

Notes on unique continuation properties for
Partial Differential Equations – Introduction
to the stability estimates for inverse problems

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*Dedicated to my whole family and especially
to my dear wife Luisella and my dear son Luigi*

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Abstract. These Notes are intended for graduate or undergraduate students who have familiarity with Lebesgue measure theory, partial differential equations, and functional analysis. The main topics covered in this work are the study of the Cauchy problem and unique continuation properties associated with partial differential equations. The primary objective is to familiarize students with stability estimates in inverse problems and quantitative estimates of unique continuation. The treatment is presented in a self-contained manner.

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Introduction

The main purpose of these Notes is to introduce the study of unique continuation properties and stability estimates for inverse problems for partial differential equations (PDEs). The topics covered in these Notes are all chosen with proximity to inverse problems in mind, but we believe that none of the topics should be neglected in the training of those interested in PDEs, especially with regard to the study of unique continuation properties. Despite the existence of excellent review articles and books on the subject, there is a lack of a truly introductory book starting from minimum basics for a graduate or undergraduate student, who should have some familiarity with Lebesgue measure theory, basic elements of functional analysis, and a first introductory course on PDEs.

To facilitate the achievement of this purpose, we have covered other basic topics in the theory of PDEs, such as the theory of existence and regularity for second-order elliptic equations with real coefficients in Part I, and the classical theory of the Cauchy problem for equations with analytic coefficients in Part II. Part III focuses on the study of unique continuation properties for equations with non-analytic coefficients.

We provide a brief description of the present Notes in the remainder of this Introduction, with more detailed descriptions of the topics covered in the individual chapters provided in their introductions.

In Part I, in addition to Chapter 3 on Sobolev spaces, we have included Chapter 2, which serves as a connection and complement to elementary analysis topics. In Chapter 2, we recall the main definitions and theorems (without proof) of measure theory and prove some important theorems of real analysis, including the extension theorem in $C^{0,\alpha}$, the Lebesgue differentiation theorem, the Rademacher theorem, and the divergence theorem over open sets with Lipschitz boundary. Additionally, we study the distance function and the Hausdorff distance between compact sets, which is useful for studying the stability issue of inverse problems with unknown boundaries. In Chapter 4, we provide the definition and first properties of the Dirichlet-to-Neumann map, in addition to the existence and regularity L^2 theory for second-order

elliptic equations. We also introduce the inverse problem of inclusion detection and, in particular, size estimates, which have an interesting connection with the quantitative estimates of unique continuation developed in Part III. The books that inspired us the most in writing Part I are [43], [65] (Chapter 2) and [12], [23], [24] (Chapter 3, Chapter 4).

In Part II, we provide, in Chapter 5, a concise discussion of the Cauchy problem for first-order PDEs. In Chapter 6, we have given the basic properties of real analytic functions which we need in Chapter 7, where we give the formulation of the Cauchy problem for PDEs and prove the classical Cauchy-Kovalevskaya, Holmgren, and John theorems for (linear) PDEs with analytic coefficients. In Chapter 8, we apply the Holmgren and John theorems and the L^2 regularity theory to prove a uniqueness theorem for an inverse problem for the Laplace equation with unknown boundary. In Chapter 9, we introduce the concept of a well-posed problem in the sense of Hadamard, and by means of the Lax-Mizohata Theorem, we highlight the important connection between uniqueness, solvability, and continuous dependence on the data in a Cauchy problem for equations with C^∞ coefficients. In Chapter 10, we give the definition of conditional stability (or “well-posed problem in the sense of Tikhonov”) and some basic examples of conditional stability estimates for the calculus of derivatives and for the analytic continuation problem (in this area, the most famous theorem is the Hadamard three-circle inequality). Chapter 10 is a kind of “laboratory” in which we build some tools that should be kept in the toolbox of anyone who wants to study the conditional stability of not-well-posed problems in the sense of Hadamard. We conclude Part II with Chapter 11, in which we prove the John stability Theorem for the Cauchy problem for PDEs with analytic coefficients and discuss some of its consequences. The books that inspired us the most in writing Part II are [18], [23], [21], [41], [62] (Chapter 5, Chapter 6, Chapter 7), [34, Ch. V], [36, Vol. II], [56] (Chapter 9), [48] [73] (Chapter 10).

In Part III, as we have already mentioned, we provide a gradual study of Carleman estimates and the main problems of unique continuation for PDEs. In Chapter 12, we extensively explain the Nirenberg Theorem [60] concerning the Cauchy problem for constant operators in the principal part. From an educational point of view, one of the merits of this theorem consists of its simple proof and, conversely, in the powerful consequences that allow us to solve the question of the uniqueness of solutions to the Cauchy problem for the equation

$$\Delta u = b(x) \cdot \nabla u + c(x)u,$$

where $b = (b_1, \dots, b_n) \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $c \in L^\infty(\mathbb{R}^n)$. Furthermore, the Niren-

berg Theorem allows for addressing some standard aspects involving Carleman estimates quite easily, especially with regard to how the aforementioned estimates are used to infer the unique continuation property for PDEs. The actual presentation of the Carleman estimates is carried out in chapters 13, 14, and 15. In Chapter 13, we follow, with slight simplifications, the general and now classic approach developed by Hörmander [34], as it allows for a broad and general view of the issues concerning Carleman estimates. The main theorem of Chapter 13 is the Carleman estimate for general elliptic operators, which corresponds to theorem 8.3.1 of [34, Ch. VIII]. Chapter 13 has its natural continuation in Chapter 14, in which we initially review the proofs of Chapter 13 in the simple case of the Laplace operator. From there, we move on to dealing with second-order operators that are not necessarily elliptic. Unlike Chapter 13, where the integration by parts used to arrive at a Carleman estimate is based on a careful study of the quadratic differential forms, in Chapter 14, we adopt the Rellich identity and its natural generalization. This approach makes it easy to handle the case of second-order operators whose principal part has real Lipschitz continuous coefficients by providing a “miniaturized” proof of the uniqueness Calderón Theorem (here, Theorem 14.4.2) for operators with simple characteristics. In Chapter 14, we provide a hint to the notion of pseudoconvex functions, which is particularly simplified in the case of second-order operators with real coefficients. Finally, in Chapter 15, we prove some Carleman estimates with a singular weight for the second-order elliptic operator

$$Lu = \sum_{i,j=1}^n \partial_{x_i} (a^{ij}(x) \partial_{x_j} u),$$

where $\{a^{ij}(x)\}_{i,j=1}^n$ is a symmetric matrix whose entries are real-valued Lipschitz-continuous functions. We use the Carleman estimates to deduce the optimal three sphere inequality, the doubling inequality, and the strong unique continuation property (corollaries 15.5.3 and 15.7.8) for the equation

$$Lu = b(x) \cdot \nabla u + c(x)u,$$

where $b \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $c \in L^\infty(\mathbb{R}^n)$. Of the various proofs in the literature for such Carleman estimates, we present the proof given in [6], [7] which is based on transforming the elliptic operator into polar coordinates (Euclidean or Riemannian). We consider this elegant proof useful because it allows us to discuss the transformation into polar coordinates with respect to a Riemannian metric, which can be useful in other contexts of PDEs. In Chapter 16, we provide some brief and simple comments on the methods of log-convexity and the frequency function for studying the unique continuation property. In

this chapter, we also mention some simple applications of A_p weights in the stability and size estimates, and we conclude with the Runge property for the Laplace operator. The books that inspired us the most in writing Part III are [34] and [50].

I would like to conclude these Notes by thanking all those who provided me with useful advice on how to carry on this work, especially my friends Lorenzo Baldassari and Elisa Francini.

Part I

THE SOBOLEV SPACES AND THE BOUNDARY VALUE PROBLEMS

Chapter 1

Main notation and basic formulas

1.1 Notation

Let us denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We call multi-index any n -uple of elements of \mathbb{N}_0

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad \alpha_j \in \mathbb{N}_0, \quad j = 1, 2, \dots, n.$$

For any $\alpha \in \mathbb{N}_0^n$ we denote by

$$|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_n|, \quad \text{and} \quad \alpha! = \alpha_1! \alpha_2! \dots \alpha_n!,$$

the length (modulus) and the factorial of α , respectively. For any $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$, we set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

Let $\alpha, \beta \in \mathbb{N}_0^n$ we write $\alpha \leq \beta$ provided $\alpha_j \leq \beta_j$ for $j = 1, 2, \dots, n$ and we write $\alpha < \beta$ provided $\alpha \leq \beta$ and there exists $j_0 \in \{1, 2, \dots, n\}$ such that $\alpha_{j_0} < \beta_{j_0}$. For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we denote, unless otherwise stated, by $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and we write $x = (x', x_n)$. Similar convention will be used for the multi-indices.

For any $\alpha, \beta \in \mathbb{N}_0^n$ and $\alpha \leq \beta$, let us denote by (the binomial " β over α ")

$$\binom{\beta}{\alpha} = \frac{\beta!}{(\beta - \alpha)! \alpha!} \dots$$

Let us denote by ∂_k the operator $\frac{\partial}{\partial x_k}$, $k = 1, 2, \dots, n$ and by

$$\partial = (\partial_1, \partial_2, \dots, \partial_n) \quad (\text{the gradient operator}).$$

Hence, we set

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

To denote the gradient operator we also use the notation ∇ , however to denote ∂^α we **will not write** ∇^α . Of course, we will continue to denote by u_{x_k} (or by other standard symbols) the partial derivative of u with respect to x_k , $k = 1, 2, \dots, n$. The Hessian matrix of a smooth function u is denoted by

$$\partial^2 u = \{\partial_{jk}^2 u\}_{j,k=1}^n$$

We point out that some authors (and also in these notes in some context) reserve the notation D_k to denote the operator $\frac{1}{i} \frac{\partial}{\partial x_k}$, where $i = \sqrt{-1}$, consequently

$$D^\alpha = \left(\frac{1}{i}\right)^{|\alpha|} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

The latter notation is useful especially when an extensive use of the Fourier transform is done

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-ix \cdot \xi} dx.$$

Actually, we have

$$\widehat{D^\alpha u}(\xi) = \xi^\alpha \widehat{u}(\xi).$$

while, using the former notation, we have

$$\widehat{\partial^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi).$$

Let $\ell \in \mathbb{R}^n \setminus \{0\}$, for any $j \in \mathbb{N}_0$ we set

$$\frac{\partial^j}{\partial \ell^j} = \sum_{|\alpha|=j} \ell^\alpha \partial^\alpha,$$

(we mean $\frac{\partial^0 u}{\partial \ell^0} = u$). In particular

$$\frac{\partial}{\partial \ell} = \ell \cdot \partial = \ell \cdot \nabla.$$

As a consequence of the notations introduced above, we denote a polynomial P of degree m in the variables $\xi_1, \xi_2, \dots, \xi_n$

$$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \tag{1.1.1}$$

where $a_\alpha \in \mathbb{R}$ (or $a_\alpha \in \mathbb{C}$) for any $|\alpha| \leq m$. We say that the homogeneous polynomial

$$P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha,$$

is the **principal part of a polynomial** P provided that there exists $\alpha_0 \in \mathbb{N}_0^n$ such that $|\alpha_0| = m$ and $a_{\alpha_0} \neq 0$.

1.2 Some useful formulas

In this Section we recall some basic and useful formulas.

$$(x+y)^\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha-\beta}, \quad \forall x, y \in \mathbb{R}^n, \quad \forall \alpha \in \mathbb{N}_0^n. \quad (1.2.1)$$

$$\partial^\beta x^\alpha = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta}, & \text{for } \alpha \geq \beta, \\ 0, & \text{otherwise.} \end{cases} \quad (1.2.2)$$

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha, \quad \forall m \in \mathbb{N}_0. \quad (1.2.3)$$

$$\alpha! \leq |\alpha|! \leq n^{|\alpha|} \alpha!, \quad \forall \alpha \in \mathbb{N}_0^n. \quad (1.2.4)$$

Let us recall the following **Stirling formula**

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n e^{-n} \sqrt{n}} = \sqrt{2\pi}. \quad (1.2.5)$$

Let f be a smooth function and $m \in \mathbb{N}_0$, we have

$$\frac{d^m}{dt^m} f(x+ty) = \left(\left(\sum_{j=1}^n y_j \partial_j \right)^m f \right) (x+ty) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} y^\alpha (\partial^\alpha f) (x+ty). \quad (1.2.6)$$

We recall the **Taylor formula**, centered at $x_0 \in \mathbb{R}^n$, of a polynomial P of degree m

$$P(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha P(x_0) (x - x_0)^\alpha. \quad (1.2.7)$$

Let f and g be two smooth functions and $\alpha \in \mathbb{N}_0^n$, we have the **Leibniz formula** for the α -th derivative of the product fg

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} g. \quad (1.2.8)$$

Now we **check some formula**.

Formula (1.2.1) easily follows by the Newton binomial formula. Actually, we have

$$\begin{aligned} (x + y)^\alpha &= (x_1 + y_1)^{\alpha_1} \cdots (x_n + y_n)^{\alpha_n} = \\ &= \sum_{\beta_1 \leq \alpha_1} \binom{\alpha_1}{\beta_1} x_1^{\beta_1} y_1^{\alpha_1 - \beta_1} \cdots \sum_{\beta_n \leq \alpha_n} \binom{\alpha_n}{\beta_n} x_n^{\beta_n} y_n^{\alpha_n - \beta_n} = \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha - \beta}. \end{aligned}$$

The proof of (1.2.2) is immediate. Before checking (1.2.3) let us notice that if α and β are multi-indices such that $\alpha \leq \beta$ and $|\alpha| = |\beta|$, then $\alpha = \beta$. Let us denote

$$S(x) = \sum_{j=1}^n x_j$$

and set

$$P(x) = (S(x))^m.$$

Since P is a homogeneous polynomial of degree m we get

$$P(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha.$$

Let us show that $c_\alpha = \frac{m!}{\alpha!}$ for every multi-indices α such that $|\alpha| = m$. Let β be a multi-index satisfying $|\beta| = m$. By what we notice above and by (1.2.2), we get

$$\partial^\beta P(x) = \sum_{|\alpha|=m} c_\alpha \partial^\beta x^\alpha = c_\beta \beta!. \quad (1.2.9)$$

On the other hand,

$$\begin{aligned} \partial^\beta P(x) &= \partial^\beta S^m = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} S^m = \\ &= m \cdots (m - \beta_n + 1) \partial_1^{\beta_1} \cdots \partial_{n-1}^{\beta_{n-1}} S^{m - \beta_n} = \\ &= m \cdots (m - \beta_n - \beta_{n-1} + 1) \partial_1^{\beta_1} \cdots \partial_{n-2}^{\beta_{n-2}} S^{m - \beta_n - \beta_{n-1}} = \cdots = m!. \end{aligned} \quad (1.2.10)$$

By (1.2.9) and (1.2.10) we get

$$c_\beta \beta! = m! \quad \text{for every } \beta \in \mathbb{N}_0^n \text{ such that } |\beta| = m,$$

from which we obtain (1.2.3). Of course, formula (1.2.3) can be proved more elementarily. For instance, it can be proved by induction starting from the Newton binomial formula.

Concerning the inequality $\alpha! \leq |\alpha|!$ in (1.2.4), recalling that $h!k! \leq (h+k)!$ for every $h, k \in \mathbb{N}_0$ we get

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \leq (\alpha_1 + \alpha_2)! \alpha_3! \cdots \alpha_n! \leq \cdots \leq (\alpha_1 + \alpha_2 + \cdots + \alpha_n)! = |\alpha|!.$$

Regarding the inequality $|\alpha|! \leq n^{|\alpha|} \alpha!$ it suffices to use formula (1.2.3) and we have

$$n^{|\alpha|} = \underbrace{(1 + 1 + \cdots + 1)}_n^{|\alpha|} = \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}.$$

The first equality in (1.2.6) can be obtained by iterating the formula

$$\frac{d}{dt} f(x + ty) = \left(\sum_{j=1}^n y_j \partial_j f \right) (x + ty).$$

The second equality in (1.2.6) can be obtained by a formal development of

$$\left(\sum_{j=1}^n y_j \partial_j \right)^m$$

through (1.2.3).

Leibniz formula (1.2.8) can be easily obtained by the namesake formula for the one variable functions

$$\frac{d^k}{dt^k} (fg) = \sum_{h=0}^k \binom{k}{h} \frac{d^h f}{dt^h} \frac{d^{k-h} g}{dt^{k-h}},$$

where f and g are two smooth functions in the variable t .

Let $m \in \mathbb{N}_0$ and let a_α , $|\alpha| \leq m$, be some functions defined in an open set $\Omega \subset \mathbb{R}^n$ with values in \mathbb{R} or in \mathbb{C} . We say that the operator

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha \tag{1.2.11}$$

is a **linear differential operator of order m** in Ω , provided that there exists $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m$ such that a_α does not vanish identically in Ω . We say that the functions a_α , $|\alpha| \leq m$, are the **coefficients** of the differential operator (1.2.11).

We define the **symbol of operator** (1.2.11) as the following polynomial in the variable ξ

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) (i\xi)^\alpha. \quad (1.2.12)$$

Notice that if we write $P(x, \partial)$ as

$$\tilde{P}(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) (iD)^\alpha,$$

then the symbol (1.2.12) can be obtained by formally substituting D to ξ .

We have

$$e^{-ix \cdot \xi} \tilde{P}(x, D) e^{ix \cdot \xi} = P(x, \xi). \quad (1.2.13)$$

It might seem more natural to define the symbol of (1.2.11) as simply $\sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$, actually the contexts in which it is mostly used the definition of the symbol of a differential operator, are often the same ones in which it is convenient to use $D_j = \frac{1}{i} \partial_j$ as derivative operator. It is therefore advisable to stick to the standard definition of symbol for do not stray from the current literature. For instance, the symbol of the **Laplace operator**

$$\Delta = \sum_{j=1}^n \partial_j^2 = - \sum_{j=1}^n D_j^2$$

is given by

$$- \sum_{j=1}^n \xi_j^2,$$

the symbol of the **heat operator**

$$\sum_{j=1}^n \partial_j^2 - \partial_t = - \sum_{j=1}^n D_j^2 - iD_t,$$

is equal to

$$- \sum_{j=1}^n \xi_j^2 - i\xi_{n+1}$$

and the symbol of the **wave operator** or **d'Alembertian operator**

$$\square = \Delta - \partial_t^2 = \sum_{j=1}^n \partial_j^2 - \partial_t^2 = - \sum_{j=1}^n D_j^2 + D_t^2$$

is equal to

$$-\sum_{j=1}^n \xi_j^2 + \xi_0^2.$$

We will call the **principal part of operator** (1.2.11), the differential operator

$$P_m(x, \partial) = \sum_{|\alpha|=m} a_\alpha(x) \partial^\alpha. \quad (1.2.14)$$

In the sequel, to simplify the notations, we will concentrate our attention to the case in which the coefficients a_α are real-valued functions. However, we warn that what we will establish, in many cases, can easily be extended to the case where the coefficients a_α is a complex-valued function.

If all the coefficients of the operator $P(x, \partial)$ are constants, we will say that $P(x, \partial)$ is an **operator with constant coefficients**. In these cases, to denote the operator $P(x, \partial)$, we will just write $P(\partial)$.

We notice that, by the above definition, the symbol of the principal part of operator (1.2.11) is the homogeneous polynomial

$$P_m(x, \xi) = i^m \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha. \quad (1.2.15)$$

Conventions on the constants. In the sequel, to denote a positive constant we will use very often the letter C . We notice right now that the value of the constants will may change from line to line, but we will generally indicate the dependence of the constants by the various parameters. However, sometimes to be able to better follow the various steps, we will put an index or a sign to C and we will write $C_0, C_1, \bar{C}, \tilde{C} \dots$. We will generally omit the dependence of the various constants on the dimension of the space.

Chapter 2

Review of some function spaces and measure theory

2.1 The space C^k

Let X be a subset of \mathbb{R}^n . We will denote by $C^0(X)$ the vector space of continuous functions defined in X with values in \mathbb{R} . If $u \in C^0(X)$, we denote the **support of u** by

$$\text{supp } u := \overline{\{x \in X : u(x) \neq 0\}} \quad (\text{closure in } \mathbb{R}^n).$$

We will denote by $C_0^0(X)$ the space of continuous functions whose support is a compact set of \mathbb{R}^n contained in X .

Proposition 2.1.1. *Let $u \in C^0(X)$. Let $K \subset X$ be a compact set of \mathbb{R}^n ; for any $r > 0$ we set*

$$K_r = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq r\},$$

where $\text{dist}(x, K)$ denotes the distance of x from K . Let us suppose that there exists $r_0 > 0$ such that $K_{r_0} \subset X$. Then

$$\lim_{r \rightarrow 0} \max_{K_r} u = \max_K u. \quad (2.1.1)$$

Proof. Since K_{r_0} is a compact subset of X and $u \in C^0(X)$, u is uniformly continuous on K_{r_0} . Let $\varepsilon > 0$ and $\delta > 0$ such that

$$|u(x) - u(y)| < \varepsilon, \quad \text{for all } x, y \in K_{r_0} \text{ such that } |x - y| \leq \delta;$$

we may assume that $\delta < r_0$. Let $r \in (0, \delta]$ and let x be any point of K_r . Hence there exists $y \in K$ such that $|x - y| \leq r$. Therefore

$$u(x) < u(y) + \varepsilon \leq \max_K u + \varepsilon.$$

By the arbitrariness of x in K_r we have

$$\max_K u \leq \max_{K_r} u < \max_K u + \varepsilon,$$

which concludes the proof of (2.1.1). ■

Let $u \in C^0(X)$. We define the **modulus of continuity** of u in X

$$\omega(\delta) = \sup \{|u(x) - u(y)| : x, y \in X, |x - y| \leq \delta\}, \quad \text{for } \delta > 0. \quad (2.1.2)$$

ω is an increasing function, defined on $[0, +\infty)$ and satisfies $\omega(0) = 0$. Of course, ω may not be finite. It is easy to check that u is uniformly continuous if and only if

$$\lim_{\delta \rightarrow 0} \omega(\delta) = 0. \quad (2.1.3)$$

If the function ω is bounded, it may be convenient to use the **concave modulus of continuity** which is defined as

$$\tilde{\omega}(\delta) = \inf \{f(\delta) : f \text{ concave, } f \geq \omega, \text{ in } [0, +\infty)\}, \quad \text{for } \delta > 0. \quad (2.1.4)$$

Now we check that

$$\lim_{\delta \rightarrow 0} \tilde{\omega}(\delta) = 0. \quad (2.1.5)$$

Let us denote

$$M = \sup_{\delta \in [0, +\infty)} \omega(\delta) < +\infty.$$

If $M = 0$, (2.1.5) is trivial. Let us suppose therefore $M > 0$. Let $0 < \varepsilon < M$, from (2.1.3) it follows that there exists $\delta_0 > 0$ such that

$$0 \leq \omega(\delta) < \frac{\varepsilon}{2}, \quad \forall \delta \in [0, \delta_0].$$

Set

$$g_\varepsilon(\delta) = \begin{cases} \frac{\varepsilon}{2} + \frac{2M-\varepsilon}{2\delta_0}\delta, & \text{for } \delta \in [0, \delta_0], \\ M, & \text{for } \delta \in (\delta_0, +\infty). \end{cases}$$

It is easy to check that g_ε is concave and that $g_\varepsilon \geq \omega$ in $[0, +\infty)$. Furthermore, set

$$\delta_1 = \frac{\varepsilon\delta_0}{2M - \varepsilon},$$

it turns out

$$g_\varepsilon(\delta) < \varepsilon, \quad \forall \delta \in [0, \delta_1).$$

Therefore

$$\tilde{\omega}(\delta) < \varepsilon, \quad \forall \delta \in [0, \delta_1),$$

which gives (2.1.5).

Remark 1. Let $\tilde{\omega}$ be a concave modulus of continuity, then

$$0 < \eta_1 < \eta_2 \implies \eta_1 \tilde{\omega}\left(\frac{1}{\eta_1}\right) \leq \eta_2 \tilde{\omega}\left(\frac{1}{\eta_2}\right). \quad (2.1.6)$$

Let us check (2.1.6). From the concavity of $\tilde{\omega}$ and recalling that $\tilde{\omega}(0) = 0$ we have, for $0 < \eta_1 < \eta_2$,

$$\eta_1 \tilde{\omega}\left(\frac{1}{\eta_1}\right) = \frac{\tilde{\omega}\left(\frac{1}{\eta_1}\right) - \tilde{\omega}(0)}{\frac{1}{\eta_1} - 0} \leq \frac{\tilde{\omega}\left(\frac{1}{\eta_2}\right) - \tilde{\omega}(0)}{\frac{1}{\eta_2} - 0} = \eta_2 \tilde{\omega}\left(\frac{1}{\eta_2}\right).$$

◆

Let us denote by $C_*^0(X)$ the space of the bounded functions of $C^0(X)$, let us define the norm

$$\|u\|_{C_*^0(X)} = \sup_X |u(x)|, \quad \forall u \in C_*^0(X). \quad (2.1.7)$$

As it is well-known, the space $C_*^0(X)$ equipped with the norm (2.1.7), is a **Banach space**.

The following Theorem holds true (see [46, Corollary 1.3, Ch. 3] for a proof)

Theorem 2.1.2 (Weierstrass approximation). *Let X be a compact subset of \mathbb{R}^n . For every $u \in C^0(X)$ and for every $\varepsilon > 0$ there exists a polynomial P such that*

$$\|u - P\|_{C^0(X)} < \varepsilon.$$

We notice that by approximating the polynomial P given in the previous Theorem, by a polynomial with rational coefficients, we derive that $C^0(X)$, with compact X , is a **separable space**. Let us recall that a topological space \mathcal{S} is said to be separable if there exists a countable set $D \subset \mathcal{S}$ such that $\overline{D} = \mathcal{S}$.

The following Proposition holds true

Proposition 2.1.3. *Let \mathcal{S} be a metric space with distance d . If there exists \mathcal{Y} , uncountable subset of \mathcal{S} , and $\delta > 0$ such that*

$$d(x, y) > \delta, \quad \forall x, y \in \mathcal{Y}, \quad x \neq y, \quad (2.1.8)$$

then \mathcal{S} is not a separable space.

Proof. We argue by contradiction and we assume that \mathcal{S} is separable. Hence there exists $D = \{u_n\}_{n \in \mathbb{N}}$ such that $\overline{D} = \mathcal{S}$. Consequently, for every $x \in \mathcal{Y}$ there exists u_{n_x} such that

$$d(x, u_{n_x}) < \frac{\delta}{3}.$$

Therefore, if $x, y \in \mathcal{Y}$, $x \neq y$, the triangle inequality gives

$$d(u_{n_x}, u_{n_y}) \geq d(x, y) - d(x, u_{n_x}) - d(y, u_{n_y}) > \frac{\delta}{3}.$$

In particular, if $x, y \in \mathcal{Y}$, $x \neq y$, then $u_{n_x} \neq u_{n_y}$. Consequently, the map

$$\mathcal{Y} \ni x \rightarrow u_{n_x} \in D,$$

is injective, but this fact contradicts that \mathcal{Y} is an uncountable set. Therefore \mathcal{S} is not separable. ■

Remark 2. Let us note that the compactness assumption of X cannot be dropped for $C_*^0(X)$ to be separable. We show, for instance, that $C_*^0(\mathbb{R})$ is not separable.

For any $A \in \mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$ (where $\mathcal{P}(\mathbb{Z})$ denotes the power set of \mathbb{Z}) and any $\varepsilon \in (0, \frac{1}{2})$, define

$$u_A = \sum_{g \in A} u_g,$$

where

$$u_g(t) = \begin{cases} 1 - \varepsilon^{-1}|t - g|, & \text{for } t \in [g - \varepsilon, g + \varepsilon], \\ 0, & \text{for } t \in \mathbb{R} \setminus [g - \varepsilon, g + \varepsilon]. \end{cases}$$

We have

$$\|u_A - u_B\|_{C_*^0(\mathbb{R})} = 1, \quad \forall A, B \in \mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}, \quad A \neq B.$$

Since $\mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$ is uncountable, Proposition 2.1.3 implies that $C_*^0(\mathbb{R})$ is not separable. ♠

Generally we will be interested in the case when X is an open or the closure of an open set Ω of \mathbb{R}^n . If Ω is a bounded open set then we may consider

$$C^0(\overline{\Omega})$$

as a subspace of $C_*^0(\Omega)$ and we will denote the norm of $C^0(\overline{\Omega})$ by

$$\|u\|_{C^0(\overline{\Omega})} = \sup_{\overline{\Omega}} |u|, \quad \forall u \in C^0(\overline{\Omega}). \quad (2.1.9)$$

◆

In the sequel we will use the following classical theorems on relatively compact sets.

Theorem 2.1.4. *Let (X, d) be a complete metric space and let $Y \subset X$. Then Y is a relatively compact set (i.e., $\overline{Y} = X$) if and only if it is **totally bounded** that is, for every $\delta > 0$ there exists a finite set $\{x_1, \dots, x_N\} \subset X$ such that*

$$d(y, x_j) < \delta, \quad \forall y \in Y, \quad j = 1, \dots, N$$

or, equivalently,

$$Y \subset \bigcup_{j=1}^N B_\delta(x_j).$$

Theorem 2.1.5 (Arzelà–Ascoli). *Let Ω a bounded open set of \mathbb{R}^n and let $\{u_k\}$ be a sequence of functions belonging to $C^0(\overline{\Omega})$ such that:*

(i) $\{u_k\}$ is equibounded, i.e., there exists $M > 0$ such that

$$\|u_k\|_{C^0(\overline{\Omega})} \leq M, \quad \forall k \in \mathbb{N};$$

(ii) $\{u_k\}$ is equicontinuous, i.e., for every $\eta > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ and $x, y \in \overline{\Omega}$, then

$$|u_k(x) - u_k(y)| \leq \eta, \quad \forall k \in \mathbb{N}.$$

Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function $u \in C^0(\overline{\Omega})$ such that

$$\lim_{j \rightarrow \infty} \|u_{k_j} - u\|_{C^0(\overline{\Omega})} = 0.$$

Let $k \in \mathbb{N}$ and let Ω be an open set of \mathbb{R}^n , we will denote by $C^k(\Omega)$ the space of functions which satisfy $\partial^\alpha u \in C^0(\Omega)$ for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$. Further, we will denote by $C^k(\overline{\Omega})$ the space of the functions $u \in C^k(\Omega)$ such that, for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$, $\partial^\alpha u$ is extensible to a function $U_\alpha \in C^0(\overline{\Omega})$. Of course, if such an extension exists it is unique and we will write $\partial^\alpha u$ instead of U_α . If Ω is a bounded **open set** of \mathbb{R}^n , we define the norm on $C^k(\overline{\Omega})$ as follows

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} \sup_{\overline{\Omega}} |\partial^\alpha u|, \quad \forall u \in C^k(\overline{\Omega}). \quad (2.1.10)$$

As it is well-known, the space $C^k(\overline{\Omega})$, equipped with the norm (2.1.10) is a **Banach space**. In some contexts it turns out to be convenient to consider, instead of norm (2.1.10), an equivalent "dimensionless" norm, e.g.

$$\sum_{|\alpha| \leq k} d_0^{|\alpha|} \sup_{\overline{\Omega}} |\partial^\alpha u|, \quad \forall u \in C^k(\overline{\Omega}), \quad (2.1.11)$$

where d_0 is the diameter of Ω .

Proposition 2.1.6. *Let Ω be a bounded open set of \mathbb{R}^n and $k \in \mathbb{N}_0$, then the space $C^k(\overline{\Omega})$, with norm (2.1.10) is a separable space.*

Proof. Recall that if a topological space is separable, then every subset of it is a separable space [16, Ch. 3, Sec. 6].

We have already noticed (after Theorem 2.1.2) that $C^0(\overline{\Omega})$ is separable. We consider the case $k = 1$ (the case $k > 1$ can be treated similarly). Let Ψ be the map

$$\Psi : C^1(\overline{\Omega}) \rightarrow \mathcal{X},$$

where

$$\mathcal{X} = \underbrace{C^0(\overline{\Omega}) \times \cdots \times C^0(\overline{\Omega})}_{(n+1) \text{ - times}},$$

$$\Psi(u) = (u, \partial_1 u, \cdots, \partial_n u), \quad \forall u \in C^1(\overline{\Omega}).$$

if we equip \mathcal{X} with the norm

$$\|\mathbf{v}\|_{\mathcal{X}} = \sum_{j=0}^n \|v_j\|_{C^0(\overline{\Omega})}, \quad \forall \mathbf{v} = (v_0, v_1, \cdots, v_n) \in \mathcal{X},$$

Ψ is an isometry.

On the other hand, \mathcal{X} is a separable space as a cartesian product of separable spaces. Thus $\Psi(C^1(\overline{\Omega}))$ is separable as a subspace of \mathcal{X} and, since Ψ is an isometry, also $C^1(\overline{\Omega})$ is separable. ■

It is evident that if $k, m \in \mathbb{N}_0$ and $k < m$ then $C^m(\overline{\Omega}) \subset C^k(\overline{\Omega})$. We set

$$C_0^k(\Omega) = \{u \in C^k(\Omega) : \text{supp } u \text{ is a compact set contained in } \Omega\},$$

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega), \quad C^\infty(\overline{\Omega}) = \bigcap_{k=0}^{\infty} C^k(\overline{\Omega}), \quad C_0^\infty(\Omega) = \bigcap_{k=0}^{\infty} C_0^k(\Omega).$$

Let $k \in \mathbb{N}$ or $k = \infty$ and $\tilde{\Omega} \supset \Omega$, we will often adopt the convention of identifying $C_0^k(\Omega)$ with the space of functions u belonging to $C_0^k(\tilde{\Omega})$ and such that $\text{supp } u$ is a compact set contained in Ω .

2.2 The space $C^{k,\alpha}$

Let X be a subset of \mathbb{R}^n and $\alpha \in (0, 1]$, we will denote by $C^{0,\alpha}(X)$ the space of the functions $u \in C^0(X)$ which satisfy

$$[u]_{C^{0,\alpha}(X)} = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} : x, y \in X, x \neq y \right\} < +\infty.$$

If u is a function of $C^{0,\alpha}(X)$ it is also said that u is a **Hölder function of order α** . The number α is said the **Hölder exponent** of the space $C^{0,\alpha}(X)$. For any $u \in C^{0,\alpha}(X)$, the number $[u]_{C^{0,\alpha}(X)}$ is called the **Hölder constant** of u . If $\alpha = 1$ we will also say that u is a **Lipschitz function** in X and we call **Lipschitz constant** the number $[u]_{C^{0,1}(X)}$. We observe that if $\alpha > 1$ and X is a connected open set, then the space $C^{0,\alpha}(X)$ consists of only the constant functions (as a matter of fact, if $\alpha > 1$ then any function of $C^{0,\alpha}(X)$ is differentiable with zero gradient in X). It can be easily checked that if X is bounded, and the space $C^{0,\alpha}(X)$ is equipped with the norm

$$\|u\|_{C^{0,\alpha}(X)} = \|u\|_{C^0(X)} + [u]_{C^{0,\alpha}(X)}, \quad (2.2.1)$$

then $C^{0,\alpha}(X)$ is a **Banach space**. Sometimes it is convenient to consider, instead of the norm (2.2.1), an equivalent "dimensionless" norm, e.g.

$$\|u\|_{C^0(X)} + d_0^\alpha [u]_{C^{0,\alpha}(X)},$$

where d_0 is the diameter of X .

