J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2023-0181

Critical classes of power graphs and reconstruction of directed power graphs

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Communicated by Christopher W. Parker

Abstract. In a graph $\Gamma = (V, E)$, we consider the common closed neighbourhood of a subset of vertices and use this notion to introduce a Moore closure operator in V. We also consider the closed twin equivalence relation in which two vertices are equivalent if they have the same closed neighbourhood. Those notions are explored deeply in the case when Γ is the power graph associated with a finite group G. In that case, among the corresponding closed twin equivalence classes, we introduce the concepts of plain, compound and critical classes. The study of critical classes, together with properties of the Moore closure operator, allow us to correct a mistake in the proof of [P. J. Cameron, The power graph of a finite group, II, *J. Group Theory* **13** (2010), no. 6, 779–783; Theorem 2] and to deduce a simple algorithm to reconstruct the directed power graph of a finite group from its undirected counterpart, as asked in [P. J. Cameron, Graphs defined on groups, *Int. J. Group Theory* **11** (2022), no. 2, 53–107; Question 2].

1 Introduction

The interaction between group theory and graph theory has been known since 1878, when Cayley graphs were first defined. In the subsequent years, various graphs were associated with groups. In 1955, Brauer and Fowler introduced the commuting graph in [1], in early 2000, directed power graphs were introduced for semigroups in [14, 15], and in 2009, Chakrabarty, Ghosh and Sen [8] defined the undirected version, called simply the power graph. Given a group G, the directed power graph $\vec{\mathcal{P}}(G)$ has vertex set G and arc set

$$\{(x, y) \in G^2 : x \neq y,\ y = x^m \text{ for some } m \in \mathbb{N}\}.$$

The power graph $\mathcal{P}(G)$ is its underlying graph. The recent state of the art for the power graph is well summarized in [16].

Daniela Bubboloni is partially supported by GNSAGA of INdAM (Italy), and local funding from the Università degli Studi di Firenze. This work is also funded by the European Union – Next Generation EU, Missione 4 Componente 1, CUP B53D23009410006, PRIN 2022 – 2022PSTWLB – Group Theory and Applications.

Investigating a group G through a graph $\Gamma(G)$ associated with it allows us to focus on some specific properties of the group, reducing the complexity of the algebraic structure G to the simpler combinatorial object $\Gamma(G)$. Of course, if the reduction of information is dramatic, then the use of $\Gamma(G)$ for the study of G can be almost useless. Among the many graphs associated with groups, the power graph seems to be one of the best to reveal information about the group structure. A result among many justifying this opinion is surely [18, Theorem 15], which shows that if G is a finite simple group, then any finite group having the same power graph as G must be isomorphic to G. However it is well known that, given two groups G_1 and G_2, $\mathscr{P}(G_1) \cong \mathscr{P}(G_2)$ does not imply $G_1 \ncong G_2$. In other words, if a certain graph Γ is the power graph of some group, then Γ can be the power graph of many non-isomorphic groups. For further arguments distinguishing the power graph from other graphs associated with groups, the reader is referred to [2, Introduction].

In principle, the directed power graph should encode even more information than the power graph. Surprisingly, this is not the case, at least for finite groups. In this paper, we deal only with finite groups, and we call a graph Γ a *power graph* if there exists at least one finite group G such that $\Gamma = \mathscr{P}(G)$. As our Main Theorem, we show that, given a power graph $\Gamma = \mathscr{P}(G)$, for a certain finite group G, we can always reconstruct $\vec{\mathscr{P}}(G)$ by purely arithmetical and graph theoretical considerations, without taking into account any group theoretical information about G (see Section 6.1 for formal details).

Main Theorem. *We can reconstruct the directed power graph from any power graph.*

We also emphasize that the question of reconstructing the directed power graph from the power graph makes sense also for infinite groups and there are, in the recent literature, important contributions for certain classes of infinite groups [22]. However, in this paper, we deal only with finite groups. The proof of the Main Theorem is entirely constructive and gives rise to a precise algorithm whose description and pseudo-code are available in the arXiv version of our paper [3, Appendix]. That completely answers one of the questions recently set by P. J. Cameron [5, Question 2], which asks for a simple algorithm for reconstructing the directed power graph from the power graph.

Evidence of emerging interest in such questions includes a very recent preprint [9], dealing with an algorithm for reconstructing the directed power graph from both the enhanced power graph and the power graph. Recall that the enhanced power graph of a group G has vertex set G and two vertices are adjacent if they generate a cyclic subgroup. That algorithm is completely independent from ours and based on maximal cyclic subgroups.

The main scope of our research is theoretical and is mainly inspired by one of the most cited papers about power graphs, by P. J. Cameron [4]. We have found there many deep stimulating ideas and tried to exploit them to their maximum extent.

As a first step, we generalize a construction in [4], based on closed neighbourhoods, to any graph giving rise to what we call the *neighbourhood closure operator*. Remarkably, this operator turns out to be a Moore closure operator (Section 3). It plays a central role in our paper and, to the best of our knowledge, it does not appear elsewhere in the literature. As in [4], we consider the equivalence relation $\mathtt{N}$ which puts in relation two vertices of the power graph of a group G having the same closed neighbourhood. The study of the partition of the power graph into the corresponding equivalence classes was initiated in [4], but it is far from complete. As in [4], we split the $\mathtt{N}$-classes into two types (which we call *plain* and *compound*) according to their behaviour with respect to another equivalence relation $\diamond$, for which two vertices are equivalent if they generate the same subgroup of G. Then we introduce a further crucial type of $\mathtt{N}$-class, which we call a *critical class* (Section 5.2). The critical classes arise, in principle, from the necessity to rectify a serious mistake in one argument leading to the main result in [4].

We emphasize that the error in [4] is not rectifiable by modifying some reasoning (see Section 5.2 for details) and can be fixed only by developing radically new theoretical instruments. An intriguing question is to determine when a critical class is of plain or compound type. That task is completed using a recent result by Feng, Ma and Wang [11]. We observe that, even when all the tools are ready, the proof of the Main Theorem (Section 6.1) remains non-obvious and some delicate parts benefit from information coming from general graph theory, such as those describing the influence of the $\mathtt{N}$-classes on graph automorphisms (Proposition 1).

Finally, we stress that the contribution of our paper goes beyond the reconstruction of the directed power graph. Our general analysis of the $\mathtt{N}$-classes can be used for other research on power graphs. Moreover, the creation of a significant Moore closure for graphs seems to be a promising step towards new general research on graphs. In particular, even though the neighbourhood closure operator is not usually a Kuratowski operator, the possibility remains open, for certain classes of graphs, of obtaining an interesting topological structure.

2 Notation and basic facts

We denote by $\mathbb{N}$ the set of positive integers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $k \in \mathbb{Z}$, we set $[k] := \{x \in \mathbb{N} : x \leq k\}$ and $[k]_0 := \{x \in \mathbb{N}_0 : x \leq k\}$. Let X be a finite set. We denote by 2^X its power set and by S_X the symmetric group on X. When $X = [k]$, we simplify the notation to S_k. If $\mathcal{K} = \{X_1, \ldots, X_r\}$, with $r \in \mathbb{N}$, is a partition of X and $\psi \in S_X$, then we denote by $\psi(\mathcal{K})$ the partition of X given by $\{\psi(X_1), \dots, \psi(X_r)\}$. Given $A, B \subseteq X$, we write $X = A \mathbin{\dot\cup} B$ if $X = A \cup B$ and $A \cap B = \varnothing$. Let $\Gamma = (V, E)$ be a graph with vertex set $V \neq \varnothing$ and edge set $E \subseteq \{e \subseteq V : |e| = 2\}$. For $X \subseteq V$, we denote by Γ_X the induced subgraph of Γ with vertex set X. For $x \in V$, the closed neighbourhood of x in Γ is given by $N[x] = \{y \in V : \{y, x\} \in E\} \cup \{x\}$. Note that, for every $x, y \in V$, we have $y \in N[x]$ if and only if $x \in N[y]$. If $N[x] = V$, then x is called a *star vertex* of Γ. The set of star vertices of Γ is denoted by $\mathcal{S}$. For $x, y \in V$, we write $x\mathtt{N}y$ if $N[x] = N[y]$. The relation $\mathtt{N}$ is an equivalence relation on V, called the *closed twin relation* of Γ. Note that $x\mathtt{N}y$, with $x \neq y$, implies $\{x, y\} \in E$. We denote the $\mathtt{N}$-class of $x \in V$ by $[x]_{\mathtt{N}}$.

Proposition 1. *Let $\Gamma = (V, E)$ be a graph and $\mathcal{K}$ the partition of V into its $\mathtt{N}$-classes. Then $\times_{C \in \mathcal{K}} S_C \leq \mathrm{Aut}(\Gamma)$.*

Proof. Let $C \in \mathcal{K}$. For $\sigma \in S_C$, define $\tilde{\sigma}\colon V \to V$ by $\tilde{\sigma}(x) = x$ for all $x \in V \setminus C$, and $\tilde{\sigma}(x) = \sigma(x)$ for all $x \in C$. We show that $\tilde{\sigma}$ is a graph automorphism of Γ. Clearly, $\tilde{\sigma}$ is a bijection on V. We show that, for every $x, y \in V$ with $x \neq y$, we have $\{x, y\} \in E$ if and only if $\{\tilde{\sigma}(x), \tilde{\sigma}(y)\} \in E$. If both $x, y \in V \setminus C$, simply observe that $\{\tilde{\sigma}(x), \tilde{\sigma}(y)\} = \{x, y\}$. If both $x, y \in C$, we also have $\tilde{\sigma}(x), \tilde{\sigma}(y) \in C$, and thus both $\{x, y\} \in E$ and $\{\tilde{\sigma}(x), \tilde{\sigma}(y)\} \in E$ hold. Consider finally the case $x \in C$ and $y \in V \setminus C$. Then we have $\tilde{\sigma}(x) \in C$, $N[x] = N[\tilde{\sigma}(x)]$ and $\tilde{\sigma}(y) = y$. If $\{x, y\} \in E$, then $y \in N[x] = N[\tilde{\sigma}(x)]$ so that $\{\tilde{\sigma}(x), \tilde{\sigma}(y)\} = \{\tilde{\sigma}(x), y\} \in E$. Conversely, if $\{\tilde{\sigma}(x), \tilde{\sigma}(y)\} \in E$, then $\{\tilde{\sigma}(x), y\} \in E$ so that $y \in N[\tilde{\sigma}(x)] = N[x]$ and hence $\{x, y\} \in E$. Now, using the fact that $\mathcal{K}$ is a partition of V, we immediately get $\times_{C \in \mathcal{K}} S_C \leq \mathrm{Aut}(\Gamma)$. □

Throughout the paper, when we desire to emphasize that a set is the vertex set of a graph Γ, we indicate it by V_Γ. We do the same for the edge set and for other symbols related to the graph Γ. We now briefly recall power graphs, that is, the graphs on which our paper focuses. All the groups considered in this paper are finite. Let G be a group with identity element 1. The *power graph* of G, denoted by $\mathcal{P}(G)$, has vertex set $V := G$ and $\{x, y\} \in E$ if $x \neq y$ and there exists a positive integer m such that $x = y^m$ or $y = x^m$. The *proper power graph* of G, denoted $\mathcal{P}^*(G)$, is the subgraph of $\mathcal{P}(G)$ induced by $G \setminus \{1\}$. It is easily observed that if H is a subgroup of G, then the subgraph of $\mathcal{P}(G)$ induced by H coincides with $\mathcal{P}(H)$. A graph Γ is said to be a power graph if there exists a group G such that $\Gamma = \mathcal{P}(G)$.

