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Slice conformality and Riemann manifolds on quaternions and octonions

Graziano Gentili[1] · Jasna Prezelj[2,3,4] · Fabio Vlacci[5]

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Abstract

In this paper we establish quaternionic and octonionic analogs of the classical Riemann surfaces. The construction of these manifolds has nice peculiarities and the scrutiny of Bernhard Riemann approach to Riemann surfaces, mainly based on conformality, leads to the definition of slice conformal or slice isothermal parameterization of quaternionic or octonionic Riemann manifolds. These new classes of manifolds include slice regular quaternionic and octonionic curves, graphs of slice regular functions, the 4 and 8 dimensional spheres, the helicoidal and catenoidal 4 and 8 dimensional manifolds. Using appropriate Riemann manifolds, we also give a unified definition of the quaternionic and octonionic logarithm and n-th root function.

Keywords Slice regular functions · Conformal mappings · Riemann surfaces

Mathematics Subject Classification 30G35 · 30C35 · 30F99

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✉ Graziano Gentili
graziano.gentili@unifi.it

Jasna Prezelj
jasna.prezelj@fmf.uni-lj.si

Fabio Vlacci
fvlacci@units.it

1 DiMaI, Università di Firenze, Viale Morgagni 67/A, Florence, Italy

2 Fakulteta za matematiko in fiziko, Jadranska 19, 1000 Ljubljana, Slovenia

3 UP FAMNIT, Glagoljaška 8, Koper, Slovenia

4 IMFM, Jadranska 19, 1000 Ljubljana, Slovenia

5 DiSPeS, Università di Trieste, Piazzale Europa 1, Trieste, Italy

1 Preface

The initial project originating this paper was giving a well structured and unifying definition of the logarithm and n-th root functions in the quaternionic and octonionic settings. To this purpose, our first aim was to construct the quaternionic and octonionic analogs of the well known Riemann surface of the complex logarithm, which in the complex setting allows a complete understanding of this function and of its branches.

Indeed, the manifolds constructed with this aim revealed new, interesting and peculiar features, so that they captured the central position among the results of this paper.

We will illustrate how the project of this paper developed and, to begin with, point out that for the case of the principal branch of the logarithm, definitions were already given in the general setting of Clifford Algebras - see, e.g., [19, Definition 11.24, p. 231] - and also specialized to the case of quaternions - see, e.g., [10, Definition 3.4].

Let $\mathbb{K}$ be either the division algebra of quaternions $\mathbb{H}$ or the division algebra of octonions $\mathbb{O}$; we denote by $\dim \mathbb{K}$ the *real* dimension of $\mathbb{K}$, namely $\dim \mathbb{H} = 4$ and $\dim \mathbb{O} = 8$. Let $\mathbb{S}_{\mathbb{K}} \subset \mathbb{K}$ be the 2-sphere or, respectively, the 6-sphere of imaginary units, i.e. the sets of $I \in \mathbb{K}$ such that $I^2 = -1$. For the sake of simplicity, both in the case of quaternions and in the case of octonions we will simply write $\mathbb{S}$ instead of $\mathbb{S}_{\mathbb{K}}$ since no confusion can arise. The construction of the logarithm and its branches given in the complex case cannot be directly replicated in the quaternionic and octonionic environments. This is mainly due to the fact that the exponential function

$$\exp q = \sum_{n=1}^{\infty} \frac{q^n}{n!}$$

is an entire function (i.e., its domain of definition is $\mathbb{K}$), but cannot be used to define a covering of $\mathbb{K} \setminus \{0\}$. In fact, for all $0 \neq x \in \mathbb{R}$, the preimage of x is not a discrete set but consists of infinitely many 2 or 6 dimensional spheres. Indeed, for instance in the case $x < 0$, setting $\mathbb{S}(2k+1)\pi = \{q(2k+1)\pi : q \in \mathbb{S}\}$, we have

$$(\exp)^{-1}(x) = \{\log |x| + \mathbb{S}(2k+1)\pi : k \in \mathbb{Z}\}.$$

It follows that, contrarily to what happens in the case of the complex logarithm, no continuous branch of the quaternionic or octonionic logarithm can be defined on any open neighborhood of any strictly negative $x \in \mathbb{R}$. A similar phenomenon happens for all strictly positive $x \in \mathbb{R}$, except for the principal branch. To overcome this difficulty, we turn our attention to the construction of a 4-dimensional, respectively 8-dimensional, manifold obtained by blowing-up $\mathbb{K}$ along the real axis, and "adapting" it to become a domain of definition for the quaternionic or octonionic logarithm. Our natural approach to perform this construction passes through the recent theory of slice regular functions - see, e.g., the monograph [8] and references therein - and leads to the quaternionic and octonionic helicoidal Riemann manifolds (which are manifolds in the sense of [4]) inspired by the classical helicoidal surface of the space $\mathbb{R}^3$.

These manifolds, constructed with the purpose specified above, have new, interesting and peculiar features that attracted the attention of the authors and encouraged them to go back to the scrutiny of Bernhard Riemann approach to holomorphic functions and Riemann surfaces, which was mainly based on conformality, as in [20]. All this led to a deeper appreciation of the work of Riemann, to a nice surprise and to Definition 3.2 of *slice conformal or slice isothermal parameterization* and of *hypercomplex Riemann manifold*.

Indeed, the study of slice conformality and the investigation of quaternionic and octonionic Riemann manifolds became the true main subject of this paper.

Let $\langle\ ,\ \rangle$ denote the standard Euclidean scalar product in $\mathbb{R}^{\dim \mathbb{K}} \cong \mathbb{K}$ and, for any purely imaginary unit $I \in \mathbb{S}$, set

$$\mathbb{C}_I^{\perp} = \{q \in \mathbb{K} : \langle q, x + Iy\rangle = 0, \forall (x + Iy) \in \mathbb{C}_I\}$$

to be the orthogonal space to the slice $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$. A C^1 injective $\mathbb{R}^N$-valued immersion f defined on a suitable domain Ω of $\mathbb{K}$ is called slice conformal or slice isothermal immersion if, for any purely imaginary unit $I \in \mathbb{K}$ and any $x, y \in \mathbb{R}$, the differential $df(x + Iy)$ is such that both

$$df(x + Iy)_{|\mathbb{C}_I}$$

and

$$df(x + Iy)_{|\mathbb{C}_I^{\perp}}$$

are conformal. If this is the case, $f(\Omega)$ is called a hypercomplex Riemann manifold.

The nice surprise was that the quaternionic and octonionic spheres, the helicoidal and catenoidal manifolds, together with the natural quaternionic and octonionic curves, are all hypercomplex Riemann manifolds.

The study performed in [13] by Ghiloni and Perotti shows that the Jacobian matrix J_f of a slice regular function f is such that $\det(J_f) \geq 0$, i.e that f is orientation preserving. Slice conformality is indeed an extension of the definition of slice regularity, even in the case of $\mathbb{K}$-valued, orientation preserving immersions defined on a domain Ω of $\mathbb{K}$: for a fixed non real quaternion a, the function $f(q) = aq$ in not slice regular, but it is slice conformal (actually conformal) and orientation preserving. The following remark is basic to help placing the results of this paper in the right perspective.

Remark After recalling that the real differential df of a slice regular function $f : \Omega \to \mathbb{K}$ is conformal (if non singular) at all real points of the slice domain Ω (see, e.g., [8, Corollary 8.17.]), it is worthwhile noticing that to require that the differential df is conformal at all points of the domain of definition Ω may be too restrictive: by a classical result due to Liouville, for $n > 2$ a conformal map from a domain of $\mathbb{R}^n$ to $\mathbb{R}^n$ is a Möbius transformation.

In the paper a *standard set of curves* is applied to study the real differential of (smooth enough) injective $\mathbb{R}^N$-valued immersions f defined on suitable domains Ω of $\mathbb{K}$ (we point out that similar techniques were already introduced in [13, 17, 19]). As a result, the paper can exhibit a collection of quaternionic and octonionic Riemann manifolds, inspired by classical Riemann surfaces, which testify the interest of the approach.

Sub-manifolds of the helicoidal hypercomplex manifolds, endowed with suitable atlases which define different structures, provide a natural environment for the definition of the quaternionic and octonionic logarithm, and for their possible branches. Once done this, the construction of natural manifolds of definition for the n-th root quaternionic and octonionic functions is an easily doable step.

The paper is organized as follows. After a few preliminaries, which also subsume the approach to slice regular functions based on stem functions, Sect. 3 is dedicated to the definition and construction of classes of hypercomplex Riemann manifolds, including quaternionic and octonionic slice regular curves. This construction is based on Theorem 3.6, which studies slice conformal curves in terms of their stem functions, and calls into play the standard set of curves. In Sect. 4 we present other explicit examples of quaternionic and octonionic regular curves, which comprise the hypercomplex Riemann sphere, the helicoidal hypercomplex manifold, the catenoidal hypercomplex manifold and the study of the relations between them. Section 5 contains the presentation of natural manifolds for the definition of the quaternionic and octonionic logarithm. The same section contains the construction of the manifolds of the n-th root quaternionic and octonionic functions.

