ON THE GEODESIC PROBLEM FOR THE DIRICHLET METRIC AND THE EBIN METRIC ON THE SPACE OF SASAKIAN METRICS

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Abstract. We study the geodesic equation for the Dirichlet (gradient) metric in the space of Kähler potentials. We first solve the initial value problem for the geodesic equation of the *combination metric*, including the gradient metric. We then discuss a comparison theorem between it and the Calabi metric. As geometric motivation of the combination metric, we find that the Ebin metric restricted to the space of type II deformations of a Sasakian structure is the sum of the Calabi metric and the gradient metric.

Contents

Introduction

This is the sequel of the previous paper [13] on the *Dirichlet metric*, which here will be called gradient metric. We recall the background briefly. The idea of defining

a Riemannian structure on the space of all metrics on a fixed manifold goes back
to the sixties with the work of Ebin [19]. His work concerns the pure Riemannian
setting and, among other things, defines a weak Riemannian metric on the space
$\mathcal{M}$ of all Riemannian metrics on a fixed compact Riemannian manifold (M, g). The
geometry of the Hilbert manifold $\mathcal{M}$ was later studied by Freed and Groisser in [21]
and Gil-Medrano and Michor in [22]. In particular the curvature and the geodesics
of $\mathcal{M}$ were computed.

Let (M, ω) be a compact Kähler manifold. The space $\mathcal{H}$ of Kähler metrics
cohomologous to ω is isomorphic to the space of the Kähler potentials modulo
constants. It can be endowed with three different metrics, known as the *Donaldson-
Mabuchi-Semmes* L^2-metric (1.2), the *Calabi* metric (1.5) and the *Dirichlet (or
gradient)* metric (1.8).

The Calabi metric goes back to Calabi [10] and it was later studied by the first
author in [11] where its Levi-Civita covariant derivative is computed, it is proved
that it is of constant sectional curvature, that $\mathcal{H}$ is then isometric to a portion of a
sphere in $C^\infty(M)$ and that both the Dirichlet problem (find a geodesic connecting
two fixed points) and the Cauchy problem (find a geodesic with assigned starting
point and speed) admit smooth explicit solutions.

The gradient metric was introduced and studied in [11, 13]. Its Levi-Civita
connection, geodesic equation and curvature are written down in [13]. In this paper,
we continue to study its geometry. We solve the Cauchy problem of its geodesic
equation, so we prove it is locally well-posed, unlike the corresponding problem for
the L^2 metric, which is known to be ill-posed.

Actually, we define a more general metric, the linear combination of the three
metrics on $\mathcal{H}$ we call *combination metric* whose special instance is the *sum metric*,
i.e. the sum of the gradient and Calabi metrics.

We denote the Hölder spaces with respect to the fixed Kähler metric g by $C^{k,\alpha}(g)$.
We prove that our Cauchy problem is well-posed (See Thm. 2.2, 2.8 and 2.15).

Theorem 0.1. *On a compact Kähler manifold, for every initial Kähler potential*
φ_0, *and initial speed* ψ_0 *in* $C^{k,\alpha}(g)$, *for all* $k \geq 6$ *and* $\alpha \in (0, 1)$, *there exists, for a
small time* T, *a unique* $C^2([0, T], C^{k,\alpha}(g) \cap \mathcal{H})$ *geodesic for the combination metric,
starting from* φ_0 *with initial velocity* ψ_0. *Moreover if* (φ_0, ψ_0) *are smooth then also
the solution is.*

Furthermore, we prove a Rauch type comparison theorem of the Jacobi fields
(Theorem 2.16) between the gradient metric and the Calabi metric.

Theorem 0.2. *Let* γ_G *and* γ_C *be two geodesics of equal length with respect to
the gradient metric and the Calabi metric respectively and suppose that for every*
$X_G \in T_{\gamma_G(t)}\mathcal{H}$ *and* $X_C \in T_{\gamma_C(t)}\mathcal{H}$, *we have*

$$K_G(X_G, \gamma_G'(t)) \leq K_C(X_C, \gamma_C'(t)).$$

Let J_G *and* J_C *be the Jacobi fields along* γ_G *and* γ_C *such that*

- $J_G(0) = J_C(0) = 0$,
- $J_G'(0)$ *is orthogonal to* $\gamma_G'(0)$ *and* $J_C'(0)$ *is orthogonal to* $\gamma_C'(0)$,
- $\|J_G'(0)\| = \|J_C'(0)\|$.

then we have, for all $t \in [0, T]$,

$$\|J_G(t)\| \geq \frac{\left| \sin\left(2t\sqrt{\mathrm{vol}} \right) \right|}{\sqrt{\mathrm{vol}}}.$$

The sum metric arises from Sasakian geometry. Indeed the geometric motivation comes naturally from the space of Sasakian metrics $\mathcal{H}_S$ as follows.

Since $\mathcal{H}_K$ naturally embeds in the Ebin space $\mathcal{M}$, it is natural to ask what the restriction of the Ebin metric is. To our knowledge, the restriction of Ebin metric to subspaces of the space of Riemannian metrics was first considered by [35, (9.19), page 2485] (for the space of Kähler metrics, see [15]). In this paper we consider on $\mathcal{H}_K$ the metric given by (twice) the sum of the Calabi and the gradient metric and we will refer to it as the sum metric. Its study is justified by the fact that it arises when restricting the Ebin metric to the space of *Sasakian* metrics, introduced (and endowed with the Sasakian analogue of the Mabuchi metric) in [24, 25].

One of our results is the following.

Proposition 0.3. *The restriction of the Ebin metric of $\mathcal{M}$ to the space of Sasakian metrics is twice the sum metric.*

Moreover, Theorem 0.1 can be generalized to the Sasakian setting, leading to the corresponding statement for the restriction of the Ebin metric to the space of Sasakian metrics.

The paper is organized as follows. In section 1 we recall the main definition of the space of Kähler metrics and in section 2 we write down the Levi-Civita connection of the combination metric and study the equation of the Cauchy problem for gradient metric. Finally, in section 3 we compute the restriction of the Ebin metric on the space of Sasakian metrics, proving Prop. 0.3.

1. Preliminaries

In this section we recall the definitions of space of Riemannian and Kähler metrics and several weak Riemannian structures on them.

1.1. Ebin metric.

1.1. Ebin metric. The space of the Riemmanian metrics $\mathcal{M}$ is identified with the space $S^2_+(T^*M)$ of all symmetric positive $(0,2)$-tensors on M. The formal tangent space at a metric $g \in \mathcal{M}$ is then given by all symmetric $(0,2)$-tensors $S^2(T^*M)$. For $a, b \in T_g\mathcal{M}$, the Ebin [19] metric is defined as the pairing

$$g_E(a,b)_g = \int_M g(a,b)dv_g$$

where $g(a,b)$ is the metric g extended to $(0,2)$-tensors and dv_g is the volume form of g. From e.g. [22] one can see that the curvature is non-positive and the geodesic satisfies the equation

$$g_{tt} = g_t g^{-1} g_t + \frac{1}{4} \operatorname{tr}(g^{-1} g_t g^{-1} g_t) g - \frac{1}{2} \operatorname{tr}(g^{-1} g_t) g_t.$$

Moreover in [22] the explicit expression of the Cauchy geodesics is given.