Let $m \in \mathbb{N}$, we denote by $C^{0,\alpha}(X; \mathbb{R}^m)$ the space of the functions $u \in C^0(X; \mathbb{R}^m)$ satisfying $u_j \in C^{0,\alpha}(X)$, $j = 1, \dots, m$. We set

$$\|u\|_{C^{0,\alpha}(X; \mathbb{R}^m)} = \|u\|_{C^0(X; \mathbb{R}^m)} + [u]_{C^{0,\alpha}(X; \mathbb{R}^m)},$$

where

$$[u]_{C^{0,\alpha}(X; \mathbb{R}^m)} = \sup \left\{ \frac{|u(x) - u(y)|_{\mathbb{R}^m}}{|x - y|_{\mathbb{R}^n}^\alpha} : x, y \in X, x \neq y \right\} < +\infty,$$

where $|\cdot|_{\mathbb{R}^m}$ is the Euclidean norm in \mathbb{R}^m (in the sequel we will often omit the subscript \mathbb{R}^m from this norm).

The following Proposition holds true.

Proposition 2.2.1. *Let X be a bounded set of \mathbb{R}^n and $0 < \beta < \alpha \leq 1$ then*

$$[u]_{C^{0,\beta}(X)} \leq d_0^{\alpha-\beta} [u]_{C^{0,\alpha}(X)}, \quad (2.2.2)$$

where d_0 is the diameter of X ;

$$[u]_{C^{0,\beta}(X)} \leq \left(2 \|u\|_{C^0(X)}\right)^{1-\frac{\beta}{\alpha}} \left([u]_{C^{0,\alpha}(X)}\right)^{\frac{\beta}{\alpha}}. \quad (2.2.3)$$

In particular we have

$$C^{0,\alpha}(X) \subset C^{0,\beta}(X). \quad (2.2.4)$$

Proof. It suffices to observe that for $x, y \in X$, $x \neq y$, by (2.2.2), we have

$$\frac{|u(x) - u(y)|}{|x - y|^\beta} = \frac{|u(x) - u(y)|}{|x - y|^\alpha} |x - y|^{\alpha-\beta} \leq [u]_{C^{0,\alpha}(X)} d_0^{\alpha-\beta}.$$

Regarding (2.2.3), we first note that the case

$$[u]_{C^{0,\alpha}(X)} = 0$$

is trivial. Let us assume, then,

$$[u]_{C^{0,\alpha}(X)} \neq 0$$

and let $r > 0$ be chosen later. Let $x, y \in X$, $x \neq y$. If

$$|x - y| \leq r,$$

then

$$\frac{|u(x) - u(y)|}{|x - y|^\beta} \leq [u]_{C^{0,\alpha}(X)} r^{\alpha-\beta}.$$

If

$$|x - y| > r,$$

then

$$\frac{|u(x) - u(y)|}{|x - y|^\beta} \leq 2r^{-\beta} \|u\|_{C^0(X)}.$$

In any case, we have

$$[u]_{C^{0,\beta}(X)} \leq r^{-\beta} \max \left\{ r^\alpha [u]_{C^{0,\alpha}(X)}, 2 \|u\|_{C^0(X)} \right\}$$

and, choosing

$$r = \left(\frac{2 \|u\|_{C^0(X)}}{[u]_{C^{0,\alpha}(X)}} \right)^{1/\alpha},$$

we obtain (2.2.3). ■

Remark 1. By using the Mean Value Theorem, it is easily shown that if Ω is a bounded, convex open set of \mathbb{R}^n then

$$C^1(\bar{\Omega}) \subset C^{0,\beta}(\bar{\Omega}) \quad (2.2.5)$$

and

$$[u]_{0,1,\Omega} \leq \|\nabla u\|_{C^0(\bar{\Omega})},$$

where

$$\|\nabla u\|_{C^0(\bar{\Omega})} = \|\|\nabla u\|\|_{C^0(\bar{\Omega})}.$$

Nevertheless, for a bounded open set Ω it is not necessarily the case that the inclusion (2.2.5) holds. Let us consider, for instance, the following example. Let

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : x \leq \sqrt{|y|}, \quad x^2 + y^2 < 1 \right\},$$

$$1 < \beta < 2,$$

$$u(x, y) = \begin{cases} x^\beta \operatorname{sgn}(y), & \text{if } x \in \Omega, x > 0, \\ 0, & \text{if } x \in \Omega, x \leq 0. \end{cases}$$

We have $u \in C^1(\overline{\Omega})$, however if α satisfies $\frac{\beta}{2} < \alpha \leq 1$ then

$$u \notin C^{0,\alpha}(\overline{\Omega}).$$

As a matter of fact, if $x = \sqrt{|y|}$, we get

$$\frac{|u(x, y) - u(x, -y)|}{(2|y|)^\alpha} = 2^{1-\alpha} |y|^{\frac{\beta}{2}-\alpha} \rightarrow +\infty, \quad \text{as } y \rightarrow 0.$$

◆

Remark 2. The space $C^{0,\alpha}(\overline{\Omega})$, where $\alpha \in (0, 1]$ and Ω is a bounded open set, is not separable. Let us consider the case $n = 1$ and $\Omega = (0, 1)$ and for any $a \in (0, 1)$, let us define

$$u_a(t) = \begin{cases} 0, & \text{for } t \in [0, a), \\ (t - a)^\alpha, & \text{for } t \in [a, 1]. \end{cases}$$

We have

$$\|u_a - u_b\|_{C^{0,\alpha}([0,1])} \geq 1, \quad \forall a, b \in [0, 1] \quad a \neq b. \quad (2.2.6)$$

We check (2.2.6). Let $a, b \in [0, 1]$, $a < b$ and let us denote

$$v_{a,b} = u_a - u_b.$$

We have

$$v_{a,b}(b) = u_a(b) - u_b(b) = u_a(b) = (b - a)^\alpha$$

and

$$v_{a,b}(a) = u_a(a) - u_b(a) = 0.$$

Therefore

$$[u_a - u_b]_{C^{0,\alpha}([0,1])} = [v_{a,b}]_{C^{0,\alpha}([0,1])} \geq \frac{|v_{a,b}(b) - v_{a,b}(a)|}{|b - a|^\alpha} = 1,$$

which implies (2.2.6). Finally, from the latter and from Proposition 2.1.3 it follows that $C^{0,\alpha}([0, 1])$ is not separable. ◆

Theorem 2.2.2 (extension in $C^{0,\alpha}$). *Let X be a bounded set of \mathbb{R}^n and $u \in C^{0,\alpha}(X)$, $\alpha \in (0, 1]$, then there exists $U \in C^{0,\alpha}(\mathbb{R}^n)$ such that*

$$U(x) = u(x), \quad \forall x \in X, \quad (2.2.7)$$

$$\|U\|_{C^0(\mathbb{R}^n)} = \|u\|_{C^0(X)}, \quad (2.2.8)$$

$$[U]_{C^{0,\alpha}(\mathbb{R}^n)} = [u]_{C^{0,\alpha}(X)}. \quad (2.2.9)$$

Proof. Let us denote

$$M = \|u\|_{C^0(X)}, \quad m = [u]_{C^{0,\alpha}(X)}$$

and let us define the function

$$v(x) = \sup_{y \in X} \{u(y) - m|x - y|^\alpha\}, \quad \text{for } x \in \mathbb{R}^n.$$

We have

$$v(x) = u(x), \quad \forall x \in X. \quad (2.2.10)$$

We check (2.2.10). First note that we have trivially

$$u(x) \leq v(x), \quad \forall x \in X. \quad (2.2.11)$$

On the other hand we have

$$u(y) - u(x) \leq m|x - y|^\alpha, \quad \forall x, y \in X \quad \forall x \in X,$$

hence

$$u(y) - m|x - y|^\alpha \leq u(x), \quad \forall x, y \in X \quad \forall x \in X.$$

Consequently

$$v(x) \leq u(x), \quad \forall x \in X.$$

By the latter and by (2.2.11) we get (2.2.10).

We also notice that

$$v(x) \leq M, \quad \forall x \in \mathbb{R}^n. \quad (2.2.12)$$

Now, for any $x \in \mathbb{R}^n$ let us define

$$U(x) = \begin{cases} v(x), & \text{for } |v(x)| \leq M, \\ -M, & \text{for } v(x) < -M. \end{cases}$$

Let us note that, by (2.2.12), U is defined throughout \mathbb{R}^n . Let us note also that if $x \in X$, then (2.2.10) gives (2.2.7) and

$$\sup_X |U(x)| = M.$$

We also have

$$\sup_{\mathbb{R}^n} |U(x)| = M,$$

Concerning the latter, notice that, if $|v(x)| \leq M$ then $U(x) = v(x)$, hence $|U(x)| \leq M$ and if $v(x) < -M$ then $|U(x)| = M$.

It only remains to prove that $U \in C^{0,\alpha}(\mathbb{R}^n)$ and that (2.2.9) holds. Let, then, $x, y \in \mathbb{R}^n$ be such that $x \neq y$ and otherwise arbitrary. Let us suppose that $U(x) \neq U(y)$. For instance, let us assume

$$U(x) > U(y). \tag{2.2.13}$$

Let us check that

$$0 < U(x) - U(y) \leq v(x) - v(y). \tag{2.2.14}$$

The following cases occur.

- (a) $v(x) < -M$ and $|v(y)| \leq M$,
- (b) $v(x) < -M$ and $v(y) < -M$,
- (c) $|v(x)| \leq M$ and $|v(y)| \leq M$,
- (d) $|v(x)| \leq M$ and $v(y) < -M$.

Cases (a) and (b) cannot occur. As a matter of fact, in case (a) we would have

$$U(x) - U(y) = -M - v(y) \leq -M + M = 0,$$

that contradicts (2.2.13). In case (b) we would have

$$U(x) - U(y) = -M - (-M) = 0,$$

that contradicts (2.2.13).

In case (c) we have

$$U(x) - U(y) = v(x) - v(y).$$

Finally, in case (d) we have

$$U(x) - U(y) = v(x) - (-M) = v(x) + M < v(x) - v(y).$$

Therefore (2.2.14) holds true.

We have

$$\begin{aligned}
v(x) - v(y) &= \sup_{z \in X} \{u(z) - m|x - z|^\alpha\} - \sup_{z \in X} \{u(z) - m|y - z|^\alpha\} \leq \\
&\leq m \sup_{z \in X} \{|y - z|^\alpha - |x - z|^\alpha\} \leq \\
&\leq m \sup_{\zeta \in X} \{(|y - x| + |\zeta|)^\alpha - |\zeta|^\alpha\}.
\end{aligned} \tag{2.2.15}$$

Now, let us denote by

$$\omega(t) = t^\alpha, \quad \text{if } t \in [0, +\infty).$$

Since ω is concave, we have

$$\omega(t + h) - \omega(t) \leq \omega(h) - \omega(0) = \omega(h), \quad \forall t, h \in [0, +\infty),$$

by the just obtained inequality, by (2.2.14) and (2.2.15) we get

$$|U(x) - U(y)| \leq m|y - x|^\alpha \tag{2.2.16}$$

and since we have proved (2.2.7), we get (2.2.9).

■

A more general version of Theorem 2.2.2, valid for uniformly continuous functions, can be found in [65, Chapter 4].

Let us notice that if

$$u : X \rightarrow \mathbb{R}^m$$

is a Lipschitz continuous function, then, Theorem 2.2.2 implies that there exists an extension

$$U : \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

such that

$$[U]_{C^{0,1}(\mathbb{R}^n; \mathbb{R}^m)} \leq \sqrt{m}[u]_{C^{0,1}(X; \mathbb{R}^m)}.$$

Actually, this result can be improved. As a matter of fact, the following Theorem of **Kirszbraun** holds true, for the proof of which we refer to [51, cap. 7].

Theorem 2.2.3 (Kirszbraun). *Let $u : X \rightarrow \mathbb{R}^m$ be a Lipschitz continuous function, where $X \subset \mathbb{R}^n$, then there exists $U \in C^{0,1}(\mathbb{R}^n; \mathbb{R}^m)$ such that*

$$U = u, \quad \text{in } X$$

and

$$[U]_{C^{0,1}(\mathbb{R}^n; \mathbb{R}^m)} = [u]_{C^{0,1}(X; \mathbb{R}^m)}.$$

Let Ω be a bounded open set of \mathbb{R}^n , $k \in \mathbb{N}_0$ and $\alpha \in (0, 1]$, we denote by $C^{k,\alpha}(\overline{\Omega})$ the space of the functions $u \in C^k(\overline{\Omega})$, satisfying

$$[\partial^\beta u]_{C^{0,\alpha}(\overline{\Omega})} < +\infty, \quad \forall \beta \in \mathbb{N}_0^n, \quad |\beta| = k.$$

It is easily proven that $C^{k,\alpha}(\overline{\Omega})$, equipped with the norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + [u]_{C^{k,\alpha}(\overline{\Omega})} \quad \forall u \in C^k(\overline{\Omega}), \quad (2.2.17)$$

is a **Banach space**, where

$$[u]_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta|=k} [\partial^\beta u]_{C^{0,\alpha}(\overline{\Omega})}.$$

Sometimes, instead of the norm (2.2.17) we will consider the dimensionless norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta| \leq k} d_0^{|\beta|} \sup_{\overline{\Omega}} |\partial^\beta u| + d_0^{k+\alpha} [u]_{C^{k,\alpha}(\overline{\Omega})}.$$

where d_0 is the diameter of Ω .

We define the space $C_{loc}^{k,\alpha}(\Omega)$, $k \in \mathbb{N}_0$, $0 < \alpha \leq 1$, as the space of functions $u \in C^0(\Omega)$ such that for every bounded open set $\omega \Subset \Omega$ (i.e. $\overline{\omega}$ compact and $\overline{\omega} \subset \Omega$) we have

$$u|_\omega \in C^{k,\alpha}(\overline{\omega}).$$

2.3 Review of measure theory and L^p spaces

In this Section, we give, for the convenience of the reader, the main definitions and statements of the main theorems of the Measure Theory and of L^p spaces. Some reference texts are [65], [68] (see also lecture notes [52] and [53]).

2.3.1 Measurable sets, measurable functions, positive measures

Definition 2.3.1. Let X be a set and \mathcal{M} be a family of subsets of X with the following properties:

- (i) $X \in \mathcal{M}$,
- (ii) $E \in \mathcal{M} \implies \mathcal{C}E := X \setminus E \in \mathcal{M}$,
- (iii) $E_j \in \mathcal{M}, j \in \mathbb{N} \implies \bigcup_{j \in \mathbb{N}} E_j \in \mathcal{M}$.

\mathcal{M} is called a σ -**algebra** and the couple (X, \mathcal{M}) is called a **measurable space**.

We will be interested almost exclusively in the case where $X = \mathbb{R}^n$ and \mathcal{M} consists of the Lebesgue measurable subsets of \mathbb{R}^n .

Definition 2.3.2. Let (X, \mathcal{M}) be a measurable space and Y be a topologic space. We say that the function

$$f : X \rightarrow Y,$$

is a **measurable function**, provided we have

$$f^{-1}(A) \in \mathcal{M}, \quad \text{for every open subset } A \text{ of } Y.$$

Let us recall that if (X, \mathcal{M}) is a measurable space, Y, Z two topological spaces, $f : X \rightarrow Y$ is a measurable function and $g : Y \rightarrow Z$ is a continuous function, then $g \circ f : X \rightarrow Z$ is a measurable function.

We denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, the extended real line equipped with its usual topology.

The following theorems hold true.

Theorem 2.3.3. Let (X, \mathcal{M}) be a measurable space and $f : X \rightarrow \overline{\mathbb{R}}$. Then f is a measurable function if and only if for any $t \in \mathbb{R}$ one of the following level sets

$$f^{-1}((t, +\infty]), \quad f^{-1}([t, +\infty]), \quad f^{-1}([-\infty, t)), \quad f^{-1}([-\infty, t]),$$

is a measurable set.

Theorem 2.3.4. Let (X, \mathcal{M}) be a measurable space. Then we have

(i) If $f, g : X \rightarrow \overline{\mathbb{R}}$ are two measurable functions and $f + g$ is well defined, then $f + g$ e λf are measurable functions (we use the convention that $0 \cdot (\pm\infty) = 0$).

(ii) Let $\{f_k\}$ be a sequence of measurable functions, then

$$\sup_{k \in \mathbb{N}} f_k, \quad \inf_{k \in \mathbb{N}} f_k, \quad \liminf_{k \rightarrow \infty} f_k, \quad \limsup_{k \rightarrow \infty} f_k,$$

are measurable functions.

We define as **simple function** on the measurable space X a function

$$s : X \rightarrow \mathbb{R}$$

that assumes a finite set of values.

Theorem 2.3.5. *Let (X, \mathcal{M}) be a measurable space and $f : X \rightarrow \overline{\mathbb{R}}$ be a measurable function, then there exists a sequence $\{s_k\}$ of simple functions such that*

$$\lim_{k \rightarrow \infty} s_k(x) = f(x), \quad \forall x \in X.$$

If f is bounded then $\{s_k\}$ uniformly converges to f .

Definition 2.3.6. Let (X, \mathcal{M}) a measurable space. We say that

$$\mu : \mathcal{M} \rightarrow [0, \infty],$$

is a **positive measure** provided that we have

- (i) $\mu(\emptyset) = 0$;
- (ii) if $\{E_j\}_{j \in \mathbb{N}}$ is a countable family of measurable sets such that

$$E_i \cap E_j = \emptyset, \quad \text{for } i \neq j,$$

then we have

$$\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j).$$

The tern (X, \mathcal{M}, μ) is called a **measure space**.

If E is a Lebesgue measurable set of \mathbb{R}^n , we will denote by $|E|$ its measure.

The following theorems hold true.

Theorem 2.3.7. *Let (X, \mathcal{M}, μ) be a measure space. The following properties hold true.*

(i) if $\{E_j\}_{1 \leq j \leq N}$ is a finite family of measurable sets such that $E_i \cap E_j = \emptyset$, for $i \neq j$, then

$$\mu \left(\bigcup_{j=1}^N E_j \right) = \sum_{j=1}^N \mu(E_j);$$

(ii) if $E \subset F$ and $E, F \in \mathcal{M}$, then $\mu(E) \leq \mu(F)$;

(iii) if $\{E_j\}_{j \in \mathbb{N}}$ is a countable family of measurable sets such that $E_j \subset E_{j+1}$, for every $j \in \mathbb{N}$, then

$$\lim_{j \rightarrow \infty} \mu(E_j) = \mu \left(\bigcup_{j=1}^{\infty} E_j \right);$$

(iv) if $\{E_j\}_{j \in \mathbb{N}}$ is a countable family of measurable sets such that

$$E_{j+1} \subset E_j,$$

for every $j \in \mathbb{N}$, then

$$\lim_{j \rightarrow \infty} \mu(E_j) = \mu \left(\bigcap_{j=1}^{\infty} E_j \right).$$

Theorem 2.3.8. Let E be a Lebesgue measurable subset of \mathbb{R}^n whose measure be finite. Let $\{f_j\}$ be a sequence of measurable functions such that there exists the limit

$$\lim_{j \rightarrow \infty} f_j(x)$$

and it is finite almost everywhere. Then, for every $\varepsilon > 0$ there exists a compact set $K \subset E$ which satisfies $|E \setminus K| < \varepsilon$ and

$$f_j \rightarrow f, \text{ as } j \rightarrow \infty, \quad \text{uniformly on } K.$$

Theorem 2.3.9 (Lusin). Let E be a Lebesgue measurable subset of \mathbb{R}^n which has finite measure, and let $f : E \rightarrow \overline{\mathbb{R}}$ such that

$$|f(x)| < +\infty, \quad \text{a.e. } x \in E.$$

Then f is a measurable function in E if and only if for each $\varepsilon > 0$ there exists $K \subset E$, K closed, such that $|E \setminus K| < \varepsilon$ and $f|_K$ is a continuous function.

Now, let us define **the integral over the measure space** (X, \mathcal{M}, μ) . Let s be a nonnegative simple function

$$s(x) = \sum_{i=1}^N c_i \chi_{E_i},$$

where $\{E_j\}_{1 \leq j \leq N}$ is a finite family of measurable set pairwise disjoints and $c_j \geq 0$, $j = 1, \dots, N$. If $E \in \mathcal{M}$, we set by definition

$$\int_E s(x) d\mu = \sum_{i=1}^N c_i \mu(E \cap E_i),$$

in which the convention $0 \cdot \infty = 0$ occurs. We call $\int_E s(x)d\mu$ "the integral of s over E ".

Let us consider the measurable function

$$f : X \rightarrow [0, +\infty].$$

We call **the Lebesgue integral of f with respect to the measure μ** the following element of $\overline{\mathbb{R}}$

$$\int_E f(x)d\mu := \sup \left\{ \int_E s(x)d\mu : s \text{ simple function, } 0 \leq s \leq f \text{ in } E \right\}.$$

We say that f a summable function over E if

$$\int_E f(x)d\mu < +\infty.$$

Theorem 2.3.10. *Let (X, \mathcal{M}, μ) be a measure space and let $f, g : X \rightarrow [0, +\infty]$ $E, F \in \mathcal{M}$. The following properties hold true:*

- (i) $\int_E f d\mu = \int_X f \chi_E d\mu$;
- (ii) if $f \leq g$ in E then $\int_E f d\mu \leq \int_E g d\mu$;
- (iii) if $E \subset F$ then $\int_E f d\mu \leq \int_F f d\mu$;
- (iv) if $f = 0$ in E then $\int_E f d\mu = 0$;
- (v) if $\mu(E) = 0$ then $\int_E f d\mu = 0$.

Theorem 2.3.11 (Monotone Convergence). *Let (X, \mathcal{M}, μ) be a measure space. Let $\{f_j\}$ be a sequence of nonnegative measurable functions which satisfy*

$$f_j(x) \leq f_{j+1}(x), \quad \forall x \in X, \quad \forall j \in \mathbb{N}.$$

Then

$$\int_X \lim_{j \rightarrow \infty} f_j(x) d\mu = \lim_{j \rightarrow \infty} \int_X f_j(x) d\mu.$$

Theorem 2.3.12 (Fatou). *Let (X, \mathcal{M}, μ) be a measure space. Let $\{f_j\}$ be a sequence of nonnegative measurable functions, then we have*

$$\int_X \liminf_{j \rightarrow \infty} f_j(x) d\mu \leq \liminf_{j \rightarrow \infty} \int_X f_j(x) d\mu.$$

Theorem 2.3.13. *Let (X, \mathcal{M}, μ) be a measure space and let*

$$f, g : X \rightarrow [0, +\infty]$$

be two measurable functions, then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu;$$

$$\int_X \lambda f d\mu = \lambda \int_X f d\mu, \quad \forall \lambda \in \mathbb{R},$$

with the convention $0 \cdot \int_X f d\mu = 0$.

Theorem 2.3.14. *Let (X, \mathcal{M}, μ) be a measure space and let $\{f_j\}$ be a sequence of nonnegative measurable functions, then we have*

$$\int_X \sum_{j=1}^{\infty} f_j d\mu = \sum_{j=1}^{\infty} \int_X f_j d\mu.$$

Furthermore, recall that if

$$f : X \rightarrow [0, +\infty],$$

is a measurable function and defining

$$\nu(E) = \int_E f d\mu, \quad \forall E \in \mathcal{M},$$

ν turns out to be a measure on X .

Definition 2.3.15. Let (X, \mathcal{M}, μ) be a measure space and let

$$f : X \rightarrow \mathbb{R}.$$

We say that f is summable over X provided

$$\int_X |f| d\mu < +\infty.$$

In such a case we set

$$\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu,$$

where $f_+ = \max\{f, 0\}$, $f_- = -\min\{f, 0\}$. We denote by $\mathcal{L}^1(X)$ the class of summable functions over X .

Theorem 2.3.16. *Let (X, \mathcal{M}, μ) be a measure space and let $f \in \mathcal{L}^1(X)$, then*

$$\mu(\{x \in X : |f(x)| = +\infty\}) = 0.$$

Theorem 2.3.17. $\mathcal{L}^1(X)$ is a vector space and

$$\mathcal{L}^1(X) \ni f \rightarrow \int_X f d\mu \in \mathbb{R},$$

is a linear map. Furthermore, if $f, g \in \mathcal{L}^1(X)$ then

$$\max\{f, g\} \in \mathcal{L}^1(X)$$

and

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu, \quad \forall f \in \mathcal{L}^1(X).$$

Theorem 2.3.18. *If $f \in \mathcal{L}^1(X)$ then*

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \forall E \in \mathcal{M} \text{ and } \mu(E) < \delta \text{ we have } \int_E |f| d\mu < \varepsilon.$$

Theorem 2.3.19 (Dominated Convergence). *Let $\{f_j\}$ be a sequence of measurable functions in $\mathcal{L}^1(X)$. Let us assume*

(i)

$$\lim_{j \rightarrow \infty} f_j(x) = f(x), \quad \text{a.e. } x \in X,$$

(ii) *there exists $g \in \mathcal{L}^1(X)$ such that*

$$|f_j(x)| \leq g(x), \quad \text{a.e. } x \in X, \quad j \in \mathbb{N}.$$

Then $f \in \mathcal{L}^1(X)$ and

$$\lim_{j \rightarrow \infty} \int_X |f_j - f| d\mu = 0,$$

$$\lim_{j \rightarrow \infty} \int_X f_j d\mu = \int_X f d\mu.$$

The Monotone Convergence Theorem and the Dominated Convergence Theorem give that if $\{f_j\}$ is a sequence of measurable functions in $\mathcal{L}^1(X)$ satisfying

$$\sum_{j=1}^{\infty} \int_X |f_j| d\mu < +\infty,$$

then $\sum_{j=1}^{\infty} f_j$ converges almost everywhere to a function of $\mathcal{L}^1(X)$ and

$$\sum_{j=1}^{\infty} \int_X f_j d\mu = \int_X \sum_{j=1}^{\infty} f_j d\mu.$$

Theorem 2.3.20 (derivation under the integral sign). *Let (X, \mathcal{M}, μ) be a measure space and let A be an open set of \mathbb{R}^n . Let*

$$F : A \times X \rightarrow \mathbb{R}$$

satisfy

- (i) $F(x, \cdot) \in \mathcal{L}^1(X)$ for every $x \in A$,
- (ii) $F(\cdot, y) \in C^1(A)$ for almost every $y \in X$.

If, for any $k = 1, \dots, n$, there exist $g_k \in \mathcal{L}^1(X)$, $g_k \geq 0$ such that

$$|\partial_{x_k} F(x, y)| \leq g_k(y), \quad \forall y \in X, \quad \forall x \in A,$$

then the function

$$G(x) := \int_X F(x, y) d\mu(y), \quad x \in A$$

is of $C^1(A)$ class and we have

$$\partial_{x_k} G(x) = \int_X \partial_{x_k} F(x, y) d\mu(y), \quad \forall x \in A, \quad k = 1, \dots, n.$$

Theorem 2.3.21 (Fubini–Tonelli). *Let*

$$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$$

be a measurable function. We have

(i) If $f \geq 0$ then: $f(x, \cdot)$ is measurable for almost every $x \in \mathbb{R}^n$, in addition the function

$$\mathbb{R}^n \ni x \rightarrow \int_{\mathbb{R}^m} f(x, y) dy \in [0, +\infty],$$

is measurable over \mathbb{R}^n and we have

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y) dy \right) dx = \int_{\mathbb{R}^{n+m}} f(x, y) dx dy. \quad (2.3.1)$$

(ii) If $f \in \mathcal{L}^1(\mathbb{R}^{n+m})$ then $f(x, \cdot) \in \mathcal{L}^1(\mathbb{R}^m)$ for almost every $x \in \mathbb{R}^n$, furthermore

$$\int_{\mathbb{R}^m} f(\cdot, y) dy \in \mathcal{L}^1(\mathbb{R}^n)$$

and (2.3.1) holds true.

2.3.2 The L^p spaces

Let $p \in [1, +\infty)$ and (X, \mathcal{M}, μ) be a measurable space. We say that $f \in \mathcal{L}^p(X)$ if f is measurable and $|f|^p \in \mathcal{L}^1(X)$. $\mathcal{L}^p(X)$ is a vector space. We define $L^p(X)$ as the quotient space $(\mathcal{L}^p(X)/\sim)$ where " \sim " is the equivalence relation on $\mathcal{L}^p(X)$ defined as follows: $f \sim g$ if and only if $f = g$ almost everywhere. We equip $L^p(X)$ with the norm

$$\|f\|_{L^p(X)} = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

We say that $f \in \mathcal{L}^\infty(X)$ provided

$$\text{ess sup } |f| = \inf \{t \in \mathbb{R} : \mu(\{|f(x)| > t\}) = 0\} < +\infty.$$

We define $L^\infty(X)$ similarly as we have previously defined $L^p(X)$, $p < +\infty$. We equip $L^\infty(X)$ with the norm

$$\|f\|_{L^\infty(X)} = \text{ess sup } |f|.$$

Minkowski inequality. If $p \in [1, +\infty]$, $f, g \in L^p(X)$ then

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

Hölder inequality. Let $p \in [1, +\infty]$, let us denote by p' (the conjugate of p) the element of $[1, +\infty]$ satisfying

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If $f \in L^p(X)$ and $g \in L^{p'}(X)$ then $fg \in L^1(X)$ and

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^{p'}(X)}.$$

If $\mu(X) < +\infty$, we have

$$p_2 \geq p_1 \implies L^{p_2}(X) \subset L^{p_1}(X)$$

and the function

$$p \rightarrow \left(\frac{1}{\mu(X)} \int_X |f|^p d\mu \right)^{1/p},$$

turns out to be an increasing function (just apply Hölder inequality).

Moreover

$$\lim_{p \rightarrow \infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}.$$

Theorem 2.3.22. Let (X, \mathcal{M}, μ) be a measure space and let $p \in [1, +\infty]$. Then $L^p(X)$ is a **Banach space**. If $p = 2$, $L^2(X)$ is a **Hilbert space** equipped with the scalar product

$$(f, g)_{L^2(X)} = \int_X fg d\mu, \quad \forall f, g \in L^2(X).$$

We say that the measure space (X, \mathcal{M}, μ) is σ -finite, if there exists a countable family $\{X_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ such that

$$X = \bigcup_{j \in \mathbb{N}} X_j, \quad \text{and} \quad \mu(X_j) < +\infty.$$

Theorem 2.3.23. Let (X, \mathcal{M}, μ) be a σ -finite measure space and let $p \in [1, +\infty)$, then F is a bounded linear functional from $L^p(X)$ to \mathbb{R} if and only if there exists $g \in L^{p'}(X)$ which satisfies

$$F(f) = \int_X gf d\mu, \quad \forall f \in L^p(X).$$

Density and separability in L^p . Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set.

The following theorems hold true.

Theorem 2.3.24 (density of simple functions in $L^p(E)$, $1 \leq p \leq \infty$). If $f \in L^p(E)$ and $p \in [1, +\infty]$, then for every $\varepsilon > 0$ there exists a simple function s such that

$$\|f - s\|_{L^p(E)} < \varepsilon.$$

Theorem 2.3.25 (density of $C_0^0(E)$ in $L^p(X)$, $1 \leq p < \infty$). If $f \in L^p(E)$ and $p \in [1, +\infty)$, then for every $\varepsilon > 0$ there exists $g \in C_0^0(E)$, such that

$$\|f - g\|_{L^p(E)} < \varepsilon.$$

Theorem 2.3.26. If $p \in [1, +\infty)$, then $L^p(X)$ is a separable space. $L^\infty(X)$ is not a separable space.

Theorem 2.3.27. If $p \in [1, +\infty)$ and $f \in L^p(\mathbb{R}^n)$ then

$$\lim_{\delta \rightarrow 0} \left(\sup_{|h| < \delta} \int_{\mathbb{R}^n} |f(x-h) - f(x)|^p dx \right) = 0.$$

Reflexivity.

Let X be a normed space, let us denote by X' the space of the bounded linear functionals from X to \mathbb{R} . As it is well-known, X' is called the **dual** of X and X' turns out a Banach space (whether or not X is a Banach space) equipped with the norm

$$\|f\|_{X'} = \sup \{ \langle f, u \rangle : \|u\|_X \leq 1 \}, \quad \forall f \in X',$$

where

$$\langle f, u \rangle = f(u), \quad \forall u \in X,$$

$\langle \cdot, \cdot \rangle$ is said the "duality bracket of X' and X ".

Definition 2.3.28. Let X be a Banach space. We say that X is a **reflexive space** provided

$$\forall v \in (X')' \quad \exists u \in X, \text{ such that } \langle v, f \rangle = \langle f, u \rangle \quad \forall f \in X'.$$

Keep in mind that if $u \in X$ then the map

$$X' \ni f \rightarrow j_u(f) := \langle f, u \rangle,$$

is a bounded linear functional, hence $j_u \in (X')'$ and it can be proved (by the Hahn–Banach Theorem [12]) that

$$\|j_u\|_{(X')'} = \|u\|_X, \quad \forall u \in X. \quad (2.3.2)$$

Therefore, it is defined the map j

$$X \ni u \rightarrow j_u \in (X')'.$$

The map j is injective (applying, again, Hahn–Banach Theorem). Consequently, j is an embedding and, by (2.3.2) it is an isometry. If X is a reflexive space then j is also surjective and, by the Open Map Theorem, j^{-1} is continuous too. Ultimately, if X is a reflexive space, we can identify $(X')'$ with X by means of j . Recall that the Hilbert spaces are reflexive.

Definition 2.3.29. Let X be a Banach space and let $\{u_k\}$ be a sequence of X . We say that $\{u_k\}$ weakly converges to $u \in X$ and we write

$$u_k \rightharpoonup u, \quad \text{as } k \rightarrow \infty, \quad (\text{or } \{u_k\} \rightharpoonup u),$$

provided

$$\langle f, u_k \rangle \rightarrow \langle f, u \rangle, \quad \text{as } k \rightarrow \infty, \quad \forall f \in X'.$$

If a weak limit exists, then it is unique (it can be again proved by Hahn–Banach Theorem).

Proposition 2.3.30. Let X be a Banach space and $\{u_k\}$ be a sequence of X .

