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Non-Equilibrium Corrections to the Covariant Wigner Function in Relativistic Hydrodynamics

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Contents

Notations

In this work we adopt the natural system: $c = k_B = \hbar = 1$, with c, speed of light, k_B Boltzmann constant and $\hbar$ reduced Planck constant. Masses, temperatures and frequencies will have dimensions of an energy while lengths and times dimension of the inverse of an energy.

The Einstein index convention is used; all repeated indexes are meant to be summed all over their possible values, unless specifically stated otherwise. We will denote with a Greek letters space-time indexes; $\mu = 0, 1, 2, 3$, while with Latin letters "spatial" indexes; $i = 1, 2, 3$. The letters a, b, r, s will denote spin indexes; for spin 1/2: $a, b, r, s = -1/2, 1/2$. Uppercase letters will indicate spinorial indexes $A, B = 1, 2, 3, 4$.

We adopt the "mostly minus" convention for the flat-spacetime metric $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, hence time-like and space-like vectors will have positive and negative norm respectively. A four-vector β is denoted by its covariant components: $\beta^\mu = (\beta^0, \boldsymbol{\beta})$, where β^0 is the time-component while $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ is the spatial one. The square norm is denoted with $\beta^2 = \eta_{\mu\nu}\beta^\mu\beta^\nu = (\beta^0)^2 - |\boldsymbol{\beta}|^2$ and must not be confused with the second component of the spatial part. Versors will be denoted by a small "hat" $\hat{n}$. The scalar product between two four-vectors or two three vectors is denoted by a dot "$\cdot$". The Levi-Civita pseudotensor is defined as $\varepsilon^{0123} = +1$.

The partial derivative with respect to a space-time coordinate x is defined as $\partial_\mu = \partial/\partial x^\mu$, where $\partial_0 = \partial/\partial t$ is the derivative with respect to the time and $\partial_i = -\nabla_i$ is the gradient with respect to the space-coordinates. The covariant derivative is denoted with ∇_μ and must not be confused with the spatial gradient $\vec{\nabla}$.

Quantum operators on an Hilbert space will be denoted with a "widehat": $\widehat{O}$. The adjoint operator is denoted by a "dagger": $\dagger$.

The commutator of two operators is denoted with $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$, while the anticommutator by $\{\widehat{A}, \widehat{B}\} = \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$.

The Dirac field operator will be written without a hat as ψ. The Dirac adjoint is instead denoted by: $\overline{\psi} = \psi^\dagger\gamma^0$, where γ^μ are the Dirac matrices: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $(\gamma^0)^2 = I$. The "slashed" convention is adopted, $\not{V} = \gamma^\mu V_\mu$.

The real and imaginary part of a complex number $z = x + iy$ will be denoted by $\mathrm{Re}(z) = (z + z^*)/2$ and $\mathrm{Im}(z) = \mathrm{i}(z^* - z)/2$ where $\mathrm{i}^2 = -1$ is the imaginary unit and $z^* = x - \mathrm{i}y$ is the complex conjugate.

The symbols "Tr" will be used to indicate the trace over the quantum states of the system while "tr" will indicate the trace over the spinorial indexes of a Dirac spinor.

Other conventions and notations will be introduced where needed.

Introduction

Quantum Chromodynamics (QCD), the fundamental theory of the strong interaction, describes the dynamics of quarks and gluons through the exchange of color charge. One of its most remarkable properties is asymptotic freedom [1–3]: at high energies or short distances the coupling becomes weak, allowing perturbative methods to be applied with great success [4, 5]. Conversely, at low energies the interaction strength increases, giving rise to phenomena such as color confinement [6] and dynamical chiral symmetry breaking [7]. These intrinsically non-perturbative features of QCD are responsible for the vast majority of the visible mass in the universe, yet they remain extremely challenging to investigate analytically due to the breakdown of perturbation theory.

The study of QCD in the strongly coupled regime therefore relies on complementary approaches, including lattice QCD simulations [8, 9], effective field theories [10, 11], and phenomenological models. Each of these methods, however, suffers from intrinsic limitations: lattice QCD, for instance, faces severe difficulties in the presence of a nonzero baryon density [12], while effective theories inevitably rely on model-dependent assumptions [13]. For these reasons, the possibility of creating and probing strongly interacting matter under extreme conditions in the laboratory is of paramount importance.

Since the early 1980s, it has been expected that collisions between relativistic nuclei lead to the formation of a new state of matter composed of deconfined, extremely hot quarks and gluons, known as the quark–gluon plasma (QGP) [14–17]. The production of the QGP in the laboratory provides a unique opportunity to investigate the properties of QCD in its non-perturbative regime, including the behavior of strongly interacting matter at extreme temperatures and densities [18–20], as well as in the presence of intense electromagnetic fields [21–23].

After an initial far-from-equilibrium stage, the QGP rapidly approaches local thermal equilibrium and subsequently evolves according to the equations of relativistic hydrodynamics [24]. This hydrodynamic description has proven remarkably successful in reproducing collective flow observables, leading to the conclusion that the QGP behaves as an almost perfect fluid with an exceptionally low shear viscosity-to-entropy density ratio, close to the conjectured universal lower bound $\eta/s \simeq (1/4\pi)$ [25, 26]. A major breakthrough in QGP phenomenology occurred in 2017, when the STAR collaboration reported the first observation of a global polarization of Λ and $\overline{\Lambda}$ hyperons produced in non-central heavy-ion collisions [27]. This discovery triggered intense experimental and theoretical activity devoted to the study of global and local polarization phenomena [28–32].

The possibility that spin polarization could emerge in the products of heavy-ion collisions was first proposed in 2005 [33, 34], where it was suggested that quarks become polarized through spin–orbit coupling with the large orbital angular momentum generated in non-central collisions [35]. This polarization is then expected to seed vortical structures that persist throughout the hydrodynamic evolution of the produced plasma [36–39].

The space–time evolution of matter created in heavy-ion collisions proceeds through several distinct stages. Immediately after the collision, the system is in a highly far-from-equilibrium pre-equilibrium phase, characterized by strong fields and rapid gluon production. Within a short time scale ($\sim$1 fm/c), it approaches local equilibrium, marking the onset of the hydrodynamic QGP phase, during which the system is well described by relativistic fluid dynamics until hadronization. As the QGP cools, it undergoes a transition—a smooth crossover at small baryon density—into a hadron resonance gas, in which quarks and gluons recombine into color-neutral hadrons and a fluid description is no longer applicable. At later times, the system experiences chemical freeze-out, where inelastic reactions cease and particle abundances are fixed, followed by kinetic freeze-out, when elastic scatterings stop and particles free-stream toward the detectors. Each of these stages leaves characteristic imprints on the observed particle spectra and correlations, making a detailed reconstruction of the full evolution essential for connecting experimental measurements to theoretical descriptions of QCD matter.

From a fundamental perspective, the QGP is a quantum-relativistic system which, for most of its evolution, remains close to thermodynamic equilibrium. The natural framework to describe such systems is therefore quantum-relativistic statistical mechanics. The fundamental degrees of freedom are those of quantum fields, while the state of the system is described by an appropriate non-equilibrium statistical operator known as the *Zubarev operator*. The Zubarev operator was originally introduced to formalize the description of thermodynamic systems out of equilibrium [40], and was later adapted to the context of relativistic fluids [41, 42]. The main advantage of this approach in the description of the QGP is its applicability to strongly interacting systems and its ability, for systems close to equilibrium, to systematically separate dissipative from non-dissipative effects. This separation enables a first-principles derivation of non-ideal relativistic hydrodynamics [43] and a rigorous definition of viscous coefficients through the well-known Kubo formulae [41, 42].

To bridge the microscopic description in terms of quantum fields with the macroscopic thermodynamic picture—and taking into account that experimentally observed particles are quasi-free states emitted after freeze-out—we employ the Wigner function [44]. This object provides a crucial link between measurable observables and the underlying quantum fields, and can be interpreted as a quantum generalization of the single-particle distribution function of kinetic theory. A distinctive feature of the Wigner function is its intrinsic non-locality in space-time, as it simultaneously encodes information in both position and momentum space, consistently with the uncertainty principle. This property fundamentally distinguishes it from classical distribution functions and allows for the computation of a wide range of observables, including currents, the stress–energy tensor, particle spectra, and the spin-polarization vector.

In recent years, the thermal expectation value of the Wigner function has been the subject of extensive investigation. For systems in global equilibrium, exact analytical expressions are available [45, 46], whereas at local equilibrium exact results can be obtained only for highly symmetric configurations [47–51]. In situations where the system is close to equilibrium, linear response theory is typically employed. Non-dissipative, local-equilibrium corrections to the Wigner function have led to the prediction of thermal shear contributions to fermion spin polarization [52–54]. More recently, these calculations have been extended to second order in gradients [55, 56]. All of these studies relied on specific geometrical assumptions about the freeze-out hypersurface, which play a particularly important role for shear-induced contributions due to their strong sensitivity to its geometry. By contrast, dissipative effects have received considerably less attention [57].

The investigation of dissipative effects, which ultimately originate from the underlying microscopic dynamics of the system, is commonly carried out within the framework of kinetic theory. In this approach, the central object is the single-particle distribution function, which can be generalized to incorporate quantum effects. Its time evolution is governed by the Boltzmann equation, where interactions are modeled through binary or multi-particle collisions among quasi-particles [58–67]. This framework is particularly suited for describing the system after decoupling, in the hadron gas phase, where the QCD plasma has transitioned into a system of localized hadrons. The quasi-particle picture captures the notion of localized excitations of the underlying fields that behave approximately as particles with well-defined positions and momenta. While highly predictive in weakly coupled regimes, this description is not universally applicable. In particular, in strongly coupled systems such as the QGP, where fields are highly delocalized, quarks and gluons cannot be consistently described as point-like quasi-particles. As a consequence, the validity of kinetic theory in this regime becomes questionable.

To investigate dissipative phenomena in the QGP, one is therefore compelled to adopt a more fundamental description based directly on quantum field theory. However, since the QGP is governed by the strong interaction in a deeply non-perturbative regime, conventional perturbative techniques are inadequate. Alternative methods are required. One powerful framework is provided by holographic duality, which maps strongly coupled quantum field theories onto weakly coupled gravitational theories in higher-dimensional spacetimes. This correspondence has been successfully employed to compute transport coefficients and, more recently, to explore spin-related dissipative effects, providing insights that are inaccessible to perturbative approaches [68].

In this thesis, we develop a new expansion method, in the context of quantum-relativistic statistical mechanics that allows the computation, at any desired order in linear response theory, of the local-equilibrium correction to the Wigner function for an arbitrary shape of the decoupling hypersurface. We employ this method to derive an improved expression for the spin polarization vector of Dirac fermions produced in heavy-ion collisions. We further extend the approach to compute the full out-of-equilibrium correction—namely, the contribution arising from both the non-homogeneity of the statistical operator and genuine dissipative effects—to the scalar Wigner function. This is achieved by introducing a fully interacting expansion of the field, which enables the inclusion of generic interaction effects in the QGP phase that may influence the Wigner function. These results are important, as they enable a compelling investigation of the thermodynamic properties of the quark–gluon plasma within a quantum-relativistic framework derived directly from quantum field theory.

We demonstrate that, in general, the local equilibrium, and the full non-equilibrium correction as well, to the Wigner function exhibits qualitative differences with respect to the corresponding corrections obtained for macroscopic quantities such as the stress–energy tensor. The underlying reason is that the Wigner function is an intrinsically *non-local* object: it is defined as a bilinear functional of the quantum fields and depends explicitly on the momentum variable k. As a consequence, dissipative corrections to the Wigner function cannot, even in the hydrodynamic limit, be reduced to the familiar form of a local gradient expansion in space–time multiplied by transport coefficients.

By contrast, we show that the leading-order correction to the Wigner function departs qualitatively from this paradigm. Rather than involving gradients evaluated at a single space–time point, the correction is governed by the finite difference of thermodynamic fields evaluated at two distinct points connected by a virtual particle worldline. This remarkable feature admits a natural interpretation as a *memory effect*, reflecting the fact that the Wigner function retains information about the history of the system, rather than depending solely on the local conditions at a given point. The only dynamical input enters through the spectral function of the effective hadronic field, which characterizes how excitations are modified as they propagate through the plasma.

The implications of this result are far-reaching. First, the presence of such a correction highlights that the Wigner function encodes a richer form of non-local dynamics than is typically captured by the stress–energy tensor or other local observables. Second, the structure of the correction suggests that it may lead to an enhancement of particle production at low momentum, a feature of direct phenomenological relevance for heavy-ion collisions. Despite its potential importance, this contribution has been systematically overlooked in the existing literature, and its proper inclusion offers a qualitatively new perspective on the role of non-local quantum effects in relativistic hydrodynamics.

The structure of this thesis is organized as follows.

Chapter 1 provides a concise overview of the main properties of the quark–gluon plasma produced in heavy-ion collisions. We highlight how the geometric features of the system and the production of quasi-free final states, which are directly accessible experimentally, naturally fit within our theoretical framework based on the Wigner function.

In **Chapter 2**, we introduce the Zubarev statistical operator and present a rigorous definition of the quantum state associated with a locally thermalized relativistic quantum system. Assuming the existence of a hydrodynamic regime, we apply linear response theory and show that two distinct contributions arise in a natural way: a non-dissipative component associated with local equilibrium, and a dissipative component related to entropy production.

Chapter 3 is devoted to the definition of the Wigner operator and the corresponding Wigner function for both complex scalar and Dirac fields. We examine in detail the mathematical properties of the Wigner function and its connection to macroscopic observables. In particular, we show that the spin polarization vector of fermions produced at freeze-out can be expressed directly in terms of integrals of the Wigner function. For the specific case of scalar fields, we introduce a new mode expansion of the Wigner function in terms of generalized interacting modes. This formulation makes it possible to extend the definition of the Wigner function to interacting fields, which is essential for describing the plasma phase dominated by non-perturbative interactions.

In **Chapter 4**, we compute the local-equilibrium correction to the Wigner function for free scalar and Dirac fields by introducing a novel method that allows for the exact integration over the decoupling hypersurface, order by order in the hydrodynamic expansion. This method is then applied to derive an improved expression for the spin polarization vector at local equilibrium, revealing the presence of additional contributions that are absent under the commonly adopted assumption of a flat hypersurface geometry. The proposed expansion and integration procedure is fully general and independent of the specific choice of hypersurface, and can therefore be extended beyond the local-equilibrium regime. This chapter is based on [69].

Finally, in **Chapter 5**, we compute the full non-equilibrium correction to the Wigner function for a generally interacting scalar field, extending the method developed in the previous chapter. We show that the natural linear response expansion of the full non-equilibrium statistical operator leads to a gradient expansion of the thermodynamic fields evaluated on the *initial* hypersurface, rather than on the *instantaneous* one, as is commonly assumed. The leading-order correction to the Wigner function takes the form of a memory-effect term which, to the best of our knowledge, has not been previously identified. We discuss in detail the origin and physical interpretation of this contribution, as well as the reasons for its absence in other approaches, such as kinetic theory. Finally, we estimate its impact on the pion spectra produced in heavy-ion collisions, showing that this effect is likely to enhance the low-p_T region of the spectrum. This chapter is based on [70].

Chapter 1

Heavy-ion collisions and quark gluon plasma

In this chapter we introduce and discuss the phenomenology of heavy-ion collisions, focusing on the production and properties of the quark–gluon plasma. In particular, we introduce the concept of spin polarization and its role as a sensitive probe of strongly interacting matter at high temperature.

1.1 The physics of heavy-ion collisions

Relativistic heavy-ion collisions provide a unique laboratory for studying the properties of QCD under extreme conditions of temperature and energy density that cannot be achieved elsewhere in the laboratory [71, 72]. By colliding heavy nuclei at ultra-relativistic energies, it is possible to transiently create an extended region of deconfined strongly interacting matter, commonly referred to as the QGP.

In such collisions, the kinetic energy of the incoming nuclei is converted into internal excitation energy, leading to the dissolution of the nucleons in the overlap region. The resulting state consists of quarks and gluons liberated from their hadronic bound states. Even before the discovery of asymptotic freedom, it was realized that the hadronic mass spectrum exhibits a *limiting temperature*, known as the Hagedorn temperature, $T_{\mathrm{H}} \simeq 150$ MeV, above which the hadronic partition function diverges [73]. This temperature was later interpreted as signaling the onset of a phase transition [15]. The idea that, at sufficiently high temperature or density, hadronic matter would transform into a plasma of quarks and gluons was subsequently proposed in the mid-1970s [16, 17], motivated by the property of asymptotic freedom in QCD. In this regime, the QCD coupling decreases with increasing momentum scale, suggesting that quarks and gluons should interact weakly at high temperature. However, experimental results from RHIC and the LHC have demonstrated that the produced QGP exhibits an extremely small shear viscosity-to-entropy density ratio, close to the conjectured quantum lower bound [25], and displays strong collective behavior. These observations indicate that the QGP is a strongly coupled fluid rather than a weakly interacting gas. While relativistic hydrodynamics provides an effective description of the macroscopic evolution of the QGP, it has become increasingly clear that significant non-hydrodynamic contributions can play an important role during the early-time evolution and in the approach toward local equilibrium, particularly in rapidly expanding systems such as those created in heavy-ion collisions [74, 75].

The expansion of the fireball created in the collision is driven by strong pressure gradients that develop in the hot and dense medium. As the system expands and cools, it undergoes a smooth crossover transition from the deconfined QGP phase back to a hadronic phase, where color confinement is restored and hadronic bound states—baryons and mesons—are formed. This process, known as *hadronization* [76–82], is of particular interest because it involves intrinsically non-perturbative QCD dynamics and the emergence of color-neutral hadrons from partonic degrees of freedom. Despite significant progress, understanding the microscopic mechanisms underlying hadronization remains one of the central challenges in the study of the strong interaction.

Although the microscopic dynamics of the QGP are governed by QCD in its non-perturbative regime, the macroscopic evolution of the medium can be effectively described using relativistic fluid dynamics. The remarkable success of hydrodynamic models in reproducing the collective flow patterns observed experimentally supports the interpretation of the QGP as a nearly perfect fluid. Moreover, it is widely believed that a similar state of deconfined matter filled the early Universe a few microseconds after the Big Bang, making heavy-ion collisions a unique opportunity to recreate and study, on microscopic scales, the conditions that prevailed in the primordial Universe [83].

Since the QGP exists only for a fleeting moment before hadronizing, it cannot be observed directly. Its presence must instead be inferred from the properties of the final-state hadrons that reach the detectors. These particles carry imprints of the entire dynamical evolution of the system—from the initial impact to the final freeze-out—and therefore act as probes of deconfinement and collectivity. A broad set of observables, including particle spectra, flow coefficients, electromagnetic emissions, and polarization measurements, is employed to extract information about the formation and properties of the quark–gluon plasma.

A variety of observables have been proposed to identify the formation of a deconfined quark–gluon plasma and to characterize its properties. Among these, strangeness enhancement, entropy production, and charm suppression have played a central role in shaping our current understanding of the QGP.

- **Strangeness enhancement.** One of the earliest proposed signatures of the QGP is the enhanced production of strange hadrons in heavy-ion collisions compared to proton–proton interactions [84, 85]. In a deconfined medium, the creation of strange quarks through gluon fusion or quark–antiquark scattering is energetically favored, leading to chemical equilibration of the strange sector on time scales much shorter than those characteristic of a hadronic gas. The observed enhancement of multi-strange baryons at SPS, RHIC, and LHC energies therefore provides compelling evidence for the transient existence of a deconfined, high-temperature phase.

- **Entropy production.** The large final-state particle multiplicities observed in heavy-ion collisions reflect the substantial entropy generated during the early stages of the collision [86, 87]. Entropy production is dominated by the approach to local thermodynamic equilibrium, during which strong color fields decohere and the medium undergoes hydrodynamization. The entropy per baryon, as inferred from particle yields and hydrodynamic simulations, provides valuable information on the thermalization process, the magnitude of viscous dissipation, and the effective number of active degrees of freedom in the QGP.

- **Charm suppression.** The suppression of heavy quarkonium states, in particular the J/ψ, was proposed as a "smoking-gun" signature of deconfinement [88, 89]. In a deconfined plasma, color screening weakens the binding potential between heavy quark–antiquark pairs, leading to their dissociation at sufficiently high temperatures. The sequential melting of quarkonium states with different binding energies provides an effective thermometer for the QGP and a direct probe of its color-screening properties.

Over the past decade, it has become increasingly clear that another sensitive probe of the quark–gluon plasma is the **spin polarization and spin alignment** of emitted hadrons. Non-central heavy-ion collisions carry a large total angular momentum, part of which can be transferred to the medium, generating local vorticity and strong electromagnetic fields. If the produced system reaches, at least approximately, local thermodynamic equilibrium for spin degrees of freedom, the spins of emitted particles tend to align with the thermal vorticity of the fluid. Measurements of global and local polarization of hyperons and of the spin alignment of vector mesons [27–29, 31, 90] provide direct evidence of this phenomenon, revealing that the QGP created at RHIC is the most vortical fluid ever observed in nature. Spin polarization thus opens a new window onto the microscopic structure and dynamical evolution of the QGP, offering complementary insights into its rotational properties, dissipative behavior, and the interplay between spin, vorticity, and electromagnetic fields.

The study of the quark–gluon plasma and the exploration of the QCD phase diagram are pursued at several experimental facilities operating at different center-of-mass energies, each providing complementary information on the properties of hot and dense strongly interacting matter.

- **Super Proton Synchrotron (SPS), CERN**
  The first indications of QGP-like behavior emerged from fixed-target heavy-ion collisions at the SPS, with center-of-mass energies per nucleon pair up to $\sqrt{s_{NN}} \simeq 17$ GeV. Experiments such as NA49, NA57, and NA61/SHINE were instrumental in establishing phenomena such as strangeness enhancement and the onset of deconfinement.

- **Relativistic Heavy Ion Collider (RHIC), Brookhaven National Laboratory**
  RHIC has been operating since 2000, colliding gold ions at energies up to $\sqrt{s_{NN}} = 200$ GeV. Experiments including STAR and PHENIX have provided compelling evidence for the formation of a strongly coupled QGP exhibiting pronounced collective flow and low viscosity. RHIC also conducts Beam Energy Scan (BES) programs aimed at mapping the QCD phase diagram and searching for the possible existence of a critical point.

- **Large Hadron Collider (LHC), CERN**
  The LHC currently explores the highest-energy regime of heavy-ion collisions, with $\sqrt{s_{NN}} = 5.02$ TeV for Pb–Pb interactions. Experiments such as ALICE, ATLAS, and CMS have observed QGP signatures consistent with those at

RHIC, but at substantially higher initial temperatures and energy densities, providing precise measurements of flow coefficients, jet quenching, and heavy-flavor dynamics.

- **Future facilities: FAIR and NICA**
  The forthcoming Facility for Antiproton and Ion Research (FAIR) at GSI and the Nuclotron-based Ion Collider fAcility (NICA) at JINR will focus on the high-baryon-density, moderate-temperature region of the phase diagram, with beam energies of $\sqrt{s_{NN}} \sim 4$–11 GeV. These facilities aim to complement the RHIC Beam Energy Scan by investigating the onset of deconfinement, critical phenomena, and the properties of strongly interacting matter at finite baryon density.

Together, these facilities cover a wide range of collision energies, from a few GeV to several TeV per nucleon pair, enabling a systematic exploration of QCD matter across different temperature and baryon-density regimes. Their combined experimental programs have firmly established the existence of the quark–gluon plasma and continue to refine our understanding of its transport properties, equilibration dynamics, and connections to the early Universe.

1.1.1 The QCD phase diagram

The quark–gluon plasma represents one particular phase of strongly interacting matter, which is believed to have filled the Universe during its first few microseconds after the Big Bang [91]. However, the QCD phase structure is considerably richer, encompassing a variety of states of matter that depend on the temperature T and baryon chemical potential μ_{B}. Relativistic heavy-ion collisions provide an experimental tool to explore selected regions of this phase diagram by varying the collision energy and the size of the colliding system.

In the plane of temperature and baryon chemical potential, different phases of QCD matter occupy distinct regions [72]. At vanishing or small μ_{B}, corresponding to conditions of nearly zero net baryon density, lattice QCD calculations predict a smooth *crossover* transition between the hadronic phase and the quark–gluon plasma [92–94]. This regime is realized in ultra-relativistic heavy-ion collisions at the LHC and at the highest RHIC energies, where the colliding baryons are highly Lorentz-contracted and largely deflected toward forward rapidities, leaving the mid-rapidity region nearly baryon-symmetric. The QGP produced under these conditions is therefore characterized by extremely high temperatures and a very small net baryon density, closely resembling the thermodynamic conditions of the early Universe.

As the collision energy decreases, baryon stopping becomes increasingly significant and the produced medium acquires a finite baryon chemical potential. This enables experimental access to regions of the phase diagram where the transition between hadronic and partonic matter is expected to change its nature. Model studies suggest that, at sufficiently large μ_{B}, the crossover may turn into a first-order phase transition, terminating at a *critical end point* (CEP) [95–97]. The experimental search for this critical point is a major objective of the Beam Energy Scan program at RHIC and of future low-energy facilities such as FAIR and NICA. In this context, observables sensitive to fluctuations of conserved charges, correlations, and event-by-event variations are being intensively investigated.

From a dynamical perspective, the exploration of the QCD critical point poses additional challenges due to the intrinsically out-of-equilibrium nature of the system created in HIC. Near a critical point, the phenomenon of critical slowing down implies that the relaxation time of long-wavelength fluctuations grows rapidly and may become comparable to the lifetime of the fireball. As a consequence, critical fluctuations cannot remain in equilibrium as the system expands and cools, and their non-equilibrium dynamics can significantly influence both fluctuation observables and the bulk hydrodynamic evolution. These effects have been systematically investigated in recent years within extended hydrodynamic frameworks, such as the Hydro$^+$ formalism, which couples conventional hydrodynamics to additional slow modes describing critical fluctuations and accounts for their backreaction on the medium [98–100].

In contrast, the region of very high baryon density and relatively low temperature, which remains inaccessible to current heavy-ion experiments, is expected to host color-superconducting phases of QCD matter [101]. In these phases, quarks form Cooper pairs in close analogy with conventional superconductors, leading to the spontaneous breaking of color symmetry. Such forms of matter are believed to exist in the cores of compact stellar objects, particularly massive neutron stars, and may therefore be probed indirectly through astrophysical observations, such as neutron star mergers and measurements of mass–radius relations [102].

By varying the collision energy, heavy-ion programs effectively explore different trajectories in the (T, μ_B) plane, providing complementary insights into the thermodynamics of QCD matter. At the highest energies, such as those achieved at the LHC, the system evolves close to the $\mu_\mathrm{B} \simeq 0$ axis and undergoes a smooth crossover transition. At lower energies, the system probes regions of higher baryon density, where nontrivial critical behavior and phase coexistence may emerge. The combined efforts of collider experiments and astrophysical observations are therefore essential to achieve a comprehensive understanding of the QCD phase structure across its full range of thermodynamic conditions.

1.1.2 Geometry of an HIC

The dynamical evolution of a relativistic heavy-ion collision can be represented as a trajectory across the QCD phase diagram Fig. 1.1. Starting from two ultrarelativistic nuclei approaching each other, the system undergoes a sequence of well-defined stages, each characterized by different physical processes and relevant degrees of freedom. The final-state hadrons observed in detectors are the byproduct of this multi-stage evolution, which converts the initial kinetic energy of the colliding nuclei into a hot, dense, and collectively expanding medium.

When the two nuclei overlap, a large amount of energy density is deposited in a small space-time region. In this initial stage, the system is far from equilibrium and is often described in terms of strong classical gluon fields, known as the *Glasma* [103, 104]. This Glasma emerges from the collision of two color-glass condensates, representing the saturated gluon distributions of the incoming nuclei at high energies. It carries large longitudinal color-electric and color-magnetic fields and serves as the precursor of the thermalized quark–gluon plasma.

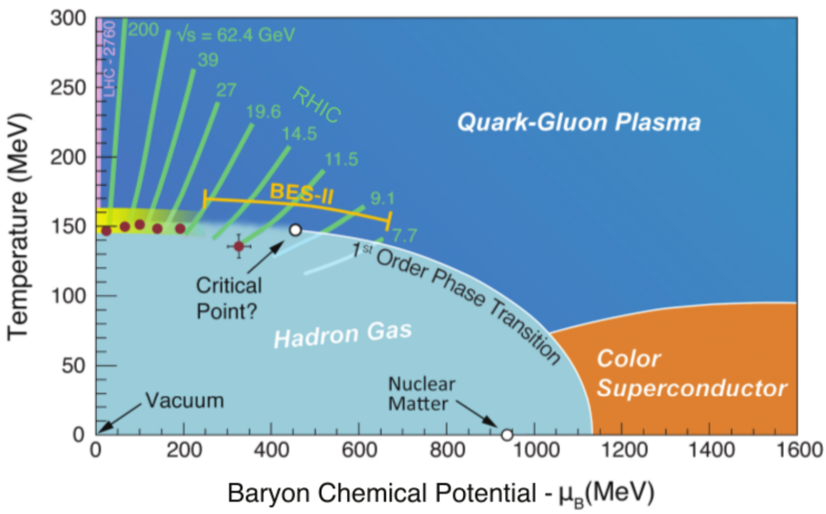


Figure 1.1: Schematic QCD phase diagram in the temperature–baryon chemical potential plane. High-energy heavy-ion collisions probe the region of high temperature and low baryon chemical potential, where the QGP is formed. The diagram also illustrates the crossover transition at small μ_B, the possible first-order phase transition at larger μ_B, and the conjectured critical point explored by the RHIC Beam Energy Scan program. (Illustration: Swagato Mukherjee, Brookhaven National Laboratory.)

In the early moments after the collision, the system also experiences extremely strong electromagnetic fields, reaching magnitudes up to $eB \sim m_\pi^2$ at LHC energies [105–108]. These fields are generated primarily by the spectator (non-participating) protons and may play an important role in inducing phenomena such as the chiral magnetic effect and spin polarization [109].

Through interactions among quarks and gluons, the initially anisotropic and out-of-equilibrium medium is expected to reach local thermal equilibrium after a short time interval, typically of order $\tau_{\rm eq} \sim 0.5$–$1\,{\rm fm}/c$, corresponding to a temperature $T_{\rm eq} \gtrsim 300\,{\rm MeV}$. At this stage, the system can be characterized by a space-like hypersurface $\Sigma_{\rm eq}$ in Minkowski space-time, across which local thermodynamic quantities such as temperature, flow velocity, and baryon density are defined (see Chapter 2).

The subsequent evolution of the medium is described effectively by the equations of relativistic (viscous) hydrodynamics, which express local energy–momentum and baryon number conservation:

$$\partial_\mu T^{\mu\nu} = 0\,, \qquad \partial_\mu j^\mu_{\rm B} = 0\,,$$

where $T^{\mu\nu}$ is the energy–momentum tensor and $j^\mu_{\rm B}$ the baryon current. From a microscopic point of view, these tensors correspond to appropriate quantum expectation values of field operators evaluated in the local thermal state of the QGP. This stage is what is properly referred to as the *quark–gluon plasma*: a deconfined, strongly coupled ensemble of quarks and gluons that behaves collectively as an almost perfect relativistic fluid.

The microscopic mechanisms underlying this rapid equilibration and the emergence of hydrodynamic behavior from far-from-equilibrium dynamics remain an active area of research, with recent studies emphasizing the role of non-hydrodynamic modes and transient excitations in governing the relaxation toward equilibrium [110].

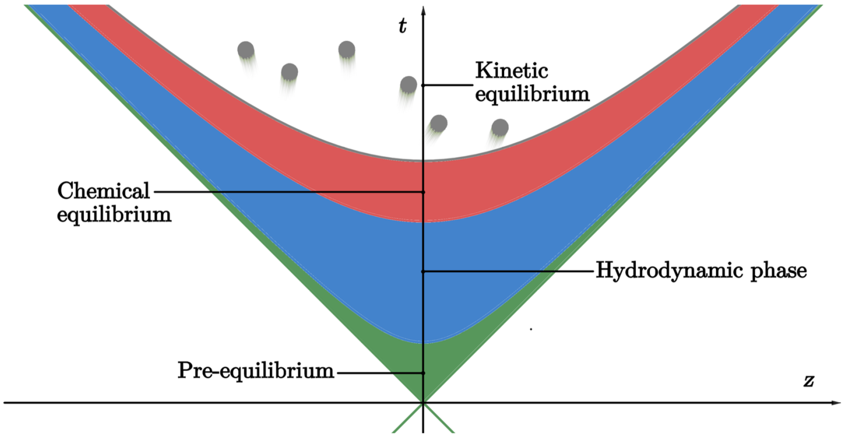


Figure 1.2: Schematic representation of the space-time region associated with a typical HIC. The QGP phase properly exists in the blue colored area where an hydrodynamic description is available. The phase transition happens between this area and the red colored one. Image from [46].

Driven by large pressure gradients, the QGP expands and cools. As the temperature decreases toward the characteristic hadronic scale, the system undergoes a transition—smooth or of crossover type depending on the trajectory in the QCD phase diagram—into a hadronic phase. This process, known as *hadronization*, marks the confinement of color degrees of freedom into bound states such as mesons and baryons. The microscopic dynamics of hadronization remain one of the most challenging open problems in QCD, as it involves the interplay between perturbative and non-perturbative processes.

After hadronization, the system enters the *hadron gas* stage. Although no longer deconfined, the medium still experiences frequent hadronic interactions that maintain partial thermal equilibrium. As the expansion continues, the mean free path of the hadrons eventually becomes comparable to or larger than the system size. At this point, inelastic collisions cease (*chemical freeze-out*), fixing the relative abundances of different hadron species. Elastic collisions persist slightly longer, until even these become negligible (*kinetic freeze-out*), after which particles free-stream toward the detectors. Two stages can be identified in the fluid-particle conversion process. The first stage is the end of the hydrodynamic approximation, when the system can no longer be approximated by a fluid, i.e. a system close to local thermodynamic equilibrium; this stage will henceforth called *decoupling*. After decoupling, the system is best seen as weakly interacting particles whose collisions drive it out of local equilibrium. The second stage occurs when these particles finally cease to interact; this stage is generally called *freeze-out*.

At very high collision energies, such as those achieved at the LHC, the distinction between the various decoupling stages becomes less pronounced, and the assumption of a sudden or instantaneous freeze-out provides a reasonable approximation. If decoupling and freeze-out are very close, as a first approximation, the residual interaction after decoupling is neglected and one just considers free particles at local thermodynamic equilibrium. The momentum distribution is obtained as an integral of the Jüttner distribution (in the relativistic regime) the so-called Cooper-Frye formula [111]:

$$E(k)\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \int_{\Sigma_D} \mathrm{d}\Sigma_\mu(x)k^\mu \, \frac{1}{\mathrm{e}^{\beta(x)\cdot p-\zeta(x)} \pm 1} \,, \tag{1.1}$$

where Σ_D is the decoupling hypersurface (the boundary between the blue and red area in Fig.1.2), $\beta(x)$ is the four-temperature vector at the point x, $\zeta(x)$ is the reduced chemical potential μ/T at the point x; the sign + applies to fermions, the - to bosons and $E(k)$ is the energy of the particle with momentum k. Yet, it is generally accepted that this distribution should include other terms: dissipative as well as quantum corrections. It is commonly believed that dissipative terms are, at the leading order, proportional to the gradients of the thermo-hydrodynamic fields (temperature, four-velocity, chemical potential) and quantum corrections proportional to the gradients squared [112–114]. In heavy ion physics, where the decoupling stage is also called *particlization* and quickly follows the transition from the Quark Gluon Plasma to hadron gas, these corrections have been addressed in several studies [115–121] and some models of them have been implemented in numerical codes [122, 123].

The single-particle momentum spectra is among the most fundamental observables in heavy-ion collisions. It provides direct information on the thermodynamic properties and collective expansion of the medium at freeze-out. In particular, the transverse momentum (p_T) spectra of identified hadrons are sensitive to the temperature, radial flow velocity, and chemical composition of the system. Light hadrons, and pions in particular, dominate the final-state particle yields and therefore play a central role in characterizing the bulk dynamics of the collision. The low-p_T region of the pion spectrum is especially important, as it is strongly influenced by collective flow and late-stage dynamics, and is expected to be most sensitive to non-equilibrium effects and long-wavelength phenomena [124]. Deviations from equilibrium expectations in this region may signal the presence of residual interactions, memory effects, or non-local dynamics persisting up to freeze-out. A precise understanding of the mechanisms governing low-p_T pion production is therefore essential for a consistent interpretation of experimental data and for establishing reliable connections between microscopic theory and macroscopic observables.

1.2 Polarization and Vorticity

When two nuclei collide with a finite impact parameter, the overlapping region of the collision carries a very large orbital angular momentum (OAM), of the order of $10^4\hbar$ at RHIC energies [35]. This angular momentum originates from the geometric asymmetry of the colliding system and is partially converted into local vorticity of the produced matter during the subsequent hydrodynamic evolution. In the seminal works by Liang and Wang [33, 34], it was first proposed that the global orbital angular momentum of the fireball could be transferred to the spins of quarks through spin–orbit coupling mechanisms intrinsic to QCD. As a result, quarks in the locally equilibrated QGP may acquire a net polarization, which is subsequently transmitted to the hadrons formed at freeze-out, in particular to hyperons such as the Λ and $\bar{\Lambda}$.

Assuming that the QGP reaches local thermodynamic equilibrium characterized by a well-defined vorticity field, the first quantitative predictions for hyperon polarization were obtained within the framework of *local-equilibrium statistical me-*

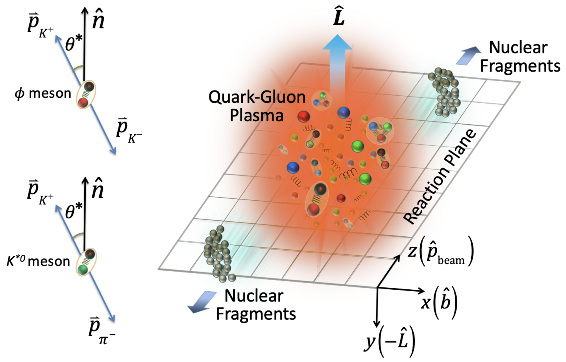


Figure 1.3: Schematic representation of the experimental coordinates apt to study non-central collisions between heavy-nucleai. The direction of the OAM is orthogonal to the *reaction plane*. The impact parameter b quantify the non-centrality of the collision. The QGP is formed in the overlap. Figure from [27].

chanics [36]. In this approach, spin degrees of freedom are populated according to the *thermal vorticity* of the medium, defined as

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) \,, \tag{1.2}$$

where $\beta_\mu = u_\mu/T$ is the inverse-temperature four-vector. The polarization vector of emitted fermions is then proportional to the thermal vorticity evaluated on the decoupling hypersurface, and is given by:

$$S^\mu_\varpi(k) = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}k_\sigma \frac{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma \cdot k \, \varpi_{\nu\rho} \, n_\mathrm{F}(k)\big(1 - n_\mathrm{F}(k)\big)}{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma \cdot k \, n_\mathrm{F}(k)} \,, \tag{1.3}$$

where n_F denotes the Fermi–Dirac distribution and the denominator corresponds to the particle number $\mathrm{N}_p(k)$. This expression establishes one of the most direct theoretical connections between relativistic hydrodynamics and experimentally measurable spin observables.

The first experimental evidence of global spin polarization in relativistic heavy-ion collisions was reported by the STAR Collaboration at RHIC [27], through measurements of the polarization of Λ and $\bar{\Lambda}$ hyperons. Remarkably, both the magnitude and the energy dependence of the observed global polarization were found to be consistent with predictions based on local-equilibrium models, providing strong evidence that the QGP behaves as a vortical, locally thermalized fluid.

However, more differential measurements of the *local* components of the polarization revealed significant discrepancies with the simple thermal-vorticity picture. In particular, the azimuthal dependence of the longitudinal polarization was observed to exhibit a sign opposite to that predicted by hydrodynamic calculations, a phenomenon commonly referred to as the *sign puzzle*. This inconsistency indicates that additional dynamical effects, beyond the ideal local-equilibrium description, contribute to the observed spin polarization.

It was later realized that an additional *non-equilibrium* but *non-dissipative* contribution arises from gradients of the thermodynamic fields [52–54]. This contribution depends on the so-called *thermal shear* tensor, defined as

$$\xi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\right) \,, \tag{1.4}$$

which corresponds to the symmetric part of the four-temperature gradient and vanishes in global equilibrium. The associated contribution is extremely sensitive to the geometry of the decoupling/freeze-out hypersurface and has the general form:

$$S^{\mu}_{\xi}(k) = -\frac{1}{4m\,\mathrm{N}_p(k)}\,\epsilon^{\mu\nu\sigma\tau}k_{\tau}k^{\rho}\int_{\Sigma_{\mathrm{D}}}\mathrm{d}\Sigma\cdot k\,n_{\mathrm{F}}(k)\big(1-n_{\mathrm{F}}(k)\big)\frac{\xi_{\sigma\rho}\hat{v}_{\nu}}{|k\cdot\hat{v}|}\,, \tag{1.5}$$

where $\hat{v}^{\mu}$ is a unit vector which is tightly related with the geometric assumption made on the shape of the hypersurface. Hence the evaluation of this term depends sensitively on the geometric modeling of the hypersurface and on the prescription adopted for $\hat{v}^{\mu}$. Different choices have been explored in the literature, including identifying $\hat{v}^{\mu}$ with the time direction $\hat{t}$ [54], with the local fluid velocity $\hat{u}$ [52]. However both approximation are not generally valid in which the assumption of $\hat{v} = \hat{t}$ is valid only if the decoupling hypersurface is a flat hyperplane whereas the choice $\hat{v} = \hat{u}$ is valid only if the decoupling hypersurface is non-vortical with both cases not true for realistic HIC.

In Chapter 4, we show that, by employing a new expansion method, an improved formula for the spin polarization can be derived that fully accounts for the non-flat geometry of the decoupling hypersurface.

Beyond local-equilibrium and non-dissipative effects, additional sources of spin polarization may arise from explicitly non-equilibrium dynamics, including spin–hydrodynamic coupling [125], quantum corrections to the Wigner function [64], electromagnetic field effects [126], and gradients of chemical potentials [127]. These mechanisms are particularly relevant for understanding the spin alignment of vector mesons, for which local-equilibrium predictions fail to reproduce the experimental observations, pointing to a richer and more intricate spin dynamics in the quark–gluon plasma.

The study of spin polarization in relativistic heavy-ion collisions has therefore evolved into a powerful tool for probing the microscopic properties of the QGP. It provides unique insight into the degree of local equilibration, the structure of vorticity and shear fields, and the possible role of dissipative and quantum corrections in spin transport.

1.2.1 Measuring the spin polarization

The first measurement of the spin polarization of Λ and $\overline{\Lambda}$ hyperons in relativistic heavy-ion collisions, reported by the STAR Collaboration in 2017 [27], marked a milestone in the study of spin-related phenomena in the quark–gluon plasma (QGP). This observation confirmed that the QGP behaves as a vortical, locally equilibrated fluid, providing direct experimental access to the rotational and thermodynamic properties of the strongly interacting medium. The result generated significant interest, as it enabled, for the first time, an experimental test of one of the central assumptions of relativistic hydrodynamics—local thermal equilibrium—in a system characterized by extreme vorticity and intense electromagnetic fields.

The polarization of Λ hyperons is experimentally determined through their self-analyzing weak decay $\Lambda \to p^+ + \pi^-$, which violates parity and therefore encodes information on the spin orientation of the parent hyperon. In the rest frame of the Λ, the angular distribution of the emitted protons is given by

$$\frac{\mathrm{d}N}{\mathrm{d}\cos\theta} = \frac{1}{2}\left[1 + \alpha_{\Lambda,\overline{\Lambda}}\,\left|\boldsymbol{P}_{\Lambda,\overline{\Lambda}}\right|\cos\theta\right], \tag{1.6}$$

where θ denotes the angle between the proton momentum and the spin polarization vector $\boldsymbol{P}_{\Lambda,\overline{\Lambda}}$, and $\alpha_\Lambda = -\alpha_{\overline{\Lambda}} = 0.732 \pm 0.014$ [128] is the weak decay parameter. The vector $\boldsymbol{P}_{\Lambda,\overline{\Lambda}}$ represents the *global polarization*, namely the average spin orientation of the emitted hyperons, and is related to the spin four-vector by [129]

$$P^\mu = \frac{S^\mu(p)}{S} \,. \tag{1.7}$$

The hyperons are emitted from the decoupling (or freeze-out) hypersurface of the collision, where they are assumed to follow a thermal distribution according to the Cooper–Frye prescription [111]. The global polarization can therefore be expressed as

$$P^\mu = 2\frac{\int \frac{\mathrm{d}^3\mathrm{k}}{E(k)}\, S^\mu(p)\left(\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma\cdot k\, n_{\mathrm{F}}(k)\right)}{\int \frac{\mathrm{d}^3\mathrm{k}}{E(k)}\, \left(\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma\cdot k\, n_{\mathrm{F}}(k)\right)} \,, \tag{1.8}$$

where $S^\mu(p)$ is the spin four-vector, n_{F} is the Fermi–Dirac distribution, and Σ_{D} denotes the decoupling hypersurface.

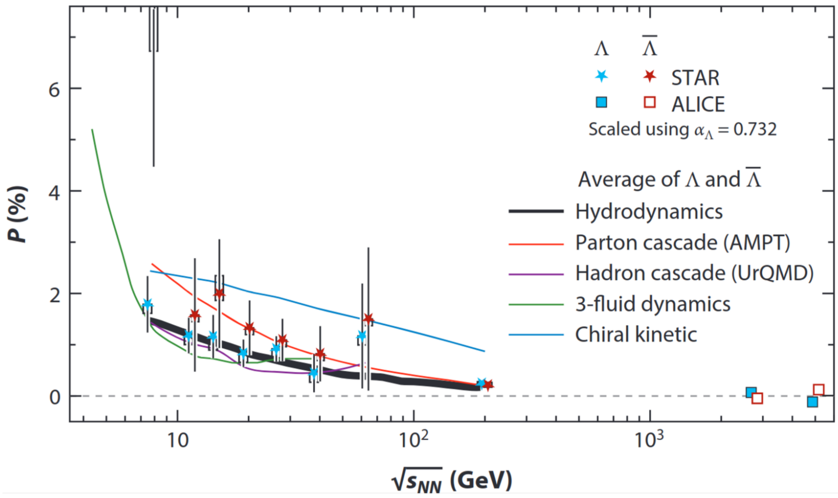


Figure 1.4: Measure of the global polarization at midrapidity. The hydrodynamic model is able to reproduce the data at all the scanned energies. Figure from [38].

Beyond the global observable, differential measurements of the polarization as a function of rapidity, transverse momentum, and azimuthal angle have revealed a rich and complex structure. These analyses probe the space–time evolution of the vortical and shear fields in the QGP and have uncovered subtle features, such as local polarization patterns and the so-called "sign puzzle" in the longitudinal component. Ongoing and future measurements at RHIC and the LHC, together with increasingly refined theoretical descriptions that incorporate shear, electromagnetic, and quantum corrections, are expected to further clarify the interplay between spin dynamics, local thermodynamic fields, and the geometry of the freeze-out hypersurface.

The principal experimental signatures supporting a thermodynamic (thermal-vorticity) origin of the observed polarization are summarized as follows:

- **Energy dependence.** The global polarization decreases with increasing collision energy, consistent with a reduction of the net vorticity in hotter and more symmetric systems.

- **Particle–antiparticle equality at high energy.** In a pure thermal-vorticity scenario, and for $\mu_B \simeq 0$, the polarization induced by thermal vorticity is independent of electric charge and baryon number, and therefore exhibits the same sign and comparable magnitude for Λ and $\overline{\Lambda}$. A significant splitting between Λ and $\overline{\Lambda}$ signals the presence of additional mechanisms, such as electromagnetic contributions coupling to the particle magnetic moment, finite-μ_B effects, or acceptance and feed-down biases.

- **Event-plane correlation and centrality dependence.** The polarization direction is expected to correlate with the global angular-momentum (event-plane) direction of the system and to increase in more peripheral collisions, where the orbital angular momentum is larger.

- **Differential patterns (rapidity, p_T, azimuth).** The detailed dependence of polarization on rapidity, transverse momentum, and azimuth provides direct information on the spatial structure of vorticity and shear fields. Deviations from thermal-vorticity expectations in these differential observables, such as the longitudinal sign puzzle, point to the presence of non-equilibrium, dissipative, quantum, or electromagnetic corrections beyond the simple local-equilibrium picture.

Chapter 2

Quantum Relativistic Statistical Mechanics

In this chapter we introduce the formalism of the non-equilibrium statistical operator. We define the expansion in linear response theory and distinguish between local-equilibrium and dissipative corrections. We then introduce a new expansion method that consistently combines both local-equilibrium and dissipative contributions, yielding an expansion of expectation values in terms of the *initial* thermodynamic fields rather than the instantaneous ones, as is usually implemented.

2.1 Density operator and global thermodynamic equilibrium

In quantum mechanics, the state of a system is described in terms of vectors in a complex Hilbert space, known as **pure states**. A pure state evolves according to the Schrödinger equation and represents a physical system for which complete information is available. However, when dealing with macroscopic systems composed of an extremely large number of microscopic constituents, it is neither practical nor meaningful to have complete knowledge of the exact quantum state of the system.

In general, one assumes that the system can be found in one of the possible microstates $|\psi_\lambda\rangle$ with probability P_λ. The state of the system is then described as a statistical mixture of these microstates [130]:

$$\widehat{\rho} = \sum_\lambda P_\lambda \, |\psi_\lambda\rangle\langle\psi_\lambda| \ , \qquad \sum_\lambda P_\lambda = 1 \ . \tag{2.1}$$

The operator $\widehat{\rho}$ is referred to as the **statistical operator** or **density operator** and represents a statistical ensemble of microstates. If the system is known to be in a specific microstate $|\psi_{\overline{\lambda}}\rangle$, then $P_{\overline{\lambda}} = 1$ and $P_{\lambda\neq\overline{\lambda}} = 0$, and Eq. (2.1) reduces to the projector onto the state $|\psi_{\overline{\lambda}}\rangle$.

Since the density operator is constructed from the possible microstates of the system, it is time-independent in the Heisenberg picture and therefore satisfies the Liouville equation:

$$\frac{\mathrm{d}\widehat{\rho}}{\mathrm{d}t} = 0 \ . \tag{2.2}$$

The density operator provides a convenient framework for computing expectation values of macroscopic observables associated with the system.

Let $\widehat{O}$ be a quantum operator representing a physical observable, such as the energy. The expectation value of $\widehat{O}$ in the microstate $|\psi_\lambda\rangle$ is given by $\langle\psi_\lambda|\widehat{O}|\psi_\lambda\rangle$. Taking into account that the system occupies this microstate with probability P_λ, the expectation value over the ensemble reads

$$\langle\widehat{O}\rangle \equiv \sum_\lambda P_\lambda \langle\psi_\lambda|\widehat{O}|\psi_\lambda\rangle \ .$$

By expanding the microstates $|\psi_\lambda\rangle$ in a complete basis $|e_i\rangle$ of the Hilbert space, this expression can be rewritten as

$$\langle\widehat{O}\rangle = \sum_i \langle e_i| \left(\sum_\lambda P_\lambda \, |\psi_\lambda\rangle\langle\psi_\lambda| \right) |e_i\rangle = \mathrm{Tr}\left(\widehat{\rho}\,\widehat{O}\right) \ .$$

Therefore, the expectation value of an observable $\widehat{O}$ for a system described by the density operator $\widehat{\rho}$ is given by

$$\langle\widehat{O}\rangle = \mathrm{Tr}\left(\widehat{\rho}\,\widehat{O}\right) \ . \tag{2.3}$$

The probabilities P_λ encode the lack of complete information about the true microscopic state of the system. This lack of information can be quantified through the **entropy** S[1]:

$$S = -\mathrm{k}_B \mathrm{Tr}\left(\widehat{\rho}\ \ln\widehat{\rho}\right) \ . \tag{2.4}$$

For macroscopic physical systems, it is neither possible to determine the probability of each microstate nor even to enumerate all possible microstates explicitly. As a consequence, the explicit form of the density operator is inferred using a thermodynamic approach.

According to the second law of thermodynamics, a system in thermodynamic equilibrium is characterized by a maximum of the entropy S. The density operator $\widehat{\rho}$ describing the equilibrium state is therefore obtained by maximizing the entropy under suitable physical constraints imposed by the environment [130].

Let us assume that the system is in thermal contact [131] with a reservoir such that N macroscopic quantities $\widehat{A}_i$ are conserved. These quantities may represent, for example, the total energy or conserved charges. Conservation here means that their expectation values are fixed, although they may fluctuate due to interactions with the reservoir. The density operator $\widehat{\rho}$ is then determined by maximizing the functional

$$F\left[\widehat{\rho}, \widehat{A}_i\right] = -S\left[\widehat{\rho}\right] - z\left[\mathrm{Tr}(\widehat{\rho}) - 1\right] - \sum_{i=1}^{N} \lambda_i \left[\mathrm{Tr}\left(\widehat{\rho}\widehat{A}_i\right) - \langle\widehat{A}_i\rangle\right] \ , \tag{2.5}$$

where the λ_i are Lagrange multipliers enforcing the constraints on the expectation values $\mathrm{Tr}(\widehat{\rho}\widehat{A}_i)$, while z ensures the normalization of the density operator.

The solution that maximizes Eq. (2.5) is given by [130]

$$\widehat{\rho} = \frac{1}{Z}\exp\left(-\sum_{i=1}^{N}\lambda_i\widehat{A}_i\right) \ , \qquad Z = \mathrm{Tr}\left[\exp\left(-\sum_{i=1}^{N}\lambda_i\widehat{A}_i\right)\right] \ , \tag{2.6}$$

[1]We reintroduce the Boltzmann constant k_B. In general, entropy is defined up to a positive multiplicative constant [130]. In order to identify information entropy with thermodynamic entropy, this constant must be chosen to be k_B.

where Z is the **partition function**, which ensures the normalization condition $\mathrm{Tr}\widehat{\rho} = 1$. If the system is in contact with a reservoir with which it can exchange both energy and particles, Eq. (2.6) reduces to the familiar grand-canonical density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left(-\beta \widehat{H} + \zeta \widehat{Q}\right) , \tag{2.7}$$

where $\widehat{H}$ denotes the Hamiltonian of the system and $\widehat{Q}$ is a conserved global charge, such as the particle number. The Lagrange multipliers $\beta = 1/T$ and $\zeta = \mu/T$ have the physical interpretation of the inverse temperature and the ratio of chemical potential to temperature, respectively.

