

## RESEARCH ARTICLE

# The N-prime graph and the Subgroup Isomorphism Problem

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**Funding information**

Università degli Studi di Firenze, Grant/Award Number: MICIU/PRX23/00370; European Union-Next Generation EU, Grant/Award Numbers: CUP B53D23009410006, PRIN 2022 2022PSTWLB

**Abstract**

We introduce a directed graph related to a group  $G$ , which we call the *N-prime graph*  $\Gamma_N(G)$  of  $G$  and is a refinement of the classical Gruenberg–Kegel graph. The vertices of  $\Gamma_N(G)$  are the primes  $p$  such that  $G$  has an element of order  $p$ , and, for distinct vertices  $p$  and  $q$ , the arc  $q \rightarrow p$  is in the graph if and only if  $G$  has a subgroup of order  $p$  whose normalizer in  $G$  has an element of order  $q$ . Generalizing some known results about the Gruenberg–Kegel graph, we prove that the group  $V(\mathbb{Z}G)$  of the units with augmentation 1 in the integral group ring  $\mathbb{Z}G$  has the same N-prime graph as  $G$  if  $G$  is a finite solvable group, and we reduce to almost simple groups the problem of whether  $\Gamma_N(V(\mathbb{Z}G)) = \Gamma_N(G)$  holds for an arbitrary finite group  $G$ . We also prove that  $\Gamma_N(V(\mathbb{Z}G)) = \Gamma_N(G)$  if the finite group  $G$  is almost simple with socle either an alternating group, or  $\text{PSL}_2(r^f)$  with  $r$  prime and  $f \leq 2$ . Finally, for a finite solvable group  $G$  we obtain some stronger results which give a contribution to the Subgroup Isomorphism Problem. More precisely, we prove that if  $V(\mathbb{Z}G)$  contains a Frobenius subgroup  $T$  with kernel of prime order and complement of prime-power order, then  $G$  contains a subgroup isomorphic to  $T$ .

**MSC 2020**

16S34 (primary) 16U60, 20C05 (secondary)

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## 1 | INTRODUCTION

This paper deals with the following general question concerning finite groups: how much information about a finite group  $G$  can be obtained from the ring-theoretical properties of its integral group ring  $\mathbb{Z}G$ ? The history of this question began when Higman proved in his Ph.D. dissertation that if  $G$  is abelian, then every group  $H$  such that  $\mathbb{Z}G$  and  $\mathbb{Z}H$  are isomorphic must be isomorphic to  $G$  [17, 18]. Whitcomb extended the latter result to the case when  $G$  is metabelian [33], and later Weiss proved the same property for nilpotent groups [32]. For a while it was believed that this could be true for every finite group but, in 2001, Hertweck discovered two nonisomorphic finite groups with isomorphic integral group rings [12]. Nevertheless, there is still much evidence that significant information about  $G$  is encoded in  $\mathbb{Z}G$ , and mostly in the group  $V(\mathbb{Z}G)$  of units of  $\mathbb{Z}G$  of augmentation 1. A detailed discussion about the interplay between properties of  $G$  and  $V(\mathbb{Z}G)$  can be found in the survey [28] and in [22].

The *First Zassenhaus Conjecture* (formulated by Zassenhaus in 1974) is an iconic problem in this research field, which has attracted the interest of many authors in the past years; it predicts that, given a finite group  $G$ , every torsion unit of  $\mathbb{Z}G$  is conjugate in  $\mathbb{Q}G$  to an element of the form  $\pm g$  for a suitable  $g \in G$ . This was proved in many special cases (see, e.g., [1, 6, 15, 32]) but, in 2018, Eisele and Margolis provided a counterexample [10]. As a related concept, we recall that the *Gruenberg–Kegel graph* of a group  $G$  (also known as the *prime graph* of  $G$ ) is the simple undirected graph  $\Gamma_{\text{GK}}(G)$  whose vertex set is the set  $\pi(G)$  of the prime numbers dividing the order of some torsion element of  $G$ , and, for distinct vertices  $p$  and  $q$ , the edge  $q - p$  is in the graph if and only if  $G$  has an element of order  $pq$ . Now, if the First Zassenhaus Conjecture holds for a given finite group  $G$ , then clearly  $V(\mathbb{Z}G)$  and  $G$  have the same Gruenberg–Kegel graph. In view of this, Kimmerle proposed in [23] the so-called *Prime Graph Question*: is it true that, for every finite group  $G$ , the graphs  $\Gamma_{\text{GK}}(V(\mathbb{Z}G))$  and  $\Gamma_{\text{GK}}(G)$  are equal? The Prime Graph Question is usually abbreviated as (PQ), referring to the common use of  $p$  and  $q$  for two different primes. It can be interpreted as a weaker form of the First Zassenhaus Conjecture, and there is no known counterexample for it. Kimmerle proved in [23] that the answer is affirmative if  $G$  is a solvable group, and in fact the problem has been reduced to *almost simple groups* by Kimmerle and Konovalov in [25] (recall that a finite group  $G$  is called almost simple if it has a unique minimal normal subgroup  $S$ , and  $S$  is a nonabelian simple group; in this situation  $S$  coincides with the *socle* of  $G$ , that is, the product of all the minimal normal subgroups of  $G$ ). Since then, several almost simple groups have been investigated from this point of view (see [28, Theorem 2.6] for a wide list of results on the Prime Graph Question for almost simple groups; see also [9]).

In this paper we introduce a different graph related to a group  $G$ , which we call the *N-prime graph*  $\Gamma_{\text{N}}(G)$  of  $G$  (here “N” is meant to suggest the concept of normalization). We define it as the simple directed graph whose vertex set is  $\pi(G)$  and, for distinct vertices  $p$  and  $q$ , the arc  $q \rightarrow p$  is in the graph if and only if there exists an element  $x$  in  $G$  of order  $p$  such that  $\langle x \rangle$  is normalized by an element of  $G$  having order  $q$ . As observed in Remark 3.1, the Gruenberg–Kegel graph of  $G$  can be obtained from  $\Gamma_{\text{N}}(G)$  by replacing every double arrow  $q \rightleftarrows p$  by the edge  $q - p$  and deleting all single arrows; on the other hand, the converse is not true because it is easy to find examples of groups having the same Gruenberg–Kegel graph but different N-prime graphs. So, this graph in principle encodes more detailed information about the structure of the group  $G$ , and we can consider the following strengthening of the Prime Graph Question.

**N-Prime Graph Question:** Is it true that, for every finite group  $G$ , the N-prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide?

We say that (NPQ) holds for a finite group  $G$  if  $\Gamma_N(V(\mathbb{Z}G)) = \Gamma_N(G)$ .

The first result of this paper is a generalization of the aforementioned reduction of Kimmerle and Konovalov for (PQ) to (NPQ).

**Theorem A.** *Let  $G$  be a finite group, and assume that the  $N$ -prime graphs of  $H$  and  $V(\mathbb{Z}H)$  coincide for every almost simple homomorphic image  $H$  of  $G$ . Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

The corresponding generalization of the main result from [23] is then derived at once.

**Corollary B.** *Let  $G$  be a finite solvable group. Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

In view of Theorem A, we explore the validity of (NPQ) for some families of almost simple groups. In particular, we prove that (NPQ) holds for rational groups, and for almost simple groups whose socle is an alternating group  $A_n$  ( $n \geq 5$ ) or a group of the form  $\text{PSL}_2(r^f)$  where  $r$  is a prime and  $f \in \{1, 2\}$  (see Theorems 3.6, 4.1, and 4.3, respectively). As a consequence, we obtain the following.

**Theorem C.** *Let  $G$  be a finite group. Assume that every almost simple homomorphic image of  $G$  is a rational group, or that its socle is an alternating group, or that its socle is isomorphic to  $\text{PSL}_2(r^f)$  for a prime  $r$  and  $f \in \{1, 2\}$ . Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

Another interesting problem in the present context, somewhat “intermediate” between the First Zassenhaus Conjecture and the Prime Graph Question, is the *Spectrum Problem*: given a finite group  $G$ , is it true that if  $V(\mathbb{Z}G)$  has an element of a given order  $n$ , then  $G$  has an element of order  $n$  as well? It has been proved by Hertweck in [13] that the answer is affirmative if the group  $G$  is solvable, and there is no known counterexample. By Corollary 4.1 in [7], we know that the answer is also affirmative if  $n$  is a prime power. Note that the Spectrum Problem (respectively, the Prime Graph Question) can be restated as follows: is it true that if  $V(\mathbb{Z}G)$  has a cyclic subgroup  $C_n$  of order  $n$  (respectively, of order  $pq$ ), then so does  $G$ ? So these questions ask for the isomorphism types of cyclic subgroups of  $G$  and  $V(\mathbb{Z}G)$ . On the other hand, (NPQ) deals with subgroups of type  $C_p \rtimes C_q$ . We can frame all these questions in the following general context.

**Subgroup Isomorphism Problem (SIP):** Let  $T$  be a finite group. Is it true that if  $G$  is a finite group such that  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $T$ , then also  $G$  has a subgroup isomorphic to  $T$ ?

As nonisomorphic finite groups may have isomorphic integral group rings, of course the (SIP) has in general a negative solution. However, the previous discussion shows that under some conditions the answer is positive. Besides the results above on the case where  $T$  is cyclic, Kimmerle observed that the (SIP) has a positive answer for  $T = C_2 \times C_2$  and Hertweck extended that to  $T = C_p \times C_p$  where  $p$  is a prime [16]. Other results for the (SIP) can be found in [8, 24, 27].

Care should be taken in considering (NPQ) as part of the (SIP). Let  $T$  be the Frobenius group  $C_p \rtimes C_q$ , where  $p$  and  $q$  are (distinct) primes. Suppose that (NPQ) holds for  $G$  and  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $T$ . If, moreover,  $G$  does not have elements of order  $pq$ , then  $G$  has a sub-

group isomorphic to  $T$ . However, if  $G$  has elements of order  $pq$ , then the existence of a subgroup of  $G$  isomorphic to  $T$  is not guaranteed by (NPQ).

Our next result shows that such a subgroup actually exists, under the assumption that  $G$  is solvable. More generally, we prove the following theorem which is a contribution to the (SIP) for  $T$  a Frobenius group of type  $C_p \rtimes C_{q^k}$ .

**Theorem D.** *Let  $G$  be a finite solvable group and let  $T$  be a Frobenius group of the form  $C_p \rtimes C_{q^k}$  with  $p$  and  $q$  primes. If  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $T$ , then so does  $G$ .*

As already mentioned, by [13], if the finite group  $G$  is solvable and  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $C_p \times C_{q^k}$ , then so does  $G$ . This, together with the previous result, solves the (SIP) for subgroups of the form  $C_p \rtimes C_{q^k}$  in the two extreme situations when the action of  $C_{q^k}$  on  $C_p$  is trivial or faithful. So we can consider the following.