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2 Preliminaries

As we said, $\mathbb{K}$ denotes either $\mathbb{H}$ or $\mathbb{O}$, i.e., the algebras of quaternions or octonions, and $\mathbb{S} \subset \mathbb{K}$ denotes, respectively, the 2-sphere or 6-sphere of imaginary units, i.e. the set of $I \in \mathbb{K}$ such that $I^2 = -1$. Given any non real $q \in \mathbb{K}$, there exist (and are uniquely determined) an imaginary unit of $\mathbb{K}$, and two real numbers x and $y > 0$, such that $q = x + Iy$. With this notation, the conjugate of q will be $\bar{q} := x - Iy$ and $|q|^2 = q\bar{q} = \bar{q}q = x^2 + y^2$. In both cases, each imaginary unit I generates (as a real algebra) a copy of the complex plane denoted by $\mathbb{C}_I$. We call such a complex plane a *slice*.

Let Ω be a *slice domain* of $\mathbb{K}$, i.e., an open and connected subset containing real points and such $\Omega_I = \Omega \cap \mathbb{C}_I$ is a domain of $\mathbb{C}_I$ for all imaginary units $I \in \mathbb{S} \subset \mathbb{K}$. The set of slice regular functions on Ω is defined using a family of Cauchy-Riemann operators (see e.g. [8, 9]).

Definition 2.1 Let $\Omega \subseteq \mathbb{K}$ be a slice domain and let $f : \Omega \to \mathbb{K}$ be a function.

If, for an imaginary unit I of $\mathbb{K}$, the restriction $f_I := f_{|\Omega_I}$ has continuous partial derivatives and

$$\bar{\partial}_I f(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) \equiv 0 \tag{2.1}$$

then f_I is called *holomorphic*. If f_I is holomorphic for all imaginary units of $\mathbb{K}$, then the function f is called *slice regular*.

If f is a slice regular function, then the *Cullen* or *slice derivative of* f is defined as

$$f'_c(x + Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x + yI).$$

It turns out that f'_c is a slice regular function (see [8]) and from (2.1) one easily obtains that $f'_c = \frac{\partial f}{\partial x}$.

The property of being holomorphic along the slices Ω_I for all imaginary units I of $\mathbb{K}$, forces slice regular functions to be affine along entire regions of each sphere of type $x + \mathbb{S}y$.

In fact, the local representation formula for quaternionic slice regular functions on slice domains (see, e.g., [6, 7]), states that, if $L, M, N \in \mathbb{S}$, with $M \neq N$, are such that $x + Ly$, $x + My$, $x + Ny$ belong to a suitable open neighborhood U of $x + Iy$ in the 2-sphere $x + \mathbb{S}y$, then the local representation formula

$$\begin{aligned} f(x + Ly) = (M - N)^{-1}\left[Mf(x + My) - Nf(x + Ny)\right] \\ +L(M - N)^{-1}\left[f(x + My) - f(x + Ny)\right] \end{aligned} \tag{2.2}$$

holds and, for $y \neq 0$, the *spherical derivative* of f is defined by

$$f'_s(x + Iy) := y^{-1}(M - N)^{-1}\left[f(x + My) - f(x + Ny)\right]. \tag{2.3}$$

Moreover, f_s' is constant in the same neighborhood U of $x+Iy$ in $x+\mathbb{S}y$ (see, e.g., [7, Definition 3.1]). The analog of this representation formula holds for octonionic slice regular functions as well (see, e.g., [11, Proposition 6], [14, Formula (5)]). A subclass of the class of slice regular functions on a slice domain $\Omega \subseteq \mathbb{K}$ particularly resembles the class of holomorphic functions of one complex variable. These functions are defined as follows: a slice regular function $f:\Omega \to \mathbb{K}$ is said to be *slice preserving* if, and only if, for all imaginary units I of $\mathbb{S} \subset \mathbb{K}$, we have that $f(\Omega_I) \subseteq \mathbb{C}_I$, (see [9] for the case of octonions).

In a while we will make use of the notion of stem function, which was defined by Ghiloni–Perotti in [11], on a class of subsets of the complex plane $\mathbb{C}$.

Definition 2.2 A subset D of $\mathbb{C}=\mathbb{R}+i\mathbb{R}$ is said to be *symmetric* (in $\mathbb{C}$) if $\overline{D}=\{\overline{z}: z\in D\}$ coincides with D. The *(axial) symmetrization* $\widetilde{E}$ of a subset E of $\mathbb{K}$ is defined by

$$\widetilde{E}=\{x+Iy : x,y\in\mathbb{R}, I\in\mathbb{S}, (x+\mathbb{S}y)\cap E\neq\varnothing\}.$$

A subset Ω of $\mathbb{K}$ is called *(axially) symmetric* (in $\mathbb{K}$) if $\widetilde{\Omega}=\Omega$.

The following definition was given in [11] in the general case of a real alternative algebra endowed with an anti-involution (or $\mathbb{R}$). For the purpose of this paper, and for the sake of simplicity, we will restrict to the cases $\mathbb{R}$, $\mathbb{H}$, $\mathbb{O}$.

Definition 2.3 Let A denote either $\mathbb{R}$ or $\mathbb{K}$ and let $A_{\mathbb{C}} := A\otimes_{\mathbb{R}}\mathbb{C}$ be the complexification of A. Let us adopt the usual representation

$$A_{\mathbb{C}}=\{x+\iota y : x,y\in A\}$$

where $\iota^2=-1$. Consider a symmetric domain D of $\mathbb{C}$.

If a function $F:D\to A_{\mathbb{C}}$ is *complex intrinsic*, that is if $F(z)=\overline{F(\overline{z})}$ for all $z\in D$, then F is called an A-*stem function* (or *stem function*) on D.

If $F:D\to A_{\mathbb{C}}$ is a stem function expressed by

$$F(z)=F_1(z)+\iota F_2(z)$$

then the function $f:\widetilde{D}\to A$

$$f(x+Iy)=F_1(x+iy)+IF_2(x+iy)$$

is called the *slice function* induced by F.

Slice regular functions on symmetric slice domains can all be induced by stem functions, as the following result states (see, e.g., [11]).

Proposition 2.4 *If a slice domain Ω in $\mathbb{K}$ is axially symmetric, then any slice regular function $f:\Omega\to\mathbb{K}$ is induced by a holomorphic stem function $F:D=\Omega_i\to\mathbb{K}_{\mathbb{C}}$.*

As we have seen, stem functions can be defined in symmetric open subsets E of $\mathbb{C}$ that do not necessarily intersect the real axis. As a consequence, holomorphic stem functions induce special slice functions, still called *slice regular functions*, defined on symmetric domains $\widetilde{E}$ of $\mathbb{K}$ which do not necessarily intersect the real axis, so generalizing the initial notion of slice regularity to the class of so called *product domains* (see e.g. [15] for the terminology and the seminal paper on stem functions [11]).

In this paper we will be mainly concerned with slice regular functions defined on slice domains of $\mathbb{K}$, which in principle can be dealt with avoiding reference to stem functions. However, by admitting on the stage the point of view of stem functions, some results may be easily extended to the case of product domains; moreover, the generation of slice regular functions through holomorphic stem functions is exactly the same for the case of quaternions and octonions, and hence such an approach has the advantage to provide a natural unified vision in the two different environments, thus simplifying technicalities and presentation.

Remark 2.5 With reference to the notations of Definition 2.3, the following facts have been proven (see, e.g., [11]):

(a) If $F : D \to A_{\mathbb{C}}$ is expressed by $F(z) = F_1(z) + \iota F_2(z)$, with $F_j : D \to A$ for $j = 1, 2$, then F is complex intrinsic if and only if

$$F_1(z) = F_1(\overline{z}) \quad \text{and} \quad F_2(z) = -F_2(\overline{z}) \ \forall z \in D. \tag{2.4}$$

(b) If we take $A = \mathbb{R}$ then the slice function f induced by the stem function F is a slice preserving function (i.e. $f(\widetilde{D}_I) \subset \mathbb{C}_I \ \forall I \in \mathbb{S}$).
(c) The local representation formula (2.2) holds, by definition, for slice functions.
(d) If $F = F_1 + \iota F_2$ is a holomorphic stem function which induces the slice regular function f, then its (complex) derivative $F' = F_1' + \iota F_2'$ is also a holomorphic stem function which induces the slice derivative f_c' of f.
(e) If f is a slice function generated by the stem function F, then for $y \neq 0$,

$$f_s'(x + Iy) = y^{-1} F_2(x + iy). \tag{2.5}$$

For most of the remaining properties of slice regular functions that will be directly used in the sequel we will mainly refer the reader to [8, 9]. As for the main applications and developments of this theory, the reader can consult [1–3, 5–7, 12, 18], and e.g. [16] for generalizations.

3 Parameterized quaternionic and octonionic Riemann manifolds

Following the case of classical parameterized surfaces and parameterized Riemann surfaces in $\mathbb{R}^N$, we will give new definitions, useful in the quaternionc and octonionic settings of slice regular functions. As customary, a differentiable map will be called *an immersion* if its differential is injective at all points of the domain of definition.

