Vietnam Journal of Mathematics (2023) 51:571–576
https://doi.org/10.1007/s10013-022-00586-z

ORIGINAL ARTICLE

A Lower Bound on the Number of Generators of a Defect Group

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Received: 4 February 2022 / Accepted: 19 July 2022 / Published online: 4 October 2022

Abstract

Assuming that the statement of the Alperin–McKay–Navarro conjecture holds for a p-block B with defect group D, we show that the number of generators of D is bounded from below by the number of height-zero characters in B fixed by a specific element of the absolute Galois group of the rational numbers.

Keywords Block · Defect group · Group generator

Mathematics Subject Classification (2010) 20C15 · 20C20

1 Introduction

One of the main themes in the Representation Theory of Finite Groups is the study of how the values of characters in a p-block B of a finite group G and the structure of a defect group of B influence each other, where p is a prime. The aim of this note is to investigate connections between the values of height-zero characters in a p-block B and generation properties of the defect groups of B.

Given a finite p-group D, recall that the size of a minimal set of generators of D is m if, and only if, $|D : \Phi(D)| = p^m$, where $\Phi(D)$ is the Frattini subgroup of D. In this note we refer to the number m as the rank of D.

If B is a p-block of a finite group G with defect group D then the height-zero characters of B, here denoted by $\mathrm{Irr}_0(B)$, are those $\chi \in \mathrm{Irr}(B)$ such that $\chi(1)_p = |G : D|_p$, where n_p is the maximal power of p dividing the integer n. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts naturally on the set $\mathrm{Irr}(G)$ preserving degrees by $\chi^\sigma(g) = \sigma(\chi(g))$ for every $g \in G$ and $\chi \in \mathrm{Irr}(G)$. Define $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ as the element that sends p-power roots of unity to their $p+1$ power and fixes roots of unity of order prime to p. Then the action of $\langle\sigma\rangle$ permutes the

To Pham Huu Tiep on the occasion of his 60th birthday.

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elements of $\mathrm{Irr}_0(B)$ for every block B. In the following, we will denote by $\mathrm{Irr}_{0,\sigma}(B)$ the set of fixed-points of $\mathrm{Irr}_0(B)$ under the action of $\langle\sigma\rangle$ and by $k_{0,\sigma}(B)$ the size of $\mathrm{Irr}_{0,\sigma}(B)$.

Recent advances on the subject [14, 18] suggest the existence of a mutual influence between $k_{0,\sigma}(B)$ and the rank of D (at least for primes $p \leq 3$). We propose the following:

$$1 \leq k_{0,\sigma}(B) \leq |D : \Phi(D)|. \tag{1}$$

The above conjectural bound is related to Brauer's $k(B)$-conjecture [1, Problem 20] and Olsson's conjecture [16]. The former predicts that $k(B) \leq |D|$, where $k(B) = |\mathrm{Irr}(B)|$ is the number of irreducible characters in the block B, and the latter that $k_0(B) \leq |D : D'|$, where $k_0(B) = |\mathrm{Irr}_0(B)|$. Notice that while $\mathrm{Irr}_0(B)$ is always non-empty, the existence of some $\langle\sigma\rangle$-fixed character in $\mathrm{Irr}_0(B)$ is not trivially guaranteed unless $D = 1$.

In Theorem 2.3 we show that Conjecture (1) holds for any p-block B with normal defect group. In particular, any p-block B satisfying the statement of the so-called Alperin–McKay–Navarro conjecture [13, Conjecture B] also satisfies the statement of Conjecture (1). Since the Alperin–McKay–Navarro conjecture is known to hold for p-solvable, sporadic, symmetric and alternating groups by [4, 5, 13, 20] and for all p-blocks with cyclic defect groups by [13, Theorem 3.4], then Theorem 2.3 implies that Conjecture (1) holds for all p-blocks of these families of groups and for all p-blocks with cyclic defect groups.