**Question.** Let  $G$  be a finite solvable group, and assume that  $V(\mathbb{Z}G)$  has a subgroup  $T$  of the form  $C_{p^h} \rtimes C_{q^k}$ , where  $p$  and  $q$  are different primes. Is it true that  $G$  has a subgroup isomorphic to  $T$ ?

The last result of this paper yields an affirmative answer under the assumption that the derived subgroup of  $G$  is cyclic.

**Theorem E.** *Let  $G$  be a finite group such that  $G'$  is cyclic, and let  $T$  be a group of the form  $C_{p^h} \rtimes C_{q^k}$  with  $p$  and  $q$  different primes. If  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $T$ , then so does  $G$ .*

## 2 | PRELIMINARY NOTATION

In this brief preliminary section we introduce some notation and concepts that will come into play.

For a finite group  $G$  and a commutative ring  $R$ , we denote by  $V(RG)$  the group of units having augmentation 1 in the group ring  $RG$ . Note that, if  $M$  is a normal subgroup of  $G$ , then the natural projection of  $G$  onto  $G/M$  can be extended by linearity to a ring homomorphism  $\varphi_M : RG \rightarrow R(G/M)$ , and the restriction of  $\varphi_M$  to  $V(RG)$  yields a group homomorphism (that we will still denote by  $\varphi_M$ ) to  $V(R(G/M))$ . We will refer to this  $\varphi_M$  as the “natural homomorphism” associated with  $M$  (both as a ring homomorphism on  $RG$  and as a group homomorphism on  $V(RG)$ ). The subscript  $M$  will usually be dropped when the context yields no ambiguity.

We also recall that, given an element  $r = \sum_{g \in G} r_g g$  in  $RG$  (where the coefficients  $r_g$  are in  $R$ ), the *partial augmentation* of  $r$  with respect to the element  $b \in G$  is

$$\varepsilon_b(r) = \sum_{g \sim b} r_g,$$

where we use the notation  $\sim$  for conjugation in  $G$  (clearly,  $\varepsilon_b(r)$  depends on the conjugacy class of  $b$  in  $G$  rather than on  $b$  itself). Observe that  $\varepsilon_b : RG \rightarrow R$  is an  $R$ -linear map. Moreover, as is easily checked, we have

$$[RG, RG] = \{r \in RG \mid \varepsilon_g(r) = 0 \text{ for every } g \in G\}$$

(see [31, (41.1), p. 237]), which implies that  $\varepsilon_b(u) = \varepsilon_b(r)$  if  $u$  and  $r$  are conjugate in  $RG$ .

A very useful result concerning the partial augmentation is the following theorem, known as the Berman–Higman theorem ([5, 17]).

**Theorem 2.1.** *Let  $G$  be a finite group. If  $v$  is a torsion unit of  $\mathbb{Z}G$ , then either  $v = \pm 1$  or  $\varepsilon_1(v) = 0$ .*

Unless otherwise stated,  $p$  and  $q$  denote two different prime integers. Then a group of the form  $C_p \rtimes C_q$  is either cyclic or a Frobenius group. In the latter case  $p \equiv 1 \pmod q$  and all the nontrivial actions of  $C_q$  on  $C_p$  yield isomorphic semidirect products  $C_p \rtimes C_q$ .

Finally, recall that a finite group  $G$  is called a *2-Frobenius group* if it has normal subgroups  $F$  and  $K$  such that  $K$  is a Frobenius group with kernel  $F$ , and  $G/F$  is a Frobenius group with kernel  $K/F$ . Such a group is necessarily solvable, and its Fitting subgroup is  $F$ .

### 3 | THE N-PRIME GRAPH QUESTION: THEOREM A

We begin our study of the N-prime graph, that was defined in the introduction, and we first clarify its relationship with the Gruenberg–Kegel graph in the following easy observation.

*Remark 3.1.* We observe that the Gruenberg–Kegel graph of a group  $G$  has the same vertex set as the N-prime graph of  $G$ , and its edges can be deduced from those of the N-prime graph as follows:  $\Gamma_{\text{GK}}(G)$  has an edge joining two vertices  $p$  and  $q$  if and only if both  $p \rightarrow q$  and  $q \rightarrow p$  are arcs of  $\Gamma_N(G)$ . Indeed, it is clear from the definitions that if  $p - q$  is an edge of  $\Gamma_{\text{GK}}(G)$ , then both  $q \rightarrow p$  and  $p \rightarrow q$  are arcs of  $\Gamma_N(G)$ . On the other hand, if  $p$  and  $q$  are not adjacent in  $\Gamma_{\text{GK}}(G)$  but  $\Gamma_N(G)$  contains the arc  $q \rightarrow p$ , then  $q$  divides  $p - 1$ ; thus, clearly  $\Gamma_N(G)$  cannot contain the arc  $p \rightarrow q$ .

An alternative way to describe  $\Gamma_{\text{GK}}(G)$  from  $\Gamma_N(G)$  is as the undirected graph with the same vertices as  $\Gamma_N(G)$  and edges  $p - q$  if and only if  $p > q$  and  $p \rightarrow q$  is an arc of  $\Gamma_N(G)$ .

However, in general  $\Gamma_N(G)$  cannot be derived from  $\Gamma_{\text{GK}}(G)$ : for example, the groups  $\text{PSL}_2(7)$  and  $\text{PSL}_2(8)$  have the same Gruenberg–Kegel graph  $(2 \ 3 \ 7)$ , but  $\Gamma_N(\text{PSL}_2(7))$  is  $(2 \rightarrow 3 \rightarrow 7)$  whereas  $\Gamma_N(\text{PSL}_2(8))$  is  $(7 \leftarrow 2 \rightarrow 3)$ . For an example involving solvable groups, we can consider a Hall  $\{3, 7\}$ -subgroup of the affine semilinear group  $\text{AG}(3^6)$ , which is a 2-Frobenius group of the form  $(C_3^6 \rtimes C_7) \rtimes C_3$ , and the Frobenius group  $C_3^6 \rtimes C_7$ : for both of them the Gruenberg–Kegel graph is disconnected, and so is the N-prime graph of the latter, but the N-prime graph of the former is  $(3 \rightarrow 7)$ . Finally,  $C_5 \times A_5$  and  $C_{25} \times A_4$  have the same cardinality and Gruenberg–Kegel graph  $2 - 5 - 3$ , while  $\Gamma_N(C_5 \times A_5) = (2 \rightleftarrows 5 \rightleftarrows 3)$  and  $\Gamma_N(C_{25} \times A_4) = (2 \rightleftarrows 5 \rightleftarrows 3)$ .

It is straightforward that, if  $H$  is a subgroup of the finite group  $G$ , then  $\Gamma_N(H)$  is a subgraph of  $\Gamma_N(G)$ , so the graph  $\Gamma_N(G)$  is well behaved with respect to subgroups. We will show next that  $\Gamma_N(G)$  behaves well also with respect to factor groups: if  $M$  is a normal subgroup of  $G$  then  $\Gamma_N(G/M)$  is a subgraph of  $\Gamma_N(G)$ . To this end, we introduce a lemma that will be useful also for other results in this paper.

**Lemma 3.2.** *Let  $p$  and  $q$  be primes, and let  $k$  be a positive integer such that  $q^k$  divides  $p - 1$ . Let  $P$  be a finite  $p$ -group and let  $\beta$  be an automorphism of order  $q^k$  of  $P$ . Then there exists  $a \in P$  such that  $|a| = p$  and  $\beta$  restricts to an automorphism of order  $q^k$  of  $\langle a \rangle$ .*

*Proof.* We argue by induction on the order of  $P$ . If the Frattini subgroup  $\Phi(P)$  of  $P$  has index  $p$  in  $P$ , then  $P$  is a cyclic group and our claim follows; thus we may assume that  $|P : \Phi(P)| > p$ . By coprimality, the action of  $\langle \beta \rangle$  on the elementary abelian  $p$ -group  $P/\Phi(P)$  is faithful (see Theorem 5.3.5 in [11]), so we can view  $P/\Phi(P)$  as a completely reducible  $\langle \beta \rangle$ -module over  $\text{GF}(p)$  having at least one faithful irreducible constituent. Denoting this constituent by  $P_0/\Phi(P)$  and setting  $|P_0/\Phi(P)| = p^n$ , we have that  $\beta$  can be identified with an element of the multiplicative group  $\text{GF}(p^n)^\times$  acting on  $P_0/\Phi(P)$  as a multiplication map (see for instance [29, Theorem 2.1]). But since  $|\beta| = q^k$  divides  $p - 1$ ,  $\beta$  is identified with an element of  $\text{GF}(p)^\times$ , thus  $|P_0/\Phi(P)|$  must be  $p$ . We conclude that  $P_0$  is a proper subgroup of  $P$  on which  $\beta$  acts as an automorphism of order  $q^k$ . The desired conclusion follows now by induction.  $\square$

**Proposition 3.3.** *Let  $G$  be a finite group and let  $M$  be a normal subgroup of  $G$ . Then  $\Gamma_N(G/M)$  is a subgraph of  $\Gamma_N(G)$ .*

*Proof.* Clearly the vertex set of  $\Gamma_N(G/M)$  is contained in that of  $\Gamma_N(G)$ . Also, let  $p, q \in \pi(G/M)$  be such that  $q \rightarrow p$  is an arc of  $\Gamma_N(G/M)$ ; this means that there exists a subgroup  $H$  of  $G$ , containing  $M$ , with  $H/M \cong C_p \rtimes C_q$ . If  $H/M$  has an element of order  $pq$ , then the same holds for  $H$  and we get the arc  $q \rightarrow p$  in  $\Gamma_N(G)$ ; thus we can assume that  $H/M$  is not a direct product, which implies that  $q$  is a divisor of  $p - 1$ . Let  $K$  be the subgroup of  $H$  containing  $M$  such that  $K/M \in \text{Syl}_p(H/M)$ , and let  $P$  be a Sylow  $p$ -subgroup of  $H$ . Since  $K$  is a normal subgroup of  $H$  whose index is coprime with  $p$ , we have that  $P$  is contained in  $K$ , and by the Frattini argument we get  $H = KN_H(P)$ . Since the  $q$ -part of  $|H|$  is strictly larger than that of  $|K|$ , we get that  $q$  is a divisor of  $|N_H(P)|$  and we can consider a subgroup  $C \subseteq N_H(P)$  having order  $q$ .