We are also interested in the directed version of the power graph. The *directed power graph* of G, denoted by $\vec{\mathcal{P}}(G)$, has vertex set $V := G$ and arc set

$A := \{(x, y) \in G^2 : x \neq y,\ y = x^m$ for some $m \in \mathbb{N}\}$. Note that $\{x, y\} \in E$ if and only if at least one of $(x, y) \in A$ and $(y, x) \in A$ holds. We say that $x, y \in G$, with $x \neq y$, are joined in $\mathcal{P}(G)$ (or in $\vec{\mathcal{P}}(G)$) if $\{x, y\} \in E$. Let $X, Y \subseteq G$. We say that X and Y are joined in $\mathcal{P}(G)$ (or in $\vec{\mathcal{P}}(G)$) if there exist $x \in X$ and $y \in Y$ that are joined. In that case, if it happens that $(x, y) \in A$, then we say that the arc joining X and Y is directed from X to Y.

We denote by ϕ the Euler totient function. Throughout the paper, we freely use the well-known fact that if $m \mid n$, then $\phi(m) \mid \phi(n)$ with equality if and only if $m = n$ or m is odd and $n = 2m$.

3 A Moore closure operator for graphs

We start our research with a section about closure operators for graphs. The results that we are going to obtain will be used later to deal with power graphs. However, they seem to be of interest for graphs in general [10, 20, 21].

Recall that an operator c on a set V is a function $c\colon 2^V \to 2^V$, and that c is called a *Moore closure operator* on V (see [19, Section 4.5. a]) if c is isotone ($A \subseteq B$ implies $c(A) \subseteq c(B)$), extensive ($A \subseteq c(A)$) and idempotent ($c^2(A) = c(A)$); or a *Kuratowski closure operator* on V (see [19, Section 5.19]) if c preserves the empty set ($c(\varnothing) = \varnothing$), is extensive, idempotent and preserves the binary unions ($c(A \cup B) = c(A) \cup c(B)$). It is immediately checked that every Kuratowski closure operator is also a Moore closure operator. Note also that Moore closures always exist because one can consider the trivial Moore closure given by $c(X) = V$ for all $X \subseteq V$. If c is a Moore closure operator, then one defines a subset X of V to be closed if $c(X) = X$. Moore closure operators are interesting in pure and applied mathematics and computer science as very well described in [7]. Non-trivial examples naturally arise in algebra. For instance, if G is a group and $c\colon 2^G \to 2^G$ is defined for every $X \subseteq G$ by $c(X) := \langle X \rangle$, then c is a Moore closure operator. Kuratowski closure operators are especially important because the closed sets of such operators give a topological structure to V.

In [4], for a subset X of vertices in a power graph, the sets $N[X] = \bigcap_{x \in X} N[x]$ and $\hat{X} = N[N[X]]$ are defined and fruitfully used when X is an $\mathtt{N}$-class. They are also considered in [22], dealing with some subsets in power graphs of infinite groups. It seems important to recognize the role of operators of such objects, generalize them to any graph and explore their properties.

Let $\Gamma = (V, E)$ be a graph. For $X \subseteq V$, we define the *common closed neighbourhood* of X by

$$N[X] := \begin{cases} \bigcap_{x \in X} N[x] & \text{if } X \neq \varnothing, \\ V & \text{if } X = \varnothing. \end{cases}$$

For instance, one has $N[V] = \mathcal{S}$. The consideration of common closed neighbourhoods in graph theory dates back at least to Lovász, in 1978, with the famous construction of the neighbour complex $\mathcal{N}(\Gamma)$, which allowed him to prove Kneser's conjecture [17]. Recall that $\mathcal{N}(\Gamma)$ has vertex set V and simplices given by those $X \subsetneq V$ having $N[X] \neq \varnothing$. We warn the reader that, typically, in the literature, the symbol $N[X]$ stands for the union of the closed neighbourhoods of the vertices in X (see, for instance, [12, Section 2.1]), and not for their intersection.

We next define, for every $X \subseteq V$, the *neighbourhood closure* of X by

$$\hat{X} := N[N[X]].$$

That gives rise to the *neighbourhood closure operator* on V, which plays a key role in our research. Note that $\hat{\varnothing} = \mathcal{S}$, $\hat{V} = V$ and $\hat{\mathcal{S}} = \mathcal{S}$.

We present now some useful properties of the common closed neighbourhoods and of the neighbourhood closure operator.

Proposition 2. *Let $\Gamma = (V, E)$ be a graph and $\mathtt{N}$ its closed twin relation. Then, for every $X \subseteq V$, the followings facts hold:*

(i) *$\hat{X} \supseteq X \cup \mathcal{S}$ and if $\hat{X} \neq \varnothing$, then $\hat{X}$ is a union of $\mathtt{N}$-classes;*

(ii) *if $A \subseteq B \subseteq V$, then $N[A] \supseteq N[B]$ and $\hat{A} \subseteq \hat{B}$;*

(iii) *$N[X] = N[\hat{X}]$ and $\widehat{(\hat{X})} = \hat{X}$;*

(iv) *if $A, B \subseteq V$, then $N[A \cup B] = N[A] \cap N[B]$ and $\widehat{A \cup B} \supseteq \hat{A} \cup \hat{B}$;*

(v) *if C is an $\mathtt{N}$-class and $y \in C$, then $N[C] = N[y]$ and*

$$\hat{C} = \bigcap_{z \in N[y]} N[z] \subseteq N[y].$$

Proof. (i) Let $X \subseteq V$. Assume first that $N[X] = \varnothing$. Then

$$\hat{X} = N[\varnothing] = V \supseteq X \cup \mathcal{S}.$$

Assume next that $N[X] \neq \varnothing$. Then $\hat{X} = \bigcap_{u \in N[X]} N[u]$. Pick $x \in X$ and let $u \in N[X]$. Then we have $u \in N[x]$ and thus also $x \in N[u]$. It follows that $x \in \hat{X}$ and hence $\hat{X} \supseteq X$. We finally note that

$$\hat{X} = \bigcap_{x \in N[X]} N[x] \supseteq \bigcap_{x \in V} N[x] = \mathcal{S}.$$

We now show that, when $\hat{X} \neq \varnothing$, $\hat{X}$ is a union of $\mathtt{N}$-classes. Let $x \in \hat{X}$. We want to show that if $v \in V$ is such that $N[x] = N[v]$, then $v \in \hat{X}$. By definition of $\hat{X}$, we have that $x \in N[z]$ for all $z \in N[X]$. Thus, for every $z \in N[X]$, we have $z \in N[x] = N[v]$ and hence $v \in N[z]$, which gives $v \in \hat{X}$.

(ii) Let $A \subseteq B \subseteq V$. We first show that $N[A] \supseteq N[B]$. If $A = \varnothing$, then we have $N[A] = V \supseteq N[B]$. If $A \neq \varnothing$, then also $B \neq \varnothing$ so that

$$N[A] = \bigcap_{x \in A} N[x] \supseteq \bigcap_{x \in B} N[x] = N[B].$$

Applying the established inequality to the subsets $N[A] \supseteq N[B]$, we now obtain $\hat{A} = N[N[A]] \subseteq N[N[B]] = \hat{B}$.

(iii) We show first that $N[X] = N[\hat{X}]$. By (i), we have $\hat{X} \supseteq X$, and hence, by (ii), we get $N[X] \supseteq N[\hat{X}]$. Assume, by contradiction, that $N[X] \supsetneq N[\hat{X}]$. Then there exists $u \in N[X] \setminus N[\hat{X}]$. In particular, $N[X] \neq \varnothing$ and $N[\hat{X}] \neq V$. Thus

$$N[\hat{X}] = \bigcap_{x \in \hat{X}} N[x],$$

and since $u \notin N[\hat{X}]$, we deduce that there exists $\hat{x} \in \hat{X}$ such that $u \notin N[\hat{x}]$. On the other hand, since $N[X] \neq \varnothing$, we have that $\hat{X} = \bigcap_{x \in N[X]} N[x]$. As a consequence, by $u \in N[X]$, we obtain $\hat{X} \subseteq N[u]$. Thus $\hat{x} \in N[u]$, that is, $u \in N[\hat{x}]$, a contradiction.

Now, we immediately deduce also that $\widehat{(\hat{X})} = N[N[\hat{X}]] = N[N[X]] = \hat{X}$.