Definition 3.1 Let n, N be natural numbers with $N \geq n$ and let Ω be a domain in $\mathbb{R}^n$. A C^1 immersion

$$f : \Omega \to \mathbb{R}^N$$

will be called a *conformal or isothermal map* if the matrix of the differential of f is conformal, i.e., if it satisfies

$${}^t df(x) df(x) = k(x) I_n$$

for a (never vanishing C^0) function $k : \Omega \to \mathbb{R}$.

Recall that, if $\langle\ ,\ \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{\dim \mathbb{K}} \cong \mathbb{K}$, then for $I \in \mathbb{S}$ the symbol $\mathbb{C}_I^\perp$ will denote the orthogonal complement of the slice $\mathbb{C}_I$.

Definition 3.2 Let Ω be a slice domain in $\mathbb{K} \cong \mathbb{R}^{\dim \mathbb{K}}$ and let $N \geq \dim \mathbb{K}$ be a natural number. Let $f : \Omega \to \mathbb{R}^N$ be a C^1 map (immersion). If for all $I \in \mathbb{S}$ and all $x, y \in \mathbb{R}$ the differential $df(x + Iy)$ is such that both

$$df(x + Iy)_{|\mathbb{C}_I}$$

and

$$df(x+Iy)_{|\mathbb{C}_I^\perp}$$

are conformal, then f will be called a *slice conformal or slice isothermal map (immersion)*.

If f is an injective immersion, then it will be called a *slice conformal or slice isothermal parameterization* and the parameterized manifold $f(\Omega)$ in $\mathbb{R}^N$ will be called a *(parameterized) hypercomplex Riemann manifold of* $\mathbb{R}^N$. *In particular, when* $\mathbb{K}=\mathbb{H}$ *we refer to it as a quaternionic Riemann manifold and in the case* $\mathbb{K}=\mathbb{O}$ *as a octonionic Riemann manifold.*

In case $f:\Omega\to\mathbb{R}^N$ itself is a conformal parameterization, then the parameterized hypercomplex Riemann manifold $f(\Omega)$ in $\mathbb{R}^N$ will be called a *special (parameterized) hypercomplex Riemann manifold* of $\mathbb{R}^N$.

The notion of parameterized quaternionic or octonionic Riemann manifold turns out to be quite natural, as the significant examples that we will present show. To construct the examples we will need a direct and easy method to compute the differential of a C^1 immersion f defined in a slice domain Ω of $\mathbb{K}\cong\mathbb{R}^{\dim\mathbb{K}}$ and with values in $\mathbb{R}^N$.

3.1 The standard set of curves and the case of the differential of a slice regular function

For $I\in\mathbb{S}$, let us consider a point $x+Iy\in\mathbb{C}_I\subset\mathbb{K}$ and choose $L\in\mathbb{S}$ orthogonal to I. In the same spirit of [13, proof of Proposition 3.1] and [17, Proposition 3.1], and with a similar purpose, we will use the following set of curves. For $y\neq 0$ set

(1) the curve $\alpha(t)=(x+t)+Iy$, such that $\alpha(0)=x+Iy$ and $\alpha'(0)=1$;
(2) the curve $\beta_I(t)=x+I(y+t)$, such that $\beta_I(0)=x+Iy$ and $\beta'(0)=I$;
(3) the curve $\Gamma_L(t)=x+\gamma(t)y$, where $\gamma(t)$ is an arc of a maximum circle C_γ of $\mathbb{S}$ such that $\gamma(0)=I$ and that $\gamma'(0)=\frac{L}{y}$; hence $\Gamma_L(0)=x+Iy$ and $\Gamma_L'(0)=L$;

Instead, when $y=0$ and so $x+Iy=x$, the first curve is

(1) $\alpha(t)=x+t$, such that $\alpha(0)=x$ and $\alpha'(0)=1$;

and the second two coherently become:

(2)–(3) $\beta_I(t)=x+It,\beta_L(t)=x+Lt$, such that $\beta_I(0)=\beta_L(0)=x$ and $\beta_I'(0)=I,\beta_L'(0)=L$.

In order to present the next definition we need to recall a well known fact: given any $I\in\mathbb{S}\subset\mathbb{K}$, then both in the case of quaternions and in the case of octonions, it is possible to complete $\{1,I\}$ to an orthonormal positively oriented standard basis

$$\{1,I,I_2,\dots,I_{\dim\mathbb{K}-1}\}$$

of the divison algebra $\mathbb{K}$ (see, e.g., [9] for the case of octonions).

Definition 3.3 *(Standard set of curves)* For any $I\in\mathbb{S}$, let us consider the point $x+Iy\in\mathbb{C}_I\subset\mathbb{K}$ and an orthonormal positively oriented standard basis $\{1,I,I_2,\dots,I_{\dim\mathbb{K}-1}\}$ of the division algebra $\mathbb{K}$. The *standard set of curves* at the point $x+Iy$ consists:

- for $y\neq 0$, of the curves $\{\alpha,\beta_I,\Gamma_{I_l},l=2,\dots,\dim\mathbb{K}-1\}$;
- for $y=0$, of the curves $\{\alpha,\beta_I,\beta_{I_l},l=2,\dots,\dim\mathbb{K}-1\}$.

We desire now to use the standard set of curves to calculate the differential df of f and to point out some of its features. Indeed, when naturally used with a slice regular function $f : \Omega \to \mathbb{K}$, defined on a slice domain Ω of $\mathbb{K}$, this set of curves reveals an easy tool to compute and directly interpret the real differential $df(x+Iy) : \mathbb{R}^{\dim \mathbb{K}} \to \mathbb{R}^{\dim \mathbb{K}}$ of the function f. But its full use will be seen in the sequel of this paper, in more general situations.

To calculate df, after fixing $I \in \mathbb{S}$, a direct computation shows that

$$\begin{aligned} df(x+Iy)1 &= df(x+Iy)\alpha'(0) = \frac{d}{dt}_{|0} f(\alpha(t)) = \frac{d}{dt}_{|0} f(x+t+Iy) \\ &= \frac{\partial f}{\partial x}(x+Iy)\,. \end{aligned}$$

Analogously, and since f is slice regular,

$$\begin{aligned} df(x+Iy)I &= df(x+Iy)\beta'(0) = \frac{d}{dt}_{|0} f(\beta(t)) \\ &= \frac{d}{dt}_{|0} f(x+I(y+t)) = \frac{\partial f}{\partial y}(x+Iy) = I\frac{\partial f}{\partial x}(x+Iy)\,. \end{aligned}$$

In particular we have that $I df(x+Iy)1 = I\frac{\partial f}{\partial x}(x+Iy) = \frac{\partial f}{\partial y}(x+Iy) = df(x+Iy)I$ and hence

$$df(x+Iy)_{|\mathbb{C}_I} = \left[f_c'(x+Iy), If_c'(x+Iy)\right].$$

Therefore, the real differential

$$df(x+Iy)_{|(\mathbb{R}+I\mathbb{R})} : \mathbb{R}^2 \to \mathbb{R}^{\dim \mathbb{K}}$$

is a conformal matrix. Let us now continue. The local representation formulas (2.2) and (2.3) yield, for any $L \in \mathbb{S}$ such that $L \perp I$

$$\begin{aligned} df(x+Iy)L &= df(x+Iy)\Gamma_L'(0) = \frac{d}{dt}_{|0} f(\Gamma_L(t)) \\ &= \frac{d}{dt}_{|0} (\gamma(t)(M-N)^{-1}\left[f(x+My) - f(x+Ny)\right]) \\ &= Ly^{-1}(M-N)^{-1}\left[f(x+My) - f(x+Ny)\right] \\ &= Lf_s'(x+Iy) \end{aligned}$$

hence we have that

$$df(x+Iy)_{|\mathbb{C}_I^{\perp}} = \left[I_2 f_s'(x+Iy), \ldots, I_{\dim \mathbb{K}-1} f_s'(x+Iy)\right].$$

Therefore, the real differential

$$df(x+Iy)_{|(\mathbb{R}+I\mathbb{R})^{\perp}} : \mathbb{R}^{\dim \mathbb{K}-2} \to \mathbb{R}^{\dim \mathbb{K}}$$

is a conformal matrix as well. Notice that even if both $df(x+Iy)_{|\mathbb{C}_I}$ and $df(x+Iy)_{|\mathbb{C}_I^{\perp}}$ are conformal, the full differential $df(x+Iy)$ may not be conformal in general.

3.2 The differential of a smooth slice function

In this subsection we exhibit the connection between conformality properties of a slice function defined in a symmetric slice domain $f : \Omega = \widetilde{D} \to \mathbb{K}$ and its stem function $F : D \to \mathbb{K}$. Since the local representation formula (2.2) holds for slice functions, then for

$y \neq 0$ we obtain, as seen above, the identity $df(x+Iy)L = Lf_s'(x+Iy) = Ly^{-1}F_2(x+iy)$ for every imaginary unit $L \perp I$; therefore, for $y \neq 0$, the restriction of the differential to the orthogonal complement of the slice $\mathbb{C}_I$ is conformal (if nonzero).