If  $C$  centralizes  $P$ , then we can easily construct an element of order  $pq$  of  $G$ . Otherwise, Lemma 3.2 yields the existence of a subgroup  $U$  of  $P$  such that  $|U| = p$  and  $C \subseteq N_G(U)$ . Now the subgroup  $UC$  yields the arc  $q \rightarrow p$  in  $\Gamma_N(G)$ , and the proof is complete.  $\square$

*Remark 3.4.* It is not true in general that if  $M$  is a normal subgroup of  $G$  and  $G/M$  contains a subgroup  $T$  isomorphic to the Frobenius group  $C_p \rtimes C_q$  where  $p \neq 2$  and  $q$  are primes, then  $G$  contains a subgroup isomorphic to  $T$ . For example, if  $p \equiv 1 \pmod{q}$  then there is a positive integer  $r$  of order  $q$  modulo  $p$ , that is,  $r \not\equiv 1 \pmod{p}$  and  $r^q \equiv 1 \pmod{p}$ . Then the center of the group  $G = \langle a, b \mid a^p = b^{q^2} = 1, a^b = a^r \rangle$  is  $M = \langle b^q \rangle$ , and  $G/M \cong T$ , but  $G$  does not have any subgroup isomorphic to  $T$ . In this case we have  $\Gamma_N(G) = (p \leftarrow q)$  and  $\Gamma_{\text{GK}}(G) = (p - q)$ , while  $\Gamma_N(G/M) = (p \leftarrow q)$  and  $\Gamma_{\text{GK}}(G/M) = (p - q)$ .

We are ready to start our discussion about the N-Prime Graph Question, which was presented in the introduction, with the following easy observation.

*Remark 3.5.* By [7, Corollary 4.1],  $\Gamma_N(G)$  and  $\Gamma_N(V(\mathbb{Z}G))$  have the same set of vertices. Moreover, it is clear that  $\Gamma_N(G)$  is a subgraph of  $\Gamma_N(V(\mathbb{Z}G))$ . So, the N-Prime Graph Question amounts to understanding if, for  $p, q$  distinct primes in  $\pi(G)$ , the existence of the arc  $q \rightarrow p$  in  $\Gamma_N(V(\mathbb{Z}G))$  implies the existence of the same arc in  $\Gamma_N(G)$ .

A finite group  $G$  is called *rational* if its ordinary character table contains only rational entries. As is well known, this is equivalent to the fact that every element  $x$  of  $G$  is conjugate in  $G$  to every generator of  $\langle x \rangle$ , which is also equivalent to the condition  $\mathbf{N}_G(\langle x \rangle)/\mathbf{C}_G(x) \cong \text{Aut}(\langle x \rangle)$  for every

$x \in G$  (see for instance the discussion preceding Lemma 2.1 in [30], or [20, V.13.7]). It has been proved in Corollary H of [2] that (PQ) holds for rational groups. Based on this, we prove that also (NPQ) holds for such groups.

**Theorem 3.6.** *Let  $G$  be a finite rational group. Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

*Proof.* Let  $p, q$  be vertices of  $\Gamma_N(V(\mathbb{Z}G))$  and assume that the arc  $q \rightarrow p$  is in  $\Gamma_N(V(\mathbb{Z}G))$ . If also  $p \rightarrow q$  is an arc of  $\Gamma_N(V(\mathbb{Z}G))$ , then we know that  $p - q$  is an edge of  $\Gamma_{\text{GK}}(V(\mathbb{Z}G))$  and the same holds for  $\Gamma_{\text{GK}}(G)$  by [2, Corollary H]. In particular, the arc  $q \rightarrow p$  is in  $\Gamma_N(G)$  and we are done. On the other hand, if  $p \rightarrow q$  is not an arc of  $\Gamma_N(V(\mathbb{Z}G))$ , then  $q$  is a divisor of  $p - 1$ . But since  $G$  is a rational group, for any element  $x \in G$  of order  $p$  we have  $|\mathbf{N}_G(\langle x \rangle) / \mathbf{C}_G(x)| = |\text{Aut}(\langle x \rangle)| = p - 1$ , so there exists an element of order  $q$  in  $\mathbf{N}_G(\langle x \rangle)$  and our claim is proved.  $\square$

The following result, which combines Theorem A and Corollary B, shows that the  $N$ -Prime Graph Question can be reduced to almost simple groups. This extends the corresponding result by Kimmerle and Konovalov for the Prime Graph Question.

**Theorem 3.7.** *Let  $G$  be a finite group, and assume that (NPQ) holds for every almost simple homomorphic image of  $G$ . Then (NPQ) holds for  $G$ . In particular, (NPQ) holds for every finite solvable group.*

*Proof.* As observed in Remark 3.5, we only have to show that, for  $p$  and  $q$  distinct primes in  $\pi(G)$ , the existence of the arc  $q \rightarrow p$  in  $\Gamma_N(V(\mathbb{Z}G))$  implies the existence of the same arc in  $\Gamma_N(G)$ . For the proof by contradiction, we will assume that  $G$  is a minimal counterexample to this fact with respect to the primes  $p$  and  $q$ . Note that, by our assumptions and by Remark 3.1, if  $X$  is any almost simple homomorphic image of  $G$  then also the Gruenberg–Kegel graphs  $\Gamma_{\text{GK}}(X)$  and  $\Gamma_{\text{GK}}(V(\mathbb{Z}X))$  coincide; therefore, by [25], we have  $\Gamma_{\text{GK}}(G) = \Gamma_{\text{GK}}(V(\mathbb{Z}G))$ , and the presence of both arcs  $q \rightleftharpoons p$  in  $\Gamma_N(V(\mathbb{Z}G))$  would imply the presence of the edge  $q - p$  in  $\Gamma_{\text{GK}}(G)$ . As a consequence, both arcs  $q \rightleftharpoons p$  would be in  $\Gamma_N(G)$ , against the fact that  $G$  is a counterexample. So  $\Gamma_N(V(\mathbb{Z}G))$  has the arc  $q \rightarrow p$  but not the arc  $p \rightarrow q$ ; in particular,  $V(\mathbb{Z}G)$  has a subgroup  $T$  which is a Frobenius group of order  $pq$ , and  $q$  is a divisor of  $p - 1$ .

Let  $M$  be a minimal normal subgroup of  $G$ , and observe that the factor group  $G/M$  satisfies our assumptions. Denoting by  $\varphi : V(\mathbb{Z}G) \rightarrow V(\mathbb{Z}(G/M))$  the natural homomorphism, if  $p$  does not divide  $|M|$ , then we get  $T \cap \ker(\varphi) = 1$  (see [31, Lemma 7.5]) and hence  $\varphi(T) \cong T$ . It follows that  $q \rightarrow p$  is an arc of  $\Gamma_N(V(\mathbb{Z}(G/M)))$  and, by our minimality assumption, it is also an arc of  $\Gamma_N(G/M)$ . But then  $q \rightarrow p$  is an arc of  $\Gamma_N(G)$  by Proposition 3.3, a contradiction. Therefore, no minimal normal subgroup of  $G$  can be a  $p'$ -group.

The discussion in the paragraph above implies that the Fitting subgroup  $\mathbf{F}(G)$  of  $G$  is a  $p$ -group, and we claim that in fact  $\mathbf{F}(G)$  must be trivial. As in the proof of Proposition 3.3, consider a subgroup  $C$  of  $G$  having order  $q$  and assume that  $\mathbf{F}(G) \neq 1$ . If  $C$  centralizes  $\mathbf{F}(G)$  then we can construct an element of order  $pq$  of  $G$ , which is not our case. Otherwise, we can use Lemma 3.2 and find a subgroup  $U$  of  $\mathbf{F}(G)$  such that  $|U| = p$  and  $C \subseteq \mathbf{N}_G(U)$ . Now the subgroup  $UC$  yields the arc  $q \rightarrow p$  in  $\Gamma_N(G)$ , again a contradiction.

To sum up, we have  $\mathbf{F}(G) = 1$  and every minimal normal subgroup of  $G$  is nonabelian of order divisible by  $p$ . Note that the last claim of the statement, concerning solvable groups, is already proved at this stage.

Our next claim is that  $G$  cannot have any minimal normal subgroup whose order is also divisible by  $q$ . In fact, assume the contrary and consider a minimal normal subgroup  $M$  of  $G$  such that  $pq$  divides  $|M|$ . Since  $M \cong S_1 \times \cdots \times S_t$  is a direct product of isomorphic nonabelian simple groups, we see that  $pq$  divides the order of each  $S_i$ . Now  $t$  must be 1, as otherwise we could produce an element of order  $pq$  in  $M$  (hence in  $G$ ) by multiplying an element of order  $p$  of  $S_1$  with an element of order  $q$  of  $S_2$ . Moreover, if such a subgroup  $M$  exists, then it must be the unique minimal normal subgroup of  $G$ . In fact, an element of order  $p$  in any other minimal normal subgroup of  $G$  would commute with an element of order  $q$  of  $M$ , again yielding the contradiction that  $G$  has an element of order  $pq$ . So  $M$  would be simple, and the unique minimal normal subgroup of  $G$ . But then  $G$  would be an almost simple group, and by our assumptions it would not be a counterexample.

As another remark, given a minimal normal subgroup  $M \cong S_1 \times \cdots \times S_t$  of  $G$ , for any fixed  $i \in \{1, \dots, t\}$  we claim that  $q$  does not divide  $|\mathbf{N}_G(S_i)|$ . Assuming the contrary, we could take an element  $y$  of order  $q$  in  $\mathbf{N}_G(S_i)$ . Recalling that  $S_i$  is a  $q'$ -group having an order divisible by  $p$ , we could find a (nontrivial) Sylow  $p$ -subgroup  $P$  of  $S_i$  such that  $y$  normalizes  $P$  (see [26, 8.2.3(a)]), and the same argument as in the third paragraph of this proof leads to a contradiction.

Now, let  $M \cong S_1 \times \cdots \times S_t$  and define

$$K = \bigcap_{i=1}^t \mathbf{N}_G(S_i);$$

in view of the paragraph above,  $K$  is a  $q'$ -group. Since the  $S_i$  are nonabelian simple groups, conjugation yields a permutation action of  $G$  on the set  $\{S_1, \dots, S_t\}$  with kernel  $K$ . So,  $G/K$  embeds in the symmetric group  $\text{Sym}(\{S_1, \dots, S_t\}) \cong \text{Sym}(t)$  and, if  $y \in G$  is an element of order  $q$ , then  $yK$  can be identified (up to renumbering) with a permutation containing the cycle  $(1, 2, \dots, q)$ . Take an element  $x_1 \in S_1$  of order  $p$  and, setting  $x_i = x_1^{y^{i-1}}$  for all  $i \in \{1, \dots, q\}$ , observe that  $x_i$  lies in  $S_i$ ; in particular, the elements  $x_i$  pairwise centralize each other. Defining  $x = x_1 \cdots x_q$ , it is not difficult to see that  $x$  and  $y$  commute, hence again we get that  $xy$  is an element of order  $pq$  in  $G$ . This is the final contradiction that completes the proof. □

The following corollary can be immediately derived from the theorem above and from Theorem 3.6.

**Corollary 3.8.** *Let  $G$  be a finite group. If every almost simple homomorphic image of  $G$  is a rational group, then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

## 4 | THE N-PRIME GRAPH QUESTION: THEOREM C

In this section, we prove that (NPQ) holds for every almost simple group whose socle is isomorphic to an alternating group, or to  $\text{PSL}_2(r^f)$  where  $r$  is a prime and  $f \in \{1, 2\}$  (Theorems 4.1 and 4.3, respectively); this, together with Theorem 3.7 and Corollary 3.8, will conclude the proof of Theorem C of the introduction.