1.2. Space of Kähler potentials. Moving on to Kähler manifolds, let (M, ω, g) be a compact Kähler manifold of complex dimension n, with ω a Kähler form and g the associated Kähler metric. By the $\partial\overline{\partial}$-Lemma, the space of all Kähler metrics cohomologous to ω can be parameterized by Kähler potentials; namely, one considers the space $\mathcal{H}$ of all smooth real-valued φ such that

$$\omega_\varphi := \omega + i\partial\overline{\partial}\varphi > 0$$

and satisfy the normalization condition [17]

$$(1.1) \qquad I(\varphi) := \int_M \varphi \frac{\omega^n}{n!} - \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \partial\varphi \wedge \overline{\partial}\varphi \wedge \frac{\omega^i}{i!} \wedge \frac{\omega_\varphi^{n-1-i}}{(n-1-i)!} = 0.$$

The tangent space of $\mathcal{H}$ at φ is then given by

$$T_\varphi\mathcal{H} = \left\{ \psi \in C^\infty(M) : \int_M \psi \frac{\omega_\varphi^n}{n!} = 0 \right\}.$$

1.3. Donaldson-Mabuchi-Semmes's L^2-metric. Donaldson, Mabuchi and Semmes [17, 30, 33] defined a pairing on the tangent space of $\mathcal{H}$ at φ given by

$$(1.2) \qquad g_M(\psi_1, \psi_2)_\varphi = \int_M \psi_1 \psi_2 \frac{\omega_\varphi^n}{n!}.$$

We shall refer to this metric as the L^2-metric. It makes $\mathcal{H}$ a non-positively curved, locally symmetric space. A geodesic φ satisfies

$$(1.3) \qquad \varphi'' - \frac{1}{2}|d\varphi'|^2_\varphi = 0$$

where $|d\varphi|^2_\varphi$ denotes the square norm of the gradient of φ' with respect to the metric ω_φ. The geodesic equation can be written down as a degenerate complex Monge-Ampère equation. It was proved by Chen [14] that there is a $C^{1,1}$ solution for the Dirichlet problem. More work on this topic was done in [1, 3, 6, 12, 16, 18, 29, 32], which is far from a complete list.

1.4. **Space of conformal volume forms.** According to the Calabi-Yau theorem, there is a bijection between $\mathcal{H}$ and the space of *conformal volume forms*

$$(1.4) \qquad \mathcal{C} = \left\{ u \in C^\infty(M) : \int_M e^u \frac{\omega^n}{n!} = \text{vol} \right\}$$

that is the space of positive smooth functions on M whose integral with respect to the initial measure is equal to the volume of M (which is constant for all metrics in $\mathcal{H}$). The map is given by

$$\mathcal{H} \ni \varphi \mapsto \log \frac{\omega_\varphi^n}{\omega_0^n},$$

where $\frac{\omega_\varphi^n}{\omega_0^n}$ represents the unique positive function f such that $\omega_\varphi^n = f \omega_0^n$. The tangent space $T_u \mathcal{C}$ is then given by

$$T_u \mathcal{C} = \left\{ v \in C^\infty(M) : \int_M v e^u \frac{\omega^n}{n!} = 0 \right\}.$$

1.5. **Calabi metric.** Calabi [10] introduced the now known *Calabi metric* as the pairing

$$(1.5) \qquad g_C(\psi_1, \psi_2)_\varphi = \int_M \Delta_\varphi \psi_1 \Delta_\varphi \psi_2 \frac{\omega_\varphi^n}{n!}$$

where, here and in the rest of the paper, the Laplacian is defined as

$$\Delta_\varphi f = (i \partial \overline{\partial} f, \omega_\varphi)_\varphi$$

i.e. the $\overline{\partial}$-Laplacian. The geometry studied in [11] is actually the one of $\mathcal{C}$, where the Calabi metric has the simpler form

$$(1.6) \qquad g_C(v_1, v_2)_u = \int_M v_1 v_2 e^u \frac{\omega^n}{n!}.$$

Back in $\mathcal{H}$, the geodesic equation is

$$(1.7) \qquad \Delta_\varphi \varphi'' - |i \partial \overline{\partial} \varphi'|_\varphi^2 + \frac{1}{2} (\Delta_\varphi \varphi')^2 + \frac{1}{2\,\text{vol}} g_C(\varphi', \varphi') = 0.$$

1.6. **Dirichlet metric.** In [11, 13], the *Dirichlet (or gradient) metric* is defined as the pairing

$$(1.8) \qquad g_G(\psi_1, \psi_2)_\varphi = \int_M (d\psi_1, d\psi_2)_\varphi d\mu_\varphi$$

that is, the global $L^2(d\mu_\varphi)$-product of the gradients of ψ_1 and ψ_2. Its geodesic equation is

$$(1.9) \qquad 2\Delta_\varphi \varphi'' - |i \partial \overline{\partial} \varphi'|_\varphi^2 + (\Delta_\varphi \varphi')^2 = 0$$

where $|i \partial \overline{\partial} \varphi'|_\varphi^2$ denotes the square norm with respect to ω_φ of the $(1,1)$-form $i \partial \overline{\partial} \varphi'$.

2. Combination metrics

We can combine together the three metrics as follows. Let α, β, γ be three non-negative constant and at least one of them positive. Consider the metric

$$(2.1) \qquad g(\psi_1, \psi_2)_\varphi = \alpha \cdot g_M(\psi_1, \psi_2)_\varphi + \beta \cdot g_G(\psi_1, \psi_2)_\varphi + \gamma \cdot g_C(\psi_1, \psi_2)_\varphi.$$

which will be referred to as the *combination metric*.

Let us prove the existence of the Levi-Civita covariant derivative for the combination metric. We can write

$$(2.2) \qquad g(\psi_1, \psi_2)_\varphi = g_C(M_\varphi \psi_1, \psi_2).$$

where

$$M_\varphi = \alpha G_\varphi^2 - \beta G_\varphi + \gamma$$

where G_φ is the Green operator associated to the Laplacian Δ_φ.

We have the following.

Proposition 2.1. *For a curve $\varphi \in \mathcal{H}$ and a section v on it, the Levi-Civita covariant derivative of the combination metric is the unique $D_t v$ that solves*

$$M_\varphi D_t \psi = [G_\varphi^2 \alpha D_t^M - \beta G_\varphi D_t^G + \gamma D_t^C]\psi$$

where D^M, D_t^G, D_t^C are the covariant derivatives of the L^2, gradient and Calabi metric.

Proof. We start by proving that M_φ is a bijection of $T_\varphi \mathcal{H}$. The injectivity holds because it defines a metric. To prove surjectivity, we see that the problem $M_\varphi u = f$ is equivalent to $Du = h$ where $D = \gamma \Delta_\varphi^2 - \beta \Delta_\varphi + \alpha$. It is elliptic and then by known results we have

$$C^\infty(M) = \ker D \oplus \text{Im}(D)$$

and by integration and the normalization condition on $T_\varphi \mathcal{H}$ we immediately see that $T_\varphi \mathcal{H} \cap \ker D = 0$, so $T_\varphi \mathcal{H} = \text{Im}(D) \cap T_\varphi \mathcal{H}$ and we obtain surjectivity.

The fact that D_t is torsion-free is evident from its definition. Let us now compute

$$\begin{aligned}
\frac{d}{dt} g(\psi, \psi) &= 2\alpha g_M(D_t^M \psi, \psi) + 2\beta g_G(D_t^G \psi, \psi) + 2\gamma g_C(D_t^C \psi, \psi) \\
&= 2\alpha g_C(G_\varphi^2 D_t^M \psi, \psi) - 2\beta g_C(G_\varphi D_t^G \psi, \psi) + 2\gamma g_C(D_t^C \psi, \psi) \\
&= 2g_C([G_\varphi^2 \alpha D_t^M - \beta G_\varphi D_t^G + \gamma D_t^C]\psi, \psi) \\
&= 2g_C(M_\varphi D_t \psi, \psi) \\
&= 2g(D_t \psi, \psi)
\end{aligned}$$

so the compatibility with the metric holds as well. $\qquad\qquad\square$

2.1. Geodesic equation of the combination metric.
The geodesic equation of the combination metric is the combination of the geodesic equations of L^2-metric, gradient metric and the Calabi metric. After rearrangement, it is written in the following form

(2.3)

$$[\alpha - \beta \Delta_\varphi + \gamma \Delta_\varphi^2]\varphi'' = \frac{\alpha}{2}|d\varphi'|_\varphi^2 + \left[\frac{\beta}{2} - \gamma \Delta_\varphi\right]|i\partial\overline{\partial}\varphi'|_\varphi^2 + \left[\frac{\beta}{2} + \frac{\gamma}{2}\Delta_\varphi\right](\Delta_\varphi \varphi')^2.$$

The key observation is that the differential order on the both sides of the geodesic equation (2.3) are the same. We will carry out in detail in the next section the study

of the geodesic equation with $\beta = \gamma = 1$ and $\alpha = 0$, the general case with $\alpha = 1$ is similar, so we omit the proof.

This observation suggest that, though the Cauchy problem of the geodesic ray with respect to the L^2-metric is ill-posed, after combining the L^2-metric with the Calabi metric and the gradient metric, the new geodesic equation is well-posed.

