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A generalization of Rohn's theorem on full-rank interval matrices

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Abstract

A general closed interval matrix is a matrix whose entries are closed connected nonempty subsets of $\mathbb{R}$, while an interval matrix is defined to be a matrix whose entries are closed bounded nonempty intervals in $\mathbb{R}$. We say that a matrix A with constant entries is contained in a general closed interval matrix μ if, for every i, j, we have that $A_{i,j} \in \mu_{i,j}$. Rohn characterized full-rank square interval matrices, that is, square interval matrices μ such that every constant matrix contained in μ is nonsingular. In this paper we generalize this result to general closed interval matrices.

1 Introduction

Let $p, q \in \mathbb{N} - \{0\}$; a $p \times q$ interval matrix is a $p \times q$ matrix whose entries are closed bounded nonempty intervals in $\mathbb{R}$. We say that a $p \times q$ matrix A with entries in $\mathbb{R}$ is contained in μ if, for every i, j, we have that $A_{i,j} \in \mu_{i,j}$.
On the other side, for any field K, a partial matrix over K is defined to be a matrix where only some of the entries are given and they are elements of K; a completion of a partial matrix is a specification in K of the unspecified entries. We say that a submatrix of a partial matrix is specified if all its entries are specified.
There are several papers both on interval matrices and on partial matrices. On partial matrices, there is a wide literature about the problem of determining the maximal and the minimal rank of

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the completions of a partial matrix. We quote, for instance, the papers [1], [11] and [3]. In the last, Cohen, Johnson, Rodman and Woerdeman determined the maximal rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices; see also [1] for the proof. The problem of determining the minimal rank of the completions of a partial matrix seems more difficult and it has been solved only in some particular cases, see for example [10] for the case of triangular matrices and the recent paper [2].
Also interval matrices have been widely studied and in particular there are several papers studying when a $p \times q$ interval matrix μ has full rank, that is when all the matrices contained in μ have rank equal to $\min\{p, q\}$. For any $p \times q$ interval matrix $\mu = ([m_{i,j}, M_{i,j}])_{i,j}$ with $m_{i,j} \leq M_{i,j}$, let C^μ,Δ^μ and $|\mu|$ be the $p \times q$ matrices such that

$$C^\mu_{i,j} = \frac{m_{i,j} + M_{i,j}}{2}, \qquad \Delta^\mu_{i,j} = \frac{M_{i,j} - m_{i,j}}{2}, \qquad |\mu|_{i,j} = \max\{|m_{i,j}|, |M_{i,j}|\}$$

for any i, j. For any nonzero natural number p, let $Y_p = \{-1, 1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x. Finally, for any $p \times p$ interval matrix $\mu = ([m_{i,j}, M_{i,j}])_{i,j}$, with $m_{i,j} \leq M_{i,j}$ for any i, j, and for any $x, y \in Y_p$, define the matrix $C^\mu_{x,y}$ as follows:

$$C^\mu_{x,y} = C^\mu - T_x\, \Delta^\mu\, T_y.$$

The following theorem characterizes full-rank square interval matrices:

Theorem 1. ***(Rohn, [4])*** *Let $\mu = ([m_{i,j}, M_{i,j}])_{i,j}$ be a $p \times p$ interval matrix, where $m_{i,j} \leq M_{i,j}$ for any i, j. Then μ is a full-rank interval matrix if and only if, for each $x, y, x', y' \in Y_p$,*

$$det(C^\mu_{x,y})\, det(C^\mu_{x',y'}) > 0.$$

See [4] and [5] for other characterizations. Finally, as to interval matrices, we want to quote also the following theorem characterizing full-rank $p \times q$ interval matrices, see [6], [7], [9]:

Theorem 2. ***(Rohn)*** *A $p \times q$ interval matrix μ with $p \geq q$ has full rank if and only if the system of inequalities*

$$|C^\mu x| \leq \Delta^\mu |x|, \qquad x \in \mathbb{R}^q$$

has only the trivial solution $x = 0$.