(iv) Let $A, B \subseteq V$. If one of A and B is empty, then $N[A \cup B] = N[A] \cap N[B]$ trivially follows. Assume that they are both nonempty. Then also $A \cup B$ is nonempty and we have

$$N[A \cup B] = \bigcap_{x \in A \cup B} N[x] = \bigcap_{x \in A} N[x] \cap \bigcap_{x \in B} N[x] = N[A] \cap N[B].$$

As a consequence, by (ii), we have

$$\widehat{A \cup B} = N[N[A \cup B]] = N[N[A] \cap N[B]] \supseteq N[N[A]] \cup N[N[B]] = \hat{A} \cup \hat{B}.$$

(v) Let $y \in C$. Then $C = [y]_{\mathtt{N}} \neq \varnothing$ so that, by the definition of the relation $\mathtt{N}$, we deduce

$$N[C] = \bigcap_{x \in [y]_{\mathtt{N}}} N[x] = N[y] \neq \varnothing.$$

Since $y \in N[y]$, it follows that $\hat{C} = N[N[y]] = \bigcap_{z \in N[y]} N[z] \subseteq N[y]$. □

Observe that, as an easy consequence of Proposition 2, the neighbourhood closure of a set of vertices is empty if and only if that set is empty and the graph admits no star. We also observe that the inclusion $\widehat{A \cup B} \supseteq \hat{A} \cup \hat{B}$ in Proposition 2 (iv) is generally proper as the following example shows.

Example 3. Let $\Gamma = (V, E)$, where $V = [5]$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{3, 4\}\}$. If $A := \{1\}$ and $B := \{5\}$, then it is immediately checked that $\widehat{A \cup B} = V$ and $\hat{A} \cup \hat{B} = V \setminus \{4\} \subsetneq \widehat{A \cup B}$.

The next result is now an immediate consequence of Proposition 2 (i)–(iii) and of Example 3.

Corollary 4. *The neighbourhood closure operator is a Moore closure operator which is not, in general, a Kuratowski closure operator.*

Clearly, there exist graphs for which the neighbourhood closure operator is a Kuratowski closure operator. For instance, this is the case for the totally disconnected graph on a set V. We set then the following problem.

Question 5. For which families of graphs is the neighbourhood closure operator also a Kuratowski closure operator?

Note that, in order to respect the preservation of the empty set, we need $\mathcal{S} = \varnothing$. But this is not enough to guarantee that our operator is a Kuratowski closure operator. Indeed, as Example 3 shows, even when the star set is empty, binary unions are not necessarily preserved. A deep enquiry into those aspects is out of scope for the present paper, but surely deserves future attention.

4 Preliminary results on power graphs

We now come back to the focus of our research, power graphs. Let G be a group and $x, y \in G$. We write $x \diamond y$ if $\langle x \rangle = \langle y \rangle$. Note that $x \diamond y$ if and only if $x = y$ or both (x, y) and (y, x) are arcs of $\vec{\mathcal{P}}(G)$. The relation $\diamond$ is clearly an equivalence relation in G. We denote the $\diamond$-class of $x \in G$ by $[x]_\diamond$. Note that $[x]_\diamond$ is the set of generators of $\langle x \rangle$. In particular, $|[x]_\diamond| = \phi(o(x))$. It is easily checked that if $\mathtt{N}$ is the closed twin relation of $\mathcal{P}(G)$, then $x \diamond y$ implies $x\mathtt{N}y$. In other words, the relation $\diamond$ is a refinement of the relation $\mathtt{N}$. As a consequence, an $\mathtt{N}$-class is a union of $\diamond$-classes. Moreover, it is immediately observed that, for every $x \in G$, we have $[x]_\diamond \subseteq [x]_\mathtt{N} \subseteq N_{\mathcal{P}(G)}[x]$ and $\langle x \rangle \subseteq N_{\mathcal{P}(G)}[x]$. In particular, two distinct elements of G in the same $\mathtt{N}$-class are joined and $\mathcal{P}(G)_{[x]_\mathtt{N}}$ is a complete graph. Note that $[1]_\mathtt{N} = \mathcal{S}_{\mathcal{P}(G)}$. For this reason, we call the $\mathtt{N}$-class of the identity element of G the *star class*. When the star class is not equal to $\{1\}$, there are well-known consequences for the structure of the group and the nature of the star class itself.

Proposition 6 ([4, Proposition 4]). *Let G be a group with $|G| = n$ such that $|\mathcal{S}_{\mathcal{P}(G)}| > 1$. Then one of the following facts holds:*

(a) *G is a cyclic p-group and $\mathcal{S}_{\mathcal{P}(G)} = G$;*

(b) *G is cyclic, not a p-group, and $\mathcal{S}_{\mathcal{P}(G)}$ consists of 1 and the generators of G, so that $|\mathcal{S}_{\mathcal{P}(G)}| = 1 + \phi(n)$;*

(c) *G is a generalized quaternion 2-group and $\mathcal{S}_{\mathcal{P}(G)}$ contains 1 and the unique involution of G, so that $|\mathcal{S}_{\mathcal{P}(G)}| = 2$.*

In particular, $\mathcal{P}(G)$ is a complete graph if and only if $G \cong C_{p^m}$ for some prime p and some $m \in \mathbb{N}_0$.

We now present a useful auxiliary result. For simplicity, from now on, when a single group G is under consideration, we completely omit all the subscripts $\mathcal{P}(G)$ in our notation and assume that $\mathtt{N}$ is the closed twin relation of $\mathcal{P}(G)$.

Lemma 7. *Let G be a group and let $x, y \in G$ be elements whose orders are powers of the same prime p, and such that $o(x) \leq o(y)$. Then $N[x] \supseteq N[y]$ if and only if x is a power of y.*

Proof. Assume that $N[x] \supseteq N[y]$. Then $y \in N[x]$, which implies $\langle x \rangle \leq \langle y \rangle$ or $\langle y \rangle \leq \langle x \rangle$. Since $o(x) \leq o(y)$, we then have $\langle x \rangle \leq \langle y \rangle$ so that x is a power of y. Conversely, assume that x is a power of y. Pick $z \in N[y]$. If y is a power of z, then x is a power of z too. If z is a power of y, then x and z belong to the cyclic p-group generated by y. Hence, since $\mathcal{P}(\langle y \rangle)$ is complete, we have $z \in N[x]$. □

5 $\mathtt{N}$-classes in power graphs

We have observed that an $\mathtt{N}$-class in a power graph is a union of $\diamond$-classes. Hence we can sensibly distinguish two types of $\mathtt{N}$-classes.

Definition 8. Let G be a group and C an $\mathtt{N}$-class. We say that C is a *class of plain type* if C is a single $\diamond$-class; C is a *class of compound type* if C is the union of at least two $\diamond$-classes.

Those classes are called, in [4], of type (a) and (b) respectively. The classes of plain type are easily characterized by the order of their elements.

Lemma 9. *Let G be a group and C an $\mathtt{N}$-class. Then C is of plain type if and only if all elements in C have the same order.*

Proof. Let C be of plain type. If $x, y \in C$, then we have $\langle x \rangle = \langle y \rangle$ and thus $o(x) = o(y)$. Assume, conversely, that the elements in C have the same order. Let $C = [x]_{\mathtt{N}}$ for some $x \in G$. Since we know that $[x]_\diamond \subseteq [x]_{\mathtt{N}}$, we need only show that $[x]_{\mathtt{N}} \subseteq [x]_\diamond$. Let $y \in [x]_{\mathtt{N}}$. Then $\{x, y\} \in E$ and $o(x) = o(y)$. Then we deduce $y \diamond x$. □

We observe that it is possible to have plain classes with elements of any fixed order $k \in \mathbb{N}$. Indeed, consider $G := \langle a \rangle \times \langle b \rangle$ with $o(a) = 2$, $o(b) = k$. It is easily checked that $[b]_{\mathtt{N}}$ is a class of plain type whose elements have order k.

Thanks to Proposition 6 and Lemma 9, we observe that $\mathcal{S}$ is of compound type if and only if $\mathcal{S} \neq \{1\}$. We now characterize the $\mathtt{N}$-classes of G different from $\mathcal{S}$ and of compound type. Our result is very little more than [4, Proposition 5]. We give full details for two reasons. First, some unspecified conditions in [4] are presumably at the origin of a gap that we are going to correct later (see the comments to Proposition 14). Second, we need to clearly introduce some notation for the sequel.

Proposition 10 ([4, Proposition 5]). *Let G be a group and C an $\mathtt{N}$-class of G, with $C \neq \mathcal{S}$. The following facts are equivalent.*

(i) *C is a compound-class.*

(ii) *If y is an element of maximum order in C, then $o(y) = p^r$ for some prime p and some integer $r \geq 2$. Moreover, there exists $s \in [r-2]_0$ such that*

$$C = \{z \in \langle y \rangle : p^{s+1} \leq o(z) \leq p^r\}.$$

In particular, we have that the number of $\diamond$-classes into which a compound $\mathtt{N}$-class C splits is $r - s \geq 2$. The orders of the elements in those $\diamond$-classes are given by $p^{s+1}, p^{s+2}, \ldots, p^r$ and the sizes of those $\diamond$-classes are

$$\phi(p^{s+1}), \phi(p^{s+2}), \ldots, \phi(p^r)$$

respectively. The ordered list (p, r, s) is uniquely determined by C.

Proof. (i) $\Rightarrow$ (ii) Let C be a compound-class. We first claim that if $x, y \in C$ are two elements of distinct order, then those orders are suitable powers, with positive integer exponent, of the same prime. Let $x, y \in C$ with $o(x) \neq o(y)$. Then, renaming if necessary, we have $o(x) < o(y)$. In particular, $x \neq y$. Moreover, both x and y are different from 1 as $C \neq \mathcal{S} = [1]_{\mathtt{N}}$. Since $N[x] = N[y]$, x and y are joined and $o(x)$ is a proper divisor of $o(y)$. So let $k := o(x)$ and $kl := o(y)$ for some $k, l \geq 2$. We show that k and l are powers of the same prime. It suffices to show that if p is a prime dividing k, then the only prime dividing l is p itself. Assume, by contradiction, that we have a prime $p \mid k$ and a prime $q \neq p$ such that $q \mid l$. Then $(kq)/p = (k/p)q \mid kl = o(y)$ and thus there exists an element $z \in \langle y \rangle$ with $o(z) = (k/p)q$. Now $z \in N[y] = N[x]$. But $o(z) \nmid o(x)$ and $o(x) \nmid o(z)$. Indeed, assume that $o(z) \mid o(x)$. Then $(k/p)q \mid k = p(k/p)$, which implies the contradiction $q \mid p$. Assume next $o(x) \mid o(z)$. Then $k \mid (k/p)q$ and thus $p(k/p) \mid q(k/p)$, which implies the contradiction $p \mid q$.

