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Exact controllability of a Faedo-Galërkin scheme for the dynamics of polymer fluids

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Abstract We describe the dynamics of fluids with scattered polymer chains through a multi-field model accounting for weakly non-local inertia and second-neighborhood interactions due to chain entanglements; viscous effects appear at both macroscopic and polymer representations. We consider a linearized version of the pertinent balance equations. For it we prove exact controllability of the pertinent Faedo-Galërkin scheme on the basis of Hilbert's uniqueness method in combination with an appropriate fixed point argument.

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1 Introduction

Exact controllability deals with stabilization: it is a way to evaluate whether it is possible to drive in a given finite time a system at rest or, more generally, in a certain region of the phase space. Its analysis rests on uniqueness results in the Hilbert space setting [16]. Its emergence in various systems, under different conditions, has been investigated variously. The set of pertinent literature is wide (the treatise [13] offers a picture of the scenario; see also [14]).

Here, we explore whether a system of balance equations with distributed controls admits an exactly controllable Faedo-Galërkin's scheme. Such a system is an approximation of balances describing the dynamics of viscous fluids with evenly distributed polymer chains. The presence of such molecules has non-trivial effects such as drag reduction [9] (see also [15], [26]), which may be controlled by varying the density of polymers in the ground liquid.

The description of this type of fluids falls within the general model-building framework for the mechanics of complex bodies [8], [19], [21], [22]. Guided by that setting, we consider observable variables representing the additional polymeric microstructure. Specifically, according to the dumbbell view on polymer chains, we choose to describe each by a head-to-tail stretchable vector.

At macroscopic continuum scale, in the instant t, the vector $\boldsymbol{\nu}$ attached at x represents the average of head-to-tail vectors associated with polymers in a multiple of the molecular mean free path, a size that does not require specification here because it is considered not perceivable at macroscopic scale so that the small neighborhood interpretation progressively fades away in the formal structure. In fact we just assign a vector field over reference or current regions for the body under analysis. In any case, placement in space and descriptor $\boldsymbol{\nu}$ of the microstructural arrangements describe the body morphology. Since we consider $\boldsymbol{\nu}$ as a kinematic-type observable variable, true interactions are associated with its time rates. Standard actions (body forces and tensions) perform power in the macroscopic shape rate of change, described by the velocity vector field with values $\mathbf{u}$. Microstructural self-actions and contact ones (represented by micro-stresses) perform power in the time rate of $\boldsymbol{\nu}$. Interactions to be considered balanced are those for which the external power is invariant under isometry-based changes in observers (or the same occurs to the so-called 'relative power' emerging in the presence of growing macroscopic defects; see [21]). The consequent statement of the balance laws is independent of constitutive structures [19], [21]. By adopting a different view, we could put at the same conceptual level derivation of balance equations and choice of constitutive variables. If we adopt such a view, we may look at fields describing the mechanical behavior sketched above as critical points of some action functional, which is just the energy when we deal with conservative processes or is a d'Alembert-type action functional when we include dissipative components of stresses. Even, we could obtain structures governing the motion (representation of interactions, pertinent balances, and constitutive restrictions) by imposing covariance to the second law, which is invariance in structure under the action of diffeomorphism-based changes in observers (a principle introduced for the second law in reference [20]).

Here, we adopt a d'Alembert-type action functional and suppose that its critical points describe motions. We refer to a body occupying the $\mathbb{T}^2$ torus in its current configuration. Our representation is purely Eulerian. After specific constitutive choices, for sufficiently smooth fields, the controlled system of balance equations that we eventually consider is

$$(I - \alpha^2 \Delta)\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = -\nabla \cdot (\mathsf{B}^{\mathsf{T}} \nabla \boldsymbol{\nu}) + \mathcal{U}(\mathcal{O}),$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\partial_t \boldsymbol{\nu} + \lambda \boldsymbol{\nu} + (\mathbf{u} \cdot \nabla)\boldsymbol{\nu} - \Delta \boldsymbol{\nu} = \nabla \cdot ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}) + \nabla \cdot (\mathsf{B} \nabla \mathbf{u}) - \Delta^2 \boldsymbol{\nu} + \mathcal{V}(\mathcal{O}),$$

where ∂_t indicates partial derivative with respect to t, while the interposed dot a scalar product, and

- $\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ is the Eulerian acceleration,
- $\alpha^2 \Delta \partial_t \mathbf{u}$ a gradient-inertia-type contribution, an indirect effect due to the vibrations of long-distance entangled molecules (effects of this type are due to microstructures that are *latent* in the sense introduced by G. Capriz [7]),
- $\nu \Delta \mathbf{u}$ the contribution of a dissipative component of the macroscopic stress,
- $\mathsf{B}^{\mathsf{T}} \nabla \boldsymbol{\nu}$ a linearization of the Ericksen stress, with B the value of a second-rank tensor field,

- $\nabla\pi$ the contribution of a reactive stress determined by the volume-preserving constraint, with $-\pi$ the standard pressure,
- $\partial_t \boldsymbol{\nu} + (\mathbf{u}\cdot\nabla)\boldsymbol{\nu}$ the Eulerian time rate of $\boldsymbol{\nu}$,
- $\lambda\boldsymbol{\nu}$ a local polymer self-action,
- $\Delta\boldsymbol{\nu}$ the contribution of a conservative microstress proportional to $\nabla\boldsymbol{\nu}$,
- $\nabla\cdot((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu})+\nabla\cdot(\mathsf{B}\nabla\mathbf{u})$ the linearization of $\Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})$, a viscous-type microstress contribution,
- $\Delta^2\boldsymbol{\nu}$ the effect ofa hyperstress proportional to $\nabla\nabla\boldsymbol{\nu}$, i.e., an effect induced by second-neighborhood interactions due to the mutual entanglement of polymers, which probability to occur grows as the density of molecules increases in the ground fluid [23].

$\mathcal{U}(\mathcal{O}) = \mathbf{U}\chi_{\mathcal{O}}$ and $\mathcal{V}(\mathcal{O}) = \mathbf{V}\chi_{\mathcal{O}}$ are vector controls, with $\mathbf{U}$ and $\mathbf{V}$ control functions, and $\chi_{\mathcal{O}}$ the characteristic function of the control domain $\mathcal{O}$. $\mathbf{V}\chi_{\mathcal{O}}$ refers to the microscopic motion of the polymers relative to the ground fluid and can be obtained in practice by varying the density of polymers. $\mathbf{U}\chi_{\mathcal{O}}$ can be obtained by pumps, which vary the pressure.

We refer to a Faedo-Galërkin scheme for the previous balances. For it, and in the pertinent finite-dimensional setting, we prove exact controllability by using the Hilbert uniqueness method in combination with an appropriate fixed point argument. In fact, we adopt a finite-dimensional approximation in space, while we leave continuous or L^p dependence on time.

The proof of controllability follows a technique used in reference [1] (see also [24]) to analyze a system of balance equations, which emerges from a proposal in references [10] and [11] for the description of micropolar fluids, with resulting equations structurally different from those that we consider here.

2 Action functional, balance equations and approximations

The region occupied by the fluid under analysis here is the torus $\mathbb{T}^2$. Take a point-valued bijective map $\phi : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ such that ϕ and its inverse ϕ^{-1} are differentiable and belong to the Sobolev space $W^{s,2}(\mathbb{T}^2, \mathbb{T}^2)$, for some s. We presume ϕ to be such that $\det(D\phi) = 1$, with $D\phi$ the derivative of ϕ. Then, we consider a family of such maps parameterized by time $t \in [0, T]$, such that they are the *identity* at 0. We also assume twice differentiability with respect to time. Then we write

$$\mathbf{u} := \tilde{\mathbf{u}}(x,t) = \tilde{\mathbf{u}}(\phi(x_*,t),t), \qquad x_* \in \mathbb{T}^2, \qquad t \in [0,T], \qquad x = \phi(x_*,t),$$

for the *velocity* in Eulerian representation. The mapping $(x,t) \longmapsto \mathbf{u} = \tilde{\mathbf{u}}(x,t) \in T_{\phi(x_*,t)}\mathbb{T}^2 \simeq \mathbb{R}^2$ defines a vector field over $\mathbb{T}^2$. We indicate by $\mathcal{D}^s_{\mathbf{u}}$ the space of volume preserving vector fields tangent to $\mathbb{T}^2$. The condition $\det(D\phi) = 1$ implies that $\mathbf{u}$ is divergence-free. Another vector field, namely

$$(x,t) \longmapsto \boldsymbol{\nu} = \tilde{\boldsymbol{\nu}}(x,t) \in T_{\phi(x_*,t)}\mathbb{T}^2 \simeq \mathbb{R}^2 \ ,$$

is also expedient here. As already declared in the Introduction, its values $\boldsymbol{\nu}$ at each event point (x,t) bring at the macroscopic scale information on the microstructural local average stretching and orientation of polymer chains in a small (so to be not perceivable at continuum scale) neighborhood of x in the instant t. We write $\mathcal{V}_{\boldsymbol{\nu}}^r$ for the pertinent space. Consequently, as a *configuration space* we choose

$$\mathcal{C} := C^\infty(0,T;\mathcal{D}_{\mathbf{u}}^s) \times C^\infty(0,T;\mathcal{V}_{\boldsymbol{\nu}}^r).$$

On it we define an *action functional* $\mathfrak{A} : \mathcal{C} \longrightarrow \mathbb{R}^+$ given by

$$\mathfrak{A}(\mathbf{u},\boldsymbol{\nu}) := \int_0^T \int_{\mathbb{T}^2} \mathcal{L}(\mathbf{u}, D\mathbf{u}, \boldsymbol{\nu}, D\boldsymbol{\nu}, D^2\boldsymbol{\nu})\, \mathrm{d}x\, \mathrm{d}t,$$

with $\mathcal{L}$ a smooth map. The functional does not account for macroscopic non-inertial bulk actions, while it includes a gradient-inertia regularization given by the presence of $D\mathbf{u}$ in the list of its entries; we consider it as an indirect consequence of the polymer entanglements, which induce at molecular scale a hyperstress associated with $D^2\boldsymbol{\nu}$, depending on the polymer density. $\mathfrak{A}$ will play a role through its first variation. To evaluate it we construct smooth test vector fields

$$\mathbf{w} := \tilde{\mathbf{w}}(\phi(x_*,t),t) = \delta\phi(x_*,t),$$

with $\tilde{\mathbf{w}}(x,0) = \tilde{\mathbf{w}}(x,T) = 0$ and $D\dot{\phi}^{-1}|_{t=0} = -D\mathbf{w}$, and

$$\boldsymbol{\varphi} := \tilde{\boldsymbol{\varphi}}(\phi(x_*,t),t) = \delta\boldsymbol{\nu}.$$

We have also

$$\delta D\boldsymbol{\nu} = D\boldsymbol{\varphi} - D\boldsymbol{\nu} D\mathbf{w}$$

and

$$\delta D^2\boldsymbol{\nu} = D^2\boldsymbol{\varphi} - D\boldsymbol{\nu} D^2\mathbf{w}$$

(for the origin of these relations see [23, Lemma 1]). Then, we consider a varied density $\mathcal{L}(\mathbf{u}+\tau_1\dot{\mathbf{w}}, D\mathbf{u}+\tau_1 D\dot{\mathbf{w}}, \boldsymbol{\nu}+\tau_2\boldsymbol{\varphi}, D\boldsymbol{\nu}+\tau_2\delta D\boldsymbol{\nu}, D^2\boldsymbol{\nu}+\tau_2\delta D^2\boldsymbol{\nu})$, where the superposed dot indicates in brief total time derivative (as usual), and we exploit the identity between Eulerian and Lagrangian representation of the velocity. Eventually, we compute the derivatives of $\mathcal{L}$ with respect to τ_1 and τ_2, evaluating them at zero.