2.1.1 Global equilibrium in relativistic systems

The operator (2.7) is obtained by maximizing the entropy (2.4) under constraints on global quantities, such as the total energy. A state described by (2.7) is therefore referred to as a **global equilibrium state**. In this case, the thermodynamic fields T and μ are constant, homogeneous, and isotropic throughout the system. The density operator (2.7) associated with the grand-canonical ensemble is, however, not manifestly covariant and is therefore valid only in a specific reference frame.

For a relativistic system, conserved charges are obtained by integrating the corresponding conserved densities, namely the stress–energy tensor $\widehat{T}^{\mu\nu}$ and the conserved currents $\widehat{j}^{\mu}_i$, where the index i runs over all independent conserved charges. For simplicity, we consider only a single global Abelian charge, which in the case of the QGP may correspond, for instance, to the baryon number. The local conservation of these densities is expressed by the well-known relations

$$\partial_\mu \widehat{T}^{\mu\nu}(x) = 0 \tag{2.8a}$$

$$\partial_\mu \widehat{j}^{\mu}(x) = 0 \, . \tag{2.8b}$$

To construct the corresponding global charges, such as the total energy, the conserved densities must be integrated over space. In order to perform this integration in a covariant manner, we introduce a foliation of space–time $\{\Sigma(\tau)\}$, defined as a family of space-like hypersurfaces Σ parametrized by a real parameter τ.

The time-like unit normal vector $\hat{n}^\mu$ to each hypersurface defines a legitimate worldline for an observer intersecting Σ, although the parameter τ does not, in general, coincide with the proper time of that observer. A necessary condition for the foliation to be well defined is provided by the Frobenius theorem, which requires the normal vector $\hat{n}^\mu$ to be irrotational:

$$\varepsilon_{\mu\nu\rho\sigma} \left(\partial^\nu \hat{n}^\rho - \partial^\rho \hat{n}^\nu\right) = 0 \, . \tag{2.9}$$

Geometrically, this condition ensures that space–time can be foliated into non-overlapping simultaneity hypersurfaces associated with a given class of observers. A natural choice is the Arnowitt–Deser–Misner (ADM) foliation (see Fig. 2.1), which we will assume to be applicable throughout the following discussion.

With this construction, the global charges of the system associated with a given

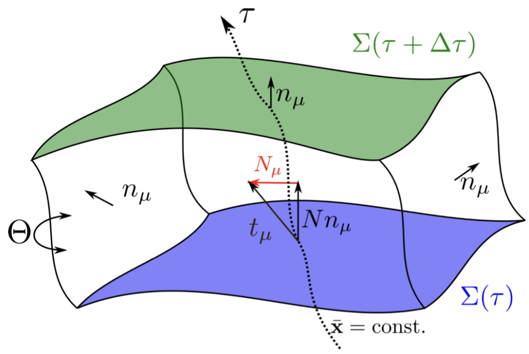


Figure 2.1: Schematic illustration of the ADM foliation of spacetime. The spacelike hypersurfaces $\Sigma(\tau)$ and $\Sigma(\tau+\Delta\tau)$ are separated by an infinitesimal interval in the foliation parameter τ. The unit normal vector n_μ defines the hypersurface geometry, while the evolution vector decomposes as $t_\mu = Nn_\mu + N_\mu$. The dashed curve represents a worldline at fixed spatial coordinates $\bar{x}$ = constant. Figure from [132]

hypersurface Σ are defined as

$$\widehat{P}^\nu = \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu} , \tag{2.10a}$$

$$\widehat{Q} = \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{j}^\mu , \tag{2.10b}$$

where $\widehat{P}^\mu$ is the four-momentum operator, $\widehat{Q}$ is the conserved charge operator, and $\mathrm{d}\Sigma_\mu \equiv \mathrm{d}\Sigma\, \hat{n}_\mu$ denotes the hypersurface element. Since the stress–energy tensor and the conserved current satisfy the conservation laws (2.8), the integrals in (2.10) are independent of the specific choice of Σ.

It is now straightforward to construct the density operator associated with a relativistic system possessing conserved four-momentum and charge (2.10). As before, the density operator is obtained by maximizing the functional (2.5), which in this case reads

$$\begin{aligned} F\left[\widehat{\rho}, \widehat{P}, \widehat{Q}\right] &= -S\left[\widehat{\rho}\right] - z\left[\mathrm{Tr}(\widehat{\rho}) - 1\right] \\ &- \int_\Sigma \mathrm{d}\Sigma_\mu \beta_\nu \left[\mathrm{Tr}\left(\widehat{\rho}\widehat{T}^{\mu\nu}\right) - \langle \widehat{T}^{\mu\nu}\rangle\right] \\ &+ \int_\Sigma \mathrm{d}\Sigma_\mu \zeta \left[\mathrm{Tr}\left(\widehat{\rho}\widehat{j}^\mu\right) - \langle \widehat{j}^\mu\rangle\right] , \end{aligned} \tag{2.11}$$

where, since the system is in equilibrium, the functional—and hence its solution—must depend only on the integrated charges $\widehat{P}$ and $\widehat{Q}$.

The density operator $\widehat{\rho}$ that maximizes the functional above is given by

$$\widehat{\rho} = \frac{1}{Z} \exp\left(- \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\beta_\nu + \int_\Sigma \mathrm{d}\Sigma_\mu \widehat{j}^\mu \zeta \right) . \tag{2.12}$$

The fields β_ν and ζ act as Lagrange multipliers. Their physical interpretation is that of the **four-temperature** and the **reduced chemical potential**, respectively:

$$\beta_\mu = \frac{u_\mu}{T} , \qquad \zeta = \frac{\mu}{T} , \tag{2.13}$$

where u_μ is the four-velocity satisfying $u \cdot u = 1$, T is the proper temperature measured in the comoving frame, and μ is the chemical potential associated with the conserved charge [133].

At first sight, the solution (2.12) appears to depend explicitly on the choice of the hypersurface Σ. However, global equilibrium requires the state of the system to depend only on the integrated charges (2.10). The necessary and sufficient condition for (2.12) to be independent of Σ is that the integrand has vanishing divergence:

$$\partial_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) = 0 \, .$$

Using the conservation laws (2.8), this condition reduces to

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \, , \qquad \partial_\mu \zeta = 0 \, , \tag{2.14}$$

that is, the reduced chemical potential must be constant, while the four-temperature field must be a Killing vector. Equation (2.14) defines the state of **generalized global equilibrium**. The independence of Σ further implies that the operator (2.12) satisfies the Liouville equation (2.2), as expected for an equilibrium state.

Restricting to the case of Minkowski space–time, the Killing equation (2.14) admits the general solution

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \, , \tag{2.15}$$

where b_μ is a constant four-vector and $\varpi_{\mu\nu}$ is the **thermal vorticity**, defined as the antisymmetric part of the gradient of the four-temperature:

$$\varpi_{\mu\nu} = -\frac{1}{2} \left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu \right) \, . \tag{2.16}$$

With these definitions, the statistical operator (2.12) can be rewritten as

$$\widehat{\rho} = \frac{1}{Z} \exp \left(-\beta_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right) \, , \tag{2.17}$$

where $\widehat{J}^{\mu\nu}$ denotes the boost–angular momentum operator. The density operator (2.17) is therefore referred to as the **global equilibrium** density operator.

By comparing (2.17) with the non-relativistic grand-canonical operator (2.7), one sees that the two coincide in the comoving reference frame $\beta_\mu = (1/T)(1, \mathbf{0})$ and when $\varpi_{\mu\nu} = 0$. The global equilibrium state with vanishing thermal vorticity,

$$\widehat{\rho}_{\rm GE} \equiv \frac{1}{Z} \exp \left(-\beta \cdot \widehat{P} + \zeta \widehat{Q} \right) \, , \tag{2.18}$$

is therefore often—though somewhat imprecisely—referred to as the global equilibrium density operator. More accurately, it represents a particular configuration of equilibrium and is thus more properly denoted as the **homogeneous global equilibrium** operator.

In general, for relativistic systems with non-vanishing vorticity, more elaborate configurations of global equilibrium are possible, involving both rotation and acceleration [134–136][2]. Somewhat counterintuitively, in a relativistic system at global

[2]The thermal vorticity encodes both linear acceleration and rotational effects, which can give rise to thermodynamic equilibrium states associated with conserved angular momentum or boost operators. The former admits a classical interpretation, while the latter is intrinsically relativistic. In general, a non-vanishing thermal vorticity implies a combination of the two.

thermodynamic equilibrium the temperature field may be *neither homogeneous*, as in the presence of rotation, *nor constant in time*, as in the presence of uniform acceleration.

We also emphasize that, as a direct consequence of the Killing condition (2.15), the **thermal shear** tensor $\xi_{\mu\nu}$, defined as the symmetric part of the four-temperature gradient,

$$\xi_{\mu\nu} \equiv \frac{1}{2}\left(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu\right) \, , \tag{2.19}$$

vanishes identically in global equilibrium.

2.2 The Zubarev density operator

The density operator (2.17) has been obtained by requiring independence from the integration hypersurface Σ. It maximizes the entropy of the system under the constraint of fixed global charges and therefore represents a state of thermodynamic equilibrium. Equilibrium states are of interest in their own right, but they are not suitable to describe realistic dynamical systems such as the QGP produced in heavy-ion collisions. In strict global equilibrium, the macroscopic fields are time-independent in the comoving frame, entropy production vanishes, and the system cannot evolve away from equilibrium in the absence of external perturbations. Once such a state is reached, it persists indefinitely.

In heavy-ion collisions, however, the system undergoes rapid expansion, and true global equilibrium is never achieved. Nevertheless, after a short pre-equilibrium stage, the QGP approaches **local thermodynamic equilibrium** (LTE) and begins to behave as a fluid. Local equilibrium does not imply equilibrium in the global sense; rather, it means that on space–time scales large compared to microscopic mean free paths and interaction times, each infinitesimal fluid cell can be *approximately* described by equilibrium thermodynamics with slowly varying intensive fields. More concretely, there exist smooth fields $u^\mu(x)$, $T(x)$, and $\mu_a(x)$ such that, locally, thermodynamic relations remain well defined[3].

In this sense, the QGP created in heavy-ion collisions resides in a *non-equilibrium* state that is sufficiently close to equilibrium *locally* to admit a controlled thermodynamic, and ultimately hydrodynamic, description. It is therefore still possible to describe local-equilibrium properties using the language of quantum statistical mechanics and equilibrium thermodynamics. In particular, one can define an appropriate density operator $\widehat{\rho}$ that describes the non-equilibrium state of the system, known as the Zubarev operator [40, 42].

The first step in constructing the true non-equilibrium density operator $\widehat{\rho}$ consists in defining the so-called *local-equilibrium operator* $\widehat{\rho}_{\rm LE}$. Despite its name, as will become clear shortly, $\widehat{\rho}_{\rm LE}$ does not represent a physical state of the system, but rather a mathematical tool required for the definition of $\widehat{\rho}$.

In a state of local equilibrium, the system is no longer described by the operator (2.17). However, since local thermodynamic relations can still be defined, the entropy of the system is nevertheless maximized. This maximization must now be performed under constraints imposed by the *local* values of the conserved densities

[3]The concept of LTE relies on a clear separation of physical scales. This will become more transparent in the following, where we formalize and apply this concept to derive a hydrodynamic expansion for local-equilibrium corrections.

appearing in (2.8). Accordingly, the local-equilibrium operator $\widehat{\rho}_{\rm LE}$ is defined as the operator that maximizes the functional

$$
\begin{aligned}
F\left[\widehat{\rho}, \widehat{T}, \widehat{j}; \Sigma\right] &= -S\left[\widehat{\rho}\right] - z\left[\mathrm{Tr}(\widehat{\rho}) - 1\right] \\
&\quad - \int_{\Sigma} \mathrm{d}\Sigma_\mu \beta_\nu \left[\mathrm{Tr}\left(\widehat{\rho}\widehat{T}^{\mu\nu}\right) - \langle \widehat{T}^{\mu\nu}\rangle\right] \\
&\quad + \int_{\Sigma} \mathrm{d}\Sigma_\mu \zeta \left[\mathrm{Tr}\left(\widehat{\rho}\widehat{j}^{\mu}\right) - \langle \widehat{j}^{\mu}\rangle\right] .
\end{aligned} \tag{2.20}
$$

Formally, this functional coincides with (2.11), and its solution is given by

$$
\widehat{\rho}_{\rm LE}(\tau) = \frac{1}{Z_{\rm LE}(\tau)} \exp\left[-\int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^{\mu}\zeta\right)\right] . \tag{2.21}
$$

The local-equilibrium operator thus defined coincides formally with (2.12). However, since the system is not in global equilibrium, it depends on the local densities $\widehat{T}$ and $\widehat{j}$, rather than on the integrated charges. The integrand is no longer divergence-free, and the conditions (2.14) are not satisfied. As a consequence, $\widehat{\rho}_{\rm LE}(\tau)$ explicitly depends on the particular hypersurface $\Sigma(\tau)$ of the foliation. The most important implication is that the local-equilibrium operator $\widehat{\rho}_{\rm LE}$ in (2.21) does *not* satisfy the Liouville equation (2.2), and therefore cannot represent a physical state of the system.

Nevertheless, since $\widehat{\rho}_{\rm LE}$ maximizes the entropy under local constraints imposed on the hypersurface $\Sigma(\tau)$, it correctly reproduces the local thermodynamic properties of the system on $\Sigma(\tau)$. In particular, the following matching conditions hold:

$$
\hat{n}_\mu \langle \widehat{T}^{\mu\nu}\rangle_{\rm LE} \equiv \hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\widehat{T}^{\mu\nu}\right) = \hat{n}_\mu \langle \widehat{T}^{\mu\nu}\rangle, \tag{2.22a}
$$

$$
\hat{n}_\mu \langle \widehat{j}^{\mu}\rangle_{\rm LE} \equiv \hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\widehat{j}^{\mu}\right) = \hat{n}_\mu \langle \widehat{j}^{\mu}\rangle, \tag{2.22b}
$$

where $\langle \widehat{T}\rangle$ and $\langle \widehat{j}\rangle$ denote the actual, unknown non-equilibrium expectation values of the system, computed in the true non-equilibrium state that is yet to be defined.

The true non-equilibrium density operator must both satisfy the Liouville equation and maximize the entropy under constraints on the local densities. Assuming that the system is *known* to be in local thermodynamic equilibrium on a given space-like hypersurface $\Sigma(\tau_0)$, the non-equilibrium, or Zubarev, operator is defined as

$$
\widehat{\rho} \equiv \widehat{\rho}_{\rm LE}(\tau_0) = \frac{1}{Z(\tau_0)} \exp\left[-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\mathrm{d}\Sigma_\mu \widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^{\mu}\zeta\right)\right] . \tag{2.23}
$$

By construction, $\widehat{\rho}$ maximizes (2.11) and, being defined at a fixed value τ_0, trivially satisfies the Liouville equation (2.2). It therefore represents a physical state of the system. Non-equilibrium expectation values are then computed using (2.23) as

$$
\langle \widehat{T}^{\mu\nu}\rangle \equiv \mathrm{Tr}\left(\widehat{\rho}\widehat{T}^{\mu\nu}\right) , \qquad \langle \widehat{j}^{\mu}\rangle \equiv \mathrm{Tr}\left(\widehat{\rho}\widehat{j}^{\mu}\right) . \tag{2.24}
$$

The matching conditions (2.22) then become functional relations,

$$
\hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\left[\beta_\mu, \zeta, \hat{n}\right]\widehat{T}^{\mu\nu}\right) = \hat{n}_\mu \langle \widehat{T}^{\mu\nu}\rangle , \tag{2.25a}
$$

$$
\hat{n}_\mu \mathrm{Tr}\left(\widehat{\rho}_{\rm LE}\left[\beta_\mu, \zeta, \hat{n}\right]\widehat{j}^{\mu}\right) = \hat{n}_\mu \langle \widehat{j}^{\mu}\rangle , \tag{2.25b}
$$

which can be solved to determine the local-equilibrium thermodynamic fields β_μ, ζ, and $\hat{n}$. The relations (2.25) are functional equations in which the left-hand side depends not only on the local values of the thermodynamic fields, but also on all their derivatives at the given space–time point [43].

Indeed, the essential difference between the local-equilibrium operator $\widehat{\rho}_{\rm LE}$ (2.21) and the true non-equilibrium statistical operator $\widehat{\rho}$ (2.23) lies in the type of physical information each of them encodes. The operator $\widehat{\rho}_{\rm LE}$ is constructed to maximize the entropy under the constraint of fixed local expectation values of conserved densities. In this sense, $\widehat{\rho}_{\rm LE}$ reflects only the instantaneous state of the system at a given space–time point, with no memory of how that state was reached. Its validity is therefore restricted to near-equilibrium situations in which correlations and relaxation effects beyond the hydrodynamic scale can be neglected.

By contrast, the exact non-equilibrium operator $\widehat{\rho}$ contains the full dynamical information about the system. It can be shown that the definition (2.23) is equivalent to an operator that integrates over the entire past history of the system, with an infinitesimal time-smoothing kernel [40, 43, 137]. As a result, $\widehat{\rho}$ naturally encodes non-local effects in both space and time:

- **Long-range correlations:** correlations between spatially separated fluid cells that cannot be captured by an ansatz based solely on local densities.
- **Memory effects:** the statistical operator retains information about the past evolution of the system, so that dissipative processes depend not only on the instantaneous state but also on the relaxation history.
- **Irreversibility:** the infinitesimal time asymmetry inherent in the construction of $\widehat{\rho}$ ensures entropy production and the emergence of macroscopic irreversibility, which are absent in the strictly reversible operator $\widehat{\rho}_{\rm LE}$.

Therefore, while $\widehat{\rho}_{\rm LE}$ provides a practical and thermodynamically motivated approximation suitable for hydrodynamic descriptions, it neglects precisely those non-local and non-Markovian features that are essential for a fully consistent treatment of non-equilibrium statistical mechanics.

These considerations can be made explicit as follows. The Zubarev operator is defined on the hypersurface $\Sigma(\tau_0)$ in the past, at the time when the system is assumed to be in local equilibrium. It can be rewritten in terms of the present hypersurface $\Sigma(\tau)$ by applying Gauss' theorem to the space–time region Ω enclosed by $\Sigma(\tau)$ and $\Sigma(\tau_0)$ (see Fig. 2.2):

$$\begin{aligned} \widehat{\rho} = \frac{1}{Z} \exp \Bigg[- \int_{\Sigma(\tau)} \left({\rm d}\Sigma_\mu \widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \zeta \right) \\ + \int_\Omega {\rm d}\Omega \left(\widehat{T}^{\mu\nu} \partial_\mu \beta_\nu - \widehat{j}^\mu \partial_\mu \zeta \right) \Bigg] \ , \end{aligned} \tag{2.26}$$

where we have used the conservation equations (2.8) and assumed that the fields vanish at the boundaries.

In Eq. (2.26), the first term again represents a local-equilibrium contribution evaluated on the present hypersurface. The second term, however, involves an integral over the space–time volume Ω and therefore depends on the entire past history of the system through the gradients of the thermodynamic fields. Equation (2.26)

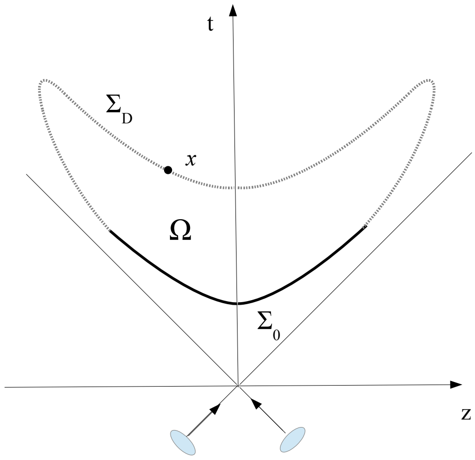


Figure 2.2: The typical initial local-equilibrium hypersurface $\Sigma_{\mathrm{EQ}} \equiv \Sigma_0$ (solid line) and the decoupling hypersurface Σ_{D} (finely dotted line) in relativistic heavy-ion collisions; the enclosed region Ω is where the quark–gluon plasma exists. On a point x on Σ_{D} one can assume that the dominant degrees of freedom are those of quasi-free hadrons.

thus provides the appropriate description of the hydrodynamic regime of the QGP, as schematically illustrated in Fig. 1.2.

As will become clearer in the following, this separation between local and history-dependent contributions is fundamental for identifying dissipative processes associated with entropy production in the system.

2.3 Linear response theory

The computation of the thermal expectation value of a generic observable $\widehat{O}(x)$ in the non-equilibrium state described by the Zubarev operator (2.23) $\widehat{\rho}$ requires evaluating the trace

$$\langle\widehat{O}(x)\rangle = Z^{-1}\mathrm{Tr}\left\{\exp\left[-\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\left(\widehat{T}^{\mu\nu}(y)\beta_\nu(y) - \widehat{j}^\mu(y)\zeta(y)\right)\right]\widehat{O}(x)\right\} , \quad (2.27)$$

where $\Sigma_0 \equiv \Sigma(\tau_0)$ denotes the space-like hypersurface on which the system is assumed to reach a local thermodynamic equilibrium configuration. The coordinates y parametrize Σ_0, and both the operators $\widehat{T}^{\mu\nu}$, $\widehat{j}^\mu$ and the thermodynamic fields are evaluated on this hypersurface.

The statistical operator $\widehat{\rho}$ does not evolve in the Heisenberg representation and is therefore fixed by the initial conditions of the system, namely by the values of the thermodynamic fields and the conserved densities on the initial hypersurface Σ_0. The observable $\widehat{O}(x)$, on the other hand, is typically evaluated at a different space–time point x, lying in the future of Σ_0. Since the statistical operator depends functionally on the fields β, ζ, and $\hat{n}$, one can write

$$\mathrm{Tr}\left[\widehat{\rho}\,\widehat{O}(x)\right] = O\left[\beta_0, \zeta_0, \hat{n}_0\right](x) \, .$$

This functional dependence can be equivalently expressed as a dependence on an infinite set of arguments. If the functions β_0, ζ_0, and $\hat{n}_0$ are infinitely differentiable, they can be replaced by the values of all their derivatives at a point x_0 on the hypersurface:

$$\mathrm{Tr}\left[\widehat{\rho}\,\widehat{O}(x)\right] = O\Big(x, \beta(x_0), \partial\beta(x_0), \dots, \zeta(x_0), \partial\zeta(x_0), \dots, \hat{n}(x_0), \partial\hat{n}(x_0), \dots\Big)\ ,$$

where the dots indicate higher-order gradients evaluated at the same point x_0. Note that, in general, the point x_0 may depend on x, the space–time point at which the observable is evaluated.

An exact evaluation of the expectation value above is generally not possible, and one must therefore resort to approximations. If the system is close to local thermodynamic equilibrium, it is near a homogeneous equilibrium configuration and fluctuations around equilibrium are small. One may then separate the equilibrium and non-equilibrium contributions, with deviations from equilibrium controlled by gradients of the thermodynamic fields, assumed to be small. This motivates truncating the expansion at a finite order in derivatives:

$$\mathrm{Tr}\left[\widehat{\rho}\,\widehat{O}(x)\right] \simeq o_0\left(x, \beta_0(x_0(x))\right) + o_1\left(x, \beta_0(x_0(x))\right)\partial\beta_0\left(x_0(x)\right) + \dots\ , \tag{2.28}$$

which provides a good approximation to (2.27). Indeed, the most direct expansion suggested by the Zubarev operator is expressed in terms of the thermodynamic fields and their gradients evaluated on the *initial* hypersurface. This follows from the fact that the Zubarev operator is completely determined by the initial values of the fields on Σ_0.

However, if the system behaves as a fluid between the initial hypersurface Σ_0 and the space–time point x at which the observable is evaluated, an alternative expansion scheme may provide a more useful approximation to the expectation value (2.27). Instead of expanding in terms of the initial thermodynamic fields, one may perform an expansion in terms of the thermodynamic fields evaluated at the same space–time point x where the observable is computed:

$$\mathrm{Tr}\left[\widehat{\rho}\,\widehat{O}(x)\right] \simeq o'_0\left(x, \beta(x)\right) + o'_1\left(x, \beta(x)\right)\partial\beta\left(x\right) + \dots\ . \tag{2.29}$$

The two expansion schemes (2.28) and (2.29) are related. In principle, the expansion in terms of *present* gradients (2.29) can be obtained from the expansion in terms of initial gradients (2.28) by accounting for the fact that the thermodynamic fields evolve according to the equations of relativistic hydrodynamics. Consequently, $\beta(x)$, $\zeta(x)$, and $\hat{n}(x)$ are themselves functionals of β_0, ζ_0, and $\hat{n}_0$. So that the coefficients $o_0,\ o_1, \dots$ in(2.28) are themselves related functionally with the coefficients $o'_0,\ o'_1, \dots$ in (2.29).

The reason why the second approach (2.29) is commonly employed in the literature is that one is often interested in expectation values of operators $\widehat{O}(x)$ that are *local* in x. In that case, an expansion in terms of the local state of the system is more natural. However, one may also be interested in expectation values of operators that are *non-local* in x (the Wigner operator introduced in the following chapter is the most important example). In such cases, it is not obvious that (2.29) provides a better approximation scheme than (2.28), which instead follows directly from the definition of the Zubarev operator.

A gradient expansion in terms of the *present gradients* of the type (2.29) emerges once one considers the expansion of the Zubarev operator after applying Gauss's theorem (2.26). Introducing the operators

$$\widehat{\mathcal{E}} \equiv -\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu\zeta\right) , \tag{2.30a}$$

$$\widehat{\mathcal{D}} \equiv \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu - \widehat{j}^\mu\partial_\mu\zeta\right) , \tag{2.30b}$$

the statistical operator (2.26) can be written as

$$\widehat{\rho} = \frac{1}{Z_{\mathcal{E}+\mathcal{D}}} \exp\left[\widehat{\mathcal{E}} + \widehat{\mathcal{D}}\right] , \qquad Z_{\mathcal{E}+\mathcal{D}} \equiv \mathrm{Tr}\left(\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}}\right) . \tag{2.31}$$

The expectation value (2.27) then becomes

$$\langle \widehat{O}(x) \rangle = \mathrm{Tr}\left[\frac{\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}}}{Z_{\mathcal{E}+\mathcal{D}}} \widehat{O}(x)\right] .$$

The operator $\widehat{\mathcal{E}}$ coincides with the exponent of the local-equilibrium statistical operator $\widehat{\rho}_{\mathrm{LE}}$ defined in (2.21). Assuming that the hydrodynamic fields vary slowly in space–time is equivalent to assuming that the integral term $\widehat{\mathcal{D}}$ represents a small deviation from the local-equilibrium expectation value

$$\langle \widehat{O}(x) \rangle_{\mathrm{LE}} \equiv \mathrm{Tr}\left[\frac{\mathrm{e}^{\widehat{\mathcal{E}}}}{Z_{\mathcal{E}}} \widehat{O}(x)\right] .$$

If $\widehat{\mathcal{D}}$ is small, Eq. (2.31) can be expanded around the reference local-equilibrium state. To this end, we rewrite (2.31) using the *Kubo identity*:

$$\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}} = \mathrm{e}^{\widehat{\mathcal{E}}} + \int_0^1 \mathrm{d}z \left[\mathrm{e}^{z(\widehat{\mathcal{E}}+\widehat{\mathcal{D}})}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right] \mathrm{e}^{\widehat{\mathcal{E}}} . \tag{2.32}$$

This identity is the starting point for deriving the linear response expansion, which in the present case coincides with an expansion in powers of $\widehat{\mathcal{D}}$. The identity can be iterated by substituting the exponential inside the integral with the right-hand side of the same equation. Assuming $\widehat{\mathcal{D}} \ll \widehat{\mathcal{E}}$ and truncating the expansion at linear order, one obtains

$$\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}} \simeq \mathrm{e}^{\widehat{\mathcal{E}}} + \int_0^1 \mathrm{d}z \left[\mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right] \mathrm{e}^{\widehat{\mathcal{E}}} .$$

In order to apply this expansion to the statistical operator (2.31), one must also take into account the expansion of the trace:

$$\begin{aligned} Z = \mathrm{Tr}\left(\mathrm{e}^{\widehat{\mathcal{E}}+\widehat{\mathcal{D}}}\right) &\simeq \mathrm{Tr}\left(\mathrm{e}^{\widehat{\mathcal{E}}}\right) + \int_0^1 \mathrm{d}z\, \mathrm{Tr}\left[\left(\mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}}\right) \mathrm{e}^{\widehat{\mathcal{E}}}\right] \\ &\equiv Z_{\mathrm{LE}} + Z_{\mathrm{LE}} \int_0^1 \mathrm{d}z \left\langle \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{\mathcal{D}}\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{\mathrm{LE}} , \end{aligned}$$

where $Z_{\rm LE} \equiv {\rm Tr}({\rm e}^{\widehat{\mathcal{E}}})$. Substituting this result into the Kubo identity yields

$$\begin{aligned}\widehat{\rho} &\simeq \left[{\rm e}^{\widehat{\mathcal{E}}} + \int_0^1 {\rm d}z \left({\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right){\rm e}^{\widehat{\mathcal{E}}}\right]\left[Z_{\rm LE} + Z_{\rm LE}\int_0^1 {\rm d}z \left\langle {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE}\right]^{-1} \\ &\simeq \frac{{\rm e}^{\widehat{\mathcal{E}}}}{Z_{\rm LE}} + \int_0^1 {\rm d}z \left({\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right)\frac{{\rm e}^{\widehat{\mathcal{E}}}}{Z_{\rm LE}} - \int_0^1 {\rm d}z \left\langle {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE}\frac{{\rm e}^{\widehat{\mathcal{E}}}}{Z_{\rm LE}}\,,\end{aligned}$$

where only terms up to $\mathcal{O}(\widehat{\mathcal{D}}^2)$ have been retained. With this result, the approximate expectation value of an observable $\widehat{O}(x)$ is given by

$$\begin{aligned}\langle\widehat{O}(x)\rangle = \langle\widehat{O}(x)\rangle_{\rm LE} &+ \int_0^1 {\rm d}z \left\langle \widehat{O}(x)\,{\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE} \\ &- \langle\widehat{O}(x)\rangle_{\rm LE}\int_0^1 {\rm d}z \left\langle {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{\mathcal{D}}\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{\rm LE} + \mathcal{O}(\widehat{\mathcal{D}}^2)\,.\end{aligned}$$

The full expectation value can therefore be written as

$$\langle\widehat{O}(x)\rangle \simeq \langle\widehat{O}(x)\rangle_{\rm LE} + \Delta O_{\rm diss}(x)\,, \tag{2.33}$$

where $\langle\widehat{O}(x)\rangle_{\rm LE} \equiv {\rm Tr}(\widehat{\rho}_{\rm LE}\widehat{O}(x))$ is the local-equilibrium expectation value, while the dissipative correction is given by

$$\begin{aligned}\Delta O_{\rm diss}(x) = &\int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\,\partial_\mu\beta_\nu(y)\left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{T}^{\mu\nu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{c,{\rm LE}} \\ &- \int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\,\partial_\mu\zeta(y)\left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{j}^{\mu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{c,{\rm LE}}\,,\end{aligned} \tag{2.34}$$

where the connected part of an expectation value is defined as

$$\langle\widehat{O}_1(x),\ \widehat{O}_2(y)\rangle_{c,{\rm LE}} \equiv \langle\widehat{O}_1(x)\widehat{O}_2(y)\rangle_{\rm LE} - \langle\widehat{O}_1(x)\rangle_{\rm LE}\langle\widehat{O}_2(y)\rangle_{\rm LE}\,.$$

The dissipative correction (2.34) can be expressed in terms of correlation functions between the observable $\widehat{O}(x)$ and the conserved densities of the system:

$$C^{\mu\nu}_{O,T}(y,x) \equiv \int_0^1 {\rm d}z \left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{T}^{\mu\nu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{c,{\rm LE}}\,, \tag{2.35a}$$

$$C^{\mu}_{O,j}(y,x) \equiv \int_0^1 {\rm d}z \left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\,\widehat{j}^{\mu}(y)\,{\rm e}^{-z\widehat{\mathcal{E}}}\right\rangle_{c,{\rm LE}}\,. \tag{2.35b}$$

For typical local observables $\widehat{O}(x)$, the correlation functions (2.35) are sharply peaked around $y \simeq x$, with a characteristic width ℓ_O determined by microscopic scales such as the temperature, particle masses, and interaction lengths. Under the assumption of local thermodynamic equilibrium, the characteristic variation scales of the thermodynamic fields,

$$\lambda_\beta \equiv \frac{\beta}{|\partial\beta|}\,, \qquad \lambda_\zeta \equiv \frac{\zeta}{|\partial\zeta|}\,, \tag{2.36}$$

are much larger, namely

$$\ell_O \ll \lambda_\beta\,, \qquad \ell_O \ll \lambda_\zeta\,. \tag{2.37}$$

As a consequence, the correlators (2.35) are dominated by contributions from a neighborhood of the same space–time point x, and the dissipative correction (2.34) is well approximated by

$$\begin{aligned}\Delta O_{\text{diss}}(x) \simeq \partial_\mu \beta_\nu(x) \int_0^1 \mathrm{d}z \int_\Omega \mathrm{d}^4 y \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{LE}} \\ - \partial_\mu \zeta(x) \int_0^1 \mathrm{d}z \int_\Omega \mathrm{d}^4 y \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{j}^{\mu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{LE}} .\end{aligned} \tag{2.38}$$

This expression makes explicit that the dissipative correction can be written in terms of local-equilibrium correlation functions between the observable $\widehat{O}(x)$ and the conserved densities, multiplied by gradients of the thermodynamic fields evaluated at the same space–time point x. These correlators are directly related to dissipative transport coefficients, such as the shear viscosity η.

The local-equilibrium expectation value can itself be approximated within linear response theory. Indeed, the assumption of slowly varying thermodynamic fields also applies to the local-equilibrium operator (2.21), allowing one to interpret the local-equilibrium state as a perturbation of a homogeneous global-equilibrium configuration. Introducing

$$\beta_\nu(y) = \beta_\nu(x) + [\beta_\nu(y) - \beta_\nu(x)] \equiv \beta_\nu(x) + \Delta\beta_\nu(y,x) \ , \tag{2.39a}$$

$$\zeta(y) = \zeta(x) + [\zeta(y) - \zeta(x)] \equiv \zeta(x) + \Delta\zeta(y,x) \ , \tag{2.39b}$$

the local-equilibrium statistical operator (2.21) can be rewritten as

$$\begin{aligned}\widehat{\rho}_{\mathrm{LE}} = \frac{1}{Z_{\mathrm{LE}}} \exp\Big[- \beta(x)\cdot \widehat{P} + \zeta(x)\widehat{Q} \\ + \int_\Sigma \mathrm{d}\Sigma_\mu(y) \Big(\Delta\beta_\nu(y,x) \widehat{T}^{\mu\nu}(y) - \Delta\zeta(y,x) \widehat{j}^{\mu}(y) \Big) \Big] .\end{aligned} \tag{2.40}$$

The first term in the exponent coincides with that of the global-equilibrium statistical operator. One can therefore define

$$\widehat{\mathcal{E}}_{\mathrm{GE}} \equiv -\beta(x)\cdot\widehat{P} + \zeta(x)\widehat{Q} \ , \tag{2.41a}$$

$$\Delta\widehat{\mathcal{E}} \equiv \int_\Sigma \mathrm{d}\Sigma_\mu(y) \Big(\Delta\beta_\nu(y,x)\widehat{T}^{\mu\nu}(y) - \Delta\zeta(y,x)\widehat{j}^{\mu}(y) \Big) \ . \tag{2.41b}$$

With these definitions, the local-equilibrium statistical operator takes the form

$$\widehat{\rho}_{\mathrm{LE}} = \frac{1}{Z_{\mathrm{LE}}} \exp\left[\widehat{\mathcal{E}}_{\mathrm{GE}} + \Delta\widehat{\mathcal{E}}\right] \ , \qquad Z_{\mathrm{LE}} = \mathrm{Tr}\left(\mathrm{e}^{\widehat{\mathcal{E}}_{\mathrm{GE}} + \Delta\widehat{\mathcal{E}}} \right) \ . \tag{2.42}$$

Assuming slowly varying thermodynamic fields is equivalent to assuming $\Delta\beta \ll \beta$ and $\Delta\zeta \ll \zeta$. Under this assumption, one may apply the same linear-response expansion discussed previously, using $\widehat{\mathcal{E}}_{\mathrm{GE}}$ as the reference state and $\Delta\widehat{\mathcal{E}}$ as a perturbation. The local-equilibrium expectation value then reads

$$\langle \widehat{O}(x) \rangle_{\mathrm{LE}} \simeq \langle \widehat{O}(x) \rangle_{\mathrm{GE}} + \Delta O_{\mathrm{LE}}(x) \ , \tag{2.43}$$

where the deviation from global equilibrium is given by

$$\begin{aligned}\Delta O_{\mathrm{LE}}(x) = -\int_0^1 \mathrm{d}z \int_\Sigma \mathrm{d}\Sigma_\mu(y)\, \Delta\beta_\nu(y,x) \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{GE}} \\ + \int_0^1 \mathrm{d}z \int_\Sigma \mathrm{d}\Sigma_\mu(y)\, \Delta\zeta(y,x) \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{j}^{\mu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{GE}} .\end{aligned} \tag{2.44}$$

With this the dissipative correction (2.34) is thus given by:

$$\begin{aligned}\Delta O_{\rm diss}(x) = &\int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\, \partial_\mu \beta_\nu(y) \left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, {\rm e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,{\rm GE}} \\ &- \int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\, \partial_\mu \zeta(y) \left\langle \widehat{O}(x),\ {\rm e}^{z\widehat{\mathcal{E}}}\, \widehat{j}^{\mu}(y)\, {\rm e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,{\rm GE}} ,\end{aligned} \tag{2.45}$$

Combining (2.44) and (2.45), the non-equilibrium expectation value in linear response theory can finally be written as

$$\langle \widehat{O}(x) \rangle \simeq \langle \widehat{O}(x) \rangle_{\rm GE} + \Delta O_{\rm LE}(x) + \Delta O_{\rm diss}(x)\,. \tag{2.46}$$

As will be shown in the next chapter, this expansion—together with the explicit expressions for $\Delta O_{\rm diss}(x)$ and $\Delta O_{\rm LE}(x)$ given in (2.34) and (2.44)—yields an expansion of the full non-equilibrium expectation value in terms of gradients of the thermodynamic fields evaluated at the *same point* x at which the operator is computed. This corresponds precisely to an expansion of the form (2.29).

The above expansion has been widely employed to compute various quantities, such as constitutive relations for the stress–energy tensor, dissipative transport coefficients, and local thermodynamic corrections to the spin polarization. However, this method relies on a crucial assumption for the dissipative term (2.34), namely that the correlation functions (2.35) are sharply peaked around $y \simeq x$, so that the approximation (2.38) holds. This assumption has often been tacitly adopted in many computations and is generally justified for local operators such as the stress–energy tensor. Nevertheless, when one is interested in observables that are intrinsically non-local in space–time, such as the Wigner function introduced later in this work, this assumption may fail and, in general, does not hold. In such cases, an expansion of the form (2.46) is no longer guaranteed to provide the most appropriate approximation scheme.

For this reason, we propose an alternative expansion scheme, which ultimately leads to an expansion in terms of the *initial gradients* (2.28) and remains valid independently of any assumption about the locality of the correlation functions.

We begin by reconsidering the non-equilibrium operator $\widehat{\rho}$ from its definition (2.23). The operator depends explicitly on the initial hypersurface Σ_0, whereas the observable $\widehat{O}(x)$ is evaluated on a different hypersurface lying in the causal future of Σ_0. We define the deviation of the thermodynamic fields from their initial values as in (2.39), where now y is a point on the initial hypersurface Σ_0. With this choice, the non-equilibrium operator (2.23) can be written as

$$\begin{aligned}\widehat{\rho} = \frac{1}{Z} \exp\Big[&- \beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q} \\ &+ \int_{\Sigma_0} {\rm d}\Sigma_\mu(y) \Big(\Delta\beta_\nu(y,x) \widehat{T}^{\mu\nu}(y) - \Delta\zeta(y,x) \widehat{j}^\mu(y) \Big) \Big]\,.\end{aligned} \tag{2.47}$$

This expression is formally analogous to (2.40). The crucial difference is that the integration is now performed over the initial hypersurface Σ_0, and the points y and x may in general be widely separated in space–time. Provided that $\Delta\beta$ and $\Delta\zeta$ remain sufficiently small, one can follow the same steps used in the derivation of the local-equilibrium correction within linear response theory and write the non-equilibrium expectation value as

$$\langle \widehat{O}(x) \rangle \simeq \langle \widehat{O}(x) \rangle_{\rm GE} + \Delta O(x)\,, \tag{2.48}$$

with $\Delta O(x)$ *formally* equivalent to (2.44), with the sole replacement of Σ by Σ_0:

$$\begin{aligned}\Delta O(x) = &-\int_0^1 \mathrm{d}z \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\, \Delta\beta_\nu(y,x) \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{GE}} \\ &+\int_0^1 \mathrm{d}z \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\, \Delta\zeta(y,x) \left\langle \widehat{O}(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}}\, \widehat{j}^{\mu}(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\mathrm{GE}} .\end{aligned} \tag{2.49}$$

Despite their formal similarity, the expansions (2.49) and (2.44) have a fundamentally different physical interpretation. This becomes particularly clear when comparing (2.46) with (2.48):

$$\Delta O(x) = \Delta O_{\mathrm{LE}}(x) + \Delta O_{\mathrm{diss}}(x)\,, \tag{2.50}$$

which shows that (2.49) simultaneously contains both the local-equilibrium and the dissipative deviations from the reference equilibrium state.

Finally, when applying this framework to the QGP produced in heavy-ion collisions, it is important to note that observables are typically computed at the decoupling hypersurface, where the relevant degrees of freedom are quasi-free hadronic fields. By contrast, on the initial equilibrium hypersurface the system is in the plasma phase, and the fundamental degrees of freedom are those of deconfined quarks and gluons. As a consequence, the expressions for the conserved densities on the two hypersurfaces are, in general, markedly different and encode the distinct interaction regimes characterizing the plasma and hadronic phases.

One can indeed recover (2.48) by applying Gauss' theorem to (2.49). However, since in (2.49) the integration is not performed over a four-dimensional volume but rather over a three-dimensional hypersurface, the point y can be far from x. Consequently, the assumption of localized correlations is not justified in general.

2.4 Pseudo-gauge transformations

We conclude this chapter with an important observation concerning the Zubarev operator (2.23), which has significant implications for spin-related phenomena.

The constraints (2.22) are not the most general ones. If the system undergoes rotation, both in orbital and internal (spin) space, then in principle the total angular momentum operator must also be constrained. This operator is defined as

$$\widehat{\mathcal{J}}^{\lambda,\mu\nu}(x) = x^\mu \widehat{T}^{\lambda\nu}(x) - x^\nu \widehat{T}^{\lambda\mu}(x) + \widehat{\mathcal{S}}^{\lambda,\mu\nu}(x), \qquad \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} = \widehat{T}^{\nu\mu} - \widehat{T}^{\mu\nu}, \tag{2.51}$$

where the presence of a non-vanishing spin tensor $\widehat{\mathcal{S}}$ leads to an additional constraint of the form

$$\hat{n}_\mu \langle \widehat{\mathcal{S}}^{\mu,\lambda\nu} \rangle_{\mathrm{LE}} = \hat{n}_\mu \langle \widehat{\mathcal{S}}^{\mu,\lambda\nu} \rangle, \tag{2.52}$$

with an associated Lagrange multiplier $\Omega_{\lambda\nu}$, commonly referred to as the *spin potential.*

The inclusion of a spin potential is a long-debated issue and is closely related to the fact that the stress–energy tensor and the spin tensor are not uniquely defined, but can be modified by a *pseudo-gauge transformation*:

$$\begin{aligned}\widehat{T}^{\mu\nu}_{\mathrm{ptg}}(x) &= \widehat{T}^{\mu\nu}(x) + \frac{1}{2}\partial_\lambda \left(\widehat{\Phi}^{\lambda,\mu\nu}(x) - \widehat{\Phi}^{\mu,\lambda\nu}(x) - \widehat{\Phi}^{\nu,\lambda\mu}(x) \right), \\ \widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\mathrm{ptg}}(x) &= \widehat{\mathcal{S}}^{\lambda,\mu\nu}(x) - \widehat{\Phi}^{\lambda,\mu\nu}(x) + \partial_\rho \widehat{Z}^{\mu\nu,\lambda\rho}(x),\end{aligned} \tag{2.53}$$

where $\widehat{\Phi}^{\lambda,\mu\nu} = -\widehat{\Phi}^{\lambda,\nu\mu}$ and $\widehat{Z}^{\mu\nu,\lambda\rho} = -\widehat{Z}^{\nu\mu,\lambda\rho} = -\widehat{Z}^{\mu\nu,\rho\lambda}$ are known as *superpotentials*. The transformed tensors $\widehat{T}_{\rm ptg}$ and $\widehat{\mathcal{S}}_{\rm ptg}$ define the same global, integrated conserved charges as the original tensors $\widehat{T}$ and $\widehat{\mathcal{S}}$. Consequently, for a system in global equilibrium, the corresponding density operator coincides with (2.17).

Out of equilibrium, however, different choices of pseudo-gauge generally lead to different non-equilibrium statistical operators, which take the form

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}_{\rm ptg}\beta_\nu - \widehat{j}^\mu\zeta + \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu,\lambda\nu}_{\rm ptg}\right)\right]. \tag{2.54}$$

Choosing the stress–energy and spin tensors derived from Noether's theorem,

$$\widehat{T}^{\mu\nu}_{\rm C} = \frac{\partial\widehat{\mathcal{L}}}{\partial(\partial_\mu\widehat{\phi}^A)}\partial^\nu\widehat{\phi}^A - g^{\mu\nu}\widehat{\mathcal{L}}, \tag{2.55a}$$

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\rm C} = \frac{\partial\widehat{\mathcal{L}}}{\partial(\partial_\lambda\widehat{\phi}^A)}\left(\Sigma^{\mu\nu}\right)_{AB}\widehat{\phi}^B, \tag{2.55b}$$

defines the so-called *canonical pseudo-gauge*. Starting from (2.54), one can recover the density operator (2.21) by performing the pseudo-gauge transformation

$$\widehat{\Phi}^{\lambda,\mu\nu}(x) = \widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\rm C}(x), \qquad \widehat{Z}^{\lambda\sigma,\mu\nu}(x) = 0, \tag{2.56}$$

for which the spin tensor vanishes and the stress–energy tensor becomes symmetric. This choice is known as the *Belinfante pseudo-gauge*, and the corresponding stress–energy tensor reads

$$\widehat{T}^{\mu\nu}_{\rm B}(x) = \widehat{T}^{\mu\nu}_{\rm C}(x) + \frac{1}{2}\partial_\lambda\left(\widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\rm C}(x) - \widehat{\mathcal{S}}^{\mu,\lambda\nu}_{\rm C}(x) - \widehat{\mathcal{S}}^{\nu,\lambda\mu}_{\rm C}(x)\right). \tag{2.57}$$

In the following, we will always work in the Belinfante pseudo-gauge. Accordingly, the local equilibrium state is described by (2.21), the stress–energy tensor is symmetric, and the spin tensor vanishes.

The dependence of the density operator—and therefore of all expectation values—on the choice of pseudo-gauge has received considerable attention in recent years. This issue is particularly relevant for spin polarization, since different pseudo-gauges can lead to different predictions for the polarization vector [138]. Moreover, the inclusion of a non-vanishing spin potential has been shown to play a central role in the formulation of spin hydrodynamics [109, 139–142].

A notable recent development is the construction of a density operator that is manifestly pseudo-gauge invariant [143], which turns out to coincide with the operator obtained using the Belinfante energy–momentum and spin tensors. This provides a further justification for adopting the Belinfante pseudo-gauge throughout this work.

Chapter 3

The Covariant Wigner function

In the previous chapter we introduced the non-equilibrium statistical operator, which provides a consistent definition of the quantum state of a relativistic system in local thermodynamic equilibrium. Under the assumption of a hydrodynamic regime, we then applied linear response theory and identified the local-equilibrium contributions together with the dissipative corrections to the observables of the system.

In the present chapter we turn to a fundamental tool that establishes a direct connection between the microscopic description of matter in terms of quantum fields and its macroscopic characterization within hydrodynamics: the Wigner function.

We begin by introducing the Wigner operator and the corresponding Wigner function for a complex scalar field. We will demonstrate that thermodynamic expectation values of observables can be expressed as integrals over four-momentum of the Wigner function, thereby allowing for its interpretation as the quantum analogue of the particle distribution function familiar from kinetic theory.

Subsequently, we extend the formalism to the case of spin-1/2 fermionic fields. On this basis, we will further employ the Wigner function to define the spin polarization vector, which constitutes a key observable for the phenomenology of relativistic heavy-ion collisions.

In particular we will present a first principle derivation of the Wigner function for interacting to include the effects of interactions, which is a fundamental step in order to treat dissipative effects.

3.1 The Wigner function: Scalar field

We begin by considering the simplest case: a free, non-interacting, complex scalar field. This provides the natural starting point for constructing the Wigner-function formalism, as all essential concepts—quantization, field decomposition, and the connection to kinetic theory—can be introduced in their cleanest form. The results obtained in this limit will later serve as a reference point for understanding how interactions modify the picture.

For a free complex scalar field, the dynamic is governed by the Lagrangian density:

$$\widehat{\mathcal{L}} = \partial_\mu \widehat{\phi}^\dagger \partial^\mu \widehat{\phi} - m^2 \widehat{\phi}^\dagger \widehat{\phi} \,, \tag{3.1}$$

which leads, via the Euler–Lagrange equations, to the free Klein–Gordon equation:

$$\left(\Box + m^2\right) \widehat{\phi}(x) = 0 \,, \tag{3.2}$$

where $\Box = \partial_\mu \partial^\mu$ denotes the d'Alembert operator. This equation describes relativistic, spin-0 particles of mass m, with the complex nature of $\widehat{\phi}$ ensuring that both particle and antiparticle degrees of freedom are present.

The general solution to Eq. (3.2) is a superposition of positive- and negative-frequency plane waves. Accordingly, the quantized field operator admits the standard momentum-space expansion:

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{2E_p} \left(\widehat{a}(p) e^{-\mathrm{i}p\cdot x} + \widehat{b}^\dagger(p) e^{\mathrm{i}p\cdot x} \right) , \tag{3.3}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$ is the on-shell energy.

Canonical quantization is implemented by imposing the equal-time commutation relations

$$\left[\widehat{\phi}(t, \mathbf{x}),\, \widehat{\pi}^\dagger(t, \mathbf{y}) \right] = \mathrm{i}\, \delta^{(3)}(\mathbf{x} - \mathbf{y}) , \tag{3.4}$$

where the conjugate momentum is $\widehat{\pi} = \partial_t \widehat{\phi}$. With these definitions, the Fourier coefficients $\widehat{a}(p)$ and $\widehat{b}^\dagger(p)$ act respectively as annihilation and creation operators for particles and antiparticles, while their adjoints $\widehat{a}^\dagger(p)$ and $\widehat{b}(p)$ perform the opposite roles. They satisfy the canonical commutation algebra:

$$[\widehat{a}(p), \widehat{a}^\dagger(p')] = [\widehat{b}(p), \widehat{b}^\dagger(p')] = 2E_p\, \delta^{(3)}(\mathbf{p} - \mathbf{p}') , \tag{3.5a}$$

$$[\widehat{a}(p), \widehat{b}(p')] = [\widehat{a}^\dagger(p), \widehat{b}(p')] = [\widehat{a}(p), \widehat{b}^\dagger(p')] = [\widehat{a}^\dagger(p), \widehat{b}^\dagger(p')] = 0 , \tag{3.5b}$$

$$[\widehat{a}(p), \widehat{a}(p')] = [\widehat{a}^\dagger(p), \widehat{a}^\dagger(p')] = [\widehat{b}(p), \widehat{b}(p')] = [\widehat{b}^\dagger(p), \widehat{b}^\dagger(p')] = 0 . \tag{3.5c}$$

From the Lagrangian density (3.1), Noether's theorem identifies the conserved currents associated with spacetime and internal symmetries. In particular, we obtain the canonical energy–momentum tensor and the conserved $U(1)$ current:

$$\widehat{T}^{\mu\nu} = \partial^\mu \widehat{\phi}^\dagger\, \partial^\nu \widehat{\phi} - g^{\mu\nu} \widehat{\mathcal{L}} , \tag{3.6a}$$

$$\widehat{j}^\mu = \mathrm{i} \left(\widehat{\phi}^\dagger \partial^\mu \widehat{\phi} - \widehat{\phi}\, \partial^\mu \widehat{\phi}^\dagger \right) . \tag{3.6b}$$

For a complex scalar field, the canonical stress–energy tensor is already symmetric, and the spin tensor vanishes identically due to the absence of intrinsic spin degrees of freedom. Consequently, the stress–energy tensor coincides with its Belinfante-improved form. Note that from the field expansion (3.34) it is immediate to prove:

$$\widehat{\phi}(x + y) = e^{\mathrm{i}\widehat{P}\cdot y}\, \widehat{\phi}(x)\, e^{-\mathrm{i}\widehat{P}\cdot y} , \tag{3.7a}$$

$$\widehat{\phi}(x) = \mathrm{e}^{-\mathrm{i}\varphi \widehat{Q}} \widehat{\phi}(x) \mathrm{e}^{\mathrm{i}\varphi \widehat{Q}} \tag{3.7b}$$

which in turns implies:

$$\mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\, \widehat{a}(p)\, \mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{\mathrm{i}p\cdot y}\, \widehat{a}(p) , \qquad e^{\mathrm{i}\widehat{P}\cdot y}\, \widehat{a}^\dagger(p)\, e^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{-\mathrm{i}p\cdot y}\, \widehat{a}^\dagger(p) , \tag{3.8a}$$

$$\mathrm{e}^{\mathrm{i}\widehat{P}\cdot y}\, \widehat{b}(p)\, \mathrm{e}^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{\mathrm{i}p\cdot y}\, \widehat{b}(p) , \qquad e^{\mathrm{i}\widehat{P}\cdot y}\, \widehat{b}^\dagger(p)\, e^{-\mathrm{i}\widehat{P}\cdot y} = \mathrm{e}^{-\mathrm{i}p\cdot y}\, \widehat{b}^\dagger(p) , \tag{3.8b}$$

and:

$$\mathrm{e}^{-\mathrm{i}\varphi \widehat{Q}} \widehat{a}(p) \mathrm{e}^{\mathrm{i}\varphi \widehat{Q}} = \mathrm{e}^{\mathrm{i}\varphi}\, \widehat{a}(p) , \tag{3.9a}$$

$$\mathrm{e}^{-\mathrm{i}\varphi \widehat{Q}}\, \widehat{b}^\dagger(p) \mathrm{e}^{\mathrm{i}\varphi \widehat{Q}} = \mathrm{e}^{\mathrm{i}\varphi} \widehat{b}^\dagger(p) , \tag{3.9b}$$

The Wigner operator provides the bridge between the microscopic quantum field description and the quasi-classical kinetic representation. For a complex scalar field, it is defined as

$$\widehat{W}(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4 s\, e^{-\mathrm{i}s\cdot k} : \widehat{\phi}^\dagger(x+\frac{s}{2})\,\widehat{\phi}(x-\frac{s}{2}) : , \tag{3.10}$$

where $:\bullet:$ denotes normal ordering. Although $\widehat{W}(x,k)$ depends on the single spacetime point x, it is intrinsically *non-local*: for each momentum k, it involves field operators evaluated at two distinct spacetime points, $x \pm s/2$. Large values of s correspond to long-range quantum correlations, implying that the Wigner operator can probe widely separated regions of spacetime. In the context of HIC, this non-locality is especially significant, since the field can reside in very different dynamical regimes depending on the spacetime point considered.

