcal{H}_n$, we can study the same block by replacing κ with (κ_2, κ_1) and replacing each bipartition λ in B with $\lambda^\leftrightarrow$. The validity of the statement of Theorem 4.0.1 for B is then independent of which way we view B.

4.6 Branching rules

Now we summarise some essential background on the Brundan–Kleshchev branching rules; a reference for these is [BK]. Recall that the content of a bipartition is the multiset of the residues of its nodes, and that two bipartitions lie in the same block if and only if they have the same content. So we may define the content $\mathcal{C}(B)$ of a block B to be the content of any bipartition in B.

Now suppose M is an $\mathcal{H}_n$-module lying in a block B, and $i \in \mathbb{Z}/e\mathbb{Z}$. If there is a block A of $\mathcal{H}_{n-1}$ such that $\mathcal{C}(A)$ is obtained by removing a copy of i from $\mathcal{C}(B)$, then we define $e_i M$ to be the block component of $M \downarrow_{\mathcal{H}_{n-1}}$ lying in A; otherwise, we set $e_i M = 0$.

The functors e_i are defined for all n, so we can define powers e_i^r. In fact, it is possible to define *divided powers*: for any $\mathcal{H}_n$-module M and for any r there is an $\mathcal{H}_{n-r}$-module $\mathrm{e}_i^{(r)}M$ such that $\mathrm{e}_i^r M \cong (\mathrm{e}_i^{(r)}M)^{\oplus r!}$. For any non-zero module M, we set $\epsilon_i M = \max\{r \mid \mathrm{e}_i^r M \neq 0\}$.

The following results are essential tools in the study of decomposition numbers. Recall that we write $\mathrm{nor}_i(\mu)$ for the number of normal i-nodes of a bipartition μ and $\mathrm{rem}_i(\lambda)$ for the number of removable i-nodes of λ.

Proposition 4.6.1.

1. *Suppose λ is a bipartition of n. Then $\mathrm{e}_i^{(r)}\mathrm{S}^\lambda$ has a filtration in which the factors are the Specht modules S^ν, for all bipartitions ν that can be obtained from λ by removing r removable i-nodes. In particular, $\epsilon_i \mathrm{S}^\lambda = \mathrm{rem}_i(\lambda)$, and $\mathrm{e}_i^{(\epsilon_i \mathrm{S}^\lambda)}\mathrm{S}^\lambda$ is isomorphic to the Specht module S^{λ^-}, where λ^- is the bipartition obtained by removing all the removable i-nodes from λ.*

2. *Suppose μ is a restricted bipartition of n. Then $\epsilon_i \mathrm{D}^\mu = \mathrm{nor}_i(\mu)$ and $\mathrm{e}_i^{(\epsilon_i \mathrm{D}^\mu)}\mathrm{D}^\mu \cong \mathrm{D}^{\mu^-}$, where μ^- is the bipartition obtained by removing all the normal i-nodes from μ.*

4.7 The cyclotomic Jantzen–Schaper formula

In this section we see the cyclotomic Jantzen–Schaper formula introduced by James and Mathas [JM1]. This will be crucial in the following chapters. We describe a special case here, specialising to the case of Iwahori–Hecke algebras of type B.

First we note that since every field is a splitting field for $\mathcal{H}_n$ [M2, Theorem 3.13], we can extend the field $\mathbb{F}$ without affecting the representation theory. So we assume that the subfield of $\mathbb{F}$ generated by q is a proper subfield, and we fix an element x of $\mathbb{F}$ lying outside this subfield.

Let $\hat{q}$ be an indeterminate over $\mathbb{F}$, and let $R = \mathbb{F}[\hat{q}^{\pm 1}]$. Choose a bicharge (k_1, k_2) for κ for which $|k_1 - k_2| > n$. Let $\mathfrak{p}$ be the prime ideal in R generated by $\hat{q} - q$. We want to consider the Iwahori–Hecke algebra of type B over R with parameters $\hat{q}, \hat{Q}_1, \hat{Q}_2$, where

$$\hat{Q}_1 = \hat{q}^{k_1}, \qquad \hat{Q}_2 = \hat{q}^{k_2} + x(\hat{q} - q).$$

Now given a node $(r, c, a) \in \mathbb{N}^2 \times \{1, 2\}$ define

$$\widehat{\mathrm{res}}(r, c, a) = \hat{q}^{c-r}\hat{Q}_a \in R.$$

Now suppose λ and ν are bipartitions of n with $\lambda \rhd \nu$. Let $G(\lambda, \nu)$ be the set of all pairs (L, N) such that

- L is a rim hook of λ and N a rim hook of ν,
- $[\lambda] \setminus L = [\nu] \setminus N$, and
- $\mathrm{res}(\mathrm{hand}(L)) = \mathrm{res}(\mathrm{hand}(N))$.

Given a pair $(L, N) \in G(\lambda, \nu)$, define $\epsilon_{LN} = (-1)^{\mathrm{ll}(L) - \mathrm{ll}(N)}$, and let

$$j_{\lambda\nu} = \prod_{(L,N) \in G(\lambda,\nu)} \left(\widehat{\mathrm{res}}(\mathrm{hand}(L)) - \widehat{\mathrm{res}}(\mathrm{hand}(N))\right)^{\epsilon_{LN}}.$$

Now for any pair of bipartitions (λ, μ) with μ restricted, we define

$$J_{\lambda\mu} = \sum_{\nu \lhd \lambda} \nu_{\mathfrak{p}}(j_{\lambda\nu})[\mathrm{S}^{\nu} : \mathrm{D}^{\mu}],$$

where $\nu_{\mathfrak{p}}(\cdot)$ denotes the p-adic valuation on the ring R. The following statement is a special case of the Jantzen–Schaper formula [JM1, Theorem 4.6].

Theorem 4.7.1 [JM1, Theorem 4.6]. *Suppose λ and μ are bipartitions of n with μ restricted. Then the decomposition number $[\mathrm{S}^{\lambda} : \mathrm{D}^{\mu}]$ is at most $J_{\lambda\mu}$, and it is non-zero if and only if $J_{\lambda\mu}$ is non-zero.*

Our particular choice of $\hat{q}, \hat{Q}_1, \hat{Q}_2$ means that the values $\nu_{\mathfrak{p}}(j_{\lambda\nu})$ will be easy to compute, and will not depend on the characteristic of $\mathbb{F}$ for the cases we will need to consider. In particular, suppose (r, c, a) and (s, d, b) are nodes. Then it is easy to check that:

- $\diamond$ if $a \neq b$, then

$$\nu_{\mathfrak{p}}(\widehat{\mathrm{res}}(r, c, a) - \widehat{\mathrm{res}}(s, d, b)) = \begin{cases} 1 & \text{if } \mathrm{res}(r, c, a) = \mathrm{res}(s, d, b), \\ 0 & \text{otherwise;} \end{cases}$$

- $\diamond$ if $a = b$, then

$$\nu_{\mathfrak{p}}(\widehat{\mathrm{res}}(r, c, a) - \widehat{\mathrm{res}}(s, d, b)) = \begin{cases} 1 & \text{if } c - r = d - s \pm e, \\ 0 & \text{if } c - r \not\equiv d - s \pmod{e}. \end{cases}$$

We can use the Jantzen–Schaper formula to refine Proposition 4.3.1 and Corollary 4.5.2, by using a coarser order than the dominance order. We write $\lambda \blacktriangleright \nu$ if $\lambda \rhd \nu$ and $\nu_{\mathfrak{p}}(j_{\lambda\nu}) \neq 0$, and we extend $\blacktriangleright$ reflexively and transitively to give a partial order, which we call the *Jantzen–Schaper dominance* order; note that this order depends on our fixed parameters e and κ. It is easy to see that the usual dominance order is a refinement of the Jantzen–Schaper dominance order, and that conjugation of bipartitions reverses the Jantzen–Schaper dominance order. Moreover, Theorem 4.7.1 allows us to strengthen Proposition 4.3.1 and Corollary 4.5.2.

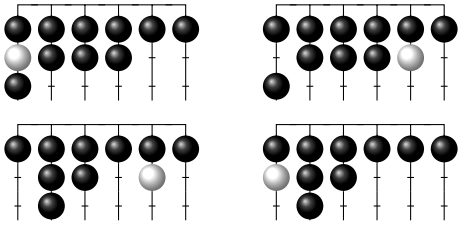
Proposition 4.7.2. *Suppose λ and μ are bipartitions with μ restricted and $[\mathrm{S}^{\lambda} : \mathrm{D}^{\mu}] > 0$, then $\mu^{\diamond} \trianglerighteq \lambda \trianglerighteq \mu$ in the Jantzen–Schaper dominance order.*

Proof. The fact that $\lambda \trianglerighteq \mu$ is immediate from Theorem 4.7.1. This then implies that $\mu^{\diamond} \trianglerighteq \lambda$, using Proposition 4.5.1 and the fact that the Jantzen–Schaper dominance order is reversed by conjugation. □

With this result in mind, we will use the Jantzen–Schaper order from now on, so we will write $\trianglerighteq$ for the Jantzen–Schaper order. Although this order is harder to work with, it will help us to narrow down the cases we need to consider. Note that when trying to decide whether $\lambda \trianglerighteq \mu$ in the Jantzen–Schaper order, the usual dominance order can be used as a "first pass": if $\lambda \not\trianglerighteq \mu$ in the usual dominance order, then certainly $\lambda \not\trianglerighteq \mu$ in the Jantzen–Schaper order. To examine the Jantzen–Schaper order more closely, it will be helpful to be able to visualise it on the abacus. Take two bipartitions λ, ν, and take the abacus configurations for these bipartitions. In order to obtain $\nu_{\mathfrak{p}}(j_{\lambda\nu}) \neq 0$, we need to be able to

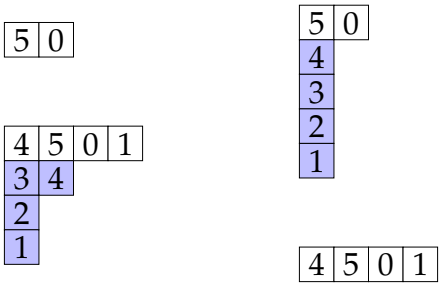
get from $\mathrm{Ab}_e(\lambda)$ to $\mathrm{Ab}_e(\nu)$ by moving a bead from position b_1 to position c_1 on component i_1, and a bead from position c_2 to position b_2 on component i_2, where $i_1, i_2 \in \{1,2\}$, $b_1 - c_1 = b_2 - c_2 > 0$, and $b_1 \equiv b_2 \pmod{e}$. We then have $\lambda \rhd \nu$ if and only if either $b_1 > b_2$ or $i_1 < i_2$. The leg length of the hook L removed from λ is the number of beads between positions b_1 and c_1, and the leg length of the hook removed from ν is the number of beads between positions b_2 and c_2.

For example, take $e = 6$, $\kappa = (\bar{5}, \bar{4})$, $\lambda = \left(2 \mid 4, 2, 1^2\right)$ and $\nu = \left(2, 1^4 \mid 4\right)$. Taking the bicharge $(11, 10)$, we obtain the following abacus configurations. (We draw the beads that have moved in white.)



λ $\qquad$ ν

Taking $b_1 = b_2 = 10$, $c_1 = c_2 = 6$, $i_1 = 2$ and $i_2 = 1$, we see that $\nu_{\mathfrak{p}}(j_{\lambda\nu}) \neq 0$ and $\lambda \lhd \nu$. Alternatively, we can see this by looking at the Young diagrams (in which we label nodes with their residues, and shade the involved rim hooks).



λ $\qquad$ ν

Chapter 5

Blocks of weight 3

The present chapter is devoted to giving a description of the bipartitions in a weight 3 block of $\mathcal{H}_n$ and to set the stage for the proof of Theorem 4.0.1.

5.1 Weight and the abacus

In this section we summarise results from [F1] showing how to calculate the weight of a bipartition from the abacus, and then use these to describe bipartitions of weight 3.

Suppose λ is a bipartition, and construct $\mathrm{Ab}(\lambda)$ the abacus configuration for λ with e runners and bicharge (k_1, k_2). We compute the weight of λ recursively, in three stages.

First suppose there is a bead in $\mathrm{Ab}(\lambda)$ with an unoccupied position immediately above. Define a new abacus configuration (and hence a new bipartition λ^-) by moving the bead up into this empty space (this corresponds to removing a rim e-hook from λ). Then [F1, Corollary 3.4] gives $\mathrm{wt}(\lambda^-) = \mathrm{wt}(\lambda) - 2$.

Applying this repeatedly, we can assume that λ is a *bicore*; that is, it has no rim e-hooks. Now for each $x \in \mathbb{Z}/e\mathbb{Z}$ define $\gamma_x(\lambda)$ as the number of beads on runner x in component 1 minus the number of beads on runner x in component 2 of $\mathrm{Ab}(\lambda)$. (Note that $\gamma_x(\lambda)$ depends on the choice of bicharge (k_1, k_2), but the difference $\gamma_x(\lambda) - \gamma_y(\lambda)$ for $x, y \in \mathbb{Z}/e\mathbb{Z}$ does not.) Now for $x, y \in \mathbb{Z}/e\mathbb{Z}$ with $x \neq y$ define the bicore $s_{xy}(\lambda)$ by moving a bead from component 1 to component 2 on runner x, and a bead from component 2 to component 1 on runner y of $\mathrm{Ab}(\lambda)$. Then [F1, Lemma 3.7] shows that $\mathrm{wt}(s_{xy}(\lambda)) = \mathrm{wt}(\lambda) - 2(\gamma_x(\lambda) - \gamma_y(\lambda) - 2)$.