Obviously the problem of partial matrices in the real case and the one of interval matrices are connected; in fact we can consider matrices whose entries are closed connected nonempty subsets of $\mathbb{R}$; these matrices generalize both the interval matrices and the partial matrices. We call a matrix whose entries are closed connected nonempty subsets of $\mathbb{R}$ a "general closed interval matrix".
In this paper we generalize Rohn's result on full-rank square interval matrices to general closed interval matrices, see Theorem 16 for the precise statement.

2 Notation and some recalls

• Let $\mathbb{R}_{>0}$ be the set $\{x \in \mathbb{R} |\ x > 0\}$ and let $\mathbb{R}_{\geq 0}$ be the set $\{x \in \mathbb{R} |\ x \geq 0\}$; we define analogously $\mathbb{R}_{<0}$ and $\mathbb{R}_{\leq 0}$.
• Throughout the paper let $p, q \in \mathbb{N} - \{0\}$.
• Let Σ_p be the set of the permutations on $\{1, ..., p\}$. For any permutation σ, we denote the sign of σ by $\epsilon(\sigma)$.
• Let $M(p \times q, \mathbb{R})$ denote the set of the $p \times q$ matrices with entries in $\mathbb{R}$. For any $A \in M(p \times q, \mathbb{R})$, let $rk(A)$ denote the rank of A and let $A^{(j)}$ be the j-th column of A.
• For any $p \times q$ general closed interval matrix μ, any $\{i_1, \ldots, i_s\} \subset \{1, \ldots, p\}$ and any $\{j_1, \ldots, j_r\} \subset \{1, \ldots, q\}$, we denote by $\mu_{\widehat{i_1, \ldots, i_s}, \widehat{j_1, \ldots, j_r}}$ the matrix obtained from μ by deleting the rows $i_1, \ldots, i_s$ and the columns $j_1, \ldots, j_r$.

Definition 3. *A* ***general interval matrix*** *is a matrix whose entries are connected nonempty subsets of* $\mathbb{R}$.
A ***general closed interval matrix*** *is a matrix whose entries are closed connected nonempty subsets of* $\mathbb{R}$.
An ***interval matrix*** *is a matrix whose entries are closed bounded nonempty intervals of* $\mathbb{R}$.
Let μ *be a* $p \times q$ *general interval matrix. As we have already said, given a matrix* $A \in M(p \times q, \mathbb{R})$, *we say that* $A \in \mu$ *if* $a_{i,j} \in \mu_{i,j}$ *for any* i, j.
We define

$$mrk(\mu) = min\{rk(A)|\ A \in \mu\},$$

$$Mrk(\mu) = max\{rk(A)|\ A \in \mu\}.$$

We call them respectively ***minimal rank*** *and* ***maximal rank*** *of* μ. *Moreover, we define*

$$rkRange(\mu) = \{rk(A)|\ A \in \mu\};$$

we call the set above the ***rank range*** *of* μ.
We say that the entry i, j *of* μ *is a* ***constant*** *if* $\mu_{i,j}$ *is a subset of* $\mathbb{R}$ *given by only one element.*
We say that the entry i, j *of* μ *is* ***bounded*** *if* $\mu_{i,j} = [a, b]$ *for some* $a, b \in \mathbb{R}$; *we say that the entry* i, j *of* μ *is* ***half-bounded*** *if either* $\mu_{i,j} = [a, +\infty)$ *(**left-bounded**) or* $\mu_{i,j} = (-\infty, a]$ *(**right-bounded**) for some* $a \in \mathbb{R}$.

Remark 4. *Let* μ *be an interval matrix. Observe that*

$$rkRange(\mu) = [mrk(\mu), Mrk(\mu)] \cap \mathbb{N}.$$

See [8] for a proof.