Inserting the plane-wave expansion (3.3) into Eq. (3.10) yields

$$\begin{aligned}\widehat{W}(x,k) = \frac{2}{(2\pi)^3}&\int \frac{\mathrm{d}^3 p}{2E_p}\int\frac{\mathrm{d}^3 p'}{2E_{p'}}\\ &\Bigg\{ e^{\mathrm{i}x\cdot(p-p')}\Big[\delta^4\Big(k-\tfrac{p+p'}{2}\Big)\widehat{a}^\dagger(p)\widehat{a}(p') + \delta^4\Big(k+\tfrac{p+p'}{2}\Big)\widehat{b}^\dagger(p)\widehat{b}(p')\Big]\\ &+\delta^4\Big(k-\tfrac{p-p'}{2}\Big)\Big[e^{\mathrm{i}x\cdot(p+p')}\widehat{a}^\dagger(p)\widehat{b}^\dagger(p') + e^{-\mathrm{i}x\cdot(p+p')}\widehat{a}(p)\widehat{b}(p')\Big]\Bigg\} .\end{aligned} \tag{3.11}$$

This expression shows that the Wigner operator naturally decomposes into three distinct contributions:

$$\widehat{W}(x,k) = \widehat{W}_+(x,k) + \widehat{W}_-(x,k) + \widehat{W}_S(x,k) , \tag{3.12}$$

with:

$$\widehat{W}_\pm(x,k) \equiv \widehat{W}(x,k)\,\theta(\pm k^0)\,\theta(k^2), \qquad \widehat{W}_S(x,k)\equiv \widehat{W}(x,k)\,\theta(-k^2) . \tag{3.13}$$

The terms $\widehat{W}_+$ and $\widehat{W}_-$ describe, respectively, the particle and antiparticle contributions associated with time-like momenta, whereas $\widehat{W}_S$ encodes purely quantum interference effects corresponding to spacelike four-momenta ($k^2 < 0$) [44].

The formula (3.11) can be worked out restoring the for-dimensional integral using the on-shell measure:

$$\int\frac{\mathrm{d}^3 p}{2E_p}\int\frac{\mathrm{d}^3 p'}{2E_{p'}} = \int \mathrm{d}^4 p\int \mathrm{d}^4 p'\delta\left(p^2-m^2\right)\delta\left(p'^{\,2}-m^2\right) .$$

Then, changing variables:

$$P \equiv \frac{p+p'}{2}, \quad q\equiv p-p' , \tag{3.14}$$

and using that the Jacobian is just 1, the double integral over the on-shell momenta turns out to be:

$$\begin{aligned}&\int \mathrm{d}^4 p\int \mathrm{d}^4 p'\delta\left(p^2-m^2\right)\delta\left(p'^{\,2}-m^2\right)\\ &=\int \mathrm{d}^4 P\int \mathrm{d}^4 q\,\delta\left(P^2+\frac{q^2}{4}-P\cdot q\right)\delta\left(P^2+\frac{q^2}{4}+P\cdot q\right) .\end{aligned}$$

Plugging the above into (3.11) we get that the particle term of the Wigner function is given by:

$$\widehat{W}^{+}(x,k) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4P\mathrm{d}^4q\,\delta\left(P^2+\frac{q^2}{4}-P\cdot q-m^2\right)\delta\left(P^2+\frac{q^2}{4}+P\cdot q-m^2\right) \times \mathrm{e}^{\mathrm{i}x\cdot q}\delta^4(k-P)\,\widehat{a}^{\dagger}(p)\widehat{a}(p')\,.$$

Integrating over d^4P sets $P=k$ and defining:

$$k_{\pm}\equiv k\pm\frac{q}{2}\,, \qquad k_{\pm}^2=m^2\,, \tag{3.15}$$

we obtain:

$$\widehat{W}^{+}(x,k) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}q\cdot x}\,\delta\left(k^2+\frac{q^2}{4}-P\cdot q-m^2\right) \times\delta\left(k^2+\frac{q^2}{4}+P\cdot q-m^2\right)\widehat{a}^{\dagger}(k_+)\widehat{a}(k_-)\,.$$

The above expression can be further worked out observing that the combinations of the two delta functions can be written as:

$$\delta\left(k^2+\frac{q^2}{4}-P\cdot q-m^2\right)\delta\left(k^2+\frac{q^2}{4}+P\cdot q-m^2\right) = \frac{1}{2}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\delta\left(k\cdot q\right)\,,$$

so that we finally get:

$$\widehat{W}^{+}(x,k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\mathrm{e}^{\mathrm{i}q\cdot x}\,\delta\left(k\cdot q\right)\delta\left(k^2+\frac{q^2}{4}-m^2\right)\widehat{a}^{\dagger}(k_+)\widehat{a}(k_-)\,. \tag{3.16}$$

For a given quantum state $\widehat{\rho}$ (pure or mixed), the *Wigner function* is defined as the expectation value of the Wigner operator:

$$W(x,k)\equiv \mathrm{Tr}\left(\widehat{\rho}\,\widehat{W}(x,k)\right)\,. \tag{3.17}$$

By construction, $W(x,k)$ is a real-valued function, but it is not positive definite. It therefore cannot be interpreted as a genuine probability distribution in phase space. Instead, it serves as a quasi-distribution function, encoding both classical statistical information and intrinsically quantum interference effects.

Applying the field equation (3.2), one finds that the *free* Wigner function satisfies the covariant evolution equation

$$\left[\frac{1}{4}\Box-(k^2-m^2)+\mathrm{i}\,k\cdot\partial\right]W(x,k)=0\,. \tag{3.18}$$

Separating this equation into its real and imaginary parts leads to the coupled constraints:

$$k\cdot\partial\,W(x,k)=0\,, \qquad \frac{1}{4}\Box W(x,k)=(k^2-m^2)\,W(x,k)\,. \tag{3.19}$$

The first equation expresses the *free streaming* of the Wigner function in phase space which solution is:

$$W^{+}(x,k) = W_0^{+}\left(x_0^0(k),\, \mathbf{x} - \frac{\mathbf{k}}{k^0}\left(x^0 - x_0^0\right),\, k\right) , \tag{3.20}$$

where x_0 is the intersection point of the characteristic line drawn from the point x and an initial hypersurface Σ_0 and W_0^{+} the Wigner function at Σ_0, while the second acts as a generalized mass-shell constraint. Quantum effects thus appear through the off-shell dependence of $W(x,k)$, which is not strictly confined to $k^2 = m^2$.

From Eq. (3.19), one sees that the four-vector $k^\mu W(x,k)$ is divergenceless, implying that its flux through any spacelike hypersurface Σ is conserved:

$$\int_\Sigma \mathrm{d}\Sigma_\mu(x)\, k^\mu\, W(x,k) , \tag{3.21}$$

and thus is independent on the choice of Σ, provided appropriate boundary conditions are satisfied. Choosing Σ as the flat hyperplane $x^0 = 0$, one finds that integrating over $\mathrm{d}^3\mathrm{x}$ projects the Wigner function onto on-shell momenta, recovering the familiar particle and antiparticle components. After performing the integrations, one obtains

$$\int_\Sigma \mathrm{d}\Sigma\cdot k\, W(x,k) = \delta(k^2 - m^2)\left[\theta(k^0)\,\langle \widehat{a}^\dagger(k)\widehat{a}(k)\rangle + \theta(-k^0)\,\langle \widehat{b}(k)\widehat{b}(k)\rangle\right] ,$$

showing explicitly that the physical contributions of the Wigner function lie on the mass shell. Note that this conlcusion holds given that $W(x,k)$ satisfies (3.18) which assumes free evolution for the fields.

Although $W(x,k)$ cannot be interpreted as a probability distribution, it provides a direct link between the microscopic field dynamics and macroscopic observables. In particular, the expectation values of conserved densities can be expressed as momentum moments of the Wigner function:

$$\langle : \widehat{T}^{\mu\nu}(x) : \rangle = \int \mathrm{d}^4k\, \left[k^\mu k^\nu - \tfrac{1}{4}\left(\partial^\mu\partial^\nu - g^{\mu\nu}\Box\right)\right] W(x,k) , \tag{3.22a}$$

$$\langle : \widehat{j}^\mu(x) : \rangle = \int \mathrm{d}^4k\, k^\mu\, W(x,k) . \tag{3.22b}$$

These relations formally establish the Wigner function as the central object mediating between the quantum field description and hydrodynamic or kinetic observables. Note that the above relations are also valid at operatorial level, i.e replacing the Wigner function with the Wigner operator (3.10).

Despite the fact that $W(x,k)$ is not a classical distribution function, Eqs. (3.22) suggest a natural correspondence between the Wigner function and the single-particle distribution functions used in kinetic theory. By inserting the free-field expansion of the Wigner operator (3.11) into Eq. (3.22) and retaining only the particle branch, we obtain

$$\langle : \widehat{j}_+(x) : \rangle = \int \frac{\mathrm{d}^3\mathrm{p}}{E(p)}\,\mathrm{Re}\left[\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathrm{p}'}{2E(p')}\, e^{\mathrm{i}(p-p')\cdot x}\langle \widehat{a}^\dagger(p)\widehat{a}(p')\rangle\right] .$$

The quantity inside the brackets defines a *complex phase-space distribution*, from which the usual (real) single-particle distribution function is obtained as

$$f_c(x,p) \equiv \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathrm{p}'}{2E(p')}\, e^{\mathrm{i}(p-p')\cdot x} \langle \widehat{a}^\dagger(p)\widehat{a}(p')\rangle \;, \qquad f(x,p) = \mathrm{Re}[f_c(x,p)] \;.$$

A similar definition applies to antiparticles, with a corresponding function $\overline{f}_c(x,p)$. Assuming that mixed terms vanish, Eq. (3.22b) then yields

$$\langle : \widehat{j}^\mu_+(x) : \rangle = \int \mathrm{d}^4\mathrm{k}\, k^\mu\, W(x,k) = \int \frac{\mathrm{d}^3\mathrm{p}}{2E(p)}\, p^\mu \big[f(x,p) + \overline{f}(x,p)\big], \tag{3.23}$$

thereby relating the covariant integral of the Wigner function to the usual phase-space expression of the current in kinetic theory [44].

The time component of $\widehat{j}^\mu$ represents the conserved charge density, and integrating it over a spacelike hypersurface gives the total number of particles,

$$\mathrm{N}_p = \int_\Sigma \mathrm{d}\Sigma_\mu(x)\, \widehat{j}^\mu_+(x) = \int \mathrm{d}^3\mathrm{k} \int \mathrm{d}^3\mathrm{x}\, f(x,k) \;. \tag{3.24}$$

Differentiating with respect to momentum yields the particle spectrum,

$$\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \int \mathrm{d}^3x\, f(x,k) = \frac{1}{2E_k} \langle \widehat{a}^\dagger(k)\widehat{a}(k)\rangle \;, \tag{3.25}$$

confirming that $f(x,k)$ represents the phase-space particle density.

Integrating Eq. (3.23) over a hypersurface Σ, we obtain the equivalent expression in terms of the Wigner function:

$$\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \int \mathrm{d}k^0 \int_\Sigma \mathrm{d}\Sigma_\mu\, k^\mu\, W^+(x,k) = \frac{1}{2E(k)} \langle \widehat{a}^\dagger(k)\widehat{a}(k)\rangle \;. \tag{3.26}$$

The expression above, plays a central role in the phenomenological description of particle production in relativistic heavy-ion collisions. It provides a direct link between the microscopic quantum description encoded in the Wigner function and the observable momentum spectrum of emitted particles and thus expresses the relation between the Wigner function and the Cooper-Frye formula (1.1). Due to the on-shellness of k, the hypersurface Σ is arbitrary and is often choose as the space–time boundary at which the system transitions from an interacting, hydrodynamically evolving medium to a free-streaming gas of asymptotic hadrons.

In the context of HICs, this hypersurface is most naturally identified with the **freeze-out** hypersurface Σ_{FO}. At this stage of the evolution, the hadronic medium becomes sufficiently dilute that further interactions among its constituents cease to be relevant. The measured hadron spectra in experiments are therefore interpreted as originating from this hypersurface.

However, one must carefully distinguish between the hadrons that are produced *on* the decoupling surface and those that are eventually *observed* in detectors. The actual observables correspond to **asymptotic free out-states**, obtained after the full time evolution of the system once all residual interactions have faded.

In the following section, we shall extend this discussion to the interacting-field case, showing how the inclusion of source terms and self-interactions modifies the definition and interpretation of the Wigner function, and how these effects influence the resulting particle spectrum.

3.1.1 The interacting case

We now turn to the case of *interacting fields*. While the assumption of free fields is sufficient to describe systems in or near local equilibrium, it fails to capture the physics of *dissipation*, which is inherently tied to interactions among the underlying degrees of freedom. Moreover, even the apparent equilibrium contributions—both local and global—should, in principle, depend on the presence of interactions, since the Wigner operator is intrinsically non-local. As we shall see, this non-locality ensures that interaction effects remain encoded in the field correlators even when the Wigner function is evaluated on the decoupling or freeze-out hypersurface.

For these reasons, it is necessary to extend the definition of the Wigner function to include the case of interacting quantum fields. This generalization modifies several properties that the Wigner function satisfies in the free-field limit and provides a consistent framework for describing both equilibrium and non-equilibrium (dissipative) dynamics in an interacting system.

A generic interacting scalar field satisfies a generalized Klein–Gordon equation of the form

$$\left(\Box + m^2\right) \widehat{\phi}(x) = \widehat{J}(x) \; , \tag{3.27}$$

where $\widehat{J}(x)$ denotes a generic interaction operator. The source term $\widehat{J}(x)$ depends on the field $\widehat{\phi}(x)$ itself as well as on its couplings to other quantum or classical fields.

The presence of interactions alters the structure of the Fock space associated with the field, and the Minkowski vacuum $|0\rangle$ is replaced by the interacting vacuum $|\Theta\rangle$, which satisfies *interacting-field* rather than free-field conditions. The notion of single-particle excitations, creation and annihilation operators, and their corresponding commutation relations must therefore be redefined.

Although the interacting field is no longer a solution of the free equation of motion, it can still be expanded in Fourier transform:

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \, e^{-\mathrm{i}p\cdot x} \widehat{\phi}_{\mathrm{F}}(p) \; ,$$

however p is not a on-shell momentum, $\mathrm{p}^2 \neq m^2$, due to the source term in (3.27). We then split the integration in two branches based on the sign of p^0, and the Fourier transform can be written as:

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \; \theta(p^0) \, \left(e^{-\mathrm{i}p\cdot x} \widehat{\phi}_{\mathrm{F}}(p) + e^{\mathrm{i}p\cdot x} \widehat{\phi}_{\mathrm{F}}(-p) \right) \; . \tag{3.28}$$

The variable p can be understood as a generalized of-shell momentum. The off-shell disperion relations can be expressed as:

$$\mathrm{p}^2 = (p^0)^2 - \mathbf{p}^2 \equiv M^2 \; , \tag{3.29}$$

with M^2 that can be either positive or negative. The Fourier expansion of the field can now be expressed in a form which makes evident the limit of non-interacting theory. First we change integration variable from p^0 to the generalized off-shell mass:

$$\int \mathrm{d}^3\mathrm{p} \int_{-\infty}^{+\infty} \mathrm{d}p^0 \; \theta(p^0) = \int \mathrm{d}^3\mathrm{p} \int_{-\mathbf{p}^2}^{+\infty} \frac{\mathrm{d}M^2}{2E(M, \mathbf{p})} \; ,$$

with $E(M, \mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}$ off-shell energy. Second we introduce the spectral function $\varrho(p)$:

$$\varrho(p) = \varrho\left(M^2, \mathbf{p}\right) \equiv \int \mathrm{d}^4x \, e^{\mathrm{i}p\cdot x} \, \mathrm{Tr}\left(\widehat{\rho}\left[\widehat{\phi}(x), \widehat{\phi}^\dagger(0)\right]\right) , \tag{3.30}$$

with $\widehat{\rho}$ state of the system, either pure or mixed. We can finally introduce the following *generalized* creation/annihialtion operators:

$$\widehat{A}(p) \equiv \frac{1}{(2\pi)^{3/2}\,\varrho(p)} \widehat{\phi}_{\mathrm{F}}(p) , \qquad \widehat{B}^\dagger(p) \equiv \frac{1}{(2\pi)^{3/2}\,\varrho(p)} \widehat{\phi}_{\mathrm{F}}(-p) , \tag{3.31}$$

so that the field can be formally written as:

$$\widehat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3\mathrm{p} \int_{-\mathbf{p}^2}^{+\infty} \frac{\mathrm{d}M^2}{2\sqrt{M^2 + \mathbf{p}^2}} \frac{\varrho(p)}{2\pi} \left(e^{-\mathrm{i}p\cdot x}\widehat{A}(p) + e^{\mathrm{i}p\cdot x}\widehat{B}^\dagger(p)\right) \tag{3.32}$$

The spectral function depends on the quantum state, namely the density operator $\widehat{\rho}$, so the extraction of the factor ϱ makes the formal expansion (3.32) apparently state-dependent. Yet, the field operator is not state-dependent because the products $\rho\widehat{A}$ and $\rho\widehat{B}$ are clearly state-independent. In particular, given that we will be interested in studying a system in a thermal state, a convenient normalization choice is to take the spectral function in (3.32) as the one computed in a state of global thermodynamic equilibrium (2.18).

Note that, if one plugs in the (3.32) the free field spectral function:

$$\varrho_{\mathrm{free}}(p) = 2\pi \, \mathrm{sign}(p^0)\delta(\mathrm{p}^2 - m^2) = 2\pi \, \mathrm{sign}(p^0)\delta(M^2 - m^2) , \tag{3.33}$$

the free field expansion in plane waves (3.3) is recovered with $\widehat{A}(p) = \widehat{a}(p)$ and $\widehat{B}(p) = \widehat{b}(p)$. Of course, it should be kept in mind that in the interacting case the operators $\widehat{A}(p)$ and $\widehat{B}(p)$ do not fulfill the commutation relations (3.5). Also, if $\widehat{\phi}$ represents the *pion field*, the generalized operators (3.31) do not commute with the field operators pertaining to the other particles, such as the Kaon field. However, due to the transformation's property of the field under translations (3.7) which are still valid in presence of interactions, it requires that also the generalized creation and annihilation operators must transform as:

$$e^{\mathrm{i}\widehat{P}\cdot y}\,\widehat{A}(p)\,e^{-\mathrm{i}\widehat{P}\cdot y} = e^{\mathrm{i}p\cdot y}\,\widehat{A}(p) , \qquad e^{\mathrm{i}\widehat{P}\cdot y}\,\widehat{A}^\dagger(p)\,e^{-\mathrm{i}\widehat{P}\cdot y} = e^{-\mathrm{i}p\cdot y}\,\widehat{A}^\dagger(p) , \tag{3.34a}$$

$$e^{\mathrm{i}\widehat{P}\cdot y}\,\widehat{B}(p)\,e^{-\mathrm{i}\widehat{P}\cdot y} = e^{\mathrm{i}p\cdot y}\,\widehat{B}(p) , \qquad e^{\mathrm{i}\widehat{P}\cdot y}\,\widehat{B}^\dagger(p)\,e^{-\mathrm{i}\widehat{P}\cdot y} = e^{-\mathrm{i}p\cdot y}\,\widehat{B}^\dagger(p) , \tag{3.34b}$$

and:

$$\mathrm{e}^{-\mathrm{i}\varphi\widehat{Q}}\widehat{A}(p)\mathrm{e}^{\mathrm{i}\varphi\widehat{Q}} = \mathrm{e}^{\mathrm{i}\varphi}\widehat{A}(p) , \tag{3.35a}$$

$$\mathrm{e}^{-\mathrm{i}\varphi\widehat{Q}}\widehat{B}^\dagger(p)\mathrm{e}^{\mathrm{i}\varphi\widehat{Q}} = \mathrm{e}^{\mathrm{i}\varphi}\widehat{B}^\dagger(p) , \tag{3.35b}$$

in analogy with the transformation relations (3.8) and (3.9).

The definition of the Wigner operator (3.10) remains valid also in the interacting case. Substituting the expansion (3.32), we can express it in terms of the generalized creation and annihilation operators as:

$$\begin{aligned}\widehat{W}(x,k) = &\frac{2}{(2\pi)^5}\int \mathrm{d}^3\mathrm{p}\,\mathrm{d}^3\mathrm{p}' \int_{-\mathbf{p}^2}^{+\infty}\frac{\mathrm{d}M^2}{2E(p)}\int_{-\mathbf{p}'^2}^{+\infty}\frac{\mathrm{d}M'^{\,2}}{2E(p')}\varrho(p)\varrho(p') \\ &\times\left\{e^{\mathrm{i}x\cdot(p-p')}\delta^4\left(k-\frac{p+p'}{2}\right)\widehat{A}^\dagger(p)\widehat{A}(p')\right. \\ &+e^{-\mathrm{i}x\cdot(p-p')}\delta^4\left(k+\frac{p+p'}{2}\right)\widehat{B}^\dagger(p)\widehat{B}(p') \\ &\left.+\,\delta^4\left(k-\frac{p-p'}{2}\right)\left[e^{\mathrm{i}x\cdot(p+p')}\widehat{A}^\dagger(p)\widehat{B}^\dagger(p')+e^{-\mathrm{i}x\cdot(p+p')}\widehat{B}(p')\widehat{A}(p)\right]\right\}.\end{aligned} \tag{3.36}$$

The above expression is then formally equivalent, despite the presence of the mixed terms, to the one obtained in the free-case and thus represents the most general expansion for the Wigner operator of an interacting scalar field theory. Note that taking the free limit of the spectral functions (3.33) the (3.36) reduces to (3.11).

The Wigner operator can be formally separated in particle, anti-particle and mixed term as in the free case (3.12) where the different branches are selected based on the sign of k^2 and k^0 (3.13). However, selecting the *particle* branch:

$$\widehat{W}^+(x,k) = \theta(\mathrm{k}^2)\theta(k^0)\widehat{W}(x,k) \ ,$$

due to the off-shellnesses of $p,\ p'$ in the interacting fields then the mixed contributions $\widehat{A}^\dagger\widehat{B}^\dagger$ and $\widehat{B}\widehat{A}$ are not discarded:

$$\begin{aligned}\widehat{W}^+(x,k) = &\frac{2}{(2\pi)^5}\int \mathrm{d}^3\mathrm{p}\,\mathrm{d}^3\mathrm{p}' \int_{-\mathbf{p}^2}^{+\infty}\frac{\mathrm{d}M^2}{2E(p)}\int_{-\mathbf{p}'^2}^{+\infty}\frac{\mathrm{d}M'^{\,2}}{2E(p')}\theta(p^0)\theta(p'^{\,0}) \\ &\times\varrho(p)\varrho(p')\left\{e^{\mathrm{i}x\cdot(p-p')}\left[\delta^4\left(k-\frac{p+p'}{2}\right)\widehat{A}^\dagger(p)\widehat{A}(p')\right]\right. \\ &\left.+\,\delta^4\left(k-\frac{p-p'}{2}\right)\left[e^{\mathrm{i}x\cdot(p+p')}\widehat{A}^\dagger(p)\widehat{B}^\dagger(p')+e^{-\mathrm{i}x\cdot(p+p')}\widehat{B}(p')\widehat{A}(p)\right]\right\}.\end{aligned}$$

This expression can be written in a more convenient form. First, we use the general identity

$$\int \mathrm{d}^3\mathrm{p}\int_{-\mathbf{p}^2}^{+\infty}\frac{\mathrm{d}M^2}{2\sqrt{M^2+\mathbf{p}^2}} = \int \mathrm{d}^4p\int_{-\mathbf{p}^2}^{+\infty}\mathrm{d}M^2\,\delta\big(\mathrm{p}^2-M^2\big) \ .$$

Then, introducing the definitions of Eq. (3.15) and setting $p\equiv P_+$ and $p'\equiv P_-$, the measure over the particle branch becomes

$$\begin{aligned}\frac{\mathrm{d}^3\mathrm{p}}{2E(M,\mathbf{p})}\frac{\mathrm{d}^3\mathrm{p}'}{2E(M',\mathbf{p}')} = &\,\mathrm{d}^4P\,\mathrm{d}^4q\,\delta\left(P^2+\frac{q^2}{4}+P\cdot q-M^2\right) \\ &\times\delta\left(P^2+\frac{q^2}{4}-P\cdot q-M'^2\right) \ .\end{aligned}$$

Then the particle branch turns out to be:

$$\widehat{W}^+(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4P\,\mathrm{d}^4q \int_{-\mathbf{p}^2}^{+\infty}\mathrm{d}M^2 \int_{-\mathbf{p}'^2}^{+\infty}\mathrm{d}M'^{\,2}\varrho(p)\varrho(p')\delta\left(P^2+\frac{q^2}{4}-M^2\right)$$
$$\times\,\theta(P_+^0)\theta(P_-^0)\delta\left(P^2+\frac{q^2}{4}-P\cdot q-M'^{\,2}\right)\left\{e^{\mathrm{i}x\cdot q}\left[\delta^4(k-P)\,\widehat{A}^\dagger(p)\widehat{A}(p')\right]\right.$$
$$\left.+\,\delta^4(k-q)\left[e^{\mathrm{i}2x\cdot P}\widehat{A}^\dagger(p)\widehat{B}^\dagger(p')+e^{-\mathrm{i}2x\cdot P}\widehat{B}(p')\widehat{A}(p)\right]\right\}.$$

The $\delta^4(k-P)$ fixes $P=k$ in the $\widehat{A}^\dagger\widehat{A}$ channel allows to perform the integration over d^4P setting $P=k$, while in the mixed channel the $\delta^4(k-q)$ allows to integrate over d^4q fixing $q=k$:

$$\widehat{W}^+(x,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{p}^2}^{+\infty}\mathrm{d}M^2 \int_{-\mathbf{p}'^{\,2}}^{+\infty}\mathrm{d}M'^{\,2}\varrho(p)\varrho(p')\widehat{A}^\dagger(p)\widehat{A}(p')e^{\mathrm{i}x\cdot q}$$
$$\times\,\delta\left(\mathrm{k}^2+\frac{\mathrm{q}^2}{4}+k\cdot q-M^2\right)\delta\left(\mathrm{k}^2+\frac{\mathrm{q}^2}{4}-k\cdot q-M'^2\right)\theta(p)\theta(p')$$
$$+\,\frac{2}{(2\pi)^5}\int \mathrm{d}^4P \int_{-\mathbf{p}^2}^{+\infty}\mathrm{d}M^2 \int_{-\mathbf{p}'^{\,2}}^{+\infty}\mathrm{d}M'^{\,2}\varrho(p)\varrho(p')\theta(p)\theta(p')$$
$$\times\,\delta\left(\mathrm{k}^2+\frac{P^2}{4}+k\cdot P-M^2\right)\delta\left(\mathrm{k}^2+\frac{P^2}{4}-k\cdot P-M'^2\right)$$
$$\times\left[\widehat{A}^\dagger(p)\widehat{B}^\dagger(p')e^{\mathrm{i}2x\cdot P}+\widehat{B}(p')\widehat{A}(p)e^{-\mathrm{i}2x\cdot P}\right].$$

In the first integral, being $k=P=(p+p')/2$ we define:

$$k_+ \equiv k+\frac{q}{2}\,,\qquad k_- \equiv k-\frac{q}{2}\,, \tag{3.37}$$

while in the second, changing variable to $P\equiv 2q$, we define:

$$q_+ = q+k\,,\quad q_- = q-k\,, \tag{3.38}$$

so that we can write the particle branch of the Wigner function as:

$$\widehat{W}^+(x,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{k}_+^2}^{+\infty}\mathrm{d}M^2 \int_{-\mathbf{k}_-^2}^{+\infty}\mathrm{d}M'^{\,2}\varrho(k_+)\varrho(k_-)\widehat{A}^\dagger(k_+)\widehat{A}(k_-)e^{\mathrm{i}x\cdot q}$$
$$\times\,\delta\left(\mathrm{k}^2+\frac{\mathrm{q}^2}{4}+k\cdot q-M^2\right)\delta\left(\mathrm{k}^2+\frac{\mathrm{q}^2}{4}-k\cdot q-M'^2\right)\theta(k_+^0)\theta(k_-^0)$$
$$+\,\frac{2^5}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{q}_+^2}^{+\infty}\mathrm{d}M^2 \int_{-\mathbf{q}_-^2}^{+\infty}\mathrm{d}M'^{\,2}\varrho(q_+)\varrho(q_-)\theta(q_+^0)\theta(q_-^0)$$
$$\times\,\delta\left(\mathrm{k}^2+\frac{q^2}{4}+k\cdot q-M^2\right)\delta\left(\mathrm{k}^2+\frac{q^2}{4}-k\cdot q-M'^2\right)$$
$$\times\left[\widehat{A}^\dagger(q_+)\widehat{B}^\dagger(q_-)e^{\mathrm{i}x\cdot q}+\widehat{B}(q_+)\widehat{A}(q_-)e^{-\mathrm{i}x\cdot q}\right].$$

Note that the integration limits of $\mathrm{d}M^2$ and $\mathrm{d}M'^2$ depend on q through $\mathbf{k}_\pm$ and $\mathbf{q}_\pm$, therefore the integrations over the generalized masses must be performed first. It is convenient to rename the integration variable in the second integral $q \equiv 2P$ and obtain:

$$\begin{aligned}
\widehat{W}^+(x,k) &= \frac{2}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{k}_+^2}^{+\infty}\mathrm{d}M^2\int_{-\mathbf{k}_-^2}^{+\infty}\mathrm{d}M'^2\Big\{\varrho(k_+)\varrho(k_-)\,\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\,\mathrm{e}^{\mathrm{i}x\cdot q}\\
&\times\delta\left(k^2+\frac{q^2}{4}+k\cdot q-M^2\right)\delta\left(k^2+\frac{q^2}{4}-k\cdot q-M'^2\right)\theta(k_+^0)\theta(k_-^0)\Big\}\\
&+\frac{2}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{k}_+^2}^{+\infty}\mathrm{d}M^2\int_{-\mathbf{k}_-^2}^{+\infty}\mathrm{d}M'^2\Big\{\varrho(k_+)\varrho(-k_-)\,\left[\mathrm{e}^{\mathrm{i}x\cdot q}\widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-)+\mathrm{h.c.}\right]\\
&\times\delta\left(k^2+\frac{q^2}{4}+k\cdot q-M^2\right)\delta\left(k^2+\frac{q^2}{4}-k\cdot q-M'^2\right)\theta(k_+^0)\theta(-k_-^0)\Big\}\,.
\end{aligned}$$

Finally, we can use the (3.15) to rewrite the arguments of the delta distributions and integrate in M^2, M'^2:

$$\begin{aligned}
\widehat{W}^+(x,k) &= \frac{2}{(2\pi)^5}\int \mathrm{d}^4q \int_{-\mathbf{k}_+^2}^{+\infty}\mathrm{d}M^2\int_{-\mathbf{k}_-^2}^{+\infty}\mathrm{d}M'^2\delta\left(k_+^2-M^2\right)\delta(k_-^2-M'^2)\\
&\times\Bigg\{\varrho(k_+)\varrho(k_-)\,\mathrm{e}^{\mathrm{i}x\cdot q}\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\theta(k_+^0)\theta(k_-^0)+\varrho(k_+)\varrho(-k_-)\\
&\times\left[\mathrm{e}^{\mathrm{i}x\cdot q}\widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-)+\mathrm{h.c.}\right]\theta(k_+^0)\theta(-k_-^0)\Bigg\}\,,
\end{aligned}$$

which is finally equal to:

$$\begin{aligned}
\widehat{W}^+(x,k) = \frac{2}{(2\pi)^5}\int \mathrm{d}^4q\ \mathrm{e}^{\mathrm{i}x\cdot q}\Big\{&\varrho(k_+)\varrho(k_-)\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\theta(k_+^0)\theta(k_-^0)\\
&+\varrho(k_+)\varrho(-k_-)\left[\widehat{A}^\dagger(k_+)\widehat{B}^\dagger(-k_-)+\mathrm{h.c.}\right]\theta(k_+^0)\theta(-k_-^0)\Big\}\,.
\end{aligned}\tag{3.39}$$

Note again that, assuming free fields, implies that the spectral function reduces to (3.33) so that the integrals over M and M' collapses ad the momenta p, p' satisfy the on-shell condition. Then $\theta(k^0)\theta(-k^0)$ is not satisfied and (3.39) reduces to the free expressions (3.16).

Regardless of the presence of interactions, the stress-energy tensor and the conserved current can still be expressed in terms of the Wigner function through Eqs. (3.22). However the relation (3.26) must be revisit if one takes into account the interactions. The interacting Wigner operator does not satisfy the relation (3.18) thus the operator $k\cdot\partial\widehat{W}$ is not conserved and its integral over a space-like hypersurface now depends on it.

It is still possible to relate the Wigner function with the spectra of the produced hadrons. What actually is observed in the detector are the asymptotic free states then a proper generalization of (3.26) is:

$$\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \lim_{t\to+\infty}\int \mathrm{d}k^0\int_{\Sigma(t)}\mathrm{d}\Sigma_\mu k^\mu\, W^+(x,k) = \frac{1}{2E_k}\langle\widehat{a}^\dagger_{\mathrm{out}}(k)\widehat{a}_{\mathrm{out}}(k)\rangle\,,\tag{3.40}$$

where $\widehat{a}_{\mathrm{out}}$ and $\widehat{a}^\dagger_{\mathrm{out}}$ denote the annihilation and creation operators for the asymptotic out-states. These operators encode the long-time behavior of the quantum field and determine the physical particle content measured far from the interaction region. By definition, the out-states correspond to free-streaming fields, for which the interaction term in the equation of motion (3.27) vanishes. The relation between a generic interacting field and an asymptotic out-field can be established through the **Yang–Feldman equation** [144]:

$$\widehat{\phi}(x) = \widehat{\phi}_{\rm out}(x) - \int d^4y\, \Delta_{\rm adv}(x-y)\, \widehat{J}(y) \;, \tag{3.41}$$

where $\Delta_{\rm adv}$ denotes the advanced propagator of the full interacting theory. In the asymptotic future one recovers the known LSZ relation:

$$\lim_{t\to\infty} \widehat{\phi}(x) = \widehat{\phi}_{\rm out}(x) \;. \tag{3.42}$$

The main issue concerning the equation (3.40) is that we are practically unable to reckon all scattering processes necessary to calculate the limit for $t \to +\infty$. What one can more easily do is to determine the Wigner function at the decoupling stage, i.e. when the fluid ceases to exist but not all interaction processes have ended. So, if $\Sigma_{\rm D}$ is the decoupling hypersurface and using the Gauss theorem, one can recast the (3.40) as:

$$\frac{{\rm dN}_p}{{\rm d^3k}} = \int {\rm d}k^0 \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu k^\mu \, W^+(x,k) + \int {\rm d}k^0 \int_{\Upsilon} {\rm d}^4x \; k^\mu \partial_\mu W^+(x,k) \tag{3.43}$$

where Υ is the space-time region encompassed by the hypersurfaces Σ_D and $\Sigma(t \to +\infty)$ (see Fig. 3.1).

The first term on the right hand side supposedly provides the dominant part of the spectrum while the second term is a correction induced by the scattering processes in the dilute, post-decoupling phase; it can be estimated by using relativistic kinetic theory.

Indeed we can prove that the neglection of the Υ term in (3.43) is equivalent to neglect all the hadron gas interactions and thus to approximate the decoupling as an instantaneous transition from a coupled phase to a free-streaming evolution

Inverting the Yang-Feldman relation (3.41) the out-field as the interacting field plus a convolution of the propagator with the source term $\widehat{J}$. To directly connect the two, one can employ the Klein–Gordon inner product:

$$\left(\widehat{\phi}_1, \widehat{\phi}_2\right)_{\rm KG} \equiv -{\rm i}\int_\Sigma {\rm d}\Sigma_\mu(x) \left[\widehat{\phi}_1^\dagger(x)\, \partial^\mu \widehat{\phi}_2(x) - (\partial^\mu \widehat{\phi}_1^\dagger(x))\, \widehat{\phi}_2(x)\right], \tag{3.44}$$

which, for $\widehat{\phi}_{1,2}$ satisfying Eq. (3.27), is independent of the specific choice of the hypersurface Σ [145].

The out-field represents the asymptotic, freely evolving field and can be expanded according to Eq. (3.3) as

$$\widehat{\phi}(x) = \int {\rm d^3p} \left[f_p(x)\, e^{-{\rm i}p\cdot x}\, \widehat{a}_{\rm out}(p) + f_p^*(x)\, e^{{\rm i}p\cdot x}\, \widehat{b}^\dagger_{\rm out}(p)\right], \qquad f_p(x) \equiv \frac{e^{-{\rm i}p\cdot x}}{(2\pi)^{3/2}\, 2E_p} \;.$$

The corresponding annihilation operator can be expressed in terms of the interacting field through Eq. (3.41), by projecting with the Klein–Gordon inner product (3.44):

$$\widehat{a}_{\rm out}(p) = \left(f_p, \widehat{\phi}\right)_{\rm KG} + \left(f_p, \Delta_{\rm adv} * \widehat{J}\right)_{\rm KG} \;,$$

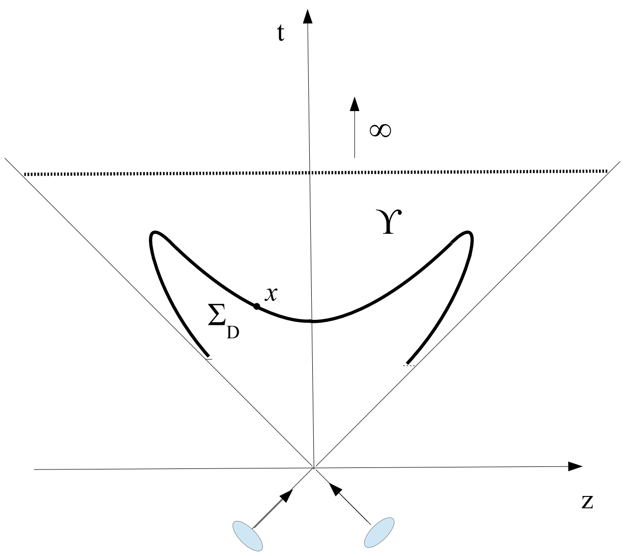


Figure 3.1: A typical shape of a decoupling hypersurface Σ_D in a relativistic nuclear collision in a space-time diagram. The fluid decouples at Σ_D and the produced particles interact through collisions in the region Υ until all interaction cease and the spectra freeze out. The particles are eventually observed in the asymptotic future $t \to \infty$.

where the symbol $*$ denotes the convolution product. The projection of the free mode onto the convolution term can be expressed through the kernel

$$K(y) \equiv \int_\Sigma \mathrm{d}\Sigma_\mu(x) \left[f_p(x)\, \partial^\mu \Delta_{\mathrm{adv}}(x-y) - \Delta_{\mathrm{adv}}(x-y)\, \partial^\mu f_p^*(x) \right],$$

so that the out-state particle number takes the form

$$\begin{aligned} \left\langle \widehat{a}^\dagger_{\mathrm{out}}(p)\, \widehat{a}_{\mathrm{out}}(p) \right\rangle = & \left\langle \big(f_p, \widehat{\phi}\big)^\dagger_{\mathrm{KG}} \big(f_p, \widehat{\phi}\big)_{\mathrm{KG}} \right\rangle \\ & + \int \mathrm{d}^4y\, \mathrm{d}^4y'\, K(y)\, K(y') \left\langle \widehat{J}^\dagger(y)\, \widehat{J}(y') \right\rangle \\ & + \int \mathrm{d}^4y\, K(y) \left\langle \big(f_p, \widehat{\phi}\big)^\dagger_{\mathrm{KG}} \widehat{J}(y) \right\rangle + \text{c.c.}\ . \end{aligned} \tag{3.45}$$

When this formalism is applied to the specific geometry of a heavy-ion collision and to the spectrum of hadrons produced at decoupling, one finds that the standard treatment based on free fields, as dictated by Eq. (3.26), effectively neglects three distinct sources of possible corrections.

The first correction originates from the fact that, at decoupling, the field $\widehat{\phi}$ still carries information about the interactions within the plasma phase. Such contributions are typically ignored when computing local equilibrium corrections, since those depend only on the field values at the decoupling hypersurface. However, because the Wigner function is intrinsically non-local, any attempt to include dissipative corrections necessarily involves field correlations extending into regions where interactions are not only present but dominant.

The second correction arises from the source term $\widehat{J}\widehat{J}$, which encodes processes of particle production or annihilation occurring in the hadron gas phase. This contribution is usually negligible, as the production of heavy hadrons is strongly suppressed.

The third correction represents a mixing between the interacting field and the source term. It corresponds to scattering and collision processes that result from the residual non-free evolution within the hadron gas.

For simplicity—and consistently with high-energy collisions, where the decoupling and freeze-out hypersurfaces nearly coincide—we assume a *sharp freeze-out*. In this picture, spacetime naturally divides into two regions: the interacting QCD plasma prior to $\Sigma_{\rm D}$, and the free-streaming region beyond it. When the Wigner function is evaluated at a point $x \in \Sigma_{\rm D}$, its bi-local structure implies that one of its endpoints may probe the plasma phase inside the past light cone of x, where interactions remain non-vanishing and typically dominant. Conversely, endpoints within the future light cone of x lie in the free region, where interactions vanish identically.

Under this approximation, the source term $\widehat{J}$ is nonzero only in the causal past of decoupling—that is, within the plasma phase. Consequently,

$$\Delta_{\rm adv}(x-y) = 0 \qquad \text{for } (x-y)^0 < 0,$$

so that $\Delta_{\rm adv}(x-y)$ has support only when $y^0 \geq x^0$, i.e. when the source point y lies in the causal *future* of x. As a result, if x belongs to the decoupling hypersurface $\Sigma_{\rm D}$ and the interaction current $\widehat{J}(y)$ has support only within the QCD plasma at earlier times $y^0 < x^0$, the convolution term in Eq. (3.41) vanishes:

$$\int d^4y\, \Delta_{\rm adv}(x-y)\, \widehat{J}(y) = 0 \qquad \text{for } x \in \Sigma_{\rm D}.$$

Hence, on the decoupling hypersurface we obtain the exact identity

$$\widehat{\phi}(x) = \widehat{\phi}_{\rm out}(x), \qquad x \in \Sigma_{\rm D}\ .$$

This effectively corresponds to retaining only the first term in Eq. (3.41) and neglecting all interaction effects within the hadron gas phase. Thus, the projection onto asymptotic out-states—and consequently, the particle spectrum—can be computed directly from the interacting field evaluated on $\Sigma_{\rm D}$, without any residual interaction.

It is important to note, however, that assuming the field to be free immediately after freeze-out and employing the free-field expansion of the Wigner function are *not* equivalent statements. This discrepancy arises from the non-local nature of the Wigner operator. Even if, for $x+s/2$ lying in the future light cone of $\Sigma_{\rm D}$, the field is assumed to be free, the Wigner function also depends on the field evaluated at $x-s/2$, which—if s is sufficiently large—may lie deep within the plasma phase. Hence, adopting the free-field expansion for the Wigner operator amounts to neglecting all plasma-phase interactions, which, in the context of dissipative corrections, are not only present but dominant.

Before passing to the spectrum modifications we derive the form of the expectation values of combinations of $\widehat{A}(p)$ and $\widehat{B}(p)$ operators at global thermodynamic equilibrium, with a density operator (2.18) justifying our choice for the spectral function ϱ in the expansion (3.39) with the spectral function computed on the equilibrium state with four-temperature β:

$$
\begin{aligned}
\langle\widehat{A}^\dagger(p)\widehat{A}(p')\rangle_{\rm GE} &= \frac{1}{(2\pi)^3}\frac{1}{\varrho(p)\varrho(p')}\langle\widehat{\phi}^\dagger_F(p)\widehat{\phi}_F(p')\rangle_{\rm GE}\\
&= \frac{1}{(2\pi)^3}\frac{1}{\varrho(p)\varrho(p')}\int {\rm d}^4x\,{\rm d}^4x'{\rm e}^{-{\rm i}p\cdot x}{\rm e}^{{\rm i}p'\cdot x'}\langle\widehat{\phi}^\dagger(x)\widehat{\phi}(x')\rangle_{\rm GE}\\
&= \frac{1}{(2\pi)^3}\frac{1}{\varrho(p)\varrho(p')}\int {\rm d}^4x\,{\rm d}^4x'{\rm e}^{-{\rm i}p\cdot x}{\rm e}^{{\rm i}p'\cdot x'}\langle\widehat{\phi}^\dagger(0)\widehat{\phi}(x'-x)\rangle_{\rm GE}\\
&= \frac{1}{(2\pi)^3}\frac{1}{\varrho(p)\varrho(p')}\int {\rm d}^4x\,{\rm d}^4y\;{\rm e}^{-{\rm i}(p-p')\cdot x}{\rm e}^{{\rm i}p'\cdot y}\langle\widehat{\phi}^\dagger(0)\widehat{\phi}(y)\rangle_{\rm GE}\\
&= \frac{2\pi}{\varrho(p)\varrho(p')}\delta^4(p-p')\int {\rm d}^4y\;{\rm e}^{{\rm i}p'\cdot y}\langle\widehat{\phi}^\dagger(0)\widehat{\phi}(y)\rangle_{\rm GE}\;,
\end{aligned}
$$

where we have taken advantage of translational invariance of the density operator and changed the integration variable from $x'-x=y$. Now, by using the definition of the lesser Wightman function at global equilibrium:

$$
\mathcal{G}^<_{\rm GE}(q) \equiv \int {\rm d}^4y\;{\rm e}^{{\rm i}q\cdot y}\langle\widehat{\phi}^\dagger(0)\widehat{\phi}(y)\rangle_{\rm GE}\;,
$$

and its known relation with the spectral function [146]:

$$
\mathcal{G}^<_{\rm GE}(q) = \frac{1}{{\rm e}^{\beta\cdot q-\zeta}-1}\,\varrho_{\rm GE}(q) = n_{\rm B}(q)\,\varrho_{\rm GE}(q)\;,
$$

we get the equation:

$$
\begin{aligned}
\langle\widehat{A}^\dagger(p)\widehat{A}(p')\rangle_{\rm GE} &= \frac{2\pi}{\varrho^2(p)}\theta(p^0)\theta(p'^0)\delta^4(p-p')n_B(p)\varrho_{\rm GE}(p)\\
&= (\text{if } \varrho(p)=\varrho_{\rm GE}(p))\;\;\frac{2\pi}{\varrho(p)}\theta(p^0)\theta(p'^0)2p^0\delta(p^2-p'^2)\delta^3(\mathbf{p}-\mathbf{p}')n_B(p)\;.
\end{aligned}
\tag{3.46}
$$

Thus, in order to obtain the last simple form of the expectation values in (3.46), the spectral function in the field expansion (3.32) must be the one calculated with the density operator (2.18); only in this case a cancellation between the spectral function in the numerator and denominator occurs.

Similarly, we can obtain:

$$
\langle\widehat{A}^\dagger(p)\widehat{B}^\dagger(p')\rangle_{\rm GE} = \frac{2\pi}{\varrho(p')}\theta(p^0)\theta(p'^0)\delta^4(p+p')n_B(p) = 0\;, \tag{3.47}
$$

whence, from the hermiticity of the density operator:

$$
\langle\widehat{B}(p)\widehat{A}(p')\rangle_{\rm GE} = \langle\widehat{A}^\dagger(p')\widehat{B}^\dagger(p)\rangle^*_{\rm GE} = 0\;. \tag{3.48}
$$

3.2 The Wigner function: Dirac field

We now turn to the case of relativistic fields with spin 1/2, namely the Dirac field. We will only considers the free case, for which the results obtained for the complex scalar field can be naturally extended to this case.

The Lagrangian density for a free spinor field ψ is given by:

$$\widehat{\mathcal{L}} = \overline{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi\ . \tag{3.49}$$

The corresponding equations of motion read are the well known Dirac equation and can be split in two independent equations for the spinor ψ and its Dirac adjoint $\overline{\psi}$:

$$(\mathrm{i}\overrightarrow{\partial\!\!\!/} - m)\psi(x) = 0\,, \qquad \overline{\psi}(x)(\mathrm{i}\overleftarrow{\partial\!\!\!/} - m) = 0\ , \tag{3.50}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$ and the arrows indicate the direction of action of the derivative. The adjoint spinor is defined as $\overline{\psi} \equiv \psi^\dagger\gamma^0$ where γ^μ, with $\mu = 0, 1, 2, 3, 4$, are the Dirac gamma matrices (see appendix A for a summary of their properties).

The general solution can then be expanded in plane waves as

$$\psi^A(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3p}{2E_p}\sum_{r=\pm}\left[\widehat{a}_r(p)\,u_r^A(p)\,e^{-\mathrm{i}p\cdot x} + \widehat{b}_r^\dagger(p)\,v_r^A(p)\,e^{+\mathrm{i}p\cdot x}\right]\ , \tag{3.51}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$. In the expansion of the field ψ, $u_r(p)$ and $v_r(p)$ are the spinor basis functions corresponding to particles and antiparticles, respectively. The index $r = \pm$ labels the spin (or helicity) projection. The operators $\widehat{a}_r(p)$ and $\widehat{b}_r(p)$ annihilate a particle or antiparticle with momentum p and spin r, and satisfy the canonical anticommutation relations:

$$\{\widehat{a}_r(p), \widehat{a}_s^\dagger(p')\} = \{\widehat{b}_r(p), \widehat{b}_s^\dagger(p')\} = 2E_p\,\delta^3(\mathbf{p} - \mathbf{p}')\,\delta_{rs}\ , \tag{3.52a}$$

$$\{\widehat{a}_r(p), \widehat{b}_s(p')\} = \{\widehat{a}_r(p), \widehat{b}_s^\dagger(p')\} = \{\widehat{a}_r^\dagger(p), \widehat{b}_s(p')\} = \{\widehat{a}_r^\dagger(p), \widehat{b}_s^\dagger(p')\} = 0\ , \tag{3.52b}$$

with all remaining anticommutators vanishing.

The spinors $u_r(p)$ and $v_r(p)$ obey the Dirac equations:

$$(p\!\!\!/ - m)u_r(p) = 0\ , \qquad \overline{u}_r(p)(p\!\!\!/ - m) = 0\ , \tag{3.53a}$$

$$(p\!\!\!/ + m)v_r(p) = 0, \qquad \overline{v}_r(p)(p\!\!\!/ + m) = 0\ , \tag{3.53b}$$

and are normalized according to

$$\overline{u}_r(p)u_s(p) = -\overline{v}_r(p)v_s(p) = 2m\,\delta_{rs}\ . \tag{3.54}$$

They form a complete orthonormal basis in the space of Dirac spinors, satisfying

$$\sum_{r=\pm} u_r(p)\overline{u}_r(p) = (p\!\!\!/ + m)\ , \qquad \sum_{r=\pm} v_r(p)\overline{v}_r(p) = (p\!\!\!/ - m)\ . \tag{3.55}$$

These relations ensure that $u_r(p)$ and $v_r(p)$ describe positive- and negative-energy solutions of the free Dirac equation, respectively, thus completing the canonical quantization of the free spin-1/2 field.

From the Lagrangian density (3.49), and by applying Noether's theorem, one obtains the expressions for the canonical stress-energy tensor and the conserved $U(1)$ current associated with the global phase symmetry of the Dirac field:

$$\begin{aligned}\widehat{T}^{\mu\nu}_{\mathrm{C}}(x) &= \frac{\mathrm{i}}{2}\overline{\psi}(x)\gamma^\mu\overleftrightarrow{\partial\!\!\!/}^{\nu}\psi(x)\ ,\\ \widehat{j}^\mu(x) &= \overline{\psi}(x)\gamma^\mu\psi(x)\ .\end{aligned} \tag{3.56}$$

In addition to these quantities, a spin-1/2 field carries an intrinsic angular momentum contribution. This gives rise to a non-vanishing spin tensor $\widehat{\mathcal{S}}$, which can be directly computed from the Lagrangian (3.49) and the Noether procedure, and reads [129]:

$$\widehat{\mathcal{S}}_{\mathrm{C}}^{\lambda,\mu\nu}(x) = \frac{\mathrm{i}}{2}\overline{\psi}(x)\left\{\gamma^{\lambda}, \Sigma^{\mu\nu}\right\}\psi(x)\,, \tag{3.57}$$

where $\Sigma^{\mu\nu} = (\mathrm{i}/4)\left[\gamma^{\mu}, \gamma^{\nu}\right]$ are the generators of Lorentz transformations in the spinor representation. The presence of this spin tensor implies that the canonical stress-energy tensor $\widehat{T}_{\mathrm{C}}^{\mu\nu}$ is in general not symmetric under the exchange $\mu \leftrightarrow \nu$.