2.2. **Local well-posedness of the geodesic equation.**

2.2.1. *Existence and uniqueness.* Recall the definition of the space of Kähler potentials

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega + i\partial\overline{\partial}\varphi > 0, I(\varphi) = 0\}.$$

We are aiming to solve the geodesic equation with $\beta = \gamma = 1$ and $\alpha = 0$, i.e. the equation

$$(2.4) \qquad (\Delta_\varphi - I)\left((\Delta_\varphi \varphi')' + \frac{1}{2}(\Delta_\varphi \varphi')^2\right) - \frac{1}{2}|i\partial\overline{\partial}\varphi'|_\varphi^2 = 0.$$

We rewrite it as a system

$$(2.5) \qquad \begin{cases} \varphi' = \psi \\ \psi' = L_\varphi(\psi) := \Delta_\varphi^{-1}\left[\frac{1}{2}(\Delta_\varphi - I)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2 + |i\partial\overline{\partial}\psi|_\varphi^2 + \frac{1}{2}(\Delta_\varphi \psi)^2\right] \end{cases}$$

with the initial data $\varphi(0) = \varphi_0, \psi(0) = \psi_0 \in C^{k,\alpha}(g)$.

Take a constant $\delta > 0$ such that $\omega + i\partial\overline{\partial}\varphi_0 \geq 2\delta\omega$. Let us introduce also the following function spaces

$$\mathcal{H}^{k,\alpha} = \{\varphi \in C^{k,\alpha}(g) : \omega + i\partial\overline{\partial}\varphi > 0, I(\varphi) = 0\}$$

and

$$\mathcal{H}_\delta^{k,\alpha} = \{\varphi \in C^{k,\alpha}(g) : \omega + i\partial\overline{\partial}\varphi \geq \delta\omega, I(\varphi) = 0\},$$

where $k \geq 2$ and $\alpha \in (0,1)$.

The aim of this subsection is to prove the following.

Theorem 2.2. *For every integer $k \geq 6$ and $\alpha \in (0,1)$ and initial data $\varphi_0 \in \mathcal{H}_\delta^{k,\alpha}$ and $\psi_0 \in T_{\varphi_0}\mathcal{H}^{k,\alpha}$ there exists a positive ε and a curve $\varphi \in C^2((-\varepsilon, \varepsilon), \mathcal{H}_\delta^{k,\alpha})$ which is the unique solution of (2.4) with initial data (φ_0, ψ_0).*

We need the following lemma.

Lemma 2.3 (Schauder estimates, see [4, p. 463]). *Let P be an elliptic linear operator of order 2 acting on the Hölder space $C^{k+2,\alpha}(g)$. Then for $u \in C^{k+2,\alpha}(g)$ we have*

$$\|u\|_{C^{k+2,\alpha}(g)} \leq c_1\|Pu\|_{C^{k,\alpha}(g)} + c_2\|u\|_{L^\infty}$$

where c_1 depends only on the $C^{k,\alpha}(g)$-norm of the coefficients of P and, if u is $L^2(g)$-orthogonal to $\ker P$, then $c_2 = 0$.

The structure of the system (2.5) suggests to consider the following complete metric space

$$(2.6) \qquad X = C^2([-\varepsilon, \varepsilon], \mathcal{H}_\delta^{k,\alpha}) \times C^2([-\varepsilon, \varepsilon], C^{k,\alpha}(g))$$

as the function space where we are going to look for solutions of our system. The norm that we consider is defined for $\psi \in C^2([-\varepsilon, \varepsilon], C^{k,\alpha}(g))$ as

$$|\psi|_{k,\alpha} := \sup_{t \in [-\varepsilon, \varepsilon]} \|\psi(t, \cdot)\|_{C^{k,\alpha}(g)},$$

and in the product space, the norm of any element $(\varphi, \psi) \in X$ is

$$|(\varphi, \psi)|_{k,\alpha} := |\varphi|_{k,\alpha} + |\psi|_{k,\alpha}.$$

We work in an appropriate metric ball in X obtained by the following lemma.

Lemma 2.4. *There exists $r > 0$ such that if $\varphi \in C^{k,\alpha}(g)$ is such that $\|\varphi - \varphi_0\|_{C^{k,\alpha}(g)} < r$ then $\varphi \in \mathcal{H}_\delta^{k,\alpha}$.*

Proof. Being $k \geq 2$ we have $\|\varphi - \varphi_0\|_{C^{2,\alpha}(g)} \leq \|\varphi - \varphi_0\|_{C^{k,\alpha}(g)} < r$. Then

$$\begin{aligned}
g_\varphi &= g_\varphi - g_{\varphi_0} + g_{\varphi_0} \\
&\geq -\|\varphi - \varphi_0\|_{C^{2,\alpha}(g)} g + 2\delta g \\
&\geq (2\delta - r)g
\end{aligned}$$

which is strictly bigger than δg for $r < \delta$. $\qquad\square$

We consider the operator

$$(2.7) \qquad T(\varphi, \psi) = \left(\varphi_0 + \int_0^t \psi(s)ds, \psi_0 + \int_0^t (L_\varphi(\psi))(s)ds \right).$$

Let us now fix $r > 0$ as in Lemma 2.4. We have the following proposition, but first let us isolate a lemma.

Lemma 2.5. *There exist a positive C depending only on r and g such that*

$$\|g_\varphi^{a\bar{b}}\|_{C^{k,\alpha}(g)} \leq C.$$

Proof. For fixed a, b it holds $\|g_\varphi^{a\bar{b}}\|_{C^{k,\alpha}(g)} \leq \|g_\varphi^{-1}\|_{C^{k,\alpha}(g)}$ where the norm is intended as operator norm. Then by the sub-multiplicative property we have $\|g_\varphi^{-1}\|_{C^{k,\alpha}(g)} \leq \|g_\varphi\|_{C^{k,\alpha}(g)}^{-1}$ and by estimate in the proof of Lemma 2.4 we have that $\|g_\varphi\|_{C^{k,\alpha}(g)}^{-1} \leq (2\delta - r)^{-1}\|g\|_{C^{k,\alpha}(g)} =: C(r, g)$. $\qquad\square$

Proposition 2.6. *For any $(\varphi_0, \psi_0) \in \mathcal{H}_\delta^{k,\alpha} \times C^{k,\alpha}(g)$ there exists $\varepsilon > 0$ such that the metric ball $B_r(\varphi_0, \psi_0) \subset X$ centered in (φ_0, ψ_0) of radius r is mapped into itself by T.*

Proof. We need to estimate $|T(\varphi_0, \psi_0) - (\varphi_0, \psi_0)|_{k,\alpha}$. Let us estimate the first component

$$\begin{aligned}
\left| \varphi_0 + \int_0^t \psi(s)ds - \varphi_0 \right|_{k,\alpha} &= \sup_{t \in [-\varepsilon, \varepsilon]} \left\| \int_0^t \psi(s)ds \right\|_{C^{k,\alpha}(g)} \\
&\leq \sup_{t \in [-\varepsilon, \varepsilon]} \int_0^t \|\psi(s)\|_{C^{k,\alpha}(g)} ds \\
&\leq \sup_{t \in [-\varepsilon, \varepsilon]} \int_0^t \sup_{s \in [-\varepsilon, \varepsilon]} \|\psi(s)\|_{C^{k,\alpha}(g)} ds \\
&\leq \varepsilon \cdot (|\psi_0|_{k,\alpha} + |\psi - \psi_0|_{k,\alpha}) \\
&\leq \varepsilon \cdot (|\psi_0|_{k,\alpha} + r).
\end{aligned}$$

As for the second component, it is clear it is enough to estimate $\|L_\varphi(\psi)\|_{C^{k,\alpha}(g)}$ for every t.