We assume that a d'Alembert-type principle given by

$$\delta\mathfrak{A} + \int_0^T \int_{\mathbb{T}^2} (\boldsymbol{\sigma}^d \cdot D\mathbf{w} + \mathbf{z}^d \cdot \boldsymbol{\varphi} + \mathbf{S}^d \cdot D\boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{1}$$

for any choice of the test fields, selects the *physically admissible motions*. The second-rank tensors $\boldsymbol{\sigma}^d$ and $\mathbf{S}^d$ are dissipative stresses, the former pertaining to the macroscopic motion, the latter peculiar of molecular entanglements; $\mathbf{z}^d$ is a dissipative polymer self-action.

We momentarily look at C^2 fields and substitute D with the gradient ∇, referring to orthonormal frames. In this setting, the integral functional $\mathfrak{A}$ admits linear first Gateaux differential. Under these conditions, and since the torus has no boundary, the repeated use of integration by parts changes the principle (1) into

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^2} \Bigg(\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + \mathbf{z}^d - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla \boldsymbol{\nu}} + \mathbf{S}^d - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla^2 \boldsymbol{\nu}} \right) \right) \cdot \boldsymbol{\varphi} \\ - \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \mathbf{u}} \right) - \nabla \cdot \boldsymbol{\sigma}^d \right. \\ \left. - \nabla \cdot \left(\nabla \boldsymbol{\nu}^{\mathsf{T}} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} - \nabla \cdot \left(\nabla \boldsymbol{\nu}^{\mathsf{T}} \frac{\partial \mathcal{L}}{\partial \nabla^2 \boldsymbol{\nu}} \right) \right) \right) \cdot \mathbf{w} \Bigg) \, \mathrm{d}x \, \mathrm{d}t = 0,\end{aligned}$$

with the presumption that it holds for any choice of $\boldsymbol{\varphi}$ and $\mathbf{w}$. The arbitrariness of $\boldsymbol{\varphi}$ implies

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} + \mathbf{z}^d - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla \boldsymbol{\nu}} + \mathbf{S}^d - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla^2 \boldsymbol{\nu}} \right) = 0,$$

while the one of $\mathbf{w}$ implies that the term multiplying it must be orthogonal to all divergence-free fields, i.e., it must be of gradient type:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla\mathbf{u}}\right) - \nabla\cdot\boldsymbol{\sigma}^d - \nabla\cdot\left(\nabla\boldsymbol{\nu}^{\mathsf{T}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nu}} - \nabla\cdot\left(\nabla\boldsymbol{\nu}^{\mathsf{T}}\frac{\partial\mathcal{L}}{\partial\nabla^2\boldsymbol{\nu}}\right)\right) = -\nabla\pi,$$

where $-\pi$ is the pressure.

We characterize the dissipative nature of $\boldsymbol{\sigma}^d$, $\mathbf{S}^d$, and $\mathbf{z}^d$ by presuming that they satisfy per se the local dissipation inequality

$$\boldsymbol{\sigma}^d\cdot\nabla\mathbf{u} + \mathbf{z}^d\cdot\dot{\boldsymbol{\nu}} + \mathbf{S}^d\cdot\nabla\dot{\boldsymbol{\nu}} \geq 0, \tag{2}$$

for any choice of $\nabla\dot{\mathbf{u}}$, $\dot{\boldsymbol{\nu}}$, and $\nabla\dot{\boldsymbol{\nu}}$. The equality sign holds only when $\nabla\dot{\mathbf{u}}$, $\dot{\boldsymbol{\nu}}$, and $\nabla\dot{\boldsymbol{\nu}}$ vanish. A possible solution of the previous inequality is

$$\boldsymbol{\sigma}^d = \nu\nabla\mathbf{u}, \qquad \mathbf{z}^d = \kappa_1\dot{\boldsymbol{\nu}}, \qquad \mathbf{S}^d = \kappa_2\nabla\dot{\boldsymbol{\nu}},$$

where ν, κ_1, and κ_2 are values of positive functions depending in principle on the state variables and their gradients, besides x and t per se. Here we take them to be just *positive constants*. Since $\dot{\boldsymbol{\nu}} = \partial_t\boldsymbol{\nu} + (\mathbf{u}\cdot\nabla)\boldsymbol{\nu}$, previous balances become

$$\begin{aligned}\kappa_1(\partial_t\boldsymbol{\nu} + (\mathbf{u}\cdot\nabla)\boldsymbol{\nu}) + \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nu}} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\boldsymbol{\nu}}\right)&\\ = \nabla\cdot\left(\kappa_2\nabla(\partial_t\boldsymbol{\nu} + (\mathbf{u}\cdot\nabla)\boldsymbol{\nu}) - \nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla^2\boldsymbol{\nu}}\right)&,\end{aligned} \tag{3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla\mathbf{u}}\right) - \nu\Delta\mathbf{u} + \nabla\pi = \nabla\cdot\left(\nabla\boldsymbol{\nu}^{\mathsf{T}}\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nu}} - \nabla\cdot\left(\nabla\boldsymbol{\nu}^{\mathsf{T}}\frac{\partial\mathcal{L}}{\partial\nabla^2\boldsymbol{\nu}}\right)\right). \tag{4}$$

Then we make the following constitutive choices and approximations:

– $\kappa_1 = 1$.

- $\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nu}} = \lambda \boldsymbol{\nu}$, a term representing a polymer self action; we do not consider external fields acting directly over the polymeric chains as it occurs for polarizable polymers under the action of external electric fields.
- We take $\kappa_2 = 1$ and neglect $\nabla \partial_t \boldsymbol{\nu}$ presuming that the viscous-type effect at polymer scale is due essentially to the convective term $\nabla\big((\mathbf{u} \cdot \nabla)\boldsymbol{\nu}\big)$, as a consequence of chain dragging.
- As a function of $\mathbf{u}$ we assume that $\mathcal{L}$ is a kinetic energy with a gradient type term, i.e., $\frac{1}{2}(\rho|\mathbf{u}|^2 + \alpha^2|\nabla \mathbf{u}|^2)$ and take $\rho = 1$. Consequently, the term $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{u}} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \mathbf{u}}\right)$ becomes $\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \alpha^2 \Delta(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u})$. However, since we attribute the regularization induced by $|\nabla \mathbf{u}|^2$ to non-local effects due to chain entanglements, we find it reasonable to exclude the contribution $\Delta((\mathbf{u} \cdot \nabla)\mathbf{u})$ because we believe it could play a role only at high concentration of polymers, a circumstance where other effects could be dominant.
- After a similar argument, we neglect the micro-hyperstress contribution $\nabla \cdot \left(\nabla \boldsymbol{\nu}^{\mathsf{T}} \frac{\partial \mathcal{L}}{\partial \nabla^2 \boldsymbol{\nu}}\right)$ to the Ericksen stress.
- $\frac{\partial \mathcal{L}}{\partial \nabla \boldsymbol{\nu}} \propto \nabla \boldsymbol{\nu}$ with proportionality coefficient equal to 1.
- $\frac{\partial \mathcal{L}}{\partial \nabla^2 \boldsymbol{\nu}} \propto \nabla^2 \boldsymbol{\nu}$, with proportionality coefficient equal to 1.

  (These two last assumptions imply that the energy is of Dirichlet type with respect to $\nabla \boldsymbol{\nu}$ and $\nabla^2 \boldsymbol{\nu}$.)

Under these assumptions, equations (3) and (4) reduce respectively to

$$\partial_t \boldsymbol{\nu} + \lambda \boldsymbol{\nu} + (\mathbf{u} \cdot \nabla)\boldsymbol{\nu} - \Delta \boldsymbol{\nu} = \Delta((\mathbf{u} \cdot \nabla)\boldsymbol{\nu}) - \Delta^2 \boldsymbol{\nu}$$

and

$$(I-\alpha^2\Delta)\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla\pi=-\nabla\cdot(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\nu}).$$

Then, we introduce a second-rank tensor field $\mathsf{B}\in C(0,T;H^1(\mathbb{T}^2))^4$, $T>0$. We use it in the following linearizations:

$$\nabla\cdot(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\nu})\approx\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),$$

and

$$\begin{aligned}\Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})&=\nabla\cdot((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu})+\nabla\cdot(\nabla\boldsymbol{\nu}\nabla\mathbf{u})\\&\approx\nabla\cdot((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu})+\nabla\cdot(\mathsf{B}\nabla\mathbf{u}).\end{aligned}$$

With them, the system of balance equations reduces to

$$(I-\alpha^2\Delta)\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla\pi=-\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\tag{5}$$

$$\nabla\cdot\mathbf{u}=0,\tag{6}$$

$$\partial_t\boldsymbol{\nu}+\lambda\boldsymbol{\nu}+(\mathbf{u}\cdot\nabla)\boldsymbol{\nu}-\Delta\boldsymbol{\nu}=\nabla\cdot((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu})+\nabla\cdot(\mathsf{B}\nabla\mathbf{u})-\Delta^2\boldsymbol{\nu},\tag{7}$$

with

$$\mathbf{u}(0)=\mathbf{u}_0\text{ and }\boldsymbol{\nu}(0)=\boldsymbol{\nu}_0.\tag{8}$$

For this system of partial differential equations, already mentioned in the Introduction, controls a part, we analyze the exact controllability of a pertinent Faedo-Galërkin approximation, once we introduce controls, which have a concrete meaning already motivated from a physical viewpoint.

When we adopt for complex fluids an Eulerian representation, the presence of Ericksen's stress is unavoidable because $\boldsymbol{\nu}=\tilde{\boldsymbol{\nu}}(\phi(x_*,t),t)$. It is nonlinear term per se; here we consider just the effects of its linearization. Moreover, once we consider microstructural component of the stress, a term like $\Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})$, which is a divergence of $\nabla((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})$, emerges naturally, as we have shown in deriving the balance equations. Then, in the linearization process we have to account for both terms.

3 Background material

For $p \geq 1$, by $L^p = L^p(\mathbb{T}^2)$ we indicate the usual Lebesgue space with norm $\|\cdot\|_p$. When $p = 2$, we use the notation $\|\cdot\| := \|\cdot\|_{L^2}$ and denote by $(\,\cdot\,,\,\cdot\,)$ the related inner product. Moreover, with k a non-negative integer and $p \geq 1$, we denote by $W^{k,p} := W^{k,p}(\mathbb{T}^2)$ the usual Sobolev space with norm $\|\cdot\|_{k,p}$ (using $\|\cdot\|_k$ when $p = 2$). We write $W^{-1,p'} := W^{-1,p'}(\mathbb{T}^2)$, $p' = p/(p-1)$, for the dual of $W^{1,p}(\mathbb{T}^2)$ with norm $\|\cdot\|_{-1,p'}$.

Let X be a real Banach space with norm $\|\cdot\|_X$. We will use the spaces $W^{k,p}(0,T;X)$, with norm denoted by $\|\cdot\|_{W^{k,p}(0,T;X)}$. For $k = 0$, $W^{0,p}(0,T;X) = L^p(0,T;X)$ are the standard Bochner spaces.

Also, $(L^p)^n := L^p(\mathbb{T}^2, \mathbb{R}^n)$, $p \geq 1$, is the function space of vector-valued L^2-maps. Similarly, $(W^{k,p})^n := (W^{k,p}(\mathbb{T}^2))^n$ is the usual Sobolev space of vector-valued maps with components in $W^{k,p}$, while $(H^s)^n$ is the space of vector-valued maps with components in $H^s := W^{s,2} \cap \{w \,:\, \nabla\cdot w = 0\}$.