Now choose $x \in C$ such that $o(x)$ is minimum in C and $y \in C$ such that $o(y)$ is maximum in C. Since C is compound, by Lemma 9, those orders are distinct. Moreover, since $C \neq \mathcal{S}$, those orders are different from 1. Hence we have $o(x) = p^{s+1}$ for some integer $s \geq 0$ and $o(y) = p^r$ for some integer $r \geq s + 2$. Pick $z \in C$. Since $z, y \in C$, y and z are joined and, since $o(z) \leq o(y)$, we have that z is a power of y. Moreover, $o(z) \geq p^{s+1}$. This shows that

$$C \subseteq \{z \in \langle y \rangle : p^{s+1} \leq o(z) \leq p^r\}.$$

Next let $z \in \langle y \rangle$ with $o(z) \geq p^{s+1} = o(x)$. Then we have $z \in N[y] = N[x]$, and hence $z = x \in C$ or z and x are joined. In this latter case, we necessarily have that x is a power of z. Moreover, x and z are of prime power order so that, by Lemma 7, we obtain $N[x] \supseteq N[z] \supseteq N[y] = N[x]$ and thus $N[x] = N[z]$, so that $z \in C$. Thus $C = \{z \in \langle y \rangle : p^{s+1} \leq o(z) \leq p^r\}$.

(ii) $\Rightarrow$ (i) In C, there exist elements with different order. Thus, by Lemma 9, C cannot be plain and hence it is compound. □

Definition 11. If $C \neq \mathcal{S}$ is an $\mathtt{N}$-class of compound type for a group G, then we call an element $y \in C$ of maximum order a *root* of C. Moreover, we call the ordered list (p, r, s), described in Proposition 10, the *parameters of* C.

Recall that p is a prime, r and s are integer with $r \geq 2$ and $s \in [r - 2]_0$.

5.1 Examples of classes of compound type

In [6, Proposition 2], Cameron and Ghosh prove, translated within our language, that if C is a compound class in an abelian group G, then $C = \mathcal{S}$. This is not true, in general, for non-abelian groups, as shown by the following example.

Example 12. The $\mathtt{N}$-classes of the elements of order 4 in S_4 are of compound type with parameters $(p, r, s) = (2, 2, 0)$. For instance,

$$[(1234)]_{\mathtt{N}} = \{(1234), (1432), (13)(24)\}$$

is the union of the two $\diamond$-classes

$$[(1234)]_{\diamond} = \{(1234), (1432)\} \quad \text{and} \quad [(13)(24)]_{\diamond} = \{(13)(24)\}.$$

This example shows that $s = 0$ can occur as a parameter of a compound class. That possibility, wrongly denied in [4], is not sporadic at all. We can indeed easily construct an infinite family of groups admitting $\mathtt{N}$-classes of compound type with $s = 0$. Recall that the dihedral group D_{2n} of order $2n$, for $n \geq 2$, is defined by

$$\mathrm{D}_{2n} := \langle a, b \mid a^n = 1 = b^2,\ b^{-1}ab = a^{-1}\rangle. \tag{5.1}$$

Now consider the choice $n = p^r$ for a prime p and $r \in \mathbb{N}$, with $r \geq 2$. It is easily checked that $\mathcal{P}(\mathrm{D}_{2p^r})$ is composed of the complete graph K_{p^r} on the vertices in $\langle a \rangle$ and p^r further vertices given by the involutions in $\mathrm{D}_{2n} \setminus \langle a \rangle$ joined only with 1. Therefore, we have that $[a]_{\mathtt{N}} = \langle a \rangle \setminus \{1\}$ is of compound type, since it contains elements of different orders. Moreover, its parameters are $(p, r, 0)$.

The construction of an infinite family of groups admitting $\mathtt{N}$-classes of compound type with $s = 1$ is a bit more tricky. Consider, for $n \geq 4$, the quasidihedral group of order 2^n defined by

$$\mathrm{QD}_{2^n} := \langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ b^{-1}ab = a^{-1+2^{n-2}} \rangle$$

(see, for instance, [13, Satz I.14.9]). Then it is easily checked that its proper power graph $\mathcal{P}^*(\mathrm{QD}_{2^n})$ is composed of a clique of order $2^{n-1} - 1$ given by the graph induced by $\langle a \rangle$, 2^{n-3} triangles having the vertex $a^{2^{n-2}}$ in common and all the 2^{n-2} remaining vertices isolated. Then the class $[a]_{\mathtt{N}}$ is compound with parameters $(2, n-1, 1)$. Figure 1 shows this in the $n = 4$ case.

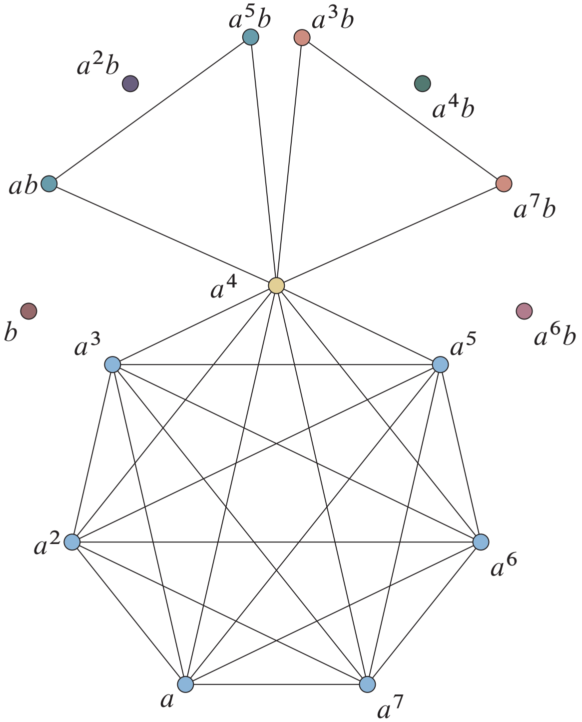


Figure 1. $\mathcal{P}^*(\mathrm{QD}_{16})$: vertices with different colours are in distinct $\mathtt{N}$-classes.

5.2 Critical classes

The Moore closure operator defined in Section 3 is a very useful tool for the study of $\mathtt{N}$-classes. The proposition below is essentially extracted from [4]. We state it formally and prove it making use of the neighbourhood closure operator.

Proposition 13. *Let G be a group and $C \neq \mathcal{S}$ an $\mathtt{N}$-class of compound type, root $y \in G$ and parameters (p, r, s). Then $|C| = p^r - p^s$ and $\hat{C} = \langle y \rangle$. In particular, $|\hat{C}| = p^r$.*

Proof. By Proposition 10, we have $C \subseteq \langle y \rangle =: Y$ and

$$|C| = |Y| - |\{z \in Y : o(z) \leq p^s\}| = p^r - p^s.$$

We first show that $Y \subseteq \hat{C}$. Let $a \in Y$. Then a is a power of y and, by Lemma 7, we have $N[a] \supseteq N[y]$. By Proposition 2 (v), recalling that $y \in C$, we deduce

$$N[C] = N[y] \subseteq \bigcap_{a \in Y} N[a] = N[Y].$$

Therefore, by Proposition 2 (i)–(ii), we get $Y \subseteq \hat{Y} = N[N[Y]] \subseteq N[N[C]] = \hat{C}$.

We next show that $Y = \hat{C}$. Suppose, by contradiction, that there exists some $u \in \hat{C} \setminus Y$. Assume first that the order of u is not a power of p. By Proposition 2 (v), we have $u \in \hat{C} \subseteq N[y]$. But $u \neq y$, because the order of y is a power of p. It follows that u and y are joined. Then $\langle u \rangle > \langle y \rangle$ since $u \notin Y$. Thus there exist $t \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $m \geq 2$ and $\gcd(m, p) = 1$ such that $o(u) = p^{r+t}m$. Now note that $o(u^{p^t m}) = o(y) = p^r$. Thus $u^{p^t m}$ and y are generators of the unique subgroup of order p^r inside the cyclic group $\langle u \rangle$. As a consequence, there exists $k \in \mathbb{N}$ with $\gcd(k, p) = 1$ such that $u^{p^t mk} = y$. Thus

$$y^p = (u^{p^t mk})^p = u^{p^{t+1} mk}.$$

Now $o(y^p) = p^{r-1}$ and, since y is a root for C, we have $y^p \in C$ so that $y \mathtt{N} y^p$ holds. It follows that $N[y] = N[y^p]$. We see that this is impossible by considering $w := u^{p^{t+1}}$ and showing that $w \in N[y^p] \setminus N[y]$. Note first that

$$o(w) = p^{r-1}m \notin \{p^{r-1}, p^r\},$$

and thus $w \notin \{y^p, y\}$. Since $y^p = w^{mk}$, we have that $w \in N[y^p]$. On the other hand, $w \notin N[y]$ because $o(y) = p^r \nmid o(w)$ and $o(w) \nmid o(y)$.

Assume next that the order of u is a power of p. By Proposition 2 (v), we have that $u \in \hat{C} \setminus Y \subseteq N[y] \setminus Y$. Thus $u \neq y$, u is joined to y and necessarily y is a power of u. Since $u \notin Y$, we have that $o(u) > o(y)$. Hence, by Lemma 7, we have $N[u] \subseteq N[y]$. Assume that $N[u] = N[y]$. Then $u \in C \subseteq Y$, a contradiction. Thus the inclusion is proper. As a consequence, there exists $w \in N[y]$ such that $w \notin N[u]$. It follows that $u \notin N[w]$, which implies $u \notin \hat{C} = \bigcap_{z \in N[y]} N[z]$, a contradiction. □

Remarkably, Proposition 13 points out that if $C \neq \mathcal{S}$ is an $\mathtt{N}$-class of compound type, then some strong arithmetic restrictions on $|C|$ and $|\hat{C}|$ arise. As a consequence, an $\mathtt{N}$-class different from $\mathcal{S}$ that does not satisfy those restrictions is necessarily of plain type. However, as we show in the next proposition, those arithmetic restrictions can be satisfied also by an $\mathtt{N}$-class of plain type in a very specific case that we can completely clarify.