We have

- (i) if $\{u_k\}$ weakly converges then it is bounded;
- (ii) if $\{u_k\}$ weakly converges to u then

$$\|u\|_X \leq \liminf_{k \rightarrow \infty} \|u_k\|_X.$$

Proof. Let us prove (i), since for every $f \in X'$, $\{\langle f, u_k \rangle\}$ is a converging sequence of \mathbb{R} , it is bounded, that is

$$\sup_{k \in \mathbb{N}} |\langle f, u_k \rangle| \leq C(f) < +\infty, \quad \forall f \in X'.$$

Now, applying the Banach–Steinhaus Theorem (see [53]) to the map

$$X' \ni f \rightarrow T_k(f) := \langle f, u_k \rangle \in \mathbb{R}, k \in \mathbb{N}$$

we have

$$\sup_{k \in \mathbb{N}} \|u_k\|_X = \sup_{k \in \mathbb{N}} \|T_k\|_{(X')'} < +\infty.$$

Let us prove (ii). Since $\{u_k\} \rightharpoonup u$, $k \rightarrow \infty$, by (i) we get

$$\sup_{k \in \mathbb{N}} \|u_k\|_X < +\infty.$$

Moreover, for every $f \in X'$, we have

$$\begin{aligned} \langle f, u \rangle &= \lim_{k \rightarrow \infty} \langle f, u_k \rangle = \liminf_{k \rightarrow \infty} \langle f, u_k \rangle \leq \\ &\leq \liminf_{k \rightarrow \infty} \|f\|_{X'} \|u_k\|_X = \\ &= \|f\|_{X'} \liminf_{k \rightarrow \infty} \|u_k\|_X. \end{aligned}$$

Therefore

$$\|u\|_X \leq \liminf_{k \rightarrow \infty} \|u_k\|_X.$$

■

We recall

Theorem 2.3.31 (Banach–Alaoglu). *Let X be a reflexive Banach and let $\{u_k\}$ be a bounded sequence of X . Then there exists a subsequence $\{u_{k_j}\}$ which weakly converges.*

Theorem 2.3.32. *Let E be a measurable subset of \mathbb{R}^n . Then, $L^p(E)$ is a reflexive space, for every $p \in (1, +\infty)$.*

Convolution.

Let f, g be two measurable functions defined in \mathbb{R}^n with values in \mathbb{R} . Let $x \in \mathbb{R}^n$, if the function of the variable y , $f(x - y)g(y)$ is summable, we set

$$(f \star g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

If the function $(f \star g)(x)$ is defined for almost every $x \in \mathbb{R}^n$, we will call it the **convolution product** (or, simply, the **convolution**) of f and g .

Theorem 2.3.33 (the Young inequality). Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where

$$\frac{1}{p} + \frac{1}{q} \geq 1,$$

then $f \star g \in L^r(\mathbb{R}^n)$ where

$$r = \frac{1}{\frac{1}{p} + \frac{1}{q} - 1}$$

and we have

$$\|f \star g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfy

- (i) $\text{supp } \eta \subset B_1$,
- (ii) $\eta \geq 0$
- (iii) $\int_{\mathbb{R}^n} \eta(x) dx = 1$.

η is named a **mollifier**. For instance, let

$$\tilde{\eta}(s) = \begin{cases} c_n \exp\left\{-\frac{1}{1-4s^2}\right\}, & \text{for } s \in [0, 1/2), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$c_n = \left(\int_0^{1/2} s^{n-1} \exp\left\{-\frac{1}{1-4s^2}\right\} ds \right)^{-1},$$

then

$$\eta(x) = \tilde{\eta}(|x|),$$

is a mollifier.

Here and in the sequel we set, for $\varepsilon > 0$,

$$\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\varepsilon^{-1}x).$$

Theorem 2.3.34. (i) if $f \in L^p(\mathbb{R}^n)$, $p \in [1, +\infty)$, then

$$\eta_\varepsilon \star f \rightarrow f, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } L^p(\mathbb{R}^n).$$

(ii) If f is uniformly continuous and bounded in \mathbb{R}^n , then

$$\eta_\varepsilon \star f \rightarrow f, \quad \text{as } \varepsilon \rightarrow 0 \quad \text{uniformly in } \mathbb{R}^n.$$

Remark. Let E be a measurable set of \mathbb{R}^n , $f \in L^p(E)$, where $p \in [1, +\infty)$, then

$$\|\eta_\varepsilon \star (f\chi_E) - f\|_{L^p(E)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$\eta_\varepsilon \star f = \eta_\varepsilon \star (f\chi_E) = \int_E \eta_\varepsilon(x-y)f(y)dy.$$

◆

Theorem 2.3.35. Let Ω be an open set of \mathbb{R}^n , $f \in L^p(\Omega)$, where $p \in [1, +\infty)$, then

$$\eta_\varepsilon \star f \in C^\infty(\Omega),$$

$$\partial^\alpha (\eta_\varepsilon \star f) = (\partial^\alpha \eta_\varepsilon) \star f$$

and

$$\eta_\varepsilon \star f \rightarrow f, \quad \text{in } L^p(\Omega), \quad \text{as } \varepsilon \rightarrow 0.$$

Hence $C^\infty(\Omega)$ is dense in $L^p(\Omega)$.

We also have

Theorem 2.3.36 (density of $C_0^\infty(\Omega)$ in $L^p(\Omega)$, $1 \leq p < +\infty$). Let Ω be an open set of \mathbb{R}^n , $p \in [1, +\infty)$, then $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Let Ω be an open set of \mathbb{R}^n and $p \in [1, +\infty]$. We denote by $L_{loc}^p(\Omega)$ the space of measurable functions defined on Ω such that for every compact set K we have $f|_K \in L^p(K)$. Let $\{u_k\}$ be a sequence of $L_{loc}^p(\Omega)$, we write

$$u_k \rightarrow u, \quad \text{as } k \rightarrow \infty, \quad \text{in } L_{loc}^p(\Omega),$$

provided $u \in L_{loc}^p(\Omega)$ and for every compact $K \subset \Omega$, we have

$$(u_k)|_K \rightarrow u|_K, \quad \text{as } k \rightarrow \infty, \quad \text{in } L^p(K).$$

Let us define, for any $\varepsilon > 0$,

$$\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}.$$

If $f \in L_{loc}^1(\Omega)$, then $(\eta_\varepsilon \star f)(x)$ is defined for every $x \in \Omega_\varepsilon$, and we may rephrase theorems 2.3.34 and 2.3.35 as follows.

Theorem 2.3.37. *Let Ω be an open set of \mathbb{R}^n and $f \in L^1_{loc}(\Omega)$. Then*

- (i) $\eta_\varepsilon \star f \in C^\infty(\Omega_\varepsilon)$;
- (ii) if $f \in C^0(\Omega)$ then $\eta_\varepsilon \star f \rightarrow f$, uniformly on the compact sets of Ω as $\varepsilon \rightarrow 0$;
- (iii) if $p \in [1, +\infty)$ and $f \in L^p_{loc}(\Omega)$ then

$$\eta_\varepsilon \star f \rightarrow f, \quad \text{as } \varepsilon \rightarrow 0, \text{ in } L^p_{loc}(\Omega).$$

Let f be a measurable function defined on \mathbb{R}^n and let

$$\mathcal{O} = \{A \subset \mathbb{R}^n : A \text{ open and } f = 0 \text{ in } A \text{ a.e.}\},$$

the set

$$\text{supp } f = \mathbb{R}^n \setminus \bigcup_{A \in \mathcal{O}} A$$

is named the **essential support** of f . Hereafter, if there is no ambiguity, instead of "essential support of f " we will simply say "support of f ". We recall that then the essential support of a function if $u \in C^0(\mathbb{R}^n)$ is equal to the support defined in Section 2.1.

2.4 Partition of unity

Let us start by the following

Lemma 2.4.1. *Let Ω be an open set of \mathbb{R}^n and K be a compact set contained in Ω , then there exists $\varphi \in C^\infty_0(\mathbb{R}^n)$ such that $\text{supp } \varphi \subset \Omega$, $0 \leq \varphi \leq 1$ and $\varphi = 1$ in a neighborhood of K .*

Proof. For any $\varepsilon > 0$, let us denote by

$$K^{(\varepsilon)} = \{x \in \mathbb{R}^n : \text{dist}(x, K) \leq \varepsilon\}.$$

Let ε_0 and ε_1 satisfy

$$0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_0 + \varepsilon_1 < \text{dist}(K, \mathbb{R}^n \setminus \Omega).$$

Let us define

$$\varphi(x) = \int_{K^{(\varepsilon_1)}} \eta_{\varepsilon_0}(x - y) dy.$$

It can be easily checked that $\varphi \in C^\infty_0(\mathbb{R}^n)$,

$$\text{supp } \varphi \subset K^{(\varepsilon_0 + \varepsilon_1)} \subset \Omega$$

and

$$\varphi(x) = 1, \quad \forall x \in K^{(\varepsilon_0)}.$$

■

Lemma 2.4.2. *Let K be a compact set of \mathbb{R}^n and let V_1, V_2, \dots, V_l be some open sets of \mathbb{R}^n satisfying*

$$K \subset \bigcup_{j=1}^l V_j.$$

Then there exist the functions $\zeta_1, \dots, \zeta_l \in C_0^\infty(\mathbb{R}^n)$ which satisfy

$$\text{supp } \zeta_j \subset V_j, \quad j = 1, \dots, l,$$

$$0 \leq \zeta_j, \quad j = 1, \dots, l; \quad \sum_{j=1}^l \zeta_j \leq 1, \quad \text{on } \mathbb{R}^n$$

$$\sum_{j=1}^l \zeta_j = 1, \quad \text{in a neighborhood of } K.$$

Proof. Let us denote, for any $\varepsilon > 0$ and $V \subset \mathbb{R}^n$,

$$V_\varepsilon = \{x \in V : \text{dist}(x, \partial V) > \varepsilon\}.$$

We have

$$K \subset \bigcup_{\varepsilon > 0} \bigcup_{j=1}^l V_{\varepsilon, j}$$

and by the compactness of K it follows that there exists $\varepsilon_0 > 0$ such that

$$K \subset \bigcup_{j=1}^l V_{\varepsilon_0, j} \subset \bigcup_{j=1}^l \overline{V_{\varepsilon_0, j}} \subset \bigcup_{j=1}^l V_j,$$

(because $\overline{V_{\varepsilon_0, j}} \subset V_j$). Hence, denoting

$$K_j = K \cap \overline{V_{\varepsilon_0, j}}, \quad j = 1, \dots, l$$

we have immediately that K_j is a compact set, $K_j \subset V_j$, for any $j = 1, \dots, l$ and

$$K \subset \bigcup_{j=1}^l K_j.$$

By Lemma 2.4.1, we derive that for every $j \in \{1, \dots, l\}$ there exist $\varphi_j \in C_0^\infty(V_j)$ satisfying

$$0 \leq \varphi_j \leq 1, \quad \varphi_j = 1, \text{ in a neighborhood, } W_j, \text{ of } K_j.$$

Now, defining

$$\zeta_1 = \varphi_1, \quad \zeta_2 = \varphi_2(1 - \varphi_1), \quad \dots, \quad \zeta_j = \varphi_j(1 - \varphi_1) \cdots (1 - \varphi_{j-1}),$$

we get

$$\begin{aligned} \sum_{j=0}^l \zeta_j &= \varphi_1 + \varphi_2(1 - \varphi_1) + \cdots + \varphi_l(1 - \varphi_1) \cdots (1 - \varphi_{l-1}) = \\ &= 1 - (1 - \varphi_1) + \varphi_2(1 - \varphi_1) + \cdots + \varphi_l(1 - \varphi_1) \cdots (1 - \varphi_{l-1}) = \\ &= 1 - (1 - \varphi_1)(1 - \varphi_2) + \varphi_3(1 - \varphi_1)(1 - \varphi_2) + \cdots + \varphi_l(1 - \varphi_1) \cdots (1 - \varphi_{l-1}) = \\ &= 1 - (1 - \varphi_1)(1 - \varphi_2) \cdots (1 - \varphi_l). \end{aligned}$$

Therefore, if

$$x \in \bigcup_{j=1}^l W_j,$$

there exists $\bar{j} \in \{1, \dots, l\}$ such that $x \in W_{\bar{j}}$, hence $\varphi_{\bar{j}}(x) = 1$ and

$$\sum_{j=0}^l \zeta_j(x) = 1, \quad \forall x \in \bigcup_{j=1}^l W_j.$$

Since $\bigcup_{j=1}^l W_j$ is a neighborhood of K , the Lemma is proved. ■

In what follows, we will say that the set of functions $\varphi_1, \dots, \varphi_l$ is a **partition of the unity subordinate to the covering** $\{V_j\}_{1 \leq j \leq l}$.

Theorem 2.4.3 (partition of unity). *Let Ω be an open set of \mathbb{R}^n and let V_1, V_2, \dots, V_l be open sets of \mathbb{R}^n satisfying*

$$\partial\Omega \subset \bigcup_{j=1}^l V_j.$$

Then there exist the functions $\zeta_0, \zeta_1, \dots, \zeta_l \in C^\infty(\mathbb{R}^n)$ such that

$$(\zeta_0)|_\Omega \in C_0^\infty(\Omega),$$

$$\text{supp } \zeta_0 \subset \mathbb{R}^n \setminus \partial\Omega; \quad \text{supp } \zeta_j \subset V_j, \quad j = 1, \dots, l,$$

$$\sum_{j=0}^l \zeta_j = 1, \quad \text{on } \mathbb{R}^n; \quad 0 \leq \zeta_j \leq 1, \quad j = 0, 1, \dots, l.$$

Proof. Let us consider a partition of unity subordinate to the covering, $\{V_j\}_{1 \leq j \leq l}$, of $\partial\Omega$. Let us denote by

$$\mathcal{O}_j = \text{supp } \varphi_j, \quad j = 1, \dots, l$$

and set

$$V_0 = \mathbb{R}^n \setminus \bigcup_{j=1}^l \mathcal{O}_j.$$

Let

$$\zeta_0 = 1 - \sum_{j=1}^l \zeta_j.$$

We get trivially $\zeta_0 \in C_0^\infty(\mathbb{R}^n)$ and

$$\sum_{j=0}^l \zeta_j(x) = 1, \quad \forall x \in \mathbb{R}^n,$$

$$\text{supp } \zeta_0 \subset \mathbb{R}^n \setminus \partial\Omega.$$

Moreover, by Lemma 2.4.2 there exists an open neighborhood, \mathcal{U} , of $\partial\Omega$ such that

$$\sum_{j=1}^l \zeta_j(x) = 1, \quad \forall x \in \mathcal{U}.$$

Hence

$$\zeta_0(x) = 0, \quad \forall x \in \mathcal{U}.$$

Therefore

$$\text{supp}(\zeta_0)|_\Omega \subset \bar{\Omega} \setminus \mathcal{U} \subset \Omega,$$

this implies

$$(\zeta_0)|_\Omega \in C_0^\infty(\Omega)$$

concluding the proof. ■

Remark. By Theorem 2.4.3 it is evident that $V_0 \cap \Omega, V_1, \dots, V_l$ is a covering of $\bar{\Omega}$ and $(\zeta_0)|_\Omega, \zeta_1, \dots, \zeta_l$ is a partition of the unity subordinate to that covering. ♦

2.5 The Lebesgue differentiation Theorem

In this Section we prove the following

Theorem 2.5.1 (Lebesgue differentiation). *If $f \in L^1_{loc}(\mathbb{R}^n)$ then*

$$\lim_{r \rightarrow 0} \int_{B_r(x)} f(y) dy = f(x), \quad \text{a.e. } x \in \mathbb{R}^n, \quad (2.5.1)$$

where

$$\int_{B_r(x)} f(y) dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy.$$

In order to prove Theorem 2.5.1 we need some preliminary lemmas and propositions. We start by

Lemma 2.5.2 (Covering). *Let E be a Lebesgue measurable subset of \mathbb{R}^n and let \mathcal{B} be a family of balls of \mathbb{R}^n satisfying*

$$E \subset \bigcup_{B \in \mathcal{B}} B.$$

and,

$$\sup_{B \in \mathcal{B}} d(B) < +\infty,$$

where $d(B)$ is the diameter of B . Then there exists a countable (or finite) family, $\{B_k\}_{k \in \Lambda} \subset \mathcal{B}$ which satisfies

$$B_j \cap B_k = \emptyset, \quad \text{for } j \neq k, \quad j, k \in \Lambda$$

and

$$\sum_{k \in \Lambda} |B_k| \geq 5^{-n} |E|.$$

Proof of the Lemma. Firstly we construct the family $\mathcal{B}_0 := \{B_k\}_{k \in \Lambda}$. Let $B_1 \in \mathcal{B}$ satisfy

$$d(B_1) \geq \frac{1}{2} \sup \{d(B) : B \in \mathcal{B}\}$$

and set

$$d_1 = \sup \{d(B) : B \in \mathcal{B}, B \cap B_1 = \emptyset\} < +\infty.$$

Let us consider the set

$$\mathcal{F}_1 = \left\{ B \in \mathcal{B} : B \cap B_1 = \emptyset, d(B) \geq \frac{1}{2}d_1 \right\}.$$

If $\mathcal{F}_1 = \emptyset$, then we choose $\mathcal{B}_0 := \{B_1\}$; otherwise, if $\mathcal{F}_1 \neq \emptyset$, we choose $B_2 \in \mathcal{F}_1$ and we continue the process. Let us suppose we have chosen B_1, \dots, B_i , let us consider the family

$$\mathcal{F}_i = \left\{ B \in \mathcal{B} : B \cap \bigcup_{j=1}^i B_j = \emptyset, d(B) \geq \frac{1}{2}d_i \right\},$$

where

$$d_i = \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^i B_j = \emptyset \right\}.$$

If $\mathcal{F}_i = \emptyset$, then we choose $\mathcal{B}_0 := \{B_1, \dots, B_i\}$; otherwise, if $\mathcal{F}_i \neq \emptyset$, then we choose $B_{i+1} \in \mathcal{F}_i$ as above and continue the process. All in all, we construct a finite or infinite, countable family, $\{B_i\}_{i \in \Lambda} \subset \mathcal{B}$ such that

$$B_j \cap B_k = \emptyset, \quad \text{for } j \neq k, j, k \in \Lambda$$

and

$$d(B_i) \geq \frac{1}{2} \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^{i-1} B_j = \emptyset \right\}, \quad \text{for } i \geq 2.$$

Let us consider the case in which \mathcal{B}_0 is finite. Let

$$\mathcal{B}_0 = \{B_1, \dots, B_k\}. \tag{2.5.2}$$

Let us denote by

$$\begin{aligned} \mathcal{A} &= \left\{ B \in \mathcal{B} : B \cap \bigcup_{j=1}^k B_j = \emptyset \right\}, \\ \mathcal{C} &= \mathcal{B} \setminus \mathcal{A} = \left\{ B \in \mathcal{B} : B \cap \bigcup_{j=1}^k B_j \neq \emptyset \right\}, \\ A &= \bigcup_{B \in \mathcal{A}} B, \\ C &= \bigcup_{B \in \mathcal{C}} B. \end{aligned}$$

We have

$$E \subset \bigcup_{B \in \mathcal{B}} B = A \cup C. \quad (2.5.3)$$

Claim. We have

$$\mathcal{A} = \emptyset.$$

Proof of Claim. We argue by contradiction. Let us assume that $\mathcal{A} \neq \emptyset$. Let us first observe that it cannot occur that

$$d(B) < \frac{1}{2} \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^k B_j = \emptyset \right\}, \quad \forall B \in \mathcal{A}$$

otherwise we would have

$$\begin{aligned} 0 < \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^k B_j = \emptyset \right\} &\leq \\ &\leq \frac{1}{2} \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^k B_j = \emptyset \right\}, \end{aligned} \quad (2.5.4)$$

Which is evidently absurd.

Therefore, there exists $\tilde{B} \in \mathcal{A}$ such that

$$d(\tilde{B}) \geq \frac{1}{2} \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^k B_j = \emptyset \right\}.$$

In particular, we have $\tilde{B} \notin \{B_1, \dots, B_k\}$ and, consequently, the process of construction of \mathcal{B}_0 does not stop, but we had assumed the opposite (i.e., \mathcal{B}_0 finite family consisting of k elements) and thus we have a contradiction.

By what is proved in the Claim and by (2.5.3), we have

$$E \subset C. \quad (2.5.5)$$

Now, let us prove

$$E \subset \bigcup_{j=1}^k B_j^*, \quad (2.5.6)$$

where B_j^* denotes the ball having the same center as B_j with radius equal to 5 times the radius of B_j . Let $x \in E$, then (2.5.5) implies that there exists $\hat{B} \in \mathcal{B}$ satisfy

$$\hat{B} \cap \bigcup_{j=1}^k B_j \neq \emptyset.$$

Let $j_0 \in \{1, \dots, k\}$ such that

$$\begin{aligned} x &\in \hat{B}, \\ \hat{B} \cap B_{j_0} &\neq \emptyset \end{aligned}$$

and

$$\hat{B} \cap \bigcup_{j=1}^{j_0-1} B_j = \emptyset,$$

(if $j_0 = 1$, $\bigcup_{j=1}^{j_0-1} B_j$ is the empty set). By the third relationship we have

$$d(B_{j_0}) \geq \frac{1}{2}d(\hat{B}).$$

By the latter and by $\hat{B} \cap B_{j_0} \neq \emptyset$ we easily obtain

$$\hat{B} \subset B_{j_0}^*.$$

Hence

$$x \in \bigcup_{j=1}^k B_j^*,$$

and (2.5.6) is proved. Moreover we have

$$|E| \leq \sum_{j=1}^k |B_j^*| = 5^n \sum_{j=1}^k |B_j|.$$

Now let us consider the case where \mathcal{B}_0 is infinite. Hence, in such a case we have $\mathcal{B}_0 = \{B_k\}_{k \in \mathbb{N}}$. If

$$\sum_{k=1}^{\infty} |B_k| = +\infty,$$

there is nothing to prove. Let us assume that

$$\sum_{k=1}^{\infty} |B_k| < +\infty,$$

which implies

$$\lim_{k \rightarrow \infty} d(B_k) = 0.$$

If $\tilde{B} \in \mathcal{B}$ and

$$k_0 = \min \left\{ j \in \mathbb{N} : d(B_{j+1}) < \frac{1}{2}d(\tilde{B}) \right\}, \quad (2.5.7)$$

then

$$\tilde{B} \cap \bigcup_{i=1}^{k_0} B_i \neq \emptyset. \quad (2.5.8)$$

To prove (2.5.8) it suffices to notice that if it were

$$\tilde{B} \cap \bigcup_{i=1}^{k_0} B_i = \emptyset,$$

we would have

$$d(B_{k_0+1}) \geq \frac{1}{2} \sup \left\{ d(B) : B \in \mathcal{B}, B \cap \bigcup_{j=1}^{k_0} B_j = \emptyset \right\} \geq \frac{1}{2}d(\tilde{B}),$$

which contradicts (2.5.7).

Since (2.5.8) holds true, we set

$$j_0 = \min \left\{ j \in \{1, \dots, k_0\} : B_j \cap \tilde{B} \neq \emptyset \right\} \quad (2.5.9)$$

obtaining

$$\tilde{B} \cap \bigcup_{j=1}^{j_0-1} B_j = \emptyset.$$

Hence (by (2.5.9) and by the definition of \mathcal{B}_0)

$$\begin{cases} \tilde{B} \cap B_{j_0} \neq \emptyset, \\ d(B_{j_0}) \geq \frac{1}{2}d(\tilde{B}). \end{cases}$$

From which it follows that for every $B \in \mathcal{B}$ there exists $B_{j_0} \in \mathcal{B}_0$ such that $B \subset B_{j_0}^*$. Therefore, arguing as in the finite case, we have

$$E \subset \bigcup_{B \in \mathcal{B}} B = \bigcup_{j=1}^{\infty} B_j^*$$

and by the latter the thesis follows. ■

Now we introduce the notion of **maximal function**. Let $f \in L^1(\mathbb{R}^n)$, the following function is called the maximal function associated to f

$$M(f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy, \quad \forall x \in \mathbb{R}^n. \quad (2.5.10)$$

Let us observe that $M(f)$ is a measurable function. More precisely, the following Proposition holds true

Proposition 2.5.3. *If $f \in L^1(\mathbb{R}^n)$ then $M(f)$ is a lower semicontinuous function.*

Proof. If f is identically 0, we have $M(f) \equiv 0$. Let us assume that f is not identically 0. Hence

$$M(f)(x) > 0, \quad \forall x \in \mathbb{R}^n. \quad (2.5.11)$$

Fix $t \geq 0$ and let us prove that

$$A = \{x \in \mathbb{R}^n : M(f)(x) > t\}$$

is an open set.

In the case where $t = 0$, we have $A = \mathbb{R}^n$. In the case where $t > 0$, let $x_0 \in A$ and $0 < \varepsilon < M(f)(x_0) - t$. By the definition of $M(f)$, there exists $r_\varepsilon > 0$ such that

$$\frac{1}{|B_{r_\varepsilon}(x_0)|} \int_{B_{r_\varepsilon}(x_0)} |f(y)| dy > M(f)(x_0) - \varepsilon > t.$$

Now, let $0 < \eta < M(f)(x_0) - t - \varepsilon$. Since $f \in L^1(\mathbb{R}^n)$, there exists $\delta > 0$ such that if $|x_0 - x| < \delta$ then

$$\left| \int_{B_{r_\varepsilon}(x_0)} |f(y)| dy - \int_{B_{r_\varepsilon}(x)} |f(y)| dy \right| < \eta |B_{r_\varepsilon}|.$$

Hence

$$\begin{aligned} M(f)(x) &\geq \frac{1}{|B_{r_\varepsilon}(x)|} \int_{B_{r_\varepsilon}(x)} |f(y)| dy > \\ &> \frac{1}{|B_{r_\varepsilon}(x_0)|} \int_{B_{r_\varepsilon}(x_0)} |f(y)| dy - \eta > \\ &> M(f)(x_0) - \varepsilon - \eta > t, \end{aligned}$$

which implies

$$B_\delta(x_0) \subset A.$$

Therefore A is open. ■

Lemma 2.5.4. *Let $f \in L^1(\mathbb{R}^n)$ and $M(f)$ its maximal function, then*

$$|\{x \in \mathbb{R}^n : M(f)(x) > t\}| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f(y)| dy, \quad (2.5.12)$$

Proof. Set

$$E_t = \{x \in \mathbb{R}^n : M(f)(x) > t\}.$$

If $x \in E_t$, then $M(f)(x) > t$. Hence there exists $r_x > 0$ such that

$$\int_{B_{r_x}(x)} |f(y)| dy > t |B_{r_x}(x)|.$$

For the sake of brevity, set $B_x = B_{r_x}(x)$ so that we have

$$\frac{1}{t} \int_{B_x} |f(y)| dy > |B_x|. \quad (2.5.13)$$

Now, we have trivially that $\{B_x\}_{x \in E_t}$ is a covering of E_t ; moreover (2.5.13) and $f \in L^1(\mathbb{R}^n)$ give

$$\sup_{x \in E_t} |B_x| < +\infty.$$

Thus, the assumptions of Lemma 2.5.2 are satisfied and therefore there exists a finite or countable, pairwise disjoint family of balls, $\{B_k\}_{k \in \Lambda}$, such that

$$\sum_{k \in \Lambda} |B_k| \geq 5^{-n} |E_t|.$$

Now, recalling that $B_k \cap B_j = \emptyset$, for any $j \neq k$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)| dy &\geq \int_{\bigcup_{k \in \Lambda} B_k} |f(y)| dy = \\ &= \sum_{k \in \Lambda} \int_{B_k} |f(y)| dy \geq \\ &\geq t \sum_{k \in \Lambda} |B_k| \geq \\ &\geq 5^{-n} t |E_t|. \end{aligned}$$

Therefore (2.5.12) is proved. ■

Remark 1. The function $M(f)$ may take the value $+\infty$, however it is almost everywhere finite. As a matter of fact, by Lemma 2.5.4 we get

$$|\{x \in \mathbb{R}^n : M(f)(x) = +\infty\}| \leq |\{x \in \mathbb{R}^n : M(f)(x) > t\}| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f(y)| dy, \quad \forall t > 0$$

hence, passing to the limit as t that goes to $+\infty$, we have

$$|\{x \in \mathbb{R}^n : M(f)(x) = +\infty\}| = 0.$$

◆

Remark 2. Inequality (2.5.12), apart from the value of the constant 5^n , cannot be improved. To show this it suffices to consider functions $f \in L^1(\mathbb{R}^n)$ which approximate the Dirac measure concentrated at 0. For instance

$$f_\varepsilon = \frac{\chi_{B_\varepsilon}}{|B_\varepsilon|}.$$

Proceeding formally (the reader takes care of the details), we consider $f = \delta(x)$ (the Dirac delta). For this choice, we have

$$M(f)(x) = \frac{1}{c_n |x|^n},$$

where c_n is the measure of unit ball of \mathbb{R}^n . Therefore

$$|\{x \in \mathbb{R}^n : M(f)(x) > t\}| = \frac{1}{t} = \frac{1}{t} \int_{\mathbb{R}^n} |f(y)| dy.$$

◆

Remark 3. Let us observe that, unless in the trivial case where f is identically equal to 0, we have

$$M(f) \notin L^1(\mathbb{R}^n).$$

In this respect, we prove

$$M(f)(x) \geq \frac{C}{|x|^n}, \quad \text{for } |x| \geq 1. \quad (2.5.14)$$

Indeed, since f does not vanish identically, there exists $t_0 > 0$ such that

$$0 < |E| < +\infty;$$

where

$$E = \{x \in \mathbb{R}^n : |f(x)| > t_0\}.$$

Let $r_0 > 0$ satisfy

$$|E \cap B_{r_0}| \geq \frac{1}{2}|E|.$$

For any $x \in \mathbb{R}^n$, we have (since $B_{r_0} \subset B_{r_0+|x|}(x)$)

$$\begin{aligned} M(f)(x) &\geq \frac{1}{|B_{r_0+|x|}(x)|} \int_{B_{r_0+|x|}(x)} |f(y)| dy \geq \\ &\geq \frac{1}{c_n (|x| + r_0)^n} \int_{B_{r_0}} |f(y)| dy \geq \\ &\geq \frac{t_0 |E|}{2c_n (|x| + r_0)^n}, \end{aligned}$$

from which we have (2.5.14) with $C = \frac{t_0 |E|}{2c_n (1+r_0)^n}$.

It can be proved that if $f \in L^p(\mathbb{R}^n)$, where $1 < p \leq +\infty$, then $M(f) \in L^p(\mathbb{R}^n)$ and

$$\|M(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p(\mathbb{R}^n). \quad (2.5.15)$$

For more insights into the maximal function, we refer to [71, Ch. 1]. \blacklozenge

Proof of Theorem 2.5.1. Provided that f is replaced by $f\chi_{B_R}$ with arbitrary R , we may assume $f \in L^1(\mathbb{R}^n)$. Let us denote

$$f_r(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy, \quad x \in \mathbb{R}^n$$

and notice that

$$f_r = \varphi_r \star f,$$

where

$$\varphi_r(x) = r^{-n} \varphi_1(r^{-1}x),$$

$$\varphi_1 = \frac{1}{|B_1|} \chi_{B_1}.$$

Hence

$$\lim_{r \rightarrow 0} \|f_r - f\|_{L^1(\mathbb{R}^n)} = 0.$$

Consequently, there exists a sequence $\{r_k\}$ such that

$$\{r_k\} \rightarrow 0^+$$

and

$$\lim_{k \rightarrow \infty} f_{r_k}(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n. \quad (2.5.16)$$

Now, let us denote

$$\Omega f(x) = \limsup_{r \rightarrow 0} f_r(x) - \liminf_{r \rightarrow 0} f_r(x).$$

By Remark 1 we get that $\limsup_{r \rightarrow 0^+} f_r(x)$, $\liminf_{r \rightarrow 0^+} f_r(x)$ are finite almost everywhere. As a matter of fact, we have

$$\left| \limsup_{r \rightarrow 0} f_r(x) \right|, \left| \liminf_{r \rightarrow 0} f_r(x) \right| \leq M(f)(x) < +\infty, \quad \text{a.e. } x \in \mathbb{R}^n. \quad (2.5.17)$$

Let us now prove that

$$\Omega f(x) = 0, \quad \text{a.e. } x \in \mathbb{R}^n. \quad (2.5.18)$$

Claim. If $g \in C_0^0(\mathbb{R}^n)$, then

$$g_r \rightarrow g, \quad \text{uniformly as } r \rightarrow 0,$$

hence

$$\Omega g(x) = 0, \quad \forall x \in \mathbb{R}^n. \quad (2.5.19)$$

Proof of Claim. Since g is a uniformly continuous function, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then

$$|g(x) - g(y)| < \varepsilon.$$

Now

$$g_r(x) - g(x) = \int_{\mathbb{R}^n} \varphi_1(z) (g(x - rz) - g(x)) dz.$$

Hence, if $0 < r < \delta$, we get

$$|g_r(x) - g(x)| \leq \int_{\mathbb{R}^n} \varphi_1(z) |g(x - rz) - g(x)| dz \leq \varepsilon, \quad \forall x \in \mathbb{R}^n.$$

Claim is proved.