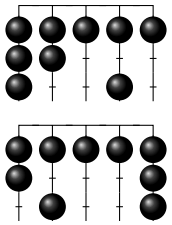
This shows that whenever there are x, y with $\gamma_x(\lambda) - \gamma_y(\lambda) \geqslant 3$, we can replace λ with a bipartition $s_{xy}(\lambda)$ of smaller weight. By doing this repeatedly (and using the fact that weight is always non-negative) we can reduce to the case where λ is a bicore with $\gamma_x(\lambda) - \gamma_y(\lambda) \leqslant 2$ for all $x, y \in \mathbb{Z}/e\mathbb{Z}$. In this case we use [F1, Proposition 3.8], which gives $\mathrm{wt}(\lambda) = \min\{|X|, |Y|\}$, where

$$X = \{\, x \in \mathbb{Z}/e\mathbb{Z} \mid \gamma_x(\lambda) - \gamma_y(\lambda) = 2, \text{ for some } y \in \mathbb{Z}/e\mathbb{Z}\},$$
$$Y = \{\, y \in \mathbb{Z}/e\mathbb{Z} \mid \gamma_x(\lambda) - \gamma_y(\lambda) = 2, \text{ for some } x \in \mathbb{Z}/e\mathbb{Z}\}.$$

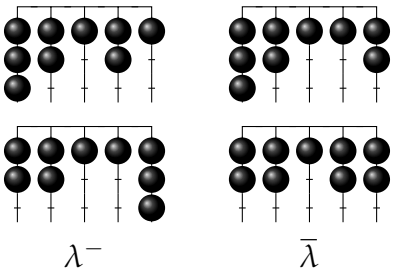
In particular, we note that $\mathrm{wt}(\lambda) = 0$ if and only if λ is a bicore with $\gamma_x(\lambda) - \gamma_y(\lambda) \leqslant 1$ for all $x, y \in \mathbb{Z}/e\mathbb{Z}$.

Example. Suppose $e = 5$, $(\kappa_1, \kappa_2) = (\bar{4}, \bar{4})$ and $(k_1, k_2) = (9, 9)$, and let λ be the bipartition

$(5,3 \mid 6,4,3)$, with the following abacus configuration.



Define a new bipartition $\lambda^- = (2,1 \mid 6,2)$ by moving all beads as up as possible in their runners, and then define $\overline{\lambda} = s_{34}(\lambda^-) = (2^2 \mid 1^2)$.



Then $\mathrm{wt}(\lambda) = \mathrm{wt}(\lambda^-) + 4$ and $\mathrm{wt}(\lambda^-) = \mathrm{wt}(\overline{\lambda}) + 2$, since $\gamma_{\overline{3}}(\lambda^-) - \gamma_{\overline{4}}(\lambda^-) = 3$. Now $\overline{\lambda}$ satisfies $\gamma_{\overline{0}}(\overline{\lambda}) - \gamma_{\overline{3}}(\overline{\lambda}) = 2$ and $\gamma_x(\overline{\lambda}) - \gamma_y(\overline{\lambda}) \leqslant 1$ for all other $x, y \in \mathbb{Z}/5\mathbb{Z}$. In the above notation we have that $|X| = |Y| = 1$ which implies $\mathrm{wt}(\overline{\lambda}) = 1$. We conclude that $\mathrm{wt}(\lambda) = 7$.

We can use these results to describe blocks of weight 3; this is very similar to the description of blocks of weight 2 in [F2, Section 2.1]. The above results show that if λ is a bipartition in a block B of weight 3, then one of the following occurs.

1a. λ has a rim e-hook, and removing this leaves a bipartition of weight 1.

1b. λ is a bicore, and there are $x, y \in \mathbb{Z}/e\mathbb{Z}$ such that $\gamma_x(\lambda) - \gamma_y(\lambda) = 3$ and $s_{xy}(\lambda)$ has weight 1.

2. λ is a bicore with $\gamma_i(\lambda) - \gamma_j(\lambda) \leqslant 2$ for all i, j, and $\min\{|X|, |Y|\} = 3$, where X and Y are defined as above.

We observe that if λ^- is obtained by removing a rim e-hook from λ, then $\delta_i(\lambda^-) = \delta_i(\lambda)$ for all i, because $\delta_i(\lambda)$ is determined by the total number of beads on each runner (across both components) in the abacus configuration for λ, which does not change when we move a bead up its runner. Similarly, if λ is a bicore then $\delta_i(s_{xy}(\lambda)) = \delta_i(\lambda)$ for all i. In view of [F1, Lemma 3.3], we then obtain $|s_{xy}(\lambda)| = n - e$. So in cases (1a) and (1b) there is a block A of some $\mathcal{H}_{n-e}$ with weight 1 and with $\delta_i(A) = \delta_i(B)$ for all i.

If B contains a bipartition satisfying (2) above, then (following [F3]) we say that B is a *core block*. By [F3, Theorem 3.1], this is equivalent to the statement that there is no block A of $\mathcal{H}_m$ for any m with weight less than 3, and with $\delta_i(A) = \delta_i(B)$ for all i.

What this means is that if B is a core block of weight 3 then all the bipartitions in B satisfy (2), while if B is a non-core block then all the bipartitions in B satisfy either (1a) or (1b).

In Section 5.3 we will see that our main theorem is easy to prove for core blocks. Instead, for non-core blocks we will need a detailed analysis which is the content of Chapters 6 to 8.

5.2 The bipartitions in a non-core weight 3 block

Suppose B is a non-core block of weight 3. In this section we show how to construct all the bipartitions in B. Suppose λ is a bipartition in B, and take $\mathrm{Ab}(\lambda)$ the abacus configuration for λ with bicharge (k_1,k_2). Then we can construct a bipartition ν lying in a block A of weight 1 with $\delta_i(\nu) = \delta_i(\lambda)$ for all i (either by removing a rim e-hook or by applying s_{xy} for some x,y). We call A the *weight* 1 *block underlying* B. Now $\gamma_x(\nu) - \gamma_y(\nu) \leqslant 2$ for all x and y, and if we define the sets X and Y as in Section 5.1 (with ν in place of λ), then $\min\{|X|,|Y|\} = 1$. We will assume that $|Y| = 1$, with the other case being similar. We set $Z = X \cup Y$ and $Z^C = \mathbb{Z}/e\mathbb{Z} \setminus Z$; note in particular that $|Z| \geqslant 2$. Now define a new abacus configuration (with bicharge (k_1+1,k_2-1)) by moving a bead from component 2 to component 1 on runner y (where $Y = \{y\}$). We denote the resulting bicore ξ_A, and call it the *nucleus* of A. Then ξ_A is independent of the choice of ν (and hence independent of the choice of λ), and the (κ_1+1,κ_2-1)-weight of ξ_A is 0. Moreover, if we define the integers $\gamma_x(\xi_A)$ using the abacus configuration with bicharge (k_1+1,k_2-1), then

$$\gamma_x(\xi_A) - \gamma_y(\xi_A) = \begin{cases} 1 & \text{if } x \in Z,\ y \in Z^C \\ -1 & \text{if } x \in Z^C,\ y \in Z \\ 0 & \text{otherwise.} \end{cases}$$

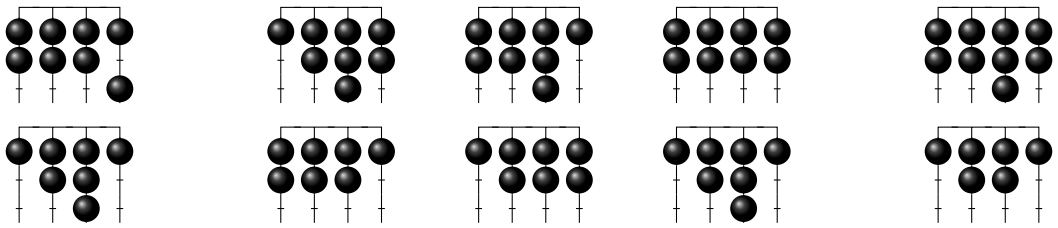
Now for each $z \in Z$ define a bipartition ν_z in A by taking the abacus configuration for ξ_A and moving a bead from component 1 to component 2 on runner z. Then the description of the bipartitions in a weight 1 block in [F1, Section 4.2.1] shows that the bipartitions ν_z for $z \in X \cup Y$ are precisely the bipartitions in A. Now the results in Section 5.1 show that each bipartition in B is obtained from a bipartition ν_z in A either by adding a rim e-hook or by applying the operator s_{xy} for some x,y for which $\gamma_x(\nu_z) - \gamma_y(\nu_z) = 1$. But the above formula for $\gamma_x(\xi_A) - \gamma_y(\xi_A)$ shows that $\gamma_x(\nu_z) - \gamma_y(\nu_z) = 1$ if and only if either $x \in Z \setminus \{z\}$ and $y \in Z^C$, or $x \in Z^C$ and $y = z$.

This enables us to describe (and label) all the bipartitions in B. These come in three families, constructed from ξ_A as follows.

- ◇ For each $z \in Z$, $x \in \mathbb{Z}/e\mathbb{Z}$ and $a \in \{1,2\}$, we obtain a bipartition $\mathrm{a}_x^{(a)}{\downarrow^z}$ from ξ_A by moving a bead from component 1 to component 2 on runner z, then moving the lowest bead on runner x on component a down one position.
- ◇ For each $x \in Z^C$, we obtain a bicore $\downarrow^x$ from ξ_A by moving a bead from component 1 to component 2 on runner x.
- ◇ For each $w,z \in Z$ with $w \neq z$ and for each $y \in Z^C$, we obtain a bicore ${\downarrow^{w,z}}{\uparrow_y}$ from ξ_A by moving beads from component 1 to component 2 on runners w and z, and then moving a bead from component 2 to component 1 on runner y.

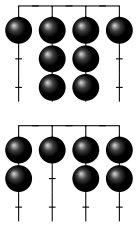
Example. Suppose $e = 4$, $(\kappa_1,\kappa_2) = (\bar{0},\bar{3})$ and $(k_1,k_2) = (8,7)$. Let B be the weight-3 block of $\mathcal{H}_{10}$ containing the bipartition $\lambda = \left(4 \mid 4,1^2\right)$. Then the underlying weight-1 block A comprises the bipartitions $\left(3,1^3 \mid \varnothing\right)$, $\left(3 \mid 1^3\right)$ and $\left(\varnothing \mid 4,1^2\right)$, and the nucleus of A is $\xi_A =$

$\left(2 \mid 1^2\right)$. These bipartitions have the following abacus configurations.



λ $\quad$ $\left(3,1^3 \mid \varnothing\right)$ $\quad$ $\left(3 \mid 1^3\right)$ $\quad$ $\left(\varnothing \mid 4,1^2\right)$ $\quad$ ξ_A

We see that $Z = \{\bar{0},\bar{2},\bar{3}\}$. So there are 28 bipartitions in B: the bipartitions $\mathrm{a}_x^{(a)}\downarrow^z$ for $z \in \{\bar{0},\bar{2},\bar{3}\}$, $x \in \mathbb{Z}/4\mathbb{Z}$ and $a \in \{1,2\}$, together with the four bipartitions $\downarrow^1$, $\downarrow^{0,2}\uparrow_1$, $\downarrow^{0,3}\uparrow_1$ and $\downarrow^{2,3}\uparrow_1$. For example, the bipartition λ above is $\mathrm{a}_3^{(1)}\downarrow^2$, while $\downarrow^{0,3}\uparrow_1$ is the bipartition $\left(3^2,1^2 \mid 1^2\right)$ shown below.



$\downarrow^{0,3}\uparrow_1$

5.3 Inductive approach to the main theorem

Now we give some results to facilitate a proof of Theorem 4.0.1 by induction. We begin with an analogue of a result for partitions in [JM2], which greatly restricts the set of bipartitions we need to consider. We give the result for blocks of weight 3 only, but a more general result along the lines of [JM2, Proposition 2.1] can easily be proved.

Lemma 5.3.1. *Suppose B is a weight 3 block of $\mathcal{H}_n$, and that Theorem 4.0.1 holds with n replaced by any smaller integer. Suppose λ, μ are bipartitions of n with μ restricted, and $i \in \mathbb{Z}/e\mathbb{Z}$.*

1. *If* $\mathrm{nor}_i(\mu) \geqslant \max\{1, \mathrm{rem}_i(\lambda)\}$, *then* $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$.
2. *If* $\mathrm{nor}_i(\mu) \geqslant \max\{1, \delta_i(B)+1\}$, *then* $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$.

Proof. Recall the notation $\epsilon_i M$ from Section 4.6, and that (from Proposition 4.6.1) $\epsilon_i \mathrm{S}^\lambda = \mathrm{rem}_i(\lambda)$ and $\epsilon_i \mathrm{D}^\mu = \mathrm{nor}_i(\mu)$.

1. Since e_i is an exact functor, if M is a module and N is a subquotient of M, then $\epsilon_i N \leqslant \epsilon_i M$. In particular, if $\mathrm{nor}_i(\mu) > \mathrm{rem}_i(\lambda)$, then $\epsilon_i \mathrm{D}^\mu > \epsilon_i \mathrm{S}^\lambda$, and therefore D^μ cannot be a composition factor of S^λ.

   Suppose instead that $\mathrm{nor}_i(\mu) = \mathrm{rem}_i(\lambda) = r$, say. It follows that $r = \epsilon_i \mathrm{S}^\lambda = \epsilon_i \mathrm{D}^\mu$. Now $\mathrm{e}_i^{(r)} \mathrm{S}^\lambda \cong \mathrm{S}^{\lambda^-}$, and $\mathrm{e}_i^{(r)} \mathrm{D}^\mu \cong \mathrm{D}^{\mu^-}$, where λ^- is obtained by removing all the removable i-nodes from λ, and μ^- is obtained by removing all the normal i-nodes from μ. Therefore

$$[\mathrm{S}^{\lambda^-} : \mathrm{D}^{\mu^-}] = [\mathrm{e}_i^{(r)} \mathrm{S}^\lambda : \mathrm{D}^{\mu^-}] = \sum_\nu [\mathrm{S}^\lambda : \mathrm{D}^\nu][\mathrm{e}_i^{(r)} \mathrm{D}^\nu : \mathrm{D}^{\mu^-}] \geqslant [\mathrm{S}^\lambda : \mathrm{D}^\mu],$$

where the sum is over all restricted bipartitions ν of n. Because $r = \mathrm{rem}_i(\lambda)$, we obtain $r \geqslant \delta_i(\lambda)$. So by Lemma 4.4.3 the weight of λ^- is at most 3. If λ^- has weight 3, then by the inductive assumption $[S^{\lambda^-} : D^{\mu^-}] \leqslant 1$. If λ^- has weight less than 3, then $[S^{\lambda^-} : D^{\mu^-}] \leqslant 1$ from Theorem 4.4.4. Either way, we obtain $[S^\lambda : D^\mu] \leqslant 1$.