Before stating the next result, we mention that, by Theorem 1.1 of [3] and Theorem A of [4], for every finite almost simple group  $G$  with a socle isomorphic to an alternating group or to  $\text{PSL}_2(r^f)$  where  $r$  is a prime and  $f \in \{1, 2\}$ , we have  $\Gamma_{\text{GK}}(V(\mathbb{Z}G)) = \Gamma_{\text{GK}}(G)$ .

**Theorem 4.1.** *Let  $G$  be a finite almost simple group whose socle is isomorphic to the alternating group  $A_n$ , for  $n \geq 5$ . Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

*Proof.* Let  $p$  and  $q$  be primes in  $\pi(G)$  and, for the sake of contradiction, suppose that  $q \rightarrow p$  is an arc of  $\Gamma_N(V(\mathbb{Z}G))$  but not of  $\Gamma_N(G)$ . Then, taking into account Theorem 3.6,  $G$  is not isomorphic to the symmetric group  $S_n$ , so we have  $G \cong A_n$  unless  $n = 6$  (and  $|G|$  divides  $4|A_n|$ ). Moreover,  $G$  does not have elements of order  $pq$  and hence, by the main theorem of [3], neither does  $V(\mathbb{Z}G)$ . Thus  $V(\mathbb{Z}G)$  has a subgroup  $T = \langle u \rangle \rtimes \langle v \rangle$  with  $|u| = p$ ,  $|v| = q$  and  $u^v = u^r$  for a suitable integer  $r$  such that  $r \not\equiv 1 \pmod p$  and  $r^q \equiv 1 \pmod p$ .

By Corollary 4.1 of [7],  $G$  has an element  $a$  of order  $p$  and, as  $p \neq 2$  and  $[G : A_n]$  divides 4, it follows that  $a \in A_n$ ; in particular we have  $n \geq p$ , and we may take  $a = (1, 2, \dots, p)$ . If  $n \geq p + 2$ , then all the elements of  $A_n$  with the same cyclic type as  $a$  lie in a single conjugacy class of  $A_n$ , hence  $a^r$  is conjugate to  $a$  in  $A_n$ . However, if  $q$  is odd, then the same conclusion holds also for  $n \in \{p, p + 1\}$ , because the map  $x \mapsto x^r$  on  $A_n$  induces a permutation of odd order on the set  $C$  of conjugacy classes of elements having order  $p$  in  $A_n$  and, in this case, we have  $|C| = 2$ . Now, if  $a^r$  is conjugate to  $a$  in  $A_n$ , then we can find an element  $b \in N_{A_n}(\langle a \rangle)$  with  $|b| = q$ , against our assumption. We conclude that  $n$  lies in  $\{p, p + 1\}$  and  $q = 2$ . Moreover,  $T$  is a dihedral group of order  $2p$  and  $a$  is not conjugate to  $a^{-1}$  in  $A_n$ , that is,  $a$  is not a real element of  $A_n$ . Note that the latter property does not hold if  $p \equiv 1 \pmod 4$ , hence  $n$  cannot be 6 and we have  $G \cong A_n$ .

Since the element  $a \in G$  is not real, we can take an irreducible character  $\chi$  of  $G$  such that  $\chi(a) \notin \mathbb{R}$ , and we denote by  $x$  the (nonzero) imaginary part of  $\chi(a)$ . On the other hand, as  $u$  is conjugate to  $u^{-1}$  in  $T$ , the linear expansion  $\widehat{\chi}$  of  $\chi$  to  $\mathbb{Z}G$  takes a real value on  $u$ .

Now,  $G$  has exactly two conjugacy classes of elements of order  $p$ , represented by  $a$  and  $a^{-1}$ . By Theorem 2.1 and [14, Theorem 2.3], if  $g \in G$  is such that  $\varepsilon_g(u) \neq 0$ , then  $g$  is conjugate to  $a$  or  $a^{-1}$ . Therefore  $\varepsilon_a(u) + \varepsilon_{a^{-1}}(u) = 1$  and

$$\widehat{\chi}(u) = \varepsilon_a(u)\chi(a) + \varepsilon_{a^{-1}}(u)\overline{\chi(a)} = \varepsilon_a(u)\chi(a) + (1 - \varepsilon_a(u))\overline{\chi(a)} = \varepsilon_a(u)(\chi(a) - \overline{\chi(a)}) + \overline{\chi(a)}.$$

Comparing the imaginary parts of the two sides, we get  $0 = x \cdot (2\varepsilon_a(u) - 1)$ ; but  $x \neq 0$ , and  $\varepsilon_a(u)$  is then an integer such that  $2\varepsilon_a(u) - 1 = 0$ . This is the final contradiction, which completes the proof. □

We move next to almost simple groups whose socle is a projective special linear group. The following general lemma will be crucial in our discussion, and it might be useful to prove that (NPQ) holds in other cases.

**Lemma 4.2.** *Let  $G$  be a finite group, let  $p$  and  $q$  be distinct primes, and let  $\zeta$  be a complex primitive  $p$ -th root of unity. Also, let  $\sigma$  be an automorphism of  $\mathbb{Q}_p = \mathbb{Q}(\zeta)$  of order  $q$ , let  $F = \{x \in \mathbb{Q}_p : \sigma(x) = x\}$  and let  $r$  be an integer such that  $\sigma(\zeta) = \zeta^r$ . Denoting by  $A$  a set of representatives for the conjugacy classes of elements of order  $p$  in  $G$ , suppose that  $G$  has a class function  $\chi : G \rightarrow \mathbb{C}$  satisfying the following conditions:*

- (a)  $\chi$  is an  $F$ -linear combination of irreducible characters of  $G$ .
- (b) For every  $a \in A$ , we have that  $[\mathbb{Q}_p : \mathbb{Q}(\chi(a))]$  is not divisible by  $q$ .
- (c) The complex numbers  $\chi(a)$ , for  $a \in A$ , are linearly independent over  $\mathbb{Q}$ .

*Then  $V(\mathbb{Z}G)$  does not have a Frobenius subgroup of the form  $C_p \rtimes C_q$ .*

*Proof.* Suppose, for the sake of contradiction, that  $V(\mathbb{Z}G)$  has a Frobenius subgroup  $T$  of the form  $C_p \rtimes C_q$ . So,  $T = \langle u \rangle \rtimes \langle v \rangle$  with  $|u| = p$ ,  $|v| = q$ , and we can assume  $u^v = u^r$ .

Observe that the  $r$ -th power map defines a permutation  $\varrho$  on the set  $\{a^G \mid a \in A\}$  (consisting of the conjugacy classes of elements having order  $p$  in  $G$ ), given by  $\varrho(a^G) = (a^r)^G$ ; as  $r^q \equiv 1 \pmod p$ , this permutation has an order dividing  $q$ . Moreover, for  $\varphi$  lying in the set  $\text{Irr}(G)$  of the irreducible complex characters of  $G$ , and for  $a \in A$ , we get  $\varphi(a) \in \mathbb{Q}_p$  and  $\sigma(\varphi(a)) = \varphi(a^r)$ . By our assumption in (a), we have  $\chi = \sum_{\varphi \in \text{Irr}(G)} c_\varphi \varphi$  for suitable  $c_\varphi \in F$ , and hence  $\sigma(\chi(a)) = \chi(a^r)$  for every  $a \in A$ . Furthermore, taking into account that  $[\mathbb{Q}_p : F] = q$ , for every  $a \in A$  the assumption in (b) yields  $\chi(a) \notin F$ ; in other words, we have  $\sigma(\chi(a)) \neq \chi(a)$  and, in particular,  $a^r$  is not conjugate to  $a$  in  $G$ . As a consequence, all the orbits of the permutation  $\varrho$  have cardinality  $q$ .

Every character  $\varphi \in \text{Irr}(G)$  extends uniquely to a linear map  $\mathbb{C}G \rightarrow \mathbb{C}$ , which we also denote by  $\varphi$ . As  $u$  and  $u^r$  are conjugate in  $V(\mathbb{Z}G)$ , and hence in  $\mathbb{C}G$ , it follows that  $\varphi(u) = \varphi(u^r)$ . Also, it is not difficult to see that we have  $\varphi(u^r) = \sigma(\varphi(u))$ . Similarly,  $\chi$  extends uniquely to a linear map  $\mathbb{C}G \rightarrow \mathbb{C}$ , which we also denote by  $\chi$ . Using the expression of  $\chi$  as a linear combination of irreducible characters of  $G$  we obtain  $\sigma(\chi(u)) = \chi(u^r) = \chi(u)$ . On the other hand, by Theorem 2.1 and [14, Theorem 2.3],  $\varepsilon_g(u) = 0$  for every  $g \in G$  of order different from  $p$ . Hence, since  $\chi$  is a class function, we have

$$\sum_{a \in A} \varepsilon_{a^r}(u)\chi(a) = \sum_{a \in A} \varepsilon_a(u)\sigma^{-1}(\chi(a)) = \sigma^{-1}(\chi(u)) = \chi(u) = \sum_{a \in A} \varepsilon_a(u)\chi(a).$$

Since the partial augmentations are integers, our assumption in (c) implies that  $\varepsilon_{a_1}(u) = \varepsilon_{a_2}(u)$  whenever  $a_1^G$  and  $a_2^G$  belong to the same  $\varrho$ -orbit. Thus, denoting by  $B$  a subset of  $A$  such that  $\{b^G \mid b \in B\}$  is a set of representatives of these orbits, we get

$$1 = \sum_{a \in A} \varepsilon_a(u) = q \sum_{b \in B} \varepsilon_b(u),$$

a contradiction. □

**Theorem 4.3.** *Let  $G$  be a finite almost simple group whose socle  $S$  is isomorphic to  $\text{PSL}_2(r^f)$ , for a prime  $r$  and  $f \in \{1, 2\}$ . Then the  $N$ -prime graphs of  $G$  and  $V(\mathbb{Z}G)$  coincide.*

*Proof.* Since  $\text{PSL}_2(2^2)$  is isomorphic to  $A_5$ , which has been already treated in Theorem 4.1, we can assume that  $r$  is an odd prime. We start by recalling that the order of  $S$  for  $r \neq 2$  is  $r^f(r^f - 1)(r^f + 1)/2$ , and the order of  $\text{Out}(S)$  is  $2f$ ; in fact, the automorphism group of  $S$  is isomorphic to  $S \langle \delta, \phi \rangle$ , where  $\delta$  is a diagonal automorphism such that  $\delta^2$  is an inner automorphism, and  $\phi$  is a field automorphism of order  $f$  (see for instance [34]). Therefore  $\pi(G) = \pi(S) = \pi^- \cup \pi^+ \cup \{r\} \cup \{2\}$ , where  $\pi^-$  and  $\pi^+$  are the sets of odd prime divisors of  $r^f - 1$  and of  $r^f + 1$ , respectively.