We also define

$$\mathcal{H} := \text{closure of } C_0^\infty(\mathbb{T}^2, \mathbb{R}^2) \cap \{\mathbf{w} \,:\, \nabla\cdot\mathbf{w} = 0\} \text{ in } (L^2)^2\,,$$

$$\mathcal{H}^s := \text{closure of } C_0^\infty(\mathbb{T}^2, \mathbb{R}^2) \cap \{\mathbf{w} \,:\, \nabla\cdot\mathbf{w} = 0\} \text{ in } (W^{s,2})^2\,,$$

and $\mathbf{H}^s := (W^{s,2})^2$ is the usual Sobolev space of vector fields with components $W^{s,2}$-functions; again $\mathbf{H} := \mathbf{H}^0$. By $\mathcal{H}^{-s}$ we indicate the space dual to $\mathcal{H}^s$. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle := \langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}^{-1},\mathcal{H}^1}$ the duality pairing between $\mathcal{H}^{-1}$ and $\mathcal{H}^1$.

We will also assume that the vector fields $\mathbf{u}$ and $\boldsymbol{\nu}$ *have null average on* $\mathbb{T}^2$. This is a key technical point because, under such an assumption, Poincaré's inequality holds true.

Remark 3.1 By multiplying $\Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})$ by the vector $\boldsymbol{\omega}$ and integrating over $\mathbb{T}^2$, we compute

$$\int_{\mathbb{T}^2} \Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})\cdot\boldsymbol{\omega}\,\mathrm{d}x = -\int_{\mathbb{T}^2} \nabla\big((\mathbf{u}\cdot\nabla)\boldsymbol{\nu}\big)\cdot\nabla\boldsymbol{\omega}\,\mathrm{d}x,$$

where $\mathbf{u}\in\mathcal{H}^1$, $\boldsymbol{\omega}\in\mathbf{H}^1$, and $\boldsymbol{\nu}\in\mathbf{H}^2$. Hence, we get

$$\int_{\mathbb{T}^2} \nabla\big((\mathbf{u}\cdot\nabla)\boldsymbol{\nu}\big)\cdot\nabla\boldsymbol{\omega}\;\mathrm{d}x = \int_{\mathbb{T}^2} \nabla\boldsymbol{\nu}\nabla\mathbf{u}\cdot\nabla\boldsymbol{\omega}\,\mathrm{d}x + \int_{\mathbb{T}^2} (\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}\cdot\nabla\boldsymbol{\omega}\,\mathrm{d}x. \quad (9)$$

The first term on the right-hand side of the above identity is such that

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla\boldsymbol{\nu}\nabla\mathbf{u}\cdot\nabla\boldsymbol{\omega}\,\mathrm{d}x &= \int_{\mathbb{T}^2} \nabla\mathbf{u}\cdot(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\omega})\,\mathrm{d}x \\ &= \int_{\mathbb{T}^2} \big(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\omega}\big)\cdot\nabla\mathbf{u}\,\mathrm{d}x,\end{aligned} \quad (10)$$

while for the second term we find

$$\int_{\mathbb{T}^2} (\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}\cdot\nabla\boldsymbol{\omega}\,\mathrm{d}x - \int_{\mathbb{T}^2} \big(\nabla\cdot\big((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}\big)\big)\cdot\boldsymbol{\omega}\,\mathrm{d}x. \quad (11)$$

By combining (9), (10), and (11), we get

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla\big((\mathbf{u}\cdot\nabla)\boldsymbol{\nu}\big)\cdot\nabla\boldsymbol{\omega}\;\mathrm{d}x + \int_{\mathbb{T}^2} \big(\nabla\cdot\big((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}\big)\big)\cdot\boldsymbol{\omega}\;\mathrm{d}x \\ = \int_{\mathbb{T}^2} \big(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\omega}\big)\cdot\nabla\mathbf{u}\;\mathrm{d}x\end{aligned} \quad (12)$$

and

$$\int_{\mathbb{T}^2} \Delta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu})\cdot\boldsymbol{\omega}\,\mathrm{d}x - \int_{\mathbb{T}^2}\left(\nabla\cdot\left((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}\right)\right)\cdot\boldsymbol{\omega}\,\mathrm{d}x \\ = -\int_{\mathbb{T}^2}\left(\nabla\boldsymbol{\nu}^{\mathsf{T}}\nabla\boldsymbol{\omega}\right)\cdot\nabla\mathbf{u}\,\mathrm{d}x. \tag{13}$$

4 Global existence and uniqueness

To show that the system (5)–to–(7) has a unique weak solution, we exploit the Faedo-Galërkin scheme, energy estimates, and compactness. We adapt the argument introduced in references [6] and [5] to the case in which we have distributed control functions **U** and **V** in equations (5) and (7), respectively.

4.1 The Faedo-Galërkin scheme

Let $\{e_1,\ldots,e_n,\ldots\}$ be a complete orthonormal system in $\mathcal{H}$ belonging to $\mathcal{H}^1$. Be also $\mathcal{H}_n$ the $n-$dimensional subspace of $\mathcal{H}$ given by $\mathrm{span}\{e_1,\ldots e_n\}$.

For any positive integer i, we denote by $(\omega_i,\pi_i)\in\mathcal{H}^2\times W^{1,2}$ the unique solution of the Stokes problem

$$\begin{aligned} &\Delta\omega+\nabla\pi=-\lambda_i\omega, && \text{in } \mathbb{T}^2,\\ &\nabla\cdot\omega=0, && \text{in } \mathbb{T}^2, \end{aligned} \tag{14}$$

with $\int_{\mathbb{T}^2}\pi dx=0$, for $i=1,2,\ldots$ and $0<\lambda_1\le\lambda_2\le\ldots\lambda_n\ldots$, with $\lambda_n\to+\infty$, as $n\to\infty$. The functions $\{w_i\}_{i=1}^{+\infty}$ determine an orthonormal basis in $\mathcal{H}$ made of the eigenfunctions of the Stokes problem (14).

We use these eigenfuctions of Stokes problem (14) for the approximation of **u** with respect to the space variables. We find reason for this choice on the circumstance that these functions span the space where solutions of the Navier-Stokes equations lie (see, e.g., [12]).

Then, we use the eigenfuctions of the Laplace operator, i.e., the solutions to Helmholtz's equation

$$\Delta\vartheta = -\kappa_j \vartheta \; , \tag{15}$$

with κ_j the $j-th$ eigenvalue, and ϑ_j the eigenfunction, for the approximation in space of $\boldsymbol{\nu}$.

In the sequel, for the sake of simplicity, we do not distinguish between the two families of eigenfunctions (namely those related to $\mathbf{u}$ and those to $\boldsymbol{\nu}$); we'll always use the same symbology at least when we refer to common function spaces. However the situation will be clear (we hope) every time (see e.g. [2], [5], for more details on the construction of an analogous Faedo-Galërkin scheme to the one we use here).

P_n denotes the orthogonal projection of $\mathcal{H}^{-1}$ to $\mathcal{H}_n$, that is, $\mathrm{P}_n\mathbf{v}^* = \sum_{i=1}^{n}\langle \mathbf{v}^*, e_i\rangle e_i = \sum_{i=1}^{n}(\mathbf{v}, e_i)e_i$. Indeed, every $\mathbf{v} \in \mathcal{H}$ is connected to a linear functional $\mathbf{v}^* \in \mathcal{H}^{-1}$ by the relation $\langle \mathbf{v}^*, \mathbf{w}\rangle = (\mathbf{v}, \mathbf{w})$, $\mathbf{w} \in \mathcal{H}^1$, thanks to Riesz's theorem. The orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n := \mathrm{span}\{e_1, \dots, e_n\}$ is given by $\mathrm{P}_n\mathbf{v} = \sum_{i=1}^{n}(\mathbf{v}, e_i)e_i$. With $B(\mathbf{u}) = B(\mathbf{u}, \mathbf{u}) = (\mathbf{u}\cdot\nabla)\mathbf{u}$, we also introduce $B_n(\mathbf{u}_n) := \mathrm{P}_n B(\mathbf{u}_n) = \mathrm{P}_n\big(\mathbf{u}_n\cdot\nabla)\mathbf{u}_n\big)$, where $\mathbf{u}_n = \mathrm{P}_n\mathbf{u}$.

Consider a complete orthonormal set $\{f_1, \dots, f_n, \dots\}$ in $\mathbf{H}$ belonging to $\mathbf{H}^2$ and write $\mathbf{H}_n$ for the $n-$dimensional subspace of $\mathbf{H}$ given by $\mathrm{span}\{f_1, \dots f_n\}$. We have $\mathbf{P}_n\mathbf{w}^* = \sum_{i=1}^{n}\langle \mathbf{w}^*, f_i\rangle_{\mathbf{H}^{-2},\mathbf{H}^2} f_i = \sum_{i=1}^{n}(\mathbf{w}, f_i)_{\mathbf{H}^1,\mathbf{H}^1} f_i$. Then, the orthogonal projection of $\mathbf{H}$ onto $\mathbf{H}_n$ is given by $\mathbf{P}_n\mathbf{w} = \sum_{i=1}^n (\mathbf{w}, f_i) f_i$. Here, both $\mathcal{H}_n$ and $\mathbf{H}_n$ are finite dimensional subspaces of $L^2(\mathbb{T}^2)^2$.

By taking these finite-dimensional projections, from equations (5)–to–(8) we obtain the following system of ordinary differential equations:

$$\big((I-\alpha^2\Delta)\partial_t\mathbf{u}_n, \mathbf{v}\big) + \nu(\nabla\mathbf{u}_n, \nabla\mathbf{v}) + \big(B_n(\mathbf{u}_n), \mathbf{v}\big) = \big(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}_n, \nabla\mathbf{v}\big), \tag{16}$$

$$\begin{aligned}(\partial_t\boldsymbol{\nu}_n, \mathbf{w}) + \lambda(\boldsymbol{\nu}, \mathbf{w}) + (\nabla\boldsymbol{\nu}_n, \nabla\mathbf{w}) + (\Delta\boldsymbol{\nu}_n, \Delta\mathbf{w}) + \big((\mathbf{u}_n\cdot\nabla)\boldsymbol{\nu}_n, \mathbf{w}\big)&\\ = -\big(\big((\mathbf{u}_n\cdot\nabla)\nabla\boldsymbol{\nu}_n\big), \nabla\mathbf{w}\big) - \big(\mathsf{B}\nabla\mathbf{u}_n, \nabla\mathbf{w}\big)&,\end{aligned} \tag{17}$$

for all $\mathbf{v} \in \mathcal{H}_n$ and all $\mathbf{w} \in \mathbf{H}_n$, with

$$\mathbf{u}_n(0) = \mathrm{P}_n\mathbf{u}_0, \quad \boldsymbol{\nu}_n(0) = \mathbf{P}_n\boldsymbol{\nu}_0. \tag{18}$$

Here and in the sequel, for the sake of conciseness, we omit the projections P_n, and $\mathbf{P}_n$ in the Faedo-Galërkin scheme (especially in its variational formulation), except cases in which rendering explicit such projections clarifies the setting.