A natural problem in this context is to study under which conditions is the upper bound in Conjecture (1) attained. Recall that a character $\chi \in \mathrm{Irr}(G)$ is p-rational if all the values of χ lie in the cyclotomic extension $\mathbb{Q}(\xi)$, where ξ is a root of unity of order $|G|_{p'}$. Here $n_{p'}$ is the p'-part of any natural number n so that $n = n_p n_{p'}$. By the definition of σ, every p-rational character of G is $\langle\sigma\rangle$-fixed.

Suppose that B is nilpotent. Then there is some $\chi \in \mathrm{Irr}_0(B)$ which is p-rational by [3, Proposition 2.6]. Following the notation of [3], $\nu \mapsto \nu * \chi$ yields a $\langle\sigma\rangle$-equivariant bijection $\mathrm{Lin}(D) \to \mathrm{Irr}_0(B)$, where $\mathrm{Lin}(D) \cong D/D'$ is the set of linear characters of D. Since the $\langle\sigma\rangle$-fixed elements in $\mathrm{Lin}(D)$ are in correspondence with the elements of $D/\Phi(D)$, it follows that $k_{0,\sigma}(B) = |D : \Phi(D)|$. Therefore the bound in Conjecture (1) is attained in nilpotent blocks. However the upper bound in Conjecture (1) is not generally attained, as shown by p-blocks with cyclic defect group (for $p \geq 5$).

In the case where $p = 2$, the relationship between $k_{0,\sigma}(B)$ and $|D : \Phi(D)|$ seems closer. As we have already noticed, if D is cyclic then $k_{0,\sigma}(B) = |D : \Phi(D)| = 2$ [3, (1.ex.3)]. Moreover, the following was conjectured in [18, Conjecture B]:

$$k_{0,\sigma}(B) = 2 \text{ if, and only if, } D \text{ is cyclic.} \tag{2}$$

If D has maximal class (that is, D is either a dihedral, semidihedral or generalized quaternion 2-group) then by celebrated work of Brauer [2, Theorem 4] and Olsson [15, Proposition 4.5] the 4 elements in $\mathrm{Irr}_0(B)$ are 2-rational, so $k_{0,\sigma}(B) = |D : \Phi(D)| = 4$. Dihedral, semidihedral and generalized quaternion groups are examples of 2-groups of rank 2. As suggested by the main result of [14], we believe that the following can be true:

$$k_{0,\sigma}(B) = 4 \text{ if, and only if, } D \text{ has rank } 2. \tag{3}$$

Unfortunately, the statements in Conjectures (2) and (3) do not seem to generalize to higher rank defect groups. If N is an elementary 2-group of rank 4 and $H \in \mathrm{Syl}_5(\mathrm{Aut}(H))$, then the group $G = N \rtimes H$ has a unique 2-block B, namely the principal one, and $k_{0,\sigma}(B) = k(G) = 2^3$.

Conjectures (2) and (3) hold for principal 2-blocks by [18, Theorem A] and [14, Theorem A]. Moreover, Conjecture (2) was proven to be a consequence of the Alperin–McKay–Navarro conjecture in [18, Theorem 1.5]. In Corollary 2.4 we show that Conjecture (3) also follows from the Alperin–McKay–Navarro conjecture.

We care to mention that, for p odd, the $\langle\sigma\rangle$-fixed characters of G are the almost p-rational characters of G defined in [6]. (We note that the definition of almost p-rational characters generalizes at the same time the classical notion of p-rationality and the notion of almost rationality defined in [8].) For $p = 2$, almost 2-rational characters are just 2-rational characters. In this case, it can be shown that the $\langle\sigma\rangle$-fixed irreducible characters of G of odd degree are 2-rational (see Theorem 2.5).