By [20, II.8.27], the group  $S$  has dihedral subgroups both of order  $r^f - 1$  and of order  $r^f + 1$ ; also, it has subgroups that are Frobenius groups with an elementary abelian kernel of order  $r^f$  and cyclic complements of order  $(r^f - 1)/2$ . In view of this fact, if  $r^f \equiv 1 \pmod 4$ , then  $\Gamma_N(S)$  contains the following arcs:  $s \rightleftharpoons t$  for any choice of  $s$  and  $t$  in  $\{2\} \cup \pi^-$  or in  $\pi^+$ , and  $2 \rightarrow s$  for any  $s \in \{r\} \cup \pi^+$ . Similarly, if  $r^f \equiv -1 \pmod 4$ , then  $\Gamma_N(S)$  contains the arcs  $s \rightleftharpoons t$  for any choice of  $s$  and  $t$  in  $\{2\} \cup \pi^+$  or in  $\pi^-$ , and  $2 \rightarrow s$  for any  $s \in \pi^-$ . Moreover, in both cases  $\Gamma_N(S)$  has the arcs  $s \rightarrow r$  where  $s \in \pi^-$  is a divisor of  $r - 1$ .

Now, all these arcs are clearly in  $\Gamma_N(G)$ , and also  $2 \rightarrow r$  is in  $\Gamma_N(G)$  unless  $r^f \equiv -1 \pmod 4$  and  $G = S$ ; in fact, it is not difficult to check that  $G$  has always a (possibly abelian) subgroup of the form  $C_r \rtimes C_2$  whenever  $G$  is not simple. Furthermore, every double arc in  $\Gamma_N(V(\mathbb{Z}G))$  is also in  $\Gamma_N(G)$  by [28, Theorem 5.5] and Remark 3.1. Since, by obvious arithmetical reasons,  $\Gamma_N(V(\mathbb{Z}G))$  cannot contain arcs of the kind  $s \rightarrow r$  for any  $s \in \pi^+$ , or  $s$  in  $\pi^-$  not dividing  $r - 1$ , the desired conclusion will be obtained by proving the following properties.

- (a)  $\Gamma_N(V(\mathbb{Z}G))$  does not contain any arc of the form  $q \rightarrow p$ , for any choice of a pair  $(p, q)$  in  $\pi^+ \times \pi^-$  or in  $\pi^- \times \pi^+$ .
- (b)  $\Gamma_N(V(\mathbb{Z}G))$  does not contain the arc  $2 \rightarrow r$  if  $G = S$  and  $r^f \equiv -1 \pmod 4$ .

We work next to show that (a) holds, and we start from the case  $G = S$ . Assuming for the moment that we have  $r^f \equiv 1 \pmod 4$ , we consider first the situation in which  $p$  lies in  $\pi^+$  and  $q$  in  $\pi^-$ . In view of the character table of  $S$ , as displayed in tab. 2 on p. 10 of [14], a set of representatives for the conjugacy classes of elements of order  $p$  in  $G$  is

$$\left\{ b^m \mid m \in \mathbb{Z}, \gcd\left(\frac{r^f + 1}{2}, m\right) = \frac{r^f + 1}{2p} \text{ and } 1 \leq m \leq \frac{r^f - 1}{4} \right\},$$

where  $b \in G$  is an element of order  $(r^f + 1)/2$ . Observe that the two conditions on the integer  $m$  yield  $m = k \cdot (r^f + 1)/2p$  for some  $k \in \mathbb{Z}$  with

$$1 \leq k \leq \frac{r^f - 1}{4} \cdot \frac{2p}{r^f + 1} \leq \frac{p}{2},$$

which implies  $k \leq (p - 1)/2$ . Moreover, setting  $\sigma$  to be a complex primitive  $\frac{r^f + 1}{2}$ -th root of unity and adopting the notation of the aforementioned table, the value of the irreducible character  $\theta_1$  (of degree  $r^f - 1$ ) on  $b^m$  is  $-(\sigma^m + \sigma^{-m}) = -(\zeta^k + \zeta^{-k})$ , where  $\zeta = \sigma^{(r^f + 1)/2p}$  is a primitive  $p$ -th root of unity.

Since  $\{\zeta^k + \zeta^{-k} \mid 1 \leq k \leq (p - 1)/2\}$  is a basis for  $\mathbb{Q}(\zeta + \zeta^{-1})$  over  $\mathbb{Q}$ , the values of  $\theta_1$  on the given set of representatives for the conjugacy classes of elements of order  $p$  are linearly independent over  $\mathbb{Q}$ , and clearly  $q$  does not divide  $2 = |\mathbb{Q}_p : \mathbb{Q}(\theta_1(b^m))|$  for any such representative  $b^m$ . We are then in a position to apply Lemma 4.2, obtaining the desired conclusion.

Still assuming  $G = S$  and  $r^f \equiv 1 \pmod 4$ , let us consider the case when  $p$  lies in  $\pi^-$  and  $q$  in  $\pi^+$ . We can apply the same method as above, this time using the irreducible character  $\chi_1 \in \text{Irr}(G)$  (of degree  $r^f + 1$ ) in place of  $\theta_1$ . The conjugacy classes of elements of order  $p$  are now represented by the elements  $a^\ell$ , where  $a \in G$  is an element of order  $(r^f - 1)/2$ , and  $\ell$  ranges over all integers that satisfy the following conditions:

$$\gcd\left(\frac{r^f - 1}{2}, \ell\right) = \frac{r^f - 1}{2p}, \quad \text{and} \quad 1 \leq \ell \leq \frac{r^f - 1}{4}.$$

As above, this yields  $\ell = k \cdot (r^f - 1)/2$  with  $1 \leq k \leq (p - 1)/2$ ; denoting by  $\rho$  a primitive  $\frac{r^f - 1}{2}$ -th root of unity and setting  $\zeta = \rho^{(r^f - 1)/2}$ , we see that the value of  $\chi_1$  on  $a^\ell$  is  $\zeta^k + \zeta^{-k}$ , and the desired conclusion follows by Lemma 4.2, as above.

A slight variation of our argument so far yields that (a) is true for  $G = S$  also when  $r^f \equiv -1 \pmod 4$ , both in the case  $p \in \pi^+, q \in \pi^-$  and vice versa. Thus, the claim in (a) is proved when  $G = S$ .

Let now  $G$  be any almost simple group with a socle  $S$  isomorphic to  $\text{PSL}_2(r^f)$ . If the characters  $\theta_1$  and  $\chi_1$  can be extended to characters  $\widehat{\theta}_1$  and  $\widehat{\chi}_1$  in  $\text{Irr}(G)$ , then it is easily seen that we can argue exactly as in the case  $G = S$ , using  $\widehat{\theta}_1$  and  $\widehat{\chi}_1$  in place of  $\theta_1$  and  $\chi_1$ . On the other hand, if either  $\theta_1$  or  $\chi_1$  does not extend to  $G$ , then Theorems 6.6 and 6.7 of [34] yield that  $G$  is isomorphic to one of the following groups:

- $\text{Aut}(\text{PSL}_2(r^2)) = \text{PSL}_2(r^2) \langle \delta, \phi \rangle$ ,
- $\text{PSL}_2(r^2) \langle \phi \rangle$ ,
- $\text{PSL}_2(r^2) \langle \delta\phi \rangle$ .

So, let us define  $\xi$  to be either  $\theta_1$  or  $\chi_1$ , depending on whether we are in the case  $p \in \pi^+$  and  $q \in \pi^-$  or vice versa, and assume that  $\xi$  does not extend to  $G$ . Let  $\psi$  be an irreducible character of  $G$  whose restriction to  $S = \text{PSL}_2(r^2)$  has  $\xi$  as a constituent. Since the diagonal automorphism  $\delta$  lies in the inertia subgroup of  $\xi$  in  $\text{Aut}(S)$  (see [34, p. 2]), we have that the restriction of  $\psi$  to  $S$  splits as  $\xi + \xi^\delta$ . Now, the automorphism  $\phi$  induces a permutation on the set  $C = \{C_1, \dots, C_t\}$  of  $S$ -conjugacy classes of elements of order  $p$ , where the class  $x^S \in C$  is mapped to  $(x^r)^S$ , and we consider a set  $\mathcal{F} = \{C_1, \dots, C_s\}$  of representatives for the orbits of this action. Note that, since  $\delta$  fixes all the elements of  $C$ , a set  $\{x_1, \dots, x_s\}$  of representatives for the  $S$ -classes in  $\mathcal{F}$  is also a set of representatives for the  $G$ -conjugacy classes of elements of order  $p$ .

We claim that the complex numbers  $\psi(x_1), \dots, \psi(x_s)$  are linearly independent over  $\mathbb{Q}$ . In fact, assume that we have

$$\sum_{j=1}^s q_j \psi(x_j) = 0,$$

where  $q_j$  is a rational number for every  $j \in \{1, \dots, s\}$ . Then we get

$$0 = \sum_{j=1}^s q_j (\xi(x_j) + \xi(x_j^r)),$$

and this is a linear combination with rational coefficients of values of  $\xi$  on a set of representatives for the  $S$ -conjugacy classes in  $C$  (indeed, if  $p$  lies in  $\pi^-$  then we have  $\xi(x_j) = \xi(x_j^r)$  for every  $j \in \{1, \dots, s\}$ , whereas if  $p$  lies in  $\pi^+$  then  $\xi(x_j) \neq \xi(x_j^r)$  for every  $j \in \{1, \dots, s\}$ ). Since we know these values are linearly independent over  $\mathbb{Q}$ , it easily follows that  $q_j = 0$  for all  $j \in \{1, \dots, s\}$ . Moreover, for every  $j \in \{1, \dots, s\}$ , we have that  $q$  does not divide  $|\mathbb{Q}_p : \mathbb{Q}(\psi(x_j) + \psi(x_j^r))|$  (which is either 2 or 4, depending on whether  $p$  is in  $\pi^-$  or in  $\pi^+$ , respectively), and again we are done by Lemma 4.2. This concludes the proof of the claim in (a).

As for the claim in (b), still referring to tab. 2 of [14] (and recalling that we are assuming  $r^f \equiv -1 \pmod{4}$ ), we consider the conjugacy classes of  $G$  containing elements of order  $r$ : there are two of them, represented by elements denoted by  $c$  and  $d$ . Also, we consider the irreducible character  $\eta_1$  of  $G$ . We have  $\eta_1(c) = -\frac{1}{2} + \frac{1}{2}i\sqrt{r^f}$  and  $\eta_1(d) = \overline{\eta_1(c)}$ , so these values are linearly independent over  $\mathbb{Q}$ . Moreover,  $|\mathbb{Q}(\eta_1(c)) : \mathbb{Q}|$  is clearly 2, hence  $|\mathbb{Q}_r : \mathbb{Q}(\eta_1(c))| = (r - 1)/2$  is not divisible by 2, and the same holds for  $|\mathbb{Q}_r : \mathbb{Q}(\eta_1(d))|$ . Another application of Lemma 4.2 yields that  $\Gamma_{\mathbb{N}}(V(\mathbb{Z}(G)))$  does not contain the arc  $2 \rightarrow r$ , and the proof is complete.  $\square$

We note that a first study of certain noncyclic finite subgroups of the unit group of integral group rings of  $\text{PSL}_2(r^f)$  was carried out in the Ph.D. thesis of Höfert ([19]). For instance, the claim in (b)

of the above proof (discussed in its last paragraph) can also be deduced from [19, Proposition 4.4] in the case when  $f = 1$ . We are grateful to the referee for pointing this out.