4.2 Nonlinearities

In the equation (5), for the convective term, the operator $B(\,\cdot\,)$ from $\mathcal{H}^1$ to $\mathcal{H}^{-1}$, with values $B(\mathbf{u}) = B(\mathbf{u}, \mathbf{u}) = (\mathbf{u}\cdot\nabla)\mathbf{u}$, is locally Lipschitz (actually, all the nonlinear terms are as such). Indeed, by using the constraint $\nabla\cdot\mathbf{u} = 0$, we obtain

$$\begin{aligned}\big|\big(B(\mathbf{u}), \mathbf{v}\big)\big| &\le \left|\int_{\mathbb{T}^2} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x\right| = \left|\int_{\mathbb{T}^2} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x\right|\\ &\le \|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{v}\| \le C\|\mathbf{u}\|\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|,\end{aligned}$$

on the basis of Hölder's and Ladyzhenskaya's inequalities. Thus, we get

$$\|B(\mathbf{u})\|_{\mathcal{H}^{-1}} \leq C\|\nabla \mathbf{u}\|^2, \text{ for all } \mathbf{u} \in \mathcal{H}^1,$$

and, in particular, we have that

$$\begin{aligned}\|B(\mathbf{u}) - B(\bar{\mathbf{u}})\|_{\mathcal{H}^{-1}} &\leq \|B(\mathbf{u}, \mathbf{u} - \bar{\mathbf{u}})\|_{\mathcal{H}^{-1}} + \|B(\bar{\mathbf{u}}, \mathbf{u} - \bar{\mathbf{u}})\|_{\mathcal{H}^{-1}} \\ &\leq C(\|\mathbf{u}\|_{\mathcal{H}^1} + \|\bar{\mathbf{u}}\|_{\mathcal{H}^1})\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathcal{H}^1}.\end{aligned}$$

Also, for the equation (7), since

$$\begin{aligned}\big|\big(\nabla \cdot ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}), \boldsymbol{\omega}\big)\big| = \big|\big(((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}), \nabla \boldsymbol{\omega}\big)\big| &= \left|\int_{\mathbb{T}^2} ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}) \cdot \nabla \boldsymbol{\omega} \, \mathrm{d}x\right| \\ &= \left|\int_{\mathbb{T}^2} (\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\omega} \cdot \nabla \boldsymbol{\nu} \, \mathrm{d}x\right| \\ &\leq \|\mathbf{u}\|_{L^4}\|\nabla \boldsymbol{\nu}\|_{L^4}\|\Delta \boldsymbol{\omega}\| \leq C\|\nabla \mathbf{u}\|\|\Delta \boldsymbol{\nu}\|\|\Delta \boldsymbol{\omega}\|,\end{aligned}$$

we find

$$\|\nabla \cdot ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu})\|_{\mathbf{H}^{-2}} \leq C\|\nabla \mathbf{u}\|\|\Delta \boldsymbol{\nu}\|,$$

for all $\mathbf{u} \in \mathcal{H}^1$ and for all $\boldsymbol{\omega} \in \mathbf{H}^2$.

Then, for all $\boldsymbol{\omega} \in \mathbf{H}^2$, we get

$$\begin{aligned}&\Big|\big(((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}), \nabla \boldsymbol{\omega}\big) - \big(((\bar{\mathbf{u}} \cdot \nabla)\nabla \bar{\boldsymbol{\nu}}), \nabla \boldsymbol{\omega}\big)\Big| \\ &\qquad \leq \big|\big((((\mathbf{u} - \bar{\mathbf{u}}) \cdot \nabla)\nabla \boldsymbol{\nu}, \nabla \boldsymbol{\omega}\big)\big| + \big|\big((\bar{\mathbf{u}} \cdot \nabla)\nabla(\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}), \nabla \boldsymbol{\omega}\big)\big| \\ &\qquad = \big|\big((((\mathbf{u} - \bar{\mathbf{u}}) \cdot \nabla)\nabla \boldsymbol{\omega}, \nabla \boldsymbol{\nu}\big)\big| + \big|\big((\bar{\mathbf{u}} \cdot \nabla)\nabla \boldsymbol{\omega}, \nabla(\boldsymbol{\nu} - \bar{\boldsymbol{\nu}})\big)\big| \\ &\qquad \leq \Big(\|\mathbf{u} - \bar{\mathbf{u}}\|_{L^4}\|\nabla \boldsymbol{\nu}\|_{L^4} + \|\bar{\mathbf{u}}\|_{L^4}\|\nabla(\boldsymbol{\nu} - \bar{\boldsymbol{\nu}})\|_{L^4}\Big)\|\Delta \boldsymbol{\omega}\|,\end{aligned}$$

where we used again $\nabla \cdot \mathbf{u} = 0$, and hence

$$\begin{aligned}&\|\big(\nabla \cdot ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}) - \big(\nabla \cdot ((\bar{\mathbf{u}} \cdot \nabla)\nabla \bar{\boldsymbol{\nu}})\|_{\mathbf{H}^{-2}} \\ &\qquad \leq C\big(\|\Delta \boldsymbol{\nu}\|\|\nabla(\mathbf{u} - \bar{\mathbf{u}})\| + \|\nabla \mathbf{u}\|\|\Delta(\boldsymbol{\nu} - \bar{\boldsymbol{\nu}})\|\big) \\ &\qquad \leq C\big(\|\Delta \boldsymbol{\nu}\|\|\mathbf{u} - \bar{\mathbf{u}}\|_{\mathcal{H}^1} + \|\nabla \mathbf{u}\|\|\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\|_{\mathbf{H}^2}\big),\end{aligned}$$

so that $(\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}$ is also locally Lipschitz.

Moreover, we may compute

$$\left|\left(\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\mathbf{v}\right)\right| = \left|\int_{\mathbb{T}^2}(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu})\cdot\nabla\mathbf{v}\mathrm{d}x\right| \leq \int_{\mathbb{T}^2}|\mathsf{B}||\nabla\boldsymbol{\nu}||\nabla\mathbf{v}|\mathrm{d}x$$

$$\leq \|B\|_{L^4}\|\nabla\boldsymbol{\nu}\|\|\nabla\mathbf{v}\|_{L^4} \leq C\|B\|_{L^4}\|\nabla\boldsymbol{\nu}\|\|\Delta\mathbf{v}\|,$$

which implies

$$\|\left(\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu})\|_{\mathcal{H}^{-2}} \leq \|B\|_{L^4}\|\nabla\boldsymbol{\nu}\|.$$

From which it follows that

$$\|\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}) - \nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\bar{\boldsymbol{\nu}})\|_{\mathcal{H}^{-2}} = \|\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla(\boldsymbol{\nu}-\bar{\boldsymbol{\nu}}))\|_{\mathcal{H}^{-2}}$$

$$\leq C\|B\|_{L^4}\|\nabla(\boldsymbol{\nu}-\bar{\boldsymbol{\nu}})\|\|\Delta\boldsymbol{\nu}\|.$$

By adapting these analyses to the finite dimensional approximation, with respect to the space variables, of equations (16) and (17), we infer that the nonlinear terms involved there are locally Lipschitz. Consequently, that system has a unique solution $(\mathbf{u}_n,\boldsymbol{\nu}_n)\in C([0,T];\mathcal{H}_n)\times C([0,T];\mathbf{H}_n)$. Existence is due to Carathéodory's pertinent result while uniqueness is immediate from the local Lipschitz property (Picard's theorem).

Energy estimates, compactness and passage to the limit follow from analyses in reference [6], [5]. They lead to existence, well-posedness, and uniqueness for the system (5)–to–(8).

5 Exact controllability of the Faedo-Galërkin approximation

As regards the Faedo-Galërkin approximation exact controllability, our main result here, we initially consider the controlled system (the controlled balance equations) already described in the Introduction, i.e.,

$$(I-\alpha^2\Delta)\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla\pi=-\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu})+\mathbf{U}\chi_{\mathcal{O}}, \tag{19}$$

$$\nabla\cdot\mathbf{u}=0, \tag{20}$$

$$\begin{aligned}\partial_t\boldsymbol{\nu}+\lambda\boldsymbol{\nu}+(\mathbf{u}\cdot\nabla)\boldsymbol{\nu}-\Delta\boldsymbol{\nu}\\ =\nabla\cdot((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu})+\nabla\cdot(\mathsf{B}\nabla\mathbf{u})-\Delta^2\boldsymbol{\nu}+\mathbf{V}\chi_{\mathcal{O}},\end{aligned} \tag{21}$$

with $\mathbf{u}(0)=\mathbf{u}_0$ and $\boldsymbol{\nu}(0)=\boldsymbol{\nu}_0$, where $\mathcal{O}$ is the control domain, which is supposed to be small as necessary. Also, $\mathbf{U}$ and $\mathbf{V}$ are the bulk controls—we recall—and $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$.

By multiplying, in L^2, equations (19) and (21) by $\mathbf{v}\in\mathcal{H}^1$ and $\mathbf{w}\in\mathbf{H}^1$, respectively, and integrating by parts, we get

$$\begin{aligned}\big((I-\alpha^2\Delta)\partial_t\mathbf{u},\mathbf{v}\big)+\nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})\\ =\big((\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\nabla\mathbf{v}\big)+(\mathbf{U}\chi_{\mathcal{O}},\mathbf{v}),\end{aligned} \tag{22}$$

$$\begin{aligned}(\boldsymbol{\nu}_t,\mathbf{w})+\lambda(\boldsymbol{\nu},\mathbf{w})+((\mathbf{u}\cdot\nabla)\boldsymbol{\nu},\mathbf{w})+(\nabla\boldsymbol{\nu},\nabla\mathbf{w})+(\Delta\boldsymbol{\nu},\Delta\mathbf{w})\\ =-\big(((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}),\nabla\mathbf{w}\big)-\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{w}\big)+(\mathbf{V}\chi_{\mathcal{O}},\mathbf{w}),\end{aligned} \tag{23}$$

with $\mathbf{u}(0)=\mathbf{u}_0\in\mathcal{H}^1_0$ and $\boldsymbol{\nu}(0)=\boldsymbol{\nu}_0\in\mathbf{H}^1$.

Consider the bases $\{e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ in $\mathcal{H}^1$ and $\mathbf{H}^2$, respectively, with the proviso that they are also linearly independent in $L^2(\mathcal{O})$.

The existence of these bases is guaranteed by the following result due to J. L. Lions and E. Zuazua [18], [17]:

Proposition 5.1 ([18]) *Let H_1 and H_2 be Hilbert spaces. Let $\mathfrak{L}:H_1\to H_2$ be a bounded linear operator with an infinite dimensional range. Then, there* *exists a Riesz basis* $\{\hat{e}_j\}_{j=1}^{\infty}$ *of* H_1 *such that* $\{\mathfrak{L}\hat{e}_j\}_{j=1}^{\infty}$ *are linearly independent in* H_2*.*

First we choose $H_1 = \mathcal{H}^1$ and $H_2 = L^2(\mathcal{O})$. Then, we select $H_1 = \mathbf{H}^2$ and $H_2 = L^2(\mathcal{O})$. Also, we define $E = \text{span}\{e_1, \ldots, e_n\}$ and $F = \text{span}\{f_1, \ldots, f_n\}$, and the approximating functions $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \times \{\boldsymbol{\nu}_n\}_{n\in\mathbb{N}} \subset E \times F$ (see section 4.1). Then, the space-type Faedo-Galërkin approximation scheme, in variational formulation, for the system (22)–(23) is given by

$$\big((I-\alpha^2\Delta)\partial_t\mathbf{u}_n,\mathbf{e}\big)+\nu(\nabla\mathbf{u}_n,\nabla\mathbf{e})+((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\mathbf{e}) = \big((\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}_n),\nabla\mathbf{e}\big)+(\mathbf{U}\chi_{\mathcal{O}},\mathbf{e}),$$

$$\begin{aligned}(\partial_t\boldsymbol{\nu}_n,\mathbf{f}) + \lambda(\boldsymbol{\nu}_n,\mathbf{f}) + ((\mathbf{u}_n\cdot\nabla)\boldsymbol{\nu}_n,\mathbf{f}) + (\nabla\boldsymbol{\nu}_n,\nabla\mathbf{f}) + (\Delta\boldsymbol{\nu}_n,\Delta\mathbf{f})&\\ = -\big(((\mathbf{u}_n\cdot\nabla)\nabla\boldsymbol{\nu}_n),\nabla\mathbf{f}\big) - \big(\mathsf{B}\nabla\mathbf{u}_n,\nabla\mathbf{f}\big) + (\mathbf{V}\chi_{\mathcal{O}},\mathbf{f}),&\end{aligned} \tag{24}$$

with $\mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{H}_0^1$ and $\boldsymbol{\nu}(0) = \boldsymbol{\nu}_0 \in \mathbf{H}^1$. For all $\mathbf{e} \in E$ and $\mathbf{f} \in F$. Here, with a slight abuse of notation, instead of using the proper form $\mathbf{u}_n(0) = \mathrm{P}_n\mathbf{u}_0$, and $\boldsymbol{\nu}_n(0) = \mathbf{P}_n\boldsymbol{\nu}_0$, we have set

$$\mathbf{u}_0 = \sum_{i=1}^{n}(\mathbf{u}_0, e_i)e_i \quad \text{and} \quad \boldsymbol{\nu}_0 = \sum_{i=1}^{n}(\boldsymbol{\nu}_0, f_i)f_i.$$

As a consequence of the existence results in references [6] and [5], we find that the system (24) has a unique solution with $(\mathbf{u},\boldsymbol{\nu}) \in C([0,T];E) \times C([0,T];F)$. In what follows we will use the notations $\|\cdot\|$ and $(\cdot\,,\,\cdot)$, referring to norm and scalar product on $L^2(\mathbb{T}^2)^2$ and the finite dimensional spaces E and F.