Assume now that $F \in \mathcal{C}^3(D)$. By Proposition 7(2) in [11], f is C^1 and hence we can calculate the differential $df(x+Iy)_{|\mathbb{C}_I}$:

$$df(x+Iy)1 = \partial_x F(x+iy) = \partial_x(F_1(x+iy) + IF_2(x+iy)),$$
$$df(x+Iy)I = \partial_y F(x+iy) = \partial_y(F_1(x+iy) + IF_2(x+iy)).$$

Therefore, in terms of the stem function F, for $y \neq 0$ by formula (2.5), the differential df may be written as

$$df(x+Iy) = \left[\partial_x F(x+iy) \quad \partial_y F(x+iy) \quad \frac{I_2F_2(x+iy)}{y} \ \dots \ \frac{I_{\dim \mathbb{K}-1}F_2(x+iy)}{y}\right].$$

Passing to the limit as $y \to 0$, then $\frac{F_2(x+Iy)}{y}$ tends to $\partial_y F_2(x)$ and so

$$df(x) = \left[\partial_x F(x) \quad \partial_y F(x) \quad I_2\partial_y F_2(x) \dots I_{\dim \mathbb{K}-1}\partial_y F_2(x)\right].$$

Therefore, for $y \neq 0$, $df(x+Iy)_{|\mathbb{C}_I^\perp}$ is conformal if and only if $F_2(x+iy) \neq 0$ and $df(x)_{|\mathbb{C}_I^\perp}$ is conformal if and only if $\partial_y F_2(x) \neq 0$.

In the case F is holomorphic and $y \neq 0$, the corresponding formula becomes

$$df = \left[f_c' \quad If_c' \quad I_2f_s' \dots I_{\dim \mathbb{K}-1}f_s'\right]$$

and, when $y = 0$, we have $f_s' = f_c'$ and so

$$df = \left[f_c' \quad If_c' \quad I_2f_c' \dots I_{\dim \mathbb{K}-1}f_c'\right]$$

which implies that $df(x)$ is conformal if $f_c'(x) \neq 0$.

Let's sum up these observations in the following

Proposition 3.4 *Let A denote either $\mathbb{R}$ or $\mathbb{K}$, and let $f : \widetilde{D} \to A$ be a slice function generated by a C^3 stem function F defined on a symmetric domain $D \subset \mathbb{C} = \mathbb{R} + i\mathbb{R}$. Then $df(x+Iy)_{|\mathbb{C}_I^\perp}$ is conformal if nondegenerate. Moreover,*

- (a) *if dF is conformal on D, then f is slice conformal on $\widetilde{D}$. In particular, if F is holomorphic, then f is slice regular and hence a slice conformal immersion if df has full rank;*
- (b) *if $A = \mathbb{R}$, dF is conformal, $\partial_y F_2 \neq 0$ on $\mathbb{R} \cap D$ and if $F_2 \neq 0$ on $D \setminus \mathbb{R}$, then f is slice preserving and slice conformal on $\widetilde{D}$.*

Proof We are left to consider only the case $A = \mathbb{R}$. Since f is slice preserving, then df written with respect to the decomposition $\mathbb{K} = \mathbb{C}_I \oplus {\mathbb{C}_I}^\perp$ is of the form

$$\begin{bmatrix} df(x+Iy)_{|\mathbb{C}_I} & 0 \\ 0 & df(x+Iy)_{|\mathbb{C}_I^\perp} \end{bmatrix}.$$

□

Remark 3.5 Notice that if a stem function F is conformal, it is not necessarily holomorphic. In the case $A = \mathbb{R}$, the stem function $F : D \to A_\mathbb{C}$ is conformal if and only if F is either holomorphic or antiholomorphic or, to put it differently, if and only if $df(x+Iy)_{|\mathbb{C}_I}$ is conformal on $\widetilde{D}$. Furthermore, notice that conformality of both $df(x+Iy)_{|\mathbb{C}_I}$ and $df(x+Iy)_{|\mathbb{C}_I^\perp}$ does not imply that df has full rank.

3.3 Slice conformal curves

Using Proposition 3.4, we can state the following result.

Theorem 3.6 *Let A denote either $\mathbb{R}$ or $\mathbb{K}$. Let D be a symmetric domain in $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ and $G, H : D \to A_{\mathbb{C}}$ be stem functions with $G, H \in \mathcal{C}^3(D)$. Write $G = G_1 + \iota G_2$, $H = H_1 + \iota H_2$ and let $F_1 = (G_1, H_1)$, $F_2 = (G_2, H_2)$. Let*

$$f : \widetilde{D} \to \mathbb{K} \times \mathbb{K}$$

be the slice curve induced by the map $F = (G, H) = F_1 + \iota F_2 : D \to A^2_{\mathbb{C}}$ in the following way

$$\begin{aligned} f(x + Iy) &= (G_1(x + iy) + IG_2(x + iy), H_1(x + iy) + IH_2(x + iy)) \\ &=: (g(x + Iy), h(x + Iy)). \end{aligned}$$

Assume that:

(a) *the differential dF is conformal on D;*
(b) *the partial derivative $\partial_y F_2 \neq 0$ on $\mathbb{R} \cap D$ and $F_2 \neq 0$ on $D \setminus \mathbb{R}$.*

Then $df(x + Iy)|_{|\mathbb{C}_I}$ and $df(x + Iy)|_{|\mathbb{C}_I^{\perp}}$ are both conformal. If, in addition, f is an injective immersion, then f is a slice conformal parameterization of $f(\widetilde{D})$.

In the case $A = \mathbb{R}$, if F is injective, then $F_2 \neq 0$ on $D \setminus \mathbb{R}$ is automatically fulfilled; hence if we assume that F is injective, dF is conformal on D and $\partial_y F_2 \neq 0$ on $\mathbb{R} \cap D$, then f is an injective immersion, and hence a slice conformal parameterization of $f(\widetilde{D})$.

Proof To prove the first part of the theorem, notice that by Remark 3.5 the conformality of dF implies $df_{|\mathbb{C}_I}$ conformal. The assumption (b) and Proposition 3.4 imply that $df(x + Iy)_{|\mathbb{C}_I^{\perp}}$ is conformal.

We are left to prove that, if $A = \mathbb{R}$ and F is injective, then f is injective and df has full rank. Notice that the non vanishing of F_2 off the real axis follows from the injectivity of F: indeed, at least one of the values $G_2(x + iy)$ or $H_2(x + iy)$ must be nonzero, otherwise $F(x + iy) = F(x - iy)$.

Let us first show that injectivity of F implies the injectivity of f. To this aim, consider $z = x + iy$, $w = u + iv \in D$ and assume that $f(x + Iy) = f(u + Jv)$. Then

$$\begin{aligned} G_1(x + iy) + IG_2(x + iy) &= G_1(u + iv) + JG_2(u + iv), \\ H_1(x + iy) + IH_2(x + iy) &= H_1(u + iv) + JH_2(u + iv). \end{aligned}$$

By assumption G_l and H_l, $l = 1, 2$ are real valued and by (2.4) the functions G_2, H_2 vanish at real points, so we have the following:

$$\begin{aligned} &G_1(x + iy) = G_1(u + iv) = G_1(u - iv), \\ &H_1(x + iy) = H_1(u + iv) = H_1(u - iv), \\ &J = I : G_2(x + iy) = G_2(u + iv),\ H_2(x + iy) = H_2(u + iv), \\ &J = -I : G_2(x + iy) = -G_2(u + iv),\ H_2(x + iy) = -H_2(u + iv), \\ &J \neq \pm I : G_2(x + iy) = G_2(u + iv) = H_2(x + iy) = H_2(u + iv) = 0. \end{aligned}$$

The injectivity of F excludes the last possibility unless $y = v = 0$. In this case $F(x) = F(u)$, so $x = u$. If $J = I$ then $F(x + iy) = F(u + iv)$ so $x + Iy = u + Iv$. If $J = -I$ then $G_2(x+iy) = -G_2(u+iv) = G_2(u-iv)$, $H_2(x+iy) = -H_2(u+iv) = H_2(u-iv)$. Because

G_2, H_2 are even in y we have $F(x+iy) = F(u-iv)$ which implies that $x+iy = u-iv$ and hence $x+Iy = x+(-I)(-y)$, so f is injective.

To see that the rank of df is full, notice that on the real axis $\partial_y F_2$ does not vanish by assumption, and F_2 does not vanish off the real axis. Since both $df_{|\mathbb{C}_I}$ and $df_{|\mathbb{C}_I^\perp}$ are conformal and df has the following block structure

$$df = \begin{bmatrix} df_{|\mathbb{C}_I} & df_{|\mathbb{C}_I^\perp} \end{bmatrix} = \begin{bmatrix} dg_{|\mathbb{C}_I} & 0 \\ 0 & dg_{|\mathbb{C}_I^\perp} \\ dh_{|\mathbb{C}_I} & 0 \\ 0 & dh_{|\mathbb{C}_I^\perp} \end{bmatrix},$$

the rank of df is full. □

Remark 3.7 With reference to the preceding statement and proof, notice that conformality of F does not imply conformality of G and H.