Proposition 14. *Let G be a group and C an $\mathtt{N}$-class of plain type. Assume that there exist a prime p and integers $r \geq 2$ and $s \in [r-2]_0$ such that $|C| = p^r - p^s$ and $|\hat{C}| = p^r$. Then $\hat{C} = C \cup \{1\}$, $s = 0$ and $C = [y]_\diamond$ for some $y \in G$, with $o(y) > 1$ not a prime power and such that $\phi(o(y)) = p^r - 1$.*

Proof. By Proposition 2 (i), we have $\hat{C} \supseteq C \cup \{1\}$ and $\hat{C}$ is a union of $\mathtt{N}$-classes and hence of $\diamond$-classes. Observe that $p^r < 2(p^r - p^s)$. Indeed, that inequality is equivalent to $2 < p^{r-s}$ and, surely, $p^{r-s} \geq p^2 > 2$. It follows that $|\hat{C}| < 2|C|$. As a consequence, every $\diamond$-class included in $\hat{C}$ and distinct from C must have size smaller than $|C|$. Pick $y \in C$. Since C is of plain type, we have $C = [y]_{\mathtt{N}} = [y]_\diamond$. Note that, since $|C| = p^s(p^{r-s} - 1) \geq 3$ and $[1]_\diamond = \{1\}$, we have $y \neq 1$. Then, by Lemma 9, all the elements in C have order equal to $o(y) > 1$ and $1 \notin C$. In particular, we have $C \cup \{1\} = C \mathbin{\dot\cup} \{1\}$.

Claim 1. *$\hat{C} \setminus [y]_\diamond$ contains no elements of order greater than or equal to $o(y)$.*

Assume, by contradiction, that there exists $x \in \hat{C} \setminus [y]_\diamond$ such that $o(x) \geq o(y)$. Then $x \neq y$ and, by Proposition 2 (v), we have $\hat{C} \subseteq N[y]$. Thus $x \in N[y] \setminus [y]_\diamond$. Hence, since $o(x) \geq o(y)$, we necessarily have $y \in \langle x \rangle$. Then $o(y) \mid o(x)$, which implies $\phi(o(y)) \mid \phi(o(x))$. In particular, we have that $\phi(o(y)) \leq \phi(o(x))$. Since $\hat{C}$ is a union of $\diamond$-classes, we have that $[x]_\diamond \subseteq \hat{C}$. It follows that

$$|[x]_\diamond| = \phi(o(x)) \geq \phi(o(y)) = |C|,$$

a contradiction.

Claim 2. $\hat{C} \subseteq \langle y \rangle$.

Let $x \in \hat{C}$. If $\langle x \rangle = \langle y \rangle$, then x is a power of y. If $\langle x \rangle \neq \langle y \rangle$, then $x \in \hat{C} \setminus [y]_\diamond$ so that, by Claim 1, we have $o(x) < o(y)$. Now, by Proposition 2 (v), we have $\hat{C} \subseteq N[y]$ so that $x \in N[y]$, and thus, again, x is a power of y.

Hence we have reached the following chain of inclusions:

$$C \cup \{1\} \subseteq \hat{C} \subseteq \langle y \rangle. \tag{5.2}$$

We now show that $o(y)$ cannot be a prime power. Assume, by contradiction, that $o(y) = q^t$ for some prime q and some integer $t \geq 1$.

We claim that $N[z] \supseteq \langle y \rangle$ holds true for all $z \in N[y]$. Let $z \in N[y]$. Then we have $z \in \langle y \rangle$ or $y \in \langle z \rangle$. Assume first that $o(z) \leq o(y)$. Then $z \in \langle y \rangle$. Now, by the fact that $o(y) = q^t$, it follows that $\mathscr{P}(\langle y \rangle)$ is a complete graph and thus $\langle y \rangle \subseteq N[z]$. Assume next that $o(z) > o(y)$. Then necessarily $y \in \langle z \rangle$ and hence $\langle y \rangle \leq \langle z \rangle \subseteq N[z]$.

Now, by Proposition 2 (v), we know that $\hat{C} = \bigcap_{z \in N[y]} N[z]$. Thus $\hat{C} \supseteq \langle y \rangle$ and, by (5.2), we deduce $\hat{C} = \langle y \rangle$. As a consequence, $p^r = |\hat{C}| = |\langle y \rangle| = q^t$. But then $q = p$, $t = r$, and hence $p^r - p^s = |C| = \phi(p^r) = p^r - p^{r-1}$, which implies $s = r - 1$, a contradiction.

Hence we have $o(y) = m$ for some integer $m > 1$, not a prime power.

Claim 3. $\hat{C} = C \cup \{1\}$.

By (5.2), we just need to show that $\hat{C} \subseteq C \cup \{1\}$. Let $x \in \hat{C}$. By (5.2), we have that $x = y^k$ for some $k \in \mathbb{N}$. Assume, by contradiction, that $y^k \notin C \cup \{1\}$. Then y^k is neither a generator nor the identity of the cyclic group $\langle y \rangle$ of order m. By Proposition 6 applied to $\mathscr{P}(\langle y \rangle)$, there exists $w \in \langle y \rangle$ such that $w \neq y^k$ and $\{y^k, w\} \notin E_{\mathscr{P}(\langle y \rangle)}$. Then we also have $\{y^k, w\} \notin E_{\mathscr{P}(G)}$. It follows that $w \in N[y]$, while $y^k \notin N[w]$. Since, by Proposition 2 (v), we have $\hat{C} = \bigcap_{z \in N[y]} N[z]$, we deduce $x = y^k \notin \hat{C}$, a contradiction. By $\hat{C} = C \mathbin{\dot\cup} \{1\}$, we now deduce $|\hat{C}| = |C| + 1$ and thus $p^r - p^s = |C| = |\hat{C}| - 1 = p^r - 1$, which implies $s = 0$. As a consequence, we also have $\phi(o(y)) = |C| = p^r - 1$. □

In [4, pages 782–783], it is claimed that there is no $\mathtt{N}$-class C of plain type satisfying $|C| = p^r - p^s$ and $|\hat{C}| = p^r$ for a prime p and integers r, s with $r \geq 2$ and $s \in [r-2]_0$. That is mistaken because we can exhibit examples of $\mathtt{N}$-classes of plain type satisfying those arithmetical restrictions.

Example 15. Consider $G = \mathrm{D}_{30}$ with the notation in (5.1). Let $C := [a]_{\mathtt{N}}$. Then C contains the element a of order 15 and 15 is not a prime power. We remark that, by Proposition 10, if an $\mathtt{N}$-class $C = [x]_{\mathtt{N}} \neq \mathscr{S}$ is compound, then $o(x) > 1$ is a prime power. Thus we deduce that C is of plain type. As a consequence, we have $C = [a]_{\diamond}$ and $|C| = \phi(15) = 8 = 3^2 - 1$. Defining $p := 3$, $r := 2$, $s := 0$, we see that $|C| = p^r - p^s$. In order to show $|\hat{C}| = p^r$, we prove that $\hat{C} = C \mathbin{\dot\cup} \{1\}$. Since the elements in C have order 15, clearly $1 \notin C$ and thus $C \cup \{1\} = C \mathbin{\dot\cup} \{1\}$.

By Proposition 2 (i), we have $\hat{C} \supseteq C \cup \{1\}$. Assume, by contradiction, that there exists $x \in \hat{C} \setminus (C \cup \{1\})$. Then we have $o(x) \notin \{1, 15\}$. In particular, $x \neq a$. Moreover, by Proposition 2 (v), we have $\hat{C} \subseteq N[a]$ and therefore $x \in \langle a \rangle$. Then, by Proposition 6, there exists $y \in \langle a \rangle \setminus \{x\}$ such that y is not joined to x. It follows that $y \in N[a]$ and $x \notin N[y]$ so that $x \notin \hat{C} = \bigcap_{z \in N[a]} N[z]$, a contradiction.

We are now in position to give birth to a crucial and original definition, the main player of our research.

Definition 16. Let Γ be a power graph. A *critical class* is an $\mathtt{N}$-class C such that $\hat{C} = C \mathbin{\dot\cup} \{1\}$ and there exist a prime p and an integer $r \geq 2$ with $|\hat{C}| = p^r$.

We emphasize that, in order to check if an $\mathtt{N}$-class is critical, one has to make only arithmetical or graph theoretical considerations. No group theoretical consideration is involved. We also want to emphasize that both critical classes of plain type and of compound type can arise in a power graph as shown in Examples 12 and 15. Note that $\mathcal{S}$ is never critical since $1 \in \mathcal{S}$.

It is interesting to note that a compound class C is critical if and only if $C \neq \mathcal{S}$ and C has parameters $(p, r, 0)$. This can easily be proved as follows. Assume that C is critical. Then $C \neq \mathcal{S}$ because $\mathcal{S}$ is never critical. Moreover, $\hat{C} = C \mathbin{\dot\cup} \{1\}$ and $|\hat{C}| = p^r$ for some prime p and integer $r \geq 2$. Hence we have $|C| = p^r - 1$. Then, by Proposition 13, the parameters of C are $(p, r, 0)$. Conversely, assume that $C \neq \mathcal{S}$ and the parameters of C are $(p, r, 0)$. By Proposition 13, $|\hat{C}| = p^r$ and $|C| = p^r - 1$. Now, clearly, $1 \notin C$; otherwise, $C = \mathcal{S}$. On the other hand, by Proposition 2 (i), we have $\hat{C} \supseteq C \cup \{1\}$. It follows that $\hat{C} = C \mathbin{\dot\cup} \{1\}$. Hence C is critical.

We stress that a critical class C is an $\mathtt{N}$-class, different from $\mathcal{S}$, which we cannot immediately recognize as plain or compound by arithmetical considerations of its size and the size of its closure. On the other hand, if we exclude those classes, the recognition is easy.