Now, let $f \in L^1(\mathbb{R}^n)$. Since $C_0^0(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ (Theorem 2.3.25) it follows that for any $\sigma > 0$ there exists $g \in C_0^0(\mathbb{R}^n)$ such that

$$\|f - g\|_{L^1(\mathbb{R}^n)} < \sigma.$$

Let $h = f - g$, we have trivially $f = g + h$, $\|h\|_{L^1(\mathbb{R}^n)} < \sigma$ and by (2.5.19) we get

$$\Omega f(x) \leq \Omega g(x) + \Omega h(x) = \Omega h(x), \quad \forall x \in \mathbb{R}^n.$$

Hence, for any $\eta > 0$, we have

$$|\{x \in \mathbb{R}^n : \Omega f(x) > \eta\}| \leq |\{x \in \mathbb{R}^n : \Omega h(x) > \eta\}|. \quad (2.5.20)$$

On the other hand, we have trivially

$$\Omega h(x) \leq 2M(h)(x), \quad \forall x \in \mathbb{R}^n,$$

this inequality and (2.5.20) imply

$$|\{x \in \mathbb{R}^n : \Omega f(x) > \eta\}| \leq \left| \left\{ x \in \mathbb{R}^n : M(h)(x) > \frac{\eta}{2} \right\} \right|.$$

By the latter, by (2.5.20) and by Lemma 2.5.4 we have

$$|\{x \in \mathbb{R}^n : \Omega f(x) > \eta\}| \leq \frac{2C}{\eta} \|h\|_{L^1(\mathbb{R}^n)} \leq \frac{2C\sigma}{\eta},$$

where $C = 5^n$. Hence, by choosing

$$\sigma = \eta^2,$$

we obtain

$$|\{x \in \mathbb{R}^n : \Omega f(x) > \eta\}| \leq 2C\eta, \quad (2.5.21)$$

which yields

$$|\{x \in \mathbb{R}^n : \Omega f(x) > 0\}| = \lim_{j \rightarrow \infty} \left| \left\{ x \in \mathbb{R}^n : \Omega f(x) > \frac{1}{j} \right\} \right| = 0$$

hence, (2.5.18) follows. Taking into account (2.5.17), we have that the limit

$$\lim_{r \rightarrow 0} f_r(x)$$

there exists almost everywhere. Therefore, (2.5.16) implies

$$\lim_{r \rightarrow 0} f_r(x) = \lim_{k \rightarrow \infty} f_{r_k}(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

■

Corollary 2.5.5. *Let $p \in [1, +\infty)$. If $f \in L_{loc}^p(\mathbb{R}^n)$, then*

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - f(x)|^p dy = 0, \quad \text{a.e. } x \in \mathbb{R}^n. \quad (2.5.22)$$

Proof. For any $c \in \mathbb{R}$ let us denote by D_c the subset of \mathbb{R}^n of the points x satisfying

$$\lim_{r \rightarrow 0} \left(\int_{B_r(x)} |f(y) - c|^p dy \right)^{1/p} = |f(x) - c|.$$

Set

$$E_c = \mathbb{R}^n \setminus D_c.$$

Theorem 2.5.1 implies

$$|E_c| = 0.$$

Consequently, setting

$$E = \bigcup_{q \in \mathbb{Q}} E_q,$$

we obtain

$$|E| = 0.$$

Now, let us prove that

$$\lim_{r \rightarrow 0} \left(\int_{B_r(x)} |f(y) - c|^p dy \right)^{1/p} = |f(x) - c|, \quad \forall x \in \mathbb{R}^n \setminus E, \quad \forall c \in \mathbb{R}. \quad (2.5.23)$$

Let $c \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus E$ and let $\delta > 0$ and $q \in \mathbb{Q}$ satisfy

$$|q - c| < \delta. \quad (2.5.24)$$

The triangle inequality gives

$$\begin{aligned} \left(\int_{B_r(x)} |f(y) - q|^p dy \right)^{1/p} - \delta &< \left(\int_{B_r(x)} |f(y) - c|^p dy \right)^{1/p} < \\ &< \left(\int_{B_r(x)} |f(y) - q|^p dy \right)^{1/p} + \delta. \end{aligned} \quad (2.5.25)$$

Now, let us denote

$$\Lambda'(x) = \liminf_{r \rightarrow 0} \left(\int_{B_r(x)} |f(y) - c|^p dy \right)^{1/p},$$

$$\Lambda''(x) = \limsup_{r \rightarrow 0} \left(\int_{B_r(x)} |f(y) - c|^p dy \right)^{1/p}.$$

Passing to the limit as $r \rightarrow 0$, by (2.5.25) we obtain

$$|f(x) - q| - \delta \leq \Lambda'(x) \leq \Lambda''(x) \leq |f(x) - q| + \delta. \quad (2.5.26)$$

Passing again to the limit as $\delta \rightarrow 0$ in (2.5.26) and taking into account (2.5.24), we obtain

$$\Lambda'(x) = \Lambda''(x) = |f(x) - c|, \quad \forall x \in \mathbb{R}^n \setminus E.$$

Therefore (2.5.23) holds true. Set therein $c = f(x)$ and we obtain (2.5.22).
■

2.6 The Rademacher Theorem

Let us recall the definition of **absolutely continuous function** over the interval $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$.

We say that the function

$$f : [a, b] \rightarrow \mathbb{R},$$

is absolutely continuous provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, chosen anyway a finite family of pairwise disjoint intervals $(a_j, b_j) \subset [a, b]$, $j = 1, \dots, N$ satisfying

$$\sum_{j=1}^N (b_j - a_j) < \delta,$$

we have

$$\sum_{j=1}^N |f(b_j) - f(a_j)| < \varepsilon.$$

We will denote by $AC([a, b])$ the class of the absolutely continuous functions on $[a, b]$ and we will denote by $AC_{loc}(\mathbb{R})$ the class of functions $f : \mathbb{R} \rightarrow \mathbb{R}$

such that for every interval $[a, b]$ we have $f|_{[a,b]} \in AC([a, b])$. Let us recall that if $f \in AC_{loc}(\mathbb{R})$, then f is a differentiable function almost everywhere in \mathbb{R} .

If f is a Lipschitz continuous function in \mathbb{R} , then $f \in AC_{loc}(\mathbb{R})$, hence f is a differentiable almost everywhere. The main purpose of the present Section is to extend this result to several variable Lipschitz continuous functions. Precisely we want to prove

Theorem 2.6.1 (Rademacher). *If $f \in C^{0,1}(\mathbb{R}^n)$, then f is a differentiable function almost everywhere.*

Proof. Let $v \in \mathbb{R}^n$ satisfy $|v| = 1$. Set

$$\partial_v f(x) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}, \quad \text{provided the limit exists;}$$

notice that, since f is Lipschitz continuous, if the limit above exists it is finite.

Claim 1.

$$\partial_v f(x), \quad \text{exists a.e. } x \in \mathbb{R}^n.$$

Proof of Claim 1. Let us denote

$$\bar{\partial}_v f(x) := \limsup_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

$$\underline{\partial}_v f(x) := \liminf_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Since f is a continuous function, we have

$$\bar{\partial}_v f(x) = \lim_{k \rightarrow \infty} \sup_{0 < |t| < 1/k, t \in \mathbb{Q}} \frac{f(x + tv) - f(x)}{t}$$

and

$$\underline{\partial}_v f(x) = \lim_{k \rightarrow \infty} \inf_{0 < |t| < 1/k, t \in \mathbb{Q}} \frac{f(x + tv) - f(x)}{t}.$$

hence $\bar{\partial}_v f$ and $\underline{\partial}_v f$ are measurable functions. Consequently

$$A_v := \{x \in \mathbb{R}^n : \underline{\partial}_v f(x) < \bar{\partial}_v f(x)\},$$

$$\ell_v := \{tv : t \in \mathbb{R}\}$$

are Lebesgue measurable sets.

Let us first consider the case $v = e_n$. The Fubini–Tonelli Theorem gives

$$\begin{aligned} |A_{e_n}| &= \int_{\mathbb{R}^n} \chi_{A_{e_n}}(x) dx = \\ &= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} \chi_{A_{e_n}}(x', x_n) dx_n = \\ &= \int_{\mathbb{R}^{n-1}} |(x' + \ell_{e_n}) \cap A_{e_n}|_1 dx', \end{aligned} \quad (2.6.1)$$

where $|(x' + \ell_{e_n}) \cap A_{e_n}|_1$ is the Lebesgue measure on \mathbb{R} of $(x' + \ell_{e_n}) \cap A_{e_n}$. For any fixed $x' \in \mathbb{R}^{n-1}$, we have

$$(x' + \ell_{e_n}) \cap A_{e_n} = \{x'\} \times \{t \in \mathbb{R} : \underline{\partial}_n f(x', t) < \bar{\partial}_n f(x', t)\}.$$

Now, by denoting

$$\varphi(t) = f(x', t),$$

since f is Lipschitz continuous, we have $\varphi \in AC_{loc}(\mathbb{R})$. In particular, the function φ is almost everywhere differentiable, hence

$$|(x' + \ell_{e_n}) \cap A_{e_n}|_1 = 0.$$

Therefore, by (2.6.1), we get

$$|A_{e_n}| = 0.$$

Whenever $v \neq e_n$, let us consider a rotation \mathcal{R} of \mathbb{R}^n such that

$$v = \mathcal{R}(e_n).$$

By setting

$$\tilde{f} = f \circ \mathcal{R},$$

we get (due to the invariance of the Lebesgue measure with respect to rotations)

$$|A_v| = |\mathcal{R}^{-1}(A_v)|.$$

Moreover, it is easily checked that

$$\mathcal{R}^{-1}(A_v) = \left\{ y \in \mathbb{R}^n : \underline{\partial}_n \tilde{f}(y) < \bar{\partial}_n \tilde{f}(y) \right\}.$$

Consequently, since \tilde{f} is Lipschitz continuous, from what we have previously proved, we derive

$$|A_v| = |\mathcal{R}^{-1}(A_v)| = \left| \left\{ y \in \mathbb{R}^n : \underline{\partial}_n \tilde{f}(y) < \bar{\partial}_n \tilde{f}(y) \right\} \right| = 0$$

that concludes the proof of Claim 1.

Set

$$\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x)), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Claim 2. We have

$$\partial_v f(x) = v \cdot \nabla f(x), \quad \text{a.e. } x \in \mathbb{R}^n \quad (2.6.2)$$

and

$$|\nabla f(x)| \leq L, \quad \text{a.e. } x \in \mathbb{R}^n, \quad (2.6.3)$$

where

$$L = [f]_{0,1,\mathbb{R}^n},$$

Proof of Claim 2. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be arbitrary. We easily get

$$\int_{\mathbb{R}^n} \frac{f(x+tv) - f(x)}{t} \zeta(x) dx = - \int_{\mathbb{R}^n} f(x) \frac{\zeta(x) - \zeta(x-tv)}{t} dx. \quad (2.6.4)$$

Now, since

$$\lim_{t \rightarrow 0} \frac{f(x+tv) - f(x)}{t} = \partial_v f(x), \quad \text{a.e. } x \in \mathbb{R}^n$$

and

$$\left| \frac{f(x+tv) - f(x)}{t} \zeta(x) \right| \leq L |\zeta(x)|, \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R} \setminus \{0\},$$

we have by Dominated Convergence Theorem and by (2.6.4),

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_v f(x) \zeta(x) dx &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} \frac{f(x+tv) - f(x)}{t} \zeta(x) dx = \\ &= - \lim_{t \rightarrow 0} \int_{\mathbb{R}^n} f(x) \frac{\zeta(x) - \zeta(x-tv)}{t} dx = \\ &= - \sum_{j=1}^n v_j \int_{\mathbb{R}^n} f(x) \partial_j \zeta(x) dx = \\ &= \sum_{j=1}^n v_j \int_{\mathbb{R}^n} \partial_j f(x) \zeta(x) dx. \end{aligned} \quad (2.6.5)$$

Let us justify the last equality. For any $j \in \{1, \dots, n\}$ we have that the function

$$x_j \rightarrow f(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n),$$

belongs to $AC_{loc}(\mathbb{R})$. Therefore, by considering, for instance, the case $j = n$ (the other cases are similar), we have by Fubini–Tonelli Theorem

$$\begin{aligned} & - \int_{\mathbb{R}^n} f(x) \partial_n \zeta(x) dx = - \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} f(x', x_n) \partial_n \zeta(x', x_n) dx_n = \\ & = - \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} (\partial_n (f(x', x_n) \zeta(x', x_n)) - \partial_n f(x', x_n) \zeta(x', x_n)) dx_n = \\ & = \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} \partial_n f(x', x_n) \zeta(x', x_n) dx_n = \\ & = \int_{\mathbb{R}^n} \partial_n f(x) \zeta(x) dx. \end{aligned}$$

Consequently, (2.6.5) gives

$$\int_{\mathbb{R}^n} \partial_v f(x) \zeta(x) dx = \int_{\mathbb{R}^n} (v \cdot \nabla f(x)) \zeta(x) dx, \quad \forall \zeta \in C_0^\infty(\mathbb{R}^n)$$

which yields (2.6.2). Concerning (2.6.3), it is an immediate consequence of (2.6.2) and of the Cauchy–Schwarz inequality. The proof of Claim 2 is concluded.

Let $D = \{v_k\}_{k \in \mathbb{N}}$ be such that

$$|v_k| = 1, \quad \forall k \in \mathbb{N}$$

and

$$\overline{D} = \partial B_1.$$

Moreover, let

$$x \in \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} A_{v_k}.$$

Let us prove that f is differentiable in x whereby we will conclude the proof of Theorem, because

$$\left| \bigcup_{k=1}^{\infty} A_{v_k} \right| \leq \sum_{k=1}^{\infty} |A_{v_k}| = 0.$$

Let $y \in \mathbb{R}^n \setminus \{x\}$ and set

$$w = \frac{y - x}{|y - x|}, \quad t = |y - x|,$$

we have trivially

$$y = x + tw.$$

Moreover, for any $k \in \mathbb{N}$ we have

$$\begin{aligned} & |f(y) - f(x) - \nabla f(x) \cdot (y - x)| = \\ & = |f(x + tw) - f(x) - t\nabla f(x) \cdot w| \leq \\ & \leq |f(x + tv_k) - f(x) - t\nabla f(x) \cdot w| + \\ & + |f(x + tw) - f(x + tv_k)| \leq \\ & \leq |f(x + tv_k) - f(x) - t\nabla f(x) \cdot v_k| + \\ & + t|\nabla f(x)| |v_k - w| + \\ & + |f(x + tw) - f(x + tv_k)| \leq \\ & \leq t \left| \frac{f(x + tv_k) - f(x) - t\nabla f(x) \cdot v_k}{t} \right| + \\ & + 2L|w - v_k|t. \end{aligned} \tag{2.6.6}$$

Let $\varepsilon > 0$, since D is dense in ∂B_1 , there exists k_ε such that

$$|w - v_{k_\varepsilon}| < \frac{\varepsilon}{2(2L + 1)}.$$

Since

$$x \in \mathbb{R}^n \setminus \bigcup_{k=1}^{\infty} A_{v_k},$$

we have

$$\lim_{\tau \rightarrow 0} \frac{f(x + \tau v_{k_\varepsilon}) - f(x)}{\tau} = \nabla f(x) \cdot v_{k_\varepsilon}.$$

Therefore, there exists $\delta > 0$ such that, if $0 < |\tau| < \delta$ then

$$\left| \frac{f(x + \tau v_{k_\varepsilon}) - f(x) - \tau \nabla f(x) \cdot v_{k_\varepsilon}}{\tau} \right| < \frac{\varepsilon}{2}.$$

Now taking into account (2.6.6) (and $t = |x - y|$), we have

$$|f(y) - f(x) - \nabla f(x) \cdot (y - x)| < \varepsilon |x - y|, \quad \forall y \in B_\delta(x) \setminus \{x\},$$

which gives the differentiability of f in x . ■

2.7 Description of the boundary of an open set of \mathbb{R}^n

Let $r > 0$, $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$, we denote by $B_r(x)$ and $B'_r(x')$, the open ball of \mathbb{R}^n centered in x with radius r and open ball of \mathbb{R}^{n-1} centered in x' with radius $r > 0$ respectively. We will also write B_r (B'_r) instead of $B_r(0)$ ($B'_r(0)$). For any $r, M > 0$ and any $x \in \mathbb{R}^n$, here and in the sequel, we denote by

$$Q_{r,M}(x) = B'_r(x') \times (-Mr + x_n, Mr + x_n).$$

We will write also $Q_{r,M}$ instead of $Q_{r,M}(0)$.

Definition 2.7.1. Let Ω be an open set of \mathbb{R}^n . Let r_0, M_0 be positive numbers and $m \in \mathbb{N}_0$.

(a) We say that Ω has the **boundary of C^m class with constants r_0, M_0** (or, briefly, Ω is of class C^m with constants r_0, M_0), if for every $P \in \partial\Omega$ there exists an isometry

$$\Phi_P : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

such that

$$\Phi_P(0) = P$$

and

$$\Phi_P^{-1}(\Omega) \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > g_P(x')\}, \quad (2.7.1)$$

where $g_P \in C^m(\overline{B'_{r_0}})$,

$$g_P(0) = 0, \quad |\nabla g_P(0)| = 0, \quad \text{for } m \geq 1$$

and

$$\|g_P\|_{C^m(\overline{B'_{r_0}})} \leq M_0 r_0,$$

where

$$\|g_P\|_{C^m(\overline{B'_{r_0}})} = \sum_{|\gamma| \leq m} r_0^{|\gamma|} \|\partial^\gamma g_P\|_{L^\infty(B'_{r_0})}.$$

(b) Let $\alpha \in (0, 1]$. We say that Ω has the **boundary of $C^{m,\alpha}$ class with constants r_0, M_0** (or, briefly, Ω is of class $C^{m,\alpha}$ with constants r_0, M_0) if for every $P \in \partial\Omega$ there exists an isometry

$$\Phi_P : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

such that

$$\Phi_P(0) = P,$$

and

$$\Phi_P^{-1}(\Omega) \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > g_P(x')\}, \quad (2.7.2)$$

where $g_P \in C^m(\overline{B'_{r_0}})$,

$$g_P(0) = 0, \quad |\nabla g_P(0)| = 0 \quad (\text{for } m \geq 1)$$

and

$$\|g_P\|_{C^{m,\alpha}(\overline{B'_{r_0}})} \leq M_0 r_0$$

where

$$\|g_P\|_{C^{m,\alpha}(\overline{B'_{r_0}})} = \|g_P\|_{C^m(\overline{B'_{r_0}})} + r_0^m \sum_{|\gamma|=m} r_0^{|\alpha|} [\partial^\gamma u]_{C^{0,\alpha}(\overline{B'_{r_0}})}.$$

(c) If there exists $r_0 > 0$ e $M_0 > 0$ such that $\partial\Omega$ has the boundary of class C^m ($C^{m,\alpha}$) with constants $r_0 > 0$ e $M_0 > 0$, then we say that $\partial\Omega$ has the boundary of class C^m ($C^{m,\alpha}$).

Exercise. Prove that if Ω is an open set of class C^0 then $\overset{\circ}{\Omega} = \Omega$. ♣

Let us note that the graph of the function g_P that occurs in the definition above is contained in Q_{r_0, M_0} (see Figure 2.1).

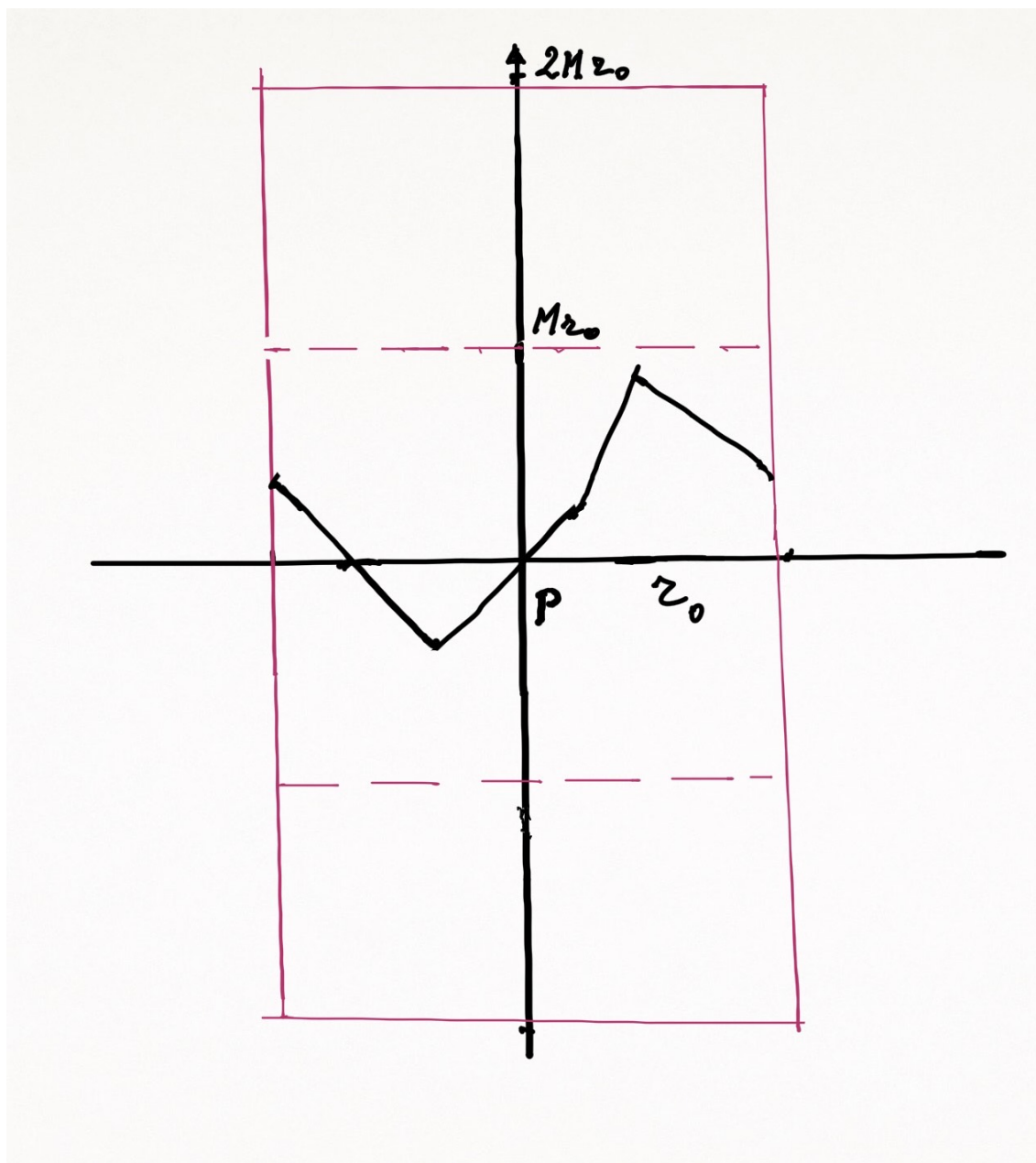


Figure 2.1:

2.8 The spaces $L^p(\partial\Omega)$

We now provide a brief review of the definition of $L^p(\partial\Omega)$ spaces, where Ω is a bounded open set of \mathbb{R}^n of class $C^{0,1}$. For any $P \in \partial\Omega$ let us denote by Φ_P an isometry which satisfies (2.7.2). Let

$$f : \partial\Omega \rightarrow \mathbb{R}.$$

We have that $\mathcal{U} = \{\Phi_{P_j}(Q_{r_0, 2M_0})\}_{1 \leq j \leq l}$ is a finite open covering of $\partial\Omega$. Let us denote

$$g_j = g_{P_j}, \quad j = 1, \dots, l,$$

and let ζ_1, \dots, ζ_l be a partition of unity subordinate to the covering \mathcal{U} . We say that $f \in L^1(\partial\Omega)$ provided that we have, for any $1 \leq j \leq l$,

$$f_j \in L^1(B'_{r_0}),$$

where

$$f_j(x') = (f \circ \Phi_{P_j})(x', g_j(x')), \quad \forall x' \in B'_{r_0}.$$

Let us denote

$$\int_{\partial\Omega} f dS = \sum_{j=1}^l \int_{\partial\Omega} f \zeta_j dS, \quad (2.8.1)$$

where

$$\int_{\partial\Omega} f \zeta_j dS = \int_{B'_{r_0}} f_j(x') \tilde{\zeta}_j(x') \sqrt{1 + |\nabla g_j(x')|^2} dx', \quad (2.8.2)$$

$$\tilde{\zeta}_j(x') = \zeta_j(\Phi_{P_j}(x', g_j(x'))).$$

Concerning the last integral in (2.8.2), take into account that it is well defined, because $g \in C^{0,1}(\overline{B'_{r_0}})$ is differentiable almost everywhere and its gradient belongs to $L^\infty(B'_{r_0}; \mathbb{R}^n)$. Let us observe that integral in (2.8.1) does not depend on the particular partition of unity that we use. As a matter of fact, if η_1, \dots, η_m is another partition of unity, then we have

$$\begin{aligned}
\sum_{j=1}^l \int_{\partial\Omega} f \zeta_j dS &= \sum_{j=1}^l \int_{\partial\Omega} \left(\sum_{k=1}^m \eta_k f \right) \zeta_j dS = \\
&= \sum_{j=1}^l \sum_{k=1}^m \int_{\partial\Omega} \zeta_j \eta_k f dS = \\
&= \sum_{k=1}^m \int_{\partial\Omega} \left(\sum_{j=1}^l \zeta_j f \right) \eta_k dS = \\
&= \sum_{k=1}^m \int_{\partial\Omega} f \eta_k dS.
\end{aligned}$$

Let us denote

$$\|u\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} |f| dS.$$

Likewise, we define $L^p(\partial\Omega)$ for $1 \leq p < +\infty$ and we set

$$\|u\|_{L^p(\partial\Omega)} = \left(\int_{\partial\Omega} |f|^p dS \right)^{1/p}. \quad (2.8.3)$$

The space $L^p(\partial\Omega)$ is a separable Banach space, and if $p = 2$, $L^2(\partial\Omega)$ is a Hilbert space.

Let $g \in C^{0,1}(\overline{B'_{r_0}})$ such that

$$\|g\|_{C^{0,1}(\overline{B'_{r_0}})} \leq M_0 r_0.$$

Let us consider the set

$$W = \{y \in Q_{r_0, 2M_0} : y_n > g(y')\};$$

We define the field of unit outward normal to the graph of g as follows

$$\nu^{(g)}(y', g(y')) = \frac{(\nabla_{y'} g(y'), -1)}{\sqrt{1 + |\nabla_{y'} g(y')|^2}}, \quad \text{a.e. } y' \in B'_{r_0}. \quad (2.8.4)$$

Let Ω be a bounded open set of \mathbb{R}^n of $C^{0,1}$ class; let $\mathcal{U} = \{\Phi_{P_j}(Q_{r_0, 2M_0})\}_{1 \leq j \leq l}$ the open covering defined above. We define the field of unit outward normal on $\partial\Omega \cap \Phi_{P_j}(Q_{r_0, 2M_0})$, $j = 1, \dots, l$, as

$$\nu(x) = (\Phi_{P_j} - P_j) (\nu^{(g_j)}(y', g_j(y'))), \quad x = \Phi_j(y', g_j(y')). \quad (2.8.5)$$

If $f \in L^1(\partial\Omega)$, we define, by (2.8.2),

$$\int_{\partial\Omega} f\nu dS = \sum_{j=1}^l \int_{\partial\Omega} f\zeta_j\nu dS. \quad (2.8.6)$$

2.9 The divergence Theorem

The purpose of this Section is to prove the following

Theorem 2.9.1 (divergence). *Let Ω be a bounded open set of \mathbb{R}^n of class $C^{0,1}$ and let $f \in C^{0,1}(\overline{\Omega})$, then we have*

$$\int_{\Omega} \nabla f dx = \int_{\partial\Omega} f\nu dS. \quad (2.9.1)$$

In order to prove Theorem 2.9.1 we need some lemmas.

Lemma 2.9.2. *Let Ω be a bounded open set of \mathbb{R}^n and let $f \in C^{0,1}(\overline{\Omega})$. Then there exists a sequence $\{f_m\}$ in $C^\infty(\mathbb{R}^n)$ satisfying*

$$\lim_{m \rightarrow \infty} \|f_m - f\|_{C^0(\overline{\Omega})} = 0, \quad (2.9.2)$$

$$\lim_{m \rightarrow \infty} \|\nabla f_m - \nabla f\|_{L^2(\Omega)} = 0. \quad (2.9.3)$$

Proof. Set

$$L = [f]_{0,1,\overline{\Omega}}.$$

Let $\tilde{f} \in C^{0,1}(\mathbb{R}^n)$ an extension of f which satisfies (see Theorem 2.2.2)

$$[\tilde{f}]_{0,1,\mathbb{R}^n} = L.$$

For any $\varepsilon > 0$, let us denote

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \tilde{f}(y) dy,$$

where η is a mollifier. We have that $f_\varepsilon \in C^\infty(\mathbb{R}^n)$. Moreover, if $x \in \Omega$, we get

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= \left| \int_{\mathbb{R}^n} \eta_\varepsilon(y) (\tilde{f}(x-y) - \tilde{f}(x)) dy \right| \leq \\ &\leq \int_{\mathbb{R}^n} \eta_\varepsilon(y) |\tilde{f}(x-y) - \tilde{f}(x)| dy \leq \\ &\leq L \int_{\mathbb{R}^n} \eta_\varepsilon(y) |y| dy \leq \\ &\leq L\varepsilon. \end{aligned}$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - f\|_{C^0(\bar{\Omega})} = 0,$$

which gives (2.9.2).

Let us prove (2.9.3). Let $1 \leq k \leq n$. Theorem 2.6.1 gives

$$\begin{aligned} \partial_{x_k} f_\varepsilon(x) &= \int_{\mathbb{R}^n} \partial_{x_k} (\eta_\varepsilon(x-y)) \tilde{f}(y) dy = \\ &= - \int_{\mathbb{R}^n} \partial_{y_k} (\eta_\varepsilon(x-y)) \tilde{f}(y) dy = \\ &= \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \partial_{y_k} \tilde{f}(y) dy. \end{aligned}$$

Now, for any $x \in \Omega$ we get

$$\begin{aligned} |\partial_{x_k} f_\varepsilon(x) - \partial_{x_k} f(x)| &\leq \int_{\mathbb{R}^n} \eta_\varepsilon(y) \left| (\partial_{x_k} \tilde{f})(x-y) - \partial_{x_k} \tilde{f}(x) \right| dy \leq \\ &\leq \left(\int_{\mathbb{R}^n} \eta_\varepsilon(y) \left| (\partial_{x_k} \tilde{f})(x-y) - \partial_{x_k} \tilde{f}(x) \right|^2 dy \right)^{1/2}. \end{aligned}$$

From which we derive

$$\begin{aligned} \int_{\Omega} |\partial_{x_k} f_\varepsilon(x) - \partial_{x_k} f(x)|^2 dx &\leq \int_{\Omega} dx \int_{B_\varepsilon} \eta_\varepsilon(y) \left| (\partial_{x_k} \tilde{f})(x-y) - \partial_{x_k} \tilde{f}(x) \right|^2 dy = \\ &= \int_{B_\varepsilon} \eta_\varepsilon(y) \int_{\Omega} \left| (\partial_{x_k} \tilde{f})(x-y) - \partial_{x_k} \tilde{f}(x) \right|^2 dx \leq \\ &\leq \sup_{|y| < \varepsilon} \int_{\Omega} \left| (\partial_{x_k} \tilde{f})(x-y) - \partial_{x_k} \tilde{f}(x) \right|^2 dx. \end{aligned}$$

By the just proved inequality and by Theorem 2.3.27 we have

$$\lim_{\varepsilon \rightarrow 0} \|\nabla f_\varepsilon - \nabla f\|_{L^2(\Omega)} = 0,$$

that gives (2.9.2). ■

For any $f \in C^{0,1}(\overline{\Omega})$ we say that a sequence $\{f_m\}$ which satisfies (2.9.2) and (2.9.3) is a **smooth approximating sequence** of f .

Lemma 2.9.3. *Let r, h be positive numbers. Let $g \in C^\infty(\overline{B'_r})$ satisfy*

$$-h < g(x') < h, \quad \forall x' \in B'_r.$$

Let $f \in C_0^\infty(Q_{r,h})$, where $Q_{r,h} = B'_r \times (-h, h)$. Denoting

$$W = \{x \in Q_{r,h} : x_n > g(x')\},$$

we have

$$\int_W \partial_n f dx = - \int_{B'_r} f(x', g(x')) dx', \quad (2.9.4a)$$

$$\int_W \partial_k f dx = \int_{B'_r} f(x', g(x')) \partial_k g(x') dx', \quad k = 1, \dots, n-1. \quad (2.9.4b)$$

Proof. Let $1 \leq k \leq n$. If $k = n$, then we have

$$\int_{g(x')}^h \partial_n f(x', x_n) dx_n = -f(x', g(x')), \quad x' \in B'_r.$$

Hence

$$\begin{aligned} \int_W \partial_n f dx &= \int_{B'_r} dx' \int_{g(x')}^h \partial_n f(x', x_n) dx_n = \\ &= - \int_{B'_r} f(x', g(x')) dx'. \end{aligned} \quad (2.9.5)$$

If $1 \leq k \leq n-1$, then we have

$$\partial_k \int_{g(x')}^h f(x', x_n) dx_n = \int_{g(x')}^h \partial_k f(x', x_n) dx_n - f(x', g(x')) \partial_k g(x'),$$

from which we get

$$\begin{aligned}
\int_W \partial_k f dx &= \int_{B'_r} dx' \int_{g(x')}^h \partial_k f(x', x_n) dx_n = \\
&= \int_{B'_r} dx' \left(\partial_k \int_{g(x')}^h f(x', x_n) dx_n \right) + \\
&+ \int_{B'_r} f(x', g(x')) \partial_k g(x') dx' = \\
&= \int_{B'_r} f(x', g(x')) \partial_k g(x') dx,
\end{aligned} \tag{2.9.6}$$

where in the fourth step we used the fact that the function

$$x' \rightarrow \int_{g(x')}^h f(x', x_n) dx_n,$$

has the support contained in B'_r . From what was obtained in (2.9.5) and by (2.9.6) we derive (2.9.4). ■

Remark. Under the same assumptions of Lemma 2.9.3, taking into account (2.8.4), we have

$$\int_W \partial_s f dx = \int_{\Gamma(g)} f \nu_s^{(g)} dS, \quad s = 1, \dots, n. \tag{2.9.7}$$

◆

Proof Theorem 2.9.1. Let us begin by considering the case where $f \in C^\infty(\mathbb{R}^n)$.

Since $\partial\Omega$ is of class $C^{0,1}$, we may assume that there exist positive numbers r_0, M_0 , such that for every $P \in \partial\Omega$ there is an isometry

$$\Phi_P : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

satisfying

$$\Phi_P(0) = P$$

and

$$\Phi_P^{-1}(\Omega) \cap Q_{r_0, 2M_0} = W_P,$$

where

$$\begin{aligned}
W_P &= \{x \in Q_{r_0, 2M_0} : x_n > g_P(x')\}, \\
g_P &\in C^{0,1}(\overline{B'_{r_0}}),
\end{aligned}$$

$$g_P(0) = 0,$$

and

$$\|g_P\|_{C^{0,1}(\overline{B'_{r_0}})} \leq M_0 r_0.$$

Since $\partial\Omega$ is compact, there exist $P_1, \dots, P_l \in \partial\Omega$ such that the family of sets $\{\Phi_{P_j}(Q_{r_0, 2M_0})\}_{1 \leq j \leq l}$ is a finite covering of $\partial\Omega$. Set

$$V_j = \Phi_{P_j}(Q_{r_0, 2M_0}), \quad j = 1, \dots, l.$$

By Theorem 2.4.3 there exist $\zeta_0, \zeta_1, \dots, \zeta_l \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$(\zeta_0)|_\Omega \in C_0^\infty(\Omega),$$

$$\text{supp } \zeta_0 \subset \mathbb{R}^n \setminus \partial\Omega; \quad \text{supp } \zeta_j \subset V_j, \quad j = 1, \dots, l,$$

$$\sum_{j=0}^l \zeta_j = 1, \quad \text{su } \mathbb{R}^n; \quad 0 \leq \zeta_j \leq 1, \quad j = 0, 1, \dots, l.$$

We have

$$\int_{\Omega} \nabla(f\zeta_0) dx = 0.$$

Hence

$$\int_{\Omega} \nabla f dx = \sum_{j=0}^l \int_{\Omega} \nabla(f\zeta_j) dx = \sum_{j=1}^l \int_{\Omega} \nabla(f\zeta_j) dx. \quad (2.9.8)$$

Now, let us fix $j \in \{1, \dots, l\}$ and let us denote by $W := W_{P_j}$, $g := g_{P_j}$ and

$$F := f\zeta_j. \quad (2.9.9)$$

Let $\{g_m\}$ be a sequence of smooth approximating of g . Set

$$\Phi_j = \Phi_{P_j}, \quad j = 1, \dots, l$$

and denoting

$$U = \Phi_j(W),$$

$$W_m = \{x \in Q_{r_0, 2M_0} : x_n > g_m(x')\},$$

$$U_m = \Phi_j(W_m),$$

we have

$$\int_{\Omega} \nabla (f\zeta_j) dx = \int_U \nabla F dx = \lim_{m \rightarrow \infty} \int_{U_m} \nabla F dx \quad (2.9.10)$$

(as a matter of fact, we have $|U_m \setminus U| \rightarrow 0$ as $m \rightarrow \infty$).