2. We show that the number $r = \mathrm{rem}_i(\lambda)$ is at most $\max\{1, \delta_i(B) + 1\}$; then in particular $r \leqslant \mathrm{nor}_i(\mu)$, and we can use part (1) of the lemma. Suppose for a contradiction that $r \geqslant 2$ and $r \geqslant \delta_i(B) + 2$, and let λ^- be the bipartition obtained from λ by removing all the removable i-nodes. Then by Lemma 4.4.3 the weight of λ^- equals $3 - r(r - \delta_i(B))$. Since by assumption $r \geqslant 2$ and $r - \delta_i(B) \geqslant 2$, this makes the weight of λ^- negative, a contradiction. □

We deduce two very useful corollaries.

Corollary 5.3.2. *Suppose B is a weight 3 block of $\mathcal{H}_n$ with $\delta_i(B) \leqslant 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$, and that Theorem 4.0.1 holds with n replaced by any smaller integer. Then Theorem 4.0.1 holds for B.*

Proof. Suppose λ, μ are bipartitions in B with μ restricted. Then $D^\mu \downarrow_{\mathcal{H}_{n-1}} \neq 0$, so $e_i D^\mu \neq 0$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. This means that $\mathrm{nor}_i(\mu) \geqslant 1$, so $\mathrm{nor}_i(\mu) \geqslant \max\{1, \delta_i(B) + 1\}$, and therefore $[S^\lambda : D^\mu] \leqslant 1$ by Lemma 5.3.1(2). □

Corollary 5.3.3. *Suppose B is a weight 3 block of $\mathcal{H}_n$, and that Theorem 4.0.1 holds with n replaced by any smaller integer. If λ is a bipartition in B and $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) \geqslant 1$ and λ has no addable i-nodes, then $[S^\lambda : D^\mu] \leqslant 1$ for all μ.*

Proof. The assumption that λ has no addable i-nodes means that $\mathrm{rem}_i(\lambda) = \delta_i(B)$, while $\mathrm{nor}_i(\mu) \geqslant \delta_i(B)$. So $\mathrm{nor}_i(\mu) \geqslant \max\{1, \mathrm{rem}_i(\lambda)\}$, and the result follows from Lemma 5.3.1(1). □

So to prove Theorem 4.0.1 by induction using Corollary 5.3.3, it suffices to consider only those λ that have at least one addable i-node for *every* i for which $\delta_i(B) \geqslant 1$. Call such λ *exceptional*. The next few lemmas will show that in most cases B has no exceptional bipartitions.

Lemma 5.3.4. *Suppose B is a core block, and that $\delta_i(B) \geqslant 1$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. Then B contains no exceptional bipartitions.*

Proof. Suppose λ is a bipartition in B. Then we claim that λ has no addable i-nodes. Assume for a contradiction that λ has at least one addable i-node. Then because $\delta_i(B) \geqslant 1$, we know that λ has at least two removable i-nodes. The two components of λ are both e-cores, and an e-core cannot have addable and removable nodes of the same residue, so the addable i-nodes of λ are in one component and the removable i-nodes in the other. We assume that the removable i-nodes are in component 1 (the other case is similar). This means that in an abacus configuration for λ there are at least two positions b on runner i such that there is a bead in position b but not in position $b - 1$. So there are at least two more beads on runner i than on runner $i - 1$ in component 1 (or at least three more, in the case $i = \bar{0}$). Similarly, in component 2 of the abacus, there is at least one position b on runner $i - 1$ such that there is a bead at position b but no bead at position $b + 1$. So there are more beads on runner $i - 1$ than on runner i in the component 2 (or at least as many beads on runner $i - 1$ as on runner i, in the case $i = \bar{0}$). We deduce that $\gamma_i(\lambda) - \gamma_{i-1}(\lambda) \geqslant 3$. But the assumption that B is a core block gives $\gamma_i(\lambda) - \gamma_{i-1}(\lambda) \leqslant 2$, a contradiction. □

In view of Lemma 5.3.4, it suffices to consider non-core blocks from now on.

Lemma 5.3.5. *Suppose $\delta_i(B) \geqslant 3$ for some $i \in \mathbb{Z}/e\mathbb{Z}$. Then B contains no exceptional bipartitions.*

Proof. If λ is a bipartition in B, then we claim that λ cannot have an addable i-node: if it did, then by Lemma 4.4.3 the weight of a bipartition obtained by adding an addable i-node to λ would be $3 - (\delta_i(\lambda) + 1)$, which is negative, a contradiction. □

Lemma 5.3.6. *Suppose there are $i, j \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B), \delta_j(B) \geqslant 1$, and $j \neq i, i \pm 1$. Then B contains no exceptional bipartitions.*

Proof. Suppose λ is an exceptional bipartition in B. Then λ has both an addable i-node $\mathfrak{m}$ and an addable j-node $\mathfrak{n}$. Define bipartitions μ, ν by $[\mu] = [\lambda] \cup \{\mathfrak{m}\}$ and $[\nu] = [\mu] \cup \{\mathfrak{n}\}$. Lemma 4.4.3 then gives $\mathrm{wt}(\mu) = 3 - (\delta_i(B) + 1) \leqslant 1$. Because $j \neq i, i \pm 1$, adding the i-node to λ does not affect the addable or removable j-nodes, so $\delta_j(\mu) = \delta_j(\lambda)$. Hence $\mathrm{wt}(\nu) = \mathrm{wt}(\mu) - (\delta_j(B) + 1) \leqslant -1$, a contradiction. □

Lemma 5.3.7. *Suppose there are $i, j \in \mathbb{Z}/e\mathbb{Z}$ such that $i \neq j$, $\delta_i(B) \geqslant 1$ and $\delta_j(B) \geqslant 2$. Then B contains no exceptional bipartitions.*

Proof. If $j \neq i \pm 1$ then the result follows from Lemma 5.3.6, so assume that $j = i + 1$ (the case $j = i - 1$ is similar). Suppose λ is an exceptional bipartition in B. Then λ has both an addable i-node $\mathfrak{m}$ and an addable $(i+1)$-node $\mathfrak{n}$. Define bipartitions μ, ν by $[\mu] = [\lambda] \cup \{\mathfrak{m}\}$ and $[\nu] = [\mu] \cup \{\mathfrak{n}\}$. Lemma 4.4.3 now gives $\mathrm{wt}(\mu) = 3 - (\delta_i(B) + 1) \leqslant 1$. Now from [F1, Lemma 3.1] we obtain $\delta_{i+1}(\mu) = \delta_{i+1}(\lambda) - 1$, so that $\mathrm{wt}(\nu) = \mathrm{wt}(\mu) - \delta_{i+1}(B) \leqslant -1$, and again we have a contradiction. □

These lemmas mean that there are only four types of non-core block B in which there can be exceptional bipartitions. These are exactly analogous to the four types of weight 3 blocks of symmetric groups considered in [JM2, Section 4]. These types can be defined according to the values $\delta_i(B)$, as follows.

I. $\delta_i(B) \leqslant 0$ for all $i \in \mathbb{Z}/e\mathbb{Z}$.

II. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \neq i$.

III. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = \delta_{i+1}(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \neq i, i+1$.

IV. There is $i \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_i(B) = 2$ while $\delta_j(B) \leqslant 0$ for all $j \neq i$.

We know from Corollary 5.3.2 that Theorem 4.0.1 holds for blocks of type I. To prove Theorem 4.0.1, we need to show that if B is a block of $\mathcal{H}_n$ of type II, III or IV and if Theorem 4.0.1 holds with n replaced by any smaller integer, then it holds for B. We address the three types of block in Chapters 6 to 8.

Chapter 6

Blocks of type II

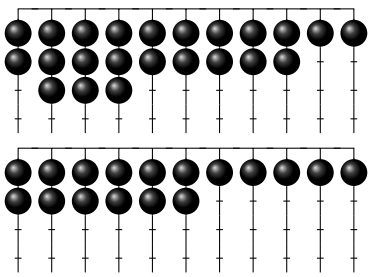
Throughout this chapter, we assume that B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \not\equiv i \pmod{e}$. Our aim is to show that if Theorem 4.0.1 holds with n replaced by any smaller integer, then it holds for B.

6.1 The abacus for a block of type II

Let ξ be the nucleus of B. Whenever we refer to $\mathrm{Ab}(\xi)$, we will mean the abacus configuration for ξ with bicharge (k_1+1, k_2-1), and whenever we refer to the residues of nodes for ξ, we mean with respect to (κ_1+1, κ_2-1). With this convention in mind, we have $\delta_i(\xi) = 1$ while $\delta_j(\xi) \leqslant 0$ for $j \not\equiv i \pmod{e}$. This means that ξ has exactly one removable node, which has residue i (because by Lemma 4.4.3 a bipartition of weight 0 cannot have an addable and a removable node of the same residue). We will assume that this removable node lies in component 1 (in the opposite case we can replace κ with (κ_2, κ_1) and appeal to the results in Section 4.5).

By considering $\mathrm{Ab}(\xi)$ and using the fact that ξ has (κ_1+1, κ_2-1)-weight 0, we find that there are integers $i \leqslant j \leqslant k \leqslant l \leqslant e+i-2$ such that in component 1 of ξ there is a removable i-node, and addable nodes of residues $j+1$ and $l+1$, while in component 2 there is just an addable node of residue $k+1$ (so in fact $\kappa_1 = \overline{j+l+1-i}$, and $\kappa_2 = \overline{k+2}$). We therefore have $Z = \{\bar{i}, \ldots, \bar{j}\} \cup \{\overline{k+1}, \ldots, \bar{l}\}$. The fact that $|Z| \geqslant 2$ implies that either $i < j$ or $k < l$.

Example. Suppose $e = 11$, $\kappa = (\bar{0}, \bar{7})$ and $\xi = \left(3^3 \;\middle|\; \varnothing\right)$, with the following abacus configuration.



$\mathrm{Ab}(\xi)$

We can take $(i,j,k,l) = (1,3,5,8)$, and $Z = \{\bar{1}, \bar{2}, \bar{3}, \bar{6}, \bar{7}, \bar{8}\}$.

6.2 Exceptional bipartitions in B

Suppose λ is a bipartition in B. From the results in Section 5.3, we need only consider Specht modules labelled by exceptional bipartitions; that is, those with addable i-nodes. By examining the abacus configurations of the bipartitions in B, we can easily classify these. There are $3|Z|$ exceptional bipartitions, which we label $\alpha_x, \beta_x, \gamma_x$ for $x \in Z$ as follows:

$$\alpha_x = \begin{cases} \mathrm{a}_i^{(2)}\downarrow^x & \text{if } x \neq i \\ \downarrow^{i-1} & \text{if } x = i, \end{cases} \qquad \beta_x = \begin{cases} \mathrm{a}_{i-1}^{(2)}\downarrow^x & \text{if } x \neq i \\ \mathrm{a}_i^{(1)}\downarrow^i & \text{if } x = i, \end{cases} \qquad \gamma_x = \begin{cases} \downarrow^{i,x}\uparrow_{i-1} & \text{if } x \neq i \\ \mathrm{a}_{i-1}^{(1)}\downarrow^i & \text{if } x = i. \end{cases}$$

It will be helpful to write these bipartitions explicitly:

$$\alpha_x = \begin{cases} \big((e+i-l)^{j-i+1}, 1^{l-i+1} \,\big|\, e+i-k-2\big) & \text{if } x = i \\ \big((e+i-l)^{j-x}, (e+i-l-1)^{x-i} \,\big|\, e+x-k-1, e+i-k, 1^{k-i}\big) & \text{if } i < x \leqslant j \\ \big((e+i-l)^{j-i+1}, 1^{l-x} \,\big|\, e+i-k-1, x-k, 1^{k-i}\big) & \text{if } k < x \leqslant l, \end{cases}$$

$$\beta_x = \begin{cases} \big((e+i-l)^{j-i+1}, 1^{l-i} \,\big|\, e+i-k-1\big) & \text{if } x = i \\ \big((e+i-l)^{j-x}, (e+i-l-1)^{x-i} \,\big|\, e+x-k-1, e+i-k-1, 1^{k-i+1}\big) & \text{if } i < x \leqslant j \\ \big((e+i-l)^{j-i+1}, 1^{l-x} \,\big|\, e+i-k-2, x-k, 1^{k-i+1}\big) & \text{if } k < x \leqslant l, \end{cases}$$

$$\gamma_x = \begin{cases} \big((e+i-l)^{j-i}, e+i-l-1, 1^{l-i+1} \,\big|\, e+i-k-1\big) & \text{if } x = i \\ \big((e+i-l)^{j-x}, (e+i-l-1)^{x-i-1}, e+i-l-2 \,\big|\, e+x-k-1, e+i-k, 1^{k-i+1}\big) & \text{if } i < x \leqslant j \\ \big((e+i-l)^{j-i}, e+i-l-1, 1^{l-x} \,\big|\, e+i-k-1, x-k, 1^{k-i+1}\big) & \text{if } k < x \leqslant l. \end{cases}$$

Each exceptional bipartition has exactly one addable i-node, and the labelling is chosen so that $\alpha_x, \beta_x, \gamma_x$ all yield the same bipartition when the addable i-node is added, with $\alpha_x \rhd \beta_x \rhd \gamma_x$.

6.3 Proving Theorem 4.0.1 in B

Our aim is to show that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$ for all bipartitions λ, μ in B, given the inductive assumption. We will do this by looking at one restricted bipartition μ at a time. The next lemma will help in narrowing down the cases we need to consider. Below, if M and N are modules, we write $M \sim N$ to mean that M and N have the same composition factors with multiplicity.

Lemma 6.3.1. *Suppose μ is a restricted bipartition in B, and that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$ for some λ. Then $\mathrm{nor}_i(\mu) = 1$, and there is some $x \in Z$ for which $[\mathrm{S}^{\alpha_x} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\beta_x} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\gamma_x} : \mathrm{D}^\mu]$ are all non-zero; in particular, $\mu^\diamond \trianglerighteq \alpha_x, \beta_x, \gamma_x \trianglerighteq \mu$.*

Proof. If λ is such that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$, then λ is exceptional, so $\lambda \in \{\alpha_x, \beta_x, \gamma_x\}$ for some x. The conclusion that μ has only one normal node comes from Lemma 5.3.1(2).