Definition 5. *Given a* $p \times p$ *interval matrix,* ν, *a* ***partial generalized diagonal*** *(**pg-diagonal** for short) of length* k *of* ν *is a* k*-uple of the kind*

$$(\nu_{i_1,j_1}, \ldots, \nu_{i_k,j_k})$$

for some $\{i_1, \dots i_k\}$ and $\{j_1, \dots, j_k\}$ subsets of $\{1, \dots, p\}$.
Its ***complementary matrix*** *is defined to be the submatrix of μ given by the rows and columns whose indices are respectively in $\{1, \dots, p\} - \{i_1, \dots, i_k\}$ and in $\{1, \dots, p\} - \{j_1, \dots, j_k\}$.*
We say that a pg-diagonal is ***totally nonconstant*** *if all its entries are not constant.*
We define $det^c(\mu)$ as follows:

$$det^c(\mu) = \sum_{\sigma \in \Sigma_p \ s.t.\ \mu_{1,\sigma(1)}, \dots, \mu_{p,\sigma(p)} \ are\ constant} \epsilon(\sigma)\, \mu_{1,\sigma(1)} \cdot \dots \cdot \mu_{p,\sigma(p)}$$

if there exists $\sigma \in \Sigma_p$ such that $\mu_{1,\sigma(1)}, \dots, \mu_{p,\sigma(p)}$ are constant; we define $det^c(\mu)$ to be equal to 0 otherwise.

The following theorem and corollary were proved in [8] for interval matrices; the same proof yields the results for general closed interval matrices:

Theorem 6. *Let μ be a $p \times p$ general closed interval matrix. Then $Mrk(\mu) < p$ if and only if the following conditions hold:*
(1) in μ there is no totally nonconstant pg-diagonal of length p,
(2) the complementary matrix of every totally nonconstant pg-diagonal of length between 0 and $p-1$ has det^c equal to 0 (in particular $det^c(\mu) = 0$).

Corollary 7. *Let μ be a general closed interval matrix. Then $Mrk(\mu)$ is the maximum of the natural numbers t such that there is a $t \times t$ submatrix of μ either with a totally nonconstant pg-diagonal of length t or with a totally nonconstant pg-diagonal of length between 0 and $t-1$ whose complementary matrix has $det^c \neq 0$.*

Notation 8. *Let μ be a general closed interval matrix.*
- *We denote by $\tilde{\mu}$ the matrix obtained from μ by replacing the entries $(-\infty, +\infty)$ with 0.*
- *We denote by $\overline{\mu}$ the matrix obtained from μ by replacing every entry of kind $[a, +\infty)$, for some $a \in \mathbb{R}$, with a and every entry of kind $(-\infty, b]$, for some $b \in \mathbb{R}$, with b.*
- *We denote by μ_l the matrix obtained from μ by replacing every entry of kind $[a, b]$, for some $a, b \in \mathbb{R}$ with $a \leq b$, with a. We denote by μ_r the matrix obtained from μ by replacing every entry of kind $[a, b]$, for some $a, b \in \mathbb{R}$ with $a \leq b$, with b.*

Definition 9. *Let μ be a general closed interval matrix.*
We say that γ is a ***vertex matrix*** *of μ if $\gamma_{i,j} \in \{m_{i,j}, M_{i,j}\}$ for any i, j such that $\mu_{i,j}$ is a bounded interval and $\gamma_{i,j} = \mu_{i,j}$ otherwise.*
We say that a vertex matrix γ of μ is ***of even type*** *if, for every 2×2 submatrix of μ such that all its entries are bounded intervals, either the number of the entries of the corresponding submatrix of γ that are equal to the minimum of the corresponding entries of μ is even or some of its entries are constant. We say that it is* ***of odd type*** *if it is not of even type.*

Example. Let

$$\mu = \begin{pmatrix} [1,2] & [2,3] & [2,+\infty) \\ [-3,4] & [-1,5] & [1,4] \end{pmatrix}.$$

Then $\gamma = \begin{pmatrix} 1 & 2 & [2,+\infty) \\ 4 & 5 & 1 \end{pmatrix}$ is a vertex matrix of μ of even type, while $\delta = \begin{pmatrix} 1 & 3 & [2,+\infty) \\ 4 & 5 & 1 \end{pmatrix}$ is a vertex matrix of μ of odd type.