Remark 3.8 A statement analogous to the one of Theorem 3.6 holds in a "n-vectorial" version, i.e., for maps $F : D \to A_{\mathbb{C}}^n$ defined by n-tuples of stem functions.

3.4 Quaternionic and octonionic slice regular curves

We will use the standard notion of curve in the quaternionic and octonionic setting.

Definition 3.9 Let $\Omega \subseteq \mathbb{K}$ be a slice domain, and let

$$f : \Omega \to \mathbb{K}^2$$
$$f(q) = (g(q), h(q))$$

be a map whose components $g, h : \Omega \to \mathbb{K}$ are slice regular functions. If f is an immersion, then f will be called *a slice regular curve (in $\mathbb{K}^2$).*

Let us now consider a slice regular curve $f : \Omega \to \mathbb{K}^2$, with slice regular components $g, h : \Omega \to \mathbb{K}$, and choose any $I \in \mathbb{S}$. Using the standard set of curves defined in Sect. 3.1, we get that the differential

$$df : \mathbb{R}^{\dim \mathbb{K}} \to \mathbb{K}^2$$

assumes the form

$$df = \begin{bmatrix} g_c' & Ig_c' & I_2 g_s' & \dots & I_{\dim \mathbb{K}-1} g_s' \\ h_c' & Ih_c' & I_2 h_s' & \dots & I_{\dim \mathbb{K}-1} h_s' \end{bmatrix}.$$

The first 2 columns of this $(2 \dim \mathbb{K}) \times \dim \mathbb{K}$ real matrix, and separately the last $(\dim \mathbb{K} - 2)$ columns of the same matrix, are orthogonal to each other and with the same norms, and hence F is slice isothermal. In conclusion we have proved

Proposition 3.10 *Let $\Omega \subseteq \mathbb{K}$ be a slice domain, and let $f : \Omega \to \mathbb{K}^2$ be a slice regular curve. If f is injective, then $f(\Omega)$ is a parameterized hypercomplex Riemann manifold in $\mathbb{K}^2$, and the map $f : \Omega \to f(\Omega)$ is a slice conformal parameterization. In particular, graphs of slice regular curves are parameterized hypercomplex Riemann manifolds in $\mathbb{K}^2$.*

As we already pointed out, in general f is (a slice conformal but) not a conformal parameterization. It is well known in fact that the slice regular functions f, g are in general not conformal at non real points of Ω (see, e.g., [8]), and hence f cannot be a conformal parameterization in general.

We end this section with a natural question, on how the quaternionic or octonionic parameter can be changed between slice regular quaternionic or octonionic curves having the same image. Indeed, let us consider $\Omega, \Omega' \subseteq \mathbb{K}$ slice domains, $f = (f_1, f_2) : \Omega \to \mathbb{K}^2$ and $g = (g_1, g_2) : \Omega' \to \mathbb{K}^2$ injective, slice regular curves with the same image $f(\Omega) = g(\Omega')$. In this situation, we may assume that locally g_1 is injective. Then the local equalities $f_1(q) = g_1(q')$ and $f_2(q) = g_2(q')$ imply

$$f_2 = g_2 \circ (g_1^{-1} \circ f_1)$$

and since $f_2, g_2 : \Omega' \to \mathbb{K}$ are slice regular functions, this functional equation is in general not valid. Nevertheless, we know that it holds if, for instance, $g_1^{-1} \circ f_1 : \Omega \to \Omega'$ is a slice preserving regular function. Hence, we can make the following

Remark 3.11 Let f and g be injective immersions having the same image $\Gamma \subseteq \mathbb{K}^2$. If a change of quaternionic or octonionic parameter between f and g is a slice preserving invertible function, then f is a slice conformal parameterization if, and only if, g is a slice conformal parameterization.

What established in this section can be directly reformulated for the case of slice regular curves $f : \Omega \to \mathbb{K}^n$ defined on slice domains $\Omega \subseteq \mathbb{K}$. To conclude, we point out that Remark 3.11 is valid in a more general setting, as explained in the next remark.

Remark 3.12 Let f be a slice isothermal parameterization having the hypercomplex Riemann manifold $\Gamma \subseteq \mathbb{R}^N$ as its image. Then, for every regular slice preserving invertible change of parameter ϕ between slice domains, the map $f \circ \phi$ is a slice isothermal parameterization for the hypercomplex Riemann manifold Γ.

The following remark should better explain the definition of slice conformal immersion that has been adopted.

Remark 3.13 Let $\{1, i, j, k\}$ be the standard basis of $\mathbb{H}$, and let $f : \mathbb{H} \to \mathbb{H}^2$ be the function

$$f(x + Iy) = (x + Iy, x + \psi(I)y)$$

where $\psi : \mathbb{S} \to \mathbb{S}$ is the odd C^∞ function defined by

$$\psi(\alpha i + \beta j + \gamma k) = \frac{\alpha^3 i + \beta j + \gamma^3 k}{\sqrt{\alpha^6 + \beta^2 + \gamma^6}},$$

i.e., when $\langle\ ,\ \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^4 \cong \mathbb{H}$, by

$$\psi(I) = \frac{\langle I, i\rangle^3 i + \langle I, j\rangle j + \langle I, k\rangle^3 k}{\sqrt{\langle I, i\rangle^6 + \langle I, j\rangle^2 + \langle I, k\rangle^6}}.$$

While applying the standard set of curves, take the point $x + Iy = x + iy$ (i.e., $I = i$) with $y \neq 0$, choose $J = j$ and use the curves

$$\Gamma_j(t) = i\cos(t/y) + j\sin(t/y) = i\exp(-k(t/y)),$$
$$\Gamma_k(t) = i\cos(t/y) + k\sin(t/y) = i\exp(j(t/y)).$$

Direct computations show that

$$
\begin{aligned}
df(x+iy)1 =&(1,1), \quad df(x+iy)i = (i,\psi(i)) = (i,i)\,,\\
df(x+iy)j =&\frac{d}{dt}_{|0}\left(x+i\exp(-k(t/y))y,\, x+\frac{i\cos^3(t/y)+j\sin(t/y)}{\sqrt{\cos^6(t/y)+\sin^2(t/y)}}y\right)\\
=&(j,j)\,,\\
df(x+iy)k =&\frac{d}{dt}_{|0}\left(x+i\exp(j(t/y))y,\, x+\frac{i\cos^3(t/y)+j\sin^3(t/y)}{\sqrt{\cos^6(t/y)+\sin^6(t/y)}}y\right)\\
=&(k,0)\,.
\end{aligned}
$$

Thus:

$$
df(x+iy) = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

and hence the last two columns have different norms. In conclusion such an f is not a slice isothermal parameterization.

4 Other examples of hypercomplex Riemann manifolds

4.1 The Riemann sphere

This example generalizes to real dimensions 4 and 8 the case of the Riemann sphere in the complex setting.

Proposition 4.1 *Let us set* $m = \dim \mathbb{K} \in \{4, 8\}$. *Consider the unit sphere* $S^m \subset \mathbb{R}^{m+1} \cong \mathbb{K} \times \mathbb{R}$ *and the inverse of the stereographic projection from the north pole* $N = (0, \dots, 0, 1)$ *of* S^m *onto the equatorial plane* $\mathbb{K} \cong \mathbb{R}^m$, *namely*

$$f : \mathbb{R}^m \cong \mathbb{K} \to S^m \setminus \{N\} \subset \mathbb{K} \times \mathbb{R} \cong \mathbb{R}^{m+1}, \tag{4.6}$$

defined by

$$f(x+Iy) = \left(\frac{2(x+Iy)}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right). \tag{4.7}$$

Then $S^m \setminus \{N\}$ *is a special parameterized hypercomplex Riemann manifold and the map* f *is a conformal parameterization. Analogous statement can be proved for the stereographic projection from the south pole* S.

Proof It is well known that the inverse of stereographic projection is conformal, the rest follows. □

We can now conclude by exhibiting the "Riemann" structures of 1-dimensional quaternionic manifolds of the spheres $S^4 \subset \mathbb{R}^5$ and $S^8 \subset \mathbb{R}^9$. In the case of S^4, this structure corresponds to that of slice quaternionic manifold, as defined in [4]. In the case of S^8, it corresponds to a natural generalization to the case of octonions.

