This is a review submitted to Mathematical Reviews/MathSciNet.

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Mathematical Reviews/MathSciNet Reviewer Number: 138582

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Title: The power of naive truth.

MR Number: MR4394765

Primary classification:

Secondary classification(s):

Review text:

Let $\mathbf{T}$ be an axiomatic theory based on a language $\mathcal{L}$ that contains a distinguished unary predicate T for truth, applying to codes $\ulcorner\varphi\urcorner$ of formulas φ of $\mathcal{L}$. Then, there are two senses in which any formula of $\mathcal{L}$ can be said to be "true". It can be that any such φ is true because it is provable in $\mathbf{T}$, or it can be that φ is proved to be true in $\mathbf{T}$, i.e., $T\ulcorner\varphi\urcorner$ is provable in $\mathbf{T}$. In the general case, results such as the arithmetical undefinability of truth make it impossible for the latter, "internal" sense of "being true" for φ (that determines what goes in the literature under the name of the *internal theory* of $\mathbf{T}$) to faithfully replicate the former, "external" sense of the expression (that determines the *external theory* of $\mathbf{T}$). The research about the logical paradoxes has made the discrepancy even more dramatic. A well-known case of conflict between the internal and the external theory is represented by the system KF (as it is commonly referred to in the literature, which corresponds to the theory Field calls KF^+ in his paper). This is the system of axioms introduced by S. Feferman [*J. Symb. Log.*, 56, 1991; MR1131728] and inspired by Kripke's fixed-point construction [Outline of a theory of truth, *J. of Philosophy*, 72, 1975]. The conflict in the case of this formal system of axioms is already at the level of the underlying logic: while the internal theory reflects Kripke's construction and has a partial logic nature, the external theory is classical instead. An attempt to fix the discrepancy between the internal and the external theory of KF was made by V. Halbach and L. Horsten [*J. Symb. Log.*, 71, 2006; MR2225901], where the authors investigate the system PKF, a direct axiomatization of Kripke's theory in partial logic.

With respect to the latter system then, the logic of the internal theory coincides with that of the external theory. However, PKF turns out to be significantly weaker, proof-theoretically speaking, than KF: while the latter was proved to be equivalent to $\mathsf{RA}_{<\epsilon_0}$, the system of ramified analysis up to the ordinal ϵ_0, PKF turned out to be equivalent to $\mathsf{RA}_{\omega^\omega}$, the system of ramified analysis up to ω^ω.

This latter fact is the main motivation of the paper under review, which can be regarded as the attempt to fix the internal vs. external theory issue without affecting the theory strength. This is achieved by means of a theory INT that faithfully interprets both the internal and the external theory of KF, and thus happens to have the same proof-theoretic strength of it. INT is obtained by extending the internal theory of KF by means of a new unary predicate constant *Scl* for "strong classicality", the idea behind it being that any sentence that falls within the scope of the predicate exhibits a classical behaviour with respect to (internal) truth. Notably, *Scl* is a classical predicate itself (i.e., it obeys the law of excluded middle). In turn, *Scl* is used to define another predicate *Strue* for "strongly classical and true" sentences, that is the key to the proof that INT faithfully interprets the external theory of KF. The result is extended to what Field calls the schematic version of KF (and which corresponds to the theory $\mathsf{Ref}^*(\mathsf{PA}(P))$ of Feferman's paper). In addition, the paper contains a lot of connections to the existing literature on the topic, and provides some insightful discussion about the philosophical value of these results.