It is easy to see that, given a $p \times p$ interval matrix μ and $x, y \in Y_p$, the matrix $C^{\mu}_{x,y}$ is a vertex matrix of μ of even type and every vertex matrix of μ of even type is equal to $C^{\mu}_{x,y}$ for some $x, y \in Y_p$. So, by using Definition 9, we can restate Rohn's theorem as follows:

Theorem 10. ***(Rohn, [4])*** *Let μ be a $p \times p$ interval matrix. Then μ is a full-rank interval matrix if and only if, for any vertex matrices A_1, A_2 of even type of μ,*

$$det(A_1)\, det(A_2) > 0.$$

3 The main result

Remark 11. *Let $p(x_1, \dots, x_n)$ be a polynomial with coefficients in $\mathbb{R}$ of degree 1 in every variable. Let $\overline{x} \in \mathbb{R}^n$. Then $p(x) \geq 0$ (respectively $p(x) > 0$) for every $x \in \overline{x} + \mathbb{R}^n_{\geq 0}$ if and only if $p(\overline{x}) \geq 0$ (respectively $p(\overline{x}) > 0$) and $\frac{\partial p}{\partial x_i}(x) \geq 0$ for every $x \in \overline{x} + \mathbb{R}^n_{\geq 0}$ and for every $i = 1, \dots, n$.*

Lemma 12. *Let μ be a general closed interval $p \times p$ matrix. Then μ is full-rank if and only if $\tilde{\mu}$ is full-rank and for every i, j such that $\mu_{i,j} = (-\infty, +\infty)$ we have that $Mrk(\mu_{\hat{i},\hat{j}}) < p - 1$.*

Proof. Suppose μ is full-rank. Then obviously $\tilde{\mu}$ is full-rank. Moreover, let i, j be such that $\mu_{i,j} = (-\infty, +\infty)$. Then the determinant of every $A \in \mu_{\hat{i},\hat{j}}$ must be zero; otherwise, suppose there exists $A \in \mu_{\hat{i},\hat{j}}$ with $det(A) \neq 0$; then, for any choice of $x_{i,s}$ in $\mu_{i,s}$ for any $s \in \{1, \dots, p\} - \{j\}$ and $x_{r,j}$ in $\mu_{r,j}$ for any $r \in \{1, \dots, p\} - \{i\}$, we can choose $x_{i,j} \in \mu_{i,j}$ such that, if we define X to be the matrix such that $X_{r,s} = x_{r,s}$ for (r, s) such that either $r = i$ or $s = j$ and $X_{\hat{i},\hat{j}} = A$, we have that the determinant of X is zero, which is absurd since μ is full-rank. So $Mrk(\mu_{\hat{i},\hat{j}}) < p - 1$ and we have proved the right-handed implication.
On the other side, let $(i_1, j_1), \dots, (i_s, j_s)$ be the indices of the entries of μ equal to $(-\infty, +\infty)$ and suppose that $\tilde{\mu}$ is full-rank and that $Mrk(\mu_{\hat{i_k},\hat{j_k}}) < p-1$ for every $k = 1, \dots, s$. Let $B \in M(p \times p, \mathbb{R})$ be such that $B \in \mu$. For every $k = 1, \dots, s$, let B_k be the matrix obtained from B by replacing the entries $(i_1, j_1), \dots, (i_k, j_k)$ with 0. Then $det(B) = det(B_1) = \ldots = det(B_s)$ since $Mrk(\mu_{\hat{i_k},\hat{j_k}}) < p-1$ for every $k = 1, \dots, s$. Obviously $B_s \in \tilde{\mu}$, so $det(B_s)$ is nonzero since $\tilde{\mu}$ is full-rank; hence $det(B)$ is nonzero and we conclude. □