Proposition 17. *Let G be a group and $C \neq \mathcal{S}$ a non-critical $\mathtt{N}$-class. Then C is compound if and only if there exist a prime p, and integers $r \geq 2$ and $s \in [r-2]_0$ such that $|C| = p^r - p^s$ and $|\hat{C}| = p^r$.*

Proof. If C is compound, by Proposition 13, there exist a prime p, and integers $r \geq 2$ and $s \in [r-2]_0$ such that $|C| = p^r - p^s$ and $|\hat{C}| = p^r$. Conversely, Proposition 14 shows that if such p, r and s exist and C is plain, then C is critical, a contradiction. □

For a better final insight on $\mathtt{N}$-classes, we need the following result by Feng, Ma and Wang [11].

Proposition 18 ([11, Lemma 3.5]). *Let G be a group and $x, y \in G$. Let $[x]_{\mathtt{N}}$ and $[y]_{\mathtt{N}}$ be two distinct $\mathtt{N}$-classes different from $\mathcal{S}$. If $\langle x\rangle < \langle y\rangle$, then $|[x]_{\mathtt{N}}| \le |[y]_{\mathtt{N}}|$, with equality if and only if both the following two conditions hold:*

(i) *both $[x]_{\mathtt{N}}$ and $[y]_{\mathtt{N}}$ are of plain type;*

(ii) *$o(y) = 2\,o(x)$ and $o(x) \ge 3$ is odd.*

We can now state and prove a result that allows us to recognize if a critical class is plain or compound, by purely graph theoretical considerations, when the star class is trivial. Such a result will be crucial for the proof of the Main Theorem.

Proposition 19. *Let G be a group with $\mathcal{S} = \{1\}$ and let $C = [y]_{\mathtt{N}}$ be a critical class. Then C is of plain type if and only if there exists $x \in G \setminus \hat{C}$ such that $|[x]_{\mathtt{N}}| \le |C|$ and $\{x, y\} \in E$.*

Proof. Note that we have $C \ne \mathcal{S}$ because $\mathcal{S}$ is never critical. Assume first that C is of plain type. Then $\hat{C}$ is formed by the generators of $\langle y\rangle$ and 1. By Proposition 14, we have $o(y) = m > 1$, not a prime power. In particular, m is not a prime and thus $m > \phi(m) + 1$. As a consequence, $\langle y\rangle \setminus \hat{C} \ne \varnothing$. Pick $x \in \langle y\rangle \setminus \hat{C}$. Then we have $\langle x\rangle < \langle y\rangle$ and thus $\{x, y\} \in E$. Note that, since $x \ne 1$, we have $[x]_{\mathtt{N}} \ne \mathcal{S} = \{1\}$. Moreover, since x does not generate $\langle y\rangle$, we also have $[x]_{\mathtt{N}} \ne [y]_{\mathtt{N}} = [y]_{\diamond}$. Hence, by Proposition 18, we deduce $|[x]_{\mathtt{N}}| \le |C|$.

Assume next that there exists $x \in G \setminus \hat{C}$ such that $|[x]_{\mathtt{N}}| \le |C| = |[y]_{\mathtt{N}}|$ and $\{x, y\} \in E$. Note that $x \ne 1$ since $x \notin \hat{C} = C \mathbin{\dot\cup} \{1\}$. As a consequence, we get $[x]_{\mathtt{N}} \ne \mathcal{S} = \{1\}$. Observe next that $[x]_{\mathtt{N}} \ne [y]_{\mathtt{N}}$ holds; otherwise, we would have $x \in C \subseteq \hat{C}$. In particular, $\langle x\rangle \ne \langle y\rangle$. Since $\{x, y\} \in E$, it follows that $\langle y\rangle < \langle x\rangle$ or $\langle x\rangle < \langle y\rangle$. If $\langle y\rangle < \langle x\rangle$, then by Proposition 18, we deduce $|[y]_{\mathtt{N}}| \le |[x]_{\mathtt{N}}|$ and hence $|[x]_{\mathtt{N}}| = |[y]_{\mathtt{N}}|$. Thus, by Proposition 18, $C = [y]_{\mathtt{N}}$ is of plain type. If $\langle x\rangle < \langle y\rangle$, then $x \in \langle y\rangle$. Suppose, by contradiction, that C is of compound type. Let z be a root of C. Then, by Propositions 10 and 13, we get $x \in \langle y\rangle \le \langle z\rangle = \hat{C}$, a contradiction. □

6 The reconstruction of the directed power graph

We now describe the $\diamond$-classes inside the directed power graph, exploiting some facts observed in [4]. The following lemma, together with other previous results, paves the way for the effective reconstruction of the directed power graph from its undirected counterpart.

Lemma 20. *Let G be a group and let X, Y be two distinct $\diamond$-classes. In $\vec{\mathcal{P}}(G)$, the following facts hold.*

(i) *The subdigraph induced by a $\diamond$-class is a complete digraph.*

(ii) *If there is at least one arc directed from X to Y, then (x, y) is an arc for all $x \in X$ and $y \in Y$. Moreover, there is no arc directed from Y to X.*

(iii) *Let X and Y be joined and $|X| > |Y|$. Then there is an arc directed from X to Y.*

(iv) *Let X and Y be joined and $1 \neq |X| = |Y|$. There is an arc directed from X to Y if and only if there exists an involution $\tau \in G$ such that $[\tau]_\diamond$ is joined with X.*

(v) *Let X and Y be joined and $1 = |X| = |Y|$. Then one of them is $[1]_\diamond$, the other is $[\tau]_\diamond$, with $\tau \in G$ an involution, and $(\tau, 1)$ is the only arc between the two $\diamond$-classes.*

Proof. Recall that A denotes the arc set of $\vec{\mathcal{P}}(G)$.

(i) This is obvious by the definition of the relation $\diamond$.

(ii) Note first that if $x \in X$ and $y \in Y$, then we cannot have in A both the arcs (x, y) and (y, x); otherwise, we would have $x \diamond y$ and then $X = Y$. Suppose now that there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in A$. Pick $x \in X$ and $y \in Y$. Then y is a power of $\bar{y}$, which is a power of $\bar{x}$, which in turn is a power of x. Hence $(x, y) \in A$.

(iii) Since the classes are joined, there exist joined vertices $x \in X$ and $y \in Y$. Since

$$\phi(o(x)) = |X| > |Y| = \phi(o(y)),$$

we deduce that $o(y) \mid o(x)$ and so y is a power of x.

(iv) Let $X = [x]_\diamond$ and $Y = [y]_\diamond$ for some $x, y \in G$. Assume first that $(x, y) \in A$. Then $y \in \langle x \rangle$ and $o(y) \mid o(x)$. Now $|X| = |Y|$ implies $\phi(o(x)) = \phi(o(y))$. This implies $o(x) = 2o(y)$, with $o(y)$ odd, since otherwise we would have $o(x) = o(y)$ and then also the contradiction $X = Y$. Observe that $\tau := x^{o(y)}$ is an involution and that $x \neq \tau$ because $|[\tau]_\diamond| = 1 \neq |[x]_\diamond|$. It follows that $(x, \tau) \in A$ and thus $[\tau]_\diamond$ and X are joined.

Conversely, assume that there exists an involution $\tau \in G$ such that $[\tau]_\diamond$ and X are joined. Then we have $|X| > |[\tau]_\diamond| = 1$ and thus, by (iii), $(x, \tau) \in A$. We show that $(x, y) \in A$. Assume, by contradiction, that $(y, x) \in A$. As before, one obtains $o(y) = 2o(x)$, with $o(x)$ odd, against the fact that $2 = o(\tau) \mid o(x)$.

(v) A $\diamond$-class of size one contains either 1 or an involution, and involutions are never joined. The result therefore follows from the definition of $\vec{\mathcal{P}}(G)$. □

6.1 The Main Theorem

We now pass to proving our Main Theorem. First we need to be precise about our terminology. It seems that, in the literature, a clear definition of reconstruction was missing.

We say that we can reconstruct the directed power graph from a power graph $\Gamma = (V, E)$ if we are able, by purely arithmetical or graph theoretical considerations, without taking into account any group theoretical information, to do one of the following:

- prove that there exists a unique group G such that $\Gamma = \mathcal{P}(G)$ and exhibit such G;
- exhibit a digraph $\vec{\Gamma}$ isomorphic to $\vec{\mathcal{P}}(G)$ for all those G such that $\Gamma = \mathcal{P}(G)$.

Note that, in the first case, G is uniquely determined and exhibited, and thus we can clearly also exhibit its directed power graph. In the second case, there could be many groups G realizing $\Gamma = \mathcal{P}(G)$, and usually, one is not able to explicitly exhibit them. The point is that, whatever those groups are, we require to be able to show a directed graph which, up to isomorphisms of directed graphs, is the directed power graph of all those groups. In particular, the directed power graphs of all those G will be isomorphic.

For the proof, we use some methods from the proof of [4, Theorem 2], correcting the mistake about critical classes and filling in some missing details. Of course, since we are proving a stronger result, the architecture of the proof and some parts of it are completely original.

Main Theorem. *We can reconstruct the directed power graph from any power graph.*

Proof. Let $\Gamma = (V, E)$ be a power graph and let $n := |V|$. Since Γ is a power graph, there exists a group G such that $\Gamma = \mathcal{P}(G)$ and $V = G$. If $n = 1$, then $G = 1$; if $n = 2$, then $G \cong C_2$. Hence, in those cases, G is uniquely determined and exhibited. Assume then that $n \geq 3$.

We consider the size of the set $\mathcal{S}$ of star vertices in Γ. By Proposition 6, if $|\mathcal{S}| > 1$, the following three possibilities arise, each of them leading to a unique and exhibited group G.

- $|\mathcal{S}| = n$. In this case, the only possibility is $G \cong C_n$ with n a prime power.
- $|\mathcal{S}| = 1 + \phi(n) \neq n$. Here we have the only possibility $G \cong C_n$, with n not a prime power.

- $|\mathcal{S}| = 2$. Note that we are not in one of the previous cases because $n \geq 3$ implies $2 \neq n$, and $2 \neq 1 + \phi(n)$. Here G is the generalized quaternion group of order n.

We now study the case $|\mathcal{S}| = 1$. Then $\mathcal{S} = \{1\}$, and we recognize which vertex of Γ is the identity element 1 of the group G. This is the genuine interesting case to deal with and it needs the whole machinery of the paper.