We have, by Lemma 2.3,

$$\|L_\varphi(\psi)\|_{C^{k,\alpha}(g)} \le \left\|\Delta_\varphi^{-1}\left[\frac{1}{2}(\Delta_\varphi - I)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2 + |i\partial\overline{\partial}\psi|_\varphi^2 + \frac{1}{2}(\Delta_\varphi\psi)^2\right]\right\|_{C^{k,\alpha}(g)}$$

$$\le C(\|\varphi\|_{C^{k,\alpha}(g)})\left\|\frac{1}{2}(\Delta_\varphi - I)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2 + |i\partial\overline{\partial}\psi|_\varphi^2 + \frac{1}{2}(\Delta_\varphi\psi)^2\right\|_{C^{k-2,\alpha}(g)}.$$

To estimate the first summand we have

$$\left\|\frac{1}{2}(\Delta_\varphi - I)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2\right\|_{C^{k-2,\alpha}(g)} \le C(\|\varphi\|_{C^{k-2,\alpha}(g)})\||i\partial\overline{\partial}\psi|_\varphi^2\|_{C^{k-4,\alpha}(g)}$$

$$\le C(r)\|g_\varphi^{i\overline{j}}g_\varphi^{k\overline{\ell}}\psi_{i\overline{\ell}}\psi_{k\overline{j}}\|_{C^{k-4,\alpha}(g)}$$

$$\le C(r)\|\psi\|_{C^{k,\alpha}(g)}$$

where in the first inequality we have used again Lemma 2.3 and in the last we have used that $\|\psi\|_{C^{k-2,\alpha}(g)} \le \|\psi\|_{C^{k,\alpha}(g)} < r$.

The second summand is estimated, similarly as before, by

$$\||i\partial\overline{\partial}\psi|_\varphi^2\|_{C^{k-2,\alpha}(g)} \le C(r)\|\psi\|_{C^{k,\alpha}(g)}.$$

The third summand is

$$\left\|\frac{1}{2}(\Delta_\varphi\psi)^2\right\|_{C^{k-2,\alpha}(g)} \le \|(\Delta_\varphi\psi)^2\|_{C^{k-2,\alpha}(g)} \le \|g_\varphi^{i\overline{j}}\psi_{i\overline{j}}\|_{C^{k-2,\alpha}(g)}^2$$

$$\le C(r)\|\psi\|_{C^{k,\alpha}(g)}.$$

So we can conclude that the second component of $|T(\varphi_0, \psi_0) - (\varphi_0, \psi_0)|_{k,\alpha}$ is estimated by $\varepsilon C(r)|\psi - \psi_0|_{k,\alpha} \le \varepsilon r C(r)$, so it is enough to choose $\varepsilon(r)$ such that $\varepsilon(r)C(r) < 1$. $\qquad\square$

Our second step is the following.

Proposition 2.7. *The map T on the metric ball $B_r(\varphi_0, \psi_0)$ is a contraction.*

Proof. For (φ, ψ) and $(\widetilde{\varphi}, \widetilde{\psi})$ in $B_r(\varphi_0, \psi_0)$, let for simplicity $\widetilde{L} = L_{\widetilde{\varphi}}$. We need to estimate $\|L(\psi) - \widetilde{L}(\widetilde{\psi})\|_{C^{k,\alpha}(g)}$. Define f and $\widetilde{f}$ such that $L(\psi) = \Delta_\varphi^{-1}f$ and $\widetilde{L}(\widetilde{\psi}) = \Delta_{\widetilde{\varphi}}^{-1}\widetilde{f}$. Then we have

$$\Delta_\varphi(L(\psi) - \widetilde{L}(\widetilde{\psi})) = f - \widetilde{f} - \Delta_\varphi\widetilde{L}(\widetilde{\psi}) + \Delta_{\widetilde{\varphi}}\widetilde{L}(\widetilde{\psi})$$

$$= f - \widetilde{f} + (g_{\widetilde{\varphi}}^{i\overline{j}} - g_\varphi^{i\overline{j}})(\widetilde{L}(\widetilde{\psi}))_{i\overline{j}}$$

so by the Schauder estimates of Lemma 2.3 we have

$$\|L(\psi) - \widetilde{L}(\widetilde{\psi})\|_{C^{k,\alpha}(g)} \le C(\|\varphi\|_{C^{k,\alpha}(g)})$$

$$\cdot \left(\|f - \widetilde{f}\|_{C^{k-2,\alpha}(g)} + \|\Delta_{\widetilde{\varphi}}\widetilde{L}(\widetilde{\psi}) - \Delta_\varphi\widetilde{L}(\widetilde{\psi})\|_{C^{k-2,\alpha}(g)}\right).$$

To estimate the second summand, let $g_s = (1-s)g_\varphi + sg_{\widetilde{\varphi}}$. Then we notice we can write

$$\Delta_{\widetilde{\varphi}}\widetilde{L}(\widetilde{\psi}) - \Delta_\varphi\widetilde{L}(\widetilde{\psi}) = -\left(\int_0^1 g_s^{i\overline{\ell}}g_s^{k\overline{j}}ds\right)\cdot(\widetilde{\varphi} - \varphi)_{k\overline{\ell}}\cdot(\widetilde{L}(\widetilde{\psi}))_{i\overline{j}}.$$

so we have

$$\|\Delta_{\widetilde{\varphi}}\widetilde{L}(\widetilde{\psi}) - \Delta_\varphi\widetilde{L}(\widetilde{\psi})\|_{C^{k-2,\alpha}(g)} \le C(\|\varphi\|_{C^{k,\alpha}(g)}, \|\widetilde{\varphi}\|_{C^{k,\alpha}(g)})\|\widetilde{\varphi} - \varphi\|_{C^{k,\alpha}(g)}\cdot\|\widetilde{L}\widetilde{\psi}\|_{C^{k,\alpha}(g)}$$

$$\le C(r)\|\widetilde{\varphi} - \varphi\|_{C^{k,\alpha}(g)}$$

where in the last inequality we have used the estimate for $\|\widetilde{L}\widetilde{\psi}\|_{C^{k,\alpha}(g)}$ from the previous proposition.

Let us now consider $\widetilde{f} - f$ which can be written as

$$(2.8) \qquad \widetilde{f} - f = \frac{1}{2}(\Delta_\varphi - 1)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2 - \frac{1}{2}(\Delta_{\widetilde{\varphi}} - 1)^{-1}|i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2$$
$$+ |i\partial\overline{\partial}\psi|_\varphi^2 - |i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2$$
$$- \frac{1}{2}(\Delta_\varphi\psi)^2 + \frac{1}{2}(\Delta_{\widetilde{\varphi}}\widetilde{\psi})^2.$$

Let $h - \widetilde{h}$ be the first summand, so we can write

$$(\Delta_\varphi - 1)(h - \widetilde{h}) = |i\partial\overline{\partial}\psi|_\varphi^2 - |i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2 + (\Delta_{\widetilde{\varphi}} - \Delta_\varphi)\widetilde{h}.$$

Again by Lemma 2.3 we have

$$\|h - \widetilde{h}\|_{C^{k-2,\alpha}(g)} \le C(\|\varphi\|_{C^{k-2,\alpha}(g)}) \cdot \||i\partial\overline{\partial}\psi|_\varphi^2 - |i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2 + (\Delta_{\widetilde{\varphi}} - \Delta_\varphi)\widetilde{h}\|_{C^{k-4,\alpha}(g)}.$$

The second summand is

$$\|(\Delta_{\widetilde{\varphi}} - \Delta_\varphi)\widetilde{h}\|_{C^{k-4,\alpha}(g)} \le \left\|-\left(\int_0^1 g_s^{i\overline{\ell}}g_s^{k\overline{j}}ds\right) \cdot (\widetilde{\varphi} - \varphi)_{k\overline{\ell}} \cdot \widetilde{h}_{i\overline{j}}\right\|_{C^{k-4,\alpha}(g)}$$
$$\le C(\|\varphi\|_{C^{k-2,\alpha}(g)}, \|\widetilde{\varphi}\|_{C^{k-2,\alpha}(g)}) \cdot \|\widetilde{\varphi} - \varphi\|_{C^{k-2,\alpha}(g)} \cdot \|\widetilde{h}\|_{C^{k-2,\alpha}(g)}.$$

By definition of $\widetilde{h}$ we estimate then

$$\|\widetilde{h}\|_{C^{k-2,\alpha}(g)} \le C(\|\widetilde{\varphi}\|_{C^{k-2,\alpha}(g)}) \cdot \||i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2\|_{C^{k-4,\alpha}(g)}$$
$$\le C(r)(\|\widetilde{\varphi}\|_{C^{k-2,\alpha}(g)} + 1)^2 \cdot \|\widetilde{\psi}\|_{C^{k-2,\alpha}(g)}^2$$
$$\le C(r).$$

So we finally have for the first summand in (2.8)

$$\left\|\frac{1}{2}(\Delta_\varphi - 1)^{-1}|i\partial\overline{\partial}\psi|_\varphi^2 - \frac{1}{2}(\Delta_{\widetilde{\varphi}} - 1)^{-1}|i\partial\overline{\partial}\widetilde{\psi}|_{\widetilde{\varphi}}^2\right\|_{C^{k-2,\alpha}(g)} \le C(r)(\|\widetilde{\varphi} - \varphi\|_{C^{k-2,\alpha}(g)} + \|\widetilde{\psi} - \psi\|_{C^{k-2,\alpha}(g)}).$$

The second summand in (2.8) is estimated by the same trick as in the previous proposition.