Lemma 13. *Let ν be a general closed interval $p \times p$ matrix with only bounded or half-bounded entries. Then ν is full-rank if and only if, for any γ vertex matrix of ν of even type, we have that γ is full-rank and $det(\overline{\gamma})$ has the same sign as $det(\overline{\nu}_l)$.*

Proof. $\Longrightarrow$ Obviously if ν is full-rank, then the determinant of all the matrices contained in ν must have the same sign (since the determinant is a continous function and the image by a continous function of a connected subset is connected), in particular $det(\overline{\gamma})$ must have the same sign as $det(\overline{\nu}_l)$.
$\Longleftarrow$ Let $A \in M(p \times p, \mathbb{R})$ be such that $A \in \nu$; define α to be the interval matrix such that $\alpha_{i,j} = \nu_{i,j}$ if $\nu_{i,j}$ is bounded and $\alpha_{i,j} = a_{i,j}$ if $\nu_{i,j}$ is half-bounded.
To prove that A is full-rank, obviously it is sufficient to prove that α is full-rank. By Rohn's theorem (see Theorem 10), to prove that α is full-rank, it is sufficient to prove that any two vertex matrices A_1, A_2 of even type of α are full-rank and the sign of their determinant is the same. Obviously, for $i = 1, 2$, the matrix A_i is contained in a vertex matrix γ_i of even type of ν; by assumption, for $i = 1, 2$, the matrix γ_i is full-rank, so the matrix A_i is full-rank and its determinant has the same sign as $det(\overline{\gamma}_i)$. Moreover, by assumption the sign of $det(\overline{\gamma}_i)$ is equal to the sign of $det(\overline{\nu}_l)$ for $i = 1, 2$; in particular $det(\overline{\gamma}_1)$ and $det(\overline{\gamma}_2)$ have the same sign, so $det(A_1)$ and $det(A_2)$ have the same sign, as we wanted to prove. $\square$

Notation 14. *Let ρ be a square general closed interval matrix. We say that $DET(\rho)$ is greater than 0 (respectively less than 0, equal to 0...) if, for any $A \in \rho$, we have that $\det(A)$ is greater than 0 (respectively less than 0, equal to 0...). More generally, given a function $f : \mathbb{R}^n \to \mathbb{R}$ for some $n \in \mathbb{N} - \{0\}$, we say that $f(x_1, \dots, x_{n-1}, DET(\rho)) \geq 0$ (respectively ≤ 0, > 0, < 0) if $f(x_1, \dots, x_{n-1}, det(A)) \geq 0$ (respectively ≤ 0, > 0, < 0) for any $A \in \rho$.*

Lemma 15. *Let ρ be a general closed interval $p \times p$ matrix with only constant or half-bounded entries. Then ρ is full-rank if and only if $det(\overline{\rho}) \neq 0$ and, for any $(i_1, j_1), \dots, (i_s, j_s) \in \{1, \dots, p\} \times \{1, \dots, p\}$ such that $i_1, \dots, i_s$ are distinct and $j_1, \dots, j_s$ are distinct and $\rho_{i_1,j_1}, \dots, \rho_{i_s,j_s}$ are half-bounded,*

$$(-1)^{i_1+j_1+\widetilde{i_2}+\widetilde{j_2}+\dots+\widetilde{i_s}+\widetilde{j_s}+\chi((i_1,j_1),\dots,(i_s,j_s))} det(\overline{\rho})\, det(\overline{\rho}_{\widehat{i_1,\dots,i_s},\widehat{j_1,\dots,j_s}}) \geq 0,$$

where:
- *the determinant of a 0×0 matrix is defined to be 1,*
- *$\chi((i_1, j_1), \dots, (i_s, j_s))$ is defined to be the number of the right-bounded intervals in $\rho_{i_1,j_1}, \dots, \rho_{i_s,j_s}$,*
- *for $t = 2, \dots, s$, we define $\widetilde{i_t}$ to be i_t minus the number of the elements among $i_1, \dots i_{t-1}$ smaller than i_t and $\widetilde{j_t}$ to be j_t minus the number of the elements among $j_1, \dots j_{t-1}$ smaller than j_t.*