5.1 Exact controllability

Definition 5.1 *System* (24) *is said to be exactly controllable at time* $T > 0$*, if for given* $(\mathbf{u}_0, \boldsymbol{\nu}_0) \in E \times F$*,* $(\mathbf{u}_T, \boldsymbol{\nu}_T) \in E \times F$*, there exists a control* $(\mathbf{U}, \mathbf{V}) \in (L^2(0,T; L^2(\mathcal{O})))^2$ *such that the solution* $(\mathbf{u}, \boldsymbol{\nu})$ *of* (24) *satisfies the conditions*

$$\mathbf{u}(\,\cdot\,, T; \mathbf{U}, \mathbf{V}) = \mathbf{u}_T \quad \textit{and} \quad \boldsymbol{\nu}(\,\cdot\,, T; \mathbf{U}, \mathbf{V}) = \boldsymbol{\nu}_T. \tag{25}$$

The *cost functional* we refer to is

$$\begin{aligned} \mathcal{J}(\mathbf{U}, \mathbf{V}) &= \frac{1}{2} \int_0^T \left(\|\mathbf{U}(t)\|^2_{L^2(\mathcal{O})} + \|\mathbf{V}(t)\|^2_{L^2(\mathcal{O})} \right) \mathrm{d}t \\ &= \frac{1}{2} \int_0^T \int_{\mathcal{O}} \left(|\mathbf{U}(t)|^2 + |\mathbf{V}(t)|^2 \right) \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{26}$$

Theorem 5.1 *For* $T > 0$*, the Faedo-Galërkin approximation given in* (24) *is exactly controllable in the sense of Definition 5.1. Moreover, the cost functional is bounded, independently of the non-linear structures in the system.*

Proof For the sake of conciseness, in the sequel we set $\mathbf{u} = \mathbf{u}_n$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_n$ (and for the same reason we omit to write explicitly the projections on finite-dimensional spaces iof the nonlinear terms) for the Faedo-Galërkin system. Then, with these notations, the approximating scheme reads

$$\begin{aligned} &(I - \alpha^2 \Delta)\partial_t \mathbf{u} + \mu(\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = -\nabla \cdot (\mathsf{B}^{\mathsf{T}} \nabla \boldsymbol{\nu}) + \mathbf{U}\chi_{\mathcal{O}}, \\ &\nabla \cdot \mathbf{u} = 0, \\ &\boldsymbol{\nu}_t + \lambda \boldsymbol{\nu} + \eta(\mathbf{u} \cdot \nabla)\boldsymbol{\nu} - \Delta \boldsymbol{\nu} \\ &\qquad = \nabla \cdot ((\mathbf{u} \cdot \nabla)\nabla \boldsymbol{\nu}) + \nabla \cdot \left(\mathsf{B} \nabla \mathbf{u} \right) - \Delta^2 \boldsymbol{\nu} + \mathbf{V}\chi_{\mathcal{O}}, \end{aligned} \tag{27}$$

with $\mathbf{u}(0) = \mathbf{u}_0$, $\boldsymbol{\nu}(0) = \boldsymbol{\nu}_0$, and $\mu, \eta \in \mathbb{R}$. For it we establish controllability of its variational counterpart, which implies the analogous result for the system (24), by taking $\mu = \eta = 1$.

In weak form, system (27) reads

$$
\begin{aligned}
&\big((I-\alpha^2\Delta)\partial_t\mathbf{u},\mathbf{e}\big)+\nu(\nabla\mathbf{u},\nabla\mathbf{e})+\mu((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{e})\\
&\qquad\qquad=\big((\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\nabla\mathbf{e}\big)+(\mathbf{U}\chi_{\mathcal{O}},\mathbf{e})\ ,\\
&(\boldsymbol{\nu}_t,\mathbf{f})+\lambda(\boldsymbol{\nu},\mathbf{f})+\eta((\mathbf{u}\cdot\nabla)\boldsymbol{\nu},\,\mathbf{f})+(\nabla\boldsymbol{\nu},\nabla\mathbf{f})+(\Delta\boldsymbol{\nu},\Delta\mathbf{f})\\
&\qquad\qquad=-\big((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu},\nabla\mathbf{f}\big)-\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{f}\big)+(\mathbf{V}\chi_{\mathcal{O}},\mathbf{f})\ ,
\end{aligned}
\tag{28}
$$

where, in the second equation, we used (10), and $\mathbf{u}(0)=\mathbf{u}_0\in E$ and $\boldsymbol{\nu}(0)=\boldsymbol{\nu}_0\in F$, for all $\mathbf{e}\in E$ and $\mathbf{f}\in F$, a structure obtained by exploiting again the identity (13). Once again, with a now recurrent abuse of notation, we have taken

$$
\mathbf{u}_0=\sum_{i=1}^{n}(\mathbf{u}_0,e_i)e_i \quad\text{and}\quad \boldsymbol{\nu}_0=\sum_{i=1}^{n}(\boldsymbol{\nu}_0,f_i)f_i .
$$

Further steps are necessary. They follow a technique developed in reference [1].

−1: Adapted linear system. Take $\mathbf{h}\in L^2(0,T;E)$ and consider the linear system

$$
\begin{aligned}
&\big((I-\alpha^2\Delta)\partial_t\mathbf{u},\mathbf{e}\big)+\nu(\nabla\mathbf{u},\nabla\mathbf{e})+\mu((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{e})\\
&\qquad\qquad=\big((\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\nabla\mathbf{e}\big)+(\mathbf{U}\chi_{\mathcal{O}},\mathbf{e})\ ,\\
&(\boldsymbol{\nu}_t,\mathbf{f})+\lambda(\boldsymbol{\nu},\mathbf{f})+\eta((\mathbf{h}\cdot\nabla)\boldsymbol{\nu},\,\mathbf{f})+(\nabla\boldsymbol{\nu},\nabla\mathbf{f})+(\Delta\boldsymbol{\nu},\Delta\mathbf{f})\\
&\qquad\qquad=-\big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\nu},\nabla\mathbf{f}\big)-\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{f}\big)+(\mathbf{V}\chi_{\mathcal{O}},\mathbf{f})\ ,
\end{aligned}
\tag{29}
$$

with $\mathbf{u}(0)=\mathbf{0}\in E$ and $\boldsymbol{\nu}(0)=\mathbf{0}\in F$. Linearity implies here uniqueness of solution with $(\mathbf{u},\boldsymbol{\nu})\in C([0,T];E)\times C([0,T];F)$ and the possibility of working with null initial data. However, the result is still valid if we take $\mathbf{u}(0)=\mathbf{u}_0\in E$ and $\boldsymbol{\nu}(0)=\boldsymbol{\nu}_0\in F$, with non-null $\mathbf{u}_0$ and $\boldsymbol{\nu}_0$.

In order to establish exact controllability in the sense of Definition 5.1 at any $T > 0$, it suffices to prove the following proposition:

Proposition 5.2 *If* $(\mathbf{g}_1, \mathbf{g}_2) \in E \times F$ *satisfies*

$$\Big(\big(\mathbf{u}(\,\cdot\,, T; \mathbf{U}, \mathbf{V}\big), \big(\boldsymbol{\nu}(\,\cdot\,, T; \mathbf{U}, \mathbf{V})\big), (\mathbf{g}_1, \mathbf{g}_2)\Big) = 0\ ,$$

for all $(\mathbf{U}, \mathbf{V}) \in \big(L^2(0, T; L^2(\mathcal{O}))\big)^2$*, we have* $(\mathbf{g}_1, \mathbf{g}_2) = (\mathbf{0}, \mathbf{0})$*.*

To this aim, we first consider the adjoint system

$$\begin{aligned}
&-(I - \alpha^2 \Delta_h)\partial_t \mathbf{p} - \nu \Delta \mathbf{p} - \mu(\mathbf{h} \cdot \nabla)\mathbf{p} + \nabla \pi(t, x) - \nabla \cdot \big((\mathsf{B}^{\mathsf{T}} \nabla \mathbf{q}\big) = \mathbf{0},\\
&-\partial_t \mathbf{q} + \lambda \mathbf{q} - \Delta \mathbf{q} + \Delta^2 \mathbf{q} - \eta(\mathbf{h} \cdot \nabla)\mathbf{q} + \nabla \cdot \big((\mathbf{h} \cdot \nabla)\nabla \mathbf{q}\big) + \nabla \cdot \big(\mathsf{B} \nabla \mathbf{p}\big) = \mathbf{0}\ ,\\
&(\nabla \cdot \mathbf{p})(t, x) = 0\ , \quad x \in \mathbb{T}^2\ , \quad t > 0\ ,
\end{aligned} \tag{30}$$

where $\mathbf{p}(T, x) = (I - \alpha^2 \Delta_h)^{-1} \mathbf{g}_1(x)$, $\mathbf{q}(T, x) = \mathbf{g}_2(x)$, $x \in \mathbb{T}^2$, and $(\mathbf{g}_1, \mathbf{g}_2) \in E \times F$. Its variational formulation is given by

$$\begin{aligned}
&-((I - \alpha^2 \Delta)\partial_t \mathbf{p}, \mathbf{e}) + \nu(\nabla \mathbf{p}, \nabla \mathbf{e}) - \mu((\mathbf{h} \cdot \nabla)\mathbf{p}, \mathbf{e}) + \big(\mathsf{B}^{\mathsf{T}} \nabla \mathbf{q}, \nabla \mathbf{e}\big) = \mathbf{0},\\
&-\big(\partial_t \mathbf{q}, \mathbf{f}\big) + \lambda(\mathbf{q}, \mathbf{f}) + (\nabla \mathbf{q}, \nabla \mathbf{f}) + (\Delta \mathbf{q}, \Delta \mathbf{f}) - \eta\big((\mathbf{h} \cdot \nabla)\mathbf{q}, \mathbf{f}\big)\\
&\qquad\qquad - \big(\mathsf{B} \nabla \mathbf{p}, \nabla \mathbf{f}\big) - \big((\mathbf{h} \cdot \nabla)\nabla \mathbf{q}, \nabla \mathbf{f}\big) = \mathbf{0}.
\end{aligned} \tag{31}$$