To obtain a symmetric stress-energy tensor—necessary for a consistent coupling with gravity and often more convenient for hydrodynamic applications—one performs a pseudo-gauge transformation (2.56). In the specific case of a Dirac spinor field, the corresponding Belinfante stress-energy tensor coincides with the symmetrized version of the canonical one:

$$\widehat{T}_{\mathrm{B}}^{\mu\nu} = \frac{1}{2}\left(\widehat{T}_{\mathrm{C}}^{\mu\nu} + \widehat{T}_{\mathrm{C}}^{\nu\mu}\right) = \frac{\mathrm{i}}{4}\overline{\psi}(x)\gamma^{\mu}\overset{\leftrightarrow}{\partial}{}^{\nu}\psi(x) + (\mu \leftrightarrow \nu)\,, \tag{3.58}$$

while, in this pseudo-gauge, the spin tensor is identically set to zero.

The definition of the Wigner operator for a spin-1/2 Dirac field follows the same logic as in the scalar case (3.10), but now carries spinorial indices. It is defined as:

$$\widehat{W}_{B}^{A}(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4 s\, e^{-\mathrm{i}s\cdot k} : \overline{\psi}_B\left(x+\frac{s}{2}\right)\psi^A\left(x-\frac{s}{2}\right) :\,, \tag{3.59}$$

which is a 4×4 complex matrix in the spinor indices A, B. It is possible to obtain a compact expression for the Wigner operator using the free field expansion (3.51) in analogy with (3.16)

$$\begin{aligned}\widehat{W}^{+}(x,k) = &\frac{1}{2(2\pi)^3}\sum_{r,r'}\int \mathrm{d}^4 q\, \mathrm{e}^{\mathrm{i}q\cdot x}\,\delta(k\cdot q)\,\delta\left(k^2 + \frac{q^2}{4} - m^2\right)\\ &\times \overline{u}_r(k_+)u_{r'}(k_-)\widehat{a}_r^{\dagger}(k_+)\widehat{a}_{r'}(k_-)\,.\end{aligned} \tag{3.60}$$

The associated Wigner function is obtained, as in the scalar case, as the expectation value of this operator in a given quantum state $\widehat{\rho}$:

$$W_{B}^{A}(x,k) = \mathrm{Tr}\left(\widehat{\rho}\,\widehat{W}_{B}^{A}(x,k)\right). \tag{3.61}$$

Thus, $W(x,k)$ itself is a 4×4 matrix in spinor space. A natural way to handle this structure is to expand it in the basis of the 16 generators of the Clifford algebra, yielding the decomposition [147]:

$$\begin{aligned}W(x,k) &= \sum_i W_i\Gamma_i\\ &= \frac{1}{4}\left[\mathcal{S} + \mathrm{i}\gamma^5\mathcal{P} + \gamma^{\sigma}\mathcal{V}_{\sigma} + \gamma^5\gamma^{\sigma}\mathcal{A}_{\sigma} + \Sigma^{\sigma\lambda}\mathcal{T}_{\sigma\lambda}\right]\,,\end{aligned} \tag{3.62}$$

where $\Gamma_i = \{I, \gamma^{\mu}, \gamma^5, \gamma^5\gamma^{\mu}, \Sigma^{\mu\nu}\}$. This decomposition is central to the quantum-kinetic description of relativistic fermions, since it allows one to identify separately the vector and axial-vector components that encode, respectively, conserved currents and spin polarization effects [61].

Each coefficient in (3.62) corresponds to a different irreducible component of the Wigner function under Lorentz transformations. Of particular interest for most applications are the scalar, vector, and axial-vector components, which are extracted as:

$$\text{Scalar}: \ \mathcal{S}(x,k) = \mathrm{tr}\left(W(x,k)\right) \ , \tag{3.63a}$$

$$\text{Vector}: \ \mathcal{V}^\mu(x,k) = \mathrm{tr}\left(\gamma^\mu W(x,k)\right) \ , \tag{3.63b}$$

$$\text{Axial}: \mathcal{A}^\mu(x,k) = \mathrm{tr}\left(\gamma^\mu\gamma^5 W(x,k)\right) \ . \tag{3.63c}$$

From the equation of motion for the field (3.50) the Wigner function for a Dirac field satisfies:

$$\left(m - \frac{\mathrm{i}}{2}\not{\partial} - \not{k}\right) W(x,k) = 0 \ . \tag{3.64}$$

The above equation can again be split in a real and imaginary part implying that the operator $k^\mu W(x,k)$ has vanishing divergence and its integral over an arbitrary space-like hypersurface is independent from the choice of it.

Analogously to the scalar field case, the macroscopic conserved quantities can be expressed directly in terms of these components of the Wigner function. In particular, the expectation values of the symmetrized Belinfante stress-energy tensor and of the conserved current read:

$$\langle : \widehat{T}^{\mu\nu}_{\mathrm{B}}(x) : \rangle = \int \mathrm{d}^4k \left[k^\nu \mathcal{V}^\mu(x,k) + k^\mu \mathcal{V}^\nu(x,k)\right] \ , \tag{3.65a}$$

$$\langle : \widehat{j}^\mu(x) : \rangle = \int \mathrm{d}^4k \mathcal{V}^\mu(x,k) \ . \tag{3.65b}$$

As for the scalar field case the above relations are also valid at operatorial level. Furthermore, the particle spectrum can also be written in terms of the scalar component of the Wigner function as:

$$E_k \frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \frac{E_k}{m}\int \mathrm{d}k^0 \int_\Sigma \mathrm{d}\Sigma \cdot k \, \mathcal{S}(x,k) \ . \tag{3.66}$$

Thus, in analogy with the scalar case, all relevant macroscopic observables—including conserved currents, energy-momentum densities, and particle spectra—can be expressed as integrals of the Wigner function over the off-shell four-momentum k. The on-shell character of physical observables is recovered once the Wigner function is integrated over a generic space-like hypersurface, ensuring consistency with relativistic quantum field theory.

3.3 Spin polarization for fermions

The spin degrees of freedom of a fermionic system are encoded in the *spin density operator* $\widehat{\Theta}(k)$, which compactly describes the statistical distribution of spin states for particles with momentum k. The associated *spin density matrix* is defined as

$$\Theta_{rs}(k) = \langle k, r|\widehat{\Theta}(k)|k, s\rangle \ , \tag{3.67}$$

where r, s label eigenvalues of the spin operator $\widehat{S}^3(k)$ along a chosen quantization axis. The matrix $\Theta_{rs}(k)$ thus encodes both probabilistic weights and quantum coherences between spin states at fixed momentum k.

For a spin-1/2 fermion, $\Theta(k)$ is a 2×2 Hermitian, positive semidefinite matrix, which can be uniquely decomposed in terms of the identity and Pauli matrices, corresponding to the irreducible representations of the $SU(2)$ algebra generated by the spin operators in the particle rest frame:

$$\Theta(k) = \frac{1}{2}\mathrm{I} + \frac{1}{2}\sum_{i=1}^{3} S^i_{\text{rest}}(k)\,\sigma^i \;, \tag{3.68}$$

where σ^i are the Pauli matrices and $S^i_{\text{rest}}(k)$ denotes the mean spin vector in the local rest frame. The polarization of the system is fully encoded in $\mathbf{S}_{\text{rest}}(k)$, while the unpolarized part corresponds to the identity component.

In quantum field theory, $\Theta_{rs}(k)$ is determined by creation and annihilation operators of fermionic modes with definite momentum and spin. Its definition is therefore meaningful only within a quasiparticle regime, where a single-particle Hilbert space exists. In strongly coupled systems, such as the non-perturbative regime of QCD, this single-particle picture ceases to apply.

The rest-frame polarization vector follows as

$$S^i_{\text{rest}}(k) = \mathrm{Tr}\left(\sigma^i \Theta(k)\right) \;, \tag{3.69}$$

while a covariant definition of the mean spin four-vector is given by

$$S^\mu(k) \equiv \mathrm{Tr}\left(\widehat{S}^\mu(k)\widehat{\Theta}(k)\right) \;, \tag{3.70}$$

where $\widehat{S}^\mu(k)$ is the relativistic spin operator associated with momentum k.

To express the spin vector in an arbitrary frame, one introduces the *standard Lorentz transformation* $[k]^\mu{}_\nu$ satisfying

$$[k]^\mu{}_\nu \, k_0^\nu = k^\mu \;, \quad |k, r\rangle = \widehat{[k]}\,|k_0, r\rangle \;, \tag{3.71}$$

with $k_0^\mu = (m, \mathbf{0})$ the rest-frame momentum. The choice of $[k]$ (canonical or helicity boosts) fixes the physical interpretation of the spin label s, since the spin generators depend on this convention. Thus, unlike momentum, spin is frame- and convention-dependent.

If a different standard Lorentz transformation $[k']$ is adopted, momentum eigenstates transform as

$$|k', r\rangle = \sum_s D^S\left([k]^{-1}[k']\right)_{sr} |k, s\rangle \;,$$

where the composite transformation $[k]^{-1}[k']$ leaves k_0 invariant and hence acts as a spatial rotation R, i.e. $[k]^{-1}[k'] \equiv R$. Consequently, the creation and annihilation operators depend on this choice:

$$\widehat{a}^\dagger_r(\mathbf{k}') = \sum_s D^S(R)_{sr}\widehat{a}^\dagger_s(\mathbf{k}) \;,$$

and the spin density matrix transforms accordingly:

$$\Theta_{ru}(k') = \sum_{r,s} D^S\left(R^{-1}\right)_{tr} \Theta_{rs}(k) D^S(R)_{tu} \;. \tag{3.72}$$

Despite this explicit dependence on $[k]$, the covariant spin vector $S^\mu(k)$ defined in (3.70) remains invariant under changes of the standard Lorentz transformation [56], ensuring a convention-independent characterization of spin polarization.

We now relate the spin density matrix (3.67) to the covariant Wigner function (3.59). As follows from (3.64), the integral of $k^\mu \widehat{W}$ over a space-like hypersurface Σ is independent of Σ provided k is on-shell:

$$\int_\Sigma \mathrm{d}\Sigma_\mu k^\mu \widehat{W}(x,k) = \frac{1}{2}\sum_{r,s} \delta\left(k^2 - m^2\right) \Big[\theta(k^0)\widehat{a}_s^\dagger(k)\widehat{a}_r(k)u_r(k)\overline{u}_s(k) \\ -\theta(-k^0)\widehat{b}_r^\dagger(-k)\widehat{b}_s(-k)v_r(-k)\overline{v}_s(-k)\Big] , \tag{3.73}$$

where mixed particle–antiparticle terms vanish due to the factor $k^0\delta(k^0)$.

This motivates the definition of the on-shell operators

$$\frac{1}{2E_k}\widehat{w}_\pm(k)\delta\left(k^0 \mp E_k\right) \equiv \int_\Sigma \mathrm{d}\Sigma_\mu k^\mu \, \widehat{W}_\pm(x,k) , \tag{3.74}$$

so that, by comparison with (3.73),

$$\widehat{w}_+(k) = \frac{1}{2}\sum_{r,s} \widehat{a}_r^\dagger(k)\widehat{a}_s(k)u_r(k)\overline{u}_s(k) , \tag{3.75}$$

with k constrained to the mass shell.

Using the completeness relations for free Dirac spinors and contracting with $\overline{u}_r(k)$ and $u_s(k)$ yields

$$\overline{u}_r(k)\widehat{w}_+(k)u_s(k) = 2m^2\widehat{a}_r^\dagger(k)\widehat{a}_s(k) , \tag{3.76}$$

establishing a direct link between $\widehat{w}_+(k)$ and the bilinear combinations of creation and annihilation operators defining the spin density matrix. Summing over spin indices and using (3.63), one recovers (3.66) [129].

Taking the expectation value of (3.76) gives

$$\Theta_{rs}(k) = \frac{\overline{u}_r(k)w_+(k)u_s(k)}{\sum_t \overline{u}_t(k)w_+(k)\,u_t(k)} ,$$

where $w_+ = \mathrm{Tr}(\widehat{\rho}\,\widehat{w}_+)$. Since k lies on-shell, this can be rewritten as

$$\Theta_{rs}(k) = \frac{\int_\Sigma \mathrm{d}\Sigma \cdot k\; \overline{u}_t(k)W_+(x,k)\,u_t(k)}{\sum_t \int_\Sigma \mathrm{d}\Sigma \cdot k\; \overline{u}_t(k)W_+(x,k)\,u_t(k)} , \tag{3.77}$$

explicitly connecting $\Theta_{rs}(k)$ to the covariant Wigner function.

A key result, following from group-theoretical properties of the Lorentz group's spinorial representation, is that the mean spin four-vector can be expressed as [129]:

$$S^\mu(k) = \frac{1}{2}\frac{\int_\Sigma \mathrm{d}\Sigma \cdot k \;\mathrm{tr}\left[\gamma^\mu\gamma^5 W(x,k)\right]}{\int_\Sigma \mathrm{d}\Sigma \cdot k\;\mathrm{tr}\left[W(x,k)\right]} , \tag{3.78}$$

or equivalently,

$$S^\mu(k) = \frac{1}{2}\frac{\int_\Sigma \mathrm{d}\Sigma \cdot k\; \mathcal{A}^\mu(x,k)}{\int_\Sigma \mathrm{d}\Sigma \cdot k\; \mathcal{S}(x,k)} , \tag{3.79}$$

where $\mathcal{A}^\mu$ and $\mathcal{S}$ are the axial and scalar components of the Wigner function defined in (3.63). Equation (3.79) provides the basis for the local-equilibrium polarization formula widely employed in phenomenological applications to relativistic heavy-ion collisions. Its interpretation and limitations in situations departing from global equilibrium, as well as the role of gradient corrections, have been analyzed in detail in the literature [129, 148].

Finally, we note that the polarization vector (3.79) derived above is formally independent of pseudo-gauge transformations. The derivation relies solely on the Wigner operator and the Lorentz structure of the spin density matrix, without invoking specific forms of the spin or energy–momentum tensors. However, when computing $S^\mu(k)$ in local thermodynamic equilibrium, the Wigner function is evaluated using the pseudo-gauge–dependent non-equilibrium statistical operator (2.23). Consequently, the spin polarization vector Consequently, the spin polarization vector inherits this dependence, and different pseudo-gauge choices yield distinct corrections to $S^\mu(k)$ [138, 149].

Chapter 4

Wigner function at Local equilibrium

In this chapter we compute the deviation of the local equilibrium state from the global homogeneous equilibrium for the Wigner function. We introduce a novel expansion method that allows for the systematic computation of this correction for an arbitrary geometry of the decoupling hypersurface. The same method can also be extended to the equilibrium hypersurface and therefore provides a natural starting point for the calculation of the full non-equilibrium correction to the Wigner function which will be explicitly computed in the next chapter.

We first carry out the computation for a scalar field and subsequently extend the analysis to the case of a Dirac field, where we will determine the local equilibrium correction to the spin polarization vector.

The key result of this chapter is the construction of a new expansion scheme that yields a gradient expansion of the local equilibrium correction to the Wigner function. Remarkably, this expansion allows one to perform the integration over the decoupling hypersurface exactly at each order, without making any geometrical assumptions about its shape.

This chapter is mainly based on the results presented in Ref. [69].

4.1 Scalar field

We begin by computing the deviation of the local equilibrium Wigner function from its global equilibrium value.

Within linear response theory, the local equilibrium expectation value of the Wigner function is obtained from (2.44) by replacing $\widehat{O}$ with the Wigner operator $\widehat{W}$, yielding

$$\langle \widehat{W}\,(x,k)\rangle_{\rm LE} \simeq \langle \widehat{W}\,(x,k)\rangle_{\rm GE} + \Delta W_{\rm LE}\,(x,k)\ . \tag{4.1}$$

The leading contribution is the global equilibrium expectation value of the Wigner function:

$$\langle W^{+}\,(x,k)\rangle_{\rm GE} = \frac{2\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\rm B}(k,x)\ , \tag{4.2}$$

which has been obtained from (3.16) using that:

$$\left\langle \widehat{a}^{\dagger}(k_+)\widehat{a}(k_-)\right\rangle_{\rm GE} = 2E(k_+)\, n_{\rm B}(k_+,x)\delta^3\left(\mathbf{k}_+ - \mathbf{k}_-\right)\ , \tag{4.3}$$

with $n_{\rm B}$ denoting the *Bose–Einstein* distribution:

$$n_{\rm B}\left(k,x\right)=\frac{1}{{\rm e}^{\beta(x)\cdot k-\zeta(x)}-1}\;. \tag{4.4}$$

We now turn to the computation of the local equilibrium correction $\Delta W_{\rm LE}$. Substituting (3.16) into (2.44) one obtains

$$\begin{aligned}\Delta W^{+}_{\rm LE}\left(x,k\right)=&-\frac{1}{\left(2\pi\right)^3}\int_{\Sigma_{\rm D}}{\rm d}\Sigma_\mu(y)\int{\rm d}^4q\left\{{\rm e}^{{\rm i}x\cdot q}\delta\left(k\cdot q\right)\delta\left(k^2+\frac{q^2}{4}-m^2\right)\right.\\&\times\theta(k^0_+)\theta(k^0_-)\int_0^1{\rm d}z\left[\Delta\beta_\nu\left(y,x\right)\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\widehat{T}^{\mu\nu}(y){\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\right.\\&\left.\left.-\Delta\zeta\left(y,x\right)\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\widehat{j}^{\mu}(y){\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\right]\right\}.\end{aligned} \tag{4.5}$$

The thermal expectation values can be simplified by exploiting (3.7) and:

$$\widehat{T}^{\mu\nu}(y)={\rm e}^{{\rm i}\widehat{P}\cdot y}\,\widehat{T}^{\mu\nu}(0)\,{\rm e}^{-{\rm i}\widehat{P}\cdot y}\;,\qquad\widehat{j}^{\mu}(y)={\rm e}^{{\rm i}\widehat{P}\cdot y}\,\widehat{j}^{\mu}(0)\,{\rm e}^{-{\rm i}\widehat{P}\cdot y}\,. \tag{4.6}$$

Combining these with (3.8) and (3.9), together with the vaninishing of the commutator $[\widehat{P}^\mu,\widehat{Q}]$ one finds:

$${\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}-{\rm i}\widehat{P}\cdot y}\,\widehat{a}^\dagger(k_+)\widehat{a}(k_-)\,{\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}+{\rm i}\widehat{P}\cdot y}={\rm e}^{{\rm i}q\cdot y}{\rm e}^{-z\beta(x)\cdot q}\,\widehat{a}^\dagger(k_+)\widehat{a}(k_-)\;. \tag{4.7}$$

As a result, the integration over z can be performed explicitly, yielding

$$\begin{aligned}\Delta W^{+}_{\rm LE}\left(x,k\right)=&\frac{1}{\left(2\pi\right)^3}\int_{\Sigma_{\rm D}}{\rm d}\Sigma_\mu(y)\int{\rm d}^4q\left\{{\rm e}^{{\rm i}q\cdot(x-y)}\theta(k^0_+)\theta(k^0_-)\delta\left(k\cdot q\right)\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\right.\\&\times\delta\left(k^2+\frac{q^2}{4}-m^2\right)\left[\Delta\beta_\nu\left(y,x\right)\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;\widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}}\right.\\&\left.\left.-\Delta\zeta\left(y,x\right)\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;\widehat{j}^{\mu}(0)\right\rangle_{c,{\rm GE}}\right]\right\}.\end{aligned} \tag{4.8}$$

At this stage, the entire spacetime dependence of the local equilibrium correction enters through the Fourier transform, with respect to q, of the thermodynamic deviations $\Delta\beta$ and $\Delta\zeta$. The remaining factors are purely thermostatic expectation values involving the conserved densities $\widehat{T}$ and $\widehat{j}$. We therefore define:

$$\Gamma^{\mu\nu}\left(k,q,\beta\right)\equiv\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;\widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}}\;, \tag{4.9a}$$

$$\Upsilon^{\mu}\left(k,q,\beta\right)\equiv\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\;\widehat{j}^{\mu}(0)\right\rangle_{c,{\rm GE}}\;. \tag{4.9b}$$

Their explicit form depends on the definition of the stress-energy tensor and conserved current. Nevertheless, their general tensorial structure is constrained by Lorentz covariance and by the symmetries of the equilibrium density operator (2.17). In the free theory both quantities can be computed exactly.

Introducing the auxiliary functions

$$G^{\mu\nu}\left(k,q,\beta\right)\equiv\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k^0_+)\theta(k^0_-)\,\Gamma^{\mu\nu}\left(k,q,\beta\right)\;, \tag{4.10a}$$

$$H^{\mu}\left(k,q,\beta\right)\equiv\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k^0_+)\theta(k^0_-)\,\Upsilon^{\mu}\left(k,q,\beta\right)\;, \tag{4.10b}$$

and

$$F_{\mu\nu}^{(\beta)}(x,q) \equiv \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\, {\rm e}^{{\rm i}q\cdot(x-y)}\, \Delta\beta_\nu(y,x) \ , \tag{4.11a}$$

$$F_{\mu}^{(\zeta)}(x,q) \equiv \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\, {\rm e}^{{\rm i}q\cdot(x-y)}\, \Delta\zeta(y,x) \ , \tag{4.11b}$$

the local equilibrium correction assumes the compact form

$$\Delta W_{\rm LE}^{+}(x,k) = \frac{1}{(2\pi)^3}\int {\rm d}^4q\,\delta(k\cdot q)\Big[G^{\mu\nu}(k,q,\beta)\,F_{\mu\nu}^{(\beta)}(x,q) - H^{\mu}(k,q,\beta)\,F_{\mu}^{(\zeta)}(x,q)\Big] \ . \tag{4.12}$$

This expression provides the natural starting point for the hydrodynamic (gradient) expansion of the local equilibrium Wigner function.

4.1.1 Hydrodynamic expansion

The structure of (4.12) makes explicit that the local equilibrium correction arises from the convolution of two qualitatively different ingredients. The functions defined in (4.10) are purely thermostatic quantities, determined by the equilibrium state and, eventually, by interactions. They are generally smooth functions of q. In contrast, the functions defined in (4.11) are Fourier transforms of slowly varying thermodynamic fields.

In the hydrodynamic regime, thermodynamic fields vary weakly over macroscopic length scales. Consequently, their Fourier transforms are sharply peaked around $q=0$. This implies that the dominant contribution to (4.12) originates from the behavior of $G^{\mu\nu}$ and H^μ in the vicinity of $q=0$. One can therefore perform a systematic expansion of the integrand in powers of q, leading to

$$\begin{aligned}\Delta W_{\rm LE}^{+}(x,k) = \frac{1}{(2\pi)^3}\sum_{N=0}^{+\infty}\frac{1}{N!}\int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y) I_N^{\nu_1\nu_2\dots\nu_N}(y-x) \\ \times\Big\{\Delta\beta_\nu(y,x)\Big[\partial^q_{\nu_1}\partial^q_{\nu_2}\dots\partial^q_{\nu_N}G^{\mu\nu}(k,q,\beta)\Big]_{q=0} \\ -\Delta\zeta(y,x)\Big[\partial^q_{\nu_1}\partial^q_{\nu_2}\dots\partial^q_{\nu_N}H^{\mu}(k,q,\beta)\Big]_{q=0}\Big\} \ ,\end{aligned} \tag{4.13}$$

where the q integrals are encoded in the rank-N tensor

$$I_N^{\nu_1\nu_2\dots\nu_N}(y-x) \equiv \int {\rm d}^4q\,\delta(k\cdot q)\ {\rm e}^{{\rm i}q\cdot(x-y)}\, q^{\nu_1}q^{\nu_2}\dots q^{\nu_N} \ . \tag{4.14}$$

This integral can be evaluated exactly for arbitrary N. Each factor of q^{ν_i} can be rewritten as a derivative acting on the exponential, yielding

$$\int {\rm d}^4q\,\delta(k\cdot q)\ {\rm e}^{{\rm i}q\cdot(x-y)}\, q^{\nu_1}q^{\nu_2}\dots q^{\nu_N} = (-{\rm i})^N\,\partial_x^{\nu_1}\partial_x^{\nu_2}\dots\partial_x^{\nu_N}\int {\rm d}^4q\,\delta(k\cdot q)\ {\rm e}^{{\rm i}q\cdot(y-x)} \ .$$

Evaluating the δ-function constraint:

$$\delta(k\cdot q) = \delta\left(k^0q^0 - \mathbf{k}\cdot\mathbf{q}\right) = \frac{1}{|k^0|}\delta\left(q^0 - \frac{\mathbf{k}\cdot\mathbf{q}}{k^0}\right) \ ,$$

and performing the q^0 integration leads to

$$\begin{aligned}\int \mathrm{d}^4q\,\delta\left(k\cdot q\right)\,\mathrm{e}^{\mathrm{i}q\cdot(y-x)} &= \frac{1}{|k^0|}\int \mathrm{d}^3q\,\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{k}\left(x^0-y^0\right)/k^0-\mathrm{i}\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{(2\pi)^3}{|k^0|}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\,.\end{aligned}$$

Thus one finds

$$I_N^{\nu_1\nu_2\ldots\nu_N}\left(y-x\right) = \frac{(2\pi)^3}{|k^0|}\left[\partial_x^{\nu_1}\partial_x^{\nu_2}\,\ldots\,\partial_x^{\nu_N}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\right]\,. \tag{4.15}$$

Substituting this result into (4.13) yields

$$\begin{aligned}\Delta W_{\mathrm{LE}}^{+}\left(x,k\right) = \frac{1}{|k^0|}\sum_{N=0}^{+\infty}\frac{(-\mathrm{i})^N}{N!}\int_{\Sigma_\mathrm{D}}\mathrm{d}\Sigma_\mu(y) \\ \times\left[\partial_x^{\nu_1}\partial_x^{\nu_2}\,\ldots\,\partial_x^{\nu_N}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\right] \\ \times\Big\{\Delta\beta_\nu\left(y,x\right)\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\ldots\partial^q_{\nu_N}G^{\mu\nu}\left(k,q,\beta\right)\right]_{q=0} \\ -\Delta\zeta\left(y,x\right)\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\ldots\partial^q_{\nu_N}H^{\mu}\left(k,q,\beta\right)\right]_{q=0}\Big\}\,.\end{aligned} \tag{4.16}$$

The presence of spacetime derivatives acting on the delta function complicates the integration over y. These derivatives can be systematically transferred onto the thermodynamic fields by repeated application of the Leibniz rule, resulting in

$$\begin{aligned}\Delta W_{\mathrm{LE}}^{+}\left(x,k\right) = \frac{1}{|k^0|}\sum_{N=0}^{+\infty}\frac{(-\mathrm{i})^N}{N!}\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\ldots\partial^q_{\nu_N}G^{\mu\nu}\left(k,q,\beta\right)\right]_{q=0} \\ \times\sum_{M=0}^{N}\frac{N!\,(-1)^M}{M!\,(N-M)!}\partial_x^{\nu_{M+1}}\ldots\partial_x^{\nu_N}\Big\{\int_{\Sigma_\mathrm{D}}\mathrm{d}\Sigma_\mu(y) \\ \times\left[\partial_x^{\nu_1}\partial_x^{\nu_2}\ldots\partial_x^{\nu_M}\Delta\beta_\nu\left(y,x\right)\right]\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\Big\} \\ -\ \text{analogous term in}\,\Delta\zeta\,.\end{aligned} \tag{4.17}$$

The decoupling hypersurface Σ_D may, in general, possess a complicated geometry. In heavy-ion collisions it is expected to contain both spacelike and timelike segments, with large curvatures in peripheral regions and an approximately hyperbolic structure near mid-rapidity. For a given spatial point $\mathbf{x}$ on Σ_D, multiple values of x^0 may exist. Nevertheless, the hypersurface can always be decomposed into single-valued branches of the form $x^0 = f_j(\mathbf{x})$, allowing the integration in (4.17) to be written as a sum over these branches. For an arbitrary function $\Xi(y,x)$ then one finds:

$$\begin{aligned}&\int_{\Sigma_\mathrm{D}}\mathrm{d}\Sigma_\mu(y)\,\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\Xi\left(y,x\right) \\ &=\sum_j\mathfrak{s}_j\int_{\Sigma_j}\mathrm{d}^3y\,\sigma_\mu^{(j)}(y)\,\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\Xi\left(y,x\right) \\ &=\sum_j\mathfrak{s}_j\sum_i\,\sigma_\mu^{(j)}(\bar{y}_k^i(x))\,\frac{|k^0|}{|k\cdot\sigma(\bar{y}_k^i(x))|}\,\Xi\left(\bar{y}_k^{(i)}(x),x\right)\,.\end{aligned}$$

Here the index j labels the different single-valued branches of the hypersurface, while i runs over all possible intersection points $\bar{y}_k$ between the hypersurface and the worldline

$$\mathbf{y} = \mathbf{x} + \frac{\mathbf{k}}{k^0}\left(y^0 - x^0\right) \ , \tag{4.18}$$

and the factor $|k^0|/(k\cdot\sigma)$ is the inverse of the determinant of the matrix:

$$\frac{\partial}{\partial y^j}\left[\mathbf{y} - \mathbf{x} - \frac{\mathbf{k}}{k^0}(f(\mathbf{y}) - x^0)\right]^i = \delta^i_j - \frac{k^i}{k^0}\frac{\partial f(\mathbf{y})}{\partial y^j} = \delta^i_j - \frac{k^i}{k^0}\sigma^j \ .$$

Although multiple intersections may exist in general, the trivial intersection $\bar{y}_k = x$ is always present, since x lies on $\Sigma_{\rm D}$. The vector

$$\sigma^{(j)}_\mu(y) = \left(1, -\frac{\partial f_j}{\partial \mathbf{y}}\right) \tag{4.19}$$

denotes the normal to the branch Σ_j, while $\mathfrak{s}_j = +1$ if $\sigma^{(j)}_\mu$ is parallel to k^μ and $\mathfrak{s}_j = -1$ otherwise.

Replacing the sum over the branches j with the sum over all the possible intersections $\bar{y}^{(i)}_k$ (including the trivial one) (4.17) turns out:

$$\begin{aligned}\Delta_{\rm LE}W^+(x,k) = {}& \sum_j \mathfrak{s}_j \sum_{N=0}^{+\infty} \frac{(-{\rm i})^N}{N!}\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\dots\partial^q_{\nu_N} G^{\mu\nu}(k,q,\beta)\right]_{q=0} \\ & \times \sum_{\bar{y}_k(x)}\sum_{M=0}^{N} \frac{N!\,(-1)^M}{M!\,(N-M)!}{\rm d}^{\nu_{M+1}}_x \dots {\rm d}^{\nu_N}_x \\ & \times \left\{\sigma_\mu\left[{\rm d}^{\nu_1}_x {\rm d}^{\nu_2}_x \dots {\rm d}^{\nu_M}_x \Delta\beta_\nu(y,x)\right]\Big|_{y=\bar{y}_k(x)}\right\} \\ & - \text{analogous term in } \Delta\zeta \ ,\end{aligned} \tag{4.20}$$

where we have introduced the total derivative:

$${\rm d}^\mu_x = \frac{\rm d}{{\rm d}x^\mu} \ ,$$

to emphasize the difference between the derivative acting on the function *before* setting $y = \bar{y}_k(x)$ (that is ∂_x) and the derivative acting on the function *after* setting $y = \bar{y}_k(x)$. The above expression can be further worked out (see appendix B) so that introducing the operator:

$$D_y(\bar{y}) \equiv -{\rm i}\Delta^{\nu\rho}(\bar{y})\partial^y_\rho\partial^q_\nu \, , \tag{4.21}$$

where $\Delta^{\nu\rho}$ is the operator:

$$\Delta^{\nu\rho}(\bar{y}) = g^{\nu\rho} - \frac{\hat{n}^\nu(\bar{y})k^\rho}{k\cdot\hat{n}(\bar{y})} \ . \tag{4.22}$$

the local equilibrium correction to the Wigner function can be finally written in the following compact form:

$$\begin{aligned}\Delta_{\mathrm{LE}} W^{+}(x,k) = &\sum_{N=0}^{+\infty} \frac{(-\mathrm{i})^N}{N!} \sum_{\bar{y}_k(x)} \Big[D_y \left(\bar{y}_k(x)\right) \Big]^N \\ &\times \left\{ \left[G^{\mu\nu}(k,q,\beta) \frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|} \right]_{\substack{q=0\\ y=\bar{y}_k(x)}} \right. \\ &\left. - \left[H^{\mu}(k,q,\beta) \frac{\hat{n}_\mu(y)\Delta\zeta(y,x)}{|k\cdot\hat{n}(y)|} \right]_{\substack{q=0\\ y=\bar{y}_k(x)}} \right\} .\end{aligned} \tag{4.23}$$

This expression shows that the local equilibrium correction to the Wigner function depends not only on the thermodynamic gradients evaluated at the spacetime point x, where the Wigner function itself is defined, but also on the gradients evaluated at the non-trivial intersection points $\bar{y}_k(x)$ between the particle worldline and the decoupling hypersurface. For a generic geometry of the decoupling hypersurface, these points need not be close to each other. The physical implications of this non-local dependence will be discussed in detail at the end of the chapter.

4.1.2 Computation of the expectation values

The expression (4.23) represents the gradient expansion of the linear response approximation to the local equilibrium correction of the Wigner function. In order to evaluate this correction order by order, it is necessary to determine explicitly the functions $G^{\mu\nu}$ and H^{μ} defined in (4.10). These functions are expressed in terms of the thermal expectation values $\Gamma^{\mu\nu}$ and Y^{μ} given in (4.9), which involve creation and annihilation operators and the conserved densities.

As a first approximation to the local equilibrium correction, we assume that both the stress-energy tensor and the conserved four-current are given by their free-field expressions. Under this assumption, the functions $G^{\mu\nu}$ and H^{μ} can be computed exactly, allowing for the explicit determination of the linear response correction to the Wigner function at each order in the gradient expansion.

We begin by considering the function $\Gamma^{\mu\nu}$,

$$\Gamma^{\mu\nu}(k,q,\beta) = \left\langle \widehat{a}^{\dagger}(k_+)\widehat{a}(k_-),\ \widehat{T}^{\mu\nu}(0) \right\rangle_{c,\mathrm{GE}} .$$

The free stress-energy tensor $\widehat{T}^{\mu\nu}$ can be expressed explicitly in terms of the creation and annihilation operators of the free scalar field by inserting the plane-wave expansion (3.16) into Eq. (3.22a). One obtains:

$$\begin{aligned}\widehat{T}^{\mu\nu}(0) &= \frac{2}{(2\pi)^3} \int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell} \int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}} \delta^4\left(w - \frac{\ell+\ell'}{2}\right) \widehat{a}^{\dagger}(\ell)\widehat{a}(\ell') \\ &\qquad \times \left[w^\mu w^\nu - \frac{1}{4}\left(\partial_x^\mu \partial_x^\nu - g^{\mu\nu}\Box_x\right) \right] \left(\mathrm{e}^{\mathrm{i}(\ell-\ell')\cdot y} \right) \Bigg|_{y=0} \\ &= \frac{2}{(2\pi)^3} \int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell} \int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}} \delta^4\left(w - \frac{\ell+\ell'}{2}\right) \widehat{a}^{\dagger}(\ell)\widehat{a}(\ell') \\ &\qquad \times \left[w^\mu w^\nu + \frac{1}{4}\left((\ell-\ell')^\mu (\ell-\ell')^\nu - g^{\mu\nu}(\ell-\ell')^2 \right) \right] .\end{aligned}$$

Here $\ell_\pm = \ell \pm w$, with $\ell_\pm^2 = m^2$ due to the on-shell condition satisfied by the momenta appearing in the creation and annihilation operators.

Substituting this expression into the definition of $\Gamma^{\mu\nu}$ yields

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell}\int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\,\delta^4\left(w - \frac{\ell+\ell'}{2}\right)$$
$$\times\left[w^\mu w^\nu + \frac{1}{4}\left((\ell-\ell')^\mu(\ell-\ell')^\nu - g^{\mu\nu}(\ell-\ell')^2\right)\right]\left\langle \widehat{a}^\dagger(k_+)\widehat{a}(k_-),\ \widehat{a}^\dagger(\ell)\widehat{a}(\ell')\right\rangle_{c,\mathrm{GE}}.$$

The connected part of the thermal expectation value involving four creation and annihilation operators can be evaluated combining (4.3) with standard techniques in thermal field theory [146] leading to:

$$\left\langle \widehat{a}^\dagger(k_+)\widehat{a}(k_-),\ \widehat{a}^\dagger(\ell)\widehat{a}(\ell')\right\rangle_{c,\mathrm{GE}} = \left\langle \widehat{a}^\dagger(k_+)\widehat{a}(\ell')\right\rangle_{\mathrm{GE}}\left\langle \widehat{a}(k_-)\widehat{a}^\dagger(\ell)\right\rangle_{\mathrm{GE}}$$
$$= 2E_{k_+}\delta^3\left(\mathbf{k}_+ - \boldsymbol{\ell}'\right)\, n_{\mathrm{B}}(x,k_+)\; 2E_{k_-}\delta^3\left(\boldsymbol{\ell} - \mathbf{k}_-\right)\left(1 + n_{\mathrm{B}}(x,k_-)\right)\ .$$

The delta functions allow one to perform the integrals over $\mathrm{d}^3\ell$ and $\mathrm{d}^3\ell'$, fixing $\ell = k_+$ and $\ell' = k_-$. One then obtains

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4 w\,\delta^4(w-k)\left[w^\mu w^\nu + \frac{1}{4}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\right]$$
$$\times\, n_{\mathrm{B}}(x,k_+)\left[1 + n_{\mathrm{B}}(x,k_-)\right]\ ,$$

where we have used $k_+ - k_- = q$ and $(k_+ + k_-)/2 = k$, as defined in (3.15). The remaining integration over $\mathrm{d}^4 w$ sets $w = k$, yielding the final result

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{2}{(2\pi)^3}\left[k^\mu k^\nu + \frac{1}{4}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\right] n_{\mathrm{B}}(x,k_+)\left[1 + n_{\mathrm{B}}(x,k_-)\right]\ .$$

The computation of the current expectation value Υ^μ proceeds analogously. Using Eqs. (3.16) and Eq. (3.22b) one finds

$$\widehat{j}^\mu(0) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell}\int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\,\delta^4\left(w - \frac{\ell+\ell'}{2}\right) w^\mu\, \widehat{a}^\dagger(\ell)\widehat{a}(\ell')\ ,$$

which leads to

$$\Upsilon^\mu(k,q,\beta) = \frac{2}{(2\pi)^3}\int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell}\int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\,\delta^4\left(w - \frac{\ell+\ell'}{2}\right)$$
$$\times\, w^\mu\,\left\langle \widehat{a}^\dagger(k_+)\widehat{a}(k_-),\ \widehat{a}^\dagger(\ell)\widehat{a}(\ell')\right\rangle_{c,\mathrm{GE}}\ ,$$

and therefore

$$\Upsilon^\mu(k,q,\beta) = \frac{2}{(2\pi)^3} k^\mu\, n_{\mathrm{B}}(x,k_+)\left[1 + n_{\mathrm{B}}(x,k_-)\right]\ .$$

In summary, for free scalar fields the functions defined in (4.9) are given explicitly by

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{2}{(2\pi)^3}\left[k^\mu k^\nu + \frac{1}{4}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\right] n_{\mathrm{B}}(k_+)\left[1 + n_{\mathrm{B}}(k_-)\right]\ , \tag{4.24a}$$

$$\Upsilon^\mu(k,q,\beta) = \frac{2}{(2\pi)^3} k^\mu\, n_{\mathrm{B}}(k_+)\left[1 + n_{\mathrm{B}}(k_-)\right]\ . \tag{4.24b}$$

4.1.3 Local equilibrium correction: Final result

Using the expansion (4.23) together with the definitions (4.10) and the explicit expressions (4.24), the local equilibrium correction to the Wigner function can be computed to any desired order in the gradient expansion.

Restricting to first order in q, the functions in (4.10) can be expanded using

$$\begin{aligned}
\Gamma^{\mu\nu}(k,q,\beta) &= \frac{2}{(2\pi)^3} k^\mu k^\nu\, n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\left(-1-\frac{1}{2}\beta_\tau(x)q^\tau\right)+\mathcal{O}(q)\,,\\
Y^{\mu}(k,q,\beta) &= \frac{2}{(2\pi)^3} k^\mu\, n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\left(-1-\frac{1}{2}\beta_\tau(x)q^\tau\right)+\mathcal{O}(q)\,,\\
\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} &= -1-\frac{1}{2}\beta_\tau(x)q^\tau+\mathcal{O}(q)\,,\\
\delta\left(k^2+\frac{q^2}{4}-m^2\right) &= \delta\left(k^2-m^2\right)+\mathcal{O}(q)\,,
\end{aligned}$$

where we have used the fact that $k_\pm = k$ at $q=0$. This leads to

$$\begin{aligned}
G^{\mu\nu}(k,q,\beta) &= -\frac{2}{(2\pi)^3} k^\mu k^\nu\,\delta\left(k^2-m^2\right)\, n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]+\mathcal{O}(q)\,,\\
\Upsilon^{\mu}(k,q,\beta) &= -\frac{2}{(2\pi)^3} k^\mu\,\delta\left(k^2-m^2\right)\, n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]+\mathcal{O}(q)\,.
\end{aligned}\tag{4.25}$$

Substituting these expressions into (4.17), one finds that the local equilibrium correction to the Wigner function, up to first order in q, is given by

$$\begin{aligned}
\Delta W^{+}_{\rm LE}(x,k) \simeq\; & -\frac{2}{(2\pi)^3}\delta\left(k^2-m^2\right) n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\\
&\times \sum_{\bar y_k(x)} \mathrm{sgn}\left(k\cdot\hat n(\bar y_k(x))\right)\left[k^\nu \Delta\beta_\nu(y,x)-\Delta\zeta(y,x)\right]\Bigg|_{y=\bar y_k(x)}\,.
\end{aligned}\tag{4.26}$$

The leading-order correction does not depend explicitly on the gradients of the thermodynamic fields, but rather on their difference between the point x and the non-trivial intersection points $\bar y_k$ between the worldline (4.18) and the decoupling hypersurface. These points may be widely separated in spacetime, and their contribution will be discussed at the end of the chapter.

If no non-trivial intersections exist—as is always the case when the decoupling hypersurface is everywhere spacelike—the leading-order contribution vanishes identically. In this situation, the first non-vanishing correction appears at second order in q, implying that the leading gradient dependence takes the schematic form

$$\partial_y^2\left[\frac{\hat n_\mu(y)\,\Delta\beta_\nu(y,x)}{|k\cdot\hat n(y)|}\right]_{y=\bar y_k(x)} \sim \Delta\beta\,\partial_y^2\hat n+\hat n\,\partial_y^2\beta+\partial_y\hat n\,\partial_y\beta\,. \tag{4.27}$$

The explicit computation of the second-order correction is presented in Appendix C. For a flat decoupling hypersurface, this structure reduces to contributions proportional to second derivatives of the four-temperature or of the reduced chemical potential, in agreement with standard hydrodynamic expectations. By contrast, for curved hypersurfaces the derivative operator also acts on the normal vector $\hat{n}$, giving rise to additional terms proportional to the thermodynamic field differences even at higher orders. Moreover, mixed contributions involving one derivative acting on $\hat{n}$ and one acting on the thermodynamic fields appear, leading to terms proportional to first-order gradients.

All such contributions are entirely neglected in standard linear-response approaches, as they survive only when the normal vector $\hat{n}$ is not constant, *i.e.*, when the decoupling hypersurface is curved. This effectively introduces an additional geometric scale into the gradient expansion. The rate of convergence of the series (4.23) is therefore expected to depend on the ratio of two characteristic length scales. The first is the correlation length ℓ_c associated with the function $G^{\mu\nu}$, defined schematically as $|\partial_q G^{\mu\nu}(q)| \,/\, |G^{\mu\nu}(q)|$, which is controlled by the microscopic scales of the quantum field and by the inverse temperature. The second is the characteristic variation length of the thermo-hydrodynamic fields defined in (2.36), together with the geometric scale associated with variations of the normal vector, $L_G = |\hat{n}| \,/\, |\partial\hat{n}|$, which depends solely on the geometry of the hypersurface. The series is therefore expected to converge rapidly provided that

$$\ell_c/\lambda_{\beta,\zeta} \ll 1 \,, \qquad \ell_c/L_G \ll 1 \,. \tag{4.28}$$

The first condition is equivalent to the scale-separation requirement underlying the hydrodynamic limit (2.37). The second condition, however, is genuinely new and originates from the non-trivial geometry of the decoupling hypersurface.

The conditions (4.28) are expected to be generally satisfied in high-energy heavy-ion collisions, although the second one has not been carefully examined so far.

In order to provide a quantitative estimate for a hypersurface we consider the casoe of a decupling hypersurface of constant proper time τ which is a good approximation for mid-rapidity region of high energetic collisions. For this case the normal vector can be expressed in terms of Milne coordinates as:

$$\hat{n}^\mu = (\cosh\xi,\, 0,\, 0,\, \sinh\xi) \,,$$

with ξ the spacetime rapidity. Hence the characteristic curvature scale is:

$$L_G = \tau \cosh^{-1}\xi \,. \tag{4.29}$$

For central Au+Au collisions at $\sqrt{s_{\rm NN}} = 200$ GeV, one finds $\tau \sim 10$ fm/c [150], corresponding to $6.5\,\text{fm} \lesssim L_G \lesssim 10\,\text{fm}$ within the interval $|\xi| < 1$. This scale is significantly larger than the typical microscopic length $\ell_c \sim 1/T \sim 1.2$ fm for $T = 160$ MeV. In more peripheral regions, where $|\xi|$ is larger, the hypersurface curvature increases and the ratio ℓ_c/L_G becomes larger; however, contributions from these regions to mid-rapidity observables are strongly suppressed by the Bose–Einstein distribution (4.4).

The emergence of this geometric scale, which is physically uncorrelated with the thermodynamic ones, represents a genuine novelty of the present expansion method. It can generate apparently spurious terms involving lower-order thermodynamic gradients mixed with higher-order curvature effects, as explicitly illustrated by Eq. (4.27) at second order. Nevertheless, the appearance of such contributions should not be surprising. As discussed in Chapter 2, both the Zubarev operator (2.23) and the local equilibrium density operator (2.21) depend explicitly on the normal vector of the defining hypersurface and on the matching condition (2.22). Consequently, when performing an expansion around a reference state, the emergence of geometric contributions of this kind is a natural and unavoidable feature of the formalism.

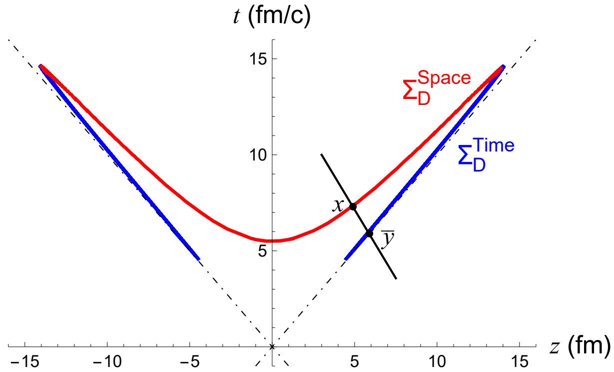


Figure 4.1: A slice of the decoupling hypersurface $\Sigma_{\rm D}$ (red and blue lines) on the transverse plane $x = y = 0$, obtained by 3+1D viscous hydrodynamic simulations with the CLVisc model at $\sqrt{s_{\rm NN}} = 27$ GeV [151–153]. The black solid line represents the worldline of a particle, which intersects $\Sigma_{\rm D}$ at two points: one on the spacelike branch and the other one on the timelike branch.

4.2 Dirac field

We now consider the case of a Dirac field. The same steps outlined previously for the scalar field can be straightforwardly extended to this case. The local equilibrium expectation value of the Wigner function in linear response theory is defined as in (4.1), where the corresponding global equilibrium expectation value reads:

$$\langle W^+(x,k)\rangle_{\rm GE} = \frac{\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\rm F}(x,k)\left(\not{k}+m\right) \ , \tag{4.30}$$

with $n_{\rm F}(k,x)$ *Fermi–Dirac* distributions function:

$$n_{\rm F}(k,x) = \frac{1}{{\rm e}^{\beta(x)\cdot k-\zeta(x)}+1} \ . \tag{4.31}$$

To compute the local equilibrium correction, we insert the field expansion of the Wigner operator (3.60) into eq. (2.44), obtaining:

$$\begin{aligned}\left[\Delta W^+_{\rm LE}(x,k)\right]^{AB} &= \frac{1}{2\,(2\pi)^3}\sum_{r,s}\int_{\Sigma_{\rm D}}{\rm d}\Sigma_\mu(y)\int {\rm d}^4q\,\left\{{\rm e}^{{\rm i}q\cdot(x-y)}\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\theta(k^0_+)\theta(k^0_-)\right.\\ &\times\delta(k\cdot q)\,\delta\left(k^2+\frac{q^2}{4}-m^2\right)u^B_s(k_-)\overline{u}^A_r(k_+)\Big[\Delta\beta_\nu(y,x)\left\langle\widehat{a}^\dagger_r(k_+)\widehat{a}_s(k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,{\rm GE}}\\ &\left.-\Delta\zeta(y,x)\left\langle\widehat{a}^\dagger_r(k_+)\widehat{a}_s(k_-),\ \widehat{j}^{\mu}(0)\right\rangle_{c,{\rm GE}}\Big]\right\} ,\end{aligned}$$

where the spinor indices A, B have been made explicit for clarity.

As in the scalar case, the above expression can be rewritten in a compact form analogous to (4.12) by defining the functions $G^{\mu\nu}$ and H^{μ} as in (4.10):

$$G^{\mu\nu}(k,q,\beta) \equiv \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{2\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k_+^0)\theta(k_-^0)\,\Gamma^{\mu\nu}(k,q,\beta)\ , \tag{4.32a}$$

$$H^{\mu}(k,q,\beta) \equiv \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{2\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k_+^0)\theta(k_-^0)\,\Upsilon^{\mu}(k,q,\beta)\ , \tag{4.32b}$$

together with the functions F, which are defined as the Fourier transforms of the thermodynamic fields in (4.11).

The only difference with respect to the scalar case is that the thermostatic expectation values $\Gamma^{\mu\nu}$ and Υ^{μ} appearing in (4.32) are now defined as:

$$\Gamma^{\mu\nu}(k,q,\beta) \equiv \sum_{r,s} u_s(k_-)\overline{u}_r(k_+)\left\langle \widehat{a}_r^{\dagger}(k_+)\widehat{a}_s(k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,\mathrm{GE}}\ , \tag{4.33a}$$

$$\Upsilon^{\mu}(k,q,\beta) \equiv \sum_{r,s} u_s(k_-)\overline{u}_r(k_+)\left\langle \widehat{a}_r^{\dagger}(k_+)\widehat{a}_s(k_-),\ \widehat{\jmath}^{\mu}(0)\right\rangle_{c,\mathrm{GE}}\ . \tag{4.33b}$$

With these definitions, the local equilibrium correction to the Wigner function for the Dirac field can be written in the same form as (4.12):

$$\begin{aligned}\Delta W^{+}_{\mathrm{LE}}(x,k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\delta(k\cdot q)\Big[&G^{\mu\nu}(k,q,\beta)\,F^{(\beta)}_{\mu\nu}(x,q)\\ &-H^{\mu}(k,q,\beta)\,F^{(\zeta)}_{\mu}(x,q)\Big]\ .\end{aligned} \tag{4.34}$$

The above expression is formally equivalent to the one obtained for the scalar field in (4.17). The difference lies in the explicit expressions of the functions $G^{\mu\nu}$ and H^{μ}, which now depend on the Dirac structure of the theory through the form of the stress–energy tensor and the four-current appearing in (4.33).

This implies that the same separation of scales holds in the hydrodynamic limit, and therefore (4.34) can be formally expanded in powers of q, as done previously in (4.23).

4.2.1 Computation of the expectation values

The tensors defined in (4.33) can be computed exactly in the case of free Dirac fields, following steps analogous to those employed for the scalar field. This allows for an explicit evaluation of the functions defined in (4.32).