For the last summand in (2.8) we have

$$\frac{1}{2}(\Delta_{\widetilde{\varphi}}\widetilde{\psi})^2 - \frac{1}{2}(\Delta_\varphi\psi)^2 = \frac{1}{2}(\Delta_\varphi\psi - \Delta_{\widetilde{\varphi}}\widetilde{\psi})(\Delta_\varphi\psi + \Delta_{\widetilde{\varphi}}\widetilde{\psi})$$
$$= \frac{1}{2}(\Delta_\varphi\psi - \Delta_{\widetilde{\varphi}}\psi + \Delta_{\widetilde{\varphi}}\psi - \Delta_{\widetilde{\varphi}}\widetilde{\psi})(\Delta_\varphi\psi + \Delta_{\widetilde{\varphi}}\widetilde{\psi}).$$

so we estimate

$$\left\|\frac{1}{2}(\Delta_{\widetilde{\varphi}}\widetilde{\psi})^2 - \frac{1}{2}(\Delta_\varphi\psi)^2\right\|_{C^{k-2,\alpha}(g)} \le \left(\|\Delta_\varphi\psi - \Delta_{\widetilde{\varphi}}\psi\|_{C^{k-2,\alpha}(g)} + \|\Delta_{\widetilde{\varphi}}\psi - \Delta_{\widetilde{\varphi}}\widetilde{\psi}\|_{C^{k-2,\alpha}(g)}\right)$$
$$\cdot \left(\|\Delta_\varphi\psi\|_{C^{k-2,\alpha}(g)} + \|\Delta_{\widetilde{\varphi}}\widetilde{\psi}\|_{C^{k-2,\alpha}(g)}\right).$$

By the estimates for the Laplacians we are able to say that this quantity is $\le C(r)(\|\widetilde{\varphi} - \varphi\|_{C^{k,\alpha}(g)} + \|\widetilde{\psi} - \psi\|_{C^{k,\alpha}(g)})$.

Again, the estimate for the norm $|\cdot|_{k,\alpha}$ is the same multiplied by ε, so again it suffices to pick $\varepsilon(r)$ such that $\varepsilon(r)C(r) < 1$. $\qquad\square$

2.2.2. *Higher regularity.* Now we explain how to obtain the smoothness of the solution of Theorem 2.2.

Theorem 2.8. *For every $\varphi_0 \in \mathcal{H}$, $\psi_0 \in T_{\varphi_0}\mathcal{H}$, there exists a positive ε and a curve $\varphi \in C^\infty((-\varepsilon, \varepsilon), \mathcal{H})$ which is the unique solution of (2.4) with smooth initial data (φ_0, ψ_0).*

We isolate the following technical lemma that can be proved by computation in local coordinates.

Lemma 2.9. *Let ∂_A be the derivative with respect to the complex coordinate z_A and let $f_A = \partial_A f$. Then the following hold*

$$(g_\varphi^{i\overline{j}})_A = -g_\varphi^{i\overline{s}}(g_{\varphi_{\overline{s}m}})_A g_\varphi^{m\overline{j}};$$

$$\partial_A(\Delta_\varphi f) = \Delta_\varphi f_A + (g_\varphi^{i\overline{j}})_A f_{i\overline{j}};$$

$$\partial_A|i\partial\overline{\partial}\psi|_\varphi^2 = 2(i\partial\overline{\partial}\psi, i\partial\overline{\partial}\psi_A) - \psi_{i\overline{j}}\psi_{k\overline{\ell}}g_\varphi^{i\overline{s}}(g_{\varphi_{\overline{s}m}})_A g_\varphi^{m\overline{\ell}}g_\varphi^{m\overline{j}} - \psi_{i\overline{j}}\psi_{k\overline{\ell}}g_\varphi^{i\overline{\ell}}g_\varphi^{k\overline{s}}(g_{\varphi_{\overline{s}m}})_A g_\varphi^{m\overline{j}}$$
$$= 2(i\partial\overline{\partial}\psi, i\partial\overline{\partial}\psi_A) + B_{\varphi\psi}\varphi_A;$$

$$\partial_A(\Delta_\varphi\psi)^2 = 2\Delta_\varphi\psi[\Delta_\varphi\psi_A + (g_\varphi^{i\overline{j}})_A\psi_{i\overline{j}}]$$

where $B_{\varphi\psi}$ is a linear operator.

We want to derive the second equation of (2.10) by deriving the equation

$$(2.9) \qquad F(\varphi, \psi) = (\Delta_\varphi - 1)\left[\Delta_\varphi\psi' - |i\partial\overline{\partial}\psi|_\varphi^2 + \frac{1}{2}(\Delta_\varphi\psi)^2\right] - \frac{1}{2}|i\partial\overline{\partial}\psi|_\varphi^2 = 0.$$

Lemma 2.10. *$\partial_A F(\varphi, \psi)$ is a linear fourth order operator on (φ_A, ψ_A). When (φ, ψ) are $C^{k,\alpha}$, the coefficients of $\partial_A F(\varphi, \psi)$ are $C^{k-4,\alpha}$.*

Proof. The derivative of the first term is, by Lemma 2.9,

$$\partial_A(\Delta_\varphi - 1)\Delta_\varphi\psi' = (\Delta_\varphi - 1)\left[\Delta_\varphi\psi'_A + (g_\varphi^{i\overline{j}})_A\psi'_{i\overline{j}}\right] + (g_\varphi^{i\overline{j}})_A(\Delta_\varphi\psi')_{i\overline{j}}$$

where we notice linearity with respect to φ_A and ψ_A.

The derivative of the second term is

$$\partial_A(\Delta_\varphi - 1)|i\partial\overline{\partial}\psi|_\varphi^2 = (\Delta_\varphi - 1)\partial_A|i\partial\overline{\partial}\psi|_\varphi^2 + (g_\varphi^{i\overline{j}})_A(|i\partial\overline{\partial}\psi|_\varphi^2)_{i\overline{j}}$$
$$= (\Delta_\varphi - 1)[2(i\partial\overline{\partial}\psi, i\partial\overline{\partial}\psi_A) + B_{\varphi\psi}\varphi_A] + (g_\varphi^{i\overline{j}})_A(|i\partial\overline{\partial}\psi|_\varphi^2)_{i\overline{j}}$$

and we notice again linearity with respect to φ_A and ψ_A.

The third and fourth terms are as in Lemma 2.9 and are linear with respect to φ_A and ψ_A as well. $\qquad\square$

Proof of Theorem 2.8. When we are given a smooth initial data (φ_0, ψ_0) and Hölder exponent (k, α) with $k \geq 6$ and $\alpha \in (0, 1)$, according to Theorem 2.2, we have a maximal lifespan $\varepsilon = \varepsilon(k + 1, \alpha)$ of the geodesic $\varphi(t) \in C^2((-\varepsilon, \varepsilon), \mathcal{H}^{k+1,\alpha})$. Meanwhile, for a less regular space (k, α), we have an other maximal lifespan $\varepsilon(k, \alpha)$. In general,

$$\varepsilon(k + 1, \alpha) \leq \varepsilon(k, \alpha).$$

Now we explore the important property of our geodesic equation and thus prove the inverse inequality $\varepsilon(k + 1, \alpha) \geq \varepsilon(k, \alpha)$.

Recall that our geodesic equation could be written down as a couple system (2.5).