Now let us deal with the last integral in (2.9.10). Since Φ_j is an isometry, there exists a matrix $A = \{a_{qs}\}_{1 \leq q, s \leq n}$ such that

$$\Phi(y) = Ay + P_j, \quad A^T A = I_n,$$

where I_n is the identity matrix $n \times n$. For the sake of brevity let us set $P := P_j$, $\Phi := \Phi_j$. We have

$$\int_{U_m} \nabla F dx = \int_{W_m} (\nabla_x F)(\Phi(y)) dy. \quad (2.9.11)$$

By setting

$$\bar{F} = F \circ \Phi, \quad (2.9.12)$$

we have

$$(\partial_y \Phi(y))^T (\nabla_x F)(\Phi(y)) = \nabla_y \bar{F}(y)$$

and, recalling that $A = (\partial_y \Phi(y))$, $A^T A = I_n$, we get

$$(\nabla_x F)(\Phi(y)) = A \nabla_y \bar{F}(y), \quad \forall y \in Q_{r_0, 2M_0}. \quad (2.9.13)$$

Now, since $\bar{F} \in C_0^\infty(Q_{r_0, 2M_0})$ and, for m large enough, the graph of g_m is contained in $Q_{r_0, 2M_0}$ (recall that $\{g_m\}$ is a smooth approximating sequence of g), we obtain, by Lemma 2.9.3 and by (2.9.13)

$$\begin{aligned} \left(\int_{W_m} (\nabla_x F)(\Phi(y)) dy \right)_q &= \left(\int_{W_m} A \nabla_y \bar{F}(y) dy \right)_q = \\ &= \sum_{s=1}^n \int_{W_m} a_{qs} \partial_s \bar{F}(y) dy = \\ &= - \int_{B'_r} a_{qn} \bar{F}(y', g_m(y')) dy' + \\ &\quad + \sum_{s=1}^{n-1} \int_{B'_r} a_{qs} \bar{F}(y', g_m(y')) \partial_s g_m(y') dy'. \end{aligned} \quad (2.9.14)$$

Since $\{g_m\}$ is a smooth approximating sequence of g , we have

$$\bar{F}(y', g_m(y')) \rightarrow \bar{F}(y', g(y')), \quad \text{as } m \rightarrow \infty \text{ in } L^\infty(B'_r)$$

and, for any $s = 1, \dots, n-1$,

$$\overline{F}(y', g_m(y')) \partial_s g_m(y') \rightarrow \overline{F}(y', g(y')) \partial_s g(y'), \text{ as } m \rightarrow \infty, \text{ in } L^2(B'_r).$$

Hence

$$\lim_{m \rightarrow \infty} \int_{B'_r} \overline{F}(y', g_m(y')) dy' = \int_{B'_r} \overline{F}(y', g(y')) dy' \quad (2.9.15)$$

and, for any $s = 1, \dots, n-1$,

$$\lim_{m \rightarrow \infty} \int_{B'_r} \overline{F}(y', g_m(y')) \partial_s g_m(y') dy' = \int_{B'_r} \overline{F}(y', g(y')) \partial_s g(y') dy'. \quad (2.9.16)$$

By (2.9.10), (2.9.14), (2.9.15) and (2.9.16), denoting by $\nu^{(g)}$ the unit outward normal to $\Gamma(g)$ (recall (2.9.9) and (2.9.12)) we have

$$\begin{aligned} \int_{\Omega} \nabla(f\zeta_j) dx &= \lim_{m \rightarrow \infty} \int_{U_m} \nabla F dx = \\ &= \lim_{m \rightarrow \infty} \int_{W_m} \nabla_x F(\Phi(y)) dy = \\ &= \int_{\Gamma(g)} \overline{F} A \nu^{(g)} dS = \\ &= \int_{\partial\Omega} f \zeta_j \nu dS. \end{aligned}$$

By what has just been obtained and by (2.9.8) we derive that if $f \in C^\infty(\mathbb{R}^n)$, then

$$\int_{\Omega} \nabla f dx = \int_{\partial\Omega} f \nu dS.$$

Finally, let us consider the case where $f \in C^{0,1}(\overline{\Omega})$. If $\{f_m\}$ is a smooth approximating sequence of f given by Lemma 2.9.2, then we have

$$\begin{aligned} \int_{\Omega} \nabla f dx &= \lim_{m \rightarrow \infty} \int_{\Omega} \nabla f_m dx = \\ &= \lim_{m \rightarrow \infty} \int_{\partial\Omega} f_m \nu dS = \\ &= \int_{\partial\Omega} f \nu dS. \end{aligned}$$

■

2.10 The Hausdorff distance

Let (X, d) be a metric space. Let us recall that the distance of a point $x \in X$ from a subset A of X , $A \neq \emptyset$, is given by

$$d(x, A) = \inf \{d(x, y) : y \in A\}. \quad (2.10.1)$$

Proposition 2.10.1. *If $A \subset X$, $A \neq \emptyset$, then we have*

$$|d(x, A) - d(y, A)| \leq d(x, y), \quad \forall x, y \in A. \quad (2.10.2)$$

In particular, the map

$$X \ni x \rightarrow d(x, A) \in \mathbb{R},$$

is Lipschitz continuous.

Proof. Let $x, y \in X$. By triangle inequality we have

$$d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z), \quad \forall z \in A,$$

from which we derive

$$d(x, A) \leq d(x, y) + d(y, A).$$

Hence

$$d(x, A) - d(y, A) \leq d(x, y)$$

and interchanging x and y , we obtain (2.10.2). ■

We denote by $\mathbf{K}(X)$ the family of nonempty compact sets of X . If $K \in \mathbf{K}(X)$ and $x \in X$ we have

$$d(x, K) = \min \{d(x, y) : y \in K\}.$$

Let $K \in \mathbf{K}(X)$. Let us denote by $S(K)$ the set of the points $x \in X$ such that

$$\{y \in K : d(x, y) = d(x, K)\}, \text{ has only one point.}$$

Trivially, we have $K \subset S(K)$ and it is well-defined map

$$p_K : S(K) \rightarrow K, \quad \text{such that } d(x, p_K(x)) = d(x, K). \quad (2.10.3)$$

If $x \in S(K)$, we call $p_K(x)$ the **point of minimum distance** of x from K or also **the projection** of x on K .

Proposition 2.10.2. *Let $K \in \mathbf{K}(X)$. Then p_K , defined by (2.10.3), is a continuous map.*

Proof. Let us argue by contradiction. Let us assume that p_K is not continuous. Consequently, there exists a point $x_0 \in S(K)$ and a sequence $\{x_n\}$ of $S(K)$ satisfying

$$\{x_n\} \rightarrow x_0 \quad (2.10.4)$$

and

$$\{p_K(x_n)\} \not\rightarrow p_K(x_0). \quad (2.10.5)$$

The latter implies that there exists $\varepsilon > 0$ and a subsequence $\{x_n^*\}$ of $\{x_n\}$ satisfying

$$d(p_K(x_n^*), p_K(x_0)) \geq \varepsilon, \quad \forall n \in \mathbb{N}. \quad (2.10.6)$$

Since for every $n \in \mathbb{N}$ we have $p_K(x_n^*) \in K$, and K is compact, there exists a subsequence $\{x_n^{**}\}$ of $\{x_n^*\}$ such that $\{p_K(x_n^{**})\}$ converges to a point $z \in K$. On the other hand, by (2.10.4) we get

$$\{x_n^{**}\} \rightarrow x_0.$$

Hence

$$d(x_0, K) = \lim_{n \rightarrow \infty} d(x_n^{**}, K) = \lim_{n \rightarrow \infty} d(x_n^{**}, p_K(x_n^{**})) = d(x_0, z).$$

Consequently

$$d(x_0, K) = d(x_0, z)$$

and by the definition of $S(K)$ we have

$$z = p_K(x_0).$$

On the other hand, (2.10.6) gives

$$d(x_0, z) = \lim_{n \rightarrow \infty} d(p_K(x_0), p_K(x_n^{**})) \geq \varepsilon.$$

We have actually reached a contradiction. Therefore the map $x \rightarrow p_K(x)$ is continuous. ■

Definition 2.10.3. For any $K_1, K_2 \in \mathbf{K}(X)$ we denote

$$\delta(K_1, K_2) = \max \{d(x, K_2) : x \in K_1\}. \quad (2.10.7)$$

Proposition 2.10.4. *If $K_1, K_2 \in \mathbf{K}(X)$, then we have*

$$K_1 \subset K_2 \Leftrightarrow \delta(K_1, K_2) = 0. \quad (2.10.8)$$

Proof. If $K_1 \subset K_2$, then

$$d(x, K_2) = 0, \quad \forall x \in K_1,$$

hence

$$\delta(K_1, K_2) = 0.$$

Conversely, if $\delta(K_1, K_2) = 0$ then

$$d(x, K_2) = 0, \quad \forall x \in K_1.$$

Since K_1 is a closed set of X , we obtain

$$x \in K_2, \quad \forall x \in K_1$$

that is $K_1 \subset K_2$. ■

Let us notice that $\delta(\cdot, \cdot)$ **does not** define a distance on $\mathbf{K}(X)$. Actually, by (2.10.8) we have that $\delta(\cdot, \cdot)$ is not symmetric and

$$\delta(K_1, K_2) = 0 \not\Rightarrow K_1 = K_2.$$

For any $K_1, K_2 \in \mathbf{K}(X)$, let us denote by

$$d_{\mathcal{H}}(K_1, K_2) = \max\{\delta(K_1, K_2), \delta(K_2, K_1)\}. \quad (2.10.9)$$

Proposition 2.10.5. *$d_{\mathcal{H}}(\cdot, \cdot)$, defined by (2.10.9), is a distance on $\mathbf{K}(X)$.*

Proof. It is obvious that $d_{\mathcal{H}}(K_1, K_2) = d_{\mathcal{H}}(K_2, K_1)$ and that $d_{\mathcal{H}}(K_1, K_2) \geq 0$ for every $K_1, K_2 \in \mathbf{K}(X)$. Furthermore, if

$$d_{\mathcal{H}}(K_1, K_2) = 0,$$

then $\delta(K_1, K_2) = 0$ and $\delta(K_2, K_1) = 0$ which imply, respectively, $K_1 \subset K_2$ e $K_2 \subset K_1$, hence $K_1 = K_2$.

It only remains to prove the triangular inequality. We begin by proving that if $K_1, K_2 \in \mathbf{K}(X)$ then for any $L \in \mathbf{K}(X)$ we have

$$\delta(K_1, K_2) \leq \delta(K_1, L) + \delta(L, K_2). \quad (2.10.10)$$

Let $x \in K_1$. For any $y \in K_2$ and for any $z \in L$ we have

$$d(x, K_2) \leq d(x, y) \leq d(x, z) + d(z, y),$$

from which we have

$$\begin{aligned} d(x, K_2) &\leq d(x, z) + d(z, K_2) \leq \\ &\leq d(x, z) + \delta(L, K_2). \end{aligned}$$

Therefore

$$\begin{aligned} d(x, K_2) &\leq d(x, L) + \delta(L, K_2) \leq \\ &\leq \delta(K_1, L) + \delta(L, K_2). \end{aligned}$$

By the latter we obtain (2.10.10) and similarly

$$\delta(K_2, K_1) \leq \delta(K_2, L) + \delta(L, K_1). \quad (2.10.11)$$

Now, let us assume, for instance, that $d_{\mathcal{H}}(K_1, K_2) = \delta(K_1, K_2)$. We have, by (2.10.10) and (2.10.11),

$$\begin{aligned} d_{\mathcal{H}}(K_1, K_2) &= \delta(K_1, K_2) \leq \\ &\leq \delta(K_1, L) + \delta(L, K_2) \leq \\ &\leq d_{\mathcal{H}}(K_1, L) + d_{\mathcal{H}}(L, K_2). \end{aligned}$$

■

Definition 2.10.6. Let $K_1, K_2 \in \mathbf{K}(X)$, we call $d_{\mathcal{H}}(K_1, K_2)$, defined by (2.10.9), the **Hausdorff distance** between K_1 and K_2 .

In Proposition 2.10.8 below we give an useful characterization of the Hausdorff distance. For this purpose we introduce the following notation. Let $r \geq 0$ and $K \in \mathbf{K}(X)$, set

$$[K]_r = \{u \in K : d(x, K) \leq r\},$$

Let us call $[K]_r$ the r -**dilation** of r of the set K .

Proposition 2.10.7. *If $K \in \mathbf{K}(X)$ and $r \geq 0$ then $[K]_r$ is a closed set of X .*

Proof. Let $\{x_n\}$ be any sequence in $[K]_r$ which satisfies

$$\lim_{n \rightarrow \infty} x_n = x_0.$$

Since

$$d(x_n, K) \leq r, \quad \forall n \in \mathbb{N},$$

passing to the limit as $n \rightarrow \infty$ and taking into account that $x \rightarrow d(x, K)$ is continuous, we have

$$d(x_0, K) = \lim_{n \rightarrow \infty} d(x_n, K) \leq r.$$

Hence $x_0 \in [K]_r$. Therefore $[K]_r$ is closed set of X . ■

Remark. It generally does not occur that $[K]_r$ is compact. For instance, let X be a infinite dimensional Hilbert space, and let $K = \{a\}$, where $a \in X$, then we have $[K]_r = \overline{B_r(a)}$ and from Functional Analysis we know that $\overline{B_r(a)}$ is not compact. ◆

Proposition 2.10.8. *Let $K_1, K_2 \in \mathbf{K}(X)$ and $r \geq 0$. Then we have*

$$d_{\mathcal{H}}(K_1, K_2) \leq r \iff \begin{cases} K_1 \subset [K_2]_r, \\ K_2 \subset [K_1]_r. \end{cases} \quad (2.10.12)$$

$$d_{\mathcal{H}}(K_1, K_2) = \min \{r \geq 0 : K_1 \subset [K_2]_r, K_2 \subset [K_1]_r\}. \quad (2.10.13)$$

Proof. The condition

$$d_{\mathcal{H}}(K_1, K_2) \leq r$$

is equivalent to $\delta(K_1, K_2) \leq r$ and $\delta(K_2, K_1) \leq r$. On the other hand

$$\delta(K_1, K_2) \leq r \iff (d(x, K_2) \leq r, \forall x \in K_1) \iff K_1 \subset [K_2]_r.$$

Similarly,

$$\delta(K_2, K_1) \leq r \iff K_2 \subset [K_1]_r.$$

From what was obtained (2.10.12) follows.

Now, we prove (2.10.13). Let us denote

$$d = d_{\mathcal{H}}(K_1, K_2)$$

and

$$\rho = \inf \{r \geq 0 : K_1 \subset [K_2]_r, K_2 \subset [K_1]_r\}.$$

(2.10.12) implies (" \implies ")

$$\rho \leq d. \quad (2.10.14)$$

On the other hand, for any $\varepsilon > 0$, we have

$$K_1 \subset [K_2]_{\rho+\varepsilon}, \quad \text{and} \quad K_2 \subset [K_1]_{\rho+\varepsilon}$$

and (2.10.12) implies (" \impliedby ")

$$d = d_{\mathcal{H}}(K_1, K_2) \leq \rho + \varepsilon.$$

Therefore, since ε is arbitrary, we obtain

$$d \leq \rho.$$

By the latter and by (2.10.14) we get

$$d = \rho,$$

from which (2.10.13) follows. ■

Example. Let us consider

$$K_1 = \overline{B_1} \setminus B_\varepsilon, \quad K_2 = \overline{B_1},$$

in \mathbb{R}^n , where $\varepsilon \in (0, 1)$.

We have

$$K_1 \subset K_2 \subset [K_2]_\varepsilon$$

and

$$K_2 \subset [K_1]_r, \quad \forall r \geq \varepsilon.$$

Hence

$$d_{\mathcal{H}}(K_1, K_2) = \varepsilon.$$

Let us see what happens with regard to

$$d_{\mathcal{H}}(\partial K_1, \partial K_2).$$

We have

$$\partial K_2 \subset \partial K_1$$

and

$$\partial K_1 \subset [\partial K_2]_r, \quad \forall r \geq 1 - \varepsilon.$$

Hence

$$d_{\mathcal{H}}(\partial K_1, \partial K_2) = 1 - \varepsilon.$$

Therefore, neither of the two relationships holds true

$$d_{\mathcal{H}}(K_1, K_2) \leq d_{\mathcal{H}}(\partial K_1, \partial K_2), \quad (2.10.15)$$

$$d_{\mathcal{H}}(\partial K_1, \partial K_2) \leq d_{\mathcal{H}}(K_1, K_2). \quad (2.10.16)$$

As a matter of fact (2.10.15) is false for $\frac{1}{2} \leq \varepsilon < 1$, and (2.10.16) is false for $0 < \varepsilon < \frac{1}{2}$. ♠

Proposition 2.10.9.

$$[K_1]_r \cup [K_2]_r = [K_1 \cup K_2]_r, \quad \forall K_1, K_2 \in \mathbf{K}(X), \forall r \geq 0.$$

Proof. Let $K_1, K_2 \in \mathbf{K}(X)$. Let us begin to prove

$$[K_1]_r \cup [K_2]_r \subset [K_1 \cup K_2]_r. \quad (2.10.17)$$

Let $x \in [K_1]_r \cup [K_2]_r$ and, for instance, let $x \in [K_1]_r$, then

$$d(x, K_1 \cup K_2) \leq d(x, K_1) \leq r,$$

which implies $x \in [K_1 \cup K_2]_r$. (2.10.17) is proved.

Now, let us prove

$$[K_1 \cup K_2]_r \subset [K_1]_r \cup [K_2]_r. \quad (2.10.18)$$

Let $x \in [K_1 \cup K_2]_r$. We have

$$d(x, K_1 \cup K_2) \leq r.$$

Let $y \in K_1 \cup K_2$ satisfy $d(x, y) = d(x, K_1 \cup K_2)$. Now, if $y \in K_1$, we have

$$d(x, K_1) \leq d(x, y) \leq r,$$

hence $x \in [K_1]_r$. Similarly, if $y \in K_2$ then $x \in [K_2]_r$. In any case $x \in [K_1]_r \cup [K_2]_r$. Hence (2.10.18) is proved. ■

The following Theorem has been proved in **Kuratowski**, [44, §15, VIII].

Theorem 2.10.10. *Let $x_0 \in X$. Let us define, for any $K \in \mathbf{K}(X)$, the function*

$$f_K(x) = d(x, K) - d(x, x_0). \quad (2.10.19)$$

We have:

(i) f_K is bounded,

(ii)

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{x \in X} |f_{K_2}(x) - f_{K_1}(x)|, \quad \forall K_1, K_2 \in \mathbf{K}(X).$$

Proof.

Let us prove (i). Let $x \in X$. We have, by the triangle inequality,

$$d(x, K) \leq d(x, y) \leq d(y, x_0) + d(x_0, x), \quad \forall y \in K.$$

Hence

$$d(x, K) \leq d(x_0, K) + d(x, x_0). \quad (2.10.20)$$

By using again the triangle inequality we have, for any $y, z \in K$

$$d(x, x_0) \leq d(x, y) + d(y, z) + d(z, x_0) \leq d(x, y) + d(z, x_0) + d(K),$$

where $d(K)$ is the diameter of K , by the last inequality we have

$$d(x, x_0) \leq d(x, K) + d(x_0, K) + d(K). \quad (2.10.21)$$

Now, (2.10.20) gives

$$f_K(x) = d(x, K) - d(x, x_0) \leq d(x_0, K)$$

and (2.10.21) gives

$$f_K(x) \geq d(x, K) - (d(x, K) + d(x_0, K) + d(K)) = -d(x_0, K) - d(K).$$

Therefore we have

$$|f_K(x)| \leq d(x_0, K) + d(K), \quad \forall x \in X.$$

Let us now prove (ii). Let $K_1, K_2 \in \mathbf{K}(X)$. It is not restrictive to assume

$$d_{\mathcal{H}}(K_1, K_2) = \delta(K_1, K_2) = \max_{x \in K_1} d(x, K_2).$$

Let $\bar{x} \in K_1$ satisfy

$$d(\bar{x}, K_2) = d_{\mathcal{H}}(K_1, K_2).$$

Since we have trivially $d(\bar{x}, K_1) = 0$, we get

$$\begin{aligned} d_{\mathcal{H}}(K_1, K_2) &= d(\bar{x}, K_2) - d(\bar{x}, K_1) = \\ &= d(\bar{x}, K_2) - d(\bar{x}, x_0) - (d(\bar{x}, K_1) - d(\bar{x}, x_0)) = \quad (2.10.22) \\ &= f_{K_2}(\bar{x}) - f_{K_1}(\bar{x}) \leq \sup_{x \in X} |f_{K_1}(x) - f_{K_2}(x)|. \end{aligned}$$

Now, for any $x \in X$ let $y \in K_1$ satisfy

$$d(x, y) = d(x, K_1).$$

We have

$$d(x, K_2) \leq d(x, y) + d(y, K_2) = d(x, K_1) + d(y, K_2).$$

Hence

$$d(x, K_2) - d(x, K_1) \leq d(y, K_2) \leq d_{\mathcal{H}}(K_1, K_2)$$

and, by interchanging K_1 with K_2 , we have.

$$d(x, K_1) - d(x, K_2) \leq d_{\mathcal{H}}(K_1, K_2).$$

Hence

$$|f_{K_1}(x) - f_{K_2}(x)| = |d(x, K_1) - d(x, K_2)| \leq d_{\mathcal{H}}(K_1, K_2), \quad \forall x \in X,$$

which implies

$$\sup_{x \in X} |f_{K_1}(x) - f_{K_2}(x)| \leq d_{\mathcal{H}}(K_1, K_2). \quad (2.10.23)$$

Finally, (2.10.22) and (2.10.23) imply (ii). ■

2.10.1 Completeness and compactness of $(\mathbf{K}(X), d_{\mathcal{H}})$

The Main Theorem that we prove in the present Section is the following one.

Theorem 2.10.11 (completeness). *If (X, d) is a complete metric space, then $(\mathbf{K}(X), d_{\mathcal{H}})$ is complete.*

In order to prove Theorem 2.10.11 we need the following Lemma.

Lemma 2.10.12. *Let (X, d) be a metric space and let $\{K_n\}$ be a Cauchy sequence in $(\mathbf{K}(X), d_{\mathcal{H}})$. Let $\{n_j\}$ be a strictly increasing sequence in \mathbb{N} and let $\{x_{n_j}\}$ be a Cauchy sequence in (X, d) satisfying*

$$x_{n_j} \in K_{n_j}, \quad \forall j \in \mathbb{N}.$$

Then there exists a Cauchy sequence $\{\bar{x}_n\}$ in (X, d) such that

$$\bar{x}_{n_j} = x_{n_j}, \quad \forall j \in \mathbb{N}, \quad \text{and} \quad \bar{x}_n \in K_n, \quad \forall n \in \mathbb{N}. \quad (2.10.24)$$

Proof. Let us define $\{\bar{x}_n\}$ as follows: if $1 \leq n \leq n_1 - 1$, then we choose \bar{x}_n satisfying

$$d(x_{n_1}, \bar{x}_n) = d(x_{n_1}, K_n),$$

if $n_j + 1 \leq n \leq n_{j+1} - 1$, $j \in \mathbb{N}$, then we choose \bar{x}_n satisfying

$$d(x_{n_j}, \bar{x}_n) = d(x_{n_j}, K_n),$$

finally, if $n = n_j$, $j \in \mathbb{N}$, then we choose

$$\bar{x}_n = x_{n_j}.$$

Notice that (2.10.24) is satisfied by construction, so we are left to prove that $\{\bar{x}_n\}$ is a Cauchy sequence.

Let us fix any $\varepsilon > 0$ and let $\nu \in \mathbb{N}$ satisfy

$$d(x_{n_j}, x_{n_h}) < \frac{\varepsilon}{3}, \quad \forall j, h \geq \nu \quad (2.10.25)$$

and

$$d_{\mathcal{H}}(K_n, K_m) < \frac{\varepsilon}{3}, \quad \forall n, m \geq n_\nu. \quad (2.10.26)$$

Let $n, m \geq n_\nu$ and let j and h be such that

$$n_j \leq n \leq n_{j+1}, \quad n_h \leq m \leq n_{h+1}.$$

By the triangle inequality we have

$$d(\bar{x}_n, \bar{x}_m) \leq d(\bar{x}_n, x_{n_j}) + d(x_{n_j}, x_{n_h}) + d(\bar{x}_m, x_{n_h}). \quad (2.10.27)$$

Now, (2.10.26) implies

$$d(x_{n_j}, \bar{x}_n) = d(x_{n_j}, K_n) \leq d_{\mathcal{H}}(K_{n_j}, K_n) < \frac{\varepsilon}{3}.$$

Similarly, we have

$$d(x_{n_h}, \bar{x}_m) < \frac{\varepsilon}{3}.$$

By these latter inequalities and by (2.10.25), (2.10.27) we get

$$d(\bar{x}_n, \bar{x}_m) < \varepsilon$$

and thereby we have also proved that $\{\bar{x}_n\}$ is a Cauchy sequence. ■

In what follows, for any sequence $\{K_n\}$ in $\mathbf{K}(X)$ and any sequence $\{x_n\}$ in X , we will write simply $\{x_n \in K_n\}$ to denote that

$$x_n \in K_n, \quad \forall n \in \mathbb{N}.$$

Proof of Theorem 2.10.11. Let $\{K_n\}$ be a Cauchy sequence in $\mathbf{K}(X)$. Let us denote

$$K = \left\{ x \in X : \text{there exists a sequence } \{x_n \in K_n\} \text{ such that } \lim_{n \rightarrow \infty} x_n = x \right\}.$$

Let us prove the following:

(a) $K \neq \emptyset$;

(b) K is closed;

(c)

$$\forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq n_\varepsilon \quad K \subset [K_n]_\varepsilon;$$

(d) K is compact;

and

$$\{K_n\} \rightarrow K, \quad \text{in } (\mathbf{K}(X), d_{\mathcal{H}}).$$

Proof of (a). Since $\{K_n\}$ is a Cauchy sequence in $(\mathbf{K}(X), d_{\mathcal{H}})$, we have that, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that

$$d_{\mathcal{H}}(K_n, K_m) < \varepsilon, \quad \forall n, m \geq n_\varepsilon. \quad (2.10.28)$$

We may assume $n_\varepsilon \in \mathbb{N}$ be strictly increasing w.r.t. ε . For any $j \in \mathbb{N}$ let

$$\varepsilon_j = \frac{1}{2^j}$$

and set $n_j = n_{\varepsilon_j}$.

Let $x_{n_1} \in K_{n_1}$ be chosen arbitrarily. Since

$$d(x_{n_1}, K_{n_2}) \leq d_{\mathcal{H}}(K_{n_1}, K_{n_2}),$$

we may choose $x_{n_2} \in K_{n_2}$ such that

$$d(x_{n_1}, x_{n_2}) = d(x_{n_1}, K_{n_2}).$$

Similarly, after choosing $x_{n_1}, \dots, x_{n_{j-1}}$, we choose

$x_{n_j} \in K_{n_j}$. More precisely, let us suppose to have already chosen $x_{n_1}, \dots, x_{n_{j-1}}$, then we choose $x_{n_j} \in K_{n_j}$ so that

$$d(x_{n_{j-1}}, x_{n_j}) = d(x_{n_{j-1}}, K_{n_j}) \leq d_{\mathcal{H}}(K_{n_{j-1}}, K_{n_j}) < \frac{1}{2^{j-1}}.$$

Now, let us prove that $\{x_{n_j}\}$ is a Cauchy sequence. For any $h > j$ the triangle inequality gives

$$d(x_{n_j}, x_{n_n}) \leq \sum_{l=j}^{h-1} d(x_{n_l}, x_{n_{l+1}}) < \sum_{l=j}^{h-1} \frac{1}{2^l} < \frac{1}{2^{j-1}}.$$

On the other hand

$$x_{n_j} \in K_{n_j}, \quad \forall j \in \mathbb{N}.$$

Hence, Lemma 2.10.12 implies that there exists a Cauchy sequence in (X, d) , $\{\bar{x}_n \in K_n\}$, which satisfies

$$\bar{x}_{n_j} = x_{n_j}, \quad \forall j \in \mathbb{N}.$$

Since (X, d) is a complete metric space, there exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} \bar{x}_n = x.$$

Hence, x belongs to K (as we have defined K). Therefore $K \neq \emptyset$.

Proof of (b). We prove that if $\{x_n\}$ is a sequence in K , which converges to x_0 , then $x_0 \in K$. By the definition of K , we have that for every $n \in \mathbb{N}$ there exists a sequence $\{y_j^{(n)}\}_{j \in \mathbb{N}}$ which satisfies

$$y_j^{(n)} \in K_j, \quad \forall j \in \mathbb{N} \quad (2.10.29)$$

and

$$y_j^{(n)} \rightarrow x_n, \quad \text{as } j \rightarrow \infty, \quad \forall n \in \mathbb{N}.$$

Since $\{x_n\}$ converges to x_0 , there exists a strictly increasing sequence in \mathbb{N} , $\{n_h\}_{h \in \mathbb{N}}$, which satisfies

$$d(x_{n_h}, x_0) < \frac{1}{h}, \quad \forall h \in \mathbb{N}. \quad (2.10.30)$$

In addition, since

$$y_j^{(n_h)} \rightarrow x_{n_h}, \quad \text{as } j \rightarrow \infty, \quad \forall h \in \mathbb{N},$$

there exists a strictly increasing sequence in \mathbb{N} , $\{m_h\}_{h \in \mathbb{N}}$, which satisfies

$$d(y_{m_h}^{(n_h)}, x_{n_h}) < \frac{2}{h}, \quad \forall h \in \mathbb{N}. \quad (2.10.31)$$

By (2.10.30) and (2.10.31) we have

$$d(y_{m_h}^{(n_h)}, x_0) < \frac{1}{h} \quad \forall h \in \mathbb{N}. \quad (2.10.32)$$

Now, let us consider the sequence $\{y_{m_h}^{(n_h)}\}_{h \in \mathbb{N}}$. Since it is convergent, it is a Cauchy sequence and, by (2.10.29), we have

$$y_{m_h}^{(n_h)} \in K_{m_h}, \quad \forall h \in \mathbb{N}.$$

Therefore by Lemma 2.10.12, there exists a Cauchy sequence, $\{\bar{y}_n\}$ which satisfies

$$\bar{y}_{m_h} = y_{m_h}^{(n_h)}, \quad \forall h \in \mathbb{N}$$

and

$$\bar{y}_n \in K_n, \quad \forall n \in \mathbb{N}.$$

Since X is a complete space and $\{\bar{y}_n\}$ is a Cauchy sequence, it converges. On the other hand, (2.10.32) implies that the subsequence $\{y_{m_h}^{(n_h)}\}_{h \in \mathbb{N}}$ converges to x_0 . Therefore the whole sequence $\{\bar{y}_n\}$ converges to x_0 and by the definition of K we have $x_0 \in K$.

Proof of (c). Let $\varepsilon > 0$ and let n_ε satisfy

$$d_{\mathcal{H}}(K_n, K_m) < \varepsilon, \quad \forall n, m \geq n_\varepsilon.$$

Proposition 2.10.8 gives

$$K_m \subset [K_n]_\varepsilon, \quad \forall n, m \geq n_\varepsilon. \quad (2.10.33)$$

Let $x \in K$. Let us prove that $x \in [K_n]_\varepsilon$ for every $n \geq n_\varepsilon$. Fix $n \geq n_\varepsilon$. By the definition of K , there exists $\{x_m \in K_m\}$ such that

$$x_m \rightarrow x, \quad \text{as } m \rightarrow \infty.$$

Now, since $x_m \in K_m \subset [K_n]_\varepsilon$ (by (2.10.33)), for every $m \geq n_\varepsilon$ and taking into account that $[K_n]_\varepsilon$ is a closed set (Proposition 2.10.7), we have $x \in [K_n]_\varepsilon$.

Proof of (d). Since X is a complete metric space and K is a closed set (by (b)), by Theorem 2.1.4, it suffices to prove that K is totally bounded. Let us argue by contradiction. Let us assume that K is not totally bounded. Hence, let us assume that there exists $\delta > 0$ and there exists a sequence $\{x_n\}$ in K so that

$$d(x_n, x_m) \geq \delta.$$

Now, by (c), there exists $\nu \in \mathbb{N}$ such that, if $n \neq m$,

$$K \subset [K_\nu]_{\frac{\delta}{4}}.$$

From which we have that for every $n \in \mathbb{N}$ there exists $y_n \in K_\nu$ such that

$$d(y_n, x_n) < \frac{\delta}{4}.$$

On the other hand, since K_ν is compact, there exists a subsequence of $\{y_n\}$, $\{y_{n_j}\}$, which converges, consequently there exists $\nu' \geq \nu$ such that

$$d(y_{n_j}, y_{n_h}) < \frac{\delta}{4}, \quad \forall j, h \geq \nu'.$$

Therefore, by the triangle inequality we have, if $j \neq h$,

$$\delta \leq d(x_{n_j}, x_{n_h}) \leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, y_{n_h}) + d(y_{n_h}, x_{n_h}) < \frac{3\delta}{4},$$

for $j, h \geq \nu'$, $j \neq h$. This is clearly a contradiction.