For the stress–energy tensor contribution, we insert (3.60) into (3.65a), obtaining:

$$\begin{aligned}\widehat{T}^{\mu\nu}(0) = \frac{1}{2(2\pi)^3}\sum_{a,b}\int \mathrm{d}^4w\int\frac{\mathrm{d}^3\ell}{2E_\ell}\int\frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\delta^4\left(w-\frac{\ell+\ell'}{2}\right)\theta(\ell_+^0)\theta(\ell_-^0)\\ \times\Big[\overline{u}_a(\ell)\left(\gamma^\mu w^\nu+\gamma^\nu w^\mu\right)u_b(\ell')\Big]\widehat{a}_a^{\dagger}(\ell)\widehat{a}_b(\ell')\ .\end{aligned}$$

Substituting this expression into (4.33a) yields:

$$\begin{aligned}\Gamma^{\mu\nu}(k,q,\beta) = \frac{1}{2(2\pi)^3}\sum_{r,s}\sum_{a,b}\int \mathrm{d}^4w\int\frac{\mathrm{d}^3\ell}{2E_\ell}\int\frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\delta^4\left(w-\frac{\ell+\ell'}{2}\right)\theta(\ell_+^0)\theta(\ell_-^0)\\ \times u_s(k_-)\overline{u}_r(k_+)\Big[\overline{u}_a(\ell)\left(\gamma^\mu w^\nu+\gamma^\nu w^\mu\right)u_b(\ell')\Big]\left\langle \widehat{a}_r^{\dagger}(k_+)\widehat{a}_s(k_-),\ \widehat{a}_a^{\dagger}(\ell)\widehat{a}_b(\ell')\right\rangle_{c,\mathrm{GE}}\ .\end{aligned}$$

The connected expectation value can be computed explicitly [146] and is given by:

$$\left\langle \widehat{a}_r^\dagger(k_+)\widehat{a}_s(k_-),\ \widehat{a}_a^\dagger(\ell)\widehat{a}_b(\ell')\right\rangle_{c,\mathrm{GE}} = 2E_{k_+}\delta^3\left(\mathbf{k}_+ - \boldsymbol{\ell}'\right)\delta_{rb}\, n_\mathrm{F}(k_+) \\ \times 2E_{k_-}\delta^3\left(\mathbf{k}_- - \boldsymbol{\ell}\right)\delta_{sa}\left[1 - n_\mathrm{F}(k_-)\right] ,$$

which allows one to perform the sums over a, b and the integrals over $\mathrm{d}^3\ell$, $\mathrm{d}^3\ell'$, and $\mathrm{d}^4 w$, setting $w = k$:

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{1}{2(2\pi)^3}\sum_{r,s}\theta(k_+^0)\theta(k_-^0)\, n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right] \\ \times u_s(k_-)\overline{u}_r(k_+)\Big[\overline{u}_b(k_+)\left(\gamma^\mu k^\nu + \gamma^\nu k^\mu\right)u_r(k_-)\Big] .$$

Finally, the sum over spin indices can be evaluated using (A.7) leading to:

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{\theta(k_+^0)\theta(k_-^0)}{2(2\pi)^3}\, n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right] \\ \times \left(\not{k}_- + m\right)\Big[\gamma^\mu k^\mu + \gamma^\nu k^\mu\Big]\left(\not{k}_+ + m\right) .$$

The computation of the current contribution proceeds analogously. One finds:

$$\widehat{j}^\mu(0) = \frac{1}{2(2\pi)^3}\sum_{a,b}\int \mathrm{d}^4 w \int \frac{\mathrm{d}^3\ell}{2E_\ell}\int \frac{\mathrm{d}^3\ell'}{2E_{\ell'}}\delta^4\left(w - \frac{\ell+\ell'}{2}\right)\theta(\ell_+^0)\theta(\ell_-^0) \\ \times \Big[\overline{u}_a(\ell)\gamma^\mu u_b(\ell')\Big]\widehat{a}_a^\dagger(\ell)\widehat{a}_b(\ell') .$$

Substituting this expression into (4.33b) and performing the integrations as before yields:

$$\Upsilon^\mu(k,q,\beta) = \frac{\theta(k_+^0)\theta(k_-^0)}{2(2\pi)^3}\, n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right]\left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right) .$$

In summary, for Dirac fields the functions defined in (4.33) take the form:

$$\Gamma^{\mu\nu}(k,q,\beta) = \frac{\theta(k_+^0)\theta(k_-^0)}{2(2\pi)^3}\, n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right] \\ \times \left(\not{k}_- + m\right)\Big[\gamma^\mu k^\nu + \gamma^\nu k^\mu\Big]\left(\not{k}_+ + m\right) , \tag{4.35a}$$

$$\Upsilon^\mu(k,q,\beta) = \frac{\theta(k_+^0)\theta(k_-^0)}{2(2\pi)^3}\, n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right]\left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right) . \tag{4.35b}$$

Combining these equations with (4.32) and using that:

$$\left(1 - \mathrm{e}^{\beta(x)\cdot q}\right) n_\mathrm{F}(k_+)\left[1 - n_\mathrm{F}(k_-)\right] = n_\mathrm{F}(k_+) - n_\mathrm{F}(k_-) ,$$

we finally obtain that the functions that define the local equilibrium correction of the Wigner function are given by:

$$G^{\mu\nu}(k,q,\beta) \equiv \frac{n_\mathrm{F}(k_+) - n_\mathrm{F}(k_-)}{4(2\pi)^3\beta(x)\cdot q}\delta\left(k^2 + \frac{q^2}{4} - m^2\right)\theta(k_+^0)\theta(k_-^0) \\ \times \left(\not{k}_- + m\right)\left[\gamma^\mu k^\nu + \gamma^\nu k^\mu\right]\left(\not{k}_+ + m\right) , \tag{4.36a}$$

$$H^\mu(k,q,\beta) \equiv \frac{n_\mathrm{F}(k_+) - n_\mathrm{F}(k_-)}{4(2\pi)^3\beta(x)\cdot q}\delta\left(k^2 + \frac{q^2}{4} - m^2\right)\theta(k_+^0)\theta(k_-^0) \\ \times \left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right) , \tag{4.36b}$$

Both functions explicitly depend on the spinorial structure through the Dirac gamma matrices. The computation of the different components of the Wigner function therefore requires evaluating the traces of the expressions above.

4.2.2 Components of the Wigner function

In order to compute the components of the Wigner function defined in (3.63), one needs to evaluate the traces of the quantities obtained above. From (4.34) the various components of the local equilibrium correction to the Wigner function read:

$$\Delta\mathcal{A}^{\sigma}_{\rm LE}(x,k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\delta(k\cdot q)\Big[\mathrm{tr}\left(\gamma^5\gamma^\sigma G^{\mu\nu}(k,q,\beta)\right)F^{(\beta)}_{\mu\nu}(x,q) - \mathrm{tr}\left(\gamma^5\gamma^\sigma H^{\mu}(k,q,\beta)\right)F^{(\zeta)}_{\mu}(x,q)\Big]\,, \tag{4.37a}$$

$$\Delta\mathcal{S}_{\rm LE}(x,k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\delta(k\cdot q)\Big[\mathrm{tr}\left(G^{\mu\nu}(k,q,\beta)\right)F^{(\beta)}_{\mu\nu}(x,q) - \mathrm{tr}\left(H^{\mu}(k,q,\beta)\right)F^{(\zeta)}_{\mu}(x,q)\Big]\,, \tag{4.37b}$$

$$\Delta\mathcal{V}^{\sigma}_{\rm LE}(x,k) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\delta(k\cdot q)\Big[\mathrm{tr}\left(\gamma^\sigma G^{\mu\nu}(k,q,\beta)\right)F^{(\beta)}_{\mu\nu}(x,q) - \mathrm{tr}\left(\gamma^\sigma H^{\mu}(k,q,\beta)\right)F^{(\zeta)}_{\mu}(x,q)\Big]\,. \tag{4.37c}$$

In the hydrodynamic regime, each of these quantities can be systematically expanded in powers of q, leading to expressions analogous to those obtained previously for the scalar case (4.23).

Scalar part

We start by computing the scalar part of the functions G and H. Using (A.7) we get:

$$\mathrm{tr}\left(\not{k}_-\gamma^\mu\right) = 2k^\mu_-\,, \qquad \mathrm{tr}\left(\gamma^\mu\not{k}_+\right) = 2k^\mu_+\,. \tag{4.38}$$

With these it is immediate to compute the two following traces:

$$\mathrm{tr}\left\{\left(\not{k}_- + m\right)\left[\gamma^\mu k^\nu + \gamma^\nu k^\mu\right]\left(\not{k}_+ + m\right)\right\} = 8m\,k^\mu k^\nu\,,$$
$$\mathrm{tr}\left[\left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right)\right] = 8m\,k^\mu\,,$$

where we used that $k_+ + k_- = 2k$. We then have:

$$\mathrm{tr}\left(G^{\mu\nu}(k,q,\beta)\right) = \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k^0_+)\theta(k^0_-) \times \frac{4m\,k^\mu k^\nu}{(2\pi)^3}n_{\rm F}(k_+)\left[1-n_{\rm F}(k_-)\right]\,, \tag{4.39a}$$

$$\mathrm{tr}\left(H^{\mu}(k,q,\beta)\right) = \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k^0_+)\theta(k^0_-) \times \frac{4m\,k^\mu}{(2\pi)^3}n_{\rm F}(k_+)\left[1-n_{\rm F}(k_-)\right]\,. \tag{4.39b}$$

Axial part

In order to compute the axial part we make use of the identities (A.7) involving γ^5 and we get:

$$\begin{aligned}
&\mathrm{tr}\left\{\gamma^5\gamma^\sigma\left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right)\right\} \\
&\quad = 4\mathrm{i}\,\varepsilon^{\sigma\alpha\mu\beta}\left(k_\alpha - \frac{q_\alpha}{2}\right)\left(k_\beta + \frac{q_\beta}{2}\right) \\
&\quad = 2\mathrm{i}\varepsilon^{\sigma\alpha\mu\beta}\left(k_\alpha q_\beta - k_\beta q_\alpha\right) \\
&\quad = -4\mathrm{i}\varepsilon^{\sigma\mu\alpha\beta}k_\alpha q_\beta\ ,
\end{aligned}$$

where we used that $k_+ - k_- = q$ and that the Levi-Civita pseudo-tensor is totally antisymmetric. In the same way we get:

$$\begin{aligned}
&\mathrm{tr}\left\{\gamma^5\gamma^\sigma\left(\not{k}_- + m\right)\left[\gamma^\mu k^\nu + \gamma^\nu k^\mu\right]\left(\not{k}_+ + m\right)\right\} \\
&\quad = -4\mathrm{i}\left(k^\nu\varepsilon^{\sigma\mu\alpha\beta} + k^\mu\varepsilon^{\sigma\nu\alpha\beta}\right)k_\alpha q_\beta\ ,
\end{aligned}$$

so that the axial components of the tensors (4.36) is finally given by:

$$\begin{aligned}
\mathrm{tr}\left(\gamma^5\gamma^\sigma G^{\mu\nu}\left(k,q,\beta\right)\right) &= \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2 + \frac{q^2}{4} - m^2\right)\theta(k_+^0)\theta(k_-^0) \qquad (4.40a) \\
&\times \frac{2\mathrm{i}}{(2\pi)^3}n_\mathrm{F}(k_+)\left[1-n_\mathrm{F}(k_-)\right]\left(k^\nu\varepsilon^{\mu\sigma\alpha\beta} + k^\mu\varepsilon^{\nu\sigma\alpha\beta}\right)k_\alpha q_\beta\ , \\
\mathrm{tr}\left(\gamma^5\gamma^\sigma H^{\mu}\left(k,q,\beta\right)\right) &= \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2 + \frac{q^2}{4} - m^2\right)\theta(k_+^0)\theta(k_-^0) \qquad (4.40b) \\
&\times \frac{2\mathrm{i}}{(2\pi)^3}n_\mathrm{F}(k_+)\left[1-n_\mathrm{F}(k_-)\right]\varepsilon^{\mu\sigma\alpha\beta}k_\alpha q_\beta\ .
\end{aligned}$$

Vector part

We finally come to the computation of the vector part. From (A.7) we obtain:

$$\mathrm{tr}\left(\gamma^\sigma\not{k}_+\gamma^\mu\not{k}_-\right) = 4\left(k_+^\sigma k_-^\mu + k_+^\mu k_-^\sigma\right) - 4g^{\sigma\mu}k_+\cdot k_-\ ,$$

which, combined with (4.38), implies:

$$\begin{aligned}
&\mathrm{tr}\left\{\gamma^\sigma\left(\not{k}_- + m\right)\left[\gamma^\mu k^\nu + \gamma^\nu k^\mu\right]\left(\not{k}_+ + m\right)\right\} \\
&\qquad\qquad = 4m\,k^\nu\ \left[k_+^\mu k_-^\sigma + k_-^\mu k_+^\sigma - g^{\mu\sigma}\left(k_+\cdot k_- - m^2\right)\right] + \left(\mu\leftrightarrow\nu\right)\ , \\
&\mathrm{tr}\left[\gamma^\sigma\left(\not{k}_- + m\right)\gamma^\mu\left(\not{k}_+ + m\right)\right] = 4m\ \left[k_+^\mu k_-^\sigma + k_-^\mu k_+^\sigma - g^{\mu\sigma}\left(k_+\cdot k_- - m^2\right)\right]\ .
\end{aligned}$$

Using the constraint enforced by the δ function in (4.39):

$$k_+\cdot k_- - m^2 = -\frac{q^2}{2}\ ,$$

and the vector part of (4.39) finally read:

$$\mathrm{tr}\left(\gamma^\sigma G^{\mu\nu}\left(k,q,\beta\right)\right) = \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k_+^0)\theta(k_-^0) \tag{4.41a}$$
$$\times\frac{4m\,k^\nu\left(k_+^\mu k_-^\sigma + k_+^\sigma k_-^\mu - g^{\mu\sigma}\frac{q^2}{2}\right)}{(2\pi)^3} n_\mathrm{F}(k_+)\left[1-n_\mathrm{F}(k_-)\right] + (\mu\leftrightarrow\nu)\ ,$$
$$\mathrm{tr}\left(\gamma^\sigma H^{\mu}\left(k,q,\beta\right)\right) = \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta\left(k^2+\frac{q^2}{4}-m^2\right)\theta(k_+^0)\theta(k_-^0) \tag{4.41b}$$
$$\times\frac{4m\left(k_+^\mu k_-^\sigma + k_+^\sigma k_-^\mu - g^{\mu\sigma}\frac{q^2}{2}\right)}{(2\pi)^3} n_\mathrm{F}(k_+)\left[1-n_\mathrm{F}(k_-)\right]\ .$$

4.3 Spin polarization vector

From the definition (3.79), the spin polarization vector at local thermodynamic equilibrium is given by:

$$S^\sigma(k) = \frac{1}{2}\frac{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\,\mathcal{A}_\mathrm{LE}\left(x,k\right)}{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\,\mathcal{S}_\mathrm{LE}\left(x,k\right)}\ . \tag{4.42}$$

It is convenient to separate the global equilibrium contribution from the local equilibrium correction by defining:

$$\mathcal{N}^\sigma(k) \equiv \int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\left[\mathcal{A}^\sigma_\mathrm{GE}\left(x,k\right) + \Delta\mathcal{A}^\sigma_\mathrm{LE}\left(x,k\right)\right] \equiv \mathcal{N}^\sigma_\mathrm{GE} + \Delta\mathcal{N}^\sigma_\mathrm{LE}(k)\ ,$$
$$\mathcal{D}(k) \equiv \int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\ \left[\mathcal{S}_\mathrm{GE}\left(x,k\right) + \Delta\mathcal{S}_\mathrm{LE}\left(x,k\right)\right] \equiv \mathcal{D}_\mathrm{GE}(k) + \Delta\mathcal{D}_\mathrm{LE}(k)\ ,$$

Using the explicit expression for the global equilibrium Wigner function (4.30), one immediately finds:

$$\mathcal{S}_\mathrm{GE}(k) = \frac{4m\,\delta\left(k^2-m^2\right)}{(2\pi)^3}\,n_\mathrm{F}(k)\ ,$$
$$\mathcal{A}^\sigma_\mathrm{GE}(k) = 0\ ,$$

which implies:

$$\mathcal{N}^\sigma_\mathrm{GE}(k) = 0\,.$$

Therefore, the spin polarization vector vanishes at global equilibrium.

The local equilibrium contribution to the spin polarization can then be written as:

$$S^\mu(k) = \frac{1}{2}\frac{\Delta\mathcal{N}^\sigma_\mathrm{LE}(k)}{\mathcal{D}_\mathrm{GE}(k)+\Delta\mathcal{D}_\mathrm{LE}(k)}\ .$$

Assuming that local equilibrium corrections are small compared to the global equilibrium contribution,

$$\frac{\Delta\mathcal{D}_\mathrm{LE}}{\mathcal{D}_\mathrm{GE}} \ll 1\,,$$

the denominator can be expanded, yielding:

$$S^\sigma(k) \simeq \frac{1}{2}\frac{\Delta\mathcal{N}^\sigma_\mathrm{LE}(k)}{\mathcal{D}_\mathrm{GE}}\left(1-\frac{\Delta\mathcal{D}_\mathrm{LE}(k)}{\mathcal{D}_\mathrm{GE}(k)}\right) \simeq \frac{1}{2}\frac{\Delta\mathcal{N}^\sigma_\mathrm{LE}(k)}{\mathcal{D}_\mathrm{GE}(k)}\ ,$$

where all terms that are at least quadratic in the deviations from global equilibrium have been neglected[1].

Within linear response theory, the spin polarization vector therefore reduces to:

$$S^\mu(k) \simeq \frac{1}{2}\frac{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\,\Delta\mathcal{A}^\sigma_\mathrm{LE}\,(x,k)}{\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\,\mathcal{S}_\mathrm{GE}\,(x,k)}\ . \tag{4.43}$$

The denominator can be computed straightforwardly using (4.30) and (4.30), which gives:

$$\mathcal{S}_\mathrm{GE}\,(x,k) = \frac{4m\delta\,(k^2-m^2)}{(2\pi)^3} n_\mathrm{F}(k,x) \tag{4.44}$$

so that:

$$\mathcal{D}_\mathrm{GE}(k) = \frac{4m}{(2\pi)^3}\delta\left(k^2-m^2\right)\mathrm{N}_p\ , \tag{4.45}$$

where N_p denotes the total number of particles at global equilibrium:

$$\mathrm{N}_p \equiv \int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma(x)\cdot k\, n_\mathrm{F}\,(k,x)\ . \tag{4.46}$$

The numerator requires the computation of the linear response correction to the axial component of the Wigner function, which reads:

$$\begin{aligned} \Delta\mathcal{A}^\sigma_\mathrm{LE}\,(x,k) \simeq \sum_{N=0}^{1}\frac{(-\mathrm{i})^N}{N!}\sum_{\bar{y}_k(x)}\left[\Delta^{\lambda\gamma}(\bar{y}_k)\partial^q_\lambda\partial^y_\gamma\right]^N \\ \times\left\{\left[\mathrm{tr}\left(\gamma^5\gamma^\sigma G^{\mu\nu}\,(k,q,\beta)\right)\frac{\hat{n}_\mu(y)\Delta\beta_\nu\,(y,x)}{|k\cdot\hat{n}(y)|}\right]_{\substack{q=0\\ y=\bar{y}_k(x)}}\right. \\ \left. -\left[\mathrm{tr}\left(\gamma^5\gamma^\sigma H^{\mu}\,(k,q,\beta)\right)\frac{\hat{n}_\mu(y)\Delta\zeta\,(y,x)}{|k\cdot\hat{n}(y)|}\right]_{\substack{q=0\\ y=\bar{y}_k(x)}}\right\}\ . \end{aligned} \tag{4.47}$$

The traces of $G^{\mu\nu}$ and H^μ with $\gamma^5\gamma^\sigma$ are given in (4.40). At zeroth order in q, both traces vanish:

$$\mathrm{tr}\left(\gamma^5\gamma^\sigma G^{\mu\nu}\,(k,0,\beta)\right) = 0\ ,$$
$$\mathrm{tr}\left(\gamma^5\gamma^\sigma H^{\mu}\,(k,0,\beta)\right) = 0\ ,$$

implying that $\Delta\mathcal{A}^\sigma_\mathrm{LE}$ vanishes at zeroth order, and therefore the spin polarization vector does as well.

The leading non-vanishing contribution arises at first order in the gradients, corresponding to the linear term in the q-expansion. One finds, expanding (4.40) in q:

$$\begin{aligned} \partial^q_\lambda\Big[\mathrm{tr}\left(\gamma^5\gamma^\sigma G^{\mu\nu}\,(k,q,\beta)\right)\Big]_{q=0} &= \frac{2\mathrm{i}}{(2\pi)^3}\delta\left(k^2-m^2\right)\theta(k^0)n_\mathrm{F}(k)\left[1-n_\mathrm{F}(k)\right] \\ &\quad\times\frac{1}{2}\left(k^\nu g^\mu_\rho + k^\mu g^\nu_\rho\right)\varepsilon^{\sigma\rho\alpha\beta}k_\alpha\delta_{\beta\lambda}\ , \\ \partial^q_\lambda\Big[\mathrm{tr}\left(\gamma^5\gamma^\sigma H^{\mu}\,(k,q,\beta)\right)\Big]_{q=0} &= \frac{2\mathrm{i}}{(2\pi^3)}\delta\left(k^2-m^2\right)\theta(k^0)n_\mathrm{F}(k)\left[1-n_\mathrm{F}(k)\right] \\ &\quad\times\frac{1}{2}\varepsilon^{\sigma\mu\alpha\beta}k_\alpha\delta_{\beta\lambda}\ , \end{aligned}$$

[1] If one is interested in quadratic response effects, these contributions must be retained in the definition of ΔS_LE [55].

Substituting these expressions into (4.47) yields the axial component of the Wigner function at linear order in gradients:

$$\begin{aligned}\Delta\mathcal{A}^{\sigma}_{\rm LE}(x,k) \simeq & \frac{2\theta(k^0)\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\rm F}(k)\left[1-n_{\rm F}(k)\right]\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha \\ & \times \sum_{\bar{y}_k}\Delta_{\lambda\gamma}(\bar{y}_k)\Bigg\{\frac{1}{2}\left(k^\nu g^\mu_\rho + k^\mu g^\nu_\rho\right)\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} \\ & + \frac{1}{2}g^\mu_\rho\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)\Delta\zeta(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k}\Bigg\} .\end{aligned} \tag{4.48}$$

When the derivative with respect to y acts on the square brackets, it generates two types of contributions: one proportional to the gradients of the thermodynamic fields and another proportional to the gradient of the normal vector $\hat{n}_\mu$, i.e. to the curvature of the decoupling hypersurface.

We begin by analyzing the latter contribution. For the four-temperature term, the curvature contribution reads:

$$\begin{aligned}&\Delta\beta_\nu\left(\bar{y},x\right)\left(k^\nu g^\mu_\rho + k^\mu g^\nu_\rho\right)\Delta_{\lambda\gamma}(\bar{y})\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} \\ &= \Delta\beta_\rho\left(\bar{y},x\right)\Delta_{\lambda\gamma}(\bar{y})\partial^\gamma_y\Big[\mathrm{sgn}\left(k\cdot\hat{n}\right)\Big]_{y=\bar{y}_k} + k\cdot\Delta\beta\left(\bar{y},x\right)\Delta_{\lambda\gamma}(\bar{y})\partial^\gamma_y\left[\frac{\hat{n}_\rho(y)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} .\end{aligned}$$

The first term contributes only when $k\cdot\hat{n}(\bar{y})=0$, a case that has been excluded in the derivation of the q-expansion of the Wigner function. Consequently, this term can be neglected. The remaining contribution can therefore be written as:

$$\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\left(k^\nu g^\mu_\rho - k^\mu g^\nu_\rho\right)\Delta_{\lambda\gamma}(\bar{y})\Delta\beta_\nu(\bar{y},x)\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} , \tag{4.49}$$

which follows from changing the sign of the second term inside the parentheses. Using the Schouten identity:

$$\varepsilon^{\sigma\mu\alpha\lambda}k^\delta + \varepsilon^{\mu\alpha\lambda\delta}k^\sigma + \varepsilon^{\alpha\lambda\delta\sigma}k^\mu + \varepsilon^{\lambda\delta\sigma\mu}k^\alpha + \varepsilon^{\delta\sigma\mu\alpha}k^\lambda = 0 , \tag{4.50}$$

one obtains:

$$\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\left(k^\nu g^\mu_\rho - k^\mu g^\nu_\rho\right) = -\left(\varepsilon^{\mu\alpha\lambda\nu}k^\sigma + \varepsilon^{\lambda\nu\sigma\mu}k^\alpha + \varepsilon^{\nu\sigma\mu\alpha}k^\lambda\right)k_\alpha . \tag{4.51}$$

Substituting this expression into (4.49), and using the identity $k^\lambda\Delta_{\lambda\gamma}=0$, the curvature contribution reduces to:

$$\begin{aligned}&-\left(\varepsilon^{\mu\alpha\lambda\nu}k^\sigma + \varepsilon^{\lambda\nu\sigma\mu}k^\alpha\right)k_\alpha\Delta_{\lambda\gamma}(\bar{y})\Delta\beta_\nu(\bar{y},x)\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} \\ &\propto -\varepsilon^{\mu\alpha\lambda\nu}\left(k^\sigma k_\alpha - g^\sigma_\alpha k^2\right)\Delta_{\lambda\gamma}(\bar{y})\Delta\beta_\nu(\bar{y},x)\partial^\gamma_y\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k} .\end{aligned} \tag{4.52}$$

Using the definition of the projector $\Delta_{\lambda\gamma}$ in (4.22) we get:

$$\begin{aligned}\varepsilon^{\mu\alpha\lambda\nu}\Delta^{\ \gamma}_{\lambda} &= \varepsilon^{\mu\alpha\lambda\nu}\left[g^\gamma_\lambda - \frac{k^\sigma\hat{n}_\lambda(\bar{y})}{k\cdot\hat{n}(\bar{y})}\right] \\ &= \left(k^\lambda\varepsilon^{\mu\alpha\gamma\nu} - k^\gamma\varepsilon^{\mu\alpha\lambda\nu}\right)\frac{\hat{n}_\lambda(\bar{y})}{k\cdot\hat{n}(\bar{y})} \\ &= -\left(k^\mu\varepsilon^{\alpha\gamma\nu\lambda} + k^\alpha\varepsilon^{\gamma\mu\lambda\mu} + k^\nu\varepsilon^{\lambda\mu\alpha\gamma}\right)\frac{\hat{n}_\lambda(\bar{y})}{k\cdot\hat{n}(\bar{y})} ,\end{aligned}$$

where in the third line we made again use of the Schouten identity (4.50). Plugging the above in (4.52) and taking into account that:

$$\begin{aligned} k^\lambda \left(k^\mu k_\lambda - g^\mu_\lambda k^2\right) &= 0 \;, \\ k^\lambda \partial^\gamma_y \left[\frac{\hat{n}_\lambda(y)}{k\cdot\hat{n}(y)}\right]_{y=\bar{y}_k} &= \partial^\gamma_y \left[\mathrm{sgn}\left(k\cdot\hat{n}(\bar{y}_k)\right)\right] = 0 \;, \end{aligned}$$

the curvature term (4.52) finally reduces to:

$$\varepsilon^{\alpha\lambda\mu\gamma}\left(k^\sigma k_\alpha - g^\sigma_\alpha k^2\right)\frac{\hat{n}_\lambda(\bar{y})}{k\cdot\hat{n}(\bar{y})}\partial^y_\gamma\left[\frac{\hat{n}_\mu(y)}{k\cdot\hat{n}(y)}\right]_{y=\bar{y}_k} k^\nu \Delta\beta_\nu\left(\bar{y},x\right) \;. \tag{4.53}$$

Now lets define the following vector:

$$\begin{aligned} \Xi^\alpha(\bar{y}) &\equiv \varepsilon^{\alpha\lambda\mu\gamma}\frac{\hat{n}_\lambda(\bar{y})}{k\cdot\hat{n}(\bar{y})}\partial^y_\gamma\left[\frac{\hat{n}_\mu(y)}{k\cdot\hat{n}(y)}\right]_{y=\bar{y}_k} \\ &= \frac{\mathrm{sgn}\left(k\cdot\hat{n}(\bar{y})\right)}{\left(k\cdot\hat{n}(\bar{y})\right)^2}\varepsilon^{\alpha\lambda\mu\gamma}\hat{n}_\lambda(\bar{y})\partial^y_\gamma\left[\hat{n}_\mu(y)\right]_{y=\bar{y}_k} \;. \end{aligned} \tag{4.54}$$

This vector depends on the curvature of the decoupling hypersurface at the point $\bar{y}$. With this the full curvature contribution to the axial part of the Wigner function (4.53) can be written as:

$$\left(k^\sigma k_\alpha - g^\sigma_\alpha k^2\right) k^\nu \Delta\beta_\nu\left(\bar{y},x\right)\Xi^\alpha(\bar{y}) \;. \tag{4.55}$$

Now the above term is also vanishing. To prove it consider that from its definition the normal vector $\hat{n}$ can be written as:

$$\hat{n}_\mu(y) = A(y)\sigma_\mu(y) \;,$$

with A normalization factor and σ_μ normal vector to the hypersurface given in (4.19). Hence its derivative with respect to y reads:

$$\partial^y_\gamma\hat{n}_\mu(y) = A(y)\partial^y_\gamma\sigma_\mu(y) + \hat{n}_\mu(y)\partial^y_\gamma \ln A(y) \;.$$

The second term of the derivative, combined with the $\hat{n}_\lambda$ in (4.54) gives vanishing contribution due to the contraction with the Levi-Civita pseudotensor. However also the first term has vanishing contraction given that, from (4.19) one has:

$$\partial^y_\gamma\sigma_\mu = \begin{cases} 0 & \gamma = 0 \text{ or } \mu = 0 \;, \\ -\frac{\partial^2 f}{\partial y^i \partial y^j} & \gamma = i \;, \mu = j \;, \end{cases}$$

hence one conlcudes that:

$$\Xi^\alpha(\bar{y}) = 0 \;. \tag{4.56}$$

We thus have that also the second term in the curvature term does not contribute to the axial part of the Wigner function. Given that the same contribution from the vector (4.54) is present in the reduced chemical potential term, overall the curvature term does not contribute to the first order axial part of the Wigner function.

We can thus write the first order local equilibrium correction to the axial part of the Wigner function (4.48) as:

$$\Delta\mathcal{A}^{\sigma}_{\rm LE}(x,k) \simeq \frac{2\theta(k^0)\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\rm F}(k)\left[1-n_{\rm F}(k)\right]\frac{\hat{n}_\mu(y)\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha}{|k\cdot\hat{n}(y)|}\sum_{\bar{y}_k}\Delta_{\lambda\gamma}(\bar{y}_k)$$
$$\times\left\{\frac{1}{2}\left(k^\nu g^\mu_\rho + k^\mu g^\nu_\rho\right)\partial^\gamma_y\Big[\Delta\beta_\nu(y,x)\Big]_{y=\bar{y}_k} + \frac{1}{2}g^\mu_\rho\partial^\gamma_y\Big[\Delta\zeta(y,x)\Big]_{y=\bar{y}_k}\right\}. \tag{4.57}$$

Now we consider the gradients of the thermodynamic fields. For the four-temperature we have:

$$\frac{1}{2}\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\left(k^\nu g^\mu_\rho + k^\mu g^\nu_\rho\right)\Delta_{\lambda\gamma}(\bar{y})\hat{n}_\mu(\bar{y})\partial^\gamma_y\Big[\Delta\beta_\nu\left(y,x\right)\Big]_{y=\bar{y}_k}$$
$$= \frac{1}{2}\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\Delta_{\lambda\gamma}(\bar{y})\hat{n}_\rho(\bar{y})k^\nu\partial^\gamma_y\Big[\Delta\beta_\nu\left(y,x\right)\Big]_{y=\bar{y}_k}$$
$$+\frac{1}{2}\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\Delta_{\lambda\gamma}(\bar{y})|k\cdot\hat{n}(\bar{y})|\partial^\gamma_y\Big[\Delta\beta_\rho\left(y,x\right)\Big]_{y=\bar{y}_k}.$$

Now using the explicit expression of the projector $\Delta_{\lambda\gamma}$ the above reduces to:

$$\frac{1}{2}\varepsilon^{\sigma\rho\alpha\gamma}k_\alpha\hat{n}_\rho(\bar{y})k^\nu\partial^y_\gamma\Big[\Delta\beta_\nu\left(y,x\right)\Big]_{y=\bar{y}_k} + \frac{1}{2}\varepsilon^{\sigma\rho\alpha\gamma}k_\alpha|k\cdot\hat{n}(\bar{y})|\partial^y_\gamma\Big[\Delta\beta_\rho\left(y,x\right)\Big]_{y=\bar{y}_k}$$
$$-\frac{1}{2}\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\hat{n}_\lambda(\bar{y})k^\gamma\partial^y_\gamma\Big[\Delta\beta_\rho\left(y,x\right)\Big]_{y=\bar{y}_k},$$

whence, renaming the repeated indexes:

$$-\frac{1}{2}\varepsilon^{\sigma\rho\mu\nu}k_\rho\hat{n}_\mu(\bar{y})k^\lambda\left[\partial^y_\nu\beta_\lambda(y)\Big|_{y=\bar{y}} + \partial^y_\lambda\beta_\alpha(y)\Big|_{y=\bar{y}}\right]$$
$$+|k\cdot\hat{n}(\bar{y})|\frac{1}{2}\varepsilon^{\sigma\rho\mu\nu}k_\rho\partial^y_\mu\beta_\nu(y)\Big|_{y=\bar{y}}.$$

By using the definitions of the thermal vortcity and thermal shear:

$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right), \qquad \xi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu\right),$$

and reinstating the missing factor $2/|k\cdot n|$ the contribution from the gradient of the four-temperature reads:

$$-\,\mathrm{sgn}\left(k\cdot\hat{n}(\bar{y})\right)\varepsilon^{\sigma\rho\mu\nu}k_\rho\left[\varpi_{\mu\nu}(\bar{y}) + \frac{2\hat{n}_\mu(\bar{y})k^\lambda}{k\cdot\hat{n}(\bar{y})}\xi_{\nu\lambda}(\bar{y})\right]. \tag{4.58}$$

For the chemical potential term one simply has:

$$\frac{1}{2}\varepsilon^{\sigma\rho\alpha\lambda}k_\alpha\hat{n}_\rho(\bar{y})\Delta_{\lambda\gamma}(\bar{y})\partial^\gamma_y\zeta(y)\Big|_{y=\bar{y}} = -\frac{1}{2}\varepsilon^{\sigma\rho\mu\nu}k_\rho\hat{n}_\mu(\bar{y})\partial^y_\nu\zeta(y)\Big|_{y=\bar{y}},$$

which in turn implies that the contribution from the chemical potential is, reinstaiting the same missing term:

$$-\mathrm{sgn}\left(k\cdot\hat{n}(\bar{y})\right)\varepsilon^{\sigma\rho\mu\nu}k_\rho\left[\frac{\hat{n}_\mu(\bar{y})}{k\cdot\hat{n}(\bar{y})}\partial_\nu\zeta(y)\right]\Big|_{y=\bar{y}}. \tag{4.59}$$

Finally, plugging (4.58) and (4.59) in (4.57) we thus obtain the local equilibrium correction to the axial part of the Wigner function:

$$\begin{aligned}\Delta\mathcal{A}^{\sigma}_{\rm LE}(x,k) \simeq& -\frac{\theta(k^0)\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\rm F}(k)\left[1-n_{\rm F}(k)\right]\\ &\times\sum_{\bar{y}_k(x)} {\rm sgn}\left[k\cdot\hat{n}(\bar{y}_k(x))\right]\varepsilon^{\sigma\rho\mu\nu}k_\rho \\ &\times\left\{\varpi_{\mu\nu}(\bar{y}) + \frac{2\hat{n}_\mu(\bar{y})}{k\cdot\hat{n}(\bar{y})}\left[k^\lambda\xi_{\nu\lambda}(\bar{y}) - \partial_\nu\zeta(\bar{y})\right]\right\} .\end{aligned} \tag{4.60}$$

With this the numerator of (4.43) then reads:

$$\begin{aligned}\Delta\mathcal{N}^{\sigma}_{\rm LE}(k) \simeq& -\frac{\theta(k^0)\delta\left(k^2-m^2\right)}{(2\pi)^3}\varepsilon^{\sigma\rho\mu\nu}k_\rho\int_{\Sigma_{\rm D}} {\rm d}\Sigma(x)\cdot k\\ &\times\sum_{\bar{y}_k(x)} {\rm sgn}\left[k\cdot\hat{n}(\bar{y}_k(x))\right] n_{\rm F}(k,x)\left[1-n_{\rm F}(k,x)\right] \\ &\times\left\{\varpi_{\mu\nu}(\bar{y}_k(x)) + \frac{\hat{n}_\mu(\bar{y}_k(x))}{k\cdot\hat{n}(\bar{y}_k(x))}\left[2k^\lambda\xi_{\nu\lambda}(\bar{y}_k(x)) - \partial_\nu\zeta(\bar{y}_k(x))\right]\right\} .\end{aligned} \tag{4.61}$$

Finally plugging (4.45) and (4.61) in (4.43) we obtain the local equilibrium correction to the spin polarization vector:

$$\begin{aligned}S^\sigma(k) \simeq& -\frac{\varepsilon^{\sigma\rho\mu\nu}k_\rho}{8m{\rm N}_p}\int_{\Sigma_{\rm D}} {\rm d}\Sigma(x)\cdot k\, n_{\rm F}(k,x)\left[1-n_{\rm F}(k,x)\right]\\ &\times\sum_{\bar{y}_k(x)} {\rm sgn}\left[k\cdot\hat{n}(\bar{y}_k(x))\right]\left\{\varpi_{\mu\nu}(\bar{y}_k(x))\right. \\ &\left.+\frac{2\hat{n}_\mu(\bar{y}_k(x))}{k\cdot\hat{n}(\bar{y}_k(x))}\left[k^\lambda\xi_{\nu\lambda}(\bar{y}_k(x)) - \partial_\nu\zeta(\bar{y}_k(x))\right]\right\} .\end{aligned} \tag{4.62}$$

The three terms in eq. (4.62) are then identified as contributions of the thermal vorticity [154–156], the thermal shear tensor [157–160], and the spin Hall effect [127].

4.3.1 Isothermal decoupling

An important feature of the expression (4.62) for the spin polarization vector is that, under the assumption of *isothermal decoupling*, i.e. $T(x) = \text{const.}$, the contributions arising from temperature gradients vanish identically.

The thermal vorticity and thermal shear tensors can be decomposed as:

$$\varpi_{\mu\nu} = -\frac{1}{2T}\left(\partial_\mu u_\nu - \partial_\nu u_\mu\right) + \frac{1}{2T^2}\left(u_\nu\partial_\mu T - u_\mu\partial_\nu T\right) , \tag{4.63a}$$

$$\xi_{\mu\nu} = \frac{1}{2T}\left(\partial_\mu u_\nu + \partial_\nu u_\mu\right) - \frac{1}{2T^2}\left(u_\nu\partial_\mu T + u_\mu\partial_\nu T\right) . \tag{4.63b}$$

Substituting these expressions into (4.62) and separating explicitly the contributions associated with thermal vorticity and thermal shear, one obtains:

$$\begin{aligned}
S^{\sigma}_{\varpi}(k) &= -\frac{\varepsilon^{\sigma\rho\mu\nu}k_{\rho}}{8m\mathrm{N}_p}\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\, n_{\mathrm{F}}(k,x)\left[1-n_{\mathrm{F}}\left(k,x\right)\right] \\
&\quad \times \sum_{\bar{y}_k} \mathrm{sgn}(k\cdot\hat{n})\frac{1}{T}\left(\partial_{\mu}u_{\nu} + \frac{1}{T}u_{\mu}\partial_{\nu}T\right)\Big|_{y=\bar{y}_k} , \\
S^{\sigma}_{\xi}(k) &= -\frac{\varepsilon^{\sigma\rho\mu\nu}k_{\rho}}{8m\mathrm{N}_p}\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\, n_{\mathrm{F}}(k,x)\left[1-n_{\mathrm{F}}\left(k,x\right)\right] \\
&\quad \times \sum_{\bar{y}_k} \mathrm{sgn}(k\cdot\hat{n})\frac{\hat{n}_{\mu}k^{\lambda}}{T}\left(\partial_{\lambda}u_{\mu} + \partial_{\mu}u_{\lambda} - \frac{1}{T}\left(u_{\mu}\partial_{\lambda}T + u_{\lambda}\partial_{\mu}T\right)\right)\Big|_{y=\bar{y}_k} .
\end{aligned}$$

It is convenient to collect all terms proportional to gradients of the temperature. The resulting contribution reads:

$$\begin{aligned}
S^{\sigma}_{T}(k) &= -\frac{\varepsilon^{\sigma\rho\mu\nu}k_{\rho}}{8m\mathrm{N}_p}\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\, n_{\mathrm{F}}(k,x)\left[1-n_{\mathrm{F}}\left(k,x\right)\right] \\
&\quad \times \sum_{\bar{y}_k} \mathrm{sgn}(k\cdot\hat{n})\frac{1}{T^2}\left\{u_{\nu}\left[\partial_{\mu}T - \hat{n}_{\mu}\frac{k\cdot\partial T}{k\cdot\hat{n}}\right] - \frac{k\cdot u}{k\cdot\hat{n}}\hat{n}_{\mu}\partial_{\nu}T\right\}\Big|_{y=\bar{y}_k} .
\end{aligned} \tag{4.64}$$

The gradient of the temperature can be decomposed into components parallel and orthogonal to the normal vector $\hat{n}$:

$$\partial_{\mu}T = \hat{n}_{\mu}\frac{\hat{n}\cdot\partial T}{\hat{n}\cdot\hat{n}} + \partial^{\perp}_{\mu}T , \qquad \hat{n}^{\mu}\partial^{\perp}_{\mu}T = 0 .$$

The assumption of isothermal decoupling implies the absence of temperature variations along directions orthogonal to the decoupling hypersurface, i.e. $\partial^{\perp}_{\mu}T \equiv 0$. Under this condition one finds:

$$\begin{aligned}
&\partial_{\mu}T - \hat{n}_{\mu}\frac{k\cdot\partial T}{k\cdot\hat{n}} = 0 , \\
&\qquad \varepsilon^{\sigma\rho\mu\nu}\hat{n}_{\mu}\partial_{\nu}T \propto \varepsilon^{\sigma\rho\mu\nu}\hat{n}_{\mu}\hat{n}_{\nu}\,\hat{n}\cdot\partial T = 0 .
\end{aligned}$$

Substituting these relations into (4.64), one concludes that:

$$S^{\mu}_{T}(k) = 0 . \tag{4.65}$$

The assumption of isothermal decoupling implicitly requires that contributions associated with the reduced chemical potential be negligible. In general, for finite chemical potential, one has:

$$\partial_{\nu}\zeta = \frac{1}{T}\partial_{\nu}\mu - \frac{\mu}{T^2}\partial_{\nu}T ,$$

which implies the presence of an additional contribution to (4.64):

$$\begin{aligned}
S^{\sigma}_{\zeta}(k) &= \frac{\varepsilon^{\sigma\rho\mu\nu}k_{\rho}}{8m\mathrm{N}_p}\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma(x)\cdot k\, n_{\mathrm{F}}(k,x)\left[1-n_{\mathrm{F}}\left(k,x\right)\right] \\
&\quad \times \sum_{\bar{y}_k} \mathrm{sgn}(k\cdot\hat{n})\frac{\hat{n}_{\mu}}{T}\left(\partial_{\nu}\mu - \frac{\mu}{T}\partial_{\nu}T\right)\Big|_{y=\bar{y}_k} .
\end{aligned} \tag{4.66}$$

Assuming again $\partial^\perp T = 0$, only the term proportional to $\partial_\nu \mu$ survives. In general, this contribution depends both on gradients tangential to the hypersurface and on gradients orthogonal to it. Consequently, the assumption of isothermal decoupling also requires neglecting variations of the chemical potential across the decoupling hypersurface. This approximation is typically justified at high collision energies, where μ is small, but becomes increasingly inaccurate at lower energies, where finite chemical potential effects are significant and the decoupling process is generally non-isothermal.

4.4 Summary and Discussion

The equation (4.62) improves upon the existing formulae in the literature in two distinct ways. First, it provides a novel expression for the shear-induced polarization and spin-Hall effect. In contrast to previous derivations, where the factor $\hat{t}^\mu/(p\cdot\hat{t})$ in Refs. [157, 158] or $u^\mu/(k\cdot u)$ in Refs. [159, 160] (with $\hat{t}^\mu$ denoting the unit time vector in the QGP frame and u^μ the fluid four-velocity) emerges as a consequence of strong geometric approximations, it is here replaced by the fully covariant structure $\hat{n}^\mu/|k\cdot\hat{n}|$. The new expression reduces to that of Ref. [158] when $\Sigma_{\rm D}$ is a hyperplane, and to Ref. [160] when the velocity field is everywhere normal to $\Sigma_{\rm D}$. Therefore, previously obtained results are recovered as special cases corresponding to specific geometric assumptions. In this sense, the present result resolves the long-standing issue of the ambiguity in the shear-induced polarization formula, which had been highlighted in several works [161, 162].

As already observed in the case of the scalar field (4.26), the formula (4.62) also contains additional contributions arising from the non-trivial solution of Eq. (5.46), for which $\bar{y}(x,p) \neq x$. These terms are geometrically associated with particles moving inward across the hypersurface and subsequently traversing the fluid region [163–166]. Such contributions appear unphysical and may, within the present formalism, represent spurious effects caused by the use of free fields in the Wigner operator when describing the in-medium region in Eq. (3.60). While a careful analysis of these additional terms is left for future work, here we propose a practical prescription to discard them. In analogy with the classical Cooper–Frye formalism, where inward-moving particles are eliminated by introducing a cutoff $\theta(k\cdot\hat{n})$ [164–167], one may include the factor $\theta(k\cdot\hat{n}(x))\,\theta(k\cdot\hat{n}(\bar{y}))$ in Eq. (4.62). This procedure removes the additional intersections associated with inward trajectories.

Finally, Eq. (4.62) implies that both the vorticity-induced and shear-induced contributions receive terms proportional to derivatives of the β field along the normal direction $\hat{n}_\mu$. However, these contributions cancel when the two effects are combined. Consequently, if the decoupling hypersurface is isothermal and the normal vector n_μ is aligned with $\partial_\mu T$, temperature gradients do not contribute to the polarization. Within this improved formalism, the absence of temperature-gradient contributions at all orders of the expansion [157, 168, 169] therefore emerges naturally.

Chapter 5

Non-equilibrium scalar Wigner function

In this chapter we compute the full non-equilibrium correction to the scalar Wigner function by extending the method developed in the previous chapter to the case of a fully interacting theory. The inclusion of interactions is essential in order to consistently account for dissipative effects and has been discussed in detail in Chapter. 3.

In the hydrodynamic regime, we derive a gradient expansion of the Wigner function in terms of the derivatives of the *initial* thermodynamic fields, of the form (2.29). This expansion naturally includes both the local-equilibrium contribution and the dissipative corrections, together with all interaction effects. It therefore represents a direct generalization of the computation presented in Sec. 4.

This chapter is based on Ref. [70].

5.1 Non-equilibrium corrections

The full non-equilibrium correction to the Wigner function in linear response theory is obtained by replacing the Wigner operator $\widehat{W}$ with $\widehat{O}$ in the expression (2.49):

$$\langle \widehat{W}\,(x,k)\rangle \simeq \langle \widehat{W}\,(x,k)\rangle_{\rm GE} + \Delta W\,(x,k)\ . \tag{5.1}$$

The leading order term at global equilibrium has been computed for free fields in (4.2). If one more correctly consider full interacting fields the expression is modified. Taking the expectation value of (3.39) and using (3.47) and (3.48) we get:

$$\langle \widehat{W}\,(x,k)\rangle_{\rm GE} = \frac{2}{(2\pi)^4}\, n_{\rm B}(k)\, \varrho(k)\ . \tag{5.2}$$

Comparing the result with (4.2) one sees that the difference between the free and the interacting case is condensed in the spectral function at global equilibrium with four-temperature $\beta(x)$ and reduced chemical potential $\zeta(x)$. Note that in the interacting case also the global equilibrium expectation value to the Wigner function is generally off-shell, i.e $k^2 \neq m^2$.

The calculation of the linear response proceeds as for the local equilibrium term with replacing the free expansion with the interacting one. Plugging (3.39) in (5.1) we get:

$$\Delta W^{+}(x,k) = -\frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q \Big\{ \varrho(k_+)\varrho(k_-)\theta(k_+^0)\theta(k_-^0)\,\mathrm{e}^{\mathrm{i}q\cdot x} \times \int_0^1 \mathrm{d}z \Big[\Delta\beta_\nu(y,x) \left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\right\rangle_{c,\mathrm{GE}} - \Delta\zeta(y,x)\left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\widehat{j}^{\mu}(y)\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\right\rangle_{c,\mathrm{GE}} \Big]\Big\}. \tag{5.3}$$

The mixed terms depending on the thermal expectation values between $\widehat{A}^\dagger\widehat{B}^\dagger$ with $\widehat{T}^{\mu\nu}$ and $\widehat{j}^\mu$ and those with $\widehat{B}\widehat{A}$ as well has not been included because they will not play any role as we will show later.

The dependence from the hypersurface coordinate y can be take out from the thermal expectation value using (3.34) and (3.35) together with (4.6), which is valid also for interacting fields, so that, proceeding in analogy with the free case, we obtain:

$$\Delta W^{+}(x,k) = \frac{2}{(2\pi)^5}\int_{\Sigma_{\mathrm{D}}} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q \Big\{ \frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\,\mathrm{e}^{\mathrm{i}q\cdot(y-x)}\theta(k_+^0)\theta(k_-^0) \times \varrho(k_+)\varrho(k_-)\Big[\left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,\mathrm{GE}} \Delta\beta_\nu(y,x) - \left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{j}^{\mu}(0)\right\rangle_{c,\mathrm{GE}} \Delta\zeta(y,x)\Big]\Big\}. \tag{5.4}$$

The above expression can be compared with its analogous in the free case (4.8). Indeed replacing the spectral function with the free one (3.33) and taking into account the on-shellness condition of $k_\pm$, in the free case (5.4) reduces to (4.8). One may wrongly conclude that the only difference with the free case is then condensed only in the presence of the two interacting spectral functions in place of the free one. This is not the case for two key differences.

The first one is that $\widehat{T}^{\mu\nu}$ is the stress-energy tensor computed on the equilibrium hypersurface Σ_0 and not on the decoupling hypersurface Σ_{D}. This means that being in the full plasma phase it is not built in terms of fields and in particular is generally dominated by non-perturbative interactions. Hence the thermostatic expectation values cannot be computed as we did in Chapter 4.

The other difference is due to the fact that the thermal expectation value is not between the creation and annihilation operators of on-shell particles but those of off-shell interacting field excitations. This in particular implies striking differences with the free case because the possible kinetic constraints and the corresponding number of possible contributions increase exponentially.

5.2 Thermal form factors

The non-equilibrium correction (5.4) depends on the correlators:

$$\Theta^{\mu\nu}(k,q,\beta) \equiv \left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,\mathrm{GE}}, \tag{5.5a}$$

$$\Upsilon^{\mu}(k,q,\beta) \equiv \left\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{j}^{\mu}(0)\right\rangle_{c,\mathrm{GE}}, \tag{5.5b}$$

In order to compute the above expectation values one should be able to explicit the expression of the stress-energy tensor and the four-current which for the full QGP are unknown. In essence (5.5) are the extension of the gravitational and charged form factors at finite temperature and chemical potential and we will refer to them as *thermo-gravitational* and *thermo-charged* form factors.

We start considering the (5.5a). Its precise form is unknown without specific the precise dynamical underlying quantum system. However general covariance dictates that it must be a symmetric tensor built in terms of the possible vectors at our disposal, namely k^μ, q^μ and β^μ, on the pseudo-vector $a^\mu = \varepsilon^{\mu\alpha\beta\gamma}k_\alpha q_\beta \beta_\gamma$ and on the metric tensor $g^{\mu\nu}$ as well. The most general combination producing a symmetric tensor is thus:

$$\begin{aligned}\Theta^{\mu\nu}(k,q,\beta) = {} & \Theta_1(S)k^\mu k^\nu + \Theta_2(S)q^\mu q^\nu + \Theta_3(S)\beta^\mu\beta^\nu \\ & + \Theta_4(S)\left(k^\mu q^\nu + k^\nu q^\mu\right) + \Theta_5(S)\left(k^\mu\beta^\nu + k^\nu\beta^\mu\right) \\ & + \Theta_6(S)\left(q^\mu\beta^\nu + q^\nu\beta^\mu\right) + \Theta_7(S)g^{\mu\nu} + \Theta_8(S)\left(k^\mu a^\nu + k^\nu a^\mu\right) \\ & + \Theta_9(S)\left(q^\mu a^\nu + q^\nu a^\mu\right) + \Theta_{10}(S)\left(a^\mu\beta^\nu + a^\nu\beta^\mu\right) \ ,\end{aligned} \tag{5.6}$$

where the $\Theta_1(S),\dots,\Theta_{10}(S)$ are scalar coefficients depending on all the possible scalars one can build with the vectors and pseudo-vectors at disposal which we collectively denote with:

$$S = \left\{k^2,\, q^2,\, \beta^2,\, k\cdot q,\, k\cdot\beta,\, q\cdot\beta,\, \zeta\right\} \ . \tag{5.7}$$

Note that there are no pseudo-scalars one can form given that, by construction, the pseudo-vector a^μ is orthogonal to k, q and β. Also note that in (5.6) there is no term $a^\mu a^\nu$ because it is not independent from the others (see appendix D). The form of the thermal form factors (5.6) is also constrained by the properties of the equilibrium operator under complex conjugation, parity and time reversal (see appendix E):

$$\Theta^{\mu\nu}(k,q,\beta)^* = \mathrm{e}^{-\beta(x)\cdot q}\Theta^{\mu\nu}(k,-q,\beta) \ , \tag{5.8a}$$

$$\Theta^{\mu\nu}(k,q,\beta) = \mathrm{e}^{-\beta(x)\cdot q}\,\theta^\mu_\alpha\theta^\nu_\beta\,\Theta^{\alpha\beta}\left(\widetilde{k},-\widetilde{q},\widetilde{\beta}\right) \ , \tag{5.8b}$$

$$\Theta^{\mu\nu}(k,q,\beta) = \theta^\mu_\alpha\theta^\nu_\beta\,\Theta^{\alpha\beta}\left(\widetilde{k},\widetilde{q},\widetilde{\beta}\right) \ , \tag{5.8c}$$

where $\theta^\mu_\alpha = \mathrm{diag}\,(1,-1,-1,-1)$ is the transformation associated to the parity and $\widetilde{V}$ parity and time-reversal of V, four-momentum k, q or four-temperature β:

$$V = \left(V^0,\mathbf{V}\right) \mapsto \widetilde{V} = \left(V^0,-\mathbf{V}\right) \ .$$

Now taking into account that:

$$\theta^\mu_\alpha\theta^\nu_\beta\widetilde{V}_1^\alpha\widetilde{V}_2^\beta = V_1^\mu V_2^\nu \ , \qquad \theta^\mu_\alpha\theta^\nu_\beta\widetilde{V}_1^\alpha\widetilde{a}^\beta = -V_1^\mu a^\nu \ ,$$

and combining (5.6) with (5.8) we get:

$$\Theta_i(S) \in \mathbb{R} \ , \qquad \forall\, i = 1,\dots,10 \ , \tag{5.9a}$$

$$\Theta_i(S) = \mathrm{e}^{-\beta(x)\cdot q}\Theta_i(S)\Big|_{q\to -q} \qquad \forall\, i = 1,\dots,7 \ , \tag{5.9b}$$

$$\Theta_i(S) = -\Theta_i(S)\Big|_{q\to -q} = 0 \qquad \forall\, i = 8,\,9,\,10 \ , \tag{5.9c}$$

which then implies that the correlator (5.6) is real and comes down to:

$$\begin{aligned}\Theta^{\mu\nu}(k,q,\beta) = {} & \Theta_1(S)k^\mu k^\nu + \Theta_2(S)q^\mu q^\nu + \Theta_3(S)\beta^\mu\beta^\nu \\ & + \Theta_4(S)\left(k^\mu q^\nu + k^\nu q^\mu\right) + \Theta_5(S)\left(k^\mu\beta^\nu + k^\nu\beta^\mu\right) \\ & + \Theta_6(S)\left(q^\mu\beta^\nu + q^\nu\beta^\mu\right) + \Theta_7(S)g^{\mu\nu} \ .\end{aligned} \tag{5.10}$$

The remaining non-vanishing scalars Θ_i are in general unknown. However their value is constrained by general conservation laws. The (5.5a) represent the correlator with a conserved density hence is value is globally constrained. Integrating (5.5a) over an arbitrary space-like hypersurface we get using (2.10a):

$$\int_\Sigma \mathrm{d}\Sigma_\mu(y)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{\mu\nu}(y)\rangle_{c,\mathrm{GE}} = \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{P}^\nu\rangle_{c,\mathrm{GE}} \ , \tag{5.11}$$

with $\widehat{P}^\nu$ total four-momentum operator which also include all the interactions.