It admits a unique solution with $(\mathbf{p}, \mathbf{q}) \in C([0, T]; E) \times C([0, T]; F)$. Also, by taking $\mathbf{e} = \mathbf{u}$ and $\mathbf{f} = \boldsymbol{\nu}$ in the system (31) we find

$$\begin{aligned}
&-((I - \alpha^2 \Delta)\partial_t \mathbf{p}, \mathbf{u}) + \nu(\nabla \mathbf{p}, \nabla \mathbf{u}) - \mu((\mathbf{h} \cdot \nabla)\mathbf{p}, \mathbf{u}) + \big(\mathsf{B}^{\mathsf{T}} \nabla \mathbf{q}, \nabla \mathbf{u}\big) = \mathbf{0},\\
&-\big(\partial_t \mathbf{q}, \boldsymbol{\nu}\big) + \lambda(\mathbf{q}, \mathbf{f}) + (\nabla \mathbf{q}, \nabla \boldsymbol{\nu}) + (\Delta \mathbf{q}, \Delta \boldsymbol{\nu}) - \eta\big((\mathbf{h} \cdot \nabla)\mathbf{q}, \boldsymbol{\nu}\big)\\
&\qquad\qquad - \big(\mathsf{B} \nabla \mathbf{p}, \nabla \boldsymbol{\nu}\big) - \big((\mathbf{h} \cdot \nabla)\nabla \mathbf{q}, \nabla \boldsymbol{\nu}\big) = \mathbf{0}.
\end{aligned} \tag{32}$$

A direct consequence of (10) is

$$\big(\mathsf{B}^{\mathsf{T}}\nabla\mathbf{q},\nabla\mathbf{u}\big)=\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{q}\big)\ \text{ and }\big(\mathsf{B}\nabla\mathbf{p},\nabla\boldsymbol{\nu}\big)=\big(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu},\nabla\mathbf{u}\big). \tag{33}$$

Moreover, by recalling that $\mathbf{h}\in L^2(0,T;E)$, we get

$$\begin{gathered}-\mu((\mathbf{h}\cdot\nabla)\mathbf{p},\mathbf{u})=\mu((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{p}),\quad -\eta\big((\mathbf{h}\cdot\nabla)\mathbf{q},\boldsymbol{\nu}\big)=\eta\big((\mathbf{h}\cdot\nabla)\boldsymbol{\nu},\mathbf{q}\big),\quad\text{and}\\ \big((\mathbf{h}\cdot\nabla)\nabla\mathbf{q},\nabla\boldsymbol{\nu}\big)=-\big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\nu},\nabla\mathbf{q}\big).\end{gathered} \tag{34}$$

By exploiting the identities (33) and (34), from system (32), and after integration in time from 0 to T, we infer

$$\begin{aligned}-\big(\mathbf{u}(T),(I-\alpha^2\varDelta_h)\mathbf{p}(T)\big)+\int_0^T\Big[&((I-\ \alpha^2\varDelta_h)\mathbf{u}_t,\mathbf{p})+\nu(\nabla\mathbf{u},\nabla\mathbf{p})\\ &+\mu((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{p})+\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{q}\big)\Big]\,\mathrm{d}t=0,\end{aligned} \tag{35}$$

$$\begin{aligned}-\big(\boldsymbol{\nu}(T),\mathbf{q}(T)\big)+\int_0^T\Big[&(\boldsymbol{\nu}_t,\,\mathbf{q})+\lambda(\boldsymbol{\nu},\mathbf{q})+(\nabla\boldsymbol{\nu},\nabla\mathbf{q})+(\varDelta\boldsymbol{\nu},\varDelta\mathbf{q})\\ &+\eta\big((\mathbf{h}\cdot\nabla)\boldsymbol{\nu},\mathbf{q}\big)+\big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\nu},\nabla\mathbf{q}\big)-\big(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu},\nabla\mathbf{p}\big)\Big]\,\mathrm{d}t=0.\end{aligned} \tag{36}$$

By adding the two equations, we obtain

$$\begin{aligned}&-\big(\mathbf{u}(T),(I-\alpha^2\varDelta_h)\mathbf{p}(T)\big)-\big(\boldsymbol{\nu}(T),\mathbf{q}(T)\big)\\ &\quad+\int_0^T\Big[((I-\alpha^2\varDelta_h)\mathbf{u}_t,\mathbf{p})+\nu(\nabla\mathbf{u},\nabla\mathbf{p})+\mu((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{p})\\ &\qquad\qquad-\big(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu},\nabla\mathbf{p}\big)\Big]\,\mathrm{d}t\\ &\quad+\int_0^T\Big[(\boldsymbol{\nu}_t,\,\mathbf{q})+\lambda(\boldsymbol{\nu},\mathbf{q})+(\nabla\boldsymbol{\nu},\nabla\mathbf{q})+(\varDelta\boldsymbol{\nu},\varDelta\mathbf{q})+\eta\big((\mathbf{h}\cdot\nabla)\boldsymbol{\nu},\mathbf{q}\big)\\ &\qquad\qquad+\big(\mathsf{B}\nabla\mathbf{u},\nabla\mathbf{q}\big)+\big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\nu},\nabla\mathbf{q}\big)\Big]\,\mathrm{d}t=0,\end{aligned}$$

that is

$$
\begin{aligned}
&-\big(\mathbf{u}(T),(I-\alpha^2\Delta_h)\mathbf{p}(T)\big)-\big(\boldsymbol{\nu}(T),\mathbf{q}(T)\big)\\
&\quad+\int_0^T\Big((I-\alpha^2\Delta_h)\mathbf{u}_t-\nu\Delta\mathbf{u}+\mu(\mathbf{h}\cdot\nabla)\mathbf{u}+\nabla\cdot(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\,\mathbf{p}\Big)\,\mathrm{d}t\\
&\quad+\int_0^T\Big(\boldsymbol{\nu}_t+\lambda\boldsymbol{\nu}-\Delta\boldsymbol{\nu}+\Delta^2\boldsymbol{\nu}+\eta(\mathbf{h}\cdot\nabla)\boldsymbol{\nu}-\nabla\cdot(\mathsf{B}\nabla\mathbf{u})\\
&\qquad\qquad-\nabla\cdot\big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\nu}\big),\,\mathbf{q}\Big)\,\mathrm{d}t=0,
\end{aligned}
$$

Since $\mathbf{p}(T)=(I-\alpha^2\Delta_h)^{-1}\mathbf{g}_1$, and $\mathbf{q}(T)=\mathbf{g}_2$ in the previous relation, with the help of equation (29) we get

$$
\Big(\big(\mathbf{u}(T),\boldsymbol{\nu}(T)\big),\big(\mathbf{g}_1,\mathbf{g}_2\big)\Big)=\int_0^T\Big(\big(\mathbf{U}(t)\chi_{\mathcal{O}},\mathbf{V}(t)\chi_{\mathcal{O}}\big),\big(\mathbf{p}(t),\mathbf{q}(t)\big)\Big)\,\mathrm{d}t, \quad (37)
$$

after integration by parts. If Proposition (5.2) would hold true, the identity (37) would give

$$
\int_0^T\Big(\big(\mathbf{U}(t)\chi_{\mathcal{O}},\mathbf{V}(t)\chi_{\mathcal{O}}\big),\big(\mathbf{p}(t),\mathbf{q}(t)\big)\Big)\mathrm{d}t=0,
$$

for all $(\mathbf{U},\mathbf{V})\in\big(L^2(0,T;L^2(\mathcal{O}))\big)^2$, which guarantees that

$$
(\mathbf{p},\mathbf{q})=(\mathbf{0},\mathbf{0}),\ \text{in } \mathcal{O}\times(0,T). \quad (38)
$$

Since $\mathbf{p}=\sum_{i=1}^n\mathbf{p}_i(t)e_i$, $\mathbf{q}=\sum_{i=1}^n\mathbf{q}_i(t)f_i$ and the elements of $\{e_j\}_{j=1}^\infty\times\{f_j\}_{j=1}^\infty$ are linearly independent in $L^2(\mathcal{O})$ (see [18]), as a consequence of the identity (38) we obtain $\mathbf{p}_i=\mathbf{q}_i=\mathbf{0}$, for all $i=1,\dots,n$.

Therefore, $(\mathbf{p}_i(t),\mathbf{q}_i(t))=(\mathbf{0},\mathbf{0})$, $i=1,\dots,n$, and hence $(\mathbf{g}_1,\mathbf{g}_2)=(\mathbf{0},\mathbf{0})$, so that Proposition (5.2) holds true. Hence, the linear system (28) is exactly controllable.

–2: Uniform estimates. Thanks to results obtained in previous step we can define $\mathcal{M} : L^2(0,T;E) \to \mathbb{R}$ by

$$\mathcal{M}(\mathbf{h}) = \inf_{(\mathbf{u},\boldsymbol{\nu})\in\mathcal{U}_{ad}} \frac{1}{2}\int_0^T \int_{\mathcal{O}} \left(|\mathbf{U}|^2 + |\mathbf{V}|^2\right) \mathrm{d}x\, \mathrm{d}t,$$

where $\mathcal{U}_{ad}$ is the set of admissible controls defined by

$$\mathcal{U}_{ad} = \left\{(\mathbf{U},\mathbf{V}) \in \left(L^2(\mathcal{O}\times(0,T))\right)^2 \,|\, (\mathbf{u},\boldsymbol{\nu}) \text{ solves (29)–(25)}\right\}.$$

We need to prove that

$$\mathcal{M}(\mathbf{h}) \leq C,$$

with C a positive constant independent of $\mathbf{h}$, μ, and η. To this aim we use a duality argument (see, e.g., [1] and [24]). We consider the continuous linear map $L : \left(L^2(\mathcal{O}\times(0,T))\right)^2 \to E \times F$ defined by

$$L(\mathbf{U},\mathbf{V}) := \left(\mathbf{u}(\,\cdot\,,T;\mathbf{U},\mathbf{V}), \left(\boldsymbol{\nu}(\,\cdot\,,T;\mathbf{U},\mathbf{V}\right)\right),$$

and introduce the functionals

$$F_1(\mathbf{U},\mathbf{V}) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} \left(|\mathbf{U}|^2 + |\mathbf{V}|^2\right) \mathrm{d}x\mathrm{d}t, \tag{39}$$

and

$$F_2(\mathbf{g}_1,\mathbf{g}_2) = \begin{cases} 0, & \text{if } (\mathbf{g}_1,\mathbf{g}_2) = (\mathbf{u}_T,\boldsymbol{\nu}_T), \\ \\ \infty, & \text{otherwise.} \end{cases} \tag{40}$$

Thus, we can rewrite the functional $\mathcal{M}$ as

$$\mathcal{M}(\mathbf{h}) = \inf_{(\mathbf{U},\mathbf{V})\in L^2(\mathcal{O}\times(0,T))^2} \left[F_1(\mathbf{U},\mathbf{V}) + F_2(L(\mathbf{U},\mathbf{V}))\right].$$

By exploiting Fenchel's and Rockafellar's duality result [25, Theorem 31.1], we get

$$-\mathcal{M}(\mathbf{h}) = \inf_{(\mathbf{g}_1,\mathbf{g}_2)\in E\times F} \left[F_1^*(L^*(\mathbf{g}_1,\mathbf{g}_2)) + F_2^*(-(\mathbf{g}_1,\mathbf{g}_2))\right],$$

where $L^* : E\times F \to L^2(\mathcal{O}\times(0,T))^2$ is the adjoint of L. Then, by using relation (37), we obtain

$$L^*(\mathbf{g}_1,\mathbf{g}_2) = (\mathbf{p},\mathbf{q}) \text{ in } \mathcal{O}\times(0,T).$$