Proof of (e). Since we have proved (c), it suffices to prove

$$\forall \varepsilon > 0 \exists \nu_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq \nu_\varepsilon \quad K_n \subset [K]_\varepsilon. \quad (2.10.34)$$

Since $\{K_n\}$ is a Cauchy sequence we have that, for any $\varepsilon > 0$ there exists $\nu_\varepsilon \in \mathbb{N}$ such that

$$d_{\mathcal{H}}(K_n, K_m) < \frac{\varepsilon}{2}, \quad \forall n, m \geq \nu_\varepsilon. \quad (2.10.35)$$

Hence

$$K_m \subset [K_n]_{\frac{\varepsilon}{2}}, \quad \forall n, m \geq \nu_\varepsilon.$$

Let us fix $\bar{n} \geq \nu_\varepsilon$. Inequality (2.10.35) implies that there exists a strictly increasing sequence $\{n_j\}$ in \mathbb{N} such that $n_j \geq \nu_\varepsilon$, for every $j \in \mathbb{N}$, and

$$d_{\mathcal{H}}(K_{n_{j-1}}, K_{n_j}) < \frac{\varepsilon}{2j}.$$

Since $n_1, \bar{n} \geq \nu_\varepsilon$, we get by (2.10.35)

$$d_{\mathcal{H}}(K_{\bar{n}}, K_{n_1}) < \frac{\varepsilon}{2}.$$

Hence

$$K_{\bar{n}} \subset [K_{n_1}]_{\frac{\varepsilon}{2}}. \quad (2.10.36)$$

Now, let us fix $y \in K_{\bar{n}}$ and let us prove that $y \in [K]_\varepsilon$. By (2.10.36) we have

$$y \in [K_{n_1}]_{\frac{\varepsilon}{2}},$$

hence there exists $x_{n_1} \in K_{n_1}$ such that

$$d(x_{n_1}, y) < \frac{\varepsilon}{2}. \quad (2.10.37)$$

Generally speaking, since

$$K_{n_{j-1}} \subset [K_{n_j}]_{\frac{\varepsilon}{2^j}}, \quad \forall j \geq 2,$$

there exists a sequence $\{x_{n_j}\}$ which satisfies $x_{n_j} \in K_{n_j}$, for every $j \in \mathbb{N}$ and

$$d(x_{n_{j-1}}, x_{n_j}) < \frac{\varepsilon}{2^j}, \quad \forall j \in \mathbb{N}.$$

By the latter and by (2.10.37) we get

$$d(y, x_{n_j}) \leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + \cdots + d(x_{n_{j-1}}, x_{n_j}) < \varepsilon \quad (2.10.38)$$

and

$$d(x_{n_j}, x_{n_h}) \leq \sum_{l=j}^{h-1} d(x_{n_l}, x_{n_{l+1}}) < \frac{\varepsilon}{2^j}, \quad \forall h > j \geq \nu_\varepsilon \in \mathbb{N}.$$

In particular, the just obtained inequality implies that for every $\delta > 0$ there exists n_δ such that

$$d(x_{n_j}, x_{n_h}) < \delta, \quad \forall j, h \geq n_\delta.$$

Hence $\{x_{n_j}\}$ is a Cauchy sequence and it satisfies

$$x_{n_j} \in K_{n_j}, \quad \forall j \in \mathbb{N}.$$

Now, Lemma 2.10.12 implies that there exists a Cauchy sequence $\{\bar{x}_n \in K_n\}$ which satisfies

$$\bar{x}_{n_j} = x_{n_j}, \quad \forall j \in \mathbb{N}.$$

Consequently $\{\bar{x}_n \in K_n\}$ converges to a point x and such a point x , by the definition of K , belongs to K . In addition, since $\{x_{n_j}\}$ is a subsequence of $\{\bar{x}_n\}$, we have

$$\{x_{n_j}\} \rightarrow x.$$

Hence, by (2.10.38), we have

$$d(x, y) = \lim_{j \rightarrow \infty} d(y, x_{n_j}) \leq \varepsilon.$$

Therefore

$$y \in [K]_\varepsilon.$$

Hence (2.10.34) is proved. ■

Theorem 2.10.13 (compactness). *If (X, d) is a compact metric space, then $(\mathbf{K}(X), d_{\mathcal{H}})$ is a compact metric space.*

Proof. Since (X, d) is a compact space it is complete and totally bounded. On the other hand, by Theorem 2.10.11, $(\mathbf{K}(X), d_{\mathcal{H}})$ is complete, hence for proving that it is compact, it suffices to prove that $(\mathbf{K}(X), d_{\mathcal{H}})$ is totally bounded.

Let ε be any positive number, since X is totally bounded, there exists a finite set F_{ε} which satisfies

$$d(x, F_{\varepsilon}) < \varepsilon, \quad \forall x \in X. \quad (2.10.39)$$

Let $\mathcal{G}_{\varepsilon}$ be the family of all subsets of F_{ε} ($\mathcal{G}_{\varepsilon}$ is finite because F_{ε} is finite). Let $K \in \mathbf{K}(X)$. Let us consider the set

$$G = \{p \in F_{\varepsilon} : d(p, K) < \varepsilon\}.$$

We have $G \neq \emptyset$. As a matter of fact, (2.10.39) implies that if $x \in K$ then there exists $y \in F_{\varepsilon}$ such that

$$d(x, y) < \varepsilon,$$

hence

$$d(y, K) \leq d(x, y) < \varepsilon,$$

therefore $y \in G$. Notice, that we have trivially $G \in \mathcal{G}_{\varepsilon}$ and

$$\delta(G, K) = \max_{p \in G} d(p, K) < \varepsilon. \quad (2.10.40)$$

Now, let us prove

$$\delta(K, G) = \max_{x \in K} d(x, G) < \varepsilon. \quad (2.10.41)$$

Let $x \in K$. Relationship (2.10.39) implies that there exists $y \in F_{\varepsilon}$ such that

$$d(x, y) < \varepsilon,$$

consequently

$$d(y, K) \leq d(y, x) < \varepsilon.$$

Therefore $y \in G$ which yields

$$d(x, G) \leq d(x, y) < \varepsilon,$$

and (2.10.41) follows. Hence

$$d_{\mathcal{H}}(K, G) = \max \{\delta(K, G), \delta(G, K)\} < \varepsilon.$$

All in all, we have proved

$$\forall K \in \mathbf{K}(X) \quad \exists G \in \mathcal{G}_{\varepsilon} \text{ such that } d_{\mathcal{H}}(K, G) < \varepsilon,$$

which, since ε is arbitrary, implies that $\mathbf{K}(X)$ is a compact metric space. ■

2.11 The distance function

In this Section we will give some properties of the function

$$\mathbb{R}^n \ni x \rightarrow d_{\partial\Omega}(x) := d(x, \partial\Omega),$$

where Ω is a bounded open set of \mathbb{R}^n whose boundary is of class $C^{1,1}$. When there is no risk of ambiguity, we simply write $d(x)$. In Proposition 2.10.1 we have proved that $d_{\partial\Omega}(x)$ is a Lipschitz continuous function, because it satisfies the inequality

$$|d_{\partial\Omega}(x) - d_{\partial\Omega}(y)| \leq |x - y|, \quad \forall x, y \in \mathbb{R}^n. \quad (2.11.1)$$

We say that an open set A of \mathbb{R}^n enjoys the **interior ball property**, if for every point $P \in \partial A$ there exists $P' \in A$ and $r > 0$ such that

$$\overline{B_r(P')} \cap \overline{\Omega} = \{P\}.$$

We say that A enjoys the **exterior ball property** if $\mathbb{R}^n \setminus \overline{\Omega}$ enjoys the property of the interior ball.

The following Proposition holds true.

Proposition 2.11.1. *Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{1,1}$ with constants r_0, M_0 . Then Ω enjoys the interior ball property and the exterior ball property. More precisely we have what follows. Denoting by*

$$\mu_0 = \min \left\{ \frac{1}{M_0}, M_0 \right\}, \quad (2.11.2)$$

for any $P \in \partial\Omega$ and for any $r \in (0, \mu_0 r_0)$, we have

$$\overline{B_r(P - r\nu(P))} \cap \overline{\Omega} = \{P\} \quad (2.11.3)$$

and

$$\overline{B_r(P + r\nu(P))} \cap \overline{\mathbb{R}^n \setminus \Omega} = \{P\}, \quad (2.11.4)$$

where $\nu(P)$ is the unit outward normal to $\partial\Omega$ in P

Proof. Let $P \in \partial\Omega$. Let us consider a local representation of $\partial\Omega$. Hence, let us assume $P = 0$ and let us assume, up to a isometry,

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > g(x')\},$$

where $g \in C^{1,1}(\overline{B'_{r_0}})$ satisfies

$$g(0) = |\nabla g(0)| = 0$$

and

$$\|g\|_{C^{1,1}(\overline{B'_{r_0}})} \leq M_0 r_0.$$

We have

$$\nu(0) = -e_n.$$

Now, notice that

$$g(x') = \int_0^1 (1-s) \partial^2 g(sx') x' \cdot x' ds.$$

Hence

$$g(x') \leq \frac{M_0 |x'|^2}{2r_0}.$$

Therefore, in order to satisfy (2.11.3) it suffices that, besides the condition $r \leq r_0$, the following conditions are satisfied

$$\frac{M_0 |x'|^2}{2r_0} < r - \sqrt{r^2 - |x'|^2}, \quad \forall x' \in \overline{B'_r} \setminus \{0\}$$

and

$$r + \sqrt{r^2 - |x'|^2} < 2M_0 r_0, \quad \forall x' \in \overline{B'_r}.$$

It is easy to check that if $r \in (0, \mu_0 r_0)$, the above conditions are satisfied. In a similar way we proceed for the property of the exterior ball. ■

For any $\rho \in (0, \mu_0 r_0)$, set

$$S_\rho = \{x \in \Omega : d_{\partial\Omega}(x) < \rho\}. \quad (2.11.5)$$

We observe that for every $x \in S_\rho$ there exists a unique point $p(x) \in \partial\Omega$ such that

$$|x - p(x)| = d_{\partial\Omega}(x).$$

As a matter of fact, let $\bar{x} \in S_\rho$ and let $p \in \partial\Omega$ a point which satisfies

$$d_{\partial\Omega}(\bar{x}) = |\bar{x} - p|.$$

We may assume that y belongs to the graph, $\Gamma^{(g)}$, of a function $g \in C^{1,1}(\overline{B'_{r_0}})$ such that

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > g(x')\},$$

and $g(0) = |\nabla g(0)| = 0$. Since p is a minimum point on $\Gamma^{(g)}$ of the function

$$y \rightarrow \frac{1}{2} |\bar{x} - y|^2.$$

By The Lagrange Multiplier Theorem, the following conditions need to be fulfilled

$$\begin{cases} p_j - \bar{x}_j + \lambda \partial_j g(p') = 0, & 1 \leq j \leq n-1, \\ p_n - \bar{x}_n - \lambda = 0, \\ g(p') - p_n = 0. \end{cases}$$

Hence

$$p - \bar{x} = (p' - \bar{x}', p_n - \bar{x}_n) = \lambda (-\nabla_{p'} g(p'), 1),$$

which implies

$$p - \bar{x} = |p - \bar{x}| \nu(y). \quad (2.11.6)$$

This relation, in turn, implies that p is the unique point of $\partial\Omega$ that achieves the minimum distance. Indeed, let $B_\rho(z)$ be the interior ball tangent in y to $\partial\Omega$ (such a ball exists by Proposition 2.11.4), the equality (2.11.6) ensures us that \bar{x} lies on the segment of extremes z and p . Consequently, by setting $\rho_1 = |\bar{x} - p|$ we have $B_{\rho_1}(\bar{x}) \subset B_\rho(z)$. It is, therefore, evident that the distance of \bar{x} from $\partial B_\rho(z)$ is greater than or equal to ρ_1 and, recalling that $B_\rho(\bar{y})$ is an interior ball to Ω , tangent to $\partial\Omega$ at the unique point y , we obtain that

$$|\bar{x} - p| = \rho_1 \leq \rho < |\bar{x} - \xi|, \quad \forall \xi \in \partial\Omega \setminus \{y\}.$$

Therefore we have proved

Proposition 2.11.2. *If $\rho \in (0, \mu_0 r_0)$, then for any $x \in S_\rho$ there exists an unique point $p(x) \in \partial\Omega$ which attains the minimum of distance from x to $\partial\Omega$. Moreover we have*

$$x = p(x) - d_{\partial\Omega}(x) \nu(p(x)). \quad (2.11.7)$$

The following Proposition holds true

Proposition 2.11.3. *If $\rho \in (0, \mu_0 r_0)$, then*

$$S_\rho = \{y - t\nu(y) : y \in \partial\Omega, \quad 0 \leq t < \rho\}. \quad (2.11.8)$$

Proof. Proposition 2.11.2 implies

$$S_\rho \subset \{y - t\nu(y) : y \in \partial\Omega, \quad 0 \leq t < \rho\}.$$

Now, let $x = y - t\nu(y)$, where $y \in \partial\Omega$ e $0 \leq t < \rho$. We have

$$d_{\partial\Omega}(x) \leq |x - y| = t < \rho.$$

Hence $x \in S_\rho$. Therefore

$$\{y - t\nu(y) : y \in \partial\Omega, 0 \leq t < \rho\} \subset S_\rho.$$

Therefore (2.11.8) is proved. ■

Let us prove the following

Lemma 2.11.4. *There exists $\mu_1 \leq \mu_0$ such that if $\rho \in (0, \mu_1 r_0)$ then the maps*

$$S_\rho \ni x \rightarrow p(x) \in \partial\Omega, \quad \text{and} \quad S_\rho \ni x \rightarrow \nu(p(x)) \in \mathbb{S}^{n-1} \quad (2.11.9)$$

are Lipschitz continuous, where $p(x)$ is the point that realizes the minimum distance of $x \in S_\rho$ from $\partial\Omega$.

Proof. We begin by proving that the map

$$\partial\Omega \ni y \rightarrow \nu(y) \in \mathbb{S}^{n-1}, \quad (2.11.10)$$

is Lipschitz continuous. To prove this, let $y_1, y_2 \in \partial\Omega$ and distinguish two cases

$$(a) |y_1 - y_2| \geq r_0,$$

$$(b) |y_1 - y_2| < r_0.$$

In case (a), we have trivially

$$|\nu(y_1) - \nu(y_2)| \leq 2 \leq 2 \frac{|y_1 - y_2|}{r_0}. \quad (2.11.11)$$

In case (b), we may employ a local representation of $\partial\Omega$ assuming that $y_2 = 0$ and $y_1 = g(x')$ where $g \in C^{1,1}(\overline{B'_{r_0}})$ and $g(0) = |\nabla g(0)| = 0$. Hence

$$\nu(y_2) = -e_n$$

and

$$\nu(y_1) = \left(\frac{\nabla_{x'} g(x')}{\sqrt{1 + |\nabla_{x'} g(x')|^2}}, \frac{-1}{\sqrt{1 + |\nabla_{x'} g(x')|^2}} \right).$$

Now it is easy to check that

$$|\nu(y_1) - \nu(y_2)| \leq \frac{\sqrt{2}M_0}{r_0} |x'| = \frac{\sqrt{2}M_0}{r_0} |y_1 - y_2|. \quad (2.11.12)$$

Therefore, by (2.11.11) and (2.11.12) we get

$$|\nu(y_1) - \nu(y_2)| \leq \frac{M_1}{r_0} |y_1 - y_2|, \quad (2.11.13)$$

where

$$M_1 = \max \left\{ 2, \sqrt{2}M_0 \right\}.$$

Now, let us prove that $x \rightarrow p(x)$ is Lipschitz continuous. By (2.11.7) we have (we omit subscript in $d_{\partial\Omega}$)

$$p(x) = x + d(x)\nu(p(x)).$$

Therefore, recalling (2.11.1) and (2.11.13), we get

$$\begin{aligned} |p(x) - p(y)| &\leq |x - y| + d(x)|\nu(p(x)) - \nu(p(y))| + |d(x) - d(y)||\nu(p(y))| \leq \\ &\leq 2|x - y| + \rho \frac{M_1}{r_0} |p(x) - p(y)|. \end{aligned}$$

Hence

$$\left(1 - \rho \frac{M_1}{r_0}\right) |p(x) - p(y)| \leq 2|x - y|.$$

Moreover for any

$$\rho < \min \left\{ \frac{1}{2M_1}, M_0, \frac{1}{M_0} \right\}$$

we have

$$|p(x) - p(y)| \leq 4|x - y|, \quad \forall x, y \in S_\rho. \quad (2.11.14)$$

The above inequality proves that the map $x \rightarrow p(x)$ is Lipschitz continuous provided

$$\mu_1 = \min \left\{ \frac{1}{2M_1}, M_0, \frac{1}{M_0} \right\}.$$

Therefore (2.11.13) and (2.11.14) imply that $\nu(p(x))$ is Lipschitz continuous.

■

Lemma 2.11.5. *Let A be an open set of \mathbb{R}^n and let $f \in C_{loc}^{0,1}(A)$. If there exists a function $g \in C^0(A; \mathbb{R}^n)$ which satisfies*

$$\nabla f(x) = g(x), \quad \text{a.e. } x \in A, \quad (2.11.15)$$

then $f \in C^1(A)$ and

$$\nabla f(x) = g(x), \quad \forall x \in A. \quad (2.11.16)$$

Proof. Let $x_0 \in A$ and $\delta = \frac{1}{4}d(x_0, \partial A)$. For any $\varepsilon \in (0, \delta)$ let us consider the function

$$f_\varepsilon(x) = \int_A f(y)\eta_\varepsilon(x-y)dy, \quad \forall x \in B_\delta(x_0),$$

where η is a mollifier. It turns out that $f_\varepsilon \in C^\infty(\overline{B_\delta(x_0)})$ and, moreover, the divergence Theorem gives

$$\begin{aligned} \partial_j f_\varepsilon(x) &= - \int_A f(y)\partial_{y_j}\eta_\varepsilon(x-y)dy = \\ &= - \int_{B_\varepsilon(x)} [\partial_{y_j}(f(y)\eta_\varepsilon(x-y)) - \partial_{y_j}f(y)\eta_\varepsilon(x-y)] dy = \\ &= \int_{B_\varepsilon(x)} \partial_{y_j}f(y)\eta_\varepsilon(x-y)dy = \\ &= \int_A \partial_{y_j}f(y)\eta_\varepsilon(x-y)dy, \quad j = 1, \dots, n, \quad \forall x \in B_\delta(x_0). \end{aligned}$$

By what has just been obtained and by (2.11.15) we have

$$\nabla f_\varepsilon(x) = g_\varepsilon(x) := \int_A g(y)\eta_\varepsilon(x-y)dy, \quad \forall x \in B_\delta(x_0). \quad (2.11.17)$$

Let v be a vector of \mathbb{R}^n . By (2.11.17) we get

$$f_\varepsilon(x_0 + tv) - f_\varepsilon(x_0) = \int_0^t g_\varepsilon(x_0 + sv) \cdot v ds, \quad \forall t \in [-\delta, \delta]. \quad (2.11.18)$$

Now, Theorem 2.3.34 implies that f_ε and g_ε uniformly converge in $B_\delta(x_0)$. Therefore passing to the limit in (2.11.18) as $\varepsilon \rightarrow 0$, we obtain

$$f(x_0 + tv) - f(x_0) = \int_0^t g(x_0 + sv) \cdot v ds, \quad \forall t \in [-\delta, \delta].$$

On the other hand, since g is continuous, we have

$$\lim_{t \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t g(x_0 + sv) \cdot v ds = g(x_0) \cdot v.$$

Therefore

$$\frac{\partial f(x_0)}{\partial v} = g(x_0) \cdot v, \quad \forall v \in \mathbb{R}^n, \quad |v| = 1.$$

which implies

$$\nabla f(x_0) = g(x_0)$$

so that, since x_0 is arbitrary in A , we have $\nabla f = g$ in A . That, in turn, by the continuity of g implies $f \in C^1(A)$. ■

Theorem 2.11.6. *Let μ_1 be the same of Proposition 2.11.4, then we have*

$$d_{\partial\Omega} \in C^{1,1}(\overline{S_{\mu_1 r_0}})$$

and

$$\nabla d_{\partial\Omega}(x) = -\nu(p(x)), \quad \forall x \in S_{\mu_1 r_0}. \quad (2.11.19)$$

Proof. For the sake of brevity, we omit the subscript in $d_{\partial\Omega}$. Inequality (2.11.1) implies that d is almost everywhere differentiable, in addition in the points where it is differentiable we have

$$|\nabla d| \leq 1. \quad (2.11.20)$$

Let $x \in S_{\mu_1 r_0} \setminus \partial\Omega$ be a point in which d is differentiable. Let $t \in (0, d(x))$. Since $x + t\nu(p(x))$ lies on the segment of endpoints x and $p(x)$, we have

$$d(x + t\nu(p(x))) = d(x) - t.$$

Hence

$$\nabla d(x) \cdot \nu(p(x)) = \lim_{t \rightarrow 0^+} \frac{d(x + t\nu(p(x))) - d(x)}{t} = -1.$$

Therefore

$$\nabla d(x) \cdot \nu(p(x)) = -1. \quad (2.11.21)$$

Consequently we get

$$1 = |\nabla d(x) \cdot \nu(p(x))| \leq |\nabla d(x)| |\nu(p(x))| \leq 1.$$

Hence, there exists $\lambda \in \mathbb{R}$ such that $\nabla d(x) = \lambda \nu(p(x))$ and by (2.11.21) we have $\lambda = -1$. This implies

$$\nabla d(x) = -\nu(p(x)), \quad \text{a.e. } x \in S_{\mu_1 r_0} \setminus \partial\Omega. \quad (2.11.22)$$

Now, by Lemma 2.11.4 we know that $\nu(p(x))$ is Lipschitz continuous, therefore by Lemma 2.11.5 we obtain $d \in C^1(S_{\mu_1 r_0} \setminus \partial\Omega)$. Finally, exploit again (2.11.22), we have $d \in C^{1,1}(\overline{S_{\mu_1 r_0}})$. ■

Corollary 2.11.7. *For any $\rho \in (0, \mu_1 r_0)$ the boundary of the open set*

$$(\Omega)_\rho = \{x \in \Omega : d_{\partial\Omega}(x) > \rho\} \quad (2.11.23)$$

is of class $C^{1,1}$ and we have

$$\partial(\Omega)_\rho = \Gamma_\rho, \quad (2.11.24)$$

where

$$\Gamma_\rho = \{x \in \Omega : d_{\partial\Omega}(x) = \rho\}. \quad (2.11.25)$$

Moreover

$$\Gamma_\rho = \{y - \rho\nu(y) : y \in \partial\Omega\}. \quad (2.11.26)$$

Proof. We first prove (2.11.24). To prove that $\Gamma_\rho \subset \partial(\Omega)_\rho$ we argue by contradiction. Let us assume that there exists $x \in \partial(\Omega)_\rho$ such that $d_{\partial\Omega}(x) > \rho$. Consequently, x would be an interior point of $(\Omega)_\rho$. If $d_{\partial\Omega}(x) < \rho$ then x would be exterior to $(\Omega)_\rho$. Therefore, $x \in \Gamma_\rho$ and we have $\partial(\Omega)_\rho \subset \Gamma_\rho$. Now, let $x \in \Gamma_\rho$, since $\rho \in (0, \mu_1 r_0)$ (and $\mu_1 \leq \mu_0$), Proposition 2.11.2 implies that there is a unique point $p(x) \in \partial\Omega$ which attains the minimum of distance of x from $\partial\Omega$, moreover

$$x = p(x) - \rho\nu(p(x)).$$

For any $\varepsilon > 0$ small enough, we have

$$x - \varepsilon\nu(p(x)) = p(x) - (\rho + \varepsilon)\nu(p(x)) \in (\Omega)_\rho$$

and

$$x + \varepsilon\nu(p(x)) = p(x) - (\rho - \varepsilon)\nu(p(x)) \notin (\Omega)_\rho.$$

Hence $x \in \partial(\Omega)_\rho$. Therefore $\Gamma_\rho \subset \partial(\Omega)_\rho$.

In order to prove that $\partial(\Omega)_\rho$ is of class $C^{1,1}$, we exploit (2.11.24). By Theorem 2.11.6 we derive $|\nabla d_{\partial\Omega}(x)| = 1$, for every $x \in \Gamma_\rho$, and by applying Implicit Function Theorem we easily reach the assertion.

Concerning (2.11.26), let us note that if $x = y - \rho\nu(y)$, taking into account $\rho < \mu_1 r_0 \leq \mu_0 r_0$, then $d_{\partial\Omega}(x) = \rho$. Conversely, if $x \in \Gamma_\rho$ Proposition 2.11.2 gives

$$x = p(x) - d_{\partial\Omega}(x)\nu(p(x)) = (x) - \rho\nu(p(x))$$

so that, since $p(x) \in \partial\Omega$ we have $x \in \{y - \rho\nu(\rho) : y \in \partial\Omega\}$. ■

We now provide a few words about the map

$$\Phi : \partial\Omega \times (0, \mu_1 r_0) \rightarrow \mathbb{R}^n,$$

such that

$$\Phi(y, t) = y - t\nu(y), \quad \forall (y, t) \in \partial\Omega \times (0, \mu_1 r_0). \quad (2.11.27)$$

The following Proposition holds true

Proposition 2.11.8. *If Ω is a bounded open set of \mathbb{R}^n of class $C^{1,1}$ then we have:*

- (a) $\Phi(\partial\Omega \times (0, \mu_1 r_0)) = S_{\mu_1 r_0}$,
- (b) $\Phi \in C^{0,1}(\partial\Omega \times [0, \mu_1 r_0])$,
- (c) Φ is injective on $\partial\Omega \times (0, \mu_1 r_0)$ and its inverse is Lipschitz continuous map.

Proof. (a) is a consequence of Proposition 2.11.3. (b) is a consequence of Lemma 2.11.4. Now let us prove (c). Let $x \in S_{\mu_1 r_0}$ satisfy

$$x = \Phi(y, t) = y - t\nu(y), \quad (y, t) \in \partial\Omega \times (0, \mu_1 r_0).$$

By Proposition 2.11.2 and by the interior ball property we get

$$y = p(x), \quad t = d_{\partial\Omega}(x).$$

Hence

$$\Phi^{-1}(x) = p(x) - d_{\partial\Omega}(x)\nu(p(x)).$$

By the latter and by Lemma 2.11.4 it follows that Φ^{-1} is Lipschitz continuous. ■

If Ω is of class C^k , $k \geq 2$, other properties of the distance function and the map Φ can be proved. For instance, one can prove that $d_{\partial\Omega} \in C^k$ and $\Phi \in C^{k-1}$. For further details, we refer to [28, Ch. 14, Sect. 6].

We say that a continuous map

$$\gamma : [0, 1] \rightarrow A,$$

is a **continuous path in a set** $A \subset \mathbb{R}^n$ **continuous path**. Let $B \subset A$ and $x, y \in B$, if $\gamma([0, 1]) \subset B$ and $\gamma(0) = x$, $\gamma(1) = y$, we say that the path γ

joins x and y in B . If γ_1 e γ_2 are two continuous paths in A which satisfy $\gamma_1(1) = \gamma_2(0)$, we denote by $\gamma_1 \vee \gamma_2$ the following continuous path

$$(\gamma_1 \vee \gamma_2)(t) = \begin{cases} \gamma_1(2t), & \text{for } t \in [0, \frac{1}{2}), \\ \gamma_2(2t - 1), & \text{for } t \in [\frac{1}{2}, 1]. \end{cases} \quad (2.11.28)$$

If $\gamma_1, \dots, \gamma_k$ are $k \geq 2$ continuous paths in A such that $\gamma_{j-1}(1) = \gamma_j(0)$, $j = 2, \dots, k$, we set

$$\gamma_1 \vee \dots \vee \gamma_k := (\gamma_1 \vee \dots \vee \gamma_{k-1}) \vee \gamma_k.$$

We say that $\gamma_1 \vee \dots \vee \gamma_k$ is the $\gamma_1, \dots, \gamma_k$.

Proposition 2.11.9. *Let us assume that Ω and $\partial\Omega$ are connected. If $\rho \in (0, \mu_1 r_0)$, then the $(\Omega)_\rho$, defined by (2.11.23), is connected.*

Proof. Let $z, w \in (\Omega)_\rho$ and let $\varepsilon > 0$ such that

$$\rho + \varepsilon < \min \{ \mu_1 r_0, d_{\partial\Omega}(z), d_{\partial\Omega}(w) \}.$$

Then

$$\begin{aligned} z, w &\in (\Omega)_{\rho+\varepsilon}, \\ \Gamma_{\rho+\varepsilon} &\subset (\Omega)_\rho, \end{aligned} \quad (2.11.29)$$

and $\Gamma_{\rho+\varepsilon}$ is connected, as it is the image by Φ , defined in Proposition 2.11.8, of the connected set $\partial\Omega \times \{\rho + \varepsilon\}$.

Now, since Ω is connected, $\bar{\Omega}$ is also connected (path connected, because $\partial\Omega$ è of class $C^{1,1}$). Be, therefore, $x \in \partial\Omega$ and be

$$\gamma_1 : [0, 1] \rightarrow \bar{\Omega} \quad \text{and} \quad \gamma_2 : [0, 1] \rightarrow \bar{\Omega}$$

two continuous paths such that

$$\gamma_1(0) = z, \quad \gamma_1(1) = x, \quad \gamma_2(0) = x, \quad \gamma_2(1) = w.$$

Let

$$t_1 = \inf \{ t \in [0, 1] : d(\gamma_1(t), \partial\Omega) < \rho + \varepsilon \},$$

we have (because $d(\gamma_1(\cdot), \partial\Omega)$ is continuous)

$$y' := \gamma_1(t_1) \in \Gamma_{\rho+\varepsilon}.$$

Similarly, let

$$t_2 = \sup \{t \in [0, 1] : d(\gamma_2(t), \partial\Omega) < \rho + \varepsilon\},$$

we have

$$y'' := \gamma_2(t_2) \in \Gamma_{\rho+\varepsilon}.$$

Since $\Gamma_{\rho+\varepsilon}$ is connected, there exists a continuous path $\tilde{\gamma} : [0, 1] \rightarrow \Gamma_{\rho+\varepsilon}$, such that

$$\tilde{\gamma}(0) = y', \quad \tilde{\gamma}(1) = y''.$$

It is now evident that the path

$$\gamma := \gamma_1 \vee \tilde{\gamma} \vee \gamma_2,$$

is continuous and it joins z e w in $(\Omega)_\rho$. ■

Remark. In Proposition 2.11.9, the assumption that $\partial\Omega$ is connected is not necessary. The proof of this assertion may follow arguing likewise the proof of Proposition 2.11.9, taking into account that due to the boundedness of Ω and the $C^{1,1}$ character of $\partial\Omega$, the connected components of $\partial\Omega$ are finite in number. We invite the reader to develop the details. ◆

Chapter 3

The Sobolev spaces

3.1 Weak derivatives

Let us give the definition of weak derivative.

Definition 3.1.1. Let Ω be an open set of \mathbb{R}^n and $\alpha \in \mathbb{N}_0^n$. Let $u, v \in L^1_{loc}(\Omega)$. We say that v is the α -th **weak derivative** of u and we write

$$\partial^\alpha u = v,$$

if

$$\int_{\Omega} u \partial^\alpha \phi dx = (-1)^\alpha \int_{\Omega} v \phi dx, \quad \forall \phi \in C_0^\infty(\Omega). \quad (3.1.1)$$

Definition 3.1.1 is justified by the integration by parts formula that, in the case of $u \in C^{|\alpha|}(\Omega)$, gives precisely the derivative $\partial^\alpha u$ in the classical sense. For instance, if $u \in C^1(\Omega)$, we have

$$\int_{\Omega} u \partial_j \phi dx = \int_{\Omega} [\partial_j (u\phi) - \phi \partial_j u] dx = - \int_{\Omega} \phi \partial_j u dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

Proposition 3.1.2. *If $u \in L^1_{loc}(\Omega)$ admits the α -th weak derivative, it is unique (up to a set of measure zero).*

Proof. Let us assume that $v_1, v_2 \in L^1_{loc}(\Omega)$ are two α -th weak derivative of u , then

$$(-1)^\alpha \int_{\Omega} v_1 \phi dx = \int_{\Omega} u \partial^\alpha \phi dx = (-1)^\alpha \int_{\Omega} v_2 \phi dx, \quad \forall \phi \in C_0^\infty(\Omega),$$

which implies

$$\int_{\Omega} (v_1 - v_2) \phi dx = 0, \quad \forall \phi \in C_0^\infty(\Omega),$$

however, $v_1 - v_2 \in L_{loc}^1(\Omega)$, so

$$v_1 = v_2, \quad \text{a.e. in } \Omega.$$

■

Example 1. Let $\Omega = (-1, 1)$, $u(x) = |x|$; let us show that

$$u' = \text{sgn}(x) := \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -1, & \text{for } x < 0, \end{cases}, \quad \text{in the weak sense.}$$

As a matter of fact we have $\text{sgn}(\cdot) \in L^1(-1, 1)$ and

$$\begin{aligned} \int_{-1}^1 |x| \phi'(x) dx &= \int_0^1 x \phi'(x) dx - \int_{-1}^0 x \phi'(x) dx = \\ &= [x\phi(x)]_0^1 - \int_0^1 \phi(x) dx - [x\phi(x)]_{-1}^0 + \int_{-1}^0 \phi(x) dx = \\ &= - \int_{-1}^1 \text{sgn}(x) \phi(x) dx, \quad \forall \phi \in C_0^\infty(-1, 1). \end{aligned}$$

♠

Example 2. Let $\Omega = (-1, 1)$, $u(x) = \text{sgn}(x)$. Let us prove that u has **not** the weak derivative. Let us assume the contrary and be $v \in L_{loc}^1(-1, 1)$ such that

$$\int_{-1}^1 u(x) \phi'(x) dx = - \int_{-1}^1 v(x) \phi(x) dx, \quad \forall \phi \in C_0^\infty(-1, 1). \quad (3.1.2)$$

Let $\phi \in C_0^\infty(-1, 1)$ arbitrary. We have

$$\int_{-1}^1 u(x) \phi'(x) dx = \int_0^1 \phi'(x) dx - \int_{-1}^0 \phi(x) dx = -2\phi(0).$$

Taking into account (3.1.2), we get

$$\int_{-1}^1 v(x)\phi(x)dx = 2\phi(0), \quad \forall \phi \in C_0^\infty(-1, 1). \quad (3.1.3)$$

Let now $\{\phi_k\}_{k \geq 2}$ be the following sequence of functions

$$\phi_k(x) = \begin{cases} e^{k^2 - \frac{k^2}{1-k^2x^2}}, & \text{for } |x| < \frac{1}{k}, \\ 0, & \text{for } \frac{1}{k} \leq |x| < 1. \end{cases}$$

we have $\phi_k \in C_0^\infty(-1, 1)$, $\text{supp } \phi_k \subset [-\frac{1}{2}, \frac{1}{2}]$ for every $k \geq 2$. Moreover

$$\phi_k(0) = 1, \quad \text{and} \quad \lim_{k \rightarrow \infty} \phi_k(x) = 0, \quad \text{for } x \neq 0.$$

On the other hand, by (3.1.3) we have

$$2 = 2\phi_k(0) = \int_{-1}^1 v(x)\phi_k(x)dx, \quad \forall k \geq 2, \quad (3.1.4)$$

but $v \in L_{loc}^1(-1, 1)$, hence the Dominated Convergence Theorem implies

$$\lim_{k \rightarrow \infty} \int_{-1}^1 v(x)\phi_k(x)dx = 0.$$

By the latter and by (3.1.4) we reach a contradiction. ♠

3.2 Definition of the Sobolev spaces

Let us give the following

Definition 3.2.1. Let $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$ and let Ω be an open set of \mathbb{R}^n , $n \geq 1$. If $k = 0$, set

$$W^{0,p}(\Omega) = L^p(\Omega).$$

If $k \geq 1$, $W^{k,p}(\Omega)$ is the set of functions $u \in L_{loc}^1(\Omega)$ satisfying

$$\partial^\alpha u \in L^p(\Omega), \quad \text{for } |\alpha| \leq k, \quad (3.2.1)$$

where $\partial^\alpha u$ is the α -th weak derivative of u .