The important observation is that this system is of order zero. In a local coordinate chart on M, we take the derivative $\partial_A = \frac{\partial}{\partial z_A}$ on the both side of the equations and get

$$(2.10) \qquad \begin{cases} (\partial_A \varphi)' = \partial_A \psi \\ (\partial_A \psi)' = \partial_A(L_\varphi \psi). \end{cases}$$

If we manage to prove that this is a linear system in $\varphi_A = \partial_A \varphi$ and $\psi_A = \partial_A \psi$ (all other functions treated as constants) then we can argue as follows. According to Lemma 2.10, the coefficients of (2.10) are $C^{k-4,\alpha}$ and exist for $|t| < \varepsilon(k, \alpha)$. Because of its linearity and of fourth order on (φ_A, ψ_A), its $C^{k,\alpha}$ solution (φ_A, ψ_A) exists as long as the coefficients do, so we have that φ is $C^{k+1,\alpha}$ at least for $|t| < \varepsilon(k, \alpha)$, proving that $\varepsilon(k + 1, \alpha) \geq \varepsilon(k, \alpha)$. □

2.3. **Exponential map, Jacobi fields and conjugate points.** With the local well-posedness of the geodesic, we are able to define the exponential map locally at point $\varphi \in \mathcal{H}$ by

$$(2.11) \qquad \exp_\varphi(t\psi) = \gamma(t), 0 \leq t \leq \varepsilon$$

where γ is the geodesic starting from φ with initial speed ψ. Furthermore, we have the following.

Corollary 2.11. *For any $\varphi_1 \in \mathcal{H}$, there exists an $\varepsilon > 0$ so that for any $\varphi_2 \in \mathcal{H}$ with $\|\varphi_1 - \varphi_2\|_{C^{2,\alpha}} < \varepsilon$, there is a unique geodesic connecting φ_1 to φ_2 whose length is less than ε.*

Now that we have achieved the existence of smooth short-time geodesics we can move a step further to bring the definition of its Jacobi vector fields. The very definition comes from classical Riemannian geometry, see [11] for more details.

Let $\gamma : [0, \varepsilon) \to \mathcal{H}$ be a smooth geodesic for the metric connection D on $\mathcal{H}$. A Jacobi field J along γ is a map $J : [0, \varepsilon) \to T\mathcal{H}$ such that $J(t) \in T_{\gamma(t)}\mathcal{H}$ for all $t \in [0, \varepsilon)$ and moreover satisfies the Jacobi equation

$$(2.12) \qquad \frac{D^2}{dt^2}J(t) + R\left(J(t), \frac{d}{dt}\gamma(t)\right)\frac{d}{dt}\gamma(t) = 0.$$

The Jacobi field is a vector field along the geodesic $\gamma(t)$. Let $v = \frac{d}{dt}|_{t=0}\gamma(t)$ at $\gamma(0) = \varphi$, the geodesic is given by the exponential map $\gamma(t) = \exp_\varphi tv$. Then given $w \in T_\varphi\mathcal{H}$, the solution of the Jacobi equation (2.12) with initial condition $J(0) = 0$ and $J'(0) = w$ is given by

$$J(t) = d\exp_\varphi|_{tv} tw.$$

The definition of conjugate points in the infinite dimensional setting is different from the one from classical Riemannian geometry. Let $\varphi \in \mathcal{H}$, $\psi \in T_\varphi\mathcal{H}$ and let γ be the geodesic with $\gamma(0) = \varphi$ and $\gamma'(0) = \psi$. There are two notions related to conjugate points, cf. e.g. [23, 28, 31].

Definition 2.12. We say that $\gamma(1)$ is
- *monoconjugate* to φ if $d\exp_\varphi|_\psi$ is not injective;
- *epiconjugate* to φ if $d\exp_\varphi|_\psi$ is not surjective.

Remark 2.13. In order to understand the conjugate points, it turns out to further study whether $d\exp_\varphi|_\psi$ is a Fredholm operator between the tangent spaces of $\mathcal{H}$. Then the infinite dimensional version of Sard's theorem applies [34].

2.4. **Dirichlet metric and a comparison theorem.** Now we continue the study of the (Dirichlet) gradient metric.

2.4.1. *Sectional curvature for the gradient metric.* We denote $\varphi = \varphi(s,t)$ be a smooth two parameter family of curves in the space of Kähler metrics $\mathcal{H}$, and the corresponding two parameter families of curves of tangent vectors φ_t, φ_s along φ are $\mathbb{R}$-linearly independent. The sectional curvature of the gradient metric is computed in [13],

$$K_G(\varphi_s, \varphi_t)_\varphi = \frac{1}{2} \int_M |da(s,t)|^2_{g_\varphi} \frac{\omega_\varphi^n}{n!} - \frac{1}{2} \int_M (da(s,s), da(t,t))_{g_\varphi} \frac{\omega_\varphi^n}{n!},$$

where the symmetric expression $a(\sigma, \tau)$ satisfies

$$\Delta_\varphi a(\sigma, \tau) = \Delta_\varphi \varphi_\sigma \Delta_\varphi \varphi_\tau - (i\partial\overline{\partial}\varphi_\sigma, i\partial\overline{\partial}\varphi_\tau).$$

We let

$$\{\varphi_s, \varphi_t\}_\varphi = \frac{\sqrt{-1}}{2} \left(g^{i\overline{j}} \frac{\partial\varphi_s}{\partial z^i} \frac{\partial\varphi_t}{\partial z^{\overline{j}}} - g^{i\overline{j}} \frac{\partial\varphi_t}{\partial z^i} \frac{\partial\varphi_s}{\partial z^{\overline{j}}} \right) = \mathrm{Im}(\partial\varphi_s, \overline{\partial}\varphi_t)_\varphi.$$

The expression of the sectional curvature K_M for the L^2 metric is, for all linearly independent sections φ_s, φ_t,

$$K_M(\varphi_s, \varphi_t)_\varphi = -\frac{\int_M \mathrm{Im}(\partial\varphi_s, \overline{\partial}\varphi_t)_\varphi^2 \frac{\omega_\varphi^n}{n!}}{\sqrt{\int_M \varphi_s^2 \frac{\omega_\varphi^n}{n!}} \sqrt{\int_M \varphi_s^2 \frac{\omega_\varphi^n}{n!}} - \int_M \varphi_s\varphi_t \frac{\omega_\varphi^n}{n!}}.$$

Therefore, $K_M \leq 0$. On the other side, the first author proved that, for any linearly independent sections φ_s, φ_t the sectional curvature for the Calabi metric K_C is

$$K_C(\varphi_s, \varphi_t) = \frac{1}{4\,\mathrm{vol}}.$$

In a private communication, Calabi conjectured that there exists the following relation among the sectional curvatures of L^2 metric, gradient metric and Calabi metric,

$$K_M \leq K_G < K_C.$$

Remark 2.14. It would be interesting to construct examples to detect the sign of the sectional curvature of the gradient metric and determine whether this conjecture holds.

2.4.2. *Local well-posedness for the gradient metric.* On the other hand, the application of the proofs of Theorem 2.2 and 2.8 leads to the corresponding theorem of the gradient metric.

Theorem 2.15. *For every integer $k \geq 6$ and $\alpha \in (0,1)$ and initial data $\varphi_0 \in \mathcal{H}^{k,\alpha}$ and $\psi_0 \in T_{\varphi_0}\mathcal{H}^{k,\alpha}$ there exists a positive ε and a curve $\varphi \in C^2((-\varepsilon, \varepsilon), \mathcal{H}^{k,\alpha})$ which is the unique solution of the geodesic equation (1.9) with initial data (φ_0, ψ_0). Moreover, if the initial data is smooth, then the solution φ is also smooth.*

2.4.3. *Sectional curvature and stability.* The idea that the sign of the sectional curvature could be used to predict the stability of the geodesic ray goes back to Arnold [2]. Intuitively, when the sectional curvature is positive, all Jacobi fields are uniformly bounded, then under a small perturbation of the initial velocities, the geodesics remain nearby. When the sectional curvature is negative, the Jacobi fields grow exponentially in time, then the geodesic rays grow unstable. When the sectional curvature is zero, the geodesic ray is linear. For the gradient metric, the picture might be more complicated as the sign of the sectional curvature might vary along different planes. However, we are able to examine the growth of Jacobi fields along geodesics by applying the comparison theorem for infinite dimensional manifolds.