Now we claim that $[\mathrm{S}^{\alpha_x} : \mathrm{D}^\mu], [\mathrm{S}^{\beta_x} : \mathrm{D}^\mu], [\mathrm{S}^{\gamma_x} : \mathrm{D}^\mu] \geqslant 1$, which gives the result using Proposition 4.7.2. Each of $\alpha_x, \beta_x, \gamma_x$ has two removable i-nodes, and the branching rules show that there are bipartitions $\hat{\alpha}_x, \hat{\beta}_x, \hat{\gamma}_x$ of $n-1$ such that

$$\begin{aligned} \mathrm{e}_i \mathrm{S}^{\alpha_x} &\sim \mathrm{S}^{\hat{\alpha}_x} + \mathrm{S}^{\hat{\beta}_x}, \\ \mathrm{e}_i \mathrm{S}^{\beta_x} &\sim \mathrm{S}^{\hat{\alpha}_x} + \mathrm{S}^{\hat{\gamma}_x}, \\ \mathrm{e}_i \mathrm{S}^{\gamma_x} &\sim \mathrm{S}^{\hat{\beta}_x} + \mathrm{S}^{\hat{\gamma}_x}. \end{aligned}$$

Because $\mathrm{nor}_i(\mu) = 1$, there is a restricted bipartition $\tilde{\mu}$ such that $\mathrm{e}_i \mathrm{D}^\mu \cong \mathrm{D}^{\tilde{\mu}}$. Now (analogously to the proof of [F4, Proposition 2.8(2)]) we obtain

$$[\mathrm{S}^{\alpha_x} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\gamma}_x} : \mathrm{D}^{\tilde{\mu}}] = [\mathrm{S}^{\beta_x} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\beta}_x} : \mathrm{D}^{\tilde{\mu}}] = [\mathrm{S}^{\gamma_x} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\alpha}_x} : \mathrm{D}^{\tilde{\mu}}].$$

Since by assumption the decomposition numbers for weight 3 blocks of $\mathcal{H}_{n-1}$ are all 0 or 1, this means that the decomposition numbers $[\mathrm{S}^{\alpha_x} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\beta_x} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\gamma_x} : \mathrm{D}^\mu]$ differ by at most 1. So if one of them is greater than 1, then they are all positive. □

Lemma 6.3.1 greatly restricts the set of restricted bipartitions μ that we need to consider. To apply Lemma 6.3.1 we look at the Jantzen–Schaper dominance order among the exceptional bipartitions, which is easy to check given the explicit description above: if ϵ is any of the symbols α, β, γ, then

$$\epsilon_i \rhd \epsilon_{k+1} \rhd \epsilon_{k+2} \rhd \cdots \rhd \epsilon_l \rhd \epsilon_{i+1} \rhd \epsilon_{i+2} \rhd \cdots \rhd \epsilon_j.$$

In particular, the most dominant γ_x is γ_i. So to show that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$ for all μ, we need only consider bipartitions μ for which:

1. μ is restricted;
2. $\gamma_i \trianglerighteq \mu$; and
3. $\mathrm{nor}_i(\mu) = 1$, and $\mathrm{nor}_j(\mu) = 0$ for all $j \neq i$.

Analysis of the possible abacus configurations enables a classification of the bipartitions μ satisfying (2) and (3). By working through the different bipartitions in B, we can check that the only bipartitions μ satisfying these conditions are $\mathrm{a}^{(2)}_{i-1}\downarrow^i$ and $\mathrm{a}^{(2)}_i\downarrow^i$.

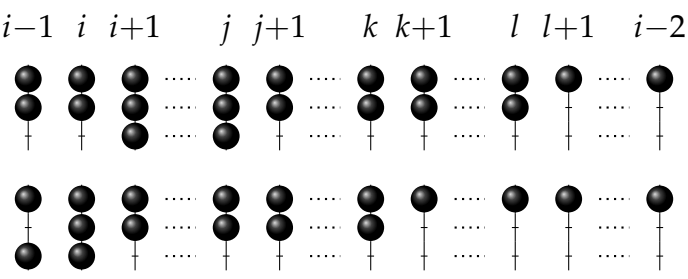
(Note that if $i < k$ then the bipartition $\mu = \mathrm{a}^{(2)}_k\downarrow^i$ does not satisfy (2): in this case

$$\mu = \big((e+i-l)^{j-i} \,\big|\, e-1, e+i-k\big),$$
$$\gamma_i = \big((e+i-l)^{j-i}, e+i-l-1, 1^{l-i+1} \,\big|\, e+i-k-1\big),$$

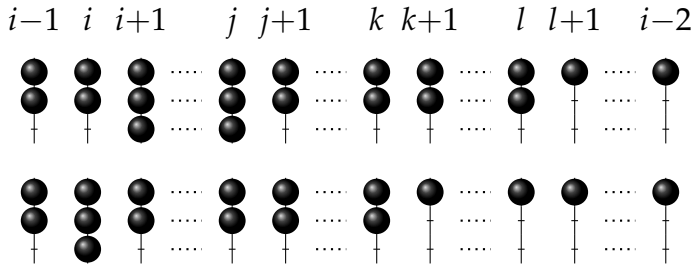
so $\gamma_i \rhd \mu$ in the usual dominance order. But by examining residues we can check that $\gamma_i \not\trianglerighteq \mu$ in the Jantzen–Schaper dominance order.)

Now we introduce condition (1). First, from Lemma 4.2.1 we know that $\mathrm{a}^{(2)}_i\downarrow^i$ is not restricted, since its second component is not e-restricted. So it remains to check when $\mu = \mathrm{a}^{(2)}_{i-1}\downarrow^i$ is restricted. We do this by removing good nodes, using the results of Section 4.2.

We begin by drawing an abacus configuration for μ (where we adjust the numbers of beads in order to make runner $i-1$ the leftmost runner).



Now we remove good nodes of residues $i, i+1, \ldots, k$, $i-1, i-2, \ldots, k+1$ in turn, to obtain the bipartition $\xi = \big((e+i-l)^{j-i} \,\big|\, e+i-k-1\big)$.

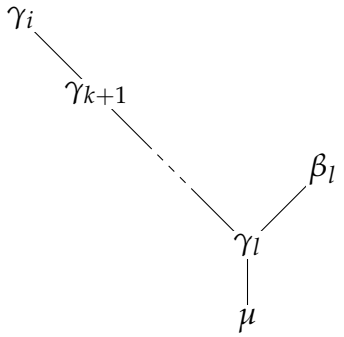


This bipartition lies in the weight 1 block underlying B, and (from the results in Section 4.2) is restricted if and only if μ is. Hence we need to understand whether ξ is restricted or not. [F1, Lemma 4.9] shows that a bipartition of weight 1 is restricted if and only if it is not the most dominant bipartition in its block. Using the explicit description of weight 1 blocks [F1, Theorem 4.4], we find all the bipartitions in the same block as ξ and we see that ξ is the most dominant bipartition in its block if and only if $k = l$. It follows that μ is restricted if and only if $k < l$.

We deal with this case using the Jantzen–Schaper formula. From Lemma 6.3.1 we need only consider exceptional bipartitions $\alpha_x, \beta_x, \gamma_x$ for which $\gamma_x \trianglerighteq \mu$, and by looking at the explicit expressions for the exceptional bipartitions, we find that this happens if and only if $x = i$ or $k < x \leqslant l$. So to show that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$ for all λ it suffices to show that

$$[\mathrm{S}^{\gamma_i} : \mathrm{D}^\mu] = [\mathrm{S}^{\gamma_{k+1}} : \mathrm{D}^\mu] = \cdots = [\mathrm{S}^{\gamma_{l-1}} : \mathrm{D}^\mu] = [\mathrm{S}^{\beta_l} : \mathrm{D}^\mu] = 0.$$

To do this using the Jantzen–Schaper order, we first find all the bipartitions λ such that $\beta_l \trianglerighteq \lambda \trianglerighteq \mu$ or $\gamma_x \trianglerighteq \lambda \trianglerighteq \mu$ for some $x \in \{i\} \cup \{k+1, \ldots, l\}$. In fact by examining the explicit forms of these bipartitions, we easily find that the only such bipartitions λ are μ, β_l, γ_i and $\gamma_{k+1}, \ldots, \gamma_l$, with the Jantzen–Schaper order on these bipartitions given by the following diagram.



The Jantzen–Schaper coefficients we need are given by

$$j_{\mu\gamma_x} = \begin{cases} (-1)^{l-x} & (k < x \leqslant l) \\ (-1)^{l-k} & (x = i), \end{cases} \qquad j_{\mu\beta_l} = -1,$$

$$j_{\gamma_l\gamma_x} = \begin{cases} (-1)^{l-x+1} & (k < x < l) \\ (-1)^{l-k+1} & (x = i), \end{cases} \qquad j_{\gamma_l\beta_l} = 1.$$

Now we deduce that $[\mathrm{S}^{\gamma_l} : \mathrm{D}^\mu] = 1$, and then (successively) $[\mathrm{S}^{\gamma_{l-1}} : \mathrm{D}^\mu] = \cdots = [\mathrm{S}^{\gamma_{k+1}} : \mathrm{D}^\mu] = [\mathrm{S}^{\gamma_i} : \mathrm{D}^\mu] = 0$ as well as $[\mathrm{S}^{\beta_l} : \mathrm{D}^\mu] = 0$.

Chapter 7

Blocks of type III

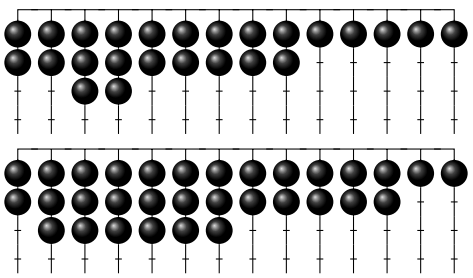
Throughout this section, we assume that B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = \delta_{i+1}(B) = 1$ while $\delta_j(B) \leqslant 0$ for all $j \not\equiv i, i+1 \pmod{e}$. Our aim is to show that if Theorem 4.0.1 holds with n replaced by any smaller integer, then it holds for B.

7.1 The abacus for a block of type III

Let ξ be the nucleus of B. As in Chapter 6, $\mathrm{Ab}(\xi)$ denotes the abacus configuration for ξ with bicharge (k_1+1, k_2-1), and the residues of nodes for ξ are with respect to the pair (κ_1+1, κ_2-1). With this convention in mind, we have $\delta_i(\xi) = \delta_{i+1}(\xi) = 1$ while $\delta_j(\xi) \leqslant 0$ for $j \not\equiv i, i+1 \pmod{e}$. This means that ξ has exactly two removable nodes, of residues i and $i+1$. We will assume that the removable $(i+1)$-node lies in component 1 (in the opposite case we can replace κ with (κ_2, κ_1) and appeal to the results in Section 4.5). This means in particular that $\gamma_{i+1}(\xi) = \gamma_i(\xi) + 1$. Now we claim that the removable i-node of ξ must lie in component 2; if not, then we obtain $\gamma_i(\xi) = \gamma_{i-1}(\xi) + 1$ and hence $\gamma_{i+1}(\xi) - \gamma_{i-1}(\xi) = 2$, and now the results in Section 5.1 show that ξ has positive weight, a contradiction.

Now we can describe $\mathrm{Ab}(\xi)$ more precisely: there are integers $i+1 \leqslant j \leqslant k \leqslant l \leqslant m \leqslant e+i-2$ such that in component 1 there is a removable node of residue $i+1$ and addable nodes of residues $j+1$ and $l+1$, while in component 2 there is a removable node of residue i and addable nodes of residues $k+1$ and $m+1$. (So in fact $\kappa_1 = \overline{j+l-i}$, and $\kappa_2 = \overline{k+m+3-i}$). We therefore have $Z = \{\overline{i+1}, \dots, \overline{j}\} \cup \{\overline{k+1}, \dots, \overline{l}\} \cup \{\overline{m+1}, \dots, \overline{i-1}\}$.

Example. Suppose $e = 14$, $\kappa = (\overline{10}, \overline{5})$ and $\xi = \left(7^2 \mid 3^6\right)$, with the following abacus configuration.

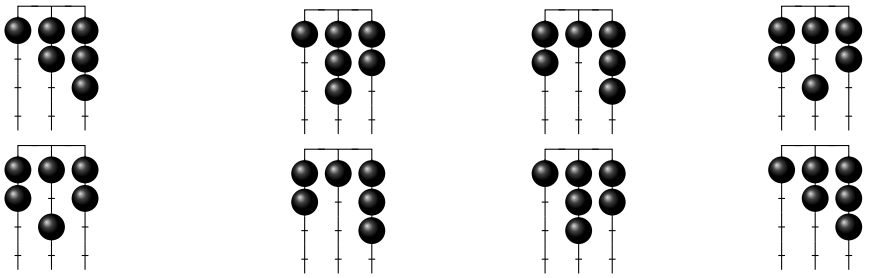


$\mathrm{Ab}(\xi)$

We can take $(i,j,k,l,m) = (1,3,6,8,11)$, and $Z = \{\overline{0}, \overline{2}, \overline{3}, \overline{7}, \overline{8}, \overline{12}, \overline{13}\}$.

7.2 Exceptional bipartitions in B

The exceptional bipartitions in B are those with both addable i-nodes and addable $(i+1)$-nodes. By examining the abacus configurations of all the bipartitions in B, we find that there are four exceptional bipartitions, which we label as follows, showing the runners $i-1, i, i+1$ of their abacus configurations:



$\alpha\beta = \mathrm{a}_i^{(2)} \downarrow^{i-1}$ $\quad$ $\alpha\gamma = \downarrow^{i-1,i+1}\uparrow_i$ $\quad$ $\gamma\alpha = \downarrow^{i}$ $\quad$ $\beta\gamma = \mathrm{a}_i^{(1)} \downarrow^{i+1}$

Observe that $\alpha\beta \rhd \alpha\gamma, \gamma\alpha \rhd \beta\gamma$ (and in fact $\alpha\gamma$ and $\gamma\alpha$ are incomparable even in the usual dominance order).