## 5 | METACYCLIC SUBGROUPS OF SOLVABLE GROUPS: THEOREMS D and E

We work next toward an improvement of Corollary B in the spirit of the Subgroup Isomorphism Problem, as described in the introduction. We will first prove that if  $G$  is a finite solvable group and  $V(\mathbb{Z}G)$  has a subgroup isomorphic to a Frobenius group of the form  $C_p \rtimes C_{q^k}$ , then so does  $G$  (Theorem D of the introduction).

We start with a preliminary result. Observe that the isomorphism type of a group of the kind  $C_p \rtimes C_{q^k}$  is uniquely determined by the order of its center, and also by the order of the kernel of the action of  $C_{q^k}$  on  $C_p$  which defines the semidirect product.

**Lemma 5.1.** *Let  $p, q$  be distinct primes,  $H$  be a finite group and  $N$  a normal elementary abelian  $q$ -subgroup of  $H$ . Assume that, for a positive integer  $k$ , the factor group  $H/N$  is a Frobenius group of the form  $C_p \rtimes C_{q^k}$ , and assume that  $b \in H$  is an element of order  $q^k$  with  $\langle b \rangle \cap N = 1$ . Then there exists a Sylow  $p$ -subgroup  $P$  of  $H$  such that  $b$  lies in  $\mathbf{N}_H(P)$ .*

*Proof.* Let  $H$  be a minimal counterexample to the statement, and let  $P = \langle a \rangle$  be a Sylow  $p$ -subgroup of  $H$ . Observe that, by the Frattini argument, we have  $H = NN_H(P)$  and, by [11, Theorem 5.2.3],  $N = [N, P] \times \mathbf{C}_N(P)$ ; therefore, setting  $M = [N, P]$  and  $C = \mathbf{C}_N(P)$ , we have  $H = MCN_H(P) = MN_H(P)$ . Note that, since  $N$  and  $P$  are both normalized by  $\mathbf{N}_H(P)$ , so is their commutator  $M$ ; it is then clear that  $M$  and  $MP$  are normal subgroups of  $H$ , because they are normalized by both  $M$  and  $\mathbf{N}_H(P)$ . Since  $M \cap \mathbf{N}_H(P) = \mathbf{N}_M(P) = \mathbf{C}_M(P) = M \cap C = 1$ , we see that  $\mathbf{N}_H(P)$  is in fact a complement for  $M$  in  $H$ .

Now, set  $H_0 = MP \langle b \rangle$ ; since  $aM$  and  $b^{q^{k-1}}M$  are elements of  $H_0/M$  that do not commute because even  $[aN, b^{q^{k-1}}N]$  is nontrivial, we have that  $H_0/M$  is isomorphic to the Frobenius group  $C_p \rtimes C_{q^k}$ . In fact,  $H_0$  is a group which satisfies our assumptions (with respect to  $M$  and the same element  $b$  of order  $q^k$ ), and if the desired conclusion holds for  $H_0$  then it clearly holds also for  $H$ . In other words, by the minimality of  $H$  as a counterexample, we must have  $H_0 = H$ . Note also that  $M$  is nontrivial, as otherwise  $H$  would not be a counterexample. Thus  $MP$  is a Frobenius group, because  $\mathbf{C}_M(P) = 1$  and so  $P$  acts fixed-point freely on  $M$ ; moreover,  $H$  is a 2-Frobenius group.

Observe that,  $H$  being a solvable group with abelian Fitting subgroup  $M$ , the conjugation action yields a faithful action of  $H/M$  on  $M$ , and  $M$  can be viewed as a module for  $H/M$  over the field  $\text{GF}(q)$  with  $q$  elements. If  $y \in H$  is an element of order  $q^k$  such that  $\langle y \rangle \cap M = 1$ , then  $yM$  generates a Frobenius complement of  $H/M$  and we are in a position to apply Theorem 15.16 of [21], obtaining that  $\dim M = q^k \dim \mathbf{C}_M(y)$  and thus  $|M| = |\mathbf{C}_M(y)|^{q^k}$ . In particular,  $\ell = |\mathbf{C}_M(y)|$  does not depend on the choice of the element  $y$  with the above properties.

As the next step, we will count the elements of order  $q^k$  in  $H$  that normalize some Sylow  $p$ -subgroup of  $H$ . Since, as observed in the first paragraph of this proof,  $\mathbf{N}_H(P)$  is a complement for  $M$  in  $H$ , we deduce that  $\mathbf{N}_H(P) \cong H/M$  is a Frobenius group of the form  $C_p \rtimes C_{q^k}$  (and clearly the same holds for the normalizer of any Sylow  $p$ -subgroup of  $H$ , because such a normalizer is conjugate to  $\mathbf{N}_H(P)$  in  $H$ ). It follows that every Sylow  $p$ -subgroup of  $H$  is normalized by precisely  $p$  (cyclic) subgroups of order  $q^k$ . Next, let us denote by  $C$  the set of the cyclic subgroups of order  $q^k$

of  $H$  that normalize some Sylow  $p$ -subgroup of  $H$ : again, as  $M$  intersects trivially the normalizer of any Sylow  $p$ -subgroup of  $H$ , for every  $Q \in C$  we have  $Q \cap M = 1$  and hence  $|\mathbf{C}_M(Q)| = \ell$ . We claim that each subgroup  $Q \in C$  normalizes precisely  $\ell$  distinct Sylow  $p$ -subgroups of  $H$ . In fact, as  $MP$  is normal in  $H$ , all the Sylow  $p$ -subgroups of  $H$  are contained in  $MP$ ; but  $MP$  is a Frobenius group, hence  $\text{Syl}_p(H)$  has cardinality  $|M|$  and consists of the  $M$ -conjugates of  $P$ . Now, assume that  $Q_0 \in C$  normalizes  $P_0 \in \text{Syl}_p(H)$  and denote by  $y$  a generator of  $Q_0$ . If, for  $m \in M$ , we have  $\langle y \rangle = Q_0 \subseteq \mathbf{N}_H(P_0^m)$ , then  $[m^{-1}, y^{-1}]$  lies in  $\mathbf{N}_H(P_0)y^{-1} \cap M = \mathbf{N}_H(P_0) \cap M = 1$  and so  $m$  lies in  $\mathbf{C}_M(y) = \mathbf{C}_M(Q_0)$ . On the other hand, it is clear that if  $m$  lies in  $\mathbf{C}_M(Q_0)$  then  $Q_0$  normalizes  $P_0^m$ , and our claim follows.

We are ready to conclude the count mentioned in the previous paragraph. Consider the bipartite graph whose vertex set is  $\text{Syl}_p(H) \cup C$  (with  $\text{Syl}_p(H)$  and  $C$  being the two parts of the graph), and where the vertices  $P_0 \in \text{Syl}_p(H)$  and  $Q_0 \in C$  are adjacent if and only if  $Q_0$  normalizes  $P_0$ . By the discussion above, the number of edges in this graph can be computed both as  $|M|p$  (which is the number of vertices in the part  $\text{Syl}_p(H)$  multiplied by the number of neighbors of each vertex in  $\text{Syl}_p(H)$ ) and  $|C|\ell$  (obtained similarly, from the point of view of the part  $C$ ). So, we have  $|C| = |M|p/\ell = \ell^{q^k-1}p$ . As each element of  $C$  contains  $q^{k-1}(q-1)$  elements of order  $q^k$ , we conclude that the number of elements of order  $q^k$  in  $H$  which normalize some Sylow  $p$ -subgroup of  $H$  is given by

$$|C|q^{k-1}(q-1) = \ell^{q^k-1}pq^{k-1}(q-1).$$

Our aim is then to show that the number above coincides with the total number of elements  $y$  of order  $q^k$  in  $H$  with  $\langle y \rangle \cap M = 1$ . So, let  $y$  be such an element: observe that  $y$  acts as an automorphism of order  $q^k$  on the  $\text{GF}(q)$ -vector space  $M$ , whose dimension we denote by  $t$ . Hence, up to conjugacy in  $\text{GL}_t(q)$ , the action of  $y$  is represented by an upper unitriangular matrix  $Y$  in the canonical Jordan form. It is easily seen that the dimension of  $\mathbf{C}_M(y)$  over  $\text{GF}(q)$  coincides with the number of Jordan blocks of  $Y$ . Denote by  $I_d, J_d$  the  $d$ -dimensional identity matrix and the Jordan block with eigenvalue 1 respectively, and set  $E = J_d - I_d$ . Then for a positive integer  $h$  we get

$$(J_d)^h = \sum_{i=0}^{d-1} \binom{h}{i} E^i$$

and, in particular, we see that  $(J_d)^{q^k}$  contains the summand  $E^{q^k}$  if  $d > q^k$ . Since in our situation  $(J_d)^{q^k}$  must be the identity matrix  $I_d$  for every Jordan block of  $Y$ , we conclude that  $d$  is at most  $q^k$  for every block of  $Y$ . But now, recalling that  $t = \dim M$  equals  $q^k \dim \mathbf{C}_M(y)$  and that  $\dim \mathbf{C}_M(y)$  is the number of Jordan blocks of  $Y$ , we conclude that the dimension of each block is precisely  $q^k$ .

Finally, for  $m \in M$  and  $r \in \mathbb{Z}$ , consider the element  $my^{-r}$  of  $H$ . Observe that  $(my^{-r})^{q^{k-1}} = m \cdot m^{y^r} \cdot m^{y^{2r}} \dots m^{y^{(q^{k-1}-1)r}} \cdot y^{q^{k-1}r}$ . So, if  $r$  is a multiple of  $q$ , we see that  $(my^{-r})^{q^{k-1}}$  lies in  $M$  and hence  $my^{-r}$  is not an element of order  $q^k$  with  $\langle my^{-r} \rangle \cap M = 1$ . On the other hand, if  $r$  is coprime to  $q$ , then  $(my^{-r})^{q^{k-1}}$  cannot lie in  $M$ , as otherwise the same would hold for  $y^{q^{k-1}r}$  and hence for  $y^{q^{k-1}}$ . As a consequence, if we want to count the elements  $z$  of order  $q^k$  lying in  $M\langle y \rangle \in \text{Syl}_q(H)$  and such that  $\langle z \rangle \cap M = 1$ , then we have to count the elements of order  $q^k$  having the form  $my^{-r}$  where  $m \in M$  and  $r$  is coprime to  $q$ . We start by focusing on the element  $my^{-1}$ .