Then with the definitions of the Jacobi equation and conjugate points in Section 2.3, we could apply Biliotti's [5] Rauch comparison theorem for weak Riemannian metrics, see [27].

Theorem 2.16. *Let γ_G and γ_C be two geodesics of equal length with respect to the gradient metric and the Calabi metric respectively and suppose that for every $X_G \in T_{\gamma_G(t)}\mathcal{H}$ and $X_C \in T_{\gamma_C(t)}\mathcal{H}$, we have*

$$K_G(X_G, \gamma_G'(t)) \leq \frac{1}{4\,\mathrm{vol}} = K_C(X_C, \gamma_C'(t)).$$

Let J_G and J_C be the Jacobi fields along γ_G and γ_C such that

- $J_G(0) = J_C(0) = 0$,
- $J_G'(0)$ *is orthogonal to* $\gamma_G'(0)$ *and* $J_C'(0)$ *is orthogonal to* $\gamma_C'(0)$,
- $\|J_G'(0)\| = \|J_C'(0)\|$.

then we have, for all $t \in [0, T]$,

$$\|J_G(t)\| \geq \frac{\left|\sin\left(2t\sqrt{\mathrm{vol}}\right)\right|}{\sqrt{\mathrm{vol}}}.$$

Proof. In Biliotti's Rauch comparison theorem, it is required that $J_C(t)$ is nowhere zero in the interval $(0, T]$ and if γ_C has most a finite number of points which are epiconjugate but not monoconjugate in $(0, T]$, this condition is satisfied for the Calabi metric, see [11]. Therefore the conclusion of the comparison theorem is that, for all $t \in [0, T]$,

$$\|J_G(t)\| \geq \|J_C(t)\|.$$

We know that, as an application of [11, Theorem 8], that

$$\|J_C(t)\| = \frac{\left|\sin\left(2t\sqrt{\mathrm{vol}}\right)\right|}{\sqrt{\mathrm{vol}}},$$

thus the resulting inequality in the proposition follows. $\square$

3. The space of Sasakian metrics

3.1. **The restricted Ebin metric.** Since the sum metric arises in the context of Sasakian geometry, in this subsection we recall the definitions of the case. A Sasakian manifold is a $(2n + 1)$-dimensional M together with a contact form η, its

Reeb field ξ, a $(1,1)$-tensor field Φ and a Riemannian metric g that makes ξ Killing, such that

$$\eta(\xi) = 1, \iota_\xi d\eta = 0$$
$$\Phi^2 = -\operatorname{id} + \xi \otimes \eta$$
$$g(\Phi\cdot, \Phi\cdot) = g + \eta \otimes \eta$$
$$d\eta = g(\Phi\cdot, \cdot)$$
$$N_\Phi + \xi \otimes d\eta = 0$$

where N_Φ is the torsion of Φ. The first four mean that M is a *contact metric manifold* and the last one means it is *normal*, see [7, Chap. 6].

The foliation defined by ξ is called *characteristic foliation*. Let $D = \ker \eta$. It is known that $(d\eta, J = \Phi|_D)$ is a *transversally Kähler* structure, as the second, third and fourth equation above say.

A form α is said to be *basic* if $\iota_\xi \alpha = 0$ and $\iota_\xi d\alpha = 0$. A function $f \in C^\infty(M)$ is basic if $\xi \cdot f = 0$. The space of smooth basic functions on M is denoted by $C_B^\infty(M)$. The transverse Kähler structure defines the transverse operators ∂, $\overline{\partial}$ and $d^c = \frac{i}{2}(\overline{\partial} - \partial)$ acting on basic forms, analogously as in complex geometry.[1] The form $d\eta$ is basic and its basic class is called *transverse Kähler class*.

Given an initial Sasakian manifold $(M, \eta_0, \xi_0, \Phi_0, g_0)$, basic functions parameterize a family of other Sasakian structures on M which share the same characteristic foliation and are in the same transverse Kähler class, in the following way. We follow the notation of [7, p. 238].

Let $\varphi \in C_B^\infty(M)$ and define $\eta_\varphi = \eta_0 + d^c\varphi$. The space of all φ's is

$$\widetilde{\mathcal{H}}_S = \{\varphi \in C_B^\infty(M) : \eta_\varphi \wedge d\eta_\varphi \neq 0\}$$

and, in analogy of the Kähler case, we consider normalized "potentials"

$$\mathcal{H}_S = \{\varphi \in \widetilde{\mathcal{H}}_S : I(\varphi) = 0\}.$$

The equation $I = 0$ is a normalization condition, similar to (1.1). We refer to [25] for the definition of I in our case, which is such that

$$T_\varphi \mathcal{H}_S = \left\{\psi \in C_B^\infty(M) : \int_M \psi \frac{1}{n!} \eta_\varphi \wedge d\eta_\varphi^n = 0\right\}.$$

These deformations are called of *type II* and it is easy to check that they leave the Reeb foliation and the transverse holomorphic structure fixed, since ξ is still the Reeb field for η_φ.

Every $\varphi \in \mathcal{H}_S$ defines a new Sasakian structure where the Reeb field and the transverse holomorphic structure are the same and

$$
\begin{aligned}
&\eta_\varphi = \eta_0 + d^c\varphi \\
(3.1) \quad &\Phi_\varphi = \Phi_0 - (\xi \otimes d^c\varphi) \circ \Phi_0 \\
&g_\varphi = d\eta_\varphi \circ (\operatorname{id} \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi.
\end{aligned}
$$

Note that one could write $g_\varphi = d\eta_\varphi \circ (\operatorname{id} \otimes \Phi_0) + \eta_\varphi \otimes \eta_\varphi$ since the endomorphism $\Phi_\varphi - \Phi_0$ has values parallel to ξ and $d\eta_\varphi$ is basic. Indeed, the range of Φ_φ is the one of Φ_0 plus a component along ξ, so if we contract it with $d\eta$ the latter vanishes.

---

[1]This definition with the $\frac{1}{2}$ is classical in Sasakian geometry and differs from the convention usually used in complex geometry $d^c = i(\overline{\partial} - \partial)$. With this convention, the relation $dd^c = i\partial\overline{\partial}$ holds on basic forms.

As in the Kähler case, these deformations keep the volume of M fixed, which will be denoted by vol.

The L^2 metric was generalized to $\mathcal{H}_S$ in [25, 26], where Guan and Zhang solved the Dirichlet problem for the geodesic equation and He provided a Sasakian analogue of Donaldson's picture about extremal metrics.

On the space $\mathcal{H}_S$ one can define the Calabi metric and the gradient metric in the same ways as in formulae (1.5) and (1.8) by using the so called *basic Laplacian* which acts on basic functions in the same way as in the Kähler case and by using the volume form $\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$ in the integrals.

In this setting, it is easy to see that the map

$$\mathcal{H}_S \ni \varphi \mapsto \log \frac{\eta_\varphi \wedge d\eta_\varphi^n}{\eta_0 \wedge d\eta_0^n}$$

maps basic functions to basic functions. The *transverse Calabi-Yau theorem* of [8] allows to prove the surjectivity of this map as in the Kähler case, more precisely between $\mathcal{H}_S$ and the space of *basic* conformal volume forms

$$\mathcal{C}_B = \left\{ u \in C_B^\infty(M) : \int_M e^u \frac{1}{n!}\eta_0 \wedge d\eta_0^n = \text{vol} \right\}.$$

As noted above, the space $\mathcal{C}$ can be defined also for Sasakian manifolds by just taking the Sasakian volume form $\frac{1}{n!}\eta_0 \wedge d\eta_0^n$ instead of the Kähler one. One might ask how the spaces $\mathcal{C}_B$ and $\mathcal{C}$ are related. Obviously $\mathcal{C}_B \subseteq \mathcal{C}$ but we can say more.

Proposition 3.1. *$\mathcal{C}_B$ is totally geodesic in $\mathcal{C}$.*

Proof. It is straightforward to verify that for any curve in $\mathcal{C}_B$ and section along it, the covariant derivative defined in [11] is still basic, meaning that the (formal) second fundamental form of $\mathcal{C}_B$ vanishes. $\qquad\square$

Let $\mathcal{M}$ be the Ebin space of all Riemannian metrics on (M, g_0, ξ_0, η_0) Sasakian of dimension $2n+1$.