2 On Conjectures (1) and (3)

For a fixed prime p, consider the set $\mathrm{Bl}(G)$ of Brauer p-blocks of G as in [12], so that $\mathrm{Bl}(G)$ induces a partition of $\mathrm{Irr}(G)$. We write $\mathrm{Irr}(B)$ to denote the subset of characters in $\mathrm{Irr}(G)$ that belong to B for $B \in \mathrm{Bl}(G)$. Every block B has associated a uniquely defined conjugacy class of p-subgroups of G, namely its defect groups. Given a block B of G with defect group D, we write $B \in \mathrm{Bl}(G|D)$ and we let $b \in \mathrm{Bl}(\mathbf{N}_G(D)|D)$ denote its Brauer first main correspondent. We write $\mathrm{Irr}_0(B)$ to denote the subset of height-zero characters in $\mathrm{Irr}(B)$, as in Section 1.

The group $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$ acts on $\mathrm{Irr}(G)$ permuting the subsets $\{\mathrm{Irr}_0(B) \mid B \in \mathrm{Bl}(G)\}$ by [12, Theorem 3.19]. Define $\mathcal{H} \leq \mathcal{G}$ to be the subgroup consisting of those elements $\tau \in \mathcal{G}$ for which there exists some fixed $f \in \mathbb{N}$ such that $\tau(\xi) = \xi^{p^f}$ for every root of unity of order prime to p in $\mathbb{Q}(e^{2\pi i/|G|})$. We write $\mathrm{Irr}_{0,\tau}(B)$ to denote the subset of τ-fixed characters in $\mathrm{Irr}_0(B)$.

The Alperin–McKay–Navarro conjecture *Let $B \in \mathrm{Bl}(G|D)$ and let $b \in \mathrm{Bl}(\mathbf{N}_G(D)|D)$ be its Brauer first main correspondent. If $\tau \in \mathcal{H}$, then*

$$|\mathrm{Irr}_{0,\tau}(B)| = |\mathrm{Irr}_{0,\tau}(b)|.$$

Notice that we can see the Galois automorphism σ defined in Section 1 as an element of $\mathcal{H}$ by restricting it to $\mathbb{Q}(e^{2\pi i/|G|})$. Actually $\langle\sigma\rangle$ acts on each $\mathrm{Irr}_0(B)$ for $B \in \mathrm{Bl}(G)$ (meaning that $\langle\sigma\rangle$ permutes trivially the set $\{\mathrm{Irr}_0(B) \mid B \in \mathrm{Bl}(G)\}$) because $\langle\sigma\rangle$ fixes every Brauer character of G.

We will see that the statements of Conjectures (1) and (3) follow from the statement of the Alperin–McKay–Navarro conjecture together with important results which already appeared in the literature. In order to do so, we recall some results on the theory of blocks with a normal defect group. We follow the notation of [12, Chapter 9]. Let $B \in \mathrm{Bl}(G|D)$ and assume that $D \trianglelefteq G$. Write $C = \mathbf{C}_G(D)$. We will denote by $b \in \mathrm{Bl}(CD|D)$ a root of B, and we will let $\theta \in \mathrm{Irr}(b)$ be the canonical character associated with B (see [12, Theorem 9.12] and the subsequent discussion). Recall that $D \subseteq \ker(\theta)$ and θ has p-defect zero when viewed as a character of CD/D (that is, $\theta(1)_p = |CD : D|_p$). In this situation, the set $\mathrm{Irr}(b)$ is parametrized by the set $\mathrm{Irr}(D)$. We may write $\mathrm{Irr}(b) = \{\theta_\lambda \mid \lambda \in \mathrm{Irr}(D)\}$, where θ_λ is defined as in [12, Theorem 9.12] and $\ker(\lambda) \subseteq \ker(\theta_\lambda)$. Then

$$\mathrm{Irr}(B) = \bigcup_{\lambda \in \mathrm{Irr}(D)} \mathrm{Irr}(G|\theta_\lambda).$$

It is not difficult to see that height-zero characters of B lie over characters θ_λ parametrized by linear characters λ of D, so that

$$\mathrm{Irr}_0(B) = \bigcup_{\lambda \in \mathrm{Irr}(D/D')} \mathrm{Irr}(G|\theta_\lambda) \subseteq \mathrm{Irr}(G/D').$$

If $N \trianglelefteq G$, we will often identify with $\mathrm{Irr}(G/N)$ the irreducible characters $\chi \in \mathrm{Irr}(G)$ with $N \subseteq \ker(\chi)$. In the above expression we are using that kind of identifications (as for $\mathrm{Lin}(D) = \mathrm{Irr}(D/D')$).