7.3 Proving Theorem 4.0.1 in B

Now we consider what conditions we need on a restricted bipartition μ in order to have $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$ for some λ.

Lemma 7.3.1. *Suppose μ is a restricted bipartition in B, and $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$ for some λ. Then $\mathrm{nor}_i(\mu) = \mathrm{nor}_{i+1}(\mu) = 1$, and at least three of the decomposition numbers $[\mathrm{S}^{\alpha\beta} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\alpha\gamma} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\gamma\alpha} : \mathrm{D}^\mu]$ and $[\mathrm{S}^{\beta\gamma} : \mathrm{D}^\mu]$ are positive. Hence $\mu^\diamond \trianglerighteq \alpha\gamma, \gamma\alpha \trianglerighteq \mu$.*

Proof. In the same way as in the proof of Lemma 6.3.1, we can apply the restriction functor e_i to show that μ has only one normal i-node and that if either of the decomposition numbers $[\mathrm{S}^{\alpha\beta} : \mathrm{D}^\mu]$ and $[\mathrm{S}^{\gamma\alpha} : \mathrm{D}^\mu]$ is greater than 1 then they are both positive, while if either of $[\mathrm{S}^{\alpha\gamma} : \mathrm{D}^\mu]$ and $[\mathrm{S}^{\beta\gamma} : \mathrm{D}^\mu]$ is greater than 1 then they are both positive. But in the same way we can use e_{i+1} to show that $\mathrm{nor}_{i+1}(\mu) = 1$ and either $[\mathrm{S}^{\alpha\beta} : \mathrm{D}^\mu]$ and $[\mathrm{S}^{\alpha\gamma} : \mathrm{D}^\mu]$ are both positive or $[\mathrm{S}^{\gamma\alpha} : \mathrm{D}^\mu]$ and $[\mathrm{S}^{\beta\gamma} : \mathrm{D}^\mu]$ are both positive. Combining these statements shows that at least three of the given decomposition numbers are positive. Now Proposition 4.7.2, together with the Jantzen–Schaper dominance order on the bipartitions $\alpha\beta, \alpha\gamma, \gamma\alpha, \beta\gamma$ gives the second statement. □

So to prove our main result for B we can focus on bipartitions μ in B for which:

1. μ is restricted;
2. $\alpha\gamma, \gamma\alpha \trianglerighteq \mu$;
3. $\mu^\diamond \trianglerighteq \alpha\gamma, \gamma\alpha$; and
4. $\mathrm{nor}_i(\mu) = \mathrm{nor}_{i+1}(\mu) = 1$, and $\mathrm{nor}_j(\mu) = 0$ for $j \neq i, i+1$.

We begin by finding the bipartitions μ for which conditions (2) and (4) are satisfied. A careful analysis of the possible abacus configurations shows that this happens in the cases given in the following table.

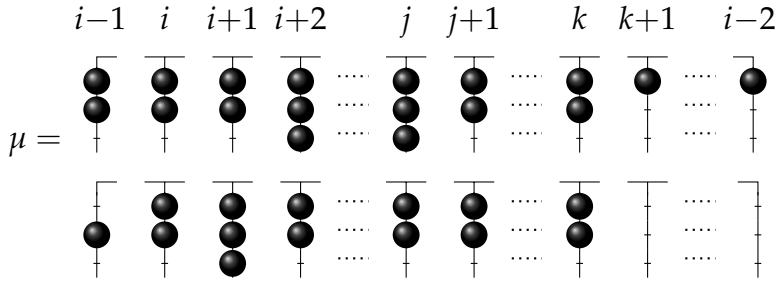
μ	conditions
$\mathrm{a}_m^{(1)}\downarrow^{i+1}$	$l < m$
$\mathrm{a}_{i-1}^{(2)}\downarrow^{i+1}$	–
$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	–
$\mathrm{a}_l^{(2)}\downarrow^{i+1}$	$k < l = m$
$\mathrm{a}_m^{(2)}\downarrow^{i+1}$	$l < m$
$\downarrow^{i+1,i+2}\uparrow_m$	$i+1 < j,\ l < m$
$\downarrow^{i+1,k+1}\uparrow_m$	$k < l < m$

(Note in particular that the bipartitions $\mu = \mathrm{a}_{i+1}^{(2)}\downarrow^{i+1}$ and $\mathrm{a}_k^{(2)}\downarrow^{i+1}$ (with $i+1 < k$) do not appear; these bipartitions satisfy (2) and (4) if the usual dominance order is used, but $\alpha\gamma \not\trianglerighteq \mu$ in the Jantzen–Schaper dominance order.)

We next want to check which of these bipartitions are restricted. To do this we use (and extend) the technique that we used for the bipartition $\mathrm{a}_{i-1}^{(2)}\downarrow^{i}$ in Section 6.3. Given a bipartition μ in the above table, we repeatedly remove good nodes until we reach a bipartition ξ which has weight 0 or 1, or has no normal nodes. If ξ has no normal nodes (but is not the empty bipartition) then ξ is not restricted, and so neither is μ. If ξ has weight 0 then it is restricted; if ξ has weight 1, then it is restricted if and only if it is not the most dominant bipartition in its block.

If we determine that μ is restricted via this technique, we can also use it to construct $\mu^\diamond$. Suppose we have reached the restricted bipartition ξ of weight 0 or 1 by removing good nodes of residues $r_1, \ldots, r_t$ in turn. The bipartition $\xi^\diamond$ can easily be identified: if ξ has weight 0 then $\xi^\diamond = \xi$, while if ξ has weight 1 then (from [F1, Lemma 4.9]) $\xi^\diamond$ is the minimal bipartition in the same block as ξ for which $\xi^\diamond \rhd \xi$. Having identified $\xi^\diamond$, we add anticogood nodes of residues $r_t, \ldots, r_1$ in turn, and by Proposition 4.5.3 we obtain $\mu^\diamond$.

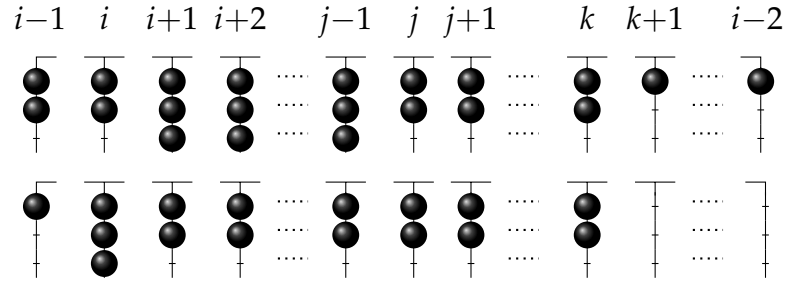
Example. Suppose $k = m$ and let $\mu = \mathrm{a}_{i-1}^{(2)}\downarrow^{i+1}$, with the following abacus configuration. (We adjust the numbers of beads in order to make runner $i-1$ the leftmost runner).



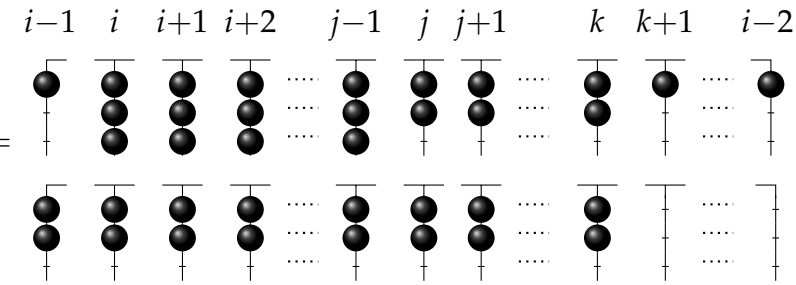
Now we remove good nodes of residues

$$i,\ i+1, i+1, i+2, i+2, \ldots, j, j,\ j+1, \ldots, k,\ i-1, i-2, \ldots, k+1$$

in turn, to obtain the bipartition ξ with the following abacus configuration.

$\xi =$ 

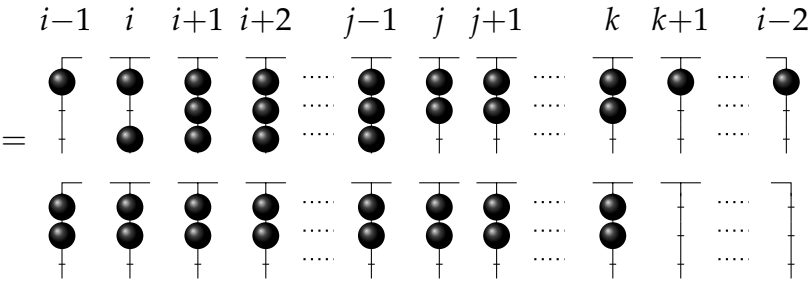
By calculating the values $\gamma_i(\xi)$ we find that ξ has weight 1. We now look at the other bipartitions in the same block as ξ, and we find that ξ is restricted, and that $\xi^\diamond$ is obtained by moving a bead from component 2 to component 1 on runner i, and from component 1 to component 2 on runner $i-1$. This implies in particular that μ is restricted.

$\xi^\diamond =$ 

Now adding anticogood nodes of residues

$$k+1, k+2, \ldots, i-1,\ k, k-1, \ldots, j+1,\ j, j, j-1, j-1, \ldots, i+1, i+1,\ i$$

in turn yields the bipartition $\mu^\diamond = \mathrm{a}_i^{(1)}{\downarrow}^{i-1}$.

$\mu^\diamond =$ 

Following this procedure for all the cases in the above table yields the following list of restricted bipartitions μ.

	μ	conditions	$\mu^\diamond$
1.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k < l < m$	$\mathrm{a}_{l+1}^{(2)}{\downarrow}^{i-1}$
2.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k < l = m$	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i-1}$
3.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k = l < m < e+i-2$	${\downarrow}^{i-1,i-2}{\uparrow}_{k+1}$
4.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k = l < m = e+i-2$	$\mathrm{a}_{k+1}^{(1)}{\downarrow}^{i-1}$
5.	$\mathrm{a}_{i-1}^{(2)}{\downarrow}^{i+1}$	$k = l = m$	$\mathrm{a}_{i}^{(1)}{\downarrow}^{i-1}$

6.	$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	$j<k<l-1$	$\downarrow^{l-1,l}\uparrow_{j+1}$
7.	$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	$j<k=l-1$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{k+1}$
8.	$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	$j<k=l$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$
9.	$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	$j=k<l$	$\mathrm{a}_{i+1}^{(1)}\downarrow^{l}$
10.	$\mathrm{a}_i^{(2)}\downarrow^{i+1}$	$j=k=l$	$\mathrm{a}_{i+1}^{(1)}\downarrow^{i-1}$
11.	$\mathrm{a}_l^{(2)}\downarrow^{i+1}$	$k<l-1,\ l=m$	$\downarrow^{l-1,l}\uparrow_{i}$
12.	$\mathrm{a}_l^{(2)}\downarrow^{i+1}$	$k=l-1,\ l=m$	$\mathrm{a}_{i}^{(1)}\downarrow^{k+1}$
13.	$\mathrm{a}_m^{(2)}\downarrow^{i+1}$	$k<l-1,\ l<m$	$\downarrow^{l-1,l}\uparrow_{i}$
14.	$\mathrm{a}_m^{(2)}\downarrow^{i+1}$	$k=l-1,\ l<m$	$\mathrm{a}_{i}^{(1)}\downarrow^{k+1}$
15.	$\mathrm{a}_m^{(2)}\downarrow^{i+1}$	$k=l<m$	$\mathrm{a}_{i}^{(1)}\downarrow^{i-1}$
16.	$\downarrow^{i+1,i+2}\uparrow_m$	$i+1<j,\ k<l<m$	$\mathrm{a}_{i}^{(2)}\downarrow^{l}$
17.	$\downarrow^{i+1,i+2}\uparrow_m$	$i+1<j,\ k=l<m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$
18.	$\downarrow^{i+1,k+1}\uparrow_m$	$k<l<m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$

Our next task is to check which of these cases satisfy $\mu^\diamond \trianglerighteq \alpha\gamma, \gamma\alpha$. We find that this happens in cases 5, 8, 10 and 15. (Note that in case 4 we need to bring into play the Jantzen–Schaper order: in this case $\mu^\diamond \trianglerighteq \alpha\gamma$ with the usual dominance order, but $\mu^\diamond \not\trianglerighteq \alpha\gamma$ with the Jantzen–Schaper order.) Now we consider the remaining four cases.

7.4 Cases 8 and 10

Assume $k=l$ and let $\mu = \mathrm{a}_i^{(2)}\downarrow^{i+1}$. In this case we will show that $[\mathrm{S}^{\alpha\gamma} : \mathrm{D}^\mu] = [\mathrm{S}^{\gamma\alpha} : \mathrm{D}^\mu] = 0$, so that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$ for all λ by Lemma 7.3.1. To do this, we use the Jantzen–Schaper formula. First we find all bipartitions in the set

$$D = \{\lambda \mid \lambda \trianglerighteq \mu \text{ and either } \alpha\gamma \trianglerighteq \lambda \text{ or } \gamma\alpha \trianglerighteq \lambda\}.$$

By examining abacus configurations, we find that

$$\begin{aligned} D = &\left\{\mathrm{a}_x^{(2)}\downarrow^{i+1} \;\middle|\; i \leqslant x \leqslant k\right\} \cup \{\downarrow^x \mid j+1 \leqslant x \leqslant k\} \\ &\cup \left\{\mathrm{a}_x^{(1)}\downarrow^{x} \;\middle|\; i+1 \leqslant x \leqslant j\right\} \cup \{\beta\gamma, \alpha\gamma, \gamma\alpha\}. \end{aligned}$$

The Jantzen–Schaper order on these bipartitions is given in Figure 7.1. Now we compute the decomposition numbers $[\mathrm{S}^\lambda : \mathrm{D}^\mu]$ for $\lambda \in D$ using the Jantzen–Schaper formula. In Table 7.3 we collate the needed Jantzen–Schaper coefficients $j_{\lambda\nu}$, and compute the bounds $J_{\lambda\mu}$, which give the decomposition numbers. (Note that for $\lambda = \mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}$, the bound $J_{\lambda\mu}$ equals 2, but we use the fact that (by the inductive assumption, and because λ is not exceptional) $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$, and therefore $[\mathrm{S}^\lambda : \mathrm{D}^\mu] = 1$.)