Also, since

$$F_1^*(\mathbf{p},\mathbf{q}) = \frac{1}{2}\int_0^T\int_{\mathcal{O}}(|\mathbf{p}|^2+|\mathbf{q}|^2)\,\mathrm{d}x\,\mathrm{d}t$$

and

$$F_2^*(-(\mathbf{g}_1,\mathbf{g}_2)) = -\big((\mathbf{g}_1,\mathbf{g}_2),(\mathbf{u}_T,\boldsymbol{\nu}_T)\big),$$

we find

$$-\mathcal{M}(\mathbf{h}) = \inf_{(\mathbf{g}_1,\mathbf{g}_2)\in E\times F}\Big[\frac{1}{2}\int_0^T\int_{\mathcal{O}}(|\mathbf{p}|^2+|\mathbf{q}|^2)\,\mathrm{d}x\,\mathrm{d}t - \big((\mathbf{g}_1,\mathbf{g}_2),(\mathbf{u}_T,\boldsymbol{\nu}_T)\big)\Big].$$

Hypotheses on E and F guarantee a norm $\|(\mathbf{e},\mathbf{f})\|_{\mathcal{O}} = \displaystyle\int_{\mathcal{O}}(|\mathbf{e}|^2+|\mathbf{f}|^2)\,\mathrm{d}x$, with $(\mathbf{e},\mathbf{f})\in E\times F$, on the product space $E\times F$. Then, since $E\times F$ is finite dimensional, we obtain

$$c\|(\mathbf{e},\mathbf{f})\|^2 \le \|(\mathbf{e},\mathbf{f})\|_{\mathcal{O}}^2 \le C\|(\mathbf{e},\mathbf{f})\|^2,\ \forall\,(\mathbf{e},\mathbf{f})\in E\times F,$$

where

$$\|(\mathbf{e},\mathbf{f})\|^2 = \int_{\mathbb{T}^2}(|\mathbf{e}|^2+|\mathbf{f}|^2)\mathrm{d}x,\ \forall\,(\mathbf{e},\mathbf{f})\in E\times F,$$

while c and C are positive constants that depend only on E and F. Hence, we obtain

$$-\mathcal{M}(\mathbf{h}) \ge \inf_{(\mathbf{g}_1,\mathbf{g}_2)\in E\times F}\left[\frac{c}{2}\int_0^T\int_{\mathbb{T}^2}\big(|\mathbf{p}|^2+|\mathbf{q}|^2\big)\mathrm{d}x\mathrm{d}t - \big((\mathbf{g}_1,\mathbf{g}_2),(\mathbf{u}_T,\boldsymbol{\nu}_T)\big)\right]. \tag{41}$$

By taking $(\mathbf{e}, \mathbf{f}) = (\mathbf{p}(t), \mathbf{q}(t))$ in the system (31) and integrating in time from t to T, we get

$$\frac{1}{2}\big(\|\mathbf{p}\|^2 + \alpha^2\|\nabla\mathbf{p}\|^2\big) + \nu\int_t^T \|\nabla\mathbf{p}\|^2\,\mathrm{d}s + \int_t^T (\mathsf{B}^\mathsf{T}\nabla\mathbf{q}, \nabla\mathbf{p})\mathrm{d}s = \frac{1}{2}\|\mathbf{g}_1\|^2, \quad (42)$$

$$\frac{1}{2}\|\mathbf{q}\|^2 + \int_t^T (\lambda\|\mathbf{q}\|^2 + \|\nabla\mathbf{q}\|^2 + \|\Delta\mathbf{q}\|^2)\mathrm{d}s - \int_t^T (\mathsf{B}\nabla\mathbf{p}, \nabla\mathbf{q})\mathrm{d}s = \frac{1}{2}\|\mathbf{g}_2\|^2\ , \quad (43)$$

since

$$\big((\mathbf{h}\cdot\nabla)\boldsymbol{\ell}, \boldsymbol{\ell}\big) = 0, \quad \text{and} \quad \big((\mathbf{h}\cdot\nabla)\nabla\boldsymbol{\omega}, \nabla\boldsymbol{\omega}\big) = 0.$$

By adding relations (42), (43), using (10), and integrating from 0 to T, we obtain

$$\frac{1}{2}\int_0^T \big(\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2 + \alpha^2\|\nabla\mathbf{p}\|^2\big)\mathrm{d}t + \int_0^T t\big(\lambda\|\mathbf{q}\|^2 + \nu\|\nabla\mathbf{p}\|^2 + \|\nabla\mathbf{q}\|^2 + \|\Delta\mathbf{q}\|^2\big)\mathrm{d}t = \frac{T}{2}\big(\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2\big).$$

Eventually, we have

$$c\|\mathbf{p}\| \le \|\nabla\mathbf{p}\| \le C\|\mathbf{p}\|\ , \quad k\|\nabla\mathbf{q}\| \le \|\Delta\mathbf{q}\| \le K\|\nabla\mathbf{q}\|\ ,$$

$$c_1\big(\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2\big) \le \big(\|\nabla\mathbf{p}\|^2 + \|\nabla\mathbf{q}\|^2\big) \le C_1\big(\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2\big)\ ,$$

for some $c, c_1 > 0$, $C, C_1 > 0$, and $k, K > 0$, all depending only on E and F, because these spaces are finite dimensional. Whence, we get the following inequality

$$\frac{T}{2}\big(\|\mathbf{g}_1\|^2 + \|\mathbf{g}_2\|^2\big) \le \big(\frac{1}{2}(1+\alpha^2 C) + (\lambda+\nu+K+1)C_1 T\big)\int_0^T \big(\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2\big)\,\mathrm{d}t\ ,$$

so that the inequality (41) becomes

$$-\mathcal{M}(\mathbf{h}) \geq \inf_{(\mathbf{g}_1,\mathbf{g}_2)\in E\times F} \left[\frac{cT\int_0^T \big(\|\mathbf{p}\|^2+\|\mathbf{q}\|^2\big)\mathrm{d}t - \big((\mathbf{g}_1,\mathbf{g}_2),(\mathbf{u}_T,\boldsymbol{\nu}_T)\big)}{(1+\alpha^2 C)+2(\lambda+\nu+K+1)C_1T)} \right].$$

Then, we obtain

$$\mathcal{M}(\mathbf{h}) \leq \frac{cT}{(1+\alpha^2 C)+2(\lambda+\nu+K+1)C_1T)}\big(\|\mathbf{u}_T\|^2+\|\boldsymbol{\nu}_T\|^2\big)$$

and the inequality (5.1) follows.

–3: Fixed point argument. Let $(\mathbf{h}) \in L^2(0,T;E)$ be given. For $(\mathbf{U},\mathbf{V}) \in L^2(0,T;L^2(\mathcal{O}))^2$, we choose the unique element $(\mathbf{U},\mathbf{V})$ such that

$$\frac{1}{2}\int_0^T\int_{\mathcal{O}} \big(|\mathbf{U}|^2+|\mathbf{V}|^2\big)\,\mathrm{d}x\,\mathrm{d}t = \mathcal{M}(\mathbf{h}).$$

Define a continuous mapping $\mathbf{h} \mapsto (\mathbf{U},\mathbf{V})$ from $L^2(0,T;E)$ to $L^2(0,T;L^2(\mathcal{O}))^2$. Denote also by $(\mathbf{u}(\mathbf{h}),\boldsymbol{\nu}(\mathbf{h}))$ the solution of system (29) with $\mathbf{U}=\mathbf{U}(\mathbf{h})$ and $\mathbf{V}=\mathbf{V}(\mathbf{h})$. Then, we take $\mathbf{e}=\mathbf{u}(t)$ and $\mathbf{f}=\boldsymbol{\nu}(t)$ in the linear system (29), so that we get

$$\frac{1}{2}\frac{d}{dx}(\|\mathbf{u}\|^2+\alpha^2\|\nabla\mathbf{u}\|^2)+\nu\|\nabla\mathbf{u}\|^2 = \big(\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu},\nabla\mathbf{u}\big)+(\mathbf{U}(t)\chi_{\mathcal{O}},\mathbf{u}(t)), \quad (44)$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dx}\|\boldsymbol{\nu}\|^2&+\lambda\|\boldsymbol{\nu}\|^2+\|\nabla\boldsymbol{\nu}\|^2+\|\Delta\boldsymbol{\nu}\|^2\\ &= -\underbrace{\big(((\mathbf{u}\cdot\nabla)\nabla\boldsymbol{\nu}),\nabla\boldsymbol{\nu}\big)}_{=0}-\big(\mathsf{B}\nabla\mathbf{u},\nabla\boldsymbol{\nu}\big)+(\mathbf{V}(t)\chi_{\mathcal{O}},\boldsymbol{\nu}(t)).\end{aligned} \quad (45)$$

By integrating in time from 0 to T, summing up the previous two relations, and using again the identity (10), we obtain

$$\begin{aligned}&\frac{1}{2}\big(\|\mathbf{u}\|^2+\|\boldsymbol{\nu}\|^2+\alpha^2\|\nabla\mathbf{u}\|^2\big)+\int_0^t(\lambda\|\boldsymbol{\nu}\|^2+\nu\|\nabla\mathbf{u}\|^2+\|\nabla\boldsymbol{\nu}\|^2+\|\nabla\boldsymbol{\nu}\|^2)\mathrm{d}s\\ &\quad\leq \|\mathbf{U}\|_{L^2((0,t)\times\mathcal{O})}\|\mathbf{u}\|_{L^2((0,t)\times\mathcal{O})}+\|\mathbf{V}\|_{L^2((0,t)\times\mathcal{O})}\|\boldsymbol{\nu}\|_{L^2((0,t)\times\mathcal{O})}.\end{aligned} \quad (46)$$

As a direct consequence of the estimate (46), when $\mathbf{h}$ varies in $L^2(0,T;E)$ the pair $(\mathbf{u},\boldsymbol{\nu})$ remains in a bounded subset $K_1 \times K_2 \subset L^2(0,T;E) \times L^2(0,T;F)$.

We need now to prove the following proposition:

Proposition 5.3 $K_1 \ni \mathbf{h} \mapsto \big(\mathbf{u}(\mathbf{h}),\boldsymbol{\nu}(\mathbf{h})\big)$ *composed with* $\big(\mathbf{u}(\mathbf{h}),\boldsymbol{\nu}(\mathbf{h})\big) \mapsto \mathbf{u}(\mathbf{h})$ *admits a fixed point in* K_1.

Since the finite-dimensional approximation adopted refers only to space variables, and we have continuous or L^p-time dependence (the Bochner spaces come into play), we still refer to infinite dimensional spaces. So, we find it expedient the use of Schauder's fixed point theorem. It states that if K_1 is a convex and closed subset of a Banach space X, any continuous and compact map $\mathcal{F} : K_1 \to K_1$ (bounded sets in K_1 are mapped into relatively compact sets) has a fixed point. Consequently, it is enough to prove that the range of $\mathbf{u}(\mathbf{h})$, when $\mathbf{h}$ spans through K_1, is relatively compact in K_1.

Proposition (5.3) is consequence of the circumstance that $\partial_t \mathbf{u}$ remains bounded in a bounded subset of $L^2(0,T;E)$, when $\mathbf{h}$ varies in K_1.