Theorem 4.2 *Let us set* $m = \dim \mathbb{K} \in \{4, 8\}$. *Let* f *and* h *be the following maps*

$$\begin{aligned} f &: \mathbb{R}^m \cong \mathbb{K} \to S^m \setminus \{N\} \subset \mathbb{K} \times \mathbb{R} \cong \mathbb{R}^{m+1} \\ f(x+Iy) &= \left(\frac{2(x+Iy)}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \end{aligned}$$

and

$$\begin{aligned} h &: \mathbb{R}^m \cong \mathbb{K} \to S^m \setminus \{S\} \subset \mathbb{K} \times \mathbb{R} \cong \mathbb{R}^{m+1} \\ h(x+Iy) &= \left(\frac{2(x-Iy)}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right). \end{aligned}$$

Then the differentiable conformal atlas $\{(\mathbb{K}, f), (\mathbb{K}, h)\}$ *endows* $S^m \subset \mathbb{R}^{m+1}$ *with a structure of slice quaternionic or slice octonionic manifold.*

Proof A direct computation shows that the transition map

$$h^{-1} \circ f = \overline{g^{-1} \circ f} : \mathbb{K} \setminus \{0\} \to \mathbb{K} \setminus \{0\}$$

has the form

$$(h^{-1} \circ f)(q) = \overline{(g^{-1} \circ f)(q)} = \frac{\bar{q}}{q^2} = \frac{1}{q}$$

and hence it is a slice regular and slice preserving function. □

4.2 The helicoidal hypercomplex manifold

This further example generalizes to quaternions and octonions the case of the helicoid in the complex setting, whose classical slice isothermal parameterization is given by $g : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ defined as $g(x+iy) = (\sinh x \cos y + i \sinh x \sin y, y)$.

Proposition 4.3 *Let the map*

$$f : \mathbb{K} \to \mathbb{K} \times \mathrm{Im}(\mathbb{K})$$

be defined by

$$f(x+Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)$$

for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. *Then* $f(\mathbb{K})$ *is a parameterized hypercomplex Riemann manifold (diffeomorphic to* $\mathbb{K}$*) and* f *is a slice isothermal parameterization. This manifold will be called* quaternionic helicoidal manifold *if* $\mathbb{K} = \mathbb{H}$ *or* octonionic helicoidal manifold *if* $\mathbb{K} = \mathbb{O}$, *and denoted by* $\mathscr{E}$.

Proof The map f is induced by the stem map

$$\begin{aligned} &F = (G, H) : \mathbb{C} \to (\mathbb{R} + \imath\mathbb{R})^2, \\ &G(x+iy) = \sinh x(\cos y + \imath \sin y), \; H(x+iy) = \imath y \end{aligned}$$

whose components are those of the classical conformal parametrization of the helicoid. We need to check that the assumptions of Theorem 3.6 hold. The injectivity of F is obvious since the last component is injective in y and the first is injective in x; moreover, $H_2(x+iy) = y \neq 0$ on $\mathbb{C} \setminus \mathbb{R}$ and $\partial_y H_2(x) = 1 \neq 0$ on $\mathbb{R}$. Now, since as we said dF is conformal, then by Theorem 3.6, the map f is a slice conformal parameterization, and the proof is complete. □

It may be interesting to see how the use of the standard set of curves leads to the explicit calculation of the differential of the slice isothermal parameterization of the helicoidal manifold

$$f(x+Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy).$$

For a fixed $x + Iy \in \mathbb{C}_I$, let us compute $df(x+Iy)1$ and $df(x+Iy)I$:

$$\begin{aligned} df(x+Iy)1 &= \frac{d}{dt}_{|0} (\sinh(x+t)\cos y + I \sinh(x+t)\sin y, Iy) \\ &= (\cosh x \cos y + I \cosh x \sin y, 0) \\ df(x+Iy)I &= \frac{d}{dt}_{|0} (\sinh x \cos(y+t) + I \sinh x \sin(y+t), I(y+t)) \\ &= (-\sinh x \sin y + I \sinh x \cos y, I). \end{aligned}$$

Moreover, from Proposition 3.4 we know that for $l = 2, \ldots, \dim \mathbb{K} - 1$

$$df(x+Iy)I_l = \left(I_l \frac{\sinh x \sin y}{y}, I_l \right).$$

In the case $\mathbb{K} = \mathbb{H}$, if we set

$$\mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$$

and

$$\begin{aligned} \mathbb{H} \times \mathrm{Im}(\mathbb{H}) &\ni (x_1 + x_2 I + x_3 J + x_4 K, y_2 I + y_3 J + y_4 K) \\ &\cong (x_1, x_2, x_3, x_4, y_2, y_3, y_4) \in \mathbb{R}^7, \end{aligned}$$

then, for $y \neq 0$, we get

$$df(x+Iy) = \begin{bmatrix} \cosh x \cos y & -\sinh x \sin y & 0 & 0 \\ \cosh x \sin y & \sinh x \cos y & 0 & 0 \\ 0 & 0 & \frac{\sinh x \sin y}{y} & 0 \\ 0 & 0 & 0 & \frac{\sinh x \sin y}{y} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and for $y = 0$, we have, taking the limit (and coherently with the use of the standard curves):

$$df(x) = \begin{bmatrix} \cosh x & 0 & 0 & 0 \\ 0 & \sinh x & 0 & 0 \\ 0 & 0 & \sinh x & 0 \\ 0 & 0 & 0 & \sinh x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As expected, $df(x+Iy)$ is slice conformal and $df(x)$ is conformal. The case $\mathbb{K} = \mathbb{O}$ is completely analogous.

4.3 The catenoidal hypercomplex manifold

The case of the catenoid in the complex setting, parameterized by the conformal map

$$g : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$$

defined by

$$g(x+iy) = (\cosh x \cos y + i \cosh x \sin y, x)$$

generalizes to quaternions and octonions as well.

Proposition 4.4 *Let the map*

$$f : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{K} \times \mathbb{R} \cong \mathbb{R}^{\dim \mathbb{K}+1}$$

be defined by

$$f(x+Iy) = (\cosh x \cos y + I \cosh x \sin y, x)$$

Then $f(\mathbb{R} \times \mathbb{S}(-\pi, \pi))$ is a parameterized hypercomplex Riemann manifold and f is a slice isothermal parameterization. This manifold will be called quaternionic catenoidal manifold *if $\mathbb{K} = \mathbb{H}$ or* octonionic catenoidal manifold *if $\mathbb{K} = \mathbb{O}$.*

Proof The map f is induced by the stem map

$$\begin{aligned} &F = (G, H) : \mathbb{R} \times (-i\pi, i\pi) \to (\mathbb{R} + \iota\mathbb{R})^2, \\ &G(x+iy) = \cosh x(\cos y + \iota \sin y), H(x+iy) = x, \end{aligned}$$

whose components are those of the classical conformal parametrization of the catenoid.

Obviously the map F is injective on $\mathbb{R} \times (-i\pi, i\pi)$, and $G_2(x, y) \neq 0$ outside the real axis; moreover the derivative $\partial_y G_2(x, 0) = \cosh x$ never vanishes. Since dF is conformal, then by Theorem 3.6, the map f is a slice conformal parameterization. This completes the proof. □

Again, it may be interesting to explicitly present the differential of the slice isothermal parameterization of the catenoidal manifold

$$f(x+Iy) = (\cosh x \cos y + I \cosh x \sin y, x)$$

which may be computed by means of the standard set of curves. In the case $\mathbb{K} = \mathbb{H}$, if we set

$$\mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$$

and

$$\mathbb{H} \times \mathbb{R} \ni (x_1 + x_2 I + x_3 J + x_4 K, y_1) \cong (x_1, x_2, x_3, x_4, y_1) \in \mathbb{R}^5$$

then, for $y \neq 0$, we have

$$df(x+Iy) = \begin{bmatrix} \sinh x \cos y & -\cosh x \sin y & 0 & 0 \\ \sinh x \sin y & \cosh x \cos y & 0 & 0 \\ 0 & 0 & \frac{\cosh x \sin y}{y} & 0 \\ 0 & 0 & 0 & \frac{\cosh x \sin y}{y} \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, for $y = 0$, we coherently obtain:

$$df(x) = \begin{bmatrix} \sinh x & 0 & 0 & 0 \\ 0 & \cosh x & 0 & 0 \\ 0 & 0 & \cosh x & 0 \\ 0 & 0 & 0 & \cosh x \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Again, $df(x+Iy)$ is slice conformal and $df(x)$ is conformal. In the case of octonions we obtain similar matrices.

As in the real case, once both naturally embedded in $\mathbb{K}^2$, the catenoidal hypercomplex manifold can be transformed to a part of an helicoidal hypercomplex manifold through a family of parameterized hypercomplex Riemann manifolds.

Let the part of the helicoidal manifold embedded in $\mathbb{K}^2$ be parameterized by $h : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{K}^2$, induced by the stem map

$$H(x+iy) := (\sinh x(\cos y + \iota \sin y), \iota y)$$

and the embedded catenoidal manifold parameterized by $c : \mathbb{R} \times \mathbb{S}(-\pi, \pi) \to \mathbb{K}^2$, induced by the stem map

$$C(x+iy) := (\cosh x(\cos y + \iota \sin y), x).$$

We claim that

$$H_\theta := H \cos\theta + C \sin\theta, \ \theta \in [0, \pi/2]$$

defines a family of conformal injective immersions with $H_0 = H$, $H_{\pi/2} = C$.