It is easy to check that $W^{k,p}(\Omega)$ is a vector space. Furthermore we define the following norms. If $1 \leq p < +\infty$, we set

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_{\Omega} |\partial^{\alpha} u|^p dx \right)^{1/p}. \quad (3.2.2)$$

If $p = +\infty$, we set

$$\|u\|_{W^{\infty,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^{\alpha} u\|_{L^{\infty}(\Omega)}. \quad (3.2.3)$$

If $p = 2$, we also set

$$H^k(\Omega) = W^{k,2}(\Omega).$$

Let us observe that $H^k(\Omega)$ is a pre-Hilbertian space equipped with the scalar product

$$(u, v)_{H^k(\Omega)} = \int_{\Omega} \sum_{|\alpha| \leq k} \partial^{\alpha} u \partial^{\alpha} v dx, \quad \forall u, v \in H^k(\Omega). \quad (3.2.4)$$

Here and in the sequel, for any $k \in \mathbb{N}$, $p \in [1, \infty]$, we denote by $W_{loc}^{k,p}(\Omega)$, ($H_{loc}^k(\Omega)$) the subspace of $L_{loc}^p(\Omega)$ ($L_{loc}^2(\Omega)$) of the functions u such that for every open set $\omega \Subset \Omega$ (i.e. $\bar{\omega} \subset \Omega$) we have $u|_{\omega} \in W^{k,p}(\omega)$ ($u|_{\omega} \in H^k(\omega)$). Let $\{u_m\}$ be a sequence in $W_{loc}^{k,p}(\Omega)$ and $u \in W_{loc}^{k,p}(\Omega)$, we say that

$$u_m \rightarrow u, \quad \text{as } m \rightarrow \infty, \text{ in } W_{loc}^{k,p}(\Omega),$$

if

$$(u_m)|_{\omega} \rightarrow u|_{\omega}, \quad \text{as } m \rightarrow \infty, \text{ in } W^{k,p}(\omega), \quad \forall \omega \Subset \Omega.$$

Exercise 1. Check that, if $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, then $W^{k,p}(\Omega)$ is a vector subspace of $L^p(\Omega)$ and $\|\cdot\|_{W^{k,p}(\Omega)}$ defines a norm on $W^{k,p}(\Omega)$. ♣

Proposition 3.2.2. *If $u \in W^{k,p}(\Omega)$, then we have*

(i) $\partial^{\alpha} u \in W^{k-|\alpha|,p}(\Omega)$ for $|\alpha| \leq k$ and $\partial^{\beta} \partial^{\alpha} u = \partial^{\alpha} \partial^{\beta} u = \partial^{\alpha+\beta} u$ for $|\alpha|+|\beta| \leq k$,

(ii) for any $\zeta \in C^{\infty}(\bar{\Omega})$ we have $\zeta u \in W^{k,p}(\Omega)$ and

$$\partial^{\alpha}(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\beta} \zeta \partial^{\alpha-\beta} u.$$

Proof. (i) Let $u \in W^{k,p}(\Omega)$, $|\alpha| \leq k$, and let β satisfy $|\beta| \leq k - |\alpha|$. For any $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \partial^\alpha u \partial^\beta \phi dx &= (-1)^{|\alpha|} \int_{\Omega} u \partial^{\alpha+\beta} \phi dx = \\ &= (-1)^{|\alpha|} (-1)^{|\alpha|+|\beta|} \int_{\Omega} \partial^{\alpha+\beta} u \phi dx = \\ &= (-1)^{|\beta|} \int_{\Omega} \partial^{\alpha+\beta} u \phi dx. \end{aligned}$$

Hence

$$\int_{\Omega} \partial^\alpha u \partial^\beta \phi dx = (-1)^{|\beta|} \int_{\Omega} \partial^{\alpha+\beta} u \phi dx, \quad \forall \phi \in C_0^\infty(\Omega),$$

consequently

$$\partial^\beta \partial^\alpha u = \partial^{\alpha+\beta} u.$$

The latter implies

$$\partial^\alpha u \in W^{k-|\alpha|,p}(\Omega),$$

for any $|\alpha| \leq k$.

(ii) Let us consider the case $|\alpha| = 1$. Let $\alpha = e_j$, for $j = 1, \dots, n$. We have, for any $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} \zeta u \partial_j \phi dx &= \int_{\Omega} u [\partial_j(\zeta \phi) - (\partial_j \zeta) \phi] dx = \\ &= - \int_{\Omega} (\partial_j u) \zeta \phi dx - \int_{\Omega} u (\partial_j \zeta) \phi dx = \\ &= - \int_{\Omega} [(\partial_j u) \zeta + u \partial_j \zeta] \phi dx. \end{aligned}$$

Now, let us notice that

$$(\partial_j u) \zeta + u \partial_j \zeta \in L^p(\Omega),$$

hence

$$\partial_j(\zeta u) = (\partial_j u) \zeta + u \partial_j \zeta, \quad \text{in the weak sense.}$$

If $|\alpha| > 1$, one proceeds by induction, and we leave the details to the reader.

■

Theorem 3.2.3 (completeness of $W^{k,p}(\Omega)$). *The space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$, equipped with the norm (3.2.2), (3.2.3), is a Banach space. If $p = 2$, $H^k(\Omega)$ is a Hilbert space.*

Proof. We limit ourselves to the case $k = 1$. Similarly it can be handle the case $k > 1$. Let $\{u_m\}$ be a Cauchy sequence in $W^{1,p}(\Omega)$. From the definition of norm of $W^{1,p}(\Omega)$ we have that

$$\{u_m\} \quad \text{and} \quad \{\partial_j u_m\}, \quad j = 1, \dots, n,$$

are Cauchy sequences in $L^p(\Omega)$. On the other hand, $L^p(\Omega)$ is complete; hence there exist $u, v_1, \dots, v_n \in L^p(\Omega)$ such that

$$u_m \rightarrow u, \quad \text{as } m \rightarrow \infty, \quad \text{in } L^p(\Omega), \quad (3.2.5a)$$

$$\partial_j u_m \rightarrow v_j, \quad \text{as } m \rightarrow \infty, \quad \text{in } L^p(\Omega), \quad j = 1, \dots, n. \quad (3.2.5b)$$

Now, (3.2.5a) and (3.2.5b) imply that, for any $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} u \partial_j \phi dx &= \lim_{m \rightarrow \infty} \int_{\Omega} u_m \partial_j \phi dx = \\ &= - \lim_{m \rightarrow \infty} \int_{\Omega} \partial_j u_m \phi dx = \\ &= - \int_{\Omega} v_j \phi dx. \end{aligned}$$

Hence

$$\partial_j u = v_j, \quad \text{for } j = 1, \dots, n.$$

Therefore by (3.2.5a) e (3.2.5b) we have

$$u_m \rightarrow u, \quad \text{as } m \rightarrow \infty, \quad \text{in } W^{1,p}(\Omega).$$

■

Proposition 3.2.4. *The space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq p < \infty$, equipped with the norm (3.2.2) is a separable space.*

Proof. The proof is similar to the one of Proposition 2.1.6. Let us consider the case $k = 1$. Let

$$\Phi : W^{1,p}(\Omega) \rightarrow L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n),$$

$$\Phi(u) = (u, \nabla u), \quad \forall u \in W^{1,p}(\Omega).$$

Φ turns out to be an isometry, provided that we equip $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n)$ by the norm

$$\left(\int_{\Omega} |v_0|^p dx + \sum_{j=1}^n \int_{\Omega} |v_j|^p dx \right)^{1/p},$$

for every $v = (v_0, v_1, \dots, v_n) \in L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n)$. Now, since $p < +\infty$, $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n)$ is separable because it is the cartesian product of separable spaces. Hence $\Phi(W^{1,p}(\Omega))$ is separable as a subspace of $L^p(\Omega) \times L^p(\Omega; \mathbb{R}^n)$ and, since Φ is an isometry, $W^{1,p}(\Omega)$ is separable too. ■

It can be proved that if $1 < p < \infty$, then $W^{1,p}(\Omega)$ is a *reflexive space*. In the sequel we will not make explicitly use this property, however for a proof we refer to [12, Proposizione IX.1].

Example 1. Let $\alpha > 0$. Let us consider

$$u(x) = \frac{1}{|x|^\alpha}.$$

We prove that $u \in W^{1,p}(B_1)$ if and only if $p < n$ and $\alpha < \frac{n}{p} - 1$.

We begin by assuming that $u \in W^{1,p}(B_1)$. Then $u \in L^p(B_1)$ and consequently $\alpha p < n$, therefore $p < \infty$. Moreover, for any $j = 1, \dots, n$ there exists $v_j \in L^p(B_1)$ such that

$$\int_{B_1} \frac{1}{|x|^\alpha} \partial_j \phi dx = - \int_{B_1} v_j \phi dx, \quad \forall \phi \in C_0^\infty(B_1).$$

In particular we have, for any $\phi \in C_0^\infty(B_1 \setminus \{0\})$,

$$- \int_{B_1} v_j \phi dx = \int_{B_1} \frac{1}{|x|^\alpha} \partial_j \phi dx = \int_{B_1} \frac{\alpha x_j}{|x|^{\alpha+2}} \phi dx.$$

Hence, for any $j = 1, \dots, n$,

$$v_j(x) = - \frac{\alpha x_j}{|x|^{\alpha+2}}, \quad \text{a.e. in } B_1.$$

Now, $v_j \in L^p(B_1)$, therefore

$$\sum_{j=1}^n \int_{B_1} \left| \frac{\alpha x_j}{|x|^{\alpha+2}} \right|^p dx < \infty. \quad (3.2.6)$$

Let us observe that if $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, then

$$\frac{1}{n^{p-1}} |a|^p \leq \sum_{j=1}^n |a_j|^p \leq n |a|^p, \quad (3.2.7)$$

(the first inequality is just a consequence of Hölder inequality, yje second is trivial). Hence (3.2.6) is satisfied if and only if

$$\int_{B_1} \frac{dx}{|x|^{(\alpha+1)p}} = \int_{B_1} \left| \frac{x}{|x|^{\alpha+2}} \right|^p dx < \infty,$$

from which we derive

$$\alpha < \frac{n}{p} - 1. \quad (3.2.8)$$

Conversely, let us assume that (3.2.8) holds true and that $p < n$. Let us show that $u \in W^{1,p}(B_1)$. Inequality (3.2.8) implies $\alpha < \frac{n}{p}$ that, in turn implies $u \in L^p(B_1)$. Now, let $\phi \in C_0^\infty(B_1)$ be arbitrary. We have, for any $j = 1, \dots, n$,

$$\begin{aligned} \int_{B_1} u \partial_j \phi dx &= \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} u \partial_j \phi dx = \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} (\partial_j(u\phi) - \phi \partial_j u) dx = \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\partial B_\varepsilon} u \phi \nu_j dS - \int_{B_1 \setminus B_\varepsilon} \phi \partial_j u dx \right\}. \end{aligned} \quad (3.2.9)$$

On the other hand we have

$$\left| \int_{\partial B_\varepsilon} u \phi \nu_j dS \right| \leq \omega_n \|\phi\|_{L^\infty(B_1)} \varepsilon^{-\alpha+n-1},$$

where ω_n is the measure of $|\partial B_1|$. Now, by (3.2.8) and $p \geq 1$ we have $\alpha < n - 1$. Hence

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{\partial B_\varepsilon} u \phi \nu_j dS \right| = 0;$$

coming back to (3.2.9) and keeping in mind that (by (3.2.8))

$$\frac{\alpha x_j}{|x|^{\alpha+2}} \in L^p(B_1) \subset L^1(B_1),$$

we get

$$\int_{B_1} u \partial_j \phi dx = \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} \frac{\alpha x_j}{|x|^{\alpha+2}} \phi dx = \int_{B_1} \frac{\alpha x_j}{|x|^{\alpha+2}} \phi dx.$$

All in all we have

$$\partial_j u = -\frac{\alpha x_j}{|x|^{\alpha+2}} \in L^p(B_1),$$

therefore $u \in W^{1,p}(B_1)$. ♠

Remark. Similarly to in Example 1, it can be proved that if

$$u \in C^0(\overline{B_1}) \cap C^1(\overline{B_1} \setminus \{0\})$$

then we have

$$u \in W^{1,p}(B_1) \iff \nabla u \in L^p(B_1),$$

here ∇u is the gradient of u in $B_1 \setminus \{0\}$ in the classic sense, Let us consider the case $n = 1$ only, because the case $n > 1$ can be treated in precisely the same way as Example 1 (the proof is left to the reader). Let $\phi \in C_0^\infty(-1, 1)$, we have

$$\begin{aligned} \int_{-1}^1 u \phi' dx &= \lim_{\varepsilon \rightarrow 0} \int_{(-1,1) \setminus [-\varepsilon, \varepsilon]} u \phi' dx = \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ -u(\varepsilon)\phi(\varepsilon) + u(-\varepsilon)\phi(-\varepsilon) - \int_{(-1,1) \setminus [-\varepsilon, \varepsilon]} u' \phi dx \right\}, \end{aligned}$$

but $u \in C^0([-1, 1])$, hence

$$\lim_{\varepsilon \rightarrow 0} (-u(\varepsilon)\phi(\varepsilon) + u(-\varepsilon)\phi(-\varepsilon)) = 0$$

and $u' \in L^p(-1, 1)$, implies

$$\int_{-1}^1 u \phi' dx = - \int_{-1}^1 u' \phi dx.$$

Therefore, u' is the weak derivative of u and $u \in W^{1,p}(-1, 1)$. ♦

3.2.1 The spaces $W_0^{k,p}(\Omega)$

We give the following

Definition 3.2.5. Let $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$ and Ω be an open set of \mathbb{R}^n , $n \geq 1$. Let us denote by

$$W_0^{k,p}(\Omega),$$

the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

Let us notice that $W_0^{0,p}(\Omega) = L^p(\Omega)$. Moreover $W_0^{k,p}(\Omega)$, equipped with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$, as it is a closed subspace of $W^{k,p}(\Omega)$, is a Banach space.

3.3 Approximation and density theorems

Let η be a mollifier, namely $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfies (i) $\text{supp } \eta \subset B_1$, (ii) $\eta \geq 0$, (iii) $\int_{\mathbb{R}^n} \eta(x) dx = 1$. Set, for any $\varepsilon > 0$,

$$\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\varepsilon^{-1}x).$$

Let Ω be an open set of \mathbb{R}^n . Set

$$\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}.$$

The following Theorem holds true

Theorem 3.3.1 (local approximation by C^∞ functions). *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty)$. Let us assume that $u \in W^{k,p}(\Omega)$. Let us denote by*

$$u^\varepsilon = \eta_\varepsilon \star u, \quad \text{in } \Omega_\varepsilon.$$

Then we have

$$u^\varepsilon \in C^\infty(\Omega_\varepsilon) \cap W_{loc}^{k,p}(\Omega_\varepsilon), \quad \forall \varepsilon > 0$$

and

$$u^\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } W_{loc}^{k,p}(\Omega).$$

Proof. The fact that $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$ is an immediate consequence of Theorem 2.3.37. Concerning $u^\varepsilon \in W_{loc}^{k,p}(\Omega_\varepsilon)$, we have by Theorem 2.3.35

$$\partial^\alpha u^\varepsilon = (\partial^\alpha \eta_\varepsilon) \star u, \quad \forall \alpha \in \mathbb{N}_0^n.$$

Now, if $|\alpha| \leq k$, we have, for any $x \in \Omega_\varepsilon$,

$$\begin{aligned} (\partial^\alpha \eta_\varepsilon) \star u &= \int_{\Omega} (\partial_x^\alpha \eta_\varepsilon)(x-y) u(y) dy = \\ &= (-1)^{|\alpha|} \int_{\Omega} \partial_y^\alpha (\eta_\varepsilon(x-y)) u(y) dy = \\ &= \int_{\Omega} \eta_\varepsilon(x-y) \partial_y^\alpha u(y) dy = \\ &= (\eta_\varepsilon \star \partial^\alpha u)(x). \end{aligned}$$

Let $\omega \Subset \Omega$, since $\partial^\alpha u \in L^p(\omega)$ we have, for any $|\alpha| \leq k$,

$$\eta_\varepsilon \star \partial^\alpha u \rightarrow \partial^\alpha u, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } L^p(\omega).$$

Therefore

$$u \rightarrow u^\varepsilon, \quad \text{as } \varepsilon \rightarrow 0, \text{ in } W^{k,p}(\omega).$$

■

Theorem 3.3.2 (Meyers – Serrin). *Let Ω be an bounded open set of \mathbb{R}^n . Let $k \in \mathbb{N}_0$, $p \in [1, +\infty)$. If $u \in W^{k,p}(\Omega)$ then there exists a sequence $\{u_m\}$ in $W^{k,p}(\Omega) \cap C^\infty(\Omega)$ which satisfies*

$$u_m \rightarrow u, \quad \text{as } m \rightarrow \infty, \text{ in } W^{k,p}(\Omega).$$

Proof. Let

$$\Omega_j = \left\{ x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{j} \right\}, \quad j \in \mathbb{N},$$

It is not restrictive to assume $\Omega_j \neq \emptyset$, for every $j \in \mathbb{N}$. We have

$$\overline{\Omega}_j \subset \Omega_{j+1}, \quad \forall j \in \mathbb{N}, \quad \bigcup_{j=1}^{\infty} \Omega_j = \Omega. \quad (3.3.1)$$

Let $\phi_j \in C^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}$, satisfy $\text{supp } \phi_j \subset \Omega_{j+1}$; $\phi_j(x) = 1$ for every $x \in \Omega_j$; $0 \leq \phi_j(x) \leq 1$, for every $x \in \mathbb{R}^n$.

For any $j \in \mathbb{N}$ we get

$$x \in \mathbb{R}^n \setminus \Omega_{j+1} \implies \phi_j(x) = 0 \leq \phi_{j+1}(x)$$

and

$$x \in \Omega_{j+1} \implies \phi_j(x) \leq 1 = \phi_{j+1}(x).$$

Hence

$$\phi_j \leq \phi_{j+1}, \quad \forall j \in \mathbb{N}, \quad \text{in } \mathbb{R}^n.$$

Set

$$\zeta_0 = \phi_2, \quad \zeta_j = \phi_{j+1} - \phi_j, \quad \forall j \in \mathbb{N}$$

and

$$V_0 = \Omega_2, \quad V_j = \Omega_{j+3} \setminus \overline{\Omega}_j, \quad \forall j \in \mathbb{N}.$$

We have

$$\zeta_j \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } \zeta_j \subset V_j, \quad \forall j \in \mathbb{N}_0.$$

Moreover

$$0 \leq \zeta_j \leq 1, \quad \forall x \in \mathbb{R}^n, \quad \forall j \in \mathbb{N}_0$$

and

$$\sum_{j=0}^{\infty} \zeta_j(x) = 1, \quad \forall x \in \Omega. \quad (3.3.2)$$

Let us check (3.3.2). Let $x \in \Omega$, by (3.3.1) we have that there exists $\bar{m} \in \mathbb{N}$ such that $x \in \Omega_j$, for every $j \geq \bar{m}$. Let $m \geq \bar{m}$, we have $x \in \Omega_m$, hence $\phi_{m+1}(x) = 1$. Consequently, we have

$$\begin{aligned} \sum_{j=0}^m \zeta_j(x) &= \zeta_0(x) + \zeta_1(x) + \cdots + \zeta_m(x) = \\ &= \phi_2(x) + (\phi_3(x) - \phi_2(x)) + \cdots + (\phi_{m+1}(x) - \phi_m(x)) = \\ &= \phi_{m+1}(x) = 1. \end{aligned}$$

Therefore we have checked (3.3.2).

Now, let $u \in W^{k,p}(\Omega)$ and let us consider the functions $\zeta_j u$, $j \in \mathbb{N}_0$. Proposition gives 3.2.2 we get $\zeta_j u \in W^{k,p}(\Omega)$, in addition

$$\zeta_j u = 0, \quad \text{in } \Omega \setminus \bar{V}_j.$$

Let us denote by $W_0 = \Omega_4$, $W_1 = \Omega_5$, $W_j = \Omega_{j+4} \setminus \bar{\Omega}_{j-1}$, $j \geq 2$. Let $\delta > 0$ be fixed and let $0 < \varepsilon_j < \frac{1}{j+4} - \frac{1}{j+3}$ satisfy

$$u^j = \eta_{\varepsilon_j} \star (\zeta_j u) \in C^\infty(\Omega) \cap W^{k,p}(\Omega),$$

we have

$$u^j = 0, \quad \text{in } \Omega \setminus \bar{W}_j.$$

Theorem 3.3.1 implies that for every $j \in \mathbb{N}_0$ there exists $\varepsilon_j > 0$ such that

$$\|u^j - \zeta_j u\|_{W^{k,p}(\Omega)} = \|u^j - \zeta_j u\|_{W^{k,p}(W_j)} \leq \frac{\delta}{2^{j+1}}, \quad j \in \mathbb{N}_0. \quad (3.3.3)$$

We now set

$$v(x) = \sum_{j=0}^{\infty} u^j(x). \quad (3.3.4)$$

Notice that, for any $x \in \Omega$, only a finite number of terms of series (3.3.4) is different from 0. Moreover, as $u^j \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$, for every $j \in \mathbb{N}_0$, we have $v \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$.

Now, taking into account that

$$u = \sum_{j=0}^{\infty} \zeta_j u,$$

for any $h \in \mathbb{N}$, (3.3.3) and (3.3.4) give

$$\begin{aligned} \|v - u\|_{W^{k,p}(\Omega_h)} &= \left\| \sum_{j=0}^{\infty} u^j - \sum_{j=0}^{\infty} \zeta_j u \right\|_{W^{k,p}(\Omega_h)} \leq \\ &\leq \sum_{j=0}^{\infty} \|u^j - \zeta_j u\|_{W^{k,p}(\Omega_h)} \leq \\ &\leq \sum_{j=0}^{\infty} \|u^j - \zeta_j u\|_{W^{k,p}(\Omega)} \leq \\ &\leq \sum_{j=0}^{\infty} \frac{\delta}{2^{j+1}} = \delta. \end{aligned}$$

All in all, we have

$$\|v - u\|_{W^{k,p}(\Omega_h)} \leq \delta, \quad \forall h \in \mathbb{N}.$$

Hence

$$\|v - u\|_{W^{k,p}(\Omega)} = \lim_{h \rightarrow \infty} \|v - u\|_{W^{k,p}(\Omega_h)} \leq \delta.$$

Therefore, the sequence

$$u_m = \sum_{j=0}^m u^j(x), \quad m \in \mathbb{N},$$

satisfies the thesis. ■

Exercise. Prove Theorem 3.3.2 without the assumption that Ω is bounded.
[Hint: consider $\Omega_j \cap B_j(x_0)$, x_0 fixed point, instead of Ω_j .] ♣

The following Theorem holds true

Theorem 3.3.3 (C^∞ approximation to the boundary). *Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ with constants r_0, M_0 . Let $u \in W^{k,p}(\Omega)$, $1 \leq p < +\infty$. Then there exists a sequence of functions $\{u_j\} \subset C^\infty(\bar{\Omega})$ such that*

$$u_j \rightarrow u, \quad \text{as } j \rightarrow \infty, \text{ in } W^{k,p}(\Omega).$$

To prepare the proof of Theorem 3.3.3, we introduce some notations and we prove a Proposition,

Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ with constants r_0, M_0 . Let $x_0 \in \partial\Omega$. We may assume (up to isometry) that $x_0 = 0$ and

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > \varphi(x')\},$$

where $\varphi \in C^{0,1}(B'_{r_0})$ satisfies

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^{0,1}(\bar{B}'_{r_0})} \leq M_0 r_0.$$

Set

$$V = \Omega \cap Q_{\frac{r_0}{2}, \frac{M_0}{2}}.$$

Let y be any point of V , we look for what $\lambda > 0$ and $\varepsilon > 0$ we have (see Figure 3.2)

$$B_\varepsilon(y^\varepsilon) \subset \Omega \cap Q_{r_0, M_0}, \quad (3.3.5)$$

where

$$y^\varepsilon = y + \varepsilon \lambda e_n,$$

1. Let us check that if ε and λ satisfy

$$\varepsilon < \frac{r_0}{2}, \quad \varepsilon(1 + \lambda) < \frac{M_0 r_0}{2}, \quad (3.3.6)$$

then we have

$$B_\varepsilon(y^\varepsilon) \subset Q_{r_0, M_0}. \quad (3.3.7)$$

Since $B_\varepsilon(y^\varepsilon) \subset B'_\varepsilon((y^\varepsilon)') \times [y_n^\varepsilon - \varepsilon, y_n^\varepsilon + \varepsilon]$, we have that the first condition of (3.3.6) implies

$$B'_\varepsilon((y^\varepsilon)') \subset B'_{r_0} \quad (3.3.8)$$

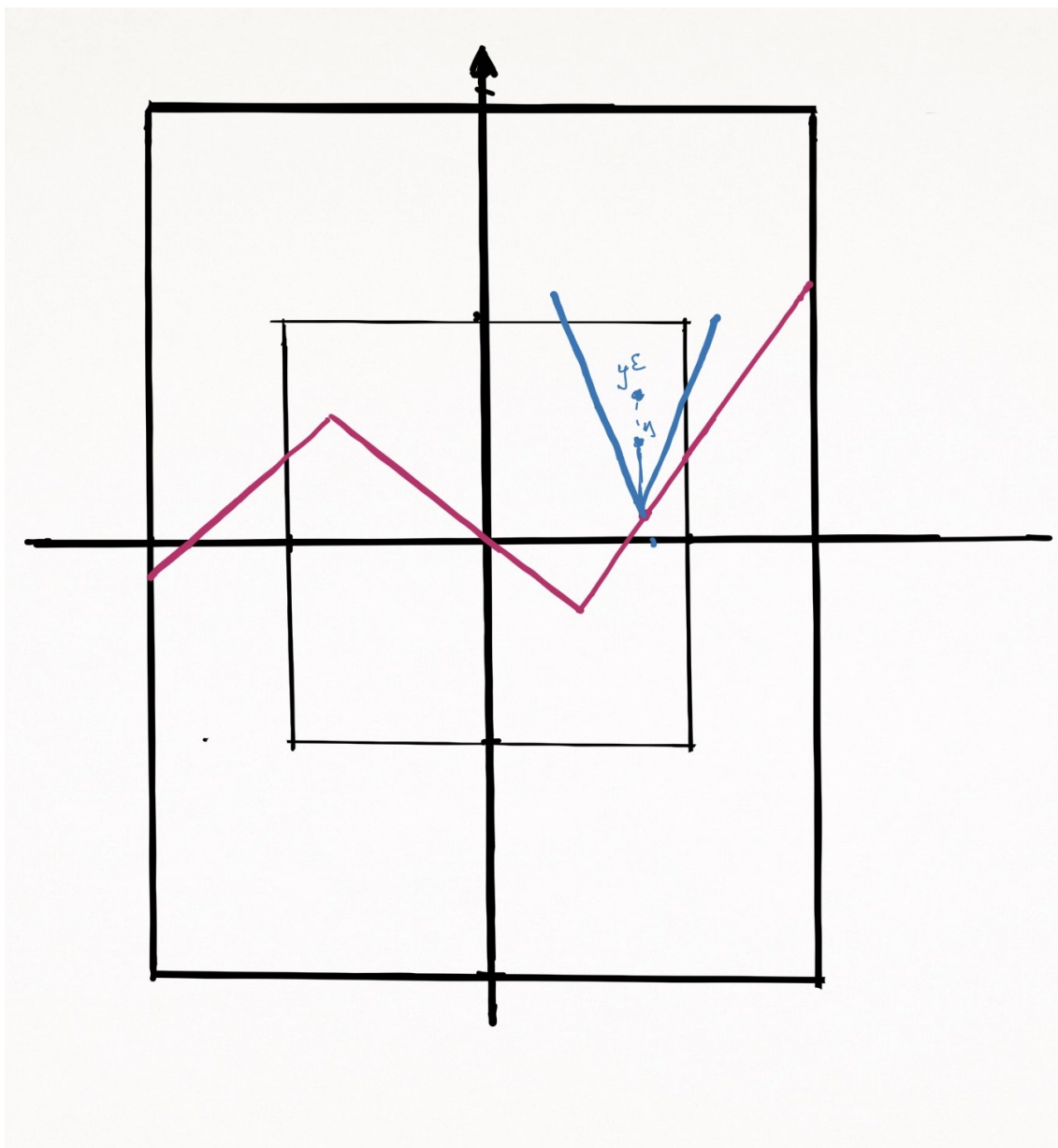


Figure 3.1:

and second condition of (3.3.6) implies

$$y_n^\varepsilon + \varepsilon \leq \frac{M_0 r_0}{2} + \varepsilon \lambda + \varepsilon < M_0 r_0$$

and, similarly,

$$y_n^\varepsilon - \varepsilon \geq -\frac{M_0 r_0}{2} + \varepsilon \lambda - \varepsilon > -M_0 r_0.$$

Hence

$$[y_n^\varepsilon - \varepsilon, y_n^\varepsilon + \varepsilon] \subset [-M_0 r_0, M_0 r_0],$$

which gives (3.3.7).

2. In order that $B_\varepsilon(y^\varepsilon) \subset \Omega \cap Q_{r_0, M_0}$, it suffices that, besides conditions (3.3.6), y^ε have a distance greater or equal to ε from the cone

$$x_n = M_0 |x' - y'| + \varphi(y').$$

Now, denoting by d_ε this distance, we have

$$\begin{aligned} d_\varepsilon &= \frac{|y_n + \varepsilon \lambda - \varphi(y')|}{\sqrt{1 + M_0^2}} = \\ &= \frac{y_n + \varepsilon \lambda - \varphi(y')}{\sqrt{1 + M_0^2}} \geq \\ &\geq \frac{\varepsilon \lambda}{\sqrt{1 + M_0^2}}. \end{aligned}$$

Hence, in order that $d_\varepsilon > \varepsilon$ it suffices that $\lambda > \sqrt{1 + M_0^2}$. Therefore, by choosing

$$\lambda = \lambda_0 := 2\sqrt{1 + M_0^2}$$

and by requiring that

$$\varepsilon < \varepsilon_0 := \min \left\{ \frac{r_0}{2}, \frac{M_0 r_0}{2}, \frac{M_0 r_0}{2\sqrt{1 + M_0^2}} \right\}$$

we obtain (3.3.5).

For any $u \in L^p(\Omega)$ and $\varepsilon < \varepsilon_0$ we set

$$u_\varepsilon(x) = u(x^\varepsilon) = u(x + \lambda_0 \varepsilon e_n), \quad \forall x \in V \quad (3.3.9)$$

and

$$v^\varepsilon(x) = \int_{B_\varepsilon(x^\varepsilon) \cap \Omega} \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) u(y) dy, \quad \forall x \in V, \quad (3.3.10)$$

where, we recall, $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(\varepsilon^{-1}x)$. Now, since $B_\varepsilon(x^\varepsilon) \subset \Omega \cap Q_{r_0, M_0}$, we have

$$\begin{aligned} v^\varepsilon(x) &= \int_{B_\varepsilon(x^\varepsilon)} \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) u(y) dy = \\ &= \int_{B_\varepsilon} \eta_\varepsilon(y) u(x + \lambda_0 \varepsilon e_n - y) dy, \quad \forall x \in V. \end{aligned} \quad (3.3.11)$$

The first equality in (3.3.11) gives

$$v^\varepsilon(x) = \int_{\Omega} \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) u(y) dy.$$

Hence

$$\partial^\alpha v^\varepsilon(x) = \int_{\Omega} \partial_x^\alpha \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) u(y) dy, \quad \forall x \in V \quad (3.3.12)$$

so that

$$\partial^\alpha v^\varepsilon \in C^\infty(\bar{V}).$$

Moreover, for any $u \in W^{k,p}(\Omega)$, we have

$$\begin{aligned} \partial^\alpha v^\varepsilon(x) &= \int_{\Omega} \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) \partial^\alpha u(y) dy = \\ &= \int_{B_\varepsilon(x^\varepsilon)} \eta_\varepsilon(x + \lambda_0 \varepsilon e_n - y) \partial^\alpha u(y) dy, \quad \forall x \in V, \end{aligned} \quad (3.3.13)$$

for every $|\alpha| \leq k$.