Given that $\widehat{T}$ is conserved the choice of Σ is arbitrary. Hence, choosing the hyperplane at $t=0$ the l.h.s of the above equation can be worked out as:

$$\begin{aligned}&\int_\Sigma \mathrm{d}\Sigma_\mu(y)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{\mu\nu}(y)\rangle_{c,\mathrm{GE}} = \int \mathrm{d}^3\mathbf{y}\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{0\nu}(0,\mathbf{y})\rangle_{c,\mathrm{GE}} \\ &= \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{0\nu}(0)\rangle_{c,\mathrm{GE}} \int \mathrm{d}^3\mathbf{y} e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{y}} \\ &= (2\pi)^3\delta^3(\mathbf{q})\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{T}^{0\nu}(0)\rangle_{c,\mathrm{GE}} \ ,\end{aligned} \tag{5.12}$$

where we used (4.6), and that $q = k_+ - k_-$. The correlator on the r.h.s of (5.11) can be related with the derivative with respect to the four-temperature of the global equilibrium operator (2.17). Taking into account (3.31) the interacting operators $\widehat{A}$ and $\widehat{B}$ are actually independent from the four-temperature β once multiplied by the spectral function, hence:

$$\varrho(k_+)\varrho(k_-)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{P}^\nu\rangle_{c,\mathrm{GE}} = -\frac{\partial}{\partial\beta_\nu}\left[\varrho(k_+)\varrho(k_-)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-)\rangle_{c,\mathrm{GE}}\right]$$

From the above equation, imposing $\varrho(k_\pm) = \varrho_{\mathrm{GE}}(k_\pm)$ with $\beta = \beta(x)$, by using the (3.46) the (5.11) and the (5.12), the following relation is obtained:

$$\begin{aligned}&\varrho(k^0_+,\mathbf{k})\varrho(k^0_-,\mathbf{k})\langle \widehat{A}^\dagger(k^0_+,\mathbf{k})\widehat{A}(k^0_-,\mathbf{k}),\widehat{T}^{0\nu}(0)\rangle_{c,\mathrm{GE}}(2\pi)^3\delta^3(\mathbf{q}) \\ &= -(2\pi)\delta^3(\mathbf{q})\delta(k^0_+ - k^0_-)\theta(k^0_+)\theta(k^0_-)\frac{\partial}{\partial\beta_\nu(x)}\left(n_{\mathrm{B}}(k)\varrho(k)\right) \ ,\end{aligned}$$

which implies:

$$\begin{aligned}&\langle \widehat{A}^\dagger(k^0_+,\mathbf{k})\widehat{A}(k^0_-,\mathbf{k}),\widehat{T}^{0\nu}(0)\rangle_{c,\mathrm{GE}} \\ &= \Theta^{0\nu}(k,q^0,\mathbf{q}=0,\beta) = \frac{\theta(k^0)}{(2\pi)^2}\frac{1}{\varrho^2(k)}\delta(q^0)\left[-\frac{\partial}{\partial\beta_\nu(x)}\left(n_B(k)\varrho(k)\right)\right] \ ,\end{aligned} \tag{5.13}$$

which vanishes for $q^0 \neq 0$ and for any value of k and $\beta(x)$. Hence all terms on the right hand side of (5.10) with $\mu = 0$ and $\nu \neq 0$ must be vanishing for $q^0 \neq 0$. This requirement constrains the coefficients Θ_i to be proportional to Dirac δ distributions such that they reduce to a $\delta(q^0)$ for $\mathbf{q} = 0$. Since these coefficients must be Lorentz scalars, we can write them in general as a sum over all possible delta distributions of the scalars S in eq. (5.7) multiplied by tensors $\Gamma_F^{\mu\nu}(k,q,\beta)$. In formulae:

$$\Theta^{\mu\nu}(k,q,\beta) = \int \mathcal{D}[F]\, \delta(F(S))\, \Gamma_F^{\mu\nu}(k,q,\beta)\,, \tag{5.14}$$

where $F(S)$ is a scalar functions such that:

$$\delta(F(S))\Big|_{\mathbf{q}=0} \propto \delta(q^0)\ ,$$

so as to fulfill the equation (5.13) and $\mathcal{D}[F]$ indicates the functional measure of these functions F. The above condition requires the functions $F(S)$ to vanish for $q=0$ and that they do not have zeroes with $q^0 \neq 0$ and $\mathbf{q}=0$, for any value of k and β. Furthermore, its derivative with respect to q^0 should not vanish in $q^0=0$:

$$\frac{\partial F(S)}{\partial q^0}\Big|_{q=0} = \frac{\partial F(S)}{\partial(q\cdot k)}\Bigg|_{q=0} k^0 + \frac{\partial F(S)}{\partial(q\cdot\beta)}\Bigg|_{q=0} \beta^0 + \lim_{q^0\to 0} \frac{\partial F_j(S)}{\partial q^2}\Bigg|_{\mathbf{q}=0} q^0 \neq 0\ , \tag{5.15}$$

which, for instance, rules out a term like $F(S) = q^2$. Nevertheless, in principle, there are infinite functions fulfilling those conditions. Assuming that the functions F are analytic in $q=0$ they can be expressed explicitly in terms of either $q\cdot k$ or $q\cdot\beta$ so that, for instance:

$$\delta(F(S)) = \delta(q\cdot k - f(S)) \left|\frac{\partial F(S)}{\partial(q\cdot k)}\right|^{-1}_{q\cdot k = f_j(S)}\ ,$$

where $f(S)$ is a function of the remaining scalars (S does not include $q\cdot k$ in the above example) that vanishes for $q=0$. This must be possible because if all the derivatives of F with respect to the above scalars involving q vanished for $q\to 0$, then the condition (5.15) would be violated. Note that the pre-factors such as $|\partial F/\partial(q\cdot k)|$ are scalars and can be re-absorbed into a re-definition of the scalar coefficients Θ_i without loss of generality. In the above case, the function $f(S)$ might include a linear term in $q\cdot\beta$ plus other terms, hence the argument of the delta distribution would be:

$$q\cdot k + Cq\cdot\beta + \text{other terms vanishing for }\ q=0$$

Similarly, if the delta distribution was $\delta(q\cdot\beta - f(S))$, the function $f(S)$ might include a linear term in $q\cdot k$. One can thus use linear combinations of $q\cdot k$ and $q\cdot\beta$ in the argument of the delta distribution and write the equation (5.14) in a different fashion combining both cases. Defining the four-vector w as the linear combination of k and β:

$$w^\mu(\vartheta) = \cos\vartheta \frac{k^\mu}{\sqrt{k^2}} + \sin\vartheta \frac{\beta^\mu}{\sqrt{\beta^2}}\ , \qquad \vartheta \in \left[0, \frac{\pi}{2}\right]\ , \tag{5.16}$$

so we can recast the equation (5.14) as:

$$\Theta^{\mu\nu}(k,q,\beta) = \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \int \mathcal{D}[f]\, \delta\left(q\cdot w(\theta) - f(S)\right)\, \Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)\ , \tag{5.17}$$

with the condition (5.15) now becoming:

$$\lim_{q\to 0} \frac{\partial}{\partial q^0} \left(q \cdot w(\vartheta) - f(S)\right) \neq 0 \ , \tag{5.18}$$

with $f(S)$ no longer including linear terms of the kind $q \cdot \beta$ and $q \cdot k$ [1]. Note that in the equation (5.16) the range of the ϑ angle is $[0, \pi/2]$ because of the requirement (5.18), which should apply for any value of k and β with both $k^0 > 0$ and $\beta^0 > 0$. The angles ϑ and the functions $f(S)$ involved in the formula (5.17) can form either a continuous or a discrete set; in case of a discrete set, the tensors $\Gamma^{\mu\nu}$ will feature delta factors such as $\delta(\vartheta - \vartheta_0)$ or delta distributions $\delta(f - f_0)$ for the functional measure $\mathcal{D}f$.

There are however some peculiar limitations on these functions owing to the continuity equation of the stress-energy tensor (2.8a). Indeed, by using Eq.(4.6), Eq.(3.46) and the ciclicity of the trace, it can be readily shown that:

$$\begin{aligned} 0 &= \frac{\partial}{\partial y^\mu} \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-), \widehat{T}^{\mu\nu}(y)\rangle_{c,\mathrm{GE}} = \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-), \widehat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}} \frac{\partial}{\partial y^\mu}\left(\mathrm{e}^{-\mathrm{i}q\cdot y}\right) \\ &= -\mathrm{i}q_\mu \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-), \widehat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}} \mathrm{e}^{-\mathrm{i}q\cdot y} \ , \end{aligned}$$

implying the transversality, or Ward identity, condition:

$$q_\mu \Theta^{\mu\nu}(k, q, \beta) = 0 \ , \qquad \forall\, k,\ q,\ \beta \ . \tag{5.19}$$

Plugging the equation (5.17) into the transversality condition leads to:

$$\delta(q \cdot w(\vartheta) - f(S)) q_\mu \Gamma^{\mu\nu}_{\vartheta,f}(k, q, \beta) = 0 \ , \qquad \forall\, \vartheta,\ f \ , \tag{5.20}$$

The combination of Eq.(5.10) and Eq. (5.17) implies that each tensor $\Gamma^{\mu\nu}_{\vartheta,f}(k, q, \beta)$ can be decomposed as:

$$\begin{aligned} \Gamma^{\mu\nu}_{\vartheta,f}(k, q, \beta) = {} & G_1(S)\, k^\mu k^\nu + G_2(S)\, q^\mu q^\nu + G_3(S)\, \beta^\mu \beta^\nu \\ & + G_4(S)\, (k^\mu q^\nu + k^\nu q^\mu) + G_5(S)\, (k^\mu \beta^\nu + k^\nu \beta^\mu) \\ & + G_6(S)\, (q^\mu \beta^\nu + q^\nu \beta^\mu) + G_7(S)\, g^{\mu\nu} \ , \end{aligned} \tag{5.21}$$

with suitable scalar coefficients $G_i(S)$. For each tensor $\Gamma^{\mu\nu}_{\vartheta,f}$ not all coefficients G_i appearing in Eq. (5.21) are independent. Indeed, the Ward identity (5.20) implies relations reducing the number of independent coefficients. These relations depend on the specific constraint imposed by the associated δ function; consequently, different tensors $\Gamma^{\mu\nu}_{\vartheta,f}$ satisfy different sets of relations among the corresponding G_i.

To systematically implement these constraints, we introduce a fundamental assumption: each tensor $\Gamma^{\mu\nu}_{\vartheta,f}$ appearing in the decomposition (5.17) is assumed to be an analytic function of the momentum q in a neighborhood of $q = 0$ (in the free case (4.24) this is indeed the case). More precisely each tensor $\Gamma^{\mu\nu}_{\vartheta,f}(k, q, \beta)$ is required to have a finite limit and be infinitely differentiable at $q = 0$. This assumption plays a crucial role in the following analysis.

Once the transversality conditions (5.19) are enforced under this assumption it then follows that, in the limit $q \to 0$, the only non-vanishing contributions arise from the terms on the right-hand side of Eq. (5.17) for which $f(S) = 0$. All remaining terms must vanish at $q = 0$. This conclusion is very important for a twofold reason:

[1] Note that, however, the function $f(S)$ may include terms such as $q^2/q \cdot k$

1. among all possible terms of the series in the equation (5.14), the delta distribution $\delta(q \cdot w(\vartheta))$ must exist in order to ensure the validity of the equation (5.13), see the derivation of the equation (5.26) below;

2. as it was shown in Chapter 4, there is a one-to-one correspondence between the order of the q expansion and the order of the hydrodynamic expansion in the gradients of the thermo-hydrodynamic fields β, ζ; the equation (5.24) tells us that all terms in the series except the one with $f(S) = 0$ do not contribute at the lowest orders of the gradient expansion.

We begin by discussing the aforementioned term: $\Gamma^{\mu\nu}_{\vartheta,0}$, corresponding to $f(S) = 0$. Contracting the tensor $\Gamma^{\mu\nu}_{\vartheta,f}$ with the four-vector q, using the decomposition (5.21) and enforcing vanishing, three constraints are obtained for the coefficients $G_i(S)$:

$$
\begin{aligned}
G_1(S) &= -\frac{q^2}{q \cdot k} G_4(S) - \frac{q \cdot \beta}{q \cdot k} G_5(S) \, , \\
G_7(S) &= -q^2 \, G_2(S) - q \cdot k \, G_4(S) - q \cdot \beta \, G_6(S) \, , \\
G_3(S) &= -\frac{q \cdot k}{q \cdot \beta} G_5(S) - \frac{q^2}{q \cdot \beta} G_6(S) \, .
\end{aligned}
$$

Plugging the above relations in (5.20) with $f(S) = 0$, imposing the constraint $q \cdot w(\vartheta) = 0$ and absorbing the denominators into a redefinition of the scalar factors so as to keep the analyticity of the scalar coefficients (for instance, one replaces G_5 and not G_1) the following expression is obtained:

$$
\begin{aligned}
\Gamma^{\mu\nu}_{\vartheta}(k, q, \beta) = \Gamma_1(\vartheta; S) \Big[& \cot\vartheta \sqrt{\frac{\beta^2}{k^2}} \, k^\mu k^\nu + \tan\vartheta \sqrt{\frac{k^2}{\beta^2}} \, \beta^\mu \beta^\nu + k^\mu \beta^\nu + k^\nu \beta^\mu \Big] \\
& + \Gamma_2(\vartheta; S) \left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) \\
& + \Gamma_3(\vartheta; S) \left[(q \cdot k)(k^\mu q^\nu + k^\nu q^\mu) - q^2 k^\mu k^\nu - (q \cdot k)^2 g^{\mu\nu} \right] \\
& + \Gamma_4(\vartheta; S) \left[(q \cdot \beta)(q^\mu \beta^\nu + q^\nu \beta^\mu) - q^2 \beta^\mu \beta^\nu - (q \cdot \beta)^2 g^{\mu\nu} \right] \, ,
\end{aligned}
$$

where the coefficients have been renamed Γ_i. The final expression of the tensor $\Gamma^{\mu\nu}_{\vartheta}$ is obtained by noting that the term multiplying the factor Γ_1 can be written as follows:

$$
\cot\vartheta \sqrt{\frac{\beta^2}{k^2}} \, k^\mu k^\nu + \tan\vartheta \sqrt{\frac{k^2}{\beta^2}} \, \beta^\mu \beta^\nu + k^\mu \beta^\nu + k^\nu \beta^\mu = \frac{\sqrt{k^2 \beta^2}}{\sin\vartheta \cos\vartheta} w^\mu(\vartheta) w^\nu(\vartheta) \, ,
$$

so that, absorbing also the pre-factor in Γ_1 we finally obtain:

$$
\begin{aligned}
\Gamma^{\mu\nu}_{\vartheta}(k, q, \beta) = \; & \Gamma_1(\vartheta; S) w^\mu(\vartheta) w^\nu(\vartheta) + \Gamma_2(\theta; S) \left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) \\
& + \Gamma_3(\vartheta; S) \left[(q \cdot k)(k^\mu q^\nu + k^\nu q^\mu) - q^2 k^\mu k^\nu - (q \cdot k)^2 g^{\mu\nu} \right] \\
& + \Gamma_4(\vartheta; S) \left[(q \cdot \beta)(q^\mu \beta^\nu + q^\nu \beta^\mu) - q^2 \beta^\mu \beta^\nu - (q \cdot \beta)^2 g^{\mu\nu} \right] \, .
\end{aligned}
\tag{5.22}
$$

with, according to the general analyticity assumption, all the coefficients Γ_i are analytic functions of the four-momentum q. These coefficients will be henceforth denoted as *thermo-gravitational form factors*. It is worth mentioning that in the limit of free fields the only delta distribution in the formula (5.17) is $\delta(q \cdot k)$ which is related to the on-shell condition. This corresponds to a single form factor for $\vartheta = 0$, that is all form factors $\Gamma_i(\vartheta; S)$ bear a $\delta(\vartheta)$ factor.

A similar calculation can be carried out if $f(S) \neq 0$ in eq. (5.17). Proceeding in the same way as in the previous case, one obtains:

$$\begin{aligned}
\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta) = \Gamma_{1,f}(\vartheta;S)\Big\{&(q\cdot k)(q\cdot\beta)w^\mu(\vartheta)w^\nu(\vartheta) \\
&- f(S)\Big[(q\cdot\beta)\frac{\cos\vartheta}{\sqrt{k^2}}k^\mu k^\nu + (q\cdot k)\frac{\sin\vartheta}{\sqrt{\beta^2}}\beta^\mu\beta^\nu\Big]\Big\} \\
&+ \Gamma_{2,f}(\vartheta;S)\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \\
&+ \Gamma_{3,f}(\vartheta;S)\left[(q\cdot k)\left(k^\mu q^\nu + k^\nu q^\mu\right) - q^2 k^\mu k^\nu - (q\cdot k)^2 g^{\mu\nu}\right] \\
&+ \Gamma_{4,f}(\vartheta;S)\left[(q\cdot\beta)\left(q^\mu\beta^\nu + q^\nu\beta^\mu\right) - q^2\beta^\mu\beta^\nu - (q\cdot\beta)^2 g^{\mu\nu}\right] .
\end{aligned} \tag{5.23}$$

Note that the factor $(q\cdot k)(q\cdot\beta)$ in Γ_1 stems from the request of analiticity of the final expressions; only for $f(S) = 0$ can that factor be reabsorbed in the definition of the form-factor. Since $f(S)$ ought to vanish for $q = 0$ and must be such that the conditions (5.15) are fulfilled, it must be at least linear in q, hence all terms in the equation (5.23) are at least quadratic in q. As a result, for $f(S) \neq 0$, $\Gamma^{\mu\nu}_{\vartheta,f}$ vanishes at $q = 0$ together with its first derivative with respect to q for any ϑ:

$$\Gamma^{\mu\nu}_{\vartheta,f}(k,0,\beta) = 0\,, \qquad \frac{\partial}{\partial q^\lambda}\Gamma^{\mu\nu}_{\vartheta,f}(\vartheta;k,q,\beta)\Big|_{q=0} = 0\,, \qquad \text{for } f(S)\neq 0\,. \tag{5.24}$$

Comparing the limit $q \to 0$ of the equations (5.22) and (5.23) we then conclude that, as confirmed in the (5.24), only when $f(S) = 0$ does a non-vanishing term appear. For $q = 0$, as it is apparent from the eq. (5.22), $\Gamma_1(\vartheta; k, 0, \beta)$ is the only relevant form factor and the thermo-gravitational correlators become, according to the equation (5.22) and (5.24):

$$\Theta^{\mu\nu}(k,0,\beta) = \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta\; w^\mu(\vartheta)w^\nu(\vartheta)\Gamma_1(\vartheta;k,0,\beta)$$

Now we can match the equations (5.17) and (5.13) to obtain:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\vartheta}{|w^0(\theta)|}\delta\left(q^0 - \frac{\mathbf{q}\cdot\boldsymbol{w}(\theta)}{w^0(\theta)}\right)\Big|_{\mathbf{q}=0}\Gamma^{0\nu}_{\vartheta}(k,0,\beta) \\
&= -\frac{\theta(k^0)}{(2\pi)^2\varrho^2(k)}\delta(q^0)\left[\frac{\partial}{\partial\beta_\nu(x)}\left(n_B(k)\varrho(k)\right)\right] ,
\end{aligned}$$

whence, taking into account the equation (5.22):

$$\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta\,\Gamma_1(\vartheta;k,0,\beta)\,w^\nu(\vartheta) = \frac{\theta(k^0)}{(2\pi)^2\varrho^2(k)}\frac{\partial}{\partial\beta^\nu}\left(n_{\mathrm{B}}(k)\varrho(k)\right) . \tag{5.25}$$

Since ϱ is a scalar function, it can only depend on $k^2,\ k\cdot\beta\ ,\beta^2$ hence:

$$\frac{\partial}{\partial\beta^\nu}\left[n_{\mathrm{B}}(k)\varrho(k)\right] = -n_{\mathrm{B}}(k)[1+n_{\mathrm{B}}(k)]\varrho(k)k^\nu + n_{\mathrm{B}}(k)\frac{\partial\varrho(k)}{\partial(k\cdot\beta)}k^\nu + 2n_{\mathrm{B}}(k)\frac{\partial\varrho(k)}{\partial\beta^2}\beta^\nu .$$

By using the definition (5.16) and equations (5.22) in the eq. (5.25), two integral constraints on the form factors $\Gamma_1(\vartheta; S)$ at $q = 0$ can be derived:

$$\frac{1}{\sqrt{k^2}} \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \, \cos\vartheta \, \Gamma_1(\vartheta; k, 0, \beta) = \frac{\theta(k^0) n_{\mathrm{B}}(k)}{(2\pi)^2 \varrho(k)} \left(1 + n_{\mathrm{B}}(k) - \frac{\partial \log \varrho(k)}{\partial (k \cdot \beta)} \right) , \quad (5.26a)$$

$$\frac{1}{\sqrt{\beta^2}} \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \, \sin\vartheta \, \Gamma_1(\vartheta; k, 0, \beta) = -\frac{2\theta(k^0)}{(2\pi)^2 \varrho(k)} n_{\mathrm{B}}(k) \frac{\partial \log \varrho(k)}{\partial \beta^2} . \quad (5.26b)$$

The value of the derivatives of the correlators in $q = 0$ is also constrained by complex conjugation, parity and time-reversal transformations (see Appendix E). It turns out that the tensor $\Gamma_\vartheta^{\mu\nu}$ in (5.22) fulfill the following relations:

$$\Gamma_\vartheta^{\mu\nu}(k, q, \beta) = \mathrm{e}^{-\beta(x)\cdot q} \Gamma_\vartheta^{\mu\nu}(k, -q, \beta) . \quad (5.27)$$

From the (5.27) it follows:

$$\frac{\partial}{\partial q^\lambda} \Gamma_\vartheta^{\mu\nu}(k, q, \beta) \Big|_{q=0} = -\frac{1}{2} \beta_\lambda(x) \Gamma_\vartheta^{\mu\nu}(k, 0, \beta) . \quad (5.28)$$

Much in the same way as for the correlator in eq. (5.13), for the correlator involving $\widehat{A}^\dagger \widehat{B}^\dagger$, the equation (3.47) implies:

$$\langle \widehat{A}^\dagger(k_+^0, \mathbf{k}) \widehat{B}^\dagger(-k_-^0, -\mathbf{k}), \ \widehat{T}^{0\nu}(0) \rangle_{c,\mathrm{GE}} = 0 . \quad (5.29)$$

For this correlator, in principle the same tensor decomposition in eq. (**??**) can be written, and from the equation (5.29), the conclusion trivially follows:

$$\langle \widehat{A}^\dagger(k_+^0, \mathbf{k}) \widehat{B}^\dagger(-k_-^0, -\mathbf{k}), \ \widehat{T}^{0\nu}(0) \rangle_{c,\mathrm{GE}} = 0 \quad \forall \, k, \, \beta \implies \Theta_{i\widehat{A}^\dagger \widehat{B}^\dagger} \equiv 0 .$$

so, the correlator $\langle \widehat{A}^\dagger(k_+) \widehat{B}^\dagger(-k_-), \ \widehat{T}^{\mu\nu}(0) \rangle_{c,\mathrm{GE}}$ and its complex conjugate vanish and do not play any role.

5.2.1 Charged form factors

A similar calculation can be carried out for the correlators involving the charged current. The correlator can be expanded in terms of the independent vectors k, q and β:

$$\langle \widehat{A}^\dagger(k_+) \widehat{A}(k_-), \ \widehat{\jmath}^\mu(0) \rangle_{c,\mathrm{GE}} \equiv Y^\mu(k, q, \beta) = Y_1(S) \, k^\mu + Y_2(S) \, q^\mu + Y_3(S) \, \beta^\mu . \quad (5.30)$$

Again a term proportional to the pseudo-vector a^μ is forbidden by parity and time-reversal while:

$$Y^\mu(k, q, \beta) = \mathrm{e}^{-\beta(x)\cdot q} \theta^\mu_\alpha Y^\alpha \left(\widetilde{k}, -\widetilde{q}, \widetilde{\beta} \right) , \quad (5.31a)$$

$$Y^\mu(k, q, \beta) = \theta^\mu_\alpha Y^\alpha \left(\widetilde{k}, \widetilde{q}, \widetilde{\beta} \right) . \quad (5.31b)$$

Since the charged current is a conserved according to the eq. (2.8b), for an arbitrary space-like hypersurface Σ we have a globally conserved charge operator:

$$\int_\Sigma \mathrm{d}\Sigma_\mu(y) \, \widehat{\jmath}^\mu(y) = \widehat{Q} . \quad (5.32)$$

Hence, integrating over the hyperplane at $t = 0$, we obtain:

$$\int_\Sigma \mathrm{d}\Sigma_\mu(y)\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{\jmath}^\mu(y)\rangle_{c,\mathrm{GE}} = \langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-),\ \widehat{Q}\rangle_{c,\mathrm{GE}}\,,$$

which in turn implies, in view of the form of the density operator (2.18) (see analogous derivation in the equations (5.12)-(5.13) above):

$$\langle \widehat{A}^\dagger\left(k^0_+,\mathbf{k}\right)\widehat{A}\left(k^0_-,\mathbf{k}\right),\ \widehat{\jmath}^{\,0}(0)\rangle_{c,\mathrm{GE}} = \frac{\theta(k^0)}{(2\pi)^2}\frac{1}{\varrho^2(k)}\delta(q^0)\frac{\partial}{\partial\zeta}\left(n_\mathrm{B}(k)\varrho(k)\right)\,. \tag{5.33}$$

From the conservation of the four-current (2.8b) the Ward identity ensues:

$$q_\mu Y^\mu\left(k,q,\beta\right) = 0\,, \qquad \forall\, k,\, q,\, \beta\,,$$

implying that the correlator (5.30) can be expanded much the same way as we have seen in eq. (5.14):

$$Y^\mu\left(k,q,\beta\right) = \int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\int\mathcal{D}[f]\,\delta\left(q\cdot w(\vartheta) - f(S)\right)\Upsilon^\mu_{\vartheta,f}\left(k,q,\beta\right)\,, \tag{5.34}$$

where S are all the scalars (5.7) and $f(S)$ is at least quadratic in q. The four-vectors $\Upsilon^\mu_{\vartheta,f}$ are assumed to be analytic functions of q so that they are finite and infinitely differentiable in $q = 0$.

Again, according to the condition (5.33), the only terms in the functional integral (5.34) which are non-vanishing for $q = 0$ are those associated with the $q\cdot w(\vartheta) = 0$ delta distributions:

$$\Upsilon^\mu_\vartheta\left(k,q,\beta\right) = \Upsilon_1(\vartheta;S)w^\mu(\vartheta) + \Upsilon^k_2(\vartheta;S)\left[\left(q\cdot k\right)q^\mu - q^2k^\mu\right]\,, \quad \text{for } f(S) = 0\,, \tag{5.35}$$

whereas all remaining $\Upsilon^\mu_{\vartheta,f}$ associated with $f(S) \neq 0$, along with their first order derivative in q, vanish for $q = 0$. Combining the (5.35) and the (5.33), we thus obtain:

$$\Upsilon_1\left(\vartheta;k,q=0,\beta\right) = -\frac{\theta(k^0)}{(2\pi)^2\varrho^2(k)}\frac{\partial}{\partial\zeta}\left(n_\mathrm{B}(k)\varrho(k)\right)\,. \tag{5.36}$$

Note that, in principle, for the four-current terms we are not able to determine separately the two coefficients in $q = 0$ like in the case of the stress energy tensor (5.26), but only their sum. However, in the free-limit, ϱ turns out to be independent from ζ and thus we can conclude that:

$$\frac{1}{\sqrt{k^2}}\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\,\cos\vartheta\,\Upsilon_1\left(\vartheta;k,0,\beta\right) = -\frac{\theta(k^0)n_\mathrm{B}(k)}{(2\pi)^2\varrho(k)}\left(1+n_\mathrm{B}(k)\right) + \Lambda\left(k,\beta\right)\,, \tag{5.37a}$$

$$\frac{1}{\sqrt{\beta^2}}\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\,\sin\vartheta\,\Upsilon_1\left(\vartheta;k,0,\beta\right) = -\frac{\theta(k^0)n_\mathrm{B}(k)}{(2\pi)^2\varrho(k)}\frac{\partial\log\varrho}{\partial\zeta} - \Lambda\left(k,\beta\right)\,, \tag{5.37b}$$

with $\Lambda\left(k,\beta\right)$ scalar function which must be vanishing.

Finally, due to the relations (5.31), using the results from appendix E, the vectors $\Upsilon_{k/\beta}$ fulfill:

$$\Upsilon^\mu_\vartheta\left(k,q,\beta\right) = \mathrm{e}^{-\beta(x)\cdot q}\,\Upsilon^\mu_\vartheta\left(k,-q,\beta\right)\,, \tag{5.38}$$

implying:

$$\partial^q_\lambda\,\Upsilon^\mu_\vartheta\left(k,q,\beta\right)\Big|_{q=0} = -\frac{1}{2}\beta_\lambda(x)\Upsilon^\mu_\vartheta\left(k,0,\beta\right)\,, \tag{5.39}$$

5.3 Hydrodynamic limit and gradient expansion

We can use the results of the foregoing Section to further develop the off-equilibrium correction of the Wigner function. With the definitions (5.5), taking into account the vanishing of the correlators involving $\widehat{A}^\dagger\widehat{B}^\dagger$ and plugging the (5.17) into the equation (5.4) a new expression is obtained:

$$\begin{aligned}\Delta W^+(x,k) = &\frac{2}{(2\pi)^5}\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \int \mathcal{D}[f] \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y) \\ &\times \int \mathrm{d}^4 q\; \varrho(k_+)\varrho(k_-)\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q\cdot(x-y)}\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} \\ &\times \delta(q\cdot w(\vartheta) - f(S))\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)\Delta\beta_\nu(y,x) - \Upsilon^{\mu}_{\vartheta,f}(k,q,\beta)\Delta\zeta(y,x)\right] .\end{aligned} \tag{5.40}$$

The formula (5.40) can be rewritten in a way which makes it apparent the effect of the hydrodynamic limit which is the extension of (4.12) to the general interacting case:

$$\begin{aligned}\Delta W^+(x,k) = &\frac{2}{(2\pi)^5}\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \int \mathcal{D}[f] \int \mathrm{d}^4 q\; \delta\left(q\cdot w(\vartheta) - f(S)\right) \\ &\times \left[G^{\mu\nu}_{\vartheta,f}\left(k,q,\beta\right) F^{(\beta)}_{\mu\nu}\left(x,q\right) - H^{\mu}_{\vartheta,f}\left(k,q,\beta\right) F^{(\zeta)}_{\mu}\left(x,q\right)\right] ,\end{aligned} \tag{5.41}$$

where:

$$G^{\mu\nu}_{\vartheta,f}\left(k,q,\beta\right) \equiv \theta(k_+^0)\theta(k_-^0)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\varrho(k_+)\varrho(k_-)\Gamma^{\mu\nu}_{\vartheta,f}\left(k,q,\beta\right) , \tag{5.42a}$$

$$H^{\mu}_{\vartheta,f}\left(k,q,\beta\right) \equiv \theta(k_+^0)\theta(k_-^0)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\varrho(k_+)\varrho(k_-)\Upsilon^{\mu}_{\vartheta,f}\left(k,q,\beta\right) , \tag{5.42b}$$

and:

$$F^{(\beta)}_{\mu\nu}\left(x,q\right) = \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\; \mathrm{e}^{iq\cdot(x-y)}\Delta\beta_\nu(y,x) , \tag{5.43a}$$

$$F^{(\zeta)}_{\mu}\left(x,q\right) = \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\; \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\Delta\zeta(y,x) . \tag{5.43b}$$

The above expressions are very similar to the one obtained for the local equilibrium case for both scalar fields and Dirac field (see chapter 4). The main difference is the presence of a infinite number of terms, each one with a different kinetic constraint, all of which multiplied by the same Fourier transform of the thermodynamic fields. Hence (5.41) is formally equivalent to (4.12) and one can deal with the integration of each term in a similar way outlined in section 4.1.1.

In the hydrodynamic limit, $\Delta\beta_\nu$, $\Delta\zeta$ and the normal vector $\hat{n}_\mu$ to the hypersurface Σ_0 are slowly varying functions in space and time, implying that the $F^{(\beta)}_{\mu\nu}$, $F^{(\zeta)}_{\mu}$, which are Fourier transform in the variable q integrated in the variable y, are functions peaked around $q^\mu = 0$. This makes it possible to obtain a good approximation of the (5.41) by expanding the functions $G^{\mu\nu}_{\vartheta,f}(q)$ and $H^{\mu}_{\vartheta,f}(q)$ around $q^\mu = 0$ in perfect analogy with the free case. The most important difference from a operational point of view is that the same procedure must in principle be performed for each different $\delta(q\cdot w - f)$.

First, we study the casese where the form factors present non-vanishing contributions at the lowest order in the q expansion, namely $f(S) = 0$. This case is exactly the same we computed for free fields with the vector $w(\vartheta)$ in place of k. Hence, denoting with the subscript $\vartheta, 0$ the term on the left hand side of (5.41) with $f = 0$ we can follow the same steps outlined in section 4.1.1 and obtain:

$$\Delta W^{+}_{\vartheta,0}(x,k) = \frac{2}{w^0(\vartheta)\,(2\pi)^2} \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \left[\partial^q_{\nu_1} \cdots \partial^q_{\nu_N} G^{\mu\nu}_{\vartheta,0}(k,q,\beta)\right]\Big|_{q=0}$$

$$\times \sum_{M=0}^{N} \frac{N!(-1)^M}{M!(N-M)!} \partial_x^{\nu_{M+1}} \cdots \partial_x^{\nu_N} \tag{5.44}$$

$$\times \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\, \delta^3\left(\mathbf{y} - \mathbf{x} - \frac{\boldsymbol{w}(\vartheta)}{w^0(\vartheta)}(y^0 - x^0)\right) \partial_x^{\nu_1} \ldots \partial_x^{\nu_M} \Delta\beta_\nu(y,x) \tag{5.45}$$

$$+ \text{ analogous term for } \Delta\zeta\ .$$

The key observation is again that the presence of the δ–function restricts the support of the integral to those points lying at the intersection between the equilibrium hypersurface and:

$$\mathbf{y} = \mathbf{x} - \frac{\boldsymbol{w}(\vartheta)}{w^0(\vartheta)}(y^0 - x^0)\ , \tag{5.46}$$

corresponding to the world-line of a free particle emitted from $x = (x^0, \mathbf{x})$ and propagating to $y = (y^0, \mathbf{y})$ with velocity $\boldsymbol{w}/w^0$. It is important to note, however, that since $k^2 \neq m^2$ the particle is off-mass-shell for every values of ϑ hence it is a virtual particle in the language of Feynman diagrams. This was the case also for the free theory where only the $\vartheta = 0$ contribution survives.

Now given that the integration is performed over the equilibrium hypersurface rather then the equilibrium one, Σ_0 is supposedly space-like hence there is at most one intersection between the world-line (5.46) and the hypersurface, that we will henceforth denote by $\bar{y}_\vartheta(x)$. Consequently, one will obtain for the $k \cdot q = 0$ kinetic branch the analogous of (4.23):

$$\Delta W^{+}_{\vartheta,0}(x,k) = \sum_{N=0}^{\infty} \frac{2\theta_\vartheta(x)}{(2\pi)^2} \frac{(-\mathrm{i})^N}{N!} \left[D_y(\bar{y}_\vartheta(x))\right]^N \left\{ \frac{\hat{n}_\mu(y)}{|w(\vartheta)\cdot\hat{n}(y)|} \right.$$

$$\left. \times \left[G^{\mu\nu}_{\vartheta,0}(k,q,\beta)\,\Delta\beta_\nu(y,x) - H^{\mu}_{\vartheta,0}(k,q,\beta)\,\Delta\zeta(y,x)\right] \right\} \Bigg|_{q=0,y=\bar{y}_\vartheta(x)} \tag{5.47}$$

where $\theta_\vartheta(x)$ is a Heaviside-like function:

$$\theta_\vartheta(x) = \begin{cases} 1 & \text{if an intersection point } \bar{y}_\vartheta(x) \text{ exists}\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

and where the differential operator D_y is defined as:

$$D_y(\bar{y}_\vartheta(x)) \equiv \Delta^{\nu\rho}(\bar{y}_\vartheta(x))\partial^y_\rho \partial^q_\nu\ , \qquad \Delta^{\nu\rho}(\bar{y}_\vartheta(x)) \equiv g^{\nu\rho} - \frac{\hat{n}^\nu(\bar{y}_\vartheta(x))k^\rho}{|k\cdot\hat{n}(\bar{y}_\vartheta(x))|}\ , \tag{5.48}$$

with $\hat{n}$ normal vector to the hypersurface Σ_0 and $\bar{y}_\vartheta(x)$ intersection between the worldline (5.46) and Σ_0. Hence it is the very same differential operator (4.22) just defined on the equilibrium hypersurface rather then on the decoupling one and computed at the intersection with the worldline (5.46) rather then with (4.18).

Note that the most important difference between (5.47) and (4.23) is the presence of θ_ϑ and of all the possible mixed contributions for $\vartheta \neq 0$. Indeed in the local equilibrium case at least one intersection is always possible, namely the trivial one $\bar{y}_{\vartheta=0} = x$. In this case x does not lie on the same hypersurface hence a trivial solution cannot exist and it is possible that no intersection, and thus no contribution, is present at all, for a given x and k.

In analogy with the local equilibrium correction then the equation (5.47) includes all linear terms in the gradients of $\Delta\beta, \Delta\zeta$ to all orders as well as gradients of the normal vector $\hat{n}$ to the hypersurface. Most importantly in (5.47) the gradients are evaluated on the initial equilibrium hypersurface Σ_0, which makes the (5.47) an expansion of the Wigner function of the kind (2.28) discussed in Section 2.3.

Again the crucial feature of (5.47) is that the order of the gradient of β, ζ, n is N, that is the order of the expansion in powers of q. Therefore, the q expansion of the functions (5.42) corresponds, order by order, to the expansion in gradients of this particular contribution to the Wigner function or, otherwise stated, the gradient expansion in the thermo-hydrodynamic (β, ζ) and geometric ($\hat{n}$) fields is generated by the expansion in q of the functions (5.42). More specifically, in the (5.47), the space-time gradients are coupled to derivatives in q of the same order for $q = 0$, according to the (5.48), so the vanishing of a q-gradient in $q = 0$ implies the vanishing of the corresponding term in the space-time gradient expansion.

In the assumption of general interacting stress energy tensor, but free fields in the Wigner function expansion, the (5.47) would be the only contribution to the full non-equilibrium correction to the Wigner function. However considering the generic interacting field expansion as we did other infinite terms arise as we had already mentioned.

The general case $f(S) \neq 0$ can be tackled by first solving the equation $q \cdot w(\vartheta) = f(S)$ with respect to q^0 so to turn the delta distribution in (5.40) into:

$$\delta(q \cdot w - f(S)) = \frac{1}{\chi(\mathbf{q})}\delta(q^0 - \varphi(\mathbf{q})) \; ;$$

the arguments k, β in the functions χ, φ are understood. In order to fulfill the requirement discussed in Section 5.2, that the delta distributions must reduce to a $\delta(q^0)$ for $\mathbf{q} = 0$, the function $\chi(\mathbf{q})$ must be non-vanishing for $\mathbf{q} = 0$ whereas $\varphi(\mathbf{q})$ is ought to vanish for $\mathbf{q} = 0$. Hence, the latter can be expanded as:

$$\varphi(\mathbf{q}) = \nabla_{\mathbf{q}}\varphi|_{\mathbf{q}=0} \cdot \mathbf{q} + R(\mathbf{q}) \equiv \mathbf{v} \cdot \mathbf{q} + R(\mathbf{q}) \; ,$$

where $R(\mathbf{q})$ is at least quadratic in the components q^i. Therefore, the single contribution $\Delta W^+_{\vartheta,f}$ in eq. (5.40) can be rewritten as:

$$\Delta W^+_{\vartheta,f}(x,k) = \frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q\ \varrho(k_+)\varrho(k_-)\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q\cdot(x-y)}\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}$$
$$\times\,\delta(q^0-\mathbf{v}_{\vartheta,f}\cdot\mathbf{q}-R_{\vartheta,f}(\mathbf{q})))\frac{1}{\chi(\mathbf{q})}\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)\Delta\beta_\nu(y,x)-\Upsilon^\mu_{\vartheta,f}(k,q,\beta)\Delta\zeta(y,x)\right]$$
$$=\frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^3\mathrm{q}\ \varrho(k_+)\varrho(k_-)\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{y}-\mathbf{x}-\mathbf{v}_{\vartheta,f}(y^0-x^0))}\mathrm{e}^{-\mathrm{i}R(\mathbf{q})(y^0-x^0)}$$
$$\times\,\frac{1}{\chi(\mathbf{q})}\left\{\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)\Delta\beta_\nu(y,x)-\Upsilon^\mu_{\vartheta,f}(k,q,\beta)\Delta\zeta(y,x)\right]\right\}\Bigg|_{q^0=\mathbf{v}_{\vartheta,f}\cdot\mathbf{q}+R_{\vartheta,f}(\mathbf{q})}.$$

To further proceed, two new vectors can be defined:

$$q' = q-r = q-(R_{\vartheta,f}(\mathbf{q}),\mathbf{0}) = (q^0-R_{\vartheta,f}(\mathbf{q}),\mathbf{q})\ , \qquad v_{\vartheta,f}=(1,\mathbf{v}_{\vartheta,f})\ .$$

The vector $r=(R_{\vartheta,f}(\mathbf{q}),\mathbf{0})$ can be seen as a function of either q or q' (since $\mathbf{q}=\mathbf{q}'$) and it has no dependence on the time component, i.e. $\partial r/\partial q^0 = \partial r/\partial q'^0 = 0$. $\Delta W^+_{\vartheta,f}(x,k)$ can be thus rewritten as:

$$\Delta W^+_{\vartheta,f}(x,k) = \frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q\ \delta(q'\cdot v_{\vartheta,f})\varrho(k_+)\varrho(k_-)\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q\cdot(x-y)}$$
$$\times\,\frac{1}{\chi(\mathbf{q})}\left\{\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q,\beta)\Delta\beta_\nu(y,x)-\Upsilon^\mu_{\vartheta,f}(k,q,\beta)\Delta\zeta(y,x)\right]\right\}.$$

The integration variable can be changed from q to q' by using the definition above, and the resulting Jacobian determinant is just 1, so that we can recast the above expression as:

$$\Delta W^+_{\vartheta,f}(x,k) = \frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q'\ \ \delta(q'\cdot v_j)\varrho(k_+)\varrho(k_-)$$
$$\times\,\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q'\cdot(x-y)}\mathrm{e}^{\mathrm{i}r(q')\cdot(x-y)}\frac{1}{\chi(\mathbf{q}')}\left\{\frac{1-\mathrm{e}^{\beta(x)\cdot(q'+r(q'))}}{\beta(x)\cdot(q'+r(q'))}\right.$$
$$\times\left.\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q'+r(q'),\beta)\Delta\beta_\nu(y,x)-\Upsilon^\mu_{\vartheta,f}(k,q'+r(q'),\beta)\Delta\zeta(y,x)\right]\right\}.$$

Then, expanding the exponential $\exp[\mathrm{i}r(q')\cdot(x-y)]$:

$$\Delta W^+_{\vartheta,f}(x,k) = \sum_{\ell=0}^{\infty}\frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q'\ \ \delta(q'\cdot v_{\vartheta,f})\varrho(k_+)\varrho(k_-)$$
$$\times\,\theta(k_+^0)\theta(k_-^0)\mathrm{e}^{\mathrm{i}q'\cdot(x-y)}(x^0-y^0)^\ell\frac{\mathrm{i}^\ell R^\ell(\mathbf{q})}{\ell!\chi(\mathbf{q}')}\left\{\frac{1-\mathrm{e}^{\beta(x)\cdot(q'+r(q'))}}{\beta(x)\cdot(q'+r(q'))}\right.$$
$$\times\left.\left[\Gamma^{\mu\nu}_{\vartheta,f}(k,q'+r(q'),\beta)\Delta\beta_\nu(y,x)-\Upsilon^\mu_{\vartheta,f}(k,q'+r(q'),\beta)\Delta\zeta(y,x)\right]\right\},$$

whence:

$$\Delta W^+_{\vartheta,f}(x,k) = \sum_{\ell=0}^{\infty}\frac{2}{(2\pi)^5}\int \mathrm{d}^4 q'\ \ \delta(q'\cdot v_j)\Big[G'^{\mu\nu}_{\vartheta,f\,(\ell)}(k,q',\beta)\,F'^{(\beta)}_{\mu\nu\,(\ell)}(q',x)$$
$$-\ H'^{\mu}_{\vartheta,f\,(\ell)}(k,q',\beta)\,F'^{(\zeta)}_{\mu\nu\,(\ell)}(q',x)\Big]\ . \qquad (5.49)$$

Each term of the series can be written in a form which is similar to the case with vanishing $f(S)\ \Delta W^+_{\vartheta,0}(x,k)$, by defining:

$$G'^{\mu\nu}_{\vartheta,f\,(\ell)}(k,q',\beta) \equiv \varrho(k_+)\varrho(k_-)\theta(k^0_+)\theta(k^0_-)\frac{\mathrm{i}^\ell R^\ell(\mathbf{q}')}{\chi(\mathbf{q}')} \times \frac{1-\mathrm{e}^{\beta(x)\cdot(q'+r(q'))}}{\beta(x)\cdot(q'+r(q'))}\Gamma^{\mu\nu}_{\vartheta,f}(k,q'+r(q'),\beta)\ , \tag{5.50a}$$

$$H'^{\mu}_{\vartheta\,(\ell)}(k,q',\beta) \equiv \varrho(k_+)\varrho(k_-)\theta(k^0_+)\theta(k^0_-)\frac{\mathrm{i}^\ell R^\ell(\mathbf{q}')}{\chi(\mathbf{q}')} \times \frac{1-\mathrm{e}^{\beta(x)\cdot(q'+r(q'))}}{\beta(x)\cdot(q'+r(q'))}\Upsilon^{\mu}_{\vartheta,f}(k,q'+r(q'),\beta)\ , \tag{5.50b}$$

and:

$$F'^{(\beta)}_{\mu\nu\,(\ell)}(x,q) = \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\ \mathrm{e}^{\mathrm{i}q'\cdot(x-y)}\Delta\beta_\nu(y,x)(x^0-y^0)^\ell\ , \tag{5.51a}$$

$$F'^{(\zeta)}_{\mu\,(\ell)}(x,q) = \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\ \mathrm{e}^{\mathrm{i}q'\cdot(x-y)}\Delta\zeta(y,x)(x^0-y^0)^\ell\ . \tag{5.51b}$$

The functions in eq. (5.51) are strongly peaked around $q'=0$ in the hydrodynamic limit, so those in (5.50) can be expanded around $q'=0$, just like in the previous cases. Because of the distribution $\delta(q'\cdot v)$ in the integral (5.49), all the conclusions previously achieved hold. The $\Delta W_{\vartheta,f}$ can be written in the form of an expansion, complicated as it may be, of the form (5.47), where the orders of the q' power expansion correspond to the gradients of the β,ζ and $\hat{n}$ fields evaluated at the intersection point between the hypersurface Σ_0 and the lines:

$$\mathbf{y} = \mathbf{x} - \mathbf{v}_{\vartheta,f}(k,\beta)(y^0-x^0)\ ,$$

where we have restored the previously understood dependence of v_j on k and β. For the functions $\Gamma_{\vartheta,f}(k,q,\beta)$ and $r(q)$ are at least quadratic in q, also the functions $G'_{\vartheta,f}(q')_{(\ell)}$ and $H'_{\vartheta,f}(q')_{(\ell)}$ turn out to be quadratic in q', implying that there are no contributions from the zeroth order and first order gradients in the (5.49).

In conclusion, stopping at the first order of the q expansion amounts to stop at the first order gradient expansion and, in this case, the only non-vanishing contributions to the off-equilibrium part of the Wigner function stems from the Γ_k and Γ_β form factors. In formulae, the correlators in the equation (5.17) read:

$$\Theta^{\mu\nu}(k,q,\beta) = \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta\,\delta\left(q\cdot w(\vartheta)\right)\,\Gamma^{\mu\nu}_\vartheta(k,q,\beta) + \int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\int_{f\neq 0}\mathcal{D}[f]\delta\left(q\cdot w(\vartheta)-f(S)\right)\mathcal{O}(q^2)\ ,$$

and the off-equilibrium correction, neglecting $\mathcal{O}(q^2)$ terms, turns out to be:

$$\Delta W^+(x,k) \simeq \frac{2}{(2\pi)^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\vartheta\sum_{N=0}^{1}\frac{(-\mathrm{i})^N\,\theta_\vartheta(x)}{N!}\left[D_y(\bar{y}_\vartheta(x))\right]^N \left\{\frac{n_\mu(y)}{|w(\vartheta)\cdot n(y)|}\left[G^{\mu\nu}_\vartheta(k,q,\beta)\,\Delta\beta_\nu(y,x) - H^\mu_\vartheta(k,q,\beta)\,\Delta\zeta(y,x)\right]\right\}\Bigg|_{\substack{q=0\\ y=\bar{y}_k(x)}}\ . \tag{5.52}$$

So, at least at first order in the gradient, the full non-equilibrium correction to the Wigner function has a general form which is very similar to the local equilibrium case (4.23), with the major differences coming from the additional terms with $\vartheta \neq 0$[2] and the general unknown form of the functions G_ϑ and H_ϑ depending on the thermo-gravitational and thermo-charged form factors.

It should be stressed, however, that if the hypersurface is curved there are contributions at the lowest order gradients (zeroth and first order) of the thermo-hydrodynamic fields β and ζ from all $N > 1$ terms, obtained by letting derivatives ∂_y in the above corrections (5.47) and (5.49) to act on the field $n^\mu(y)$. Only if the curvature of the hypersurface is small such terms can be regarded as small corrections of the main terms with $N = 0, 1$ [170]. We will henceforth disregard those terms, keeping in mind though that their potential relevance should be considered for moderately curved hypersurfaces. Usually the shape of the equilibrium hypersurface, compared with the decoupling one, in HIC is much smoother and resemples a branch of hyperbola in proper time hence this curvature term are probably less important then the one at local equilibrium computed at the decoupling.

Up to the first order in the $q-$derivatives the two functions $G_\vartheta^{\mu\nu}$ (5.42) can be expanded taking into account:

$$\begin{aligned}
&\frac{1 - \mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} = -1 - \frac{1}{2}\beta_\tau(x) q^\tau + \mathcal{O}(q^2) , \\
&\Gamma_\vartheta^{\mu\nu}(k, q, \beta) = \Gamma_\vartheta^{\mu\nu}(k, 0, \beta) + q^\tau \frac{\partial}{\partial q^\tau}\Gamma_\vartheta^{\mu\nu}(k, q, \beta)\bigg|_{q=0} + \mathcal{O}(q^2) , \\
&\varrho(k_+)\varrho(k_-) = \varrho^2(k) + \mathcal{O}(q^2) ,
\end{aligned}$$

where $k_\pm = k \pm q/2$. Similarly we can expand the tensors Υ_ϑ^μ and the functions H_ϑ^μ so the functions $G_\vartheta^{\mu\nu}$ and H_ϑ^μ , at lowest order in q, are given by: (5.26):

$$\begin{aligned}
G_\vartheta^{\mu\nu}(k, 0, \beta) &= -\varrho^2(k)\theta(k^0)\,\Gamma_1(\vartheta; k, 0, \beta)\, w^\mu(\vartheta) w^\nu(\vartheta) , \\
H_\vartheta^\mu(k, 0, \beta) &= -\varrho^2(k)\theta(k^0)\Upsilon_1(\vartheta; k, 0, \beta)\, w^\mu(\vartheta) .
\end{aligned}$$

Plugging these expressions in (5.52) and taking the $N = 0$ term we then obtain the off-equilibrium correction to the Wigner function at the leading order of the gradient expansion which reads:

$$\begin{aligned}
\Delta^{(0)} W^+(x, k) \simeq \frac{2\varrho^2(k)}{(2\pi)^2}\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta\, \theta_\vartheta(x) \bigg\{ &\Gamma_1(\vartheta; S_0)\left[w(\vartheta)\cdot\beta(x) - w(\vartheta)\cdot\beta(\bar{y}_\vartheta(x))\right] \\
&- \Upsilon_1(\vartheta; S_0)\left[\zeta(x) - \zeta(\bar{y}_\vartheta(x))\right] \bigg\} .
\end{aligned} \tag{5.53}$$

The first order correction in the gradient expansion is given by the $N = 1$ term in

[2] We stress again that the origin of these additional terms comes from having used an interacting expansion for the fields in the Wigner function and not from the fact that the stress-energy tensor is not the free one. These terms would be entirely missed if one would have used free fields in the Wigner operator as has been usually done *even* considering a general interacting stress tensor.