Proof. By Remark 11, the matrix ρ is full-rank if and only if $det(\overline{\rho}) \neq 0$ and, for any i_1, j_1 such that ρ_{i_1,j_1} is half-bounded,

$$(-1)^{i_1+j_1+\chi(i_1,j_1)}\, det(\overline{\rho})\, DET(\rho_{\hat{i_1},\hat{j_1}}) \geq 0$$

and, again by Remark 11, the last condition holds if and only if, for any i_1, j_1 such that ρ_{i_1,j_1} is half-bounded, we have that

$$(-1)^{i_1+j_1+\chi(i_1,j_1)}\, det(\overline{\rho})\, det(\overline{\rho}_{\hat{i_1},\hat{j_1}}) \geq 0$$

and, for any i_2, j_2 such that $i_1 \neq i_2$ and $j_1 \neq j_2$ and ρ_{i_2,j_2} is half-bounded, we have that

$$(-1)^{i_1+j_1+\widetilde{i_2}+\widetilde{j_2}+\chi((i_1,j_1),(i_2,j_2))}\, det(\overline{\rho})\, DET(\rho_{\widehat{i_1,i_2},\widehat{j_1,j_2}}) \geq 0$$

and so on. $\square$

Theorem 16. *Let μ be a general closed interval matrix. Then μ is full-rank if and only if the following conditions hold:*
(1) for every i,j such that $\mu_{i,j} = (-\infty, +\infty)$ we have that in $\mu_{\hat{i},\hat{j}}$ there are no totally nonconstant pg-diagonal of length $p-1$ and the complementary matrix of every totally nonconstant pg-diagonal in $\mu_{\hat{i},\hat{j}}$ of length between 0 and $p-2$ has det^c equal to 0.
(2) the product of the determinant every vertex matrix of $\overline{\tilde{\mu}}$ of even type and the determinant of $\overline{\tilde{\mu}}_l$ is positive,
(3) for every vertex matrix γ of $\tilde{\mu}$ of even type and every $(i_1, j_1), \ldots, (i_s, j_s) \in \{1, \ldots, p\} \times \{1, \ldots, p\}$ such that $i_1, \ldots, i_s$ are distinct and $j_1, \ldots, j_s$ are distinct and $\mu_{i_1,j_1}, \ldots, \mu_{i_s,j_s}$ are half-bounded, we have:

$$(-1)^{i_1+j_1+\widetilde{i_2}+\widetilde{j_2}+\ldots+\widetilde{i_s}+\widetilde{j_s}+\chi((i_1,j_1),\ldots,(i_s,j_s))} det(\overline{\tilde{\mu}}_l)\, det(\overline{\gamma}_{\widehat{i_1,\ldots,i_s},\widehat{j_1,\ldots,j_s}}) \geq 0,$$

where:
- *the determinant of a 0×0 matrix is defined to be 1,*
- *the number $\chi((i_1, j_1), \ldots, (i_s, j_s))$ is defined to be the number of right-bounded intervals in $\mu_{i_1,j_1}, \ldots, \mu_{i_s,j_s}$,*
- *for $t = 2, \ldots, s$, we define*

$$\widetilde{i_t} := i_t - \sharp\{i_r \text{ for } r = 1, \ldots, t-1 |\ i_r < i_t\}$$

$$\widetilde{j_t} := j_t - \sharp\{j_r \text{ for } r = 1, \ldots, t-1 |\ j_r < j_t\}.$$