With the methods described in the paragraph above, we see that the order of  $my^{-1}$  is  $q^k$  if and only if, in an additive notation, we have

$$m + Ym + Y^2m + \dots + Y^{q^k-1}m = 0.$$

In other words,  $my^{-1}$  has order  $q^k$  if and only if  $m$  lies in the kernel of the endomorphism of  $M$  represented by the matrix  $I_t + Y + Y^2 + \dots + Y^{q^k-1}$ . Now, it can be computed that

$$I_{q^k} + J_{q^k} + (J_{q^k})^2 + \dots + (J_{q^k})^{q^k-1} = \sum_{i=0}^{q^k-1} \left( \sum_{j=i}^{q^k-1} \binom{j}{i} \right) E^i,$$

and the  $(r, s)$ -entry of this matrix depends only on  $i = s - r$  (where  $1 \leq r \leq s \leq q^k$ ). Since we have

$$\sum_{j=i}^{q^k-1} \binom{j}{i} = \binom{q^k}{i+1}$$

(this equality, not difficult to check, is sometimes referred to as the Christmas stocking theorem), it is then clear that  $I_t + Y + Y^2 + \dots + Y^{q^k-1}$  is a block-diagonal matrix whose blocks are all equal to the  $q^k$ -dimensional matrix having all zero entries except the top-right entry, which is 1. The kernel of the endomorphism associated with this matrix has then a dimension equal to  $t - \dim \mathbf{C}_M(y)$ , and so it contains  $\ell^{q^k-1}$  elements.

Taking into account that the above calculation yields the same result if we replace  $y^{-1}$  by any generator of  $\langle y \rangle$ , we conclude that the number of elements  $z$  of order  $q^k$  lying in the Sylow  $q$ -subgroup  $M \langle y \rangle$  of  $H$  and with  $\langle z \rangle \cap M = 1$  is  $\ell^{q^k-1} q^{k-1} (q - 1)$ . Multiplying this result by the number of Sylow  $q$ -subgroups of  $H$ , we obtain a total of  $\ell^{q^k-1} p q^{k-1} (q - 1)$  elements, including our element  $b$ . The desired conclusion now follows because this number coincides with the number of elements of order  $q^k$  in  $H$  that normalize some Sylow  $p$ -subgroup of  $H$ , and the proof is complete. □

We are ready to prove a slightly stronger form of Theorem D.

**Theorem 5.2.** *Let  $G$  be a finite solvable group, let  $p, q$  be distinct primes and let  $k$  be a positive integer. Assume that  $V(\mathbb{Z}G)$  has two elements  $u, v$  such that  $|u| = p, |v| = q^k$ , and the group  $T$  generated by  $u$  and  $v$  is isomorphic to the Frobenius group  $C_p \rtimes C_{q^k}$ . Then there exist  $a, b \in G$  such that  $|a| = p, |b| = q^k, \langle a, b \rangle \cong T$  and  $\varepsilon_b(v) \neq 0$ .*

*Proof.* Let  $G$  be a counterexample of minimal order. Assume that the fitting subgroup  $\mathbf{F}(G)$  of  $G$  is not a  $p$ -group. Then we can take a minimal normal subgroup  $N$  of  $G$  which is an elementary abelian  $r$ -group for some prime  $r \neq p$ . Denoting by  $\varphi : V(\mathbb{Z}G) \rightarrow V(\mathbb{Z}(G/N))$  the natural homomorphism, we get  $T \cap \ker(\varphi) = 1$  and hence  $\varphi(T) \cong T$ . By our minimality assumption, there exist  $x, y \in G$  such that  $|x| = |\varphi(x)| = p, |\varphi(y)| = q^k, \langle \varphi(x), \varphi(y) \rangle \cong T$ , and  $\varepsilon_{\varphi(y)}(\varphi(v)) \neq 0$ .

Since

$$\varepsilon_{\varphi(y)}(\varphi(v)) = \sum_{g^G : \varphi(g) \sim \varphi(y)} \varepsilon_g(v),$$

we can find an element  $b \in G$  such that  $\varepsilon_b(v) \neq 0$  and  $bN$  is conjugate to  $yN$  in  $G/N$ . Note that, in particular,  $|b|$  is a multiple of  $q^k$ , but in fact we have  $|b| = q^k$  by [14, Theorem 2.3]. Now, let  $g \in G$  be such that  $bN = (yN)^{gN}$ . Clearly  $bN$  normalizes  $\langle x^gN \rangle$ , hence  $x^g, b \in G$  satisfy the following properties:  $|x^g| = p$ ,  $|b| = q^k$ , the subgroup  $H = \langle N, x^g, b \rangle$  is such that  $H/N \cong C_p \rtimes C_{q^k}$  is a Frobenius group, and  $\varepsilon_b(v) \neq 0$ .

If  $r \neq q$ , then, by the Schur-Zassenhaus theorem,  $N$  is complemented in  $H$  by a Hall  $\{p, q\}$ -subgroup  $H_0$ , and we can choose  $H_0$  so that  $b \in H_0 \cong H/N \cong T$ . Now, if  $a$  is an element of order  $p$  in  $H_0$ , then the elements  $a$  and  $b$  satisfy the conclusions of the statement and  $G$  is not a counterexample. On the other hand, if  $r = q$ , then we are in a position to apply Lemma 5.1 to the subgroup  $H$  (note that  $\langle b \rangle \cap N = 1$  because  $|b| = |bN| = q^k$ ), and again  $G$  is not a counterexample.

In view of the above discussion, we conclude that  $\mathbf{F}(G)$  must be a  $p$ -group. By the main theorem in [13], we can choose an element  $b \in G$  such that  $|b| = q^k$  and  $\varepsilon_b(v) \neq 0$ ; moreover,  $b$  acts as an automorphism of order  $q^k$  on  $\mathbf{F}(G)$ , because  $G$  is solvable and therefore  $\mathbf{C}_G(\mathbf{F}(G)) = \mathbf{Z}(\mathbf{F}(G))$ . But then we may apply Lemma 3.2, and again it turns out that  $G$  is not a counterexample. This is the final contradiction that completes the proof.  $\square$

Note that, by [13], if  $G$  is a finite solvable group and  $V(\mathbb{Z}G)$  has a subgroup isomorphic to  $C_p \times C_{q^k}$ , then so does  $G$ . This, together with the previous result, solves the Subgroup Isomorphism Problem for subgroups of the form  $C_p \rtimes C_{q^k}$  in the two extreme situations when the action of  $C_{q^k}$  on  $C_p$  is trivial or faithful. It is then natural to consider the following:

**Question.** Let  $G$  be a finite solvable group, and assume that  $V(\mathbb{Z}G)$  has a subgroup  $T$  of the form  $C_{p^h} \rtimes C_{q^k}$ . Is it true that  $G$  has a subgroup isomorphic to  $T$ ?

We provide next a positive answer to this question under the stronger assumption that the group  $G$  has a cyclic derived subgroup. Before stating it, we observe the following.

*Remark 5.3.* Let  $G$  be a finite group and let  $V = V(\mathbb{Z}G)$ . We observe that, if an element  $v \in V'$  is conjugate in  $\mathbb{Q}G$  to an element  $g$  of  $G$ , then  $g$  lies in fact in  $G'$ .

Indeed, consider the natural algebra homomorphism  $\varphi : \mathbb{Q}G \rightarrow \mathbb{Q}(G/G')$  and its restriction to a group homomorphism of  $V(\mathbb{Q}G)$  to  $V(\mathbb{Q}(G/G'))$ . Since  $V(\mathbb{Q}(G/G'))$  is an abelian group,  $V'$  is contained in the kernel of this group homomorphism; in particular  $\varphi(v) = \varphi(1)$ , and it easily follows that  $v - 1$  lies in the kernel of the algebra homomorphism  $\varphi$ . As this kernel is an ideal, namely the ideal generated by  $\{xy - yx \mid x, y \in G\}$ , also  $g - 1$  lies in the kernel of  $\varphi$ , that is,  $\varphi(g) - \varphi(1) = 0$  and therefore  $g \in G'$ .

**Theorem 5.4.** Let  $G$  be a finite group such that  $G'$  is cyclic, and let  $p, q$  be distinct primes. Let  $u, v$  be elements of  $V(\mathbb{Z}G)$  such that  $|u| = p^h, |v| = q^k$  (where  $h$  and  $k$  are positive integers) and  $T = \langle u, v \rangle \cong C_{p^h} \rtimes C_{q^k}$  is nonabelian. Then there exist  $a \in G'$  and  $b \in G$  such that  $\langle a, b \rangle \cong T$ , where  $b$  is conjugate to  $v$  in  $\mathbb{Q}G$ .

*Proof.* By means of contradiction we assume that  $G$  is a minimal counterexample to the statement and  $T = \langle u \rangle \rtimes \langle v \rangle$  is as in the hypothesis. Since  $T$  is nonabelian, we have that  $q$  is a divisor of  $p - 1$ .

We first claim that  $G'$  is a  $p$ -group. Otherwise  $G'$  contains a subgroup  $N$ , normal in  $G$ , whose order is a prime  $r \neq p$ . Then, denoting by  $\varphi : V = V(\mathbb{Z}G) \rightarrow V(\mathbb{Z}(G/N))$  the natural homomorphism, we get  $T \cap \ker(\varphi) = 1$  and hence  $\varphi(T) \cong T$ . By the minimality of  $G$ , there are  $a \in G'$

and  $g \in G$  such that  $\langle \varphi(a), \varphi(g) \rangle = \langle \varphi(a) \rangle \rtimes \langle \varphi(g) \rangle \cong T$  and  $\varphi(v)$  is conjugate to  $\varphi(g) = gN$  in  $\mathbb{Q}(G/N)$ . On the other hand, by the main theorem of [6], the First Zassenhaus Conjecture holds for  $G$ ; therefore  $v$  is conjugate in  $\mathbb{Q}G$  to an element  $b$  of  $G$ , so that  $bN$  and  $gN$  are both conjugate to  $\varphi(v)$  in  $\mathbb{Q}(G/N)$ . Since  $gN$  and  $bN$  are elements of  $G/N$  that are conjugate in  $\mathbb{Q}(G/N)$ , every irreducible character of  $G/N$  takes the same value on  $bN$  and  $gN$ . Hence they are in fact conjugate in  $G/N$ , and we may assume  $g = b$ . Let  $H = \langle N, a, b \rangle$ . Then  $H/N \cong T$ . If  $r \neq q$ , then  $G$  contains a Hall subgroup of the form  $\langle a \rangle \rtimes \langle b \rangle \cong T$ , a contradiction. Otherwise, since  $N$  has order  $q$  and  $p$  is larger than  $q$ , we have  $\langle N, a \rangle = N \times \langle a \rangle$ . Recalling that  $b$  is an element of order  $q^k$ , we have  $\langle N, b \rangle = N \rtimes \langle b \rangle$  and so  $H = (N \times \langle a \rangle) \rtimes \langle b \rangle$ ; now, as  $\langle aN, bN \rangle$  is isomorphic to  $T$ , then so is  $\langle a \rangle \rtimes \langle b \rangle$ , against the assumption. This finishes the proof of our claim that  $G'$  is a  $p$ -group.