We have the following

Proposition 3.3.4. *If $u \in W^{k,p}(\Omega)$ and $p \in [1, +\infty)$ then*

$$v^\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } W^{k,p}(V).$$

Proof. First of all we prove

$$v^\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \text{ in } L^p(V).$$

The triangle inequality gives

$$\|v^\varepsilon - u\|_{L^p(V)} \leq \|u_\varepsilon - u\|_{L^p(V)} + \|v^\varepsilon - u_\varepsilon\|_{L^p(V)}. \quad (3.3.14)$$

Now

$$\|u_\varepsilon - u\|_{L^p(V)}^p = \int_V |u(x + \lambda_0 \varepsilon e_n) - u(x)|^p dx$$

and, by Theorem 2.3.27, we have

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L^p(V)} = 0. \quad (3.3.15)$$

Moreover, by the second equality in (3.3.11) we have, for any $x \in V$,

$$v^\varepsilon(x) - u_\varepsilon(x) = \int_{B_\varepsilon} \eta_\varepsilon(y) (u(x + \lambda_0 \varepsilon e_n - y) - u(x + \lambda_0 \varepsilon e_n)) dy.$$

In order to prove that the second term on the right hand side in (3.3.14) goes to 0 it suffices to repeat the same steps which provide the proof of Theorem 2.3.34. For completeness, let us repeat these steps.

$$\begin{aligned} \int_V |v^\varepsilon - u_\varepsilon|^p dx &\leq \int_V \left(\int_{B_\varepsilon} \eta_\varepsilon(y) |u(x + \lambda_0 \varepsilon e_n - y) - u(x)| dy \right)^p dx = \\ &= \int_V \left(\int_{B_\varepsilon} \eta_\varepsilon^{1/p'}(y) \eta_\varepsilon^{1/p}(y) |u(x + \lambda_0 \varepsilon e_n - y) - u(x)| dy \right)^p dx \leq \\ &\leq \int_V dx \left(\int_{B_\varepsilon} \eta_\varepsilon(y) dy \right)^{p/p'} \int_{B_\varepsilon} \eta_\varepsilon(y) |u(x + \lambda_0 \varepsilon e_n - y) - u(x)|^p dy = \\ &= \int_V dx \int_{B_\varepsilon} \eta_\varepsilon(y) |u(x + \lambda_0 \varepsilon e_n - y) - u(x)|^p dy = \\ &= \int_{B_\varepsilon} \left(\eta_\varepsilon(y) \int_V |u(x + \lambda_0 \varepsilon e_n - y) - u(x)|^p dx \right) dy \leq \\ &\leq \sup_{|y| \leq \varepsilon} \int_V |u(x + \lambda_0 \varepsilon e_n - y) - u(x)|^p dx. \end{aligned}$$

All in all, we have

$$\int_V |v^\varepsilon - u_\varepsilon|^p dx \leq \sup_{|y| \leq \varepsilon} \int_V |u(x + \lambda_0 \varepsilon e_n - y) - u(x)|^p dx.$$

Theorem 2.3.27 now yields

$$\lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - u_\varepsilon\|_{L^p(V)} = 0.$$

By the latter, by (3.3.14) and by (3.3.15) we have

$$\lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - u\|_{L^p(V)} = 0.$$

If $u \in W^{k,p}(\Omega)$, we obtain from what has been proven above and from (3.3.13)

$$\partial^\alpha v^\varepsilon \rightarrow \partial^\alpha u, \quad \text{per } \varepsilon \rightarrow 0, \text{ in } L^p(V), \text{ for } |\alpha| \leq k.$$

Hence

$$v^\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \text{ in } W^{k,p}(V).$$

■

Proof of Theorem 3.3.3. Let $x_0 \in \partial\Omega$, let us denote by $\tilde{Q}_{r_0, 2M_0}(x_0)$ the cylinder isometric to $Q_{r_0, 2M_0}$ such that (2.7.2) holds. As a consequence, $\left\{ \tilde{Q}_{\frac{r_0}{2}, \frac{M_0}{2}}(x_0) \right\}_{x_0 \in \partial\Omega}$ is an open covering of the compact set $\partial\Omega$. Let

$$\left\{ \tilde{Q}_{\frac{r_0}{2}, \frac{M_0}{2}}(x_i) \right\}_{1 \leq i \leq N}$$

be a finite subcovering of $\partial\Omega$. For any $1 \leq i \leq N$ and let us denote

$$V_i = \Omega \cap \tilde{Q}_{\frac{r_0}{2}, \frac{M_0}{2}}(x_i).$$

For any fixed $\delta > 0$ let $v_i \in C^\infty(\bar{V}_i)$ be the function constructed in (3.3.10) which satisfies

$$\|v_i - u\|_{W^{k,p}(V_i)} \leq \delta. \quad (3.3.16)$$

Moreover, let $V_0 \subset \Omega$ be such that

$$\Omega \subset \bigcup_{i=0}^N V_i$$

and let $\{\zeta_i\}_{0 \leq i \leq N}$ be a partition of unity (compare Theorem 2.4.3) which satisfies $\zeta_i \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } \zeta_i \subset V_i$ per $1 \leq i \leq N$ and

$$\sum_{i=0}^N \zeta_i(x) = 1, \quad \forall x \in \Omega.$$

Let us denote by $v_0 = \zeta_0 u$ and

$$v = \sum_{i=0}^N \zeta_i v_i.$$

We have $v \in C^\infty(\overline{\Omega})$ and, taking into account (3.3.16),

$$\begin{aligned} \|\partial^\alpha v - \partial^\alpha u\|_{L^p(\Omega)} &= \left\| \sum_{i=0}^N \partial^\alpha(\zeta_i v_i) - \sum_{i=0}^N \partial^\alpha(\zeta_i u) \right\|_{L^p(\Omega)} \leq \\ &\leq \sum_{i=0}^N \|\partial^\alpha(\zeta_i v_i) - \partial^\alpha(\zeta_i u)\|_{L^p(V_i)} \leq \\ &\leq C \sum_{i=0}^N \|v_i - u\|_{W^{k,p}(V_i)} \leq \\ &\leq CN\delta, \end{aligned}$$

for every $|\alpha| \leq k$.

Therefore, if $u \in W^{k,p}(\Omega)$ then for every $\eta > 0$ there exists $v \in C^\infty(\overline{\Omega})$ such that

$$\|v - u\|_{W^{k,p}(\Omega)} < \eta.$$

The Theorem is proved. ■

We conclude this Section with some propositions and exercises.

Proposition 3.3.5. *Let Ω be a connected open set of \mathbb{R}^n and let $u \in W_{loc}^{1,1}(\Omega)$ satisfy*

$$\nabla u = 0, \quad \text{in } \Omega,$$

then u is almost everywhere equal to a constant.

Proof. Let us first consider the case in which $\Omega = B_r$, $r > 0$, and $u \in W^{1,1}(B_r)$. Let δ be any number in $(0, r)$ and let $\varepsilon \in (0, \delta)$. Set

$$u_\varepsilon(x) = \int_{B_r} \eta_\varepsilon(x-y) u(y) dy.$$

By Theorem 3.3.1 we derive that $u_\varepsilon \in C^\infty(B_{r-\delta})$ and that, for any $x \in B_{r-\delta}$,

$$\nabla u_\varepsilon(x) = - \int_{B_r} \nabla(\eta_\varepsilon(x-y)) u(y) dy = \int_{B_r} \eta_\varepsilon(x-y) \nabla_y u(y) dy = 0.$$

Hence

$$u_\varepsilon(x) = C_\varepsilon, \quad \text{in } B_{r-\delta},$$

where C_ε is a constant which depends on ε . On the other hand

$$u_\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } L^1(B_{r-\delta}).$$

Since the limit, in $L^1(B_{r-\delta})$, of a sequence of constant functions is a constant function, we have $u = \tilde{C}_\delta$, almost everywhere in $B_{r-\delta}$, where \tilde{C}_δ is a constant. Trivially, \tilde{C}_δ does not depend on δ and, as δ is arbitrary in $(0, r)$, we have that u is constant almost everywhere in B_r .

Now, let us consider the general case and let us assume that $u \in W_{loc}^{1,1}(\Omega)$. Let $\bar{x} \in \Omega$ and $B_r(\bar{x}) \Subset \Omega$, for what proved before we have that there is $C \in \mathbb{R}$ such that

$$u = C, \quad \text{a.e. in } B_r(\bar{x}). \quad (3.3.17)$$

Let y be any point of Ω , $y \neq \bar{x}$. We prove that there exists $\rho > 0$ such that

$$u = C, \quad \text{a.e. in } B_\rho(y). \quad (3.3.18)$$

Since Ω is a connected open set, there exists a continuous path $\gamma : [0, 1] \rightarrow \Omega$, γ such that $\gamma(0) = \bar{x}$, $\gamma(1) = y$. Since $\gamma([0, 1])$ is a compact, we have

$$r_0 := \text{dist}(\gamma([0, 1]), \partial\Omega) > 0.$$

Moreover, let $\rho = \min\{r_0, r\}$, we can extract a finite subcovering by the open covering $\{B_\rho(x)\}_{x \in \gamma([0, 1])}$, of $\gamma([0, 1])$. Let $\{B_\rho(x_j)\}_{1 \leq j \leq N}$ be such a finite subcovering of $\gamma([0, 1])$, where $x_j \in \gamma([0, 1])$. It is not restrictive to assume $x_1 = \bar{x}$, $x_N = y$. For this purpose it suffices, eventually, to add to the family $\{B_\rho(x_j)\}_{1 \leq j \leq N}$, the balls $B_\rho(\bar{x})$ and $B_\rho(y)$ and, rearranging the remaining points x_2, \dots, x_{N-1} , we may assume that (as $\gamma([0, 1])$ is connected)

$$B_\rho(x_j) \cap B_\rho(x_{j+1}) \neq \emptyset, \quad j = 1, \dots, N-1. \quad (3.3.19)$$

In each ball $B_\rho(x_j)$, u is constant almost everywhere and, since $B_\rho(x_j) \cap B_\rho(x_{j+1})$ has positive measure, for $j = 1, \dots, N-1$, we have by (3.3.17) that $u = C$ almost everywhere in $B_\rho(x_j)$, $j = 1, \dots, N$. Therefore we obtain (3.3.18). ■

Proposition 3.3.6. *Let $F \in C^1(\mathbb{R})$ be such that F' is bounded. Let Ω be a bounded open set of \mathbb{R}^n and let $u \in W^{1,p}(\Omega)$, $p \in [1, +\infty)$. Let us denote*

$$v := F(u),$$

we have $v \in W^{1,p}(\Omega)$ and

$$\partial_j v = F'(u) \partial_j u, \quad j = 1, \dots, n.$$

Proof. Since F' is a bounded function and Ω is a bounded set, we have $v \in L^p(\Omega)$. As a matter of fact

$$|v| \leq |F(u) - F(0)| + |F(0)| \leq \|F'\|_{L^\infty(\mathbb{R})} |u| + |F(0)| \in L^p(\Omega).$$

Now, we apply Theorem 3.3.3 and let $\{u_m\} \subset C^\infty(\Omega) \cap W^{1,p}(\Omega)$ be a sequence such that

$$u_m \rightarrow u, \quad \text{as } m \rightarrow \infty, \text{ in } W^{1,p}(\Omega). \quad (3.3.20)$$

We have

$$F(u_m) \rightarrow F(u), \quad \text{as } m \rightarrow \infty, \text{ in } L^p(\Omega). \quad (3.3.21)$$

Concerning the latter we have

$$\lim_{m \rightarrow \infty} \int_{\Omega} |F(u_m) - F(u)|^p dx \leq \|F'\|_{L^\infty(\mathbb{R})}^p \lim_{m \rightarrow \infty} \int_{\Omega} |u_m - u|^p dx = 0.$$

Now let us check that

$$F'(u_m) \partial_j u_m \rightarrow F'(u) \partial_j u, \quad \text{as } m \rightarrow \infty, \text{ in } L^p(\Omega), \quad (3.3.22)$$

for $j = 1, \dots, n$.

We have

$$\begin{aligned} \|F'(u_m) \partial_j u_m - F'(u) \partial_j u\|_{L^p(\Omega)} &\leq \|F'(u_m) (\partial_j u_m - \partial_j u)\|_{L^p(\Omega)} + \\ &\quad + \|(F'(u_m) - F'(u)) \partial_j u\|_{L^p(\Omega)} \leq \\ &\leq \|F'\|_{L^\infty(\mathbb{R})} \|\partial_j u_m - \partial_j u\|_{L^p(\Omega)} + \\ &\quad + \|(F'(u_m) - F'(u)) \partial_j u\|_{L^p(\Omega)}. \end{aligned}$$

Since (3.3.20) holds, the second-to-last term on the right goes to zero as $m \rightarrow \infty$, concerning the last term, it goes to zero by the Dominated Convergence Theorem. Thus, we have checked (3.3.22).

Now, by (3.3.21) and (3.3.22) we have

$$\begin{aligned} \int_{\Omega} v \partial_j \phi dx &= \lim_{m \rightarrow \infty} \int_{\Omega} F(u_m) \partial_j \phi dx = \\ &= - \lim_{m \rightarrow \infty} \int_{\Omega} \partial_j (F(u_m)) \phi dx = \\ &= - \lim_{m \rightarrow \infty} \int_{\Omega} F'(u_m) \partial_j u_m \phi dx = \\ &= - \int_{\Omega} F'(u) \partial_j u \phi dx, \end{aligned}$$

for every $\phi \in C_0^\infty(\Omega)$ and every $j = 1, \dots, n$. Hence

$$\partial_j v = F'(u) \partial_j u \in L^p(\Omega), \quad j = 1, \dots, n$$

so that, taking into account that $v \in L^p(\Omega)$, we have $v \in W^{1,p}(\Omega)$. ■

Proposition 3.3.7. *Let Ω be a bounded open set of \mathbb{R}^n , let $u \in W^{1,p}(\Omega)$ and $p \in [1, +\infty)$. Let us denote by $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$. We have $u_+, u_- \in W^{1,p}(\Omega)$ and*

$$\nabla u_+ = \begin{cases} \nabla u, & \text{for } u > 0, \\ 0, & \text{for } u \leq 0, \end{cases} \quad (3.3.23)$$

$$\nabla u_- = \begin{cases} 0, & \text{for } u \geq 0, \\ -\nabla u, & \text{for } u < 0, \end{cases} \quad (3.3.24)$$

$$\nabla |u| = \begin{cases} \nabla u, & \text{for } u > 0, \\ 0, & \text{for } u = 0, \\ -\nabla u, & \text{for } u < 0. \end{cases} \quad (3.3.25)$$

Proof. Let us prove (3.3.23). For any $\varepsilon > 0$ let us define

$$f_\varepsilon(t) = \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases}$$

Recalling that $u \in W^{1,p}(\Omega)$, by the Dominated Convergence Theorem we get

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |f_\varepsilon(u) - u_+|^p dx = 0,$$

As a matter of fact we have

$$\lim_{\varepsilon \rightarrow 0} |f_\varepsilon(u) - u_+|^p = 0, \quad \text{in } \Omega$$

and

$$|f_\varepsilon(u) - u_+|^p \leq 2^p (|f_\varepsilon(u)|^p + |u_+|^p) \leq 2^{p+1} |u|^p \in L^1(\Omega).$$

Now, we have

$$f'_\varepsilon(t) = \begin{cases} \frac{t}{\sqrt{t^2 + \varepsilon^2}}, & \text{for } t > 0, \\ 0, & \text{for } t \geq 0 \end{cases}$$

and

$$|f'_\varepsilon(t)| \leq 1.$$

Hence, Proposition 3.3.6 implies

$$f_\varepsilon(u) \in W^{1,p}(\Omega).$$

Now

$$\partial_j f_\varepsilon(u) = \begin{cases} \frac{u \partial_j u}{\sqrt{u^2 + \varepsilon^2}}, & \text{for } u > 0, \\ 0, & \text{for } u \geq 0, \end{cases}$$

for $j = 1, \dots, n$. Hence, for any $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \int_{\Omega} u_+ \partial_j \phi dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} f_\varepsilon(u) \partial_j \phi dx = \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{u > 0} \frac{u \partial_j u}{\sqrt{u^2 + \varepsilon^2}} \phi dx = \\ &= - \int_{u > 0} \partial_j u \phi dx \end{aligned} \quad (3.3.26)$$

in the last step we have applied the Dominated Convergence Theorem. Therefore

$$\int_{\Omega} u_+ \partial_j \phi dx = - \int_{\Omega} \partial_j u \chi_{u > 0} \phi dx, \quad \forall \phi \in C_0^\infty(\Omega),$$

from which we get (3.3.23). Concerning (3.3.24), it suffices to notice that $u_- = (-u)_+$. All in all, (3.3.25) follows by (3.3.23) and (3.3.24) (recall that $|u| = u_+ + u_-$). ■

Exercise 1. We say that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise C^1 function, provided that f satisfies what follows: f is a continuous function, it has a continuous derivative in $\mathbb{R} \setminus \{a_1, \dots, a_l\}$, where $a_j \in \mathbb{R}$ and f has the right and the left derivatives in a_j , for $j = 1, \dots, l$ and such derivatives are finite.

Prove that if f is a piecewise C^1 function, $f' \in L^\infty(\mathbb{R})$, Ω is a bounded open set of \mathbb{R}^n and $u \in W^{1,p}(\Omega)$, $p \in [1, +\infty)$, then we have $f(u) \in W^{1,p}(\Omega)$ and

$$\nabla(f(u)) = \begin{cases} f'(u)\nabla u, & \text{for } u \notin \{a_1, \dots, a_l\}, \\ 0, & \text{for } u \in \{a_1, \dots, a_l\}. \end{cases}$$

[Hint: consider preliminarily the case $l = 1$ and, in doing so, first address to the case in which $f(0) = 0$; observe that

$$f(t) = \begin{cases} f_1(t), & \text{for } t > 0, \\ f_2(t), & \text{for } t \leq 0, \end{cases}$$

where $f_1, f_2 \in C^1(\mathbb{R})$ and $f'_1, f'_2 \in L^\infty(\mathbb{R})$. Let us note that $f(t) = f_1(t_+) + f_2(-t_-)$ and use Proposition 3.3.6 ...].

Exercise 2. Let Ω be an open set of \mathbb{R}^n and let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Prove that

$$uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$$

and

$$\nabla(uv) = v\nabla u + u\nabla v. \quad (3.3.27)$$

Solving. Let $\phi \in C_0^\infty(\Omega)$ and let V be an open set such that

$$\text{supp } \phi \subset V \Subset \Omega.$$

Let

$$u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)u(y)dy, \quad v_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)v(y)dy.$$

We have $(u_\varepsilon)|_V, (v_\varepsilon)|_V \in C^\infty(\bar{V})$ and

$$\begin{aligned} \int_\Omega uv\partial_j\phi dx &= \int_V uv\partial_j\phi dx = \\ &= \lim_{\varepsilon \rightarrow 0} \int_V u_\varepsilon v_\varepsilon \partial_j\phi dx = \\ &= - \lim_{\varepsilon \rightarrow 0} \int_V [(\partial_j u_\varepsilon) v_\varepsilon + u_\varepsilon \partial_j v_\varepsilon] \phi dx = \\ &= - \int_V [(\partial_j u) v + u \partial_j v] \phi dx, \end{aligned}$$

first limit is justified by the Dominated Convergence Theorem, the second limit is justified as follows

$$\begin{aligned} \|(\partial_j u_\varepsilon) v_\varepsilon \phi - (\partial_j u) v \phi\|_{L^1(V)} &\leq \|\phi\|_{L^\infty(V)} \|v\|_{L^\infty(V)} |V|^{1/p'} \|\partial_j u_\varepsilon - \partial_j u\|_{L^p(V)} + \\ &\quad + \|\phi\|_{L^\infty(V)} \|\partial_j u\|_{L^p(V)} \|v_\varepsilon - v\|_{L^{p'}(V)} \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \end{aligned}$$

similarly we argue for $\|(\partial_j v_\varepsilon) u_\varepsilon \phi - (\partial_j v) u \phi\|_{L^1(V)}$. Hence, we get

$$\partial_j(uv) = (\partial_j v) u + (\partial_j u) v. \quad (3.3.28)$$

Since (by Hölder inequality) $(\partial_j v) u + (\partial_j u) v \in L^p(\Omega)$, we get (3.3.27). ♣

3.4 The extension theorems

Let us start by some propositions about the space $W_0^{k,p}(\Omega)$.

Proposition 3.4.1. *If $k \in \mathbb{N}$ and $p \in [1, +\infty)$ then*

$$W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n).$$

Proof. We limit ourselves to the case $k = 1$, the case $k > 1$ can be proved in a similar way and is left to the reader.

Let $R > 1$ and let $\zeta_R \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$\begin{aligned} 0 &\leq \zeta_R \leq 1, \quad \text{in } \mathbb{R}^n, \\ \zeta_R(x) &= 1, \quad \forall x \in B_R; \quad \zeta_R(x) = 0, \quad \forall x \in \mathbb{R}^n \setminus B_{2R}, \\ |\nabla \zeta_R| &\leq C, \quad \text{in } \mathbb{R}^n, \end{aligned}$$

where C is independent of R .

If $u \in W^{1,p}(\mathbb{R}^n)$, we have

$$\|u - \zeta_R u\|_{W^{1,p}(\mathbb{R}^n)} \rightarrow 0, \quad \text{as } R \rightarrow \infty. \quad (3.4.1)$$

Let us check (3.4.1).

$$\|u - \zeta_R u\|_{L^p(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n \setminus B_R)} \rightarrow 0, \quad \text{as } R \rightarrow \infty$$

and, for any $j = 1, \dots, n$,

$$\begin{aligned} \|\partial_j u - \partial_j(\zeta_R u)\|_{L^p(\mathbb{R}^n)} &= \|(1 - \zeta_R) \partial_j u - u \partial_j \zeta_R\|_{L^p(\mathbb{R}^n)} \leq \\ &\leq \|\partial_j u\|_{L^p(\mathbb{R}^n \setminus B_R)} + C \|u\|_{L^p(\mathbb{R}^n \setminus B_R)} \rightarrow 0, \quad \text{as } R \rightarrow \infty. \end{aligned}$$

In order to complete the proof, firstly we observe (by Theorem 2.3.35)

$$(\zeta_R u) \star \eta_\varepsilon \rightarrow \zeta_R u, \quad \text{as } \varepsilon \rightarrow 0, \text{ in } W^{1,p}(\mathbb{R}^n), \quad (3.4.2)$$

where η_ε a mollifier. Moreover, let δ be any positive number, let $R_0 > 1$ be such that

$$\|u - \zeta_{R_0} u\|_{W^{1,p}(\mathbb{R}^n)} < \frac{\delta}{2}$$

and let $\varepsilon_0 > 0$ be such that

$$\|(\zeta_{R_0} u) \star \eta_{\varepsilon_0} - \zeta_{R_0} u\|_{W^{1,p}(\mathbb{R}^n)} < \frac{\delta}{2}.$$

From the last two inequalities and the triangle inequality we get

$$\|(\zeta_{R_0} u) \star \eta_{\varepsilon_0} - u\|_{W^{1,p}(\mathbb{R}^n)} < \delta.$$

Since $(\zeta_{R_0} u) \star \eta_{\varepsilon_0} \in C_0^\infty(\mathbb{R}^n)$, the Proposition is proved. ■

Theorem 3.4.2 (The first Poincaré inequality). *Let Ω be a bounded open set of \mathbb{R}^n . Let $p \in [1, +\infty]$, $k \in \mathbb{N}$. The following inequality holds true, for any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k - 1$,*

$$\|\partial^\alpha u\|_{L^p(\Omega)} \leq C d^{k-|\alpha|} \sum_{|\beta|=k} \|\partial^\beta u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{k,p}(\Omega), \quad (3.4.3)$$

where d is the diameter of Ω and C depends on n and k only.

Proof. We restrict ourselves to the case $k = 1$, actually starting from this case (3.4.3) can easily be deduced by induction. It is not restrictive to assume $0 \in \Omega$ and

$$\Omega \subset [-d, d]^n.$$

Let $u \in W_0^{1,p}(\Omega)$. Let $\{u_j\}$ be a sequence in $C_0^\infty(\Omega)$ such that

$$\{u_j\} \rightarrow u, \quad \text{in } W^{1,p}(\Omega).$$

For any $p \in [1, +\infty)$ and any $j \in \mathbb{N}$, we have

$$\begin{aligned} |u_j(x)| &= |u_j(x) - u_j(x', -d)| = \\ &= \left| \int_{-d}^{x_n} \partial_y u_j(x', y) dy \right| \leq \\ &\leq \int_{-d}^d |\partial_y u_j(x', y)| dy \leq \\ &\leq (2d)^{1/p'} \left(\int_{-d}^d |\partial_y u_j(x', y)|^p dy \right)^{1/p}. \end{aligned}$$

Hence

$$|u_j(x)|^p \leq (2d)^{p-1} \int_{-d}^d |\partial_y u_j(x', y)|^p dy. \quad (3.4.4)$$

Let us integrate both the sides of (3.4.4) over $[-d, d]$ w.r.t. x_n . We get

$$\int_{-d}^d |u_j(x', x_n)|^p dx_n \leq (2d)^p \int_{-d}^d |\partial_y u_j(x', y)|^p dy.$$

Now, let us integrate both the sides of the last inequality over $[-d, d]^{n-1}$. We get

$$\left(\int_{\Omega} |u_j(x)|^p dx \right)^{1/p} \leq 2d \left(\int_{-d}^d |\partial_{x_n} u_j(x)|^p dx \right)^{1/p}.$$

Passing to the limit as $j \rightarrow \infty$, we obtain

$$\|u\|_{L^p(\Omega)} \leq Cd \|\nabla u\|_{L^p(\Omega)}.$$

If $p = +\infty$, then we have

$$|u_j(x)| = |u_j(x) - u_j(x', -d)| \leq 2d \|\partial_{x_n} u_j\|_{L^\infty(\Omega)},$$

from which, passing to the limit as $j \rightarrow \infty$, we have

$$\|u\|_{L^\infty(\Omega)} \leq Cd \|\nabla u\|_{L^\infty(\Omega)}.$$

■

Remarks.

1. From the proof of Proposition 3.4.2 it is evident that inequality (3.4.3) also holds if Ω is contained in a strip of \mathbb{R}^n of the type $\mathbb{R}^{n-1} \times [-d, d]$ or isometric to it.

2. Proposition 3.4.2 implies that, the following seminorms are actually norms on $W_0^{k,p}(\Omega)$

$$\sum_{|\beta|=k} \|\partial^\beta u\|_{L^p(\Omega)}, \quad \left(\sum_{|\beta|=k} \|\partial^\beta u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Moreover such norms are equivalent to the norm

$$\|u\|_{W^{k,p}(\Omega)}.$$

◆

Proposition 3.4.3. *Let $p \in [1, +\infty]$, $k \in \mathbb{N}$ and let Ω and $\tilde{\Omega}$ be open sets of \mathbb{R}^n such that $\Omega \subset \tilde{\Omega}$. Let $u \in W_0^{k,p}(\Omega)$.*

Denoting

$$\tilde{u} = \begin{cases} u, & \text{in } \Omega, \\ 0, & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases} \quad (3.4.5)$$

we have $\tilde{u} \in W_0^{k,p}(\tilde{\Omega})$.

Proof. Let $u \in W_0^{k,p}(\Omega)$ and let $\{u_j\} \subset C_0^\infty(\Omega)$ be a sequence such that

$$u_j \rightarrow u, \quad \text{as } j \rightarrow \infty, \text{ in } W^{k,p}(\Omega).$$

Hence, by denoting

$$\tilde{u}_j = \begin{cases} u_j, & \text{in } \Omega, \\ 0, & \text{in } \tilde{\Omega} \setminus \Omega, \end{cases} \quad (3.4.6)$$

we have $\{\tilde{u}_j\} \subset C_0^\infty(\tilde{\Omega})$ and

$$\tilde{u}_j \rightarrow \tilde{u}, \quad \text{as } j \rightarrow \infty, \text{ in } W^{k,p}(\tilde{\Omega}).$$

Therefore $\tilde{u} \in W_0^{k,p}(\tilde{\Omega})$. ■

The Main Theorem of the present Section is the following one.

Theorem 3.4.4 (extension in $W^{1,p}$). *Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ with constants r_0, M_0 . Let d_0 be the diameter of Ω . Let $\tilde{\Omega}$ be an open set of \mathbb{R}^n such that $\Omega \Subset \tilde{\Omega}$ and let $p \in [1, +\infty)$.*

Then there exists a linear bounded operator

$$E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n), \quad (3.4.7)$$

which satisfies, for any $u \in W^{1,p}(\Omega)$,

$$Eu = u, \quad \text{in } \Omega, \quad (3.4.8)$$

$$\text{supp}(Eu) \subset \tilde{\Omega}. \quad (3.4.9)$$

Moreover, there exists a constant C depending on r_0 , M_0 , d_0 , n and p only such that

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \quad (3.4.10)$$

Proof. Let $x_0 \in \partial\Omega$. We may assume (up to an isometry) that $x_0 = 0$ and

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > \varphi(x')\},$$

where $\varphi \in C^1(\overline{B'_{r_0}})$ satisfies

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^1(\overline{B'_{r_0}})} = \|\varphi\|_{C^0(\overline{B'_{r_0}})} + r_0[\varphi]_{0,1,B'_{r_0}} \leq M_0 r_0.$$

Set

$$V^+ = Q_{\frac{r_0}{4}, \frac{M_0}{4}} \cap \Omega, \quad V^- = Q_{\frac{r_0}{4}, \frac{M_0}{4}} \setminus \overline{V^+}.$$

Notice that, for every $x' \in \overline{B'_{r_0/4}}$ we have

$$|\varphi(x')| = |\varphi(x') - \varphi(0)| \leq [\varphi]_{0,1,B'_{r_0}} |x'| \leq \frac{M_0 r_0}{4}. \quad (3.4.11)$$

First, we assume that $u \in C^\infty(\overline{\Omega})$ and we define

$$\nu(x') = \frac{(\nabla\varphi(x'), -1)}{\sqrt{|\nabla_{x'}\varphi|^2 + 1}}, \quad x' \in B'_{r_0},$$

$$\bar{u}(x) = \begin{cases} u(x), & \text{in } V^+, \\ v(x), & \text{in } V^-, \end{cases}$$

where

$$v(x) = u(x', 2\varphi(x') - x_n).$$

Claim. $\bar{u} \in W^{1,p}(Q_{\frac{r_0}{4}, \frac{M_0}{4}})$ and

$$\|\bar{u}\|_{W^{1,p}(Q_{\frac{r_0}{4}, \frac{M_0}{4}})} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad (3.4.12)$$

where C depends on M_0 only.

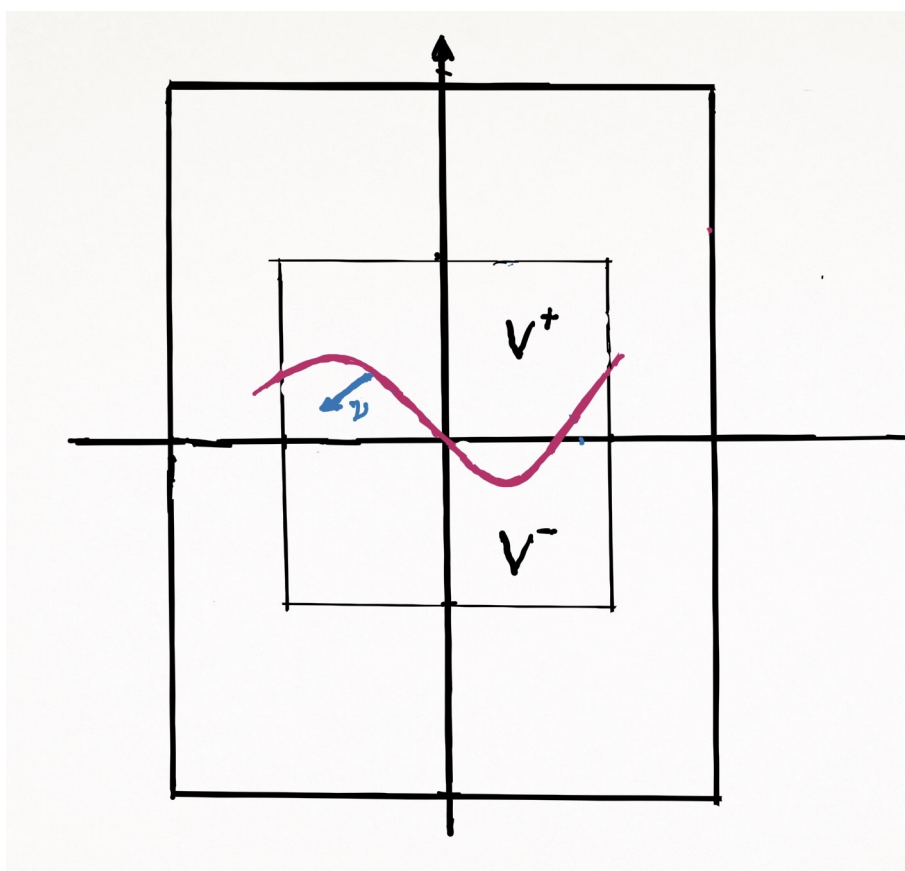


Figure 3.2:

Proof of Claim. Let $\Phi \in C_0^\infty\left(Q_{\frac{r_0}{4}, \frac{M_0}{4}}\right)$ and $1 \leq i \leq n$. Denoting by Γ the graph of $\varphi|_{B'_{r_0/4}}$, by the divergence Theorem we get

$$\begin{aligned} \int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} \bar{u} \partial_i \Phi dx &= \int_{V^+} u \partial_i \Phi dx + \int_{V^-} v \partial_i \Phi dx = \\ &= - \int_{V^+} \partial_i u \Phi dx + \int_{\Gamma} u \Phi (\nu \cdot e_i) dS - \\ &\quad - \int_{V^-} \partial_i v \Phi dx - \int_{\Gamma} v \Phi (\nu \cdot e_i) dS = \\ &= - \int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} w_i \Phi dx + \int_{\Gamma} (u - v) \Phi (\nu \cdot e_i) dS, \end{aligned}$$

where

$$w_i(x) = \begin{cases} \partial_i u(x), & \text{in } V^+, \\ \partial_i v(x), & \text{in } V^-. \end{cases} \quad (3.4.13)$$

On the other hand,

$$(u - v)(x', \varphi(x')) = 0, \quad \forall x \in B'_{r_0/4},$$

hence

$$\int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} \bar{u} \partial_i \Phi dx = - \int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} w_i \Phi dx, \quad \forall \Phi \in C_0^\infty\left(Q_{\frac{r_0}{2}, M_0}\right). \quad (3.4.14)$$

Therefore

$$\partial_i \bar{u} = w_i(x), \quad \forall x \in Q_{\frac{r_0}{4}, \frac{M_0}{4}}. \quad (3.4.15)$$

Now, let us notice that

$$\int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} |\bar{u}(x)|^p dx = \int_{V^+} |u(x)|^p dx + \int_{V^-} |v(x)|^p dx \quad (3.4.16)$$

and

$$\begin{aligned}
\int_{V^-} |v(x)|^p dx &= \int_{B'_{r_0/4}} dx' \int_{-\frac{M_0 r_0}{4}}^{\varphi(x')} |u(x', 2\varphi(x') - x_n)|^p dx_n = \\
&= \int_{B'_{r_0/4}} dx' \int_{\varphi(x')}^{2\varphi(x') + \frac{M_0 r_0}{4}} |u(x', \xi_n)|^p d\xi_n \leq \\
&\leq \int_{\Omega} |u(x)|^p dx.
\end{aligned} \tag{3.4.17}$$

In the last inequality we have used (3.4.11). By (3.4.16) and (3.4.17), we have,

$$\int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} |\bar{u}(x)|^p dx \leq 2 \int_{\Omega} |u(x)|^p dx. \tag{3.4.18}$$

Now, (3.4.13) gives

$$|\nabla \bar{u}(x)| \leq C |(\nabla u)(x', 2\varphi(x') - x_n)|, \quad \forall x \in V^-,$$

where C depends on M_0 only. Hence

$$\begin{aligned}
\int_{Q_{\frac{r_0}{4}, \frac{M_0}{4}}} |\nabla \bar{u}(x)|^p dx &\leq \int_{V^+} |\nabla u(x)|^p dx + C \int_{V^-} |(\nabla u)(x', 2\varphi(x') - x_n)|^p dx \leq \\
&\leq C \int_{\Omega} |\nabla u(x)|^p dx.
\end{aligned}$$

From the just obtained inequality e from (3.4.18) we obtain (3.4.12). Claim is proved.

Since $\partial\Omega$ is a compact set, there exist $x_{0,1}, \dots, x_{0,N} \in \partial\Omega$ such that

$$\partial\Omega \subset \bigcup_{j=1}^N \tilde{Q}_{\frac{r_0}{4}, \frac{M_0}{4}}(x_{0,j})$$

where, for any $j = 1, \dots, N$, $\tilde{Q}_{\frac{r_0}{4}, \frac{M_0}{4}}(x_{0,j})$ are suitable cylinders which are isometric to $Q_{\frac{r_0}{4}, \frac{M_0}{4}}$. Moreover, let us denote \bar{u}_j the extensions of u on $\tilde{Q}_{\frac{r_