Proof. By Lemma 12, the matrix μ is full-rank if and only if $\tilde{\mu}$ is full-rank and for every i,j such that $\mu_{i,j} = (-\infty, +\infty)$ we have that $Mrk(\mu_{\hat{i},\hat{j}}) < p-1$. By Theorem 6 this is true if and only if $\tilde{\mu}$ is full-rank and for every i,j such that $\mu_{i,j} = (-\infty, +\infty)$ we have that in $\mu_{\hat{i},\hat{j}}$ there are no totally nonconstant pg-diagonal of length $p-1$ and the complementary matrix of every totally nonconstant pg-diagonal in $\mu_{\hat{i},\hat{j}}$ of length between 0 and $p-2$ has det^c equal to 0.
Moreover, by Lemma 13 (with $\nu = \tilde{\mu}$), we have that $\tilde{\mu}$ is full-rank if and only if, for any γ vertex matrix of $\tilde{\mu}$ of even type, we have that γ is full-rank and $det(\overline{\gamma})$ has the same sign as $det(\overline{\tilde{\mu}}_l)$.
Finally, this is true if and only if (2) holds and, by Lemma 15, for any γ vertex matrix of $\tilde{\mu}$ of even type, for any $(i_1, j_1), \ldots, (i_s, j_s) \in \{1, \ldots, p\} \times \{1, \ldots, p\}$ such that $i_1, \ldots, i_s$ are distinct and $j_1, \ldots, j_s$ are distinct and $\gamma_{i_1,j_1}, \ldots, \gamma_{i_s,j_s}$ are half-bounded, we have that $det(\overline{\gamma}) \neq 0$ and

$$(-1)^{i_1+j_1+\widetilde{i_2}+\widetilde{j_2}\ldots+\widetilde{i_s}+\widetilde{j_s}+\chi((i_1,j_1),\ldots,(i_s,j_s))}\, det(\overline{\gamma})\, det(\overline{\gamma}_{\widehat{i_1,\ldots,i_s},\widehat{j_1,\ldots,j_s}}) \geq 0,$$

which, by condition (2), is equivalent to condition (3). □

Examples. 1) Let

$$\alpha = \begin{pmatrix} (-\infty,+\infty) & [1,+\infty) & 1 & 1 & 4 \\ 1 & [2,3] & 6 & 2 & 4 \\ (-\infty,2] & 0 & [1,4] & 0 & [3,6] \\ 0 & [-1,2] & 3 & 1 & 2 \\ 3 & 0 & 3 & 1 & 2 \end{pmatrix}.$$

We can easily show that α does not satisfy condition (2) of Theorem 16, in fact $\overline{\tilde{\alpha}}_l = \begin{pmatrix} 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 6 & 2 & 4 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & -1 & 3 & 1 & 2 \\ 3 & 0 & 3 & 1 & 2 \end{pmatrix}$ has negative determinant, while the following vertex matrix of even type of $\overline{\tilde{\alpha}}$ has positive determinant:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 6 & 2 & 4 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & 2 & 3 & 1 & 2 \\ 3 & 0 & 3 & 1 & 2 \end{pmatrix}$$

So α is not full-rank. In fact it contains the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 4 \\ 1 & 2 & 6 & 2 & 4 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & 6/5 & 3 & 1 & 2 \\ 3 & 0 & 3 & 1 & 2 \end{pmatrix},$$

which is not invertible.
2) Let

$$\beta = \begin{pmatrix} [2,+\infty) & 1 & 2 & (-\infty,+\infty) \\ [1,2] & 0 & 3 & 2 \\ 3 & [3,7] & 5 & 3 \\ 0 & 0 & 0 & [1,+\infty) \end{pmatrix}.$$

We can easily see that β satisfies conditions (1),(2),(3) of Theorem 16, so it is full-rank.
3) Let

$$\delta = \begin{pmatrix} (-\infty,+\infty) & 1 & 2 & (-\infty,+\infty) \\ [1,2] & [1,2] & 9 & 2 \\ 3 & [1,5] & 4 & 0 \\ 2 & [1,2] & [-1,+\infty) & 3 \end{pmatrix}.$$

Obviously it does not satisfy condition (1) of Theorem 16, in fact $\delta_{\hat{1},\hat{1}}$ contains totally nonconstant pg-diagonal whose complementary matrix has $det^c \neq 0$. So δ is not full-rank.

Open problem. A problem that naturally arises is the one of the characterization of full-rank matrices whose entries are (not necessarily closed) connected subsets of $\mathbb{R}$.

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