An explicit description of the set $\mathrm{Irr}_{0,\sigma}(B)$ when the defect group of B is normal was given in [18]. This is [18, Lemma 1.2] which we restate below.

Lemma 2.1 *Let G be a finite group and let p be a prime. Suppose that B is a block of G with a normal defect group D. Let b be a root of B with canonical character θ. Then*

$$\mathrm{Irr}_{0,\sigma}(B) = \bigcup_{\lambda \in \mathrm{Irr}(D/\Phi(D))} \mathrm{Irr}(G|\theta_\lambda) \subseteq \mathrm{Irr}(G/\Phi(D)).$$

Suppose that $N \trianglelefteq G$. If B and $\bar{B}$ are blocks of G and G/N respectively we say that B dominates $\bar{B}$ if $\mathrm{Irr}(B) \cap \mathrm{Irr}(\bar{B}) \neq \emptyset$, and in such a case $\mathrm{Irr}(\bar{B}) \subseteq \mathrm{Irr}(B)$. Every block of G/N is dominated by a unique block of G. (See [12, pp.198–199] for further details). The following key lemma was shown in [10].

Lemma 2.2 *Let B be a p-block of a finite group G with defect group D. Suppose that $D \trianglelefteq G$, then B dominates a unique block $\bar{B}$ of $G/\Phi(D)$. In particular,*

$$k_{0,\sigma}(B) = k_{0,\sigma}(\bar{B}) = k(\bar{B}).$$

Proof The first part follows by [10, Corollary 4]. Indeed if $\bar{B}$ is dominated by B, then the canonical character of $\bar{B}$ is the canonical character θ of B seen as a character of $D\mathbf{C}_G(D)/\Phi(D) = \mathbf{C}_{G/\Phi(D)}(D/\Phi(D))$ (the equality of centralizers follows by coprime action [7, Corollary 3.29]). By Lemma 2.1 we have that $\mathrm{Irr}_{0,\sigma}(B)$ can be identified with a subset of $\mathrm{Irr}(G/\Phi(D))$, hence $\mathrm{Irr}_{0,\sigma}(B) = \mathrm{Irr}_{0,\sigma}(\overline{B})$. Again by Lemma 2.1 (applied to $\bar{B}$) we see that $\mathrm{Irr}_{0,\sigma}(B) = \mathrm{Irr}_{0,\sigma}(\bar{B}) = \mathrm{Irr}(\bar{B})$, yielding $k_{0,\sigma}(B) = k_{0,\sigma}(\bar{B}) = k(\bar{B})$. □

The following statements show that blocks satisfying the statement of the Alperin–McKay–Navarro conjecture also satisfy Conjectures (1) and (3).

Theorem 2.3 *Let B be a p-block of a finite group G with defect group D. Suppose that D is normal in G. Then*

$$1 \leq k_{0,\sigma}(B) \leq |D : \Phi(D)|.$$

Moreover $k_{0,\sigma}(B) = 1$ if, and only if, $D = 1$.

Proof By Lemma 2.2, then $k_{0,\sigma}(B) = k(\bar{B})$ where $\bar{B}$ is the only block of $G/\Phi(D)$ dominated by B. In particular, $k_{0,\sigma}(B) = k(\bar{B}) = 1$ if, and only if, the defect group $D/\Phi(D)$ of $\bar{B}$ is trivial. This happens if, and only if, $D = 1$.