(5.47):

$$\Delta^{(1)}W^{+}(x,k) = -\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \frac{\theta_\vartheta(x)}{(2\pi)^2}$$
$$\times \left\{ \partial^q_\sigma G^{\mu\nu}_\vartheta(k,q,\beta)\Big|_{q=0} \Delta^{\sigma\gamma}(\bar{y}_\vartheta(x))\partial^y_\gamma \left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|w(\vartheta)\cdot\hat{n}(y)|}\right]\Bigg|_{y=\bar{y}_\vartheta(x)} \right.$$
$$\left. -\,\partial^q_\sigma H^{\mu}_\vartheta(k,q,\beta)\Big|_{q=0} \Delta^{\sigma\gamma}(\bar{y}_\vartheta(x))\partial^y_\gamma \left[\frac{\hat{n}_\mu(y)\Delta\zeta(y,x)}{|w(\vartheta)\cdot\hat{n}(y)|}\right]\Bigg|_{y=\bar{y}_\vartheta(x)} \right\}.$$

By using the relations (5.27) and (5.38) and

$$\frac{\partial}{\partial q^\sigma}\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\Bigg|_{q=0} = -\frac{1}{2}\beta_\sigma$$
$$\frac{\partial}{\partial q^\sigma}\varrho(k_+)\varrho(k_-)\Bigg|_{q=0} = 0$$

one readily obtains from the (5.42) with $f(S)=0$:

$$\partial^q_\sigma G^{\mu\nu}_\vartheta(k,q,\beta)\Big|_{q=0} = \varrho^2(k)\left[-\frac{1}{2}\beta_\sigma(x)\Gamma^{\mu\nu}_\vartheta(k,0,\beta) - \partial^q_\lambda\Gamma^{\mu\nu}_\vartheta(k,q,\beta)\Big|_{q=0}\right] = 0\ ,$$
$$\partial^q_\lambda H^{\mu\nu}(\vartheta;k,q,\beta)\Big|_{q=0} = \varrho^2(k)\left[-\frac{1}{2}\beta_\lambda(x)H^{\mu\nu}(\vartheta;k,0,\beta) - \partial^q_\lambda H^{\mu\nu}(\vartheta;k,q,\beta)\Big|_{q=0}\right] = 0\ ,$$

therefore:

$$\Delta^{(1)}W(x,k) = 0\ .$$

Higher order terms in the gradient expansion of the Wigner function are far more complicated as they depend on all the form factors and on delta channels $\delta(q\cdot w - f(S))$ with $f(S)\neq 0$ in the equation (5.41).

We conclude this section by giving the explicit expression of the zeroth-order correction (5.53) in the limiting case of free fields. Since the only kinetic constraint is $q\cdot k=0$, one has $\vartheta\equiv 0$. Combining (5.26) with (3.33), we obtain

$$\Delta^{(0)}_{\mathrm{free}}W(x,k) = \frac{\delta\left(k^2-m^2\right)}{(2\pi)^3}\theta_k(x)n_{\mathrm{B}}(k)\left(1+n_{\mathrm{B}}(k)\right)$$
$$\times\left[k\cdot\beta(x) - k\cdot\beta(\bar{y}_k(x)) - \zeta(x) + \zeta(\bar{y}_k(x))\right]\ , \tag{5.54}$$

where $\bar{y}_k(x)$ denotes the intersection between the free worldline (4.18) and the decoupling hypersurface. The function $\theta_k(x)$ vanishes for those values of k and x for which such an intersection does not exist.

5.4 Discussion

In the previous section we have obtained a gradient expansion of the off-equilibrium correction of the Wigner function where all the gradients are evaluated on the initial equilibrium hypersurface instead of the final one. In a sense, we have obtained a constitutive equation of the Wigner function - that is a relation with the geometric and the thermo-hydrodynamic fields - which is non-local in time. More precisely, we have found the leading order solution of the equation of motion of the Wigner function which parametrically depends on the initial conditions, i.e. the fields β and ζ on Σ_0. Surprisingly, according to the equation (5.53), the gradient expansion includes a zeroth order term, which depends on the finite difference between the thermo-hydrodynamic fields at the point x and over a point lying on the initial hypersurface Σ_0. This term suggests that memory effects are present in the full quantum statistical approach to the calculation of the Wigner function, hence of the momentum spectrum, at the decoupling.

It might be argued that if we had started from the traditional decomposition of the density operator (2.38) we would have obtained an expansion in gradients evaluated at the same point x of the Wigner function. As has been mentioned in Section 2.3, this result crucially depends on the shape of the correlation functions in the equation (2.35), i.e. whether it has a maximum at $y \sim x$. In this Section, we will show that this does not necessarily occur for the Wigner function and so, even if we had used the decomposition into local equilibrium and dissipative terms, we would eventually get the same expansion in terms of the initial gradients.

In this Section, we will delve into the features of the off-equilibrium correction found for the Wigner function, including the aforementioned non-locality.

5.4.1 Non-Locality of the correlators

The non-locality of the correlation function can be understood rewriting the equation (5.40) by using the definitions (5.42), as:

$$\begin{aligned}\Delta W^+(x,k) &= \frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\Delta\beta_\nu(y,x) \\ &\times \left(\int \mathrm{d}^4q \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \int \mathcal{D}[f]\,\delta(q\cdot w(\vartheta) - f(S))\mathrm{e}^{\mathrm{i}q\cdot(x-y)} G^{\mu\nu}_{\vartheta,f}(k,q,\beta)\right) ,\end{aligned} \tag{5.55}$$

where we omitted the the term in the chemical potential for the sake of simplicity. By comparison with e.g. eq. (2.49), the equation (5.55) identifies the correlation function of the Wigner operator and the stress-energy tensor operator:

$$\begin{aligned}C^{\mu\nu}_{WT}(x-y,k) &= -(2\pi)^5\int_0^1 \mathrm{d}z\,\langle \widehat{W}^+(x,k), \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\rangle_{c,\mathrm{GE}} \\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \int \mathcal{D}[f] \int \mathrm{d}^4q\,\delta(q\cdot w(\vartheta) - f(S))\mathrm{e}^{\mathrm{i}q\cdot(x-y)} G^{\mu\nu}_{\vartheta,f}(k,q,\beta)\,.\end{aligned} \tag{5.56}$$

This correlation function appears, once $\widehat{\mathcal{E}}_{\mathrm{GE}}$ is approximated by $\widehat{\mathcal{E}}$, in the equation (2.38) as well; a similar expression hold for the four-current term.

It is common wisdom that such a function decays rapidly for macroscopic distance $(x-y)^2$, that is much larger than the typical microscopic scales, such as $1/m$, $\sqrt{\beta^2}$, interactions lengths and combinations thereof. In fact, because each term of the series in the eq. (5.56) can be rearranged so as to contain $\delta(q\cdot w)$, or $\delta(q\cdot v_{\vartheta,f}(k,\beta))$ in general - as has been shown in the Section 5.3 - this is not possibly the case. Because of these distributions, the various terms of the correlation function (5.56) turn out to be constant over the lines:

$$(x-y)^\mu = \tau\, w^\mu(\vartheta) \ ,$$

for $\delta(q\cdot w)$, where τ is a real parameter. For $\delta(q\cdot v_{\vartheta,f}(k,\beta))$ also the correlation function is constant along the world-line $(x-y)^\mu = \tau v^\mu_{\vartheta,f}$, provided that the correlation function is re-defined by letting a factor $(x^0-y^0)^\ell$ with $\ell \geq 1$ to be extracted from it, see Section 5.3. For each of the contributing terms labeled by ϑ, f in equation (5.56) thus the point $y = x$ is not an isolated maximum and the intersection points between the above lines and the hypersurface Σ_0 are responsible for the largest contribution to the integral (5.55).

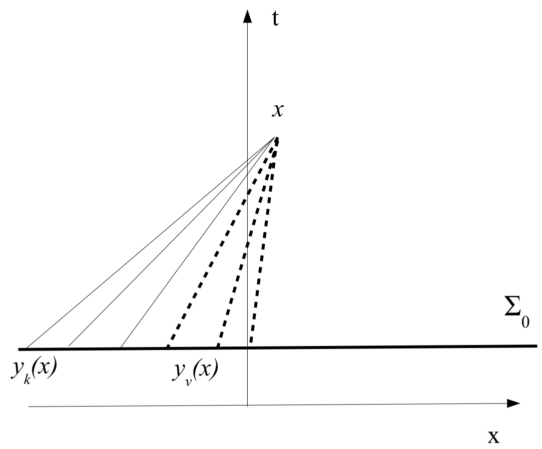


Figure 5.1: The correlation function between the Wigner operator $\widehat{W}^+(x,k)$ and stress-energy tensor (or vector current) operator in a point y features terms which are constant over the worldlines with tangent vector k (solid lines) and terms which are constant over worldlines with tangent vector $v(k,\beta)$ (dashed lines). The integration over k eventually yields correlation functions which are strongly peaked around $y = x$.

This non-locality feature is at odds with the familiar one when considering the correlation function of two local operators $\widehat{O}_1$ and $\widehat{O}_2$ depending on the quantum fields:

$$C_{O_1O_2}(x-y) = \int_0^1 \mathrm{d}z\ \langle \widehat{O}_1(x),\ \mathrm{e}^{z\widehat{\mathcal{E}}_{\rm GE}}\, \widehat{O}_2(y)\, \mathrm{e}^{-z\widehat{\mathcal{E}}_{\rm GE}} \rangle_{c,\rm GE} \ ,$$

e.g. two components of conserved currents, which has a typical maximum for $x \sim y$ with a width driven by microscopic lengths. The reason for the different behaviour between an actual local operator and the Wigner operator as to their correlation with another local operator is that the Wigner operator is not truly local, being the Fourier transform of the product of field operators in two points, see eq.(3.10). Furthermore, the Wigner operator also depends on an additional argument k besides space-time point x.

It should be emphasized that if we had used the traditional method of decomposing the density operator with the Gauss theorem, separating the local equilibrium from the dissipative contribution like in Eq.(2.26), we would not get an expansion in terms of the gradients at the point x anyway, in fact we would get the same expansion in terms of the initial gradients including the zeroth order term. An explicit calculation, starting from from Eq.(2.26) and proceeding to combine the local equilibrium with the dissipative corrections is reported in Section 2.38. The method of expanding the density operator like in eq. (2.26) is tantamount to expand the off-equilibrium correction of the Wigner operator in Eq.(5.55) with the Gauss theorem; the resulting expression is:

$$\begin{aligned}\Delta W^{+}(x,k) = &\int_{\Sigma_x} \mathrm{d}\Sigma_\mu(y)\ \Delta\beta_\nu(y,x) C^{\mu\nu}_{WT}(x-y,k)\\ &+ \int_\Omega \mathrm{d}^4 y\ \Delta\beta_\nu(y,x)\frac{\partial}{\partial y^\mu} C^{\mu\nu}_{WT}(y-x,k)\\ &+ \int_\Omega \mathrm{d}^4 y\ C^{\mu\nu}_{WT}(y-x,k)\frac{\partial}{\partial y^\mu}\Delta\beta_\nu(y,x)\ ,\end{aligned}$$

where again the contribution proportional to $\Delta\zeta$ has been neglected for the sake of simplicity. Unless the correlation function is narrow-peaked around $y \sim x$, we cannot neglect the second integral on the right hand side nor can we approximate the third integral by extracting the gradient of β in x. Therefore, a consistent elaboration of the above equation taking into account the shape of the correlation function again leads to the Eq.(5.55).

The long-distance persistence of the correlation function (5.56) makes it apparent that memory effects, which are manifest in the term (5.53), play an important role for the Wigner function in an expanding fluid. Nevertheless, it is reasonable to expect that if we generate an actual local operator by integrating the Wigner operator in the momentum variable k, a rapidly decaying correlation function in $(x-y)$ occurs and the typical behaviour is restored. This happens because one integrates the correlation functions $C^{\mu\nu}_{WT}(x-y,k)$ and $C^{\mu}_{Wj}(x-y,k)$ over an infinite set of lines with fixed k, all of them converging in x (see figure 5.1). We show that this is a likely result by working out an example for the specific case $\delta(q\cdot w - f(S)) = \delta(q\cdot k)$ and the local operator:

$$\int \mathrm{d}^4 k\ \widehat{W}(x,k) =: \widehat{\phi}^\dagger(x)\widehat{\phi}(x) :\ ,$$

considering its correlation function with the stress-energy tensor $\widehat{T}^{\mu\nu}(y)$. The constraint $\delta(q\cdot k)$ is significant because is the only one surviving in the free-limit.

A contributing term to the this function is obtained by integrating the $\widehat{W}^{+}(x,k)$ part of the Wigner operator (the anti-particle and space-like parts in eq. (3.12) should also be included) in equation (5.56), that is:

$$\begin{aligned}&-(2\pi)^5\int_0^1 \mathrm{d}z\ \langle :\widehat{\phi}^\dagger(x)\widehat{\phi}(x):,\ \mathrm{e}^{z\widehat{\mathcal{E}}_{\mathrm{GE}}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-z\widehat{\mathcal{E}}_{\mathrm{GE}}}\rangle_{c,\mathrm{GE}}\\ &= \int \mathrm{d}^4 k \int \mathrm{d}^4 q\ \mathrm{e}^{\mathrm{i}q\cdot(x-y)} G^{\mu\nu}(k,q,\beta)\ .\end{aligned}$$

Expanding the function $G(k,q,\beta)$ and proceeding like in the equation (5.44) and following we obtain:

$$\begin{aligned}
&\int \mathrm{d}^4k \int \mathrm{d}^4q \; \delta(q\cdot k)\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)} G_k^{\mu\nu}(k,q,\beta) \\
&= \sum_{N=0}^{\infty} \frac{1}{N!} \partial_x^{\nu_1}\partial_x^{\nu_2}\cdots\partial_x^{\nu_N} \int \mathrm{d}^4k \left[\partial^q_{\nu_1}\partial^q_{\nu_2}\ldots\partial^q_{\nu_N} G^{\mu\nu}(k,q,\beta)\right]\Big|_{q=0} \\
&\times \frac{(-\mathrm{i})^N}{k^0}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}(y^0-x^0)\right) \\
&= \sum_{N=0}^{\infty} \frac{(-\mathrm{i})^N}{N!} \partial_x^{\nu_1}\partial_x^{\nu_2}\cdots\partial_x^{\nu_N} \int \mathrm{d}k^0 \frac{(k^0)^2}{(y^0-x^0)^3} \\
&\times \left\{\partial^q_{\nu_1}\partial^q_{\nu_2}\ldots\partial^q_{\nu_N} G^{\mu\nu}\left[\left(k^0, \frac{k^0(\mathbf{y}-\mathbf{x})}{y^0-x^0}\right), q, \beta\right]\right\}\Big|_{q=0} .
\end{aligned}$$

The general term of last expression is difficult to work out, however the term $N=0$ can be expanded based on the equation (5.26). This term becomes:

$$\int \mathrm{d}k^0 \frac{(k^0)^2}{(y^0-x^0)^3}\theta(k^0)\varrho(k)\frac{1}{(2\pi)^2} n_{\mathrm{B}}(k\cdot\beta)\left(1+n_{\mathrm{B}}(k\cdot\beta)-\frac{\partial \log\varrho(k)}{\partial(k\cdot\beta)}\right)k^\mu k^\nu ,$$

with $\mathbf{k}=k^0(\mathbf{y}-\mathbf{x})/(y^0-x^0)$. For a free field the integral can be solved analytically and we obtain, by using the (3.33) and (5.54):

$$\frac{\theta((y-x)^2)}{4\pi}\frac{\mathrm{sgn}(y^0-x^0)m^3}{[(y-x)^2]^{5/2}} n_{\mathrm{B}}(k)\left[1+n_{\mathrm{B}}(k)\right](y-x)^\mu (y-x)^\nu , \tag{5.57}$$

where $k = m\,\mathrm{sign}(y^0-x^0)(y-x)/\sqrt{(y-x)^2}$. Apparently, the function (5.57) decays for large values of $(y-x)^2$ in all directions, as expected, with a rate dictated by the mass of the field, and it diverges for $y\to x$. In general, in the interacting case, and taking into account all terms in the expansion, it is thus reasonable to expect that the correlation functions of truly local operators are strongly peaked around $y\sim x$ in spite of the fact that those involving the Wigner operator are not.

The off-equilibrium correction to the operator $:\widehat{\phi}(x)\widehat{\phi}(x):$ can be obtained by integrating the correlation function of the Wigner operator and the stress-energy tensor in d^4k:

$$\begin{aligned}
\Delta\langle :\widehat{\phi}(x)\widehat{\phi}(x):\rangle &= \int \mathrm{d}^4k \; \Delta W(x,k) \\
&= \frac{2}{(2\pi)^5}\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\; \Delta\beta_\nu(y,x) \int \mathrm{d}^4k \; C^{\mu\nu}_{WT}(x-y,k) \\
&\equiv \int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\; \Delta\beta_\nu(y,x) C^{\mu\nu}(x-y) ,
\end{aligned}$$

plus a similar term involving $\Delta\zeta$ and $C^\mu_{Wj}(x-y)$. This integral looks odd because the correlation function is peaked around $y\sim x$, while the point y on the initial hypersurface Σ_0 is macroscopically distant from x. However, in a way similar to the deformation of a path to calculate integrals over the complex plane, one can use the

Gauss theorem to turn the integral above into:

$$\begin{aligned}\Delta\langle:\widehat{\phi}(x)\widehat{\phi}(x):\rangle = &\int_{\Sigma_x} \mathrm{d}\Sigma_\mu(y)\ \Delta\beta_\nu(y,x) C^{\mu\nu}(x-y)\\ &+\int_\Omega \mathrm{d}^4y\ \Delta\beta_\nu(y,x)\frac{\partial}{\partial y^\mu}C^{\mu\nu}(y-x)\\ &+\int_\Omega \mathrm{d}^4y\ C^{\mu\nu}(y-x)\frac{\partial}{\partial y^\mu}\Delta\beta_\nu(y,x)\ ,\end{aligned}$$

where Σ_x is a hypersurface passing through x. The first term on the right hand side corresponds to the local equilibrium correction to the global equilibrium, while the other two terms correspond to the dissipative corrections. If the function $C^{\mu\nu}(x-y)$ is highly peaked around $y \sim x$ with a width governed by microscopic quantities, whereas $\Delta\beta_\nu(y-x)$ is slowly varying, since $\Delta\beta_\nu(y=x)=0$, the rightmost integral provides the largest contribution, so that:

$$\Delta\langle:\widehat{\phi}(x)\widehat{\phi}(x):\rangle \simeq \int_\Omega \mathrm{d}^4y\ C^{\mu\nu}(y-x)\partial_\mu\Delta\beta_\nu(y,x) \simeq \partial_\mu\beta_\nu(x)\int_\Omega \mathrm{d}^4y\ C^{\mu\nu}(y-x)\ ,$$

thus recovering the kind of familiar corrections proportional to the local gradient of β, with a coefficient given by the integral of the correlation function, much like the Kubo formulae of transport coefficients.

5.4.2 Free theory Vs Interacting theory

As has been mentioned, the unexpected appearance of the zeroth order term (5.53) indicates that memory effects are present in the full quantum statistical approach to the calculation of the Wigner function, hence of the momentum spectrum, at the decoupling. An insight about its nature can be gained by considering the free field Wigner operator. In this case, the full solution is known, see equation (4.2):

$$W^+(x,k) = \mathrm{Tr}(\widehat{\rho}\,\widehat{W}^+(X,k))\ ,$$

with $\widehat{\rho}$ given by the equation (2.23) and $X \in \Sigma_0$ being the intersection point between the wordline drawn from x with velocity k/k^0. The calculation of the right hand side has been performed in Section 4.1 and its leading term reads:

$$W_0^+(X,k) \simeq \frac{1}{(2\pi)^3 E_k}\delta\left(k^0-\sqrt{\mathbf{k}^2+m^2}\right) n_\mathrm{B}(\beta(X))\ . \tag{5.58}$$

Now, suppose we set out to calculate the leading order term of Wigner function of the free field in the point x lying in the future of Σ_0, the result is a directly generalization of the local equilibrium correction (4.23) and reads:

$$\begin{aligned}W^+(x,k) \simeq &\frac{1}{(2\pi)^3}\delta\left(k^0-\sqrt{\mathbf{k}^2+m^2}\right) n_B(\beta(x))\\ &\times\Big[\theta_k(x)(1+n_\mathrm{B}(k))\left(k\cdot\beta(x)-k\cdot\beta\left(\overline{y}_k(x)\right)-\zeta(x)+\zeta(\overline{y}_k(x))\right)\Big]\ .\end{aligned}$$

The above equation should be an approximation of the exact free–streaming solution (3.20) and, indeed, it can be obtained from (3.20) and (5.58) as the leading order expansion of the Bose–Einstein distribution function (4.4) in $\Delta\beta = \beta(x) - \beta(X)$ and $\Delta\zeta = \zeta(x) - \zeta(X)$ with $X = \bar{y}_k(x)$. In formula:

$$n_{\rm B}(\beta(X)) = n_{\rm B}\left(\beta(x) - \Delta\beta\right) \simeq n_{\rm B}(\beta(x)) + n_{\rm B}(\beta(x))(1 + n_{\rm B}(\beta(x))\Delta\beta\ .$$

In the free field case, the zeroth order correction (5.53) is thus justified by the obvious fact that one should eventually reproduce, if all orders of the expansion in $\Delta\beta$ of the density operator (2.23) were worked out, the simple free-streaming solution.

Imagining to turn on the interaction coupling constants adiabatically, it is therefore reasonable to expect the zeroth order term to survive in the interacting field case and not to vanish abruptly. The somewhat surprising result is that, for the (5.53), the only difference with respect to the free case is that the particle mass is distributed according to the spectral function at finite temperature. We can thus attempt an interpretation of this term as the contribution of a free stream of virtual particles from the initial to the final hypersurface. The same interpretation can, in principle, be extended to the local-equilibrium corrections in Eq. (4.26). In particular, the contributions arising from the time-like branches in the spin-polarization vector (4.62) may be interpreted as virtual particles *exiting the plasma*. However, in order to make this interpretation quantitatively precise, one would need to extend the computation of the non-equilibrium corrections to the case of the Dirac field, which constitutes a highly non-trivial task.

5.4.3 Dissipative Vs non-dissipative

It is natural to ask whether the corrections to the Wigner function (5.53) of a dissipative or non-dissipative nature, according to the discussion presented in Section 2.3. As discussed there, the off-equilibrium correction to the Wigner function can be decomposed into a local equilibrium contribution and a genuinely dissipative one, namely

$$\Delta W^+(x,k) = \Delta W^+_{\rm LE}(x,k) + \Delta W^+_{\rm diss}(x,k)\ , \tag{5.59}$$

where the pure dissipative correction $\Delta W^+_{\rm diss}$ is obtained plugging the Wigner operator (3.10) in place of $\widehat{O}$ in Eq.(2.45) and reads:

$$\begin{aligned}\Delta W^+_{\rm diss}(x,k) = &\int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\, \partial_\mu\beta_\nu(y) \left\langle \widehat{W}^+(x,k),\ {\rm e}^{z\widehat{\mathcal{E}}}\, \widehat{T}^{\mu\nu}(y)\, {\rm e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\rm GE} \\ &- \int_0^1 {\rm d}z \int_\Omega {\rm d}^4y\, \partial_\mu\zeta(y) \left\langle \widehat{W}^+(x,k),\ {\rm e}^{z\widehat{\mathcal{E}}}\, \widehat{j}^\mu(y)\, {\rm e}^{-z\widehat{\mathcal{E}}} \right\rangle_{c,\rm GE}\ ,\end{aligned} \tag{5.60}$$

Since the dominant contribution to the Wigner function is given by the global equilibrium expression evaluated with the local four-temperature $\beta(x)$ and reduced chemical potential $\zeta(x)$, this question is particularly relevant. Indeed, it is known that non-dissipative contributions beyond global equilibrium exist, even though they vanish exactly at global equilibrium with the the shear-induced polarization being a notable example.

A key question is therefore whether the leading-order correction in $\Delta\beta$ appearing in Eq. (5.53) should be interpreted as a dissipative correction, a local equilibrium correction, or a combination of both.

In the following, we provide strong evidence supporting the interpretation of this correction as purely dissipative. The computation of the Wigner function at local thermodynamic equilibrium was performed assuming free fields, yielding the leading-order correction

$$\begin{aligned}\Delta W^{+}_{\rm LE}(x,k) \simeq \frac{2}{(2\pi)^3}\delta\left(k^2-m^2\right) n_{\rm B}(k)\left[1+n_{\rm B}(k)\right] \\ \times \sum_{\bar{y}^{(d)}_k(x)} {\rm sgn}\left(k\cdot\hat{n}(\bar{y}^{(d)}_k(x))\right)\left[k\cdot\beta(x)-k\cdot\beta(y)\right]\Big|_{y=\bar{y}^{(d)}_k(x)},\end{aligned} \tag{5.61}$$

where, for simplicity, the contribution from the chemical potential has been neglected. The superscript (d) indicates that the intersection point $\bar{y}^{(d)}_k(x)$ corresponds to the intersection between the worldline defined in Eq. (5.46) and the decoupling hypersurface $\Sigma_{\rm D}$ (see Fig.4.1).

Moreover, assuming that the spacetime point x lies on $\Sigma_{\rm D}$, the trivial intersection $\bar{y}^{(d)}_k = x$ is always present in the sum, but it yields a vanishing contribution. As discussed previously, the above expression must be properly extended to interacting fields in order to be directly compared with the full non-equilibrium correction in the free limit, which reads

$$\begin{aligned}\Delta W^{+}(x,k) \simeq \frac{2\theta_k(x)}{(2\pi)^3}\delta\left(k^2-m^2\right) n_{\rm B}(k)\left[1+n_{\rm B}(k)\right] \\ \times\left[k\cdot\beta(x)-k\cdot\beta(\bar{y}^{(0)}_k(x))\right],\end{aligned} \tag{5.62}$$

where Eq. (3.33) has been used in Eq. (5.53) and only the four-temperature contribution has been retained. In this case, if the intersection $\bar{y}^{(0)}_k(x)$ exists, it is unique and corresponds to the intersection with the equilibrium hypersurface, rather than with the decoupling hypersurface.

A direct comparison between Eqs. (5.61) and (5.62) shows that the local equilibrium correction does not contain any contribution from the values of the thermodynamic fields evaluated on the initial equilibrium hypersurface. This observation strongly suggests that such contributions must instead be of dissipative origin, arising from the four-volume integral in Eq. (5.60), which is neglected in purely local equilibrium computations.

This point can be made more precise by explicitly computing the dissipative contribution to the Wigner function and showing that the terms associated with the equilibrium hypersurface originate from $\Delta W^{+}_{\rm diss}$. Furthermore, one can verify that the sum of the dissipative correction and the local equilibrium contribution in Eq. (5.61) exactly reproduces the full non-equilibrium result in Eq. (5.62).

The dissipative correction to the Wigner function is obtained plugging the free-field expansion (3.16) in Eq. (5.60) yielding:

$$\begin{aligned}\Delta W^{+}_{\rm diss}(x,k) = \frac{1}{(2\pi)^3}\int_\Omega {\rm d}^4y\int{\rm d}^4q\left\{{\rm e}^{{\rm i}x\cdot q}\delta(k\cdot q)\,\delta\left(k^2+\frac{q^2}{4}-m^2\right)\right. \\ \left.\times\,\theta(k^0_+)\theta(k^0_-)\int_0^1{\rm d}z\,\partial_\mu\beta_\nu(y)\left\langle\widehat{a}^\dagger(k_+)\widehat{a}(k_-),\ {\rm e}^{z\widehat{\mathcal{E}}_{\rm GE}}\widehat{T}^{\mu\nu}(y){\rm e}^{-z\widehat{\mathcal{E}}_{\rm GE}}\right\rangle_{c,{\rm GE}}\right\}.\end{aligned} \tag{5.63}$$

This expression can be directly compared with the local equilibrium result in Eq. (4.5). The two expressions are formally identical, except for the replacement

$$-\int_{\Sigma_{\rm D}}{\rm d}\Sigma_\mu\,\Delta\beta_\nu(y,x)\ \longrightarrow\ \int_\Omega{\rm d}^4y\,\partial_\mu\beta_\nu(y)\,. \tag{5.64}$$

Proceeding in complete analogy with the local equilibrium case, Eq. (5.63) can be rewritten as

$$\Delta W^{+}_{\rm diss}(x,k) = -\frac{1}{(2\pi)^3}\int \mathrm{d}^4q\, \delta(k\cdot q)\, F_{\mu\nu}(q,x)\; G^{\mu\nu}(k,q,\beta)\ , \tag{5.65}$$

where the function $G^{\mu\nu}$ is defined in Eq. (4.10), while $F_{\mu\nu}$ is now given by the Fourier transform of the four-temperature gradient over the four-volume Ω:

$$F_{\mu\nu}(q,x) = \int_\Omega \mathrm{d}^4y\, \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\partial_\mu\beta_\nu(y)\ . \tag{5.66}$$

The dissipative correction in Eq. (5.65) can be analyzed in close analogy with the non-equilibrium and local equilibrium contributions. In the hydrodynamic regime, the function $F_{\mu\nu}$ is sharply peaked around $q = 0$, allowing one to expand $G^{\mu\nu}$ in powers of q. Retaining only the leading-order term yields

$$\Delta W^{+}_{\rm diss}(x,k) \simeq -\frac{G^{\mu\nu}(k,0,\beta)}{(2\pi)^3}\int_\Omega \mathrm{d}^4y \int \mathrm{d}^4q\, \delta(k\cdot q)\ \mathrm{e}^{\mathrm{i}q\cdot(x-y)}\ . \tag{5.67}$$

The integration over d^4q can be performed explicitly, as we did in Section 5.3, yielding

$$\int \mathrm{d}^4q\, \delta(k\cdot q)\ \mathrm{e}^{\mathrm{i}q\cdot(x-y)} = \frac{(2\pi)^3}{|k^0|}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\ . \tag{5.68}$$

As a result, the dissipative correction to the Wigner function takes the form

$$\begin{aligned}\Delta W^{+}_{\rm diss}(x,k) \simeq -\frac{G^{\mu\nu}(k,0,\beta)}{|k^0|}&\int_\Omega \mathrm{d}^4y\, \partial_\mu\beta_\nu(y^0,\mathbf{y})\\ &\times \delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\ .\end{aligned} \tag{5.69}$$

We now turn to a crucial difference with respect to the local equilibrium and full non-equilibrium computations discussed previously. In the present case, the integration must be performed over the four-dimensional volume Ω, rather than over a three-dimensional hypersurface. Geometrically, this region is highly non-trivial, as it corresponds to the spacetime domain enclosed between the decoupling hypersurface $\Sigma_{\rm D}$ and the equilibrium hypersurface Σ_0.

The integration over Ω can be extended to the whole Minkowski spacetime $\mathbb{R}^4$ by introducing the characteristic function

$$\theta_\Omega(y) = \begin{cases} 1 & \text{if } y\in\Omega\,,\\ 0 & \text{otherwise}\ ,\end{cases} \tag{5.70}$$

so that the four-volume integral can be written as

$$\int_{-\infty}^{+\infty}\mathrm{d}y^0\int_{\mathbb{R}^3}\mathrm{d}^3\mathbf{y}\,\theta_\Omega\left(y^0,\mathbf{y}\right)\,\partial_\mu\beta_\nu(y^0,\mathbf{y})\,\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{k}}{k^0}\left(y^0-x^0\right)\right)\ . \tag{5.71}$$

The integration over the spatial coordinates can be immediately performed using the delta function, which constrains the spacetime point y to lie along the classical worldline defined in Eq. (5.46). One thus obtains

$$\int_{-\infty}^{+\infty}\mathrm{d}y^0\,\theta_\Omega\left(y^0,\bar{\mathbf{y}}_k\right)\,\partial_\mu\beta_\nu(y)\Big|_{\mathbf{y}=\bar{\mathbf{y}}_k}\ , \tag{5.72}$$

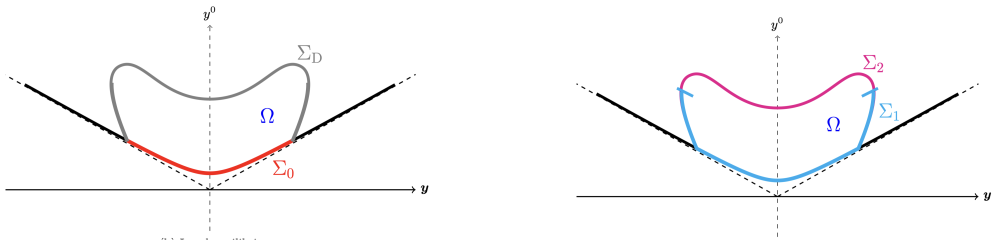
with now $\bar{\mathbf{y}}_k$ denoting a generic point along the worldline (5.46) and not only the intersection with the hypersurface as in the previous cases.

The remaining integration over the time coordinate y^0 can be treated as follows. The boundary of the spacetime region Ω can be decomposed as

$$\partial\Omega = \Sigma_{\rm D} \cup \Sigma_0 = \Sigma_1 \cup \Sigma_2 \ , \tag{5.73}$$

where Σ_2 denotes the space-like branch of the decoupling hypersurface, while Σ_1 is given by the union of the equilibrium hypersurface Σ_0 and the time-like branches of $\Sigma_{\rm D}$ (see Fig.5.4.3):

$$\Sigma_1 \equiv \Sigma_{\rm D}\Big|_{\rm t-l} \cup \Sigma_0 \ , \qquad \Sigma_2 \equiv \Sigma_{\rm D}\Big|_{\rm s-l} \ . \tag{5.74}$$



(a) (b)

Figure 5.2: Boundary of the 4D volume Ω expressed in terms of the two different parametrizations. Note that, once one specifies one of the two time-like branches, Σ_1 can be described by a single-valued function of y^0.

In this parametrization, the boundary of Ω can be described by two (possibly multi-valued) functions $f_{1,2}(\mathbf{y})$, such that the support of the function θ_Ω along the worldline is defined, for a given (x,k), by the condition

$$f_1\left(\bar{\mathbf{y}}_k(y^0)\right) \le y^0 \le f_2\left(\bar{\mathbf{y}}_k(y^0)\right) \ . \tag{5.75}$$

Let us denote by $y^0_{1,2}$ the two values of y^0 that saturate the above inequalities. The integration over $\mathrm{d}y^0$ then reduces to

$$\int_{y_1^0}^{y_2^0} \mathrm{d}y^0\, \partial_\mu\beta_\nu(y)\Big|_{\mathbf{y}=\bar{\mathbf{y}}_k} \ . \tag{5.76}$$

Using Eq. (4.10), the dissipative correction to the Wigner function can thus be written as

$$\begin{aligned}\Delta W^+_{\rm diss}(x,k) \simeq{}& \frac{2\delta\left(k^2-m^2\right)}{(2\pi)^3|k^0|} n_{\rm B}(k)\left[1+n_{\rm B}(k)\right] \\ &\times k^\mu k^\nu \int_{y_1^0}^{y_2^0} \mathrm{d}y^0\, \partial_\mu\beta_\nu(y)\Big|_{\mathbf{y}=\bar{\mathbf{y}}_k} \ .\end{aligned} \tag{5.77}$$

We can now explicitly perform the integration over y^0. Let us define

$$\bar{y}_k(x) = \left(y^0, \bar{\mathbf{y}}_k(y^0)\right) \ . \tag{5.78}$$

For $y^0 = y^0_{1,2}$, the point $\bar{y}_k(x)$ lies on the intersection between the worldline (5.46) and the boundary of Ω. Depending on which portion of the boundary is involved, this point coincides either with $\bar{y}_k^{(d)}(x)$ in Eq. (5.61) or with $\bar{y}_k^{(0)}(x)$ in Eq. (5.62).

We note that

$$k^\mu \partial_\mu \beta_\nu(\bar{y}_k) = k^0 \frac{\partial \beta_\nu(\bar{y}_k)}{\partial y^0} + \mathbf{k} \cdot \nabla_{\mathbf{y}} \beta_\nu(\bar{y}_k) \ . \tag{5.79}$$

Since $\bar{y}_k$ depends implicitly on y^0 through $\mathbf{y}_k(y^0)$, the total derivative of β_ν along the worldline reads

$$\frac{\mathrm{d}\beta_\nu(\bar{y}_k(y^0))}{\mathrm{d}y^0} = \frac{\partial \beta_\nu(\bar{y}_k(y^0))}{\partial y^0} - \frac{\partial \mathbf{y}_k}{\partial y^0} \cdot \nabla_{\mathbf{y}} \beta_\nu(\bar{y}_k(y^0)) \ . \tag{5.80}$$

Using the explicit form of the worldline (5.46), Eq. (5.79) reduces to

$$k^\mu \partial_\mu \beta_\nu(\bar{y}_k) = k^0 \frac{\mathrm{d}\beta_\nu(\bar{y}_k)}{\mathrm{d}y^0} \ . \tag{5.81}$$

Substituting this result into Eq. (5.77) allows one to integrate directly over y^0, yielding

$$\begin{aligned} \Delta W^+_{\text{diss}}(x,k) \simeq & \frac{2\delta\left(k^2-m^2\right)}{(2\pi)^3} n_{\mathrm{B}}(k)\left[1+n_{\mathrm{B}}(k)\right] \\ & \times k \cdot \left[\beta\left(\bar{y}_k(y_2^0)\right) - \beta\left(\bar{y}_k(y_1^0)\right)\right] \ , \end{aligned} \tag{5.82}$$

where we have used the fact that, for the particle contribution, $\mathrm{sgn}(k^0) = +1$.

We can now combine Eq. (5.82) with the local equilibrium contribution in Eq. (5.61) to obtain the full non-equilibrium correction:

$$\begin{aligned} \Delta W^+(x,k) = & \frac{2}{(2\pi)^3} \delta\left(k^2-m^2\right) n_{\mathrm{B}}(k)\left[1+n_{\mathrm{B}}(k)\right] \\ & \times \left\{ \sum_{\bar{y}_k^{(d)}(x)} \mathrm{sgn}\left(k \cdot \hat{n}(\bar{y}_k^{(d)}(x))\right) \left[k \cdot \beta(x) - k \cdot \beta(y)\right]\Big|_{y=\bar{y}_k^{(d)}(x)} \right. \\ & \left. + \left[k \cdot \beta\left(\bar{y}_k(y_2^0)\right) - k \cdot \beta\left(\bar{y}_k(y_1^0)\right)\right] \right\} . \end{aligned} \tag{5.83}$$

Let us now consider the case in which the point x lies on the space-like branch of the decoupling hypersurface, i.e. on Σ_2. In this situation one has $y_2^0(x) \equiv x^0$ and $\bar{y}_k(x^0) \equiv x$, so that the above expression simplifies to

$$\begin{aligned} \Delta W^+(x,k) = & \frac{2}{(2\pi)^3} \delta\left(k^2-m^2\right) n_{\mathrm{B}}(k)\left[1+n_{\mathrm{B}}(k)\right] \\ & \times \left\{ \sum_{\bar{y}_k^{(d)}(x)} \mathrm{sgn}\left(k \cdot \hat{n}(\bar{y}_k^{(d)}(x))\right) \left[k \cdot \beta(x) - k \cdot \beta(y)\right]\Big|_{y=\bar{y}_k^{(d)}(x)} \right. \\ & \left. + \left[k \cdot \beta(x) - k \cdot \beta\left(\bar{y}_k(y_1^0)\right)\right] \right\} . \end{aligned} \tag{5.84}$$

There are now two mutually exclusive possibilities for the points $\bar{y}_k(y_1^0)$:

1. The intersection between the worldline (5.46) and the boundary of Ω lies on the equilibrium hypersurface Σ_0, so that $\bar{y}_k(y_1^0) \equiv \bar{y}_k^{(0)}$. In this case, the

only intersection contributing to the local equilibrium term is the trivial one $\bar{y}_k^{(d)}(x) = x$, for which the local equilibrium contribution vanishes identically. One is therefore left with

$$\begin{aligned}\Delta W^+(x,k) = &\frac{2}{(2\pi)^3}\delta\left(k^2 - m^2\right) n_{\mathrm{B}}(k)\left[1 + n_{\mathrm{B}}(k)\right] \\ &\times \left[k\cdot\beta(x) - k\cdot\beta(\bar{y}_k^{(0)})\right] \ .\end{aligned} \tag{5.85}$$

This expression coincides exactly with the result obtained in Eq. (5.62). It originates entirely from configurations for which the local equilibrium contribution in Eq. (5.61) vanishes identically. This explicitly demonstrates that the correction in Eq. (5.62) is fully generated by the volume term and is therefore of dissipative nature.

2. The intersection between the worldline (5.46) and the boundary of Ω lies on a time-like branch of the decoupling hypersurface. In this case $\bar{y}_k(y_1^0) \equiv \bar{y}_k^{(d)}$, and the total contribution is proportional to

$$\Delta W^+(x,k) \propto \left(1 + \mathrm{sgn}\left(k\cdot\hat{n}(y)\right)\right)\Big|_{y=\bar{y}_k(y_1^0)} \left[k\cdot\beta(x) - k\cdot\beta\left(\bar{y}_k(y_1^0)\right)\right] \ . \tag{5.86}$$

This term vanishes identically for $k\cdot\hat{n} < 0$, while it survives for $k\cdot\hat{n} > 0$. Such contributions are expected to be cancelled by boundary terms arising from higher-order contributions in the volume expansion.

In conclusion, as anticipated, all contributions arising from the values of the thermodynamic fields on the equilibrium hypersurface originate entirely from the four-volume integral, i.e. from the dissipative correction. In these cases, the local equilibrium contribution vanishes exactly. This strongly supports the interpretation that the correction in Eq. (5.84), and thus in Eq. (5.53), is predominantly dissipative in nature.

Although the derivation presented here was carried out within a free-field approximation, which is not fully realistic, the central argument remains valid in an interacting theory. The only modification concerns the appearance of more intricate structures due to the non-trivial spectral function, which do not affect the integration over the spacetime volume Ω nor the conclusions drawn above.

5.4.4 Statistical field theory Vs kinetic theory

An important question, which is related to the above discussion on the nature of the zeroth order term, is whether the expansion of the Wigner function in the initial gradients can be found in a similar form in relativistic kinetic theory as well. As has been discussed in Section 2.3, in principle one can obtain an expansion of the same function $W^+(x,k)$ in the final gradients by using the Taylor expansion of the gradients, for instance:

$$\partial^{(k)}\beta(x) = \partial^{(k)}\beta(x_0) + \partial^{(k+1)}\beta(x_0)(x - x_0) + \dots \ ,$$

and the issue is which expansion (in the initial or final gradients) provides the better approximation. It is worth noting that in the above Taylor expansion a long distance is introduced at each term (i.e. $(x-y)^k$, hence only the resummation of many terms may lead to a decent approximating formula.

In classical relativistic kinetic theory, it is well known that the distribution function $f(x,k)$ expanded about local equilibrium receives corrections proportional to the gradients of the thermo-hydrodynamic fields at the same point x [171] under the assumption of separation of time scales (mean collision time $\ll$ hydrodynamic time scale) and factorization of the two-particle distribution in the collisional integral, i.e. molecular chaos hypothesis or correlation memory loss [172]:

$$\Delta f(x,k) \propto \partial(\beta,\zeta) \ .$$

If either assumption is relaxed, one has memory effects and a dependence of $f(x,k)$ on the history of the system [173–175], hence on the initial conditions. The obvious limiting example is the collision-less Boltzmann equation:

$$k^\mu \partial_\mu f(x,k) = 0 \ ,$$

where the mean collision time is infinite, the solution is analogous to the equation (3.20) for the Wigner function, the memory of the initial distribution is fully retained, and an expansion in the gradients of $\beta(x)$, even if possible in principle if the particles leave in a fluid medium, does not provide a good approximation of $f(x,p)$. In modern formulations of relativistic kinetic theory [176] memory effects are limited to a finite relaxation time but they do not involve convolution integrals in time (see also ref. [177]).

We can learn something more about the difference between the quantum statistical approach and the classical relativistic kinetic wisdom by studying the zeroth order term (5.53) in the special case where the hypersurfaces Σ_0 and Σ_{D}, passing through x, are not too far from each other. In this case we can expand the gradients of the thermo-hydrodynamic fields and retain only the leading order term. Notably, taking into account the equations (5.46):

$$\begin{aligned}
\beta^\nu(x) - \beta^\nu(\bar{y}_\vartheta(x)) &\simeq \partial_\lambda \beta^\nu(x) \frac{w^\lambda(\vartheta)}{w^0(\vartheta)} \Delta x^0(x,\vartheta) \ , \\
\zeta(x) - \zeta(\bar{y}_\vartheta(x)) &\simeq \partial_\lambda \zeta(x) \frac{w^\lambda(\vartheta)}{w^0(\vartheta)} \Delta x^0(x,\vartheta) \ ,
\end{aligned}$$

where Δx^0 is the time difference between the point x and the intersection between the world-lines (5.46) starting from x and the hypersurface Σ_0. The formula (5.53) will come down to:

$$\Delta^{(0)} W\left(x,k\right) \simeq \left\{ \frac{2\varrho^2(k)}{(2\pi)^2} \int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta \ \left[\Gamma_1\left(\vartheta;k,0,q\right) \frac{\Delta x^0\left(x,\vartheta\right)}{|w^0(\vartheta)|} w^\mu(\vartheta) w^\lambda(\vartheta) \right] \right\} \partial_\lambda \beta_\mu(x) \ ,$$

showing that the leading order correction is now proportional to the first order gradients of the fields in the point x, like in the classical relativistic kinetic theory. Indeed, if we replace the ratio $\Delta x^0/w^0$ with a small relaxation time τ_R and we express $w^\mu w^\nu$ in terms of k and β, we essentially retrieve a classical kinetic expression of the non-equilibrium correction to the distribution function in the relaxation time approximation [39]:

$$\Delta^{(0)} W\left(x,k\right) \propto k^\lambda k^\nu \partial_\lambda \beta_\nu(x) \tau_R + \dots \ . \tag{5.87}$$

This simple exercise shows that in the quantum statistical framework, the classical expressions are recovered provided that there is a *microscopically* small time distance between the current time hypersurface and the initial hypersurface where local equilibrium is previously achieved. On a macroscopic time scale, an expression such as (5.87) applies if not just the fields evolve according to Heisenberg rules (3.7a), but also the quantum state varies on a relaxation time basis. In formula, if the quantum state collapses

$$\widehat{\rho} \longrightarrow \widehat{\rho}_{\rm LE} \, , \tag{5.88}$$

every relaxation time step. This kind of decoherence in equation (5.88) makes entropy increase objective and not just an effect of restricting information to relevant observables (energy-momentum and charge density). The entropy

$$S = -{\rm Tr}(\widehat{\rho} \log \widehat{\rho}) \, ,$$

unlike in the Heisenberg picture where $\widehat{\rho}$ is fixed, does vary because of (5.88). Similarly, the Boltzmann equation with molecular chaos hypothesis (i.e. factorization of the two-particle distribution function in the collisional integral) involves an objective increase of entropy through the H-theorem.

In conclusion, it should not be surprising that rigorous quantum statistical methods where system evolves according to the Heisenberg equation and do not include additional assumptions somehow equivalent to the equation (5.88), provide off-equilibrium corrections involving a memory of the initial state. Our method of calculating the Wigner function is completely equivalent to find a parametric solution of the differential equations of quantum kinetic theory with assigned initial conditions at the Cauchy hypersurface Σ_0 (notably, the so-called Kadanoff-Baym equations [178]). Indeed, the quantum kinetic equations in their original form are non-Markovian integro-differential equations whose solution must depend on the history of the system.

5.4.5 Relaxation toward global equilibrium

Another important point to address is the reduction to global equilibrium of the expressions found. If the density operator at Σ_0 is a global equilibrium one, the four-temperature β is a Killing vector and the reduced chemical potential ζ is constant, i.e.:

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu \, , \qquad \zeta = {\rm const} \, , \tag{5.89}$$

with b and ϖ, i.e. the thermal vorticity, constant. If $\varpi \neq 0$, the correction to the leading order expression (5.1) of the Wigner function can be non-vanishing, as demonstrated in an exact calculation in ref. [179]. However, the leading order term (5.53) vanishes at global equilibrium; plugging the equations (5.89) into the correction term in the (5.53) we have $\Delta\zeta = 0$ and:

$$w(\vartheta) \cdot (\beta(x) - \beta(\bar{y}_\vartheta(x)) = w^\mu(\vartheta) \varpi_{\mu\nu} (x^\nu - \bar{y}_\vartheta(x)^\nu) = w^\mu(\vartheta) \varpi_{\mu\nu} w^\nu(\vartheta) \Delta\tau = 0 \, ,$$

where we took advantage of the fact that x and $\bar{y}$ are two events lying on the worldline whose tangent vector is proportional to w.

Another crucial problem is whether, starting from a non-equilibrium density operator such as (2.23), the expected form of the Wigner function at global equilibrium

(5.1) is achieved asymptotically in the limit $t \to +\infty$ because of dissipation. More specifically, if the system is confined within a finite region and if it evolves according to the laws of dissipative hydrodynamics, we expect the thermo-hydrodynamic fields to converge to a global equilibrium configuration where $\partial\beta$, $\partial\zeta \to 0$ (provided that the angular momentum vanishes, so thermal vorticity vanishes at equilibrium). In this case, one expects that in the same limit the off-equilibrium correction $\Delta W^+(x,k)$ vanishes thereby losing the memory of the initial state.

According to the equation (5.55), (5.56), as $\Delta\beta$ remains finite, this is the case if the correlation function vanishes for large $(x-y)^2$, which is not generally the case though, as we have discussed in this Section. Nevertheless, if Σ_0 a compact region, those terms of the correlation function (5.56) which are constant over the special worldlines obtained for $\vartheta = 0$: $(x-y)^\mu = k^\mu \tau$ and its contribution to the correction (5.53) does not survive in the limit $t \to +\infty$ because they not intersect Σ_0 except for $k^\mu \propto (1, \mathbf{0})$ (see figure 5.3). In fact, this does not necessarily apply to the general worldlines $(x-y)^\mu = w^\mu \tau$. To see it consider the particular case $\vartheta = \pi/2$ and the worldline $(x-y)^\mu = \beta^\mu \tau$. If the field β^μ has no spatial component in the limit $t \to +\infty$ and in this case its contribution to (5.53) survives, particularly the terms proportional to $\theta_{\vartheta=\pi/2}(x)$ in the equation (5.53). It remains an open question whether this term cancels out with other contributions from other worldlines ϑ or with higher order terms $\mathcal{O}(\partial^2)$ to the Wigner function, that is the terms (5.49) as well as corrections beyond linear order response.

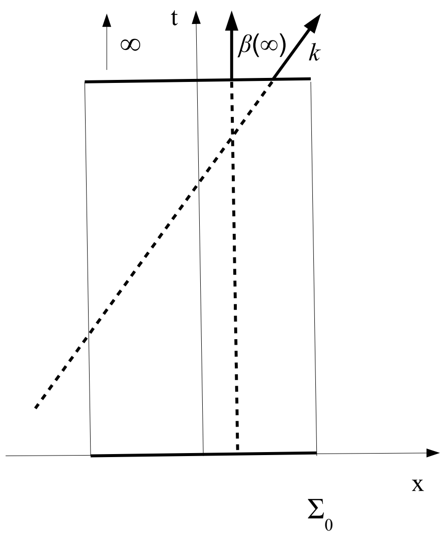


Figure 5.3: For a system with fixed volume (e.g. a gas within a vessel) in the limit $t \to +\infty$ all worldlines with a slope proportional to k eventually have no intersection with the hypersurface Σ_0 at $t = 0$. Conversely, all worldlines whose tangent four-vector is $\lim_{t\to\infty} \beta(x) = \frac{1}{T_0}(1, \mathbf{0})$ worldlines intersect Σ_0.

5.5 The momentum spectrum

We finally come to the main phenomenological consequence of the off-equilibrium correction to the Wigner function. By using the Wigner function expression up to second order gradients in eqs. (5.1) and and (5.53) in the equation (3.43), we obtain an expression of the momentum spectrum of particles at the decoupling hypersurface, i.e. before collisional corrections:

$$\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}}(\mathbf{k}) = \frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}}\Big|_0(\mathbf{k}) + \frac{2}{(2\pi)^4}\int_0^{+\infty}\mathrm{d}k^0 \int_{\Sigma_D} \mathrm{d}\Sigma(x)\cdot k\ \Delta^{(0)}W(x,k)\ , \tag{5.90}$$

where:

$$\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}}\Big|_0(\mathbf{k}) = \frac{2}{(2\pi)^4}\int_0^{+\infty}\mathrm{d}k^0 \int_{\Sigma_D} \mathrm{d}\Sigma(x)\cdot k\ \varrho(k)\, n_{\mathrm{B}}(k)\ ,$$

and $\Delta W(x,k)$ given explicitly by (5.47). The relative weight of the correction in eq. (5.90) is thus given by:

$$R(\mathbf{k}) = \frac{\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} - \frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}}\Big|_0}{\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}}\Big|_0}\ . \tag{5.91}$$

It should be reminded that the correction (5.53) is just the leading order one in the expansion of the density operator, hence it is a good approximation whenever:

$$k\cdot\Delta\beta \ll 1\ , \qquad\qquad \Delta\zeta \ll 1\ ,$$

that is for small differences between initial and final four-temperature and reduced chemical potential. In practice, if $R(\mathbf{k})$ defined above is not much smaller than 1, higher orders (quadratic response and beyond) should be considered.

The spectrum (5.90) results from the convolution, in the variable k^0, of the spectral function calculated at the decoupling temperature with the familiar Bose-Einstein distribution and thermo-hydrodynamic fields. Indeed, it can be seen as the spectrum of particles with momentum $\mathbf{k}$ and a mass distributed according to the spectral function. The space integration is carried out over a 3D hypersurface with the functions $\theta_k(x)$ and $\theta_\beta(x)$ which may cut off high momenta (see discussion below), reducing the ratio $R(\mathbf{k})$ possibly extending the range of applicability of the linear approximation.