Next, we consider two different situations, depending on whether  $\mathbf{Z}(T)$  is trivial or not. Assume first  $\mathbf{Z}(T) = 1$ , which is equivalent to the condition  $[u, v^{q^{k-1}}] \neq 1$ . As above, we take  $b$  in  $G$  which is conjugate to  $v$  in  $\mathbb{Q}G$ . If  $u^v = u^r$ , then  $r \not\equiv 1 \pmod p$ , therefore  $[u, v] = u^{r-1}$  and it has order  $p^h$ . By Remark 5.3, the order of  $G'$  is a multiple of  $p^h$  and therefore, by hypothesis,  $b^{q^{k-1}}$  commutes with  $G'$ , for otherwise there is  $a \in G'$  such that  $\langle a \rangle \rtimes \langle b \rangle \cong T$  and  $G$  would not be a counterexample. If  $g \in G$ , then  $g^{b^{q^{k-1}}} = zg$  for some  $z \in G'$ . Then  $g = g^{b^{q^k}} = z^q g$  and therefore,  $z = 1$ . This shows that  $b^{q^{k-1}}$  is central in  $G$  and hence so is  $v^{q^{k-1}}$  in  $V$ , which contradicts the assumption  $[u, v^{q^{k-1}}] \neq 1$ .

Finally, assume that  $\mathbf{Z}(T) \neq 1$ . Since  $T$  is nonabelian, we have that  $\mathbf{Z}(T)$  has order  $q^t$  for a suitable  $0 < t < k$ ; setting  $z = v^{q^{k-t}}$ , we have  $\mathbf{Z}(T) = \langle z \rangle$ . By the fact that the First Zassenhaus Conjecture holds for  $G$ , we can consider  $x \in G$  and  $\gamma$  a unit in  $\mathbb{Q}G$  such that  $(uz)^\gamma = x$ . Note also that the  $q$ -part  $x_q$  of  $x$  is  $z^\gamma$ , and the  $p$ -part  $x_p$  is  $u^\gamma$ . Since  $u$  lies in  $T' \subseteq V'$ , in view of Remark 5.3, we then have  $x_p \in G'$ . Finally, let  $b \in G$  and let  $\delta$  be a unit in  $\mathbb{Q}G$  such that  $v^\delta = b$ . Now, for any  $y \in G$ , we get

$$\varepsilon_y(x_q) = \varepsilon_y(z^\gamma) = \varepsilon_y(z^\delta) = \varepsilon_y(b^{q^{k-t}})$$

(recall that the partial augmentation with respect to a given element  $y \in G$  is a class function of  $\mathbb{Q}G$ ). It immediately follows that  $b^{q^{k-t}}$  is conjugate to  $x_q$  in  $G$ , and we take  $g \in G$  such that  $(b^{q^{k-t}})^g = x_q$ . Clearly the subgroup  $H = \langle x_p, b^g \rangle$  has a center of order at least  $q^t$ , but we claim that  $|\mathbf{Z}(H)|$  is precisely  $q^t$  (i.e.,  $H \cong T$ ). Set  $Z = \mathbf{O}_q(\mathbf{Z}(H))$ . We will first prove that  $|Z| = q^t$ , and then that  $\mathbf{Z}(H) = Z$ . Indeed,  $Z$  centralizes a nontrivial subgroup of the cyclic  $p$ -group  $G'$  and hence it centralizes  $G'$ ; on the other hand, for  $P \in \text{Syl}_p(G)$ , clearly  $Z$  acts trivially on  $P/G'$  as well, thus  $Z$  centralizes  $P$ . But as  $Z$  lies in an (abelian) Hall  $p'$ -subgroup of  $G$ , we conclude that  $Z$  lies in  $\mathbf{Z}(G)$ . If now  $|Z|$  is larger than  $q^t$ , then  $b^{q^{k-t-1}}$  would lie in  $\mathbf{Z}(G)$  and hence in  $\mathbf{Z}(\mathbb{Q}G)$ . But then  $v^{q^{k-t-1}} = (b^{q^{k-t-1}})^{\delta^{-1}} = b^{q^{k-t-1}}$  would be central in  $V$ , contradicting the fact that  $v^{q^{k-t-1}}$  does not centralize  $u$ . Therefore  $|Z| = q^t$ . As  $t < k$  and  $b^g$  has order  $q^k$ ,  $H$  is not abelian and hence  $\mathbf{Z}(H) = Z$  and  $H \cong T$ . □

Observe that, if the elements  $u$  and  $v$  in the above statement generate an abelian subgroup of order  $p^h q^k$ , then  $G$  has a subgroup isomorphic to  $\langle u, v \rangle$  by [13]. This, together with Theorem 5.4, completes the proof of Theorem E as stated in the introduction.

**ACKNOWLEDGMENTS**

This work has been done during a visit of Ángel Del Río at Dipartimento di Matematica e Informatica (DIMAI) of Università degli Studi di Firenze, funded by grant MICIU/PRX23/00370. Ángel

Del Río wishes to thank DIMAI for the hospitality and the Spanish Ministerio de Ciencia, Innovación y Universidades for the financial support. The authors also wish to thank Leo Margolis and the referee for their careful reading and very useful comments on this paper. Emanuele Pacifici and Marco Vergani are partially supported by INdAM-GNSAGA, and by the European Union-Next Generation EU, Missione 4 Componente 1, CUP B53D23009410006, PRIN 2022 2022PSTWLB - Group Theory and Applications. Ángel Del Río is partially supported by Grant PID2024-155576NB-I00 funded by MICIU/AEI/10.13039/501100011033/FEDER, UE, and by Grant 22004/PI/22 funded by Fundación Séneca of Región de Murcia, Spain.

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## REFERENCES

1. A. Bächle, A. Herman, A. Konovalov, L. Margolis, and L. G. Singh, *The status of the Zassenhaus conjecture for small groups*, Exp. Math. **27** (2018), 431–436.
2. A. Bächle, A. Kiefer, S. Maheshwary, and Á. del Río, *Gruenberg–Kegel graphs: cut groups, rational groups and the prime graph question*, Forum Math. **35** (2023), 409–429.
3. A. Bächle and L. Margolis, *An application of blocks to torsion units in group rings*, Proc. Amer. Math. Soc. **147** (2019), 4221–4231.
4. A. Bächle and L. Margolis, *On the prime graph question for integral group rings of 4-primary groups I*, Internat. J. Algebra Comput. **27** (2017), 731–767.
5. S. D. Berman, *On the equation  $x^m = 1$  in an integral group ring*, Ukrain. Mat. Z. **7** (1955), 253–261.
6. M. Caicedo, L. Margolis, and Á. del Río, *Zassenhaus conjecture for cyclic-by-abelian groups*, J. Lond. Math. Soc. **88** (2013), 65–78.
7. J. A. Cohn and D. Livingstone, *On the structure of group algebras. I*, Canad. J. Math. **17** (1965), 583–593.
8. M. A. Dokuchaev and S. O. Juriaans, *Finite subgroups in integral group rings*, Canad. J. Math. **48** (1996), 1170–1179.
9. F. Eisele and L. Margolis, *Units in group rings and blocks of Klein four or dihedral defect*, Bull. Lond. Math. Soc. **57** (2025), no. 11, 3470–3489. DOI [10.1112/blms.70164](https://doi.org/10.1112/blms.70164).
10. F. Eisele and L. Margolis, *A counterexample to the first Zassenhaus conjecture*, Adv. Math. **339** (2018), 599–641.
11. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968.
12. M. Hertweck, *A counterexample to the isomorphism problem for integral group rings*, Ann. Math. **154** (2001), 115–138.
13. M. Hertweck, *The orders of torsion units in integral group rings of finite solvable groups*, Comm. Algebra **36** (2008), 3585–3588.
14. M. Hertweck, *Partial augmentations and Brauer character values of torsion units in group rings*, arXiv:math/0612429v2, 2007.
15. M. Hertweck, *On the torsion units of some integral group rings*, Algebra Colloq. **13** (2006), 329–348.
16. M. Hertweck, *Unit groups of integral finite group rings with no noncyclic abelian finite  $p$ -subgroups*, Comm. Algebra **36** (2008), 3224–3229.
17. G. Higman, *Units in group rings*, Ph.D. thesis, University of Oxford, 1940.
18. G. Higman, *The units of group-rings*, Proc. Lond. Math. Soc. (2) **46** (1940), 231–248.

19. C. R. Höfert, *Bestimmung von Kompositionsfaktoren endlicher Gruppen aus Burnsideringen und ganzzahligen Gruppenringen*, Ph.D. thesis, University of Stuttgart, 2008.
20. B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.
21. I. M. Isaacs, *Character theory of finite groups*, Dover, New York, 1976.
22. E. Jespers and Á. del Río, *Group ring groups. Vol. 1. Orders and generic constructions of units*, De Gruyter, Berlin, 2016.
23. W. Kimmerle, *On the prime graph of the unit group of integral group rings of finite groups*, in “Groups, rings and algebras”, Contemp. Math. 420, Amer. Math. Soc., Providence, RI, 2006, 215–228.
24. W. Kimmerle, *Sylow like theorems for  $V(\mathbb{Z}G)$* , Int. J. Group Theory **4** (2015), 49–59.
25. W. Kimmerle and A. Konovalov, *On the Gruenberg–Kegel graph of integral group rings of finite groups*, Int. J. Algebra Comput. **27** (2017), 619–631.
26. H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Springer, New York, 2004.
27. L. Margolis, *Subgroup isomorphism problem for units of integral group rings*, J. Group Theory **20** (2017), 289–307.
28. L. Margolis and Á. del Río, *Finite subgroups of group rings: a survey*, Adv. Group Theory Appl. **8** (2019), 1–37.
29. O. Manz and T. R. Wolf, *Representations of solvable groups*, Cambridge University Press, Cambridge, 1993.
30. G. Navarro, *Quadratic characters in groups of odd order*, J. Algebra **322** (2009), 2586–2589.
31. S. K. Sehgal, *Units in integral group rings*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 69, Longman Scientific and Technical, Harlow, 1993.
32. A. Weiss, *Torsion units in integral group rings*, J. Reine Angew. Math. **415** (1991), 175–187.
33. A. Whitcomb, *The group ring problem*, Ph.D. thesis, University of Chicago, ProQuest LLC, Ann Arbor, MI, 1968.
34. D. L. White, *Character degrees of extensions of  $\mathrm{PSL}_2(q)$  and  $\mathrm{SL}_2(q)$* , J. Group Theory **16** (2013), 1–33.