We conclude that if $\lambda \in D$, then

$$[\mathrm{S}^\lambda : \mathrm{D}^\mu] = \begin{cases} 1 & \text{if } \lambda \in \{\mu, \beta\gamma, \mathrm{a}_{i+2}^{(2)}\downarrow^{i+1}, \mathrm{a}_{i+2}^{(1)}\downarrow^{i+2}, \mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}\} \\ 0 & \text{otherwise.} \end{cases}$$

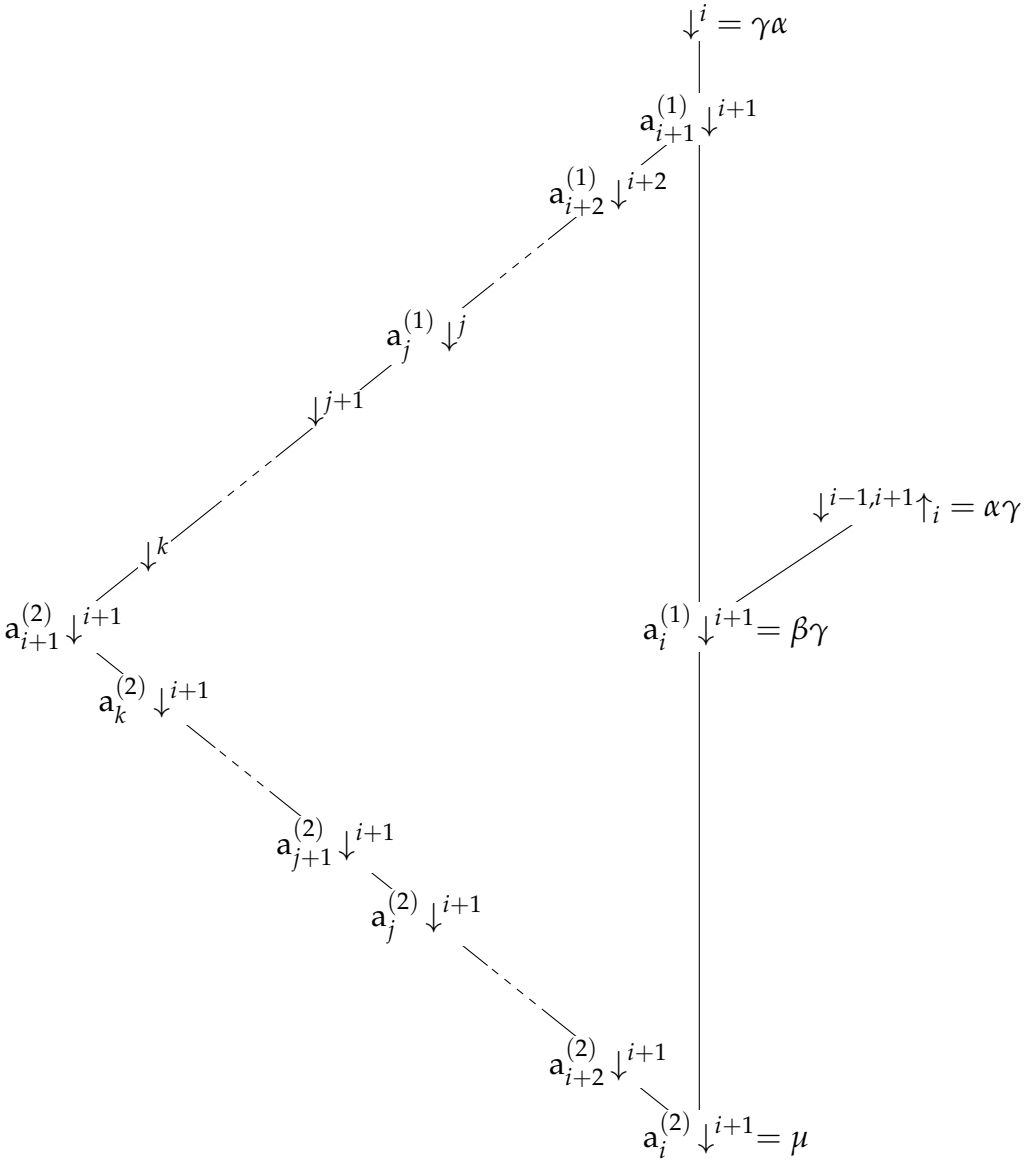


Figure 7.1: Jantzen–Schaper interval for Cases 8 and 10

Now the fact that $[S^{\alpha\gamma} : D^{\mu}] = [S^{\gamma\alpha} : D^{\mu}] = 0$, combined with Lemma 7.3.1, shows that $[S^{\lambda} : D^{\mu}] \leqslant 1$ for all λ.

7.5 Case 5

Suppose $k = l = m$, and let $\mu = a^{(2)}_{i-1}\downarrow^{i+1}$, so that $\mu^{\diamond} = a^{(1)}_{i}\downarrow^{i-1}$. In this case we use a conjugation argument. If we let B' be the block containing the bipartition $(\mu^{\diamond})'$, then B' is also a block of type III, with nucleus ξ', and can be treated in the same way as the block B,

λ	ν					$J_{\lambda\mu}$	$[S^\lambda : D^\mu]$
	μ	$\beta\gamma$	$a^{(2)}_{i+2}{\downarrow}^{i+1}$	$a^{(1)}_{i+2}{\downarrow}^{i+2}$	$a^{(1)}_{i+1}{\downarrow}^{i+1}$		
$\beta\gamma$	1					1	1
$\alpha\gamma$	-1	1				0	0
$a^{(2)}_{i+2}{\downarrow}^{i+1}$	1					1	1
$a^{(2)}_{x}{\downarrow}^{i+1}$ $(i+3 \leqslant x \leqslant k)$	$(-1)^{x-i}$		$(-1)^{x-i+1}$			0	0
$a^{(2)}_{i+1}{\downarrow}^{i+1}$	$(-1)^{k-i+1}$		$(-1)^{k-i}$			0	0
${\downarrow}^{x}$ $(k \geqslant x \geqslant j+1)$	0		0			0	0
$a^{(1)}_{x}{\downarrow}^{x}$ $(j \geqslant x \geqslant i+3)$	0		0			0	0
$a^{(1)}_{i+2}{\downarrow}^{i+2}$	0		1			1	1
$a^{(1)}_{i+1}{\downarrow}^{i+1}$	0	1	0	1		2	1
$\gamma\alpha$	-1	1	0	-1	1	0	0

Table 7.3: Jantzen–Schaper calculations for cases 8 and 10

with the integers i, j, k, l, m replaced by i', j', k', l', m', where

$$\begin{aligned} i' &= k_1 + k_2 - i - 1, \\ j' &= k_1 + k_2 + e - m, \\ k' &= k_1 + k_2 + e - l, \\ l' &= k_1 + k_2 + e - k, \\ m' &= k_1 + k_2 + e - j. \end{aligned}$$

In the block B', $(\mu^\diamond)'$ is the bipartition $a^{(2)}_{i'}{\downarrow}^{i'+1}$, with $j' = k' = l'$. So using case 10 dealt with above, we can deduce that $[S^\lambda : D^{(\mu^\diamond)'}] \leqslant 1$ for all λ, and therefore (by Proposition 4.5.1) $[S^\lambda : D^\mu] \leqslant 1$ for all λ.

7.6 Case 15

In this case we use a conjugation argument, as we did for case 5 above. In this case $(\mu^\diamond)'$ satisfies the conditions of case 8 above, therefore we can deduce that $[S^\lambda : D^\mu] \leqslant 1$ for all λ in the same way.

This deals with all cases, and so our inductive step for type III blocks is complete.

Chapter 8

Blocks of type IV

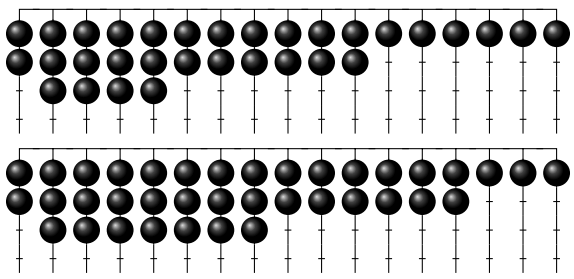
The last situation we have to investigate is when B is a block of $\mathcal{H}_n$ and $i \in \mathbb{Z}$ is such that $\delta_i(B) = 2$ and $\delta_j(B) \leqslant 0$ for every $j \not\equiv i \pmod{e}$. Again we will follow an inductive approach to show that Theorem 4.0.1 holds for B.

8.1 The abacus for a block of type IV

Take ξ the nucleus of B and consider its abacus configuration $\mathrm{Ab}(\xi)$ following the same convention of Chapters 6 and 7. Then $\gamma_i(\xi) = 2$ and $\gamma_j(\xi) \leqslant 0$ for every $j \not\equiv i \pmod{e}$ and we note that ξ has exactly two removable i-nodes (again because ξ has weight 0 and by Lemma 4.4.3 it cannot have both removable and addable nodes of the same residue). We now observe that these two removable nodes occur in different components of ξ. Assuming the contrary we would have that $\gamma_i(\xi) - \gamma_{i-1}(\xi) = 2$ but ξ has weight 0 and $\gamma_x(\xi) - \gamma_y(\xi) \leqslant 1$ for every $x, y \in \mathbb{Z}/e\mathbb{Z}$.

Similarly to the previous sections we draw the (k_1+1, k_2-1)-abacus configuration $\mathrm{Ab}(\xi)$ finding integers $i \leqslant j \leqslant k \leqslant l \leqslant m \leqslant e+i-2$ such that ξ has a removable i-node in both components and addable nodes of residue $j+1$ and $l+1$ in component 1 and of residue $k+1$ and $m+1$ in component 2 (in particular $\kappa_1 = \overline{j+l-i+1}$, and $\kappa_2 = \overline{k+m+3-i}$). Now $Z = \{\bar{i}, \ldots, \bar{j}\} \cup \{\overline{k+1}, \ldots, \bar{l}\} \cup \{\overline{m+1}, \ldots, \overline{i-1}\}$.

Example. Suppose $e = 17$, $\kappa = (\overline{14}, \bar{5})$ and $\xi = \left(7^4 \mid 4^7\right)$, with the following abacus configuration.

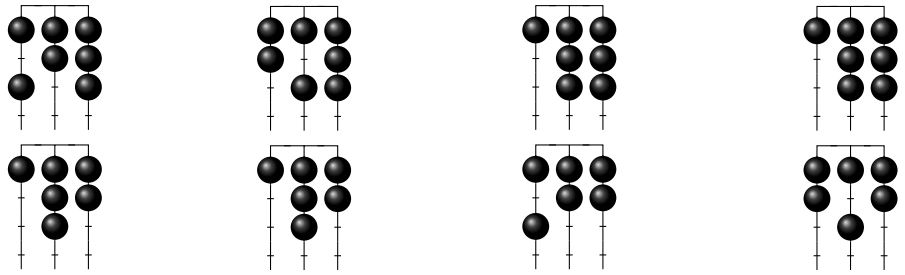


$\mathrm{Ab}(\xi)$

We can take $(i, j, k, l, m) = (1, 4, 7, 10, 13)$ then having $Z = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{8}, \bar{9}, \overline{10}, \overline{14}, \overline{15}, \overline{16}\}$.

8.2 Exceptional bipartitions in B

There are four exceptional bipartitions in B, which we label $\beta_1, \beta_2, \beta_3, \beta_4$, with abacus configurations (for runners $i-1, i, i+1$) as follows.



$\beta_1 = \mathrm{a}_{i-1}^{(1)}\downarrow^i$ $\beta_2 = \mathrm{a}_i^{(1)}\downarrow^i$ $\beta_3 = \mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$ $\beta_4 = \mathrm{a}_i^{(2)}\downarrow^{i-1}$

In contrast to Case III, these bipartitions are totally ordered by the Jantzen–Schaper dominance order; in particular $\beta_4 \rhd \beta_3 \rhd \beta_2 \rhd \beta_1$.

8.3 Proving Theorem 4.0.1 in B

Also in this situation it is useful to establish a lemma that brings out a useful property of those restricted bipartitions μ such that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$ for some λ. By Lemma 5.3.1(2) we note that such bipartitions have exactly two normal i-nodes.

Lemma 8.3.1. *Suppose μ is a restricted bipartition in B, and $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$ for some λ. Then the decomposition numbers $[\mathrm{S}^{\beta_1} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\beta_2} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\beta_3} : \mathrm{D}^\mu]$, $[\mathrm{S}^{\beta_4} : \mathrm{D}^\mu]$ are all positive; in particular $\mu^\diamond \trianglerighteq \beta_4 \rhd \beta_3 \rhd \beta_2 \rhd \beta_1 \trianglerighteq \mu$.*

Proof. Let λ be a bipartition such that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 1$. Then $\lambda = \beta_x$ for some $x \in \{1,2,3,4\}$. We follow the same idea as in Lemmas 6.3.1 and 7.3.1 with the only difference that here we need to apply the restriction functor e_i to $\mathrm{S}^{\beta_1}, \ldots, \mathrm{S}^{\beta_4}$ 'twice', namely we apply the functor e_i^2 described in Section 4.6. By the branching rules we find that there are bipartitions $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4, \tilde{\mu}$ of $n-2$ such that

$$\begin{aligned}
\mathrm{e}_i^2 \mathrm{S}^{\beta_1} &\sim (\mathrm{S}^{\hat{\beta}_1})^2 + (\mathrm{S}^{\hat{\beta}_2})^2 + (\mathrm{S}^{\hat{\beta}_3})^2,\\
\mathrm{e}_i^2 \mathrm{S}^{\beta_2} &\sim (\mathrm{S}^{\hat{\beta}_1})^2 + (\mathrm{S}^{\hat{\beta}_2})^2 + (\mathrm{S}^{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2 \mathrm{S}^{\beta_3} &\sim (\mathrm{S}^{\hat{\beta}_1})^2 + (\mathrm{S}^{\hat{\beta}_3})^2 + (\mathrm{S}^{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2 \mathrm{S}^{\beta_4} &\sim (\mathrm{S}^{\hat{\beta}_2})^2 + (\mathrm{S}^{\hat{\beta}_3})^2 + (\mathrm{S}^{\hat{\beta}_4})^2,\\
\mathrm{e}_i^2 \mathrm{D}^{\mu} &\sim \mathrm{D}^{\tilde{\mu}}.
\end{aligned}$$

Analogously as in [F4, Proposition 2.10(4)], we find

$$[\mathrm{S}^{\beta_1} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\beta}_4} : \mathrm{D}^{\tilde{\mu}}] = [\mathrm{S}^{\beta_2} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\beta}_3} : \mathrm{D}^{\tilde{\mu}}] = [\mathrm{S}^{\beta_3} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\beta}_2} : \mathrm{D}^{\tilde{\mu}}] = [\mathrm{S}^{\hat{\beta}_4} : \mathrm{D}^\mu] + [\mathrm{S}^{\hat{\beta}_1} : \mathrm{D}^{\tilde{\mu}}].$$

and we conclude by induction that if one of the decomposition numbers $[\mathrm{S}^{\beta_x} : \mathrm{D}^\mu]$ is greater than 1, then they are all positive. The last sentence follows from Proposition 4.7.2. □

Summing up, it remains to consider those bipartitions μ in B such that

1. μ is restricted;
2. $\beta_1 \trianglerighteq \mu$;
3. $\mu^\diamond \trianglerighteq \beta_4$; and
4. $\mathrm{nor}_i(\mu) = 2$, and $\mathrm{nor}_j(\mu) = 0$ for $j \neq i$.