We define an immersion $\Gamma : \mathcal{H}_S \to \mathcal{M}$ that maps $\varphi \mapsto g_\varphi$ as defined in (3.1). As in the Kähler case, it is injective. Indeed if two basic function $\varphi_1, \varphi_2 \in \mathcal{H}_S$ give rise to the same Sasakian metric, taking the corresponding transverse structures we would have $dd^c(\varphi_1 - \varphi_2) = 0$ forcing $\varphi_1 - \varphi_2 = \text{const}$. The normalization $I(\cdot) = 0$ then implies $\varphi_1 = \varphi_2$.

Let us compute the differential of Γ. Let $\varphi(t)$ be a curve in $\mathcal{H}_S$ with $\varphi(0) = \varphi$ and $\varphi'(0) = \psi \in T_\varphi \mathcal{H}_S$. Then

$$(3.2) \qquad \Gamma_{*\varphi}\psi = \left.\frac{d}{dt}\right|_{t=0} g_{\varphi(t)} = dd^c\psi(\Phi_0 \otimes \text{id}) + 2d^c\psi \odot \eta_\varphi$$

with the convention $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. For easier notation we call $\beta_\psi := dd^c\psi(\Phi_0 \otimes 1)$.

The differential of Γ is also injective. Indeed if ψ is in its kernel, then

$$0 = \Gamma_{*\varphi}\psi(\xi, \cdot) = d^c\psi,$$

forcing ψ to be zero, as it has zero integral.

On $T_g\mathcal{M}$ recall that the Ebin metric is given by, for $a, b \in T_g\mathcal{M} = \Gamma(S^2M)$,

$$g_{\text{E}}(a, b)_g = \int_M g(a, b) dv_g.$$

We want to compute the restriction of the Ebin metric on the space $\mathcal{H}_S$.

Proposition 3.2. *The restriction of the Ebin metric to $\mathcal{H}_S$ is twice the sum of the Calabi metric with the gradient metric*

$$\frac{1}{2}\Gamma^* g_{\mathrm{E}} = g_{\mathrm{C}} + g_{\mathrm{G}}$$

which we have called the sum metric.

Proof. Computing the length with respect to g_φ of the tensor in (3.2) we get

$$|\beta_\psi + 2d^c\psi \odot \eta_\varphi|_{g_\varphi}^2 = g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi \otimes \eta_\varphi, d^c\psi \otimes \eta_\varphi) + 2g_\varphi(\beta_\psi, 2d^c\psi \odot \eta_\varphi)$$

$$= g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi, d^c\psi)g_\varphi(\eta_\varphi, \eta_\varphi) + 2\beta_\psi((d^c\psi)^\sharp, \xi)$$

$$= g_\varphi(\beta_\psi, \beta_\psi) + 2g_\varphi(d^c\psi, d^c\psi)$$

using the fact that the g_φ-dual of η_φ is ξ, that the $\sharp$ is done with respect to g_φ and finally the fact that the tensor β_ψ is transverse, i.e. vanishes when evaluated on ξ.

Integrating with respect to $d\mu_\varphi$ we have

$$\langle \Gamma_{*\varphi}\psi, \Gamma_{*\varphi}\psi \rangle_\varphi = \|\beta_\psi\|_\varphi^2 + 2\|d^c\psi\|_\varphi^2$$

where the right hand side are L^2 norms with respect to the metric g_φ. The second summand is twice the gradient metric on $\mathcal{H}_S$ given by

$$g_{\mathrm{G}}(\psi, \psi) = \int_M g_\varphi(d\psi, d\psi)\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n.$$

(For a basic function, there is no difference between its Riemannian gradient and its basic gradient).

We now want to establish a useful formula that we will need in a while. Fix $\varphi \in \mathcal{H}_S$ and $h \in T_\varphi\mathcal{H}_S$ we consider the curve $\varphi(t) = \varphi + th$ which is in $\mathcal{H}_S$ for small t. We then compute for every curve $f(t) \in T_\varphi\mathcal{H}_S$,

$$0 = \frac{d}{dt}\Big|_{t=0} \int_M \Delta_{\varphi(t)}f\frac{1}{n!}\eta_{\varphi(t)} \wedge d\eta_{\varphi(t)}^n = \int_M (\Delta_\varphi f'(t) - (dd^c f, dd^c h)_\varphi + \Delta_\varphi f \Delta_\varphi h)\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n.$$

which means that

$$g_{\mathrm{C}}(f, h)_\varphi = \int_M (dd^c f, dd^c h)_\varphi \frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n.$$

Then we have, since β_ψ is the (transverse) 2-tensor associated to the basic form $dd^c\psi$, whose point-wise norms are related by $|\beta_\psi|^2 = 2|dd^c\psi|^2$,

$$g_{\mathrm{C}}(\psi, \psi) = \int_M (\Delta_\varphi \psi)^2 \frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n = \int_M (dd^c\psi, dd^c\psi)_{\omega_\varphi}\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n = \frac{1}{2}\|\beta_\psi\|_\varphi^2.$$

$\square$

3.2. The sum metric on $\mathcal{H}_S$.

Consider on $\mathcal{H}_S$ the metric $g = 2g_{\mathrm{C}} + 2g_{\mathrm{G}}$. It can be written, for $\varphi \in \mathcal{H}_S$ and $\alpha, \beta \in T_\varphi\mathcal{H}_S$,

$$g(\alpha, \beta) = 2\int_M \Delta_\varphi\alpha\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n - 2\int_M \alpha\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$$

$$= 2\int_M \Delta_\varphi(\alpha - G_\varphi\alpha)\Delta_\varphi\beta\frac{1}{n!}\eta_\varphi \wedge d\eta_\varphi^n$$

$$= g_{\mathrm{C}}(L_\varphi\alpha, \beta)$$

where $L_\varphi = 2(I - G_\varphi)$ with G_φ the Green operator associated to Δ_φ.

Note that the G_φ acting on functions with zero integral with respect to $d\mu_\varphi$ is the inverse of Δ_φ, since the projection on the space of harmonic functions is

$$H_\varphi : f \mapsto \frac{1}{\mathrm{vol}_{g_\varphi}} \int_M f \frac{1}{n!} \eta_\varphi \wedge d\eta_\varphi^n = 0$$

and because of the known relation $I = H_\varphi + \Delta_\varphi G_\varphi$.

We have the first result.

Proposition 3.3. *For any curve φ in $\mathcal{H}_S$ and any section v on φ, the only solution $D_t v$ of*

$$(3.3) \qquad \frac{1}{2} L_\varphi D_t v = D_t^C v - G_\varphi D_t^G v$$

is the Levi-Civita covariant derivative of g, i.e. it is torsion free and

$$(3.4) \qquad \frac{d}{dt} g(v,v) = 2g(D_t v, v).$$

Its proof is analogous to Proposition 2.1 and makes use of the results in [20] about transversally elliptic operators. The geodesic equation is then

$$(3.5) \qquad \Delta_\varphi^2 D_t^C \varphi' - \Delta_\varphi D_t^G \varphi' = 0$$

which is rewritten as (2.4), i.e.

$$(3.6) \qquad (\Delta_\varphi - I)\left((\Delta_\varphi \varphi')' + \frac{1}{2}(\Delta_\varphi \varphi')^2 \right) - \frac{1}{2}|i\partial\overline{\partial}\varphi'|_\varphi^2 = 0.$$

Remark 3.4. It is clear that a curve φ which is a geodesic for both the Calabi and the gradient metric would be a geodesic for our metric as well. Unfortunately there are no such nontrivial curves, as one can easily see from the equations.

3.3. **Another space of Sasakian metrics, an open problem.** Back to Sasakian geometry, it is interesting to consider also the space $\mathcal{G}$ of Sasakian structures that share the same underlying CR structure. These deformations are known as *type I* and we refer to [7, Chap. 8]. The most striking differences between $\mathcal{G}$ and the $\mathcal{H}_S$ is that the former is finite dimensional and the metrics in it do not have the same volume. Recently, it was studied by Boyer, Huang, Legendre and Tønnesen-Friedman [9] in relation to the existence of constant scalar curvature Sasakian metrics.

It would be interesting to compute the restriction of the Ebin metric to $\mathcal{G} \subset \mathcal{M}$ and study its intrinsic and extrinsic geometry.

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