By [17, Theorem 6], $k(\bar{B}) = k(e)$ where e is a p-block of a p-solvable group H with defect group isomorphic to $D/\Phi(D)$. By the solution of the $k(GV)$-problem [19] we have that $1 \leq k(e) \leq |D : \Phi(D)|$. All together, these inequalities show that $1 \leq k_{0,\sigma}(B) \leq |D : \Phi(D)|$. □

Corollary 2.4 *Let B be a 2-block of a finite group G with defect group D. Suppose that D is normal in G. Then $k_{0,\sigma}(B) = 4$ if, and only if, D has rank 2.*

Proof Assume first that $|D : \Phi(D)| = 4$. By Theorem 2.3 we have that $k_{0,\sigma}(B) \leq 4$. Since $D > 1$ we have that 2 divides $k_{0,\sigma}(B)$ by [18, Lemma 1.4]. If $k_{0,\sigma}(B) = 2$ then D is cyclic by [18, Theorem 1.5], a contradiction. Hence $k_{0,\sigma}(B) = 4$. Assume now that $k_{0,\sigma}(B) = 4$. By Lemma 2.2, let $\bar{B}$ be the only block of $G/\Phi(D)$ dominated by B so that $k_{0,\sigma}(B) = k(\bar{B}) = 4$. By [11, Corollary 1.3], the defect group of $\bar{B}$ has order 4, that is, $|D : \Phi(D)| = 4$. □

In the following, for any $\tau \in \mathcal{G}$ we denote by $\mathrm{Irr}_{2',\tau}(G)$ the set of odd-degree irreducible characters of G fixed under the action of $\langle\tau\rangle$.

Theorem 2.5 *Let G be a finite group and let $\tau \in \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q}(e^{2\pi i/|G|_{2'}}))$ be such that the subfield of $\mathbb{Q}(e^{2\pi i/|G|_2})$ fixed by $\langle\tau\rangle$ is $\mathbb{Q}(\sqrt{\varepsilon 2})$, with $\varepsilon \in \{\pm 1\}$. Then the set $\mathrm{Irr}_{2',\tau}(G)$ consists of 2-rational characters.*

Proof Let $|G| = 2^n m$ with m odd. Let $\mathbb{Q}_m = \mathbb{Q}(e^{2\pi i/m})$. A character χ is $\langle\tau\rangle$-fixed if, and only if, $\mathbb{Q}_m(\chi) = \mathbb{Q}_m(\chi(g) \mid g \in G) \subseteq \mathbb{Q}_m(\sqrt{\varepsilon 2})$.

Let $\chi \in \mathrm{Irr}_{2',\tau}(G)$. Assume that χ is not 2-rational. By [9, Theorem C] we have that $i \in \mathbb{Q}_m(\chi) \subseteq \mathbb{Q}_m(\sqrt{\varepsilon 2})$, which is absurd by elementary Galois Theory. □

Let $|G| = 2^n m$ with m odd. Let $\mathbb{Q}_m = \mathbb{Q}(e^{2\pi i/m})$. By restriction, we can see σ as an element of $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q}_m) \cong \mathrm{Gal}(\mathbb{Q}(e^{\pi i/2^{n-1}})/\mathbb{Q})$ of order 2^{n-2}. Moreover, a character χ is $\langle\sigma\rangle$-fixed if, and only if, $\mathbb{Q}_m(\chi) = \mathbb{Q}_m(\chi(g) \mid g \in G) \subseteq \mathbb{Q}_m(\sqrt{2}i)$. By Theorem 2.5, the odd-degree irreducible characters fixed under the action of $\langle\sigma\rangle$ are 2-rational. In particular, for blocks B of maximal defect, the invariant $k_{0,\sigma}(B)$ counts the number of 2-rational characters of odd-degree in B.

Acknowledgements The author acknowledges support by grants PID2020-118193GA-I00 and PID2019-103854GB-I00 funded by MCIN/AEI/10.13039/501100011033 She would like to thank Eugenio Giannelli, Gunter Malle and Gabriel Navarro for useful conversations on the topic of this note. She extends her gratitude to the referees for their valuable corrections.

Funding Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement.

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