Firstly we list all bipartitions μ which satisfy (2) and (4) and in which both components are e-restricted (which is a necessary condition for μ to be restricted, by Lemma 4.2.1).

μ	conditions
$\mathrm{a}_m^{(1)}\downarrow^i$	$l < m$
$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	–
$\mathrm{a}_m^{(2)}\downarrow^i$	$l < m$
$\downarrow^{i,i+1}\uparrow_m$	$i < j,\ l < m$
$\downarrow^{i,k+1}\uparrow_m$	$k < l < m$

Now we find which bipartitions in this list satisfy condition (1), i.e. we find the restricted bipartitions. To do this we use the same technique exploited in the previous chapters. In each case where μ is restricted we find $\mu^\diamond$ using the procedure explained in Section 7.3.

	μ	conditions	$\mu^\diamond$
1.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k < l < m$	$\mathrm{a}_{l+1}^{(2)}\downarrow^{i-1}$
2.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k < l = m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$
3.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k = l < m < e+i-2$	$\downarrow^{i-1,i-2}\uparrow_{k+1}$
4.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$k = l < m = e+i-2$	$\mathrm{a}_{k+1}^{(1)}\downarrow^{i-1}$
5.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$j < k = l = m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$
6.	$\mathrm{a}_{i-1}^{(2)}\downarrow^i$	$j = k = l = m$	$\mathrm{a}_i^{(1)}\downarrow^{i-1}$
7.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j < k < l-1,\ l < m$	$\downarrow^{l-1,l}\uparrow_{j+1}$
8.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j < k = l-1,\ l < m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{k+1}$
9.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j < k = l < m$	$\mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$
10.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j = k < l < m$	$\mathrm{a}_i^{(1)}\downarrow^{l}$
11.	$\mathrm{a}_m^{(2)}\downarrow^i$	$j = k = l < m$	$\mathrm{a}_i^{(1)}\downarrow^{i-1}$
12.	$\downarrow^{i,i+1}\uparrow_m$	$i < j,\ k < l < m$	$\mathrm{a}_i^{(2)}\downarrow^{l}$
13.	$\downarrow^{i,i+1}\uparrow_m$	$i < j,\ k = l < m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$
14.	$\downarrow^{i,k+1}\uparrow_m$	$k < l < m$	$\mathrm{a}_{i-1}^{(2)}\downarrow^{i-1}$

Finally we cut our list with condition (3). It turns out that there are four cases left: 5, 6, 9, and 11. (Note that case 4 is the only one where we need to use the Jantzen–Schaper dominance order in condition (3) instead of the usual dominance order.)

8.4 Cases 5 and 11

To address cases 5 and 11, we use a conjugation argument similar to that used for cases 5 and 15 in Chapter 7. If μ satisfies the conditions of case 5, then $\mu^\diamond = \mathrm{a}_{j+1}^{(1)}\downarrow^{i-1}$. So $(\mu^\diamond)'$ satisfies the conditions of case 11 for the block with nucleus ξ', and conversely. So, by Proposition 4.5.1, showing that $[\mathrm{S}^\lambda : \mathrm{D}^\mu] \leqslant 1$ for all λ in case 5 is equivalent to doing the same in case 11.

In fact, we will need to consider both cases simultaneously: we will obtain a partial result in case 11, and use this in our analysis of case 5 to obtain a complete result. So we assume first that μ satisfies the conditions of case 11. Our objective here will be to prove the following.

Lemma 8.4.1. *Suppose μ satisfies the conditions of case 11. Then $[\mathrm{S}^{\beta_1} : \mathrm{D}^\mu] = 1$, while $[\mathrm{S}^{\beta_2} : \mathrm{D}^\mu] \geqslant [\mathrm{S}^{\beta_3} : \mathrm{D}^\mu] \geqslant 1$.*

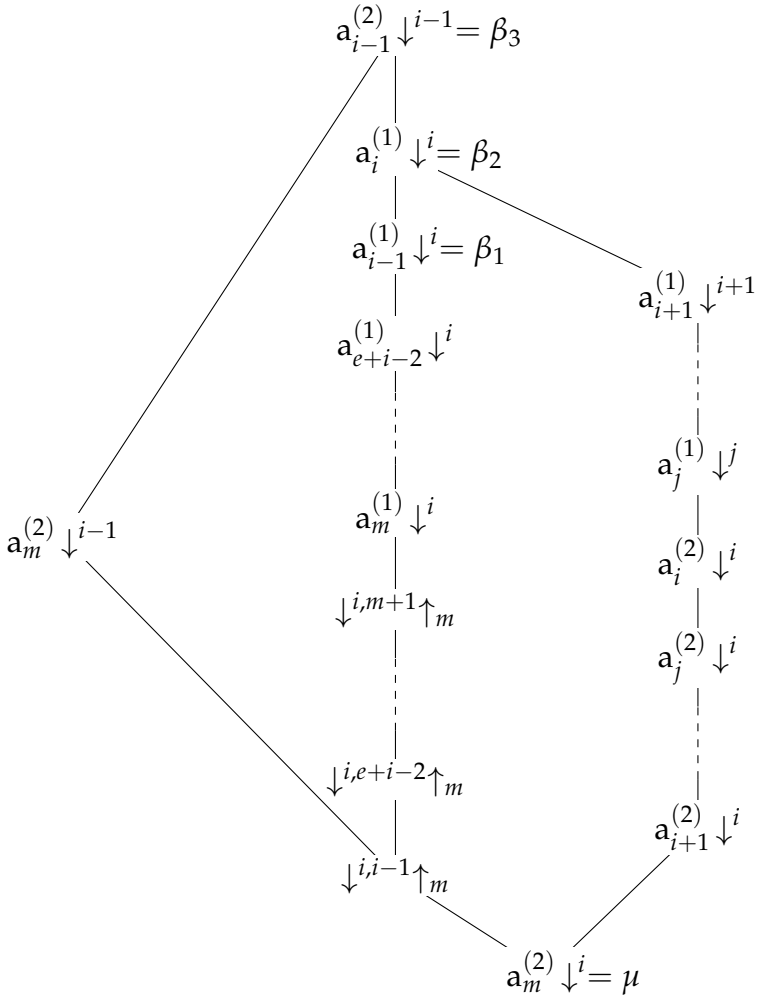


Figure 8.1: Jantzen–Schaper interval for case 11

To prove Lemma 8.4.1, we compute decomposition numbers using the Jantzen–Schaper formula. For this, we need to consider the set of bipartitions

$$D = \{\lambda \mid \beta_3 \trianglerighteq \lambda \trianglerighteq \mu\}.$$

By considering possible abacus configurations, we find that

$$D = \left\{ \mathrm{a}_x^{(2)}\downarrow^i \;\middle|\; i \leqslant x \leqslant j \right\} \cup \left\{ \mathrm{a}_x^{(1)}\downarrow^x \;\middle|\; i+1 \leqslant x \leqslant j \right\} \cup \left\{ \downarrow^{i,x}\uparrow_m \;\middle|\; m+1 \leqslant x \leqslant e+i-2 \right\}$$
$$\cup \left\{ \mathrm{a}_x^{(1)}\downarrow^i \;\middle|\; m \leqslant x \leqslant e+i-2 \right\} \cup \left\{ \mu, \mathrm{a}_m^{(2)}\downarrow^{i-1}, \downarrow^{i,i-1}\uparrow_m, \beta_1, \beta_2, \beta_3 \right\},$$

and that the Jantzen–Schaper order on these bipartitions is given in Figure 8.1. Now, as in the cases in Chapter 7 we collect in a table the integers $j_{\lambda\nu}$ and $J_{\lambda\mu}$ needed to calculate decomposition numbers $[\mathrm{S}^\lambda : \mathrm{D}^\mu]$. We obtain the results in Table 8.3, from which Lemma 8.4.1 follows.

λ	ν						$J_{\lambda\mu}$	$[\mathrm{S}^\lambda : \mathrm{D}^\mu]$
	μ	$\mathrm{a}_{i+1}^{(2)}\downarrow^i$	$\mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}$	$\downarrow^{i,i-1}\uparrow_m$	β_1	β_2		
$\mathrm{a}_{i+1}^{(2)}\downarrow^i$	1						1	1
$\mathrm{a}_x^{(2)}\downarrow^i$ $(i+2 \leqslant x \leqslant j)$	$(-1)^{x-i+1}$	$(-1)^{x-i}$					0	0
$\mathrm{a}_i^{(2)}\downarrow^i$	$(-1)^{j-i}$	$(-1)^{j-i+1}$					0	0
$\mathrm{a}_x^{(1)}\downarrow^x$ $(j \geqslant x \geqslant i+2)$	0	0					0	0
$\mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}$	0	1					1	1
$\downarrow^{i,i-1}\uparrow_m$	1						1	1
$\mathrm{a}_m^{(2)}\downarrow^{i-1}$	-1		1				0	0
$\downarrow^{i,x}\uparrow_m$ $(e+i-2 \geqslant x \geqslant m+1)$	$(-1)^{e+x-i+1}$		$(-1)^{e+x-i}$				0	0
$\mathrm{a}_m^{(1)}\downarrow^i$	$(-1)^{e+m-i+1}$		$(-1)^{e+m-i}$				0	0
$\mathrm{a}_x^{(1)}\downarrow^i$ $(m+1 \leqslant x \leqslant e+i-2)$	0		0				0	0
β_1	0			1			1	1
β_2	0	0	1	0	1		2	
β_3	0	0	-1	0	1	1	$[\mathrm{S}^{\beta_2} : \mathrm{D}^\mu]$	

Table 8.3: Jantzen–Schaper calculations for case 11

Having proved Lemma 8.4.1, we now analyse case 5 fully. Assume $j < k = l = m$ and $\mu = \mathrm{a}_{i-1}^{(2)}\downarrow^i$. First we use Proposition 4.5.1 to derive some consequences of Lemma 8.4.1: since conjugation reverses the (Jantzen–Schaper) dominance order on bipartitions, the conjugates of the bipartitions $\beta_1, \beta_2, \beta_3, \beta_4$ from case 11 are the bipartitions $\beta_4, \beta_3, \beta_2, \beta_1$ in case 5. So Proposition 4.5.1 and Lemma 8.4.1 imply that in case 5

$$[\mathrm{S}^{\beta_4} : \mathrm{D}^\mu] = 1 \qquad \text{and} \qquad 1 \leqslant [\mathrm{S}^{\beta_2} : \mathrm{D}^\mu] \leqslant [\mathrm{S}^{\beta_3} : \mathrm{D}^\mu].$$

We define τ be the bipartition $\mathrm{a}_{i+1}^{(2)}\downarrow^{i-1}$, and we want to consider the decomposition numbers $[\mathrm{S}^\lambda : \mathrm{D}^\mu]$ for bipartitions λ in the set

$$D = \{ \lambda \mid \tau \trianglerighteq \lambda \trianglerighteq \mu \}.$$

We can compute

$$D = \left\{ \mathrm{a}_x^{(2)}\downarrow^i \;\middle|\; i-1 \leqslant x \leqslant k \right\} \cup \{ \downarrow^x \mid j+1 \leqslant x \leqslant k \}$$
$$\cup \left\{ \mathrm{a}_x^{(1)}\downarrow^x \;\middle|\; i+1 \leqslant x \leqslant j \right\} \cup \{ \beta_1, \beta_2, \beta_3, \beta_4, \tau \}$$

and the Jantzen–Schaper order on the bipartitions in D is given in Figure 8.2. Now we apply the Jantzen–Schaper formula to compute decomposition numbers $[S^\lambda : D^\mu]$ for $\beta_2 \rhd \lambda \trianglerighteq \mu$. These are given in Table 8.4.

We also use the Jantzen–Schaper formula to estimate $[S^\tau : D^\mu]$: we obtain

$$J_{\tau\mu} = -[S^{a_{i+1}^{(2)}\downarrow^i} : D^\mu] + [S^{a_{i+1}^{(1)}\downarrow^{i+1}} : D^\mu] - [S^{\beta_3} : D^\mu] + [S^{\beta_4} : D^\mu] = 1 - [S^{\beta_3} : D^\mu],$$

forcing $[S^{\beta_3} : D^\mu] = 1$. Therefore we have $[S^{\beta_x} : D^\mu] = 1$ for $x = 1, 2, 3, 4$.