The differential $dH_\theta : \mathbb{C} \to \mathbb{K} \times \mathbb{K} \cong \mathbb{R}^{2\dim \mathbb{K}}$ is given by

$$dH_\theta(x+iy) = \begin{bmatrix} A \cos y & -B \sin y \\ A \sin y & B \cos y \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix},$$

where $A = (\cosh x \cos\theta + \sinh x \sin\theta)$ and $B = (\cosh x \sin\theta + \sinh x \cos\theta)$. It is obvious that the columns are orthogonal to each other, and a direct computation shows that their norms are equal. If we write $H_\theta = (F_\theta, G_\theta)$, then $G_\theta(x+iy) = x \sin\theta + \iota y \cos\theta$. If $x \sin\theta + \iota y \cos\theta = u \sin\theta + \iota v \cos\theta$, either $\theta = 0$ (and then we have the stem function for the helicoidal manifold, for which the injectivity has already been proved), or $x = u$. Then either $\theta = \pi/2$ (in which case we have the stem map for the catenoidal surface) or else $\iota y = \iota v$ so $x + iy = u + iv$. As a consequence, $G_{\theta,2}$ is injective for all $\theta \in (0, \pi/2)$. Because $\partial_y G_{\theta,2}(x) = \cos\theta \neq 0$ on the real axis for all $\theta \in (0, \pi/2)$, all the conditions of Theorem 3.6 are fulfilled and hence H_θ induces a family of slice conformal injective immersions.

5 The hypercomplex logarithm and *n*-th root

5.1 The hypercomplex logarithm

To define the complex logarithm one usually uses either the helicoid or the graph of the exponential function. Since we have shown that the latter in case of $\mathbb{K}$ is a hypercomplex manifold, the logarithm can be defined using the projection on the second coordinate (compare Remark 5.7).

We will show here how the helicoidal hypercomplex manifold defined in the previous section can be adapted to be the natural domain for the definition of a quaternionic logarithm. Compared to the logarithm defined by the graph of exponential function, this definition facilitates the identification of the argument and is therefore easier to use in the constructions which include continuations of the logarithm.

Proposition 5.1 *Let* $f : \mathbb{K} \to \mathbb{K} \times \mathrm{Im}(\mathbb{K})$ *be the map defined by*

$$f(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)$$

for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. *Let* $\mathbb{K}^+ = \{q \in \mathbb{K} : \mathrm{Re}\, q > 0\}$, *and set* $\mathscr{E}^+_{\mathbb{K}} := f(\mathbb{K}^+)$.

The $\mathscr{E}^+_{\mathbb{K}}$*-exponential map*

$$E : \mathbb{K} \to \mathscr{E}^+_{\mathbb{K}} \subset \mathbb{K} \times \mathrm{Im}(\mathbb{K})$$

defined by:

$$E(x + Iy) = (\exp(x + Iy), Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$$

is an immersion and a diffeomorphism between $\mathbb{K}$ *and* $\mathscr{E}^+_{\mathbb{K}}$. *In the case of quaternions, it endows* $\mathscr{E}^+_{\mathbb{H}}$ *with a structure of slice quaternionic manifold (see, e.g., [4]), which is different from the structure of hypercomplex Riemann manifold defined in Proposition 4.3. However, this manifold will be denoted simply by* $\mathscr{E}^+_{\mathbb{K}}$, *and called* the logarithm manifold.

Proof The proof replicates part of the one of Proposition 4.3. □

Remark 5.2 (a) If $\pi : \mathbb{K} \times \mathrm{Im}(\mathbb{K}) \to \mathbb{K}$ denotes the projection on the first factor, then by definition the following equality holds

$$(\pi \circ E)(q) = \exp(q)$$

for all $q \in \mathbb{K}$.

(b) Unlike what happens in the complex setting, the map $\pi : \mathscr{E}^+_{\mathbb{K}} \to \mathbb{K}$ is not a covering. It is not an open map as well, due to the fact that $\exp : \mathbb{K} \to \mathbb{K}$ is not an open map (it has a non–empty degenerate set consisting of spheres).

We will now define the $\mathscr{E}^+_{\mathbb{K}}$-logarithm on $\mathscr{E}^+_{\mathbb{K}}$ and exhibit some of its properties.

Definition 5.3 Let $\mathscr{E}^+_{\mathbb{K}}$ be the logarithm manifold. The $\mathscr{E}^+_{\mathbb{K}}$-logarithm

$$L : \mathscr{E}^+_{\mathbb{K}} \subset \mathbb{K} \times \operatorname{Im}(\mathbb{K}) \to \mathbb{K}$$

is defined as follows, in terms of the real logarithm log,

$$L(q, p) := \log |q| + p,$$

where p is called the *argument* of q, and denoted by $\operatorname{Arg}(q)$: hence

$$L(q, p) := \log |q| + \operatorname{Arg}(q).$$

Indeed, if $(q, p) \in \mathscr{E}^+_{\mathbb{K}}$, then $q = r \exp p$ for $r = |q|$ and L can be rewritten as

$$L(r \exp p, p) = \log r + p.$$

The following result and definition explain why the logarithm manifold is a natural domain of definition for the $\mathscr{E}^+_{\mathbb{K}}$-logarithm. Indeed, this hypercomplex manifold plays the role of an "adapted" blow-up of $\mathbb{K}$ at points $x \in \mathbb{R}$ with $x \neq 0$.

Proposition 5.4 *The map*

$$L : \mathscr{E}^+_{\mathbb{K}} \to \mathbb{K}$$

is the inverse of the $\mathscr{E}^+_{\mathbb{K}}$-exponential E, and a diffeomorphism from $\mathscr{E}^+_{\mathbb{K}}$ to $\mathbb{K}$.

Proof Let us read the $\mathscr{E}^+_{\mathbb{K}}$-logarithm through the parameterization

$$E(x + Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$$

of $\mathscr{E}^+_{\mathbb{K}}$. By composition we get that $L \circ E$ becomes the identity map of $\mathbb{K}$

$$x + Iy \mapsto (\exp x(\cos y + I \sin y), Iy) \mapsto \log(\exp x) + Iy = x + Iy\,.$$

Analogously, $E \circ L$ becomes the identity map of $\mathscr{E}^+_{\mathbb{K}}$

$$(r \exp p, p) \mapsto \log r + p \mapsto (\exp(\log r) \exp p, p) \mapsto (r \exp p, p).$$

The assertion is now proved. □

As a consequence, in the case of quaternions, the map L is a slice regular map from the logarithm manifold $\mathscr{E}^+_{\mathbb{H}}$ to $\mathbb{H}$, with respect to the structure of slice regular manifold induced by E on $\mathscr{E}^+_{\mathbb{H}}$ (see, e.g., [4]). We point out that the definition of the $\mathscr{E}^+_{\mathbb{K}}$-logarithm L is not referred to the structure of helicoidal Riemann manifold defined on $\mathscr{E}_{\mathbb{K}}$ in Proposition 4.3.

Definition 5.5 Let $\pi : \mathscr{E}^+_{\mathbb{K}} \subset \mathbb{K} \times \operatorname{Im}(\mathbb{K}) \to \mathbb{K} \setminus \{0\}$ denote the natural projection

$$(q, p) \mapsto q$$

and let $\Omega \subset \mathscr{E}^+_{\mathbb{K}}$ be a path connected subset such that $\pi_{|\Omega}$ is injective. Then, the map

$$\log_{\mathbb{K}} : \pi(\Omega) \to \mathbb{K}$$

defined by

$$\log_{\mathbb{K}} q = L(\pi^{-1}_{|\Omega}(q))$$

is called a *branch or a determination of the hypercomplex logarithm* on $\pi(\Omega)$.

Notice that, as expected, with the notations of Definition 5.5 we have that

$$\exp(\log_{\mathbb{K}} q) = \pi(E(L(\pi_{|\Omega}^{-1}(q)))) = \pi(\pi_{|\Omega}^{-1}(q)) = q$$

for all q in $\pi(\Omega)$.

Remark 5.6 It is important to notice that, unlike what happens in the case of the complex logarithm, and with the exception of the principal branch (see, e.g., [10, Definition 3.4]), no continuous branch of the hypercomplex logarithm can be defined on any open set $A \subset \mathbb{K} \setminus \{0\}$ which contains a strictly positive real point x_0, and hence a small segment $(x_0 - \epsilon, x_0 + \epsilon) \subset \mathbb{R}^+$. Indeed, for any $I \in \mathbb{S}$, on each slice A_I, the branches of the hypercomplex logarithm coincide with those of the complex logarithm of the slice $\mathbb{C}_I$. As a consequence, there is no choice of $J \in \mathbb{S}$ along $(x_0 - \epsilon, x_0 + \epsilon) \subset \mathbb{R}^+$ which can make a (non principal) branch of the hypercomplex logarithm a continuous function.

On the other hand, if $A \subset \mathbb{C}_I \setminus \{0\} \subset \mathbb{K} \setminus \{0\}$ is simply connected, any continuous branch of the hypercomplex logarithm along A coincides with the appropriate branch of the complex logarithm along A. In particular, this happens when $\alpha : [-a, a] \to \mathbb{C}_I \setminus \{0\} \subset \mathbb{K} \setminus \{0\}$ is a continuous curve having its image in a small disc Δ centered at a non zero real point x with $\Delta \subset \mathbb{C}_I \setminus \{0\}$, and such that $\alpha(0) = x$. We will address this issue in a forthcoming paper.