Let $\mathcal{K}$ be the partition of $V \setminus \{1\}$ into $\mathtt{N}$-classes. We show that, given a class in $\mathcal{K}$, we can decide if it is of plain or compound type by arithmetical or graph theoretical considerations, without taking into account any group theoretical information.

Pick then $C = [y]_{\mathtt{N}} \in \mathcal{K}$. Assume first that C is critical. Then, by Proposition 19, C is plain if and only if there exists $x \in V \setminus \hat{C}$ such that $|[x]_{\mathtt{N}}| \leq |C|$ and $\{x, y\} \in E$. Assume next that C is not critical. Then, by Proposition 17, C is compound if and only if there exist a prime p and integers $r \geq 2$ and $s \in [r-2]_0$ such that $|C| = p^r - p^s$ and $|\hat{C}| = p^r$.

In $\mathcal{K}$, denote by $\mathcal{K}_{\mathfrak{P}}$ the set of plain classes and by $\mathcal{K}_{\mathfrak{C}}$ the set of compound classes. Of course, we have $\mathcal{K} = \mathcal{K}_{\mathfrak{P}} \cup \mathcal{K}_{\mathfrak{C}}$.

Let $C \in \mathcal{K}_{\mathfrak{C}}$. By Proposition 13, we have parameters (p, r, s) associated with C, where p is a prime, $r \geq 2$ and $s \in [r-2]_0$. Recall that $|C| = p^r - p^s$ and $|\hat{C}| = p^r$ and note that

$$p^r - p^s = \sum_{i=s+1}^{r} (p^i - p^{i-1}) = \sum_{i=s+1}^{r} \phi(p^i).$$

We now partition C, arbitrarily, into $r - s \geq 2$ subsets $X_i(C)$ of sizes $\phi(p^i)$ for $s+1 \leq i \leq r$. Let $\mathcal{K}_C := \{X_i(C) : s+1 \leq i \leq r\}$ be this partition of C.

By Proposition 10, we know that $C \in \mathcal{K}_{\mathfrak{C}}$ also admits the partition $\diamond_C$ given by the $r - s$ $\diamond$-classes of C, and that the sizes of those $\diamond$-classes are $\phi(p^i)$ for $s+1 \leq i \leq r$. Thus $\mathcal{K}_C$ and $\diamond_C$ are two partitions of C with the same number of subsets of the same sizes. As a consequence, there clearly exists $\psi_C \in S_C$ such that $\psi_C(X_i(C))$ is a $\diamond$-class of C of size $\phi(p^i)$ for all $s+1 \leq i \leq r$, and $\diamond_C = \{\psi_C(X) : X \in \mathcal{K}_C\} = \psi_C(\mathcal{K}_C)$.

Next define the partition $\mathcal{K}_\diamond$ of $V \setminus \{1\}$ given by $\mathcal{K}_\diamond := \mathcal{K}_{\mathfrak{P}} \cup \bigcup_{C \in \mathcal{K}_{\mathfrak{C}}} \mathcal{K}_C$. By the above argument, there exists $\psi_{\mathfrak{C}} \in \times_{C \in \mathcal{K}_{\mathfrak{C}}} S_C$ such that $\psi_{\mathfrak{C}}(\bigcup_{C \in \mathcal{K}_{\mathfrak{C}}} \mathcal{K}_C)$ is the partition of the set $\bigcup_{C \in \mathcal{K}_{\mathfrak{C}}} C$ into its $\diamond$-classes. Completing the permutation $\psi_{\mathfrak{C}}$ to a permutation of V which fixes the $\diamond$-classes in $\mathcal{K}_{\mathfrak{P}}$ and 1, we obtain $\psi \in \times_{C \in \mathcal{K} \cup \{\mathcal{S}\}} S_C$ such that $\psi(\mathcal{K}_\diamond)$ is the partition of $V \setminus \{1\}$ in $\diamond$-classes.

Now, by Proposition 1, the group $\times_{C \in \mathcal{K} \cup \{\mathcal{S}\}} S_C$ is a group of automorphisms for the power graph Γ. Hence the above argument proves that $\mathcal{K}_\diamond$ is, up to the graph isomorphism ψ, the partition of $V \setminus \{1\}$ in $\diamond$-classes.

We are now ready to define a set of arcs $A_{\vec{\Gamma}} \subseteq V \times V$. For every $\{x, y\} \in E$, we are going to set $(x, y) \in A_{\vec{\Gamma}}$ or $(y, x) \in A_{\vec{\Gamma}}$ (or both).

Let $\{x, y\} \in E$. Assume first that $1 \in \{x, y\}$, say $y = 1$. Then we set $(x, 1) \in A_{\vec{\Gamma}}$. Assume next that $1 \notin \{x, y\}$. If there exists $C \in \mathcal{K}_\diamond$ such that $x, y \in C$, then we set both $(x, y) \in A_{\vec{\Gamma}}$ and $(y, x) \in A_{\vec{\Gamma}}$. If $x \in X$, $y \in Y$, with $X, Y \in \mathcal{K}_\diamond$ and $X \neq Y$, we make a choice based on the computation of $|X|$ and $|Y|$. If $|X| > |Y|$, then we set $(x, y) \in A_{\vec{\Gamma}}$. Assume next that $|X| = |Y| \neq 1$. Then we also have $|\psi(X)| = |\psi(Y)| \neq 1$ and, by definition of ψ, $\psi(X)$ and $\psi(Y)$ are $\diamond$-classes. Thus Lemma 20 implies that there exists $Z \in \mathcal{K}_\diamond$ such that $\psi(Z) = [\tau]_\diamond$, with $\tau \in G$ an involution and $\psi(Z)$ is joined with exactly one of $\psi(X)$ or $\psi(Y)$. Hence $|Z| = 1$ and, since ψ is a graph isomorphism, Z is joined with exactly one of X or Y. We set $(x, y) \in A_{\vec{\Gamma}}$ if X is joined with Z, and we set $(y, x) \in A_{\vec{\Gamma}}$ if Y is joined with Z. Finally, we suppose, by contradiction, that $|X| = |Y| = 1$. Then, by Lemma 20, one of the $\diamond$-classes $\psi(X)$ and $\psi(Y)$ must be $\mathcal{S} = \{1\}$, contradicting the fact that $\psi(X), \psi(Y) \in \mathcal{K}_\diamond$, a partition of $V \setminus \{1\}$.

We now define $\vec{\Gamma} := (V, A_{\vec{\Gamma}})$. We claim that ψ is a digraph isomorphism between $\vec{\Gamma}$ and the directed power graph $\vec{\mathcal{P}}(G) = (G, A)$. First observe that ψ is, by definition, a bijection between the vertex sets $V = G$ of the two digraphs. We show that $(x, y) \in A_{\vec{\Gamma}}$ if and only if $(\psi(x), \psi(y)) \in A$. Assume first that x, y belong to the same $C \in \mathcal{K}_\diamond$. This happens if and only if $\psi(x)$ and $\psi(y)$ belong to the same $\diamond$-class $\psi(C)$. Hence $(x, y) \in A_{\vec{\Gamma}}$ if and only if $(\psi(x), \psi(y)) \in A$, by the construction above and by Lemma 20 (i). Assume next that $x \in X$, $y \in Y$, with $X, Y \in \mathcal{K}_\diamond$ and $X \neq Y$. This happens if and only if $\psi(x) \in \psi(X)$, $\psi(y) \in \psi(Y)$ with $\psi(X) \neq \psi(Y)$ $\diamond$-classes.

Then, by the definition of $A_{\vec{\Gamma}}$ and by Lemma 20 (iii)–(iv), one of the following holds.

- $|X| > |Y|$ if and only if $|\psi(X)| > |\psi(Y)|$ and hence $(x, y) \in A_{\vec{\Gamma}}$ if and only if $(\psi(x), \psi(y)) \in A$.
- $|X| = |Y| \neq 1$ and there exists $Z \in \mathcal{K}_\diamond$ such that $|Z| = 1$ and Z is joined to X if and only if the same is true when we replace X, Y and Z with, respectively, $\psi(X), \psi(Y)$ and $\psi(Z)$. Thus it follows that $(x, y) \in A_{\vec{\Gamma}}$ if and only if $(\psi(x), \psi(y)) \in A$.

It remains to consider the arcs in $A_{\vec{\Gamma}}$ incident to 1. By the construction of $\vec{\Gamma}$, those arcs are of the form $(x, 1)$ for $x \in G \setminus \{1\}$, and obviously,

$$(\psi(x), \psi(1)) = (\psi(x), 1) \in A \quad \text{for all } x \in G \setminus \{1\}. \qquad \square$$

As a corollary, we immediately deduce the following result. It appeared in [4], as the main theorem, with a step of the proof affected by a mistake, as explained in Section 5.2. Thanks to the Main Theorem, we can completely confirm its validity.

Corollary 21 ([4, Theorem 2]). *If G_1 and G_2 are finite groups whose power graphs are isomorphic, then their directed power graphs are also isomorphic.*

We emphasize that the Main Theorem expresses a stronger result than Corollary 21. In particular, it allows us to give a clear answer to the request, posed in 2022 by P.J. Cameron [5, Question 2], for a “simple” algorithm able to reconstruct the directed power graph of a group from its power graph. Indeed, its proof is constructive and can be explicitly converted into an algorithm. The reader interested in the description of the algorithm and its pseudo-code are referred to the appendix in the arXiv version [3] of this paper. Here, we prefer to illustrate how the algorithm works on some enlightening examples and in the framework of a game. This approach should make clear that the Main Theorem expresses quite a surprising result: we can reconstruct the directed version of a power graph, without any knowledge about the possible groups of which it is the power graph.

6.2 Examples of reconstruction of the directed power graph

Imagine we play mathematics with a friend. She has a finite group G in her hands and computes the power graph $\Gamma := \mathcal{P}(G)$. Then she hides G inside an inaccessible black-box and lets us see only Γ. She challenges us to guess the shape of $\vec{\mathcal{P}}(G)$.

The game is quite boring if the graph shown has a star set with more than one vertex. In that case, we can easily guess what the group G in the black-box is, by Proposition 6, and then obtain its directed power graph.