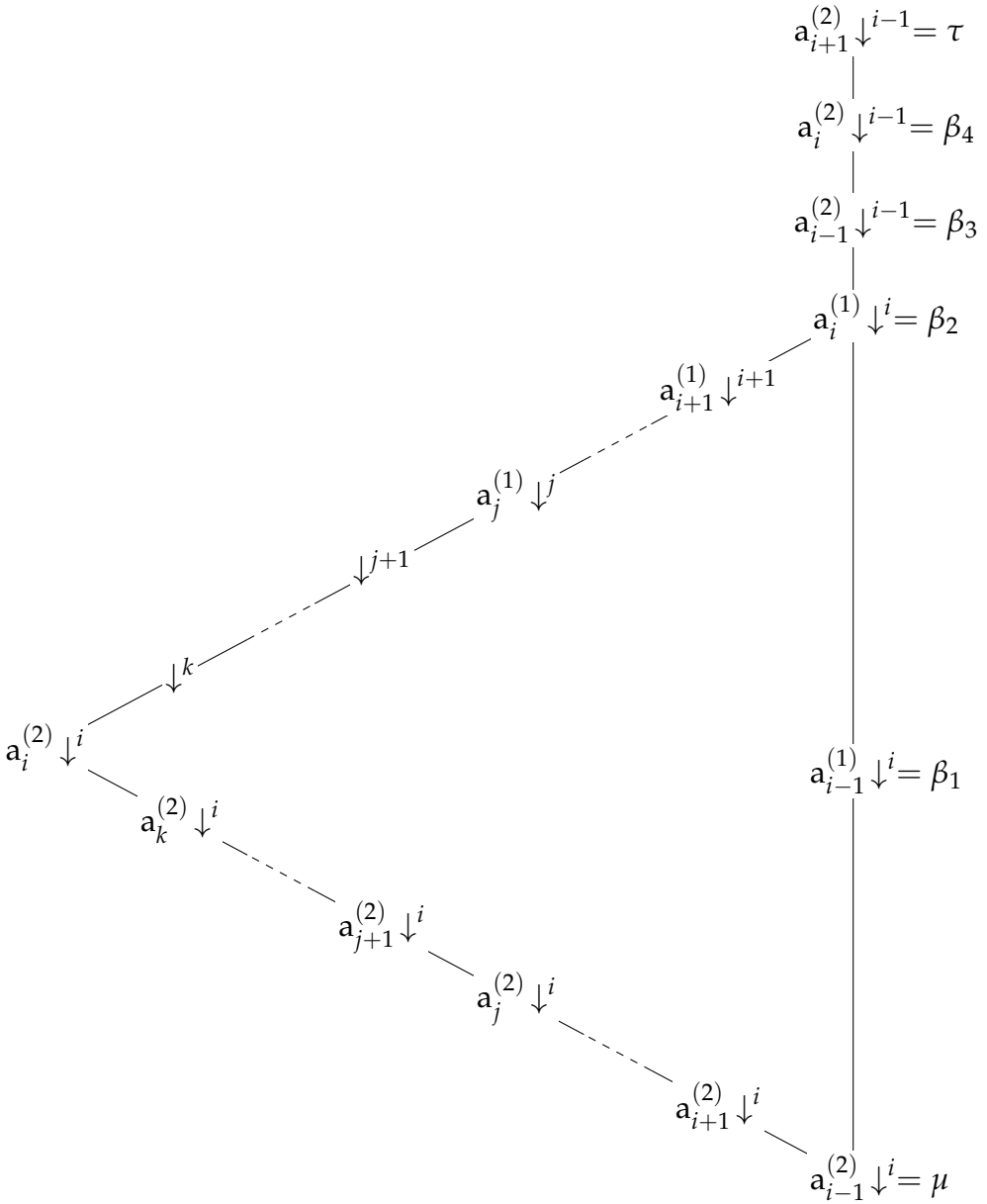


Figure 8.2: Jantzen–Schaper interval for case 5

λ	ν		$J_{\lambda\mu}$	$[S^\lambda : D^\mu]$
	μ	$a^{(2)}_{i+1}{\downarrow^i}$		
$a^{(2)}_{i+1}{\downarrow^i}$	1		1	1
$a^{(2)}_x{\downarrow^i}$ $(i+2 \leqslant x \leqslant k)$	$(-1)^{x-i+1}$	$(-1)^{x-i}$	0	0
$a^{(2)}_i{\downarrow^i}$	$(-1)^{k-i}$	$(-1)^{k-i+1}$	0	0
$\downarrow^x$ $(k \geqslant x \geqslant j+1)$	0	0	0	0
$a^{(1)}_x{\downarrow^x}$ $(j \geqslant x \geqslant i+2)$	0	0	0	0
$a^{(1)}_{i+1}{\downarrow^{i+1}}$	0	1	1	1
β_1	1	1	1	

Table 8.4: Jantzen–Schaper calculations for case 5

8.5 Case 9

The analysis of case 9 is very similar to the cases in Section 8.4. We begin by proving an exact analogue of Lemma 8.4.1 for case 9. The calculation is very similar to the calculation used to prove Lemma 8.4.1: if $\beta_3 \trianglerighteq \lambda \triangleright \mu$, then we find that

$$J_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \in \{a^{(2)}_{i+1}{\downarrow^i},\ a^{(1)}_{i+1}{\downarrow^{i+1}},\ {\downarrow^{i,i-1}}{\uparrow_m},\ \beta_1\}, \\ 2 & \text{if } \lambda = \beta_2, \\ [S^{\beta_2} : D^\mu] & \text{if } \lambda = \beta_3, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that if $i = j$ then a slight modification is necessary: the bipartition $\downarrow^{i+1}$ replaces $a^{(1)}_{i+1}{\downarrow^{i+1}}$ in the above formula.)

Having established the analogue of Lemma 8.4.1, we apply duality: case 9 is self-dual, so we deduce that

$$[S^{\beta_1} : D^\mu] = 1, \qquad [S^{\beta_2} : D^\mu] = [S^{\beta_3} : D^\mu] \geqslant 1, \qquad [S^{\beta_4} : D^\mu] = 1$$

whenever μ satisfies the conditions of case 9. We complete case 9 in the same way as case 5 above, letting τ be the bipartition $a^{(2)}_{i+1}{\downarrow^{i-1}}$. As in case 5, we find that $J_{\tau\mu} = 1 - [S^{\beta_3} : D^\mu]$, forcing $[S^{\beta_3} : D^\mu] = 1$.

8.6 Case 6

This is the last case we have to deal with. Assume that $j = k = l = m$ and $\mu = a^{(2)}_{i-1}{\downarrow^i}$. In this case we also begin by proving a direct analogue of Lemma 8.4.1. This is proved as in the previous cases: the bipartitions λ for which $\beta_3 \trianglerighteq \lambda \trianglerighteq \mu$ are shown in Figure 8.3 (ignoring the red portion of the diagram), and calculating Jantzen–Schaper coefficients as in previous

cases we find that if $\beta_3 \trianglerighteq \lambda \triangleright \mu$, then

$$J_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda \in \{\mathrm{a}_{i+1}^{(2)}\downarrow^i,\ \mathrm{a}_{i+1}^{(1)}\downarrow^{i+1}, \beta_1\}, \\ 2 & \text{if } \lambda = \beta_2, \\ [\mathrm{S}^{\beta_2} : \mathrm{D}^{\mu}] & \text{if } \lambda = \beta_3, \\ 0 & \text{otherwise.} \end{cases}$$

So an analogue of Lemma 8.4.1 follows. As with case 9, this case is self-dual, so we deduce that

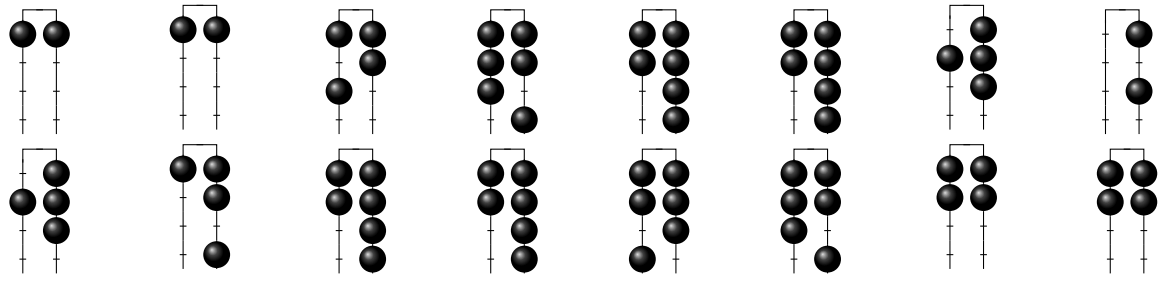
$$[\mathrm{S}^{\beta_1} : \mathrm{D}^{\mu}] = 1, \qquad [\mathrm{S}^{\beta_2} : \mathrm{D}^{\mu}] = [\mathrm{S}^{\beta_3} : \mathrm{D}^{\mu}] \geqslant 1, \qquad [\mathrm{S}^{\beta_4} : \mathrm{D}^{\mu}] = 1$$

whenever μ satisfies the conditions of case 6.

Now assume that $j < e + i - 2$. With this assumption we complete case 6 in the same way as in the preceding cases. Let τ be the bipartition $\mathrm{a}_i^{(2)}\downarrow^{e+i-2}$. Figure 8.3 (now including the red part) shows the Jantzen–Schaper order on bipartitions λ for which $\tau \trianglerighteq \lambda \trianglerighteq \mu$.

As in the preceding cases, we find that $J_{\tau\mu} = 1 - [\mathrm{S}^{\beta_3} : \mathrm{D}^{\mu}]$, forcing $[\mathrm{S}^{\beta_3} : \mathrm{D}^{\mu}] = [\mathrm{S}^{\beta_2} : \mathrm{D}^{\mu}] = 1$, so that we are done when $j < e + i - 2$. By duality, case 6 is also complete when $i < j$.

Then the only situation left to be checked is when $i = j = k = l = m = e + i - 2$. In this case $e = 2$ and B is the block of $\mathcal{H}_5$ which contains eight Specht modules labelled by the following bipartitions.



μ	$\mathrm{a}_i^{(2)}\downarrow^i$	β_1	β_2	β_3	β_4	$\mathrm{a}_{i-1}^{(1)}\downarrow^{i-1}$	$\mu^\diamond$
$(\varnothing \mid 2,1^3)$	$(\varnothing \mid 4,1)$	$(1^2 \mid 2,1)$	$(2 \mid 2,1)$	$(2,1 \mid 1^2)$	$(2,1 \mid 2)$	$(2,1^3 \mid \varnothing)$	$(4,1 \mid \varnothing)$

and two simple modules labelled by μ and β_1. It is easy to solve this case by hand. We use a dimension argument here. First we note that the two Specht modules S^{β_1} and S^{β_2} have the same dimension. From our calculations above we know that

$$[\mathrm{S}^{\beta_1} : \mathrm{D}^{\mu}] = 1,$$

while Proposition 4.7.2 and a very simple calculation with the Jantzen–Schaper formula show that

$$[\mathrm{S}^{\beta_1} : \mathrm{D}^{\beta_1}] = [\mathrm{S}^{\beta_2} : \mathrm{D}^{\beta_1}] = 1,$$

forcing $[\mathrm{S}^{\beta_2} : \mathrm{D}^{\mu}] = 1$.

This completes the proof of Case IV and of Theorem 4.0.1.

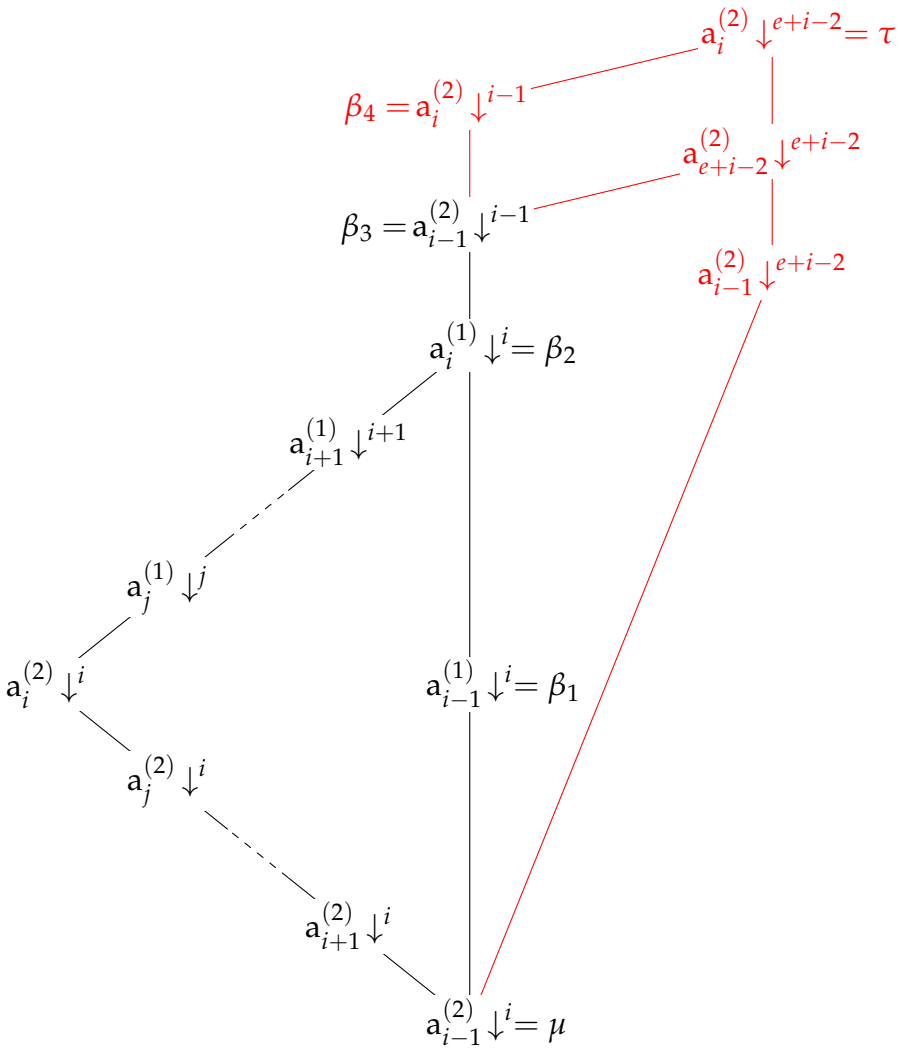


Figure 8.3: Jantzen–Schaper interval for case 6

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