In the limit of a quasi-free spectral function (3.33) using (5.53), (5.90) simplifies to:

$$\begin{aligned} E_k\frac{\mathrm{dN}_p}{\mathrm{d}^3\mathrm{k}} = \frac{1}{(2\pi)^3}\int_{\Sigma_D} &\mathrm{d}\Sigma(x)\cdot k\ n_{\mathrm{B}}(k)\Big[1+ \\ &+ \theta_k(x)\,(1+n_B(k))\,(k\cdot\beta(x) - k\cdot\beta(\overline{y}_k(x)) - \zeta(x) + \zeta(\overline{y}_k(x)))\Big]\ , \end{aligned} \tag{5.92}$$

whose leading term is precisely the Cooper-Frye formula (1.1). The theta-function $\theta_k(x)$ is non-vanishing for the k, x such that the worldline (5.46) with $\vartheta = 0$ (i.e with velocity proportional to $\mathbf{k}/k^0$) intersect the equilibrium hypersurface. Note that the very same result can be obtained in principle also for the antiparticles, i.e $\mathrm{dN}_{\bar{p}}/\mathrm{d}^3\mathrm{k}$.

An important feature of the formulae (5.90) or its simplified version (5.92) is the function $\theta_k(x)$ which cuts off momenta whose corresponding worldline does not intersect the hypersurface Σ_0; this is shown in the figure 5.4 for the typical longitudinal projection of the initial hypersurface and the decoupling hypersurface. To

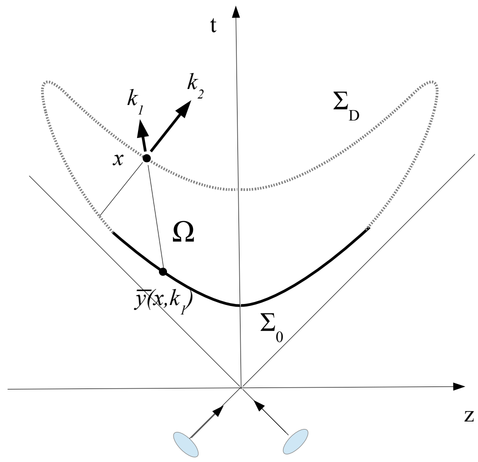


Figure 5.4: In a relativistic nuclear collision (see also figure 2.2) particles with (off-shell) four-momentum k_2 do not receive zero-order correction from the Wigner function $W^+(x,k)$ at x over the decoupling hypersurface because the world-line does not intersect the initial hypersurface Σ_0; the converse is true for the momentum k_1.

quantify the effect of this cutoff on the momentum spectrum we can use a simple argument, which applies at vanishing momentum component along the beam line, that is $p_z = 0$, and in the transverse projection of the expansion. Setting $\zeta = 0$ for simplicity, the region weighing the most for the integration in both the numerator and denominator of the equation (5.91), is the one where $\beta \cdot k$ is the smallest because of the exponential factor $\exp[-\beta \cdot k]$. Therefore, taking into account that the transverse component of the initial flow velocity $\mathbf{v}_{Ti}$ vanishes, for $p_z = 0$ one has, for the correction in the numerator of (5.91):

$$k \cdot \Delta\beta = k \cdot \beta_f - k \cdot \beta_i = \varepsilon \left(\frac{\gamma_f}{T_f} - \frac{\gamma_i}{T_i} \right) - \frac{\mathbf{p}_T \cdot \gamma_f \mathbf{v}_{Tf}}{T_f} \ ,$$

and, according to the above argument, one can take $\mathbf{p}_T$ collinear to $\mathbf{v}_T$, so as to get:

$$k \cdot \Delta\beta \longrightarrow \varepsilon \left(\frac{\gamma_f}{T_f} - \frac{\gamma_i}{T_i} \right) - \frac{p_T \gamma_f v_{Tf}}{T_f} \ . \tag{5.93}$$

For $p_T = 0$ this quantity is positive:

$$k \cdot \Delta\beta = m \left(\frac{\gamma_f}{T_f} - \frac{\gamma_i}{T_i} \right) > 0 \ ,$$

because, in general we have $\gamma_f \geq \gamma_i$ and $T_i > T_f$. At $p_T = 0$ the derivative of the function on the right hand side of eq. (5.93) is negative, so that the ratio R presumably decreases until p_T reaches a critical value, which may well be beyond the natural geometric cut-off. Therefore, in relativistic heavy ion collision, an enhancement of the transverse momentum spectrum at low transverse momenta is expected due to the correction term in eq. (5.90). Indeed, an excess of pions at low p_T [180, 181] at very high energy is a long-standing phenomenological issue which has been addressed in several papers in literature [121, 182, 183]; it is still premature to say that this correction can account for this phenomenon, however this is an effect going into the right direction.

5.6 Summary

In summary, we have derived the dissipative corrections, up to linear order in the gradients of the thermo-hydrodynamic fields, to the Wigner function and to the single-particle momentum spectrum of scalar particles emitted from an expanding decoupling fluid that is initially in local thermodynamic equilibrium .

We have performed an *ab initio* calculation of the Wigner function within the framework of statistical quantum field theory by employing the appropriate density operator and a novel approximation scheme introduced in our recent work [69]. The calculation has been carried out for a generally interacting scalar quantum field. Moreover, no specific assumptions have been made regarding the microscopic structure of the stress-energy tensor and four-current operators which include the contribution of all the remaining interacting fields. Retaining the full generality led us to parametrize correlators of the field Fourier transforms with thermal-gravitational and thermal-charged form factors, for which the only assumption made has been the requirement of analyticity in a four-momentum variable. We have shown that the leading-order expansion of the non-equilibrium contribution to the density operator within linear response theory naturally gives rise to a series involving gradients of the hydrodynamic fields evaluated on the initial local-equilibrium hypersurface, rather than on the final decoupling hypersurface, as it is customary in classical kinetic theory. The emergence of gradients at the initial hypersurface is a direct consequence of the long-distance persistence of the correlation function between the Wigner operator and the stress-energy tensor and current operators, that we have discussed in detail, entailing a memory of the initial state.

The leading contribution in the resulting expansion is a zeroth-order correction proportional to the difference between the hydrodynamic fields evaluated at the decoupling point x and at the intersection of the initial hypersurface with the worldline passing through x and having a tangent four-vector proportional to the four-momentum argument of the Wigner function. On the other hand, the first-order gradient contribution identically vanishes. The zeroth-order term provides a clear manifestation of the memory of the initial state and, at least for the main contribution, has a clear counterpart in the free field limit, where its interpretation is straightforward in the free-streaming solution of the Wigner function. Its survival in the interacting case (weighted by the spectral function and that can be interpreted as a term related to the free propagation of virtual particles) should not be surprising in the limit of a weakly interacting theory; in a strongly interacting theory its relative importance for the off-equilibrium correction to the Wigner function depends on how large the coefficients of all higher-order gradients as well as all the contributions of the terms beyond linear response.

The zeroth order term reduces to a first-order gradient correction of the familiar form encountered in relativistic kinetic theory only when the initial and decoupling hypersurfaces are microscopically close. More precisely, it is necessary that their separation is of the order of the classical relaxation time within a kinetic description. This observation suggests that, within a fully quantum-mechanical statistical framework, the suppression of initial-state memory effects needs an internal process of quantum decoherence, whereby the density operator undergoes a continuous reduction toward a local-equilibrium form and the time evolution becomes effectively non-unitary. However, such a mechanism is not expected to take place in an isolated system. Our results therefore represent the proper quantum-mechanical prediction for the physical situation under consideration.

Finally, the resulting momentum spectrum acquires corrections proportional to integrals over the decoupling hypersurface of the aforementioned field differences. These corrections induce a distortion of the spectrum, which is expected to be most pronounced at low momenta. A quantitative assessment of the magnitude of these effects, as well as an evaluation of the validity of the linear-gradient approximation in such systems, calls for a dedicated numerical investigation. We finally note that the derived expressions hold at finite chemical potential and do not rely on any approximation for the geometry of the decoupling hypersurface. Consequently, the formalism applies to collisions over a broad range of energies.

Conclusions and Outlook

The local-equilibrium and full out-of-equilibrium corrections to the Wigner function of a relativistic system have been computed at all orders in linear response theory. We introduced a new expansion method based on the hydrodynamic limit, which allows for a systematic gradient expansion of the local-equilibrium expectation value of the Wigner operator. This approach makes it possible to perform the integration over the decoupling hypersurface order by order, *without any geometrical approximation.*

We carried out the same analysis for the Dirac field and derived an improved formula for the spin polarization vector at local thermodynamic equilibrium. The new method enables the computation of the spin polarization through an exact integration over the physical decoupling hypersurface. The resulting expression identifies a natural vector to be coupled to the shear tensor, given by the normal vector to the hypersurface, and additionally accounts for possible contributions arising from distant points associated with time-like branches of the decoupling hypersurface.

The expansion method was subsequently extended to compute not only the local-equilibrium correction, but also the full out-of-equilibrium correction in linear response theory for scalar fields. In order to incorporate the effects of the plasma phase and the intrinsic non-locality of the Wigner function, we introduced an interacting expansion of the quantum field, together with a parametrization of the stress–energy tensor and current expectation values in terms of *thermo-charged gravitational form factors.* The inclusion of interactions leads, in principle, to an infinite set of kinetic constraints and, correspondingly, to an infinite number of form factors. However, at the lowest order in the hydrodynamic expansion, only two kinetic constraints contribute, and these are strongly restricted by global conservation laws.

Future perspective

The improved expression for the local-equilibrium contribution to the spin polarization presented in this work constitutes a generalization of previous results [157–160]. Beyond its formal significance, this result calls for a dedicated numerical investigation, in particular to quantify the extent to which the detailed geometry of the decoupling hypersurface affects the resulting polarization signal in realistic heavy-ion collision scenarios. Since the underlying formalism is independent of the particle spin, a natural and important extension of the present analysis is its application to the spin alignment of vector mesons.

Related considerations apply to the full non-equilibrium correction to the scalar Wigner function derived in this thesis. A quantitative numerical study of its impact on pion spectra is particularly timely, as it may shed light on the experimentally observed enhancement in the low-p_T region. At the same time, extending the present framework to fields with spin is essential in order to assess the relative importance of dissipative and non-equilibrium effects in the spin polarization vector. Such an extension, however, is considerably more challenging than in the local-equilibrium case, as it requires a systematic treatment of the interacting expansion of the Wigner function and of the associated thermo-gravitational form factors, which become substantially more involved.

Appendix A

Dirac gamma matrices

The Dirac matrices γ^μ, with $\mu = 0, 1, 2, 3$, furnish a representation of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{A.1}$$

where $\eta^{\mu\nu}$ denotes the Minkowski metric.

A possible representation for the gamma matrices is the Weyl (or chiral) representation, which is particularly convenient for the discussion of chirality and axial quantities. In this representation the gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{I}_{2\times 2} \\ \mathrm{I}_{2\times 2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{A.2}$$

where σ^i $(i = 1, 2, 3)$ are the Pauli matrices and $I_{2\times 2}$ denotes the 2×2 identity matrix.

With these definitions, the Dirac matrices satisfy the following hermiticity properties:

$$(\gamma^0)^\dagger = \gamma^0 \, , \qquad (\gamma^j)^\dagger = -\gamma^j \, . \tag{A.3}$$

These relations ensure that the Dirac Hamiltonian constructed from γ^μ is Hermitian.

The chirality matrix γ^5 is defined as

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathrm{I}_{2\times 2} & 0 \\ 0 & \mathrm{I}_{2\times 2} \end{pmatrix}, \tag{A.4}$$

which explicitly shows that γ^5 projects onto left- and right-handed components in the Weyl representation.

Equivalently, γ^5 can be written in a manifestly Lorentz-covariant form as

$$\gamma^5 = \frac{1}{4!}\,\varepsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta, \qquad \varepsilon^{0123} = +1 \, , \tag{A.5}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor.

The matrix γ^5 anticommutes with all Dirac matrices and is Hermitian:

$$(\gamma^5)^\dagger = \gamma^5 \, , \qquad \{\gamma^5, \gamma^\mu\} = 0 \, . \tag{A.6}$$

These properties play a central role in the definition of axial currents and in the computation of spin polarization observables.

The Dirac gamma matrices satisfy a number of useful trace identities. In particular, the following traces vanish:

$$\mathrm{tr}(\gamma^\sigma) = 0\ , \tag{A.7a}$$
$$\mathrm{tr}(\gamma^5) = 0\ , \tag{A.7b}$$
$$\mathrm{tr}\left(\gamma^{\sigma_1}\dots\gamma^{\sigma_{2n+1}}\right) = 0\ , \tag{A.7c}$$
$$\mathrm{tr}(\gamma^5\gamma^{\sigma_1}\gamma^{\sigma_2}) = 0\ , \tag{A.7d}$$
$$\mathrm{tr}\left(\gamma^5\gamma^{\sigma_1}\dots\gamma^{\sigma_{2n+1}}\right) = 0\ , \tag{A.7e}$$

reflecting the fact that the trace of an odd number of gamma matrices vanishes, both with and without the insertion of γ^5.

Conversely, the following traces are non-vanishing and are frequently used in practical calculations:

$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}\ , \tag{A.8a}$$
$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\right) = 4g^{\mu\nu}g^{\sigma\rho} - 4g^{\mu\sigma}g^{\nu\rho} + 4g^{\mu\rho}g^{\nu\sigma}\ , \tag{A.8b}$$
$$\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^5\right) = -4\mathrm{i}\varepsilon^{\mu\nu\sigma\rho}\ . \tag{A.8c}$$

These identities are extensively used throughout this work in the evaluation of traces appearing in the Wigner-function formalism and in the computation of spin-dependent observables.

Appendix B

Hypersurface integral

Lets start from the expression (4.17):

$$\begin{aligned}\Delta W^{+}_{\rm LE}(x,k) = & \sum_j \mathfrak{s}_j \sum_{N=0}^{+\infty} \frac{(-\mathrm{i})^N}{N!} \Big[\partial^q_{\nu_1}\partial^q_{\nu_2}\dots\partial^q_{\nu_N} G^{\mu\nu}(k,q,\beta)\Big]_{q=0} \\ & \times \sum_{\bar{y}_k(x)} \sum_{M=0}^{N} \frac{N!\,(-1)^M}{M!\,(N-M)!} \mathrm{d}_x^{\nu_{M+1}}\dots\mathrm{d}_x^{\nu_N} \\ & \times \left\{ \sigma_\mu \Big[\mathrm{d}_x^{\nu_1}\mathrm{d}_x^{\nu_2}\dots\mathrm{d}_x^{\nu_M}\Delta\beta_\nu(y,x)\Big]\Big|_{y=\bar{y}_k(x)} \right\} \\ & - \text{analogous term in}\,\Delta\zeta\ ,\end{aligned} \tag{B.1}$$

where we have introduced the total derivative:

$$\mathrm{d}_x^\mu = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\ ,$$

to emphasize the difference between the derivative acting on the function *before* setting $y = \bar{y}_k(x)$ (that is ∂_x) and the derivative acting on the function *after* setting $y = \bar{y}_k(x)$. Replacing σ with the unit vector normal to the hypersurface:

$$n_\mu = \frac{\sigma_\mu}{\sqrt{|\sigma\cdot\sigma|}}\ ,$$

the equation (B.1) is converted to:

$$\begin{aligned}\Delta W^{+}_{\rm LE}(x,k) \ = \ & \sum_j \mathfrak{s}_j \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \left[\partial^q_{\nu_1}\dots\partial^q_{\nu_N} G^{\mu\nu}(q)\right]\big|_{q=0} \sum_{M=0}^{N} \frac{N!(-1)^M}{M!(N-M)!} \\ & \times \mathrm{d}_x^{\nu_{M+1}}\dots\mathrm{d}_x^{\nu_N} \left[\frac{\hat{n}_\mu}{|k\cdot\hat{n}|}\partial_x^{\nu_1}\dots\partial_x^{\nu_M}\Delta\beta_\nu(y,x)\right]\Bigg|_{y=\bar{y}(x,k)} \\ & - \text{analogous term for}\ \Delta\zeta\ ,\end{aligned} \tag{B.2}$$

Now, the derivative of the function $y^\nu = \bar{y}^\nu_k(x)$ is obtained by taking into account that:

$$\bar{\mathbf{y}}_k - \frac{\mathbf{k}}{k^0}\bar{y}^0_k = \mathbf{x} - \frac{\mathbf{k}}{k^0}x^0\ ,$$

with $\bar{y}^0_k = f_k(\bar{\mathbf{y}}_k)$. Taking partial derivatives with respect to x^μ of the above equation:

$$\frac{\partial \bar{y}^j_k}{\partial x^\mu} - \frac{k^j}{k^0}\frac{\partial \bar{y}^0_k}{\partial \bar{y}^l_k}\frac{\partial \bar{y}^l_{k,i}}{\partial x^\mu} = \left[\delta^j_l + \frac{k^j}{k^0}\sigma_l\right]\frac{\partial \bar{y}^l_k}{\partial x^\mu} = \delta^j_\mu - \frac{k^j}{k^0}\delta^0_\mu\ ,$$

where $j, l = 1, 2, 3$ and where $\sigma_\mu(\bar{y}_k)$ is the normal vector of Σ_D at the spacetime point $\bar{y}_k$,

$$\sigma_\mu(\bar{y}_k) = \left(1, -\frac{\partial f_k(\bar{\mathbf{y}}_k)}{\partial \bar{\mathbf{y}}_k}\right) .$$

The 3×3 matrix:

$$A^j_l = \left[\delta^j_l + \frac{k^j}{k^0}\sigma_l\right] ,$$

can be inverted and one obtains:

$$\frac{\partial \bar{y}^j_{k,i}}{\partial x^\mu} = \left(A^{-1}\right)^j{}_l \left(\delta^l_\mu - \frac{k^l}{k^0}\delta^0_\mu\right) = \left(\delta^j_l - \frac{k^j \sigma_l}{k \cdot \sigma}\right)\left(\delta^l_\mu - \frac{k^l}{k^0}g^0_\mu\right) . \tag{B.3}$$

Similarly, one can calculate the derivative of y^0 with respect to x:

$$\frac{\partial \bar{y}^0_k}{\partial x^\mu} = \frac{\partial \bar{y}^0_k}{\partial \bar{y}^j_k}\frac{\partial \bar{y}^j_k}{\partial x^\mu} = -\sigma_j\left(\delta^j_\mu - \frac{k^j\sigma_\mu}{k \cdot \sigma}\right) ,$$

whence we obtain:

$$\frac{\partial \bar{y}^0_k}{\partial x^0} = \sigma_0 - \frac{k^0}{k \cdot \sigma} , \qquad \frac{\partial \bar{y}^0_k}{\partial x^m} = -\frac{k^0 \sigma_m}{k \cdot \sigma} . \tag{B.4}$$

The equations (B.3) and (B.4) can be written in a compact form as:

$$\frac{\partial \bar{y}^\nu_k}{\partial x^\mu} = \delta^\nu_\mu - \frac{k^\nu \sigma_\mu(\bar{y}_k)}{k \cdot \sigma(\bar{y}_k)} \equiv \Delta^\nu_\mu(\bar{y}_k) .$$

Finally, using the chain rule for the derivative of an implicit function:

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu} g(x, y(x)) = \frac{\partial}{\partial x^\mu} g(x, y(x)) + \Delta^\nu_\mu \left.\frac{\partial g(x,y)}{\partial y^\nu}\right|_{y=y(x)} ,$$

the equation (5.44) can be rewritten as:

$$\begin{aligned}
\Delta W^+(x,k) = &\sum_{N=0}^{\infty} \frac{(-\mathrm{i})^N}{N!} \left[\partial^q_{\nu_1} \ldots \partial^q_{\nu_N} G^{\mu\nu}(q)\right]\Big|_{q=0} \\
&\times \sum_{M=0}^{N} \frac{N!(-1)^M}{M!(N-M)!} \left[\partial^{\nu_{M+1}}_x + \Delta^{\alpha_{M+1}\rho_{M+1}}(\bar{y}_k)\partial^y_{\rho_{M+1}}\right] \\
&\times \ldots \left[\partial^{\nu_n}_x + \Delta^{\nu_n\rho_n}(\bar{y}_k)\partial^y_{\rho_n}\right] \partial^{\nu_1}_x \cdots \partial^{\nu_M}_x \left.\frac{n_\mu(y)}{|k \cdot\ n(y)|}\Delta\beta_\nu(y,x)\right|_{y=\bar{y}_k(x)} \\
&- \text{analogous term for } \Delta\zeta .
\end{aligned}$$

It is convenient to introduce the following differential operator:

$$D_y(\bar{y}_k) \equiv \Delta^{\nu\rho}(\bar{y}_k)\partial^y_\rho \partial^q_\nu , \tag{B.5}$$

By using the binomial theorem, the last expression can be finally recast as:

$$\begin{aligned}
\Delta W^+(x,k) = &\sum_{N=0}^{\infty} \frac{(-\mathrm{i})^N}{N!} \left(D_y\left(\bar{y}_k(x)\right)\right)^N \\
&\times \left.\left[G^{\mu\nu}(q)\frac{n_\mu(y)}{|k \cdot n(y)|}\Delta\beta_\nu(y,x)\right]\right|_{q=0, y=\bar{y}_k(x)} \\
&+ \text{ analogous term for } \Delta\zeta ,
\end{aligned}$$

which coincides with eq. (4.23) in the main text. Note that the differential operator (B.5) acts only on the square bracket and not on the projector Δ which is already computed on the intersection $\bar{y}_k(x)$ hence (B.5) never acts upon itself.

Appendix C

Second order local equilibrium corrections

In this appendix we perform the complete second order computation of the local equilibrium correction to the Wigner function:

$$\begin{aligned}\Delta^{(2)}W^{+}_{\rm LE}(x,k) = &\frac{1}{2}\sum_{\bar{y}_k(x)}\Big[D_y\left(\bar{y}_k(x)\right)\Big]^2 \\ &\times\left\{\left[G^{\mu\nu}(k,q,\beta)\,\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{\substack{q=0\\ y=\bar{y}_k(x)}}\right. \\ &\left. -\left[H^{\mu}(k,q,\beta)\,\frac{\hat{n}_\mu(y)\Delta\zeta(y,x)}{|k\cdot\hat{n}(y)|}\right]_{\substack{q=0\\ y=\bar{y}_k(x)}}\right\}.\end{aligned} \tag{C.1}$$

In order to compute the above expression we first need to compute the second derivative with respect to q of the functions $G^{\mu\nu}$ and H^{μ} defined in (4.10). Expanding the exponential factor and the δ function up to the second order in q we have:

$$\begin{aligned}\frac{1-{\rm e}^{\beta(x)\cdot q}}{\beta(x)\cdot q} &\simeq -1-\frac{1}{2}\beta_\lambda(x)q^\lambda-\frac{1}{6}\beta_{\lambda_1}(x)\beta_{\lambda_2}(x)q^{\lambda_1}q^{\lambda_2}\,,\\ \delta\left(k^2+\frac{q^2}{4}-m^2\right) &\simeq \delta\left(k^2-m^2\right)+\frac{1}{4}\delta'\left(k^2-m^2\right)q^2\,,\end{aligned}$$

where δ' indicates the derivative of the Dirac δ function with respect to its own argument and it is intended to be a distribution in k.

These can be combined with the expansion of the function $G^{\mu\nu}$ giving:

$$\begin{aligned}\partial^q_{\lambda_1}\partial^q_{\lambda_2}\Big[G^{\mu\nu}(k,q,\beta)\Big]_{q=0} = &-\frac{1}{2}\delta\left(k^2-m^2\right)\left[\partial^q_{\lambda_1}\partial^q_{\lambda_2}\Gamma^{\mu\nu}(k,q,\beta)\Big|_{q=0}\right. \\ &\left.-\frac{1}{6}\beta_{\lambda_1}(x)\beta_{\lambda_2}(x)\Gamma^{\mu\nu}(k,0,\beta)\right] \\ &-\frac{1}{4}\delta'\left(k^2-m^2\right)g_{\lambda_1\lambda_2}\Gamma^{\mu\nu}(k,0,\beta)\ .\end{aligned} \tag{C.2}$$

Thanks to the above relation we can separate the second order local equilibrium correction in two different terms, one on-shell:

$$
\begin{aligned}
\Delta^{(2)}_{\rm on} W^{+}_{\rm LE}(x,k) = & -\frac{\delta\left(k^2-m^2\right)}{4}\left[\partial^q_{\lambda_1}\partial^q_{\lambda_2}\Gamma^{\mu\nu}(k,q,\beta)\Big|_{q=0}\right.\\
& \left. -\frac{1}{6}\beta_{\lambda_1}(x)\beta_{\lambda_2}(x)\Gamma^{\mu\nu}(k,0,\beta)\right]\\
& \times\sum_{\bar{y}_k(x)}\Delta^{\lambda_1\gamma_1}(\bar{y})\Delta^{\lambda_2\gamma_2}(\bar{y})\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)},
\end{aligned}
\tag{C.3}
$$

and the other off-shell:

$$
\begin{aligned}
\Delta^{(2)}_{\rm off} W^{+}_{\rm LE} = & -\frac{\delta'\left(k^2-m^2\right)}{8}g_{\lambda_1\lambda_2}\Gamma^{\mu\nu}(k,0,\beta)\\
& \times\sum_{\bar{y}_k(x)}\Delta^{\lambda_1\gamma_1}(\bar{y})\Delta^{\lambda_2\gamma_2}(\bar{y})\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)},
\end{aligned}
\tag{C.4}
$$

plus analogous terms in the reduced chemical potential.

The value of $\Gamma^{\mu\nu}(k,q,\beta)$ has been computed in (4.24) and reads:

$$
\Gamma^{\mu\nu}(k,q,\beta) = \frac{2}{(2\pi)^3}\left[k^\mu k^\nu + \frac{1}{4}\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)\right] n_{\rm B}(k_+)\left[1+n_{\rm B}(k_-)\right] . \tag{C.5}
$$

This implies:

$$
\Gamma^{\mu\nu}(k,0,\beta) = \frac{2}{(2\pi)^3}k^\mu k^\nu n_{\rm B}(k)\left[1+n_{\rm B}(k)\right] , \tag{C.6}
$$

and:

$$
\begin{aligned}
\partial^q_{\lambda_1}\partial^q_{\lambda_2}\Gamma^{\mu\nu}(k,q,\beta)\Big|_{q=0} = & \frac{n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]}{(2\pi^3)}\left[k^\mu k^\nu \beta_{\lambda_1}(x)\beta_{\lambda_2}(x)\right.\\
& \left. -\frac{1}{2}\left(\delta^\mu_{\lambda_1}\delta^\nu_{\lambda_2}+\delta^\nu_{\lambda_1}\delta^\mu_{\lambda_2}-2g^{\mu\nu}g_{\lambda_1\lambda_2}\right)\right] ,
\end{aligned}
\tag{C.7}
$$

where we used that the first derivative of the term in the square brackets of (C.5) vanishes for $q=0$. Overall the on-shell term reads:

$$
\begin{aligned}
\Delta^{(2)}_{\rm on} W^{+}(x,k) = & -\frac{\delta(k^2-m^2)}{4(2\pi)^3}n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\\
& \times\left[\frac{1}{3}k^\mu k^\nu\beta_{\lambda_1}(x)\beta_{\lambda_2}(x)-\frac{1}{2}\left(\delta^\mu_{\lambda_1}\delta^\nu_{\lambda_2}+\delta^\nu_{\lambda_1}\delta^\mu_{\lambda_2}-2g^{\mu\nu}g_{\lambda_1\lambda_2}\right)\right]\\
& \times\sum_{\bar{y}_k(x)}\Delta^{\lambda_1\gamma_1}(\bar{y})\Delta^{\lambda_2\gamma_2}(\bar{y})\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)} .
\end{aligned}
\tag{C.8}
$$

Now taking the contractions with the differential operator $\Delta^{\lambda_1\gamma_1}\Delta^{\lambda_2\gamma_2}\partial^y_{\gamma_1}\partial^y_{\gamma_2}$ and introducing the operators:

$$
\begin{aligned}
\mathcal{B}^{\gamma_1\gamma_2}(y,x) \equiv & \beta_{\lambda_1}(x)\beta_{\lambda_2}(x)\Delta^{\lambda_1\gamma_1}(y)\Delta^{\lambda_2\gamma_2}\\
= & \beta^{\gamma_1}(x)\beta^{\gamma_2}(x)+\left(\frac{\beta(x)\cdot\hat{n}(y)}{|k\cdot\hat{n}(y)|}\right)^2 k^{\gamma_1}k^{\gamma_2}\\
& -\frac{2\beta(x)\cdot\hat{n}(y)}{|k\cdot\hat{n}(y)|}\beta^{\gamma_1}(x)k^{\gamma_2} ,
\end{aligned}
\tag{C.9}
$$

and:

$$\mathcal{K}^{\mu\nu,\gamma_1\gamma_2}(y) \equiv \frac{k^{\gamma_1}k^{\gamma_2}}{(k\cdot\hat{n}(y))^2}\left[(\hat{n}(y))^2 g^{\mu\nu} - \hat{n}^\mu(y)\hat{n}^\nu(y)\right] + 2\left(g^{\gamma_1\mu}g^{\gamma_2\nu} - g^{\mu\nu}g^{\gamma_1\gamma_2}\right)$$
$$+ \frac{1}{|k\cdot\hat{n}(y)|}\left[2g^{\mu\nu}\hat{n}^{\gamma_1}(y)k^{\gamma_2} - g^{\gamma_1\mu}\hat{n}^\nu(y)k^{\gamma_2} - g^{\gamma_2\nu}\hat{n}^\mu(y)k^{\gamma_1}\right], \quad \text{(C.10)}$$

the on-shell contribution can written as:

$$\Delta^{(2)}_{\rm on} W^+(x,k)\Big|_\beta = -\frac{\delta(k^2-m^2)}{4(2\pi)^3} n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\sum_{\bar{y}_k(x)}\Bigg\{\frac{1}{3}{\rm sgn}(k\cdot\hat{n}(\bar{y}_k)$$
$$\times\,\mathcal{B}^{\gamma_1\gamma_2}(\bar{y}_k,x)\,\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[k\cdot\beta(y,x)\right]\Big|_{y=\bar{y}_k(x)} \quad \text{(C.11)}$$
$$-\,\mathcal{K}^{\mu\nu,\gamma_1\gamma_2}(\bar{y}_k)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)}\Bigg\}.$$

Note that term contracted with $\mathcal{K}$ gives rise to three terms:

$$\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)} = \frac{\hat{n}_\mu(\bar{y}_k)}{|k\cdot\hat{n}(\bar{y}_k)|}\partial^y_{\gamma_1}\partial^y_{\gamma_2}\beta_\nu(y)\Big|_{y=\bar{y}_k}$$
$$+\,2\partial^y_{\gamma_1}\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]\Big|_{y=\bar{y}_k}\partial^y_{\gamma_2}\beta_\nu(y)\Big|_{y=\bar{y}_k}$$
$$+\,\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)}{|k\cdot\hat{n}(y)|}\right]\Big|_{y=\bar{y}_k}\Delta\beta_\nu(y,x)\Big|_{y=\bar{y}_k},$$

where another term proportional to $\Delta\beta_\nu$ appears again (that is a term independent from the gradient of the thermodynamic fields). As already mentioned in the main text for mildly curved hypersurfaces higher order terms in the q–expansion produces new terms depending on $\Delta\beta_\nu$ which may be comparable with the leading order one.

The computation of the off-shell term is much simpler and leads to:

$$\Delta^{(2)}_{\rm off} W^+_{\rm LE} = -\frac{\delta'(k^2-m^2)}{4(2\pi)^3} n_{\rm B}(k)\left[1+n_{\rm B}(k_+)\right] g_{\lambda_1\lambda_2}k^\mu k^\nu$$
$$\times\sum_{\bar{y}_k(x)}\Delta^{\lambda_1\gamma_1}(\bar{y})\Delta^{\lambda_2\gamma_2}(\bar{y})\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_\mu(y)\Delta\beta_\nu(y,x)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)}, \quad \text{(C.12)}$$

Taking the contraction between the metric tensor and the projectors:

$$g_{\lambda_1\lambda_2}\Delta^{\lambda_1\gamma_1}(\bar{y})\Delta^{\lambda_2\gamma_2}(\bar{y}) = \Delta^{\lambda\gamma_1}(\bar{y})\Delta_\lambda^{\ \gamma_2}(\bar{y})$$
$$= g^{\gamma_1\gamma_2} - \frac{\hat{n}^{\gamma_1}(y)k^{\gamma_2}}{|k\cdot\hat{n}(y)|} - \frac{\hat{n}^{\gamma_2}(y)k^{\gamma_1}}{|k\cdot\hat{n}(y)|} + \frac{k^{\gamma_1}k^{\gamma_2}}{(k\cdot\hat{n}(y))^2}$$
$$\equiv \mathcal{P}^{\gamma_1\gamma_2}(y),$$

the off-shell contribution reads:

$$\Delta^{(2)}_{\rm off} W^+_{\rm LE}\Big|_\beta = -\frac{\delta'(k^2-m^2)}{4(2\pi)^3} n_{\rm B}(k)\left[1+n_{\rm B}(k_+)\right]$$
$$\times\sum_{\bar{y}_k}{\rm sgn}(k\cdot\hat{n}(\bar{y}_k))\mathcal{P}^{\gamma_1\gamma_2}(\bar{y}_k)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[k\cdot\beta(y)\right]\Big|_{y=\bar{y}_k}. \quad \text{(C.13)}$$

The same computation can be made for the chemical potential term for which:

$$H^{\mu}\left(k,0,\beta\right)=\frac{2}{(2\pi)^3}k^{\mu}\,n_{\rm B}(k)\left[1+n_{\rm B}(k)\right] \tag{C.14}$$

and:

$$\partial^q_{\lambda_1}\partial^q_{\lambda}\left[H^{\mu}\left(k,q,\beta\right)\right]_{q=0}=\frac{1}{2}\beta_{\lambda_1}(x)\beta_{\lambda_2}(x)H^{\mu}\left(k,0,\beta\right)\ . \tag{C.15}$$

With these we have:

$$\begin{aligned}\Delta_{\rm on}W^{+}_{\rm LE}\left(x,k\right)\Big|_{\zeta}=&-\frac{\delta\left(k^2-m^2\right)}{4(2\pi)^3}n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\sum_{\bar{y}_k(x)}\\&\times\left\{\frac{{\rm sgn}(k\cdot\hat{n}(\bar{y}_k))}{3}\mathcal{B}^{\gamma_1\gamma_2}\left(\bar{y}_k,x\right)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\zeta(y)\right]\Big|_{y=\bar{y}_k}\right\}\ ,\end{aligned} \tag{C.16}$$

and

$$\begin{aligned}\Delta_{\rm of}W^{+}_{\rm LE}\left(x,k\right)\Big|_{\zeta}=&-\frac{\delta'\left(k^2-m^2\right)}{4(2\pi)^3}n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\sum_{\bar{y}_k(x)}\\&\times\left\{{\rm sgn}(k\cdot\hat{n}(\bar{y}_k))\mathcal{P}^{\gamma_1\gamma_2}\left(\bar{y}_k,\right)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\zeta(y)\right]\Big|_{y=\bar{y}_k}\right\}\ .\end{aligned} \tag{C.17}$$

Overall the second order correctionto the Wigner function reads:

$$\Delta^{(2)}W^{+}_{\rm LE}\left(x,k\right)=\Delta^{(2)}_{\rm on}W^{+}_{\rm LE}\left(x,k\right)+\Delta^{(2)}_{\rm of}W^{+}_{\rm LE}\left(x,k\right)\ , \tag{C.18}$$

where:

$$\begin{aligned}\Delta^{(2)}_{\rm on}W^{+}\left(x,k\right)=&-\frac{\delta(k^2-m^2)}{4(2\pi)^3}n_{\rm B}(k)\left[1+n_{\rm B}(k)\right]\sum_{\bar{y}_k(x)}\Bigg\{\frac{1}{3}{\rm sgn}(k\cdot\hat{n}(\bar{y}_k)\\&\times\mathcal{B}^{\gamma_1\gamma_2}\left(\bar{y}_k,x\right)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\Big[k\cdot\beta\left(y,x\right)-\zeta\left(y,x\right)\Big]\Big|_{y=\bar{y}_k(x)}\\&-\mathcal{K}^{\mu\nu,\gamma_1\gamma_2}(\bar{y}_k)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[\frac{\hat{n}_{\mu}(y)\Delta\beta_{\nu}\left(y,x\right)}{|k\cdot\hat{n}(y)|}\right]_{y=\bar{y}_k(x)}\Bigg\}\ ,\end{aligned} \tag{C.19}$$

and:

$$\begin{aligned}\Delta^{(2)}_{\rm off}W^{+}_{\rm LE}=&-\frac{\delta'\left(k^2-m^2\right)}{4(2\pi)^3}n_{\rm B}(k)\left[1+n_{\rm B}(k_+)\right]\sum_{\bar{y}_k}{\rm sgn}(k\cdot\hat{n}(\bar{y}_k))\\&\times\mathcal{P}^{\gamma_1\gamma_2}(\bar{y}_k)\partial^y_{\gamma_1}\partial^y_{\gamma_2}\left[k\cdot\beta(y)-\zeta(y)\right]\Big|_{y=\bar{y}_k}\ .\end{aligned} \tag{C.20}$$

Appendix D

Decomposition of $a^\mu a^\nu$

We show that the tensor $a^\mu a^\nu$, where $a^\mu \equiv \epsilon^{\mu\rho\sigma\tau} k_\rho q_\sigma \beta_\tau$, can be expressed in terms of other symmetric tensors built with k, q and β. To make notation compact, we introduce the four-vectors $\bar{q}^\mu$and $\bar{\beta}^\mu$ such that $k^\mu, \bar{q}^\mu, \bar{\beta}^\mu$ are perpendicular to each other: as:

$$
\begin{aligned}
\bar{q}^\mu &= \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k \cdot k} \right) q_\nu \ , \\
\bar{\beta}^\mu &= \left(g^{\mu\nu} - \frac{\bar{q}^\mu \bar{q}^\nu}{\bar{q} \cdot \bar{q}} \right) \left(g_{\nu\rho} - \frac{k_\nu k_\rho}{k \cdot k} \right) \beta^\rho \ .
\end{aligned} \tag{D.1}
$$

Thereby a^μ can be written as:

$$
a^\mu = \epsilon^{\mu\rho\sigma\tau} k_\rho \bar{q}_\sigma \bar{\beta}_\tau \ ;
$$

thus:

$$
a^\mu a^\nu = \epsilon^{\mu\rho\sigma\tau} k_\rho \bar{q}_\sigma \bar{\beta}_\tau \epsilon^{\nu\alpha\lambda\xi} k_\alpha \bar{q}_\lambda \bar{\beta}_\xi = \epsilon^{\mu\rho\sigma\tau} k_\rho \tilde{q}_\sigma \bar{\beta}_\tau \epsilon^{\nu}{}_{\alpha}{}^{\lambda\xi} k^\alpha \bar{q}_\lambda \bar{\beta}_\xi \ . \tag{D.2}
$$

With the help of the Schouten identity (4.50), we can write:

$$
\epsilon^{\mu\rho\sigma\tau} k^\alpha = - \left(\epsilon^{\rho\sigma\tau\alpha} k^\mu + \epsilon^{\sigma\tau\alpha\mu} k^\rho + \epsilon^{\tau\alpha\mu\rho} k^\sigma + \epsilon^{\alpha\mu\rho\sigma} k^\tau \right) \ ,
$$

hence the (D.2) can be rewritten as:

$$
a^\mu a^\nu = - \left(\epsilon^{\rho\sigma\tau\alpha} k^\mu + \epsilon^{\sigma\tau\alpha\mu} k^\rho + \epsilon^{\tau\alpha\mu\rho} k^\sigma + \epsilon^{\alpha\mu\rho\sigma} k^\tau \right) \epsilon^{\nu}{}_{\alpha}{}^{\lambda\xi} k_\rho \bar{q}_\sigma \bar{\beta}_\tau \bar{q}_\lambda \bar{\beta}_\xi \ .
$$

The contraction of two Levi-Civita symbols in the above equation can be expanded as:

$$
\begin{aligned}
&\epsilon^{\rho\sigma\tau\alpha} \epsilon^{\nu}{}_{\alpha}{}^{\lambda\xi} = \epsilon^{\alpha\rho\sigma\tau} \epsilon_{\alpha}{}^{\nu\lambda\xi} \\
&= -g^{\rho\nu}(g^{\sigma\lambda} g^{\tau\xi} - g^{\sigma\xi} g^{\tau\lambda}) - g^{\rho\lambda}(g^{\sigma\xi} g^{\tau\nu} - g^{\sigma\nu} g^{\tau\xi}) - g^{\rho\xi}(g^{\sigma\nu} g^{\tau\lambda} - g^{\sigma\lambda} g^{\tau\nu}) \ .
\end{aligned}
$$

Since $k \cdot \bar{q} = k \cdot \bar{\beta} = \bar{q} \cdot \bar{\beta} = 0$, the (D.2) can be finally cast in the following form:

$$
\begin{aligned}
a^\mu a^\nu &= - \left(\epsilon^{\alpha\rho\sigma\tau} \epsilon_{\alpha}{}^{\nu\lambda\xi} k^\mu - \epsilon^{\alpha\sigma\tau\mu} \epsilon_{\alpha}{}^{\nu\lambda\xi} k^\rho + \epsilon^{\alpha\tau\mu\rho} \epsilon_{\alpha}{}^{\nu\lambda\xi} k^\sigma - \epsilon^{\alpha\mu\rho\sigma} \epsilon_{\alpha}^{\nu\lambda\xi} k^\tau \right) k_\rho \bar{q}_\sigma \bar{\beta}_\tau \bar{q}_\lambda \bar{\beta}_\xi \\
&= \bar{q}^2 \bar{\beta}^2 k^\mu k^\nu + k^2 \bar{\beta}^2 \bar{q}^\mu \bar{q}^\nu + k^2 \bar{q}^2 \bar{\beta}^\mu \bar{\beta}^\nu - k^2 \bar{q}^2 \bar{\beta}^2 g^{\mu\nu} \ .
\end{aligned}
$$

Since $\bar{q}^\mu$ and $\bar{\beta}^\mu$ are defined as linear combinations of k^μ, q^μ, and β^μ in equation (D.1), we can further expand $\bar{q}^\mu \bar{q}^\nu$ and $\bar{\beta}^\mu \bar{\beta}^\nu$ in terms of $k^\mu k^\nu$, $q^\mu q^\nu$, $\beta^\mu \beta^\nu$, $k^\mu q^\nu + k^\nu q^\mu$, $k^\mu \beta^\nu + k^\nu \beta^\mu$ and $q^\mu \beta^\nu + q^\nu \beta^\mu$. As a consequence, $a^\mu a^\nu$ turns out to be a linear combination of these symmetric tensors and $g^{\mu\nu}$.

Appendix E

Complex conjugation, time-reversal and parity

The correlators in the equation (5.5) are constrained by the properties of the density operator, creation/annihilation operators and stress-energy tensor operator under discrete transformations: complex conjugation, time-reversal and parity. The correlator is defined as:

$$\begin{aligned}\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-), \widehat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}} = & \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta(x)\cdot\widehat{P}+\zeta(x)\widehat{Q}}\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\widehat{T}^{\mu\nu}(0)\right) \\ & - \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta(x)\cdot\widehat{P}+\zeta(x)\widehat{Q}}\widehat{A}^\dagger(k_+)\widehat{A}(k_-)\right) \\ & \times \frac{1}{Z}\mathrm{Tr}\left(\mathrm{e}^{-\beta(x)\cdot\widehat{P}+\zeta(x)\widehat{Q}}\,\widehat{T}^{\mu\nu}(0)\right) .\end{aligned} \tag{E.1}$$

Taking the complex conjugate of both sides, using $\mathrm{Tr}(\widehat{O})^* = \mathrm{Tr}(\widehat{O}^\dagger)$ and the relations (3.34), (3.35) one obtains:

$$\langle \widehat{A}^\dagger(k_+)\widehat{A}(k_-), \widehat{T}^{\mu\nu}(0)\rangle^*_{c,\mathrm{GE}} = \langle \widehat{A}^\dagger(k_-)\widehat{A}(k_+), \widehat{T}^{\mu\nu}(0)\rangle_{c,\mathrm{GE}}\,\mathrm{e}^{-\beta(x)\cdot q}$$

where $q = k_+ - k_-$. Hence, according to the definition (5.5):

$$\Theta^{\mu\nu}(k, q, \beta)^* = \mathrm{e}^{-\beta(x)\cdot q}\,\Theta^{\mu\nu}(k, -q, \beta) . \tag{E.2}$$

We now come to the time-reversal and parity transformations. At operator level time-reversal and parity are described by an involutive anti-unitary $\widehat{\mathcal{T}}$ and a unitary operator $\widehat{\Pi}$ respectively:

$$\widehat{\mathcal{T}}^\dagger = \widehat{\mathcal{T}}^{-1}, \quad \widehat{\mathcal{T}}^2 = \mathrm{I}, \quad \widehat{\mathcal{T}}\left(\alpha\,|\mathcal{H}\rangle\right) = \alpha^*\widehat{\mathcal{T}}\,|\mathcal{H}\rangle , \tag{E.3a}$$

$$\widehat{\Pi}^\dagger = \widehat{\Pi}^{-1}, \quad \widehat{\Pi}^2 = \mathrm{I}, \quad \widehat{\Pi}\left(\alpha\,|\mathcal{H}\rangle\right) = \alpha\widehat{\Pi}\,|\mathcal{H}\rangle , \tag{E.3b}$$

with $|\mathcal{H}\rangle$ is a complex vector on an Hilbert space and $\alpha \in \mathbb{C}$ is a complex number. The field transforms under time-reversal and parity as follows:

$$\widehat{\mathcal{T}}\widehat{\phi}(x)\widehat{\mathcal{T}} = \eta_T\widehat{\phi}(-x^0, \mathbf{x}) \qquad \widehat{\Pi}\widehat{\phi}(x)\widehat{\Pi} = \eta_\Pi\widehat{\phi}(x^0, -\mathbf{x})$$

where η_T and η_Π are phase factors $= \pm 1$. Hence, from (3.32), it follows:

$$\widehat{\mathcal{T}}\,\widehat{A}(k)\,\widehat{\mathcal{T}} = \eta_T\widehat{A}(\widetilde{k}), \qquad \widehat{\mathcal{T}}\,\widehat{A}^\dagger(k)\,\widehat{\mathcal{T}} = \eta_T^*\widehat{A}^\dagger(\widetilde{k}) , \tag{E.4a}$$

$$\widehat{\Pi}\,\widehat{A}(k)\,\widehat{\Pi} = \eta_\Pi\widehat{A}(\widetilde{k}), \qquad \widehat{\Pi}\,\widehat{A}^\dagger(k)\,\widehat{\Pi} = \eta_\Pi^*\widehat{A}^\dagger(\widetilde{k}) , \tag{E.4b}$$

where $\widetilde{k}$ is the time-reversal/parity transformed of the four-momentum k:

$$k = \left(k^0, \mathbf{k}\right) \mapsto \widetilde{k} = \left(k^0, -\mathbf{k}\right) \ . \tag{E.5}$$

The stress-energy tensor operator in $x = 0$ transforms under time reversal as:

$$\widehat{\mathcal{T}} \widehat{T}^{\mu\nu}(0) \widehat{\mathcal{T}} = \theta^\mu_\alpha \theta^\nu_\beta \widehat{T}^{\alpha\beta}(0) \ ,$$

and likewise for parity, with $\theta^\mu_\alpha = \mathrm{diag}\,(1, -1, -1, -1)$. In turn, the density operator at global equilibrium is such that:

$$\widehat{\mathcal{T}} \widehat{\rho}_{\mathrm{GE}}(\beta, \zeta) \widehat{\mathcal{T}} = \widehat{\rho}_{\mathrm{GE}}(\widetilde{\beta}, \zeta) \ , \tag{E.6}$$

and likewise for parity, where $\widetilde{\beta}$ is defined the same way as $\widetilde{k}$ in eq. (E.5). From the (E.6) and the general relation:

$$\mathrm{Tr}(\widehat{O}^\dagger) = \mathrm{Tr}(\widehat{O})^* = \mathrm{Tr}(\widehat{\mathcal{T}} \widehat{O} \widehat{\mathcal{T}}) \ ,$$

with $\widehat{O}$ any operator, the following relation can be obtained for the correlator in Eq. (5.5):

$$\Theta^{\mu\nu}\left(k, q, \beta\right) = \mathrm{e}^{-\beta(x)\cdot q} \theta^\mu_\alpha \theta^\nu_\beta \, \Theta^{\alpha\beta}\left(\widetilde{k}, -\widetilde{q}, \widetilde{\beta}\right) \ . \tag{E.7}$$

Likewise, for parity, being $\widehat{\Pi}$ linear, one has:

$$\mathrm{Tr}(\widehat{O}) = \mathrm{Tr}(\widehat{\Pi}\, \widehat{O}\, \widehat{\Pi}) \ ,$$

and correspondingly:

$$\Theta^{\mu\nu}\left(k, q, \beta\right) = \theta^\mu_\alpha \theta^\nu_\beta \, \Theta^{\alpha\beta}\left(\widetilde{k}, \widetilde{q}, \widetilde{\beta}\right) \ . \tag{E.8}$$

The extension of the relations (E.2), (E.7), and (E.8) to the tensor coefficients $\Gamma^{\mu\nu}_{\vartheta,f}$ appearing in Eq. (5.17) is not straightforward. The scalar arguments S defined in Eq. (5.7) are invariant under the replacement of all four-vectors by their tilde-transformed counterparts, corresponding to parity or time-reversal transformations. However, they are not, in general, invariant under the transformation $q \mapsto -q$. As a consequence, while the constraints imposed by a δ function and by the same δ function with tilde-transformed arguments coincide, this is not generally the case when the sign of q is also sign-reversed.

For the parity transformations, the invariance of arguments of the δ functions and the scalar product $q \cdot w$, allows one to extend the relation (E.8) to each tensor $\Gamma^{\mu\nu}_{\vartheta,f}$ independently, yielding:

$$\Gamma^{\mu\nu}_{\vartheta,f}\left(k, q, \beta\right) = \theta^\mu_\alpha \, \theta^\nu_\beta \, \Gamma^{\alpha\beta}_{\vartheta,f}\left(\widetilde{k}, \widetilde{q}, \widetilde{\beta}\right) \ .$$

On the other hand, upon plugging the expansion (5.17) into the relations (E.2) and (E.7), it is found that the transformation $q \mapsto -q$ changes, in general, the argument of the delta distribution, mapping the tensor $\Gamma_{\vartheta,f}$ into a different term. In symbols:

$$\Gamma^{\mu\nu}_{\vartheta,f}\left(k, q, \beta\right) = \mathrm{e}^{-\beta(x)\cdot q} \, \Gamma^{\mu\nu}_{\vartheta,\underline{f}}\left(k, -q, \beta\right) \ ,$$

$$\Gamma^{\mu\nu}_{\vartheta,f}\left(k, q, \beta\right) = \mathrm{e}^{-\beta(x)\cdot q} \, \theta^\mu_\alpha \, \theta^\nu_\beta \, \Gamma^{\alpha\beta}_{\vartheta,\underline{f}}\left(\widetilde{k}, -\widetilde{q}, \widetilde{\beta}\right) \ .$$

where $\Gamma_{\vartheta,\underline{f}}$ denotes the tensor in the expansion (5.17) associated with the δ function fulfilling:

$$\delta\big(q\cdot w(\vartheta)+\underline{f}(S)\big)\Big|_{q\mapsto -q}=\delta(q\cdot w(\vartheta)-f(S))\ ,$$

where we used that $q\cdot w$ changes sign and that the δ functions is even. For this reason the particular case $f(S)=0$ is invariant for $q\mapsto -q$ and we thus get:

$$\Gamma^{\mu\nu}_{\vartheta,0}\,(k,q,\beta)^* = \mathrm{e}^{-\beta(x)\cdot q}\,\Gamma^{\mu\nu}_{\vartheta,0}\,(k,-q,\beta)\ , \tag{E.9a}$$

$$\Gamma^{\mu\nu}_{\vartheta,0}\,(k,q,\beta) = \mathrm{e}^{-\beta(x)\cdot q}\,\theta^{\mu}_{\alpha}\,\theta^{\nu}_{\beta}\,\Gamma^{\alpha\beta}_{\vartheta,0}\left(\widetilde{k},-\widetilde{q},\widetilde{\beta}\right)\ , \tag{E.9b}$$

$$\Gamma^{\mu\nu}_{\vartheta,0}\,(k,q,\beta) = \theta^{\mu}_{\alpha}\,\theta^{\nu}_{\beta}\,\Gamma^{\alpha\beta}_{\vartheta,0}\left(\widetilde{k},\widetilde{q},\widetilde{\beta}\right)\ . \tag{E.9c}$$

Finally, since the four-current operator $\widehat{j}^{\mu}(0)$ is Hermitian and transforms under parity and time reversal as:

$$\widehat{j}^{\mu}(0)\mapsto \theta^{\mu}_{\alpha}\,\widehat{j}^{\alpha}(0)\,,$$

the relations (E.2), (E.7), and (E.8) can be straightforwardly extended to the current expectation values, yielding

$$Y^{\mu}\,(k,q,\beta)^* = \mathrm{e}^{-\beta(x)\cdot q}\,Y^{\mu}\,(k,-q,\beta)\ , \tag{E.10a}$$

$$Y^{\mu}\,(k,q,\beta) = \mathrm{e}^{-\beta(x)\cdot q}\,\theta^{\mu}_{\alpha}\,Y^{\alpha}\left(\widetilde{k},-\widetilde{q},\widetilde{\beta}\right)\ , \tag{E.10b}$$

$$Y^{\mu}\,(k,q,\beta) = \theta^{\mu}_{\alpha}\,Y^{\alpha}\left(\widetilde{k},\widetilde{q},\widetilde{\beta}\right)\ . \tag{E.10c}$$

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