We conclude this section by pointing out a different possible definition of the hypercomplex Riemann manifold on which to define the hypercomplex logarithm.

Remark 5.7 The definition of a hypercomplex logarithm could be given, alternatively, using the graph of the exponential function

$$\Gamma(\exp) = \{(q, \exp q) : q \in \mathbb{K}\}$$

which has a natural structure of hypercomplex Riemann manifold (see Sect. 3.4), with the function $f(q) = (q, \exp q)$ as a slice isothermal parameterization. Indeed the logarithm could be defined as the slice regular function from the "reversed" graph $\Lambda(\exp) = \{(\exp w, w) : w \in \mathbb{K}\}$ onto $\mathbb{K}$, coinciding with the projection onto the second factor. The advantage of the approach that we actually adopted in this paper stays also in that it calls into scenery the helicoidal and logarithm manifolds, which more closely follow the path of the complex setting.

5.2 The hypercomplex *n*-th root

To give a proper definition of the n-th root function over the quaternions and octonions, we will first of all define a suitable hypercomplex Riemann manifold, which will be useful to find a possible domain for such a function.

Proposition 5.8 *Let $n \in \mathbb{N}$, with $n > 1$, and let the map*

$$f : \mathbb{R} \times \mathbb{S}(-\pi n, \pi n) \to \mathbb{K} \times \mathbb{K} \cong \mathbb{R}^{2 \dim \mathbb{K}}$$

be defined by

$$f(x + Iy) = \left(\sinh x \cos y + I \sinh x \sin y, n \exp\left(I \frac{y}{n}\right)\right)$$

for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. Then $f(\mathbb{R} \times \mathbb{S}(-\pi n, \pi n))$ is a parameterized Riemann hypercomplex manifold (diffeomorphic to $\mathbb{R} \times \mathbb{S}(-\pi n, \pi n)$) and f is a slice isothermal parameterization. This manifold will be denoted by $\mathcal{Q}_{\mathbb{K}}(n)$.

Proof The map $f = (g, h)$ is induced by the stem map

$$F = (G, H) : \mathbb{R} \times (-i\pi n, i\pi n) \to (\mathbb{R} + \iota\mathbb{R})^2,$$
$$G(x + iy) = \sinh x \cos y + \iota \sinh x \sin y, \; H(x + iy) = n \exp\left(\iota \frac{y}{n}\right))$$

whose components are those of the classical conformal parameterization of the Riemann surface of the n-th root. The map F is C^∞ and injective: indeed $H(x + iy) = H(u + iv)$ implies

$$\exp\left(\iota \frac{y}{n}\right) = \exp\left(\iota \frac{v}{n}\right)$$

whence $\frac{y-v}{n} = 2\pi m$ for some integer m. Hence $y - v = 2\pi nm$ implies $y = v$. Since G is injective in x, we now deduce $x = u$. The injectivity of F is then proved. Because $H_2(x + iy) = n \sin \frac{y}{n}$, we have $\partial_y H_2(x) = 1 \neq 0$.

Since, as we said, dF is conformal, then by Theorem 3.6 the map f is a slice conformal parameterization, and the proof is complete. □

Again, it is of interest to explicitly compute the differential of the slice conformal parameterization

$$f(x + Iy) = \left(\sinh x \cos y + I \sinh x \sin y, n \exp\left(I \frac{y}{n}\right)\right).$$

Since the first component of f has already been analyzed in Sect. 4.2, we will only compute the differential of the function $n \exp(I \frac{y}{n})$:

$$\begin{aligned} dh(x + Iy)1 &= \frac{d}{dt}_{|0} \left(n \exp\left(I \frac{y}{n}\right)\right) = 0\,, \\ dh(x + Iy)I &= \frac{d}{dt}_{|0} n \exp\left(I \frac{y + t}{n}\right) \\ &= -\sin\left(\frac{y}{n}\right) + I \cos\left(\frac{y}{n}\right). \end{aligned}$$

In the case $\mathbb{K} = \mathbb{H}$, if we set

$$\mathbb{H} \ni x_1 + x_2 I + x_3 J + x_4 K \cong (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$$

and

$$\begin{aligned} \mathbb{H} \times \mathbb{H} \ni &(x_1 + x_2 I + x_3 J + x_4 K, y_1 + y_2 I + y_3 J + y_4 K) \\ &\cong (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{R}^8 \end{aligned}$$

then, for $y \neq 0$, we have

$$df(x + Iy) = \begin{bmatrix} \cosh x \cos y & -\sinh x \sin y & 0 & 0 \\ \cosh x \sin y & \sinh x \cos y & 0 & 0 \\ 0 & 0 & \frac{\sinh x \sin y}{y} & 0 \\ 0 & 0 & 0 & \frac{\sinh x \sin y}{y} \\ 0 & -\sin(\frac{y}{n}) & 0 & 0 \\ 0 & \cos(\frac{y}{n}) & 0 & 0 \\ 0 & 0 & \frac{n \sin(y/n)}{y} & 0 \\ 0 & 0 & 0 & \frac{n \sin(y/n)}{y} \end{bmatrix}$$

and for $y = 0$, we coherently obtain:

$$df(x) = \begin{bmatrix} \cosh x & 0 & 0 & 0 \\ 0 & \sinh x & 0 & 0 \\ 0 & 0 & \sinh x & 0 \\ 0 & 0 & 0 & \sinh x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As expected, $df(x+Iy)$ is slice conformal and $df(x)$ is conformal. The situation in the case $\mathbb{K} = \mathbb{O}$ is totally analogous.

We will now see how to use $\mathcal{Q}_{\mathbb{K}}(n)$ to construct an appropriate domain for the quaternionic or octonionic n-th root function.

Proposition 5.9 *Let* $f(x+Iy) = (\sinh x \cos y + I \sinh x \sin y, n \exp(I\frac{y}{n}))$ *be as in Proposition 5.8, and let us set*

$$\mathcal{Q}_{\mathbb{K}}^{+}(n) := f(\mathbb{R}^{+} \times \mathbb{S}(-\pi n, \pi n)).$$

The map

$$\phi_n : \mathbb{R}^{+} \times \mathbb{S}(-\pi n, \pi n)) \to \mathcal{Q}_{\mathbb{K}}^{+}(n)$$

defined by

$$\phi_n(x+Iy) = \left(\exp(x+Iy), n \exp\left(I\frac{y}{n}\right)\right)$$

is an injective immersion and a diffeomorphism between $\mathbb{R}^{+} \times \mathbb{S}(-\pi n, \pi n)$ *and* $\mathcal{Q}_{\mathbb{K}}^{+}(n)$. *Indeed, in the case of quaternions,* ϕ_n *defines on* $\mathcal{Q}_{\mathbb{H}}^{+}(n)$ *a structure of slice regular manifold (see* [4]*) different from the one induced by the contruction of Proposition 5.8. However, this manifold will be denoted simply by* $\mathcal{Q}_{\mathbb{K}}^{+}(n)$*, and called* the n-th root manifold.

Proof The proof replicates the one used to establish Proposition 5.8 □

We will now define the hypercomplex n-th root on the n-th root manifold, and establish some of its properties.

Definition 5.10 Let $n \in \mathbb{N}$, with $n > 1$ and let $\mathcal{Q}_{\mathbb{K}}^{+}(n)$ be the n-th root manifold. The n-th root

$$R_n : \mathbb{K} \times \mathbb{K} \supset \mathcal{Q}_{\mathbb{K}}^{+}(n) \to \mathbb{K}$$

is defined as follows, for all $r \in \mathbb{R}^{+}$ and $p \in \mathbb{S}(-\pi n, \pi n)$:

$$R_n\left(r \exp p, n \exp\left(\frac{p}{n}\right)\right) = \sqrt[n]{r} \exp\left(\frac{p}{n}\right)$$

or directly (and equivalently), for all $(q, s) \in \mathcal{Q}_{\mathbb{K}}^{+}(n)$, by

$$R_n(q, s) = \sqrt[n]{|q|}\ \frac{s}{n}.$$

Indeed, this last formulation of the definition extends in a natural fashion, to $\overline{\mathcal{Q}_{\mathbb{K}}^{+}(n)} = f((\mathbb{R}^{+} \cup \{0\}) \times \mathbb{S}[-\pi n, \pi n])$ as

$$R_n(0, s) = 0$$

and

$$R_n(r, -n) = -\sqrt[n]{r}$$

for all $s \in nS^3$ and all $r \geq 0$.

As stated in Proposition 5.9, and analogously to what happens in the case of the logaritm, the definition of the n-th root function is not referred to the structure of hypercomplex Riemann manifold defined on $\mathcal{Q}_{\mathbb{K}}(n)$ in Proposition 5.8. Indeed, the structure that is naturally involved with the n-th root functions is the one defined in Proposition 5.9.

As it clearly appears, there is natural space and interest for the study of differential geometry of hypercomplex Riemann manifolds and, in particular, for the study of their curvature, of their mean curvature and minimality. This will be the subject of a forthcoming paper.

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