So suppose she opts for something like $G = \mathrm{D}_{30}$. The graph that she shows us is that in Figure 2, but without colours. We put colours to distinguish the different nature of the vertices. The white vertex is clearly the only star vertex; hence it is the identity of the group. The orange vertices are joined only with the identity, so they are involutions. We split the remaining vertices into $\mathtt{N}$-classes just by examining the edges in the graph. We end up with three $\mathtt{N}$-classes, each containing only vertices of one colour: blue, yellow and magenta. We now want to understand if those classes are of plain or of compound type. Let x, z and y be, respectively, a blue, yellow and magenta vertex. Observing the graph, we are able to compute the neighbourhood closure of $[x]_{\mathtt{N}}$ and $[z]_{\mathtt{N}}$. We obtain

$$\widehat{[x]_{\mathtt{N}}} = [x]_{\mathtt{N}} \cup [y]_{\mathtt{N}} \cup \{1\} \quad \text{and} \quad \widehat{[z]_{\mathtt{N}}} = [z]_{\mathtt{N}} \cup [y]_{\mathtt{N}} \cup \{1\};$$

hence $[x]_{\mathtt{N}}$ and $[z]_{\mathtt{N}}$ are not critical classes. Since we have $|[x]_{\mathtt{N}}| = 4$, $|\widehat{[x]_{\mathtt{N}}}| = 12$, $|[z]_{\mathtt{N}}| = 2$, $|\widehat{[z]_{\mathtt{N}}}| = 10$, neither $[x]_{\mathtt{N}}$ nor $[z]_{\mathtt{N}}$ are of compound type, by Proposition 13. Now that the simple cases are treated, let us focus on the class $[y]_{\mathtt{N}}$ of magenta vertices. As before, we compute the neighbourhood closure obtaining $\widehat{[y]_{\mathtt{N}}} = [y]_{\mathtt{N}} \,\dot\cup\, \{1\}$. Evaluating $|\widehat{[y]_{\mathtt{N}}}| = 9 = 3^2$ as well, we see that $[y]_{\mathtt{N}}$ is a critical class, just by recalling Definition 16. In order to understand the nature of $[y]_{\mathtt{N}}$, we look outside $\widehat{[y]_{\mathtt{N}}}$. It is straightforward to check the following facts for x, one of the blue vertices:

(1) $x \in G \setminus \widehat{[y]_{\mathtt{N}}}$;

(2) $4 = |[x]_{\mathtt{N}}| \leq |[y]_{\mathtt{N}}| = 8$;

(3) $\{x, y\} \in E$.

Therefore, by Proposition 19, we deduce that $[y]_{\mathtt{N}}$ is of plain type. Summarizing, we have $[x]_{\mathtt{N}} = [x]_{\diamond}$, $[z]_{\mathtt{N}} = [z]_{\diamond}$ and $[y]_{\mathtt{N}} = [y]_{\diamond}$.

We now have the partition formed by $[x]_{\diamond}$, $[z]_{\diamond}$, $[y]_{\diamond}$ and a further sixteen $\diamond$-classes, each containing the identity or an involution. That is the partition of V in $\diamond$-classes.

The game is not over yet, because we need to exhibit a digraph, but we are in the home stretch. We replace all the edges joining a vertex with the identity with arcs of the type $(v, 1)$ for $v \in V \setminus \{1\}$. For all the other edges, we follow Lemma 20. In particular, we use (i) and (iii). Edges between vertices of the same $\diamond$-class are replaced by two arcs, one for both the directions. The remaining edges are those between the magenta vertices and the blue or yellow vertices. Without hesitating, we replace all those edges with arcs starting in the magenta vertices and ending in the blue or yellow vertices, because $|[y]_{\mathtt{N}}| = 8$, $|[x]_{\mathtt{N}}| = 4$ and $|[z]_{\mathtt{N}}| = 2$ hold.

Our friend asks for a rematch. This time she prepares for us the graph $\mathcal{P}(\mathrm{D}_{18})$ in Figure 3 (a). As before, imagine it without colours. This time a delicate situation, which we did not face in the previous case, arises. The white and orange vertices are, as before, the identity and the involutions. For the remaining vertices, it is easily checked, by a study of the closed neighbourhoods, that they belong to a unique $\mathtt{N}$-class C. Since $\hat{C} = C \cup \{1\}$ and $|\hat{C}| = 9$, we have that C is a critical class. Note that we have $|C| = 8 = 3^2 - 1$ and $|\hat{C}| = 9 = 3^2$ as in the previous case. Without too much effort, it is checked that there exists no $x \in G \setminus \hat{C}$ joined with a vertex of C, and hence, by Proposition 19, C is of compound type.

As a consequence, there exists $y \in C$ with $o(y) = 3^2$ such that

$$C = \{z \in \langle y \rangle : 3 \leq o(z) \leq 3^2\}.$$

Here we face the delicate situation. We have seen that knowing the partition in $\diamond$-classes allows us to use Lemma 20. But, whenever a compound class appears, we cannot directly see the $\diamond$-partition. However, we can find it up to a graph isomorphism. We arbitrarily partition the vertices of C in two sets, one formed by 6 elements and one by 2 elements. In Figure 3 (a), such a partition is revealed by the two colours magenta and blue. In C, the partition in $\diamond$-classes has exactly two sets with the same sizes as those in our partition. By an argument in the proof of

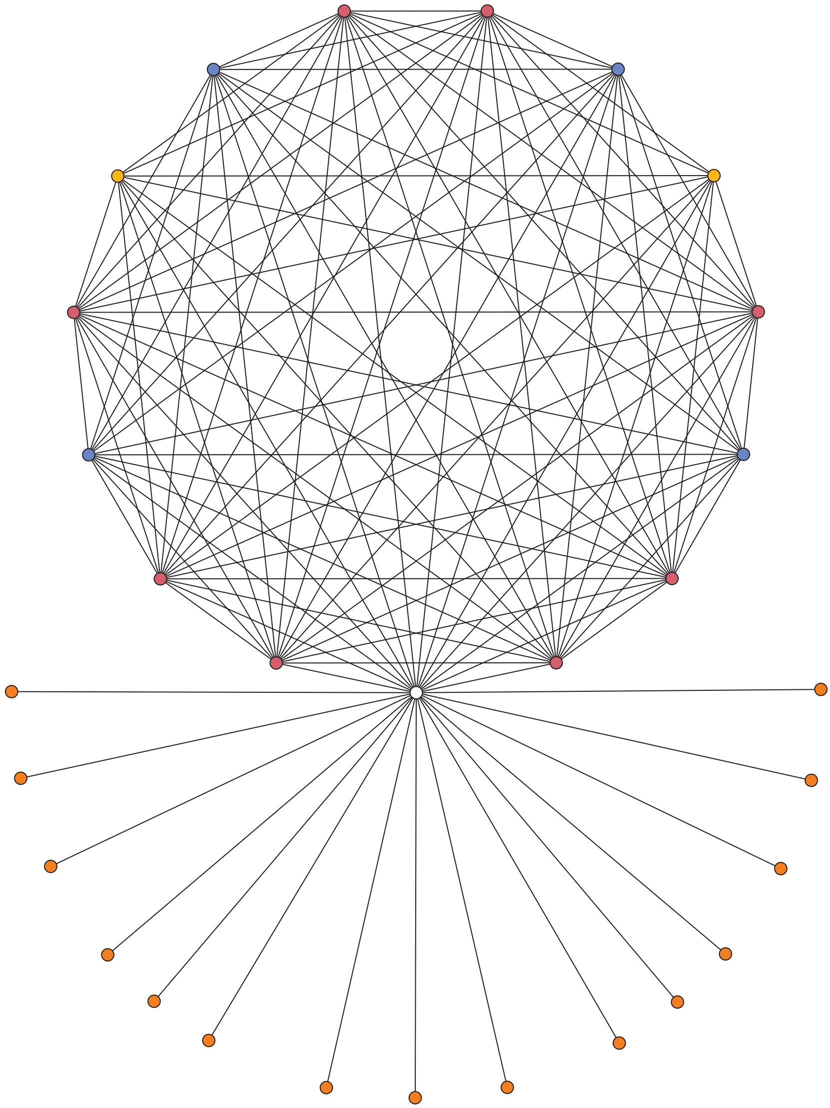
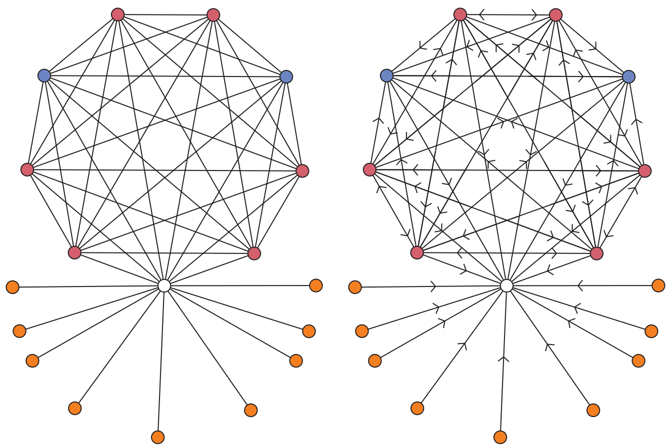


Figure 2. $\mathcal{P}(\mathrm{D}_{30})$

the Main Theorem, our arbitrary partition of the vertices in C is, up to a graph isomorphism, just the partition of C into $\diamond$-classes.

We now have, up to isomorphism, the partition of G into $\diamond$-classes. It is formed by the sets of the partition above and the singletons containing each an involution or 1. Mimicking the instructions in Lemma 20, we now assign the directions on



(a) $\mathcal{P}(D_{18})$ (b) $\vec{\mathcal{P}}(D_{18})$

Figure 3. Example of reconstruction of the directed power graph from a power graph: elements of the same order have the same colour.

the edges, obtaining the digraph shown in Figure 3 (b). By the Main Theorem, that directed graph is just $\vec{\mathcal{P}}(D_{18})$.

Acknowledgments. We wish to thank an anonymous referee for the care in dealing with our paper and the many suggestions for improving clarity and readability. We also wish to thank Peter Cameron for his encouragement to deepen the topics of his paper [4], Dikran Dikranjan, José Cáceres and Josef Šlapal for illuminating conversations about Moore closure operators for graphs.

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Received September 28, 2023; revised October 13, 2024

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