Then, from the first equation in system (29), we compute

$$
\begin{aligned}
|\big((I-\alpha^2\Delta)\partial_t\mathbf{u},\mathbf{e}\big)| &\le \nu\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\| + \mu\|\mathbf{h}\|_{L^4}\|\nabla\mathbf{u}\|\|\mathbf{e}\|_{L^4}\\
&\qquad + |\big((\mathsf{B}^{\mathsf{T}}\nabla\boldsymbol{\nu}),\nabla\mathbf{e}\big)| + |(\mathbf{U}\chi_{\mathcal{O}},\mathbf{e})|\\
&\le \nu\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\| + \mu\|\nabla\mathbf{h}\|\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\| + \|\mathsf{B}\|_{L^4}\|\nabla\boldsymbol{\nu}\|_{L^4}\|\nabla\mathbf{e}\|\\
&\qquad + \|\mathbf{U}\|_{L^2(\mathcal{O})}\|\mathbf{e}\|_{L^2(\mathcal{O})}\\
&\le \nu\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\| + \mu\|\nabla\mathbf{h}\|\|\nabla\mathbf{u}\|\|\nabla\mathbf{e}\| + \|\mathsf{B}\|_{L^4}\|\Delta\boldsymbol{\nu}\|\|\nabla\mathbf{e}\|\\
&\qquad + \|\mathbf{U}\|_{L^2(\mathcal{O})}\|\mathbf{e}\|\\
&\le C\big(\nu\|\nabla\mathbf{u}\| + \mu\|\mathbf{h}\|\|\nabla\mathbf{u}\| + \|\mathsf{B}_{L^4}\|\|\nabla\boldsymbol{\nu}\|\\
&\qquad + \|\mathbf{U}\|_{L^2(\mathcal{O})}\big)\|\mathbf{e}\|,\ \forall\mathbf{e}\in E,
\end{aligned}
$$

a result determined by the embeddings $W^{1,2}(\mathbb{T}^2)\hookrightarrow L^4(\mathbb{T}^2)$ and $W^{2,2}(\mathbb{T}^2)\hookrightarrow W^{1,4}(\mathbb{T}^2)$ and the equivalence of norms in the finite-dimensional spaces E and F.

The norms $\|\mathbf{v}\|$ and $\|(I-\alpha^2\Delta)^{\frac{1}{2}}\mathbf{v}\|$ are equivalent. Indeed, we compute

$$
\|(I-\alpha^2\Delta)^{\frac{1}{2}}\mathbf{v}\|^2 = \|((I-\alpha^2\Delta)\mathbf{v},\mathbf{v})\| = \|\mathbf{v}\|^2 + \alpha^2\|\nabla\mathbf{v}\|^2 \le \big(\frac{1}{\lambda_1}+\alpha^2\big)\|\mathbf{v}\|^2_{\mathcal{H}^1},
$$

on the basis of Poincaré's inequality. Also, we estimate

$$
\alpha^2\|\mathbf{v}\|^2_{\mathcal{H}^1} \le (1+\alpha^2)(\|\mathbf{v}\|+\alpha^2\|\nabla\mathbf{v}\|) = (1+\alpha^2)\|(I-\alpha^2\Delta)^{\frac{1}{2}}\mathbf{v}\|^2.
$$

By using previous inequalities, we get

$$
\|(I-\alpha^2\Delta)^{\frac{1}{2}}\partial_t\mathbf{u}\|^2 \le C\big(\nu\|\nabla\mathbf{u}\|+\mu\|\mathbf{h}\|\|\nabla\mathbf{u}\|+\|\mathsf{B}\|_{L^4}\|\nabla\boldsymbol{\nu}\|+\|\mathbf{U}\|_{L^2(\mathcal{O})}\big). \quad (47)
$$

By combining the inequalities (47) and (5.1), the use of Schauder's fixed point theorem allows us to obtain that the map $\mathbf{h}\mapsto(\mathbf{u}(\mathbf{h}),\boldsymbol{\nu}(\mathbf{h}))\mapsto\mathbf{u}(\mathbf{h})$, admits a fixed point in K_1. So, if $\mathbf{h}$ is such a point, since system (29) is exactly controllable in time $T > 0$, we directly obtain the exact controllability of the system (28). Also, system (28) is exactly controllable for any $(\mu, \eta) \in \mathbb{R}^2$, and in particular for $\mu = \eta = 1$, so we obtain the exact controllability of system (24). This concludes the proof.

Although not directly related to the claimed fixed point result, we can also provide an estimate for $\boldsymbol{\nu}_t$. In fact, from the second equation in system (29), we compute

$$
\begin{aligned}
|(\boldsymbol{\nu}_t, \mathbf{f})| \leq\ & \lambda \|\boldsymbol{\nu}\| \|\mathbf{f}\| + \eta \|\mathbf{h}\|_{L^4} \|\nabla \boldsymbol{\nu}\| \|\mathbf{f}\|_{L^4} + \|\nabla \boldsymbol{\nu}\| \|\nabla \mathbf{f}\| + \|\Delta \boldsymbol{\nu}\| \|\Delta \mathbf{f}\| \\
& + \|\mathsf{B}\|_{L^4} \|\nabla \mathbf{u}\| \|\nabla \mathbf{f}\|_{L^4} + \|\mathbf{h}\|_{L^4} \|\Delta \boldsymbol{\nu}\| \|\nabla \mathbf{f}\|_{L^4} + \|\mathbf{V}\|_{L^2(\mathcal{O})} \|\mathbf{f}\|_{L^2(\mathcal{O})} \\
\leq\ & \lambda \|\boldsymbol{\nu}\| \|\mathbf{f}\| + \eta \|\nabla \mathbf{h}\| \|\nabla \boldsymbol{\nu}\| \|\nabla \mathbf{f}\| + \|\nabla \boldsymbol{\nu}\| \|\nabla \mathbf{f}\| + \|\Delta \boldsymbol{\nu}\| \|\Delta \mathbf{f}\| \\
& + \|\mathsf{B}\|_{L^4} \|\nabla \mathbf{u}\| \|\Delta \mathbf{f}\| + \|\nabla \mathbf{h}\| \|\Delta \boldsymbol{\nu}\| \|\Delta \mathbf{f}\| + \|\mathbf{V}\|_{L^2(\mathcal{O})} \|\mathbf{f}\| \\
\leq\ & \big(\lambda \|\boldsymbol{\nu}\| + \eta \|\nabla \mathbf{h}\| \|\nabla \boldsymbol{\nu}\| + \|\nabla \boldsymbol{\nu}\| + \|\Delta \boldsymbol{\nu}\| + \|\mathsf{B}\|_{L^4} \|\nabla \mathbf{u}\| \\
& + \|\mathbf{V}\|_{L^2(\mathcal{O})} \big) \|\mathbf{f}\| \,.
\end{aligned}
$$

Hence, we find

$$
\|\boldsymbol{\nu}_t\| \leq C \big(\lambda \|\boldsymbol{\nu}\| + \eta \|\mathbf{h}\| \|\nabla \boldsymbol{\nu}\| + \|\nabla \boldsymbol{\nu}\| + \|\Delta \boldsymbol{\nu}\| + \|\mathsf{B}\|_{L^4} \|\nabla \mathbf{u}\| + \|\mathbf{V}\|_{L^2(\mathcal{O})} \big),
$$

where we exploited again norm equivalence on E and F.

6 Concluding remarks

Although based on a know technique, our analysis explores its value for (and in a sense extension to) a physically significant model structure for the dynamics of complex fluids with vector-type microstructure implying a gyroscopic-type non-trivial nonlinearity. The controllability of a space-discretized scheme opens the path to explicit numerical evaluations of the controlled pertinent flows, with possible (and profitable) consequences in the design of potential scientific experiments or industrial processes.

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References

1. Araruna F. D., Chaves-Silva F. W., Rojas-Medar M. A. (2010), Exact controllability of Galërkin's approximations of micropolar fluids, *Proc. Amer. Math. Soc.*, **138**, 1361–1370.
2. Bisconti L., On the convergence of an approximate deconvolution model to the 3D mean Boussinesq equations, *Math. Meth. Appl. Sci.*, **38**, 1437–1450.
3. Bisconti L., Catania D., On the exact controllability of Galërkin schemes for a 3D fluid with memory, fractional order viscosity and horizontal filtering, *Submitted.*
4. Bisconti L., Mariano P. M. (2017), Existence results in the linear dynamics of quasicrystals with phason diffusion and nonlinear gyroscopic effects, *SIAM Multiscale Model. Simul.*, **15**, 745–767.
5. Bisconti L., Mariano P. M. (2020), Global existence and regularity for the dynamics of viscous oriented fluids, *AIMS Math.*, **5**, 79–95.

6. Bisconti L., Mariano P. M., Vespri V. (2018), Existence and regularity for a model of viscous oriented fluid accounting for second-neighbor spin-to-spin interactions, *J. Math. Fluid Mech.*, **20**, 655–682.

7. Capriz G. (1985), Continua with latent microstructure, *Arch. Rational Mech. Anal.*, **90**, 43-56.

8. Capriz G. (1989), *Continua with microstructure*, Springer Verlag, Heidelberg.

9. de Gennes P. G. (1990), *Introduction to polymer dynamics*, Lezioni Lincee, Cambridge University Press, Cambridge.

10. Eringen A. C. (1964), Simple microfluids, *Int. J. Eng. Sci.*, **2**, 205–217.

11. Eringen A. C. (1966), Theory of micropolar fluids, *J. Math. Mech.*, **16**, 1–18.

12. Foias C., Manley O., Rosa R., Temam R. (2001), *Navier–Stokes equations and turbulence*, Cambridge University Press, Cambridge.

13. Glowinski R., Lions J. L. (2008), *Exact and approximate controllability for distributed parameter systems: a numerical approach*, Cambridge University Press, Cambridge.

14. Li T., Wang K., Gu Q. (2016), *Exact boundary controllability of nodal profile dor quasilinear hyperbolic systems*, Springer-Verlag, New York.

15. Likos C. N. (2001), Effective interactions in soft condensed matter physics, *Phys. Rep.*, **348**, 267–439.

16. Lions J. L. (1988), Exact controllability, stabilization and perturbations for distributed systems, *SIAM Rev.*, **30**, 1–68.

17. Lions J. L., Zuazua E. (1998), Exact boundary controllability of Galërkin approximations of Navier–Stokes equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **26**, 605–621.

18. Lions J. L., Zuazua E. (1997), On the cost of controlling unstable systems: The case of boundary controls, *J. Anal. Math.*, **73**, 225–249.

19. Mariano P. M. (2002), Multifield theories in mechanics of solids, *Adv. Appl. Mech.*, **38**, 1–93.

20. Mariano P. M. (2013), Covariance in plasticity, *Proc. Royal Soc. London A*, **469**, art. n. 20130073.

21. Mariano P. M. (2014), Mechanics of material mutations, *Adv. Appl. Mech.*, **47**, 1–91.

22. Mariano P. M. (2016), Trends and challenges in the mechanics of complex materials: a view, *Phil. Trans. Royal Soc. London A*, **374**, art. n. 20150341.

23. Mariano P. M. (2017), Second-neighbor interactions in classical field theories: invariance of the relative power and covariance, *Math. Meth. Appl. Sci.*, **40**, 1316–1332.

24. Mohan M. T. (2020), On the three dimensional Kelvin-Voigt fluids: global solvability, exponential stability and exact controllability of Galërkin approximations, *Evol. Equ. Control Theory*, **9**, 301–339.

25. Rockafellar R. T. (1997), *Convex Analysis*, reprint of the 1970 original, Princeton University Press, Princeton, NJ.

26. Suen J. K. C., Joo Y. L., Armstrong R. C. (2002), Molecular orientation effects in viscoelasticity, *Ann. Rev. Fluid Mech.*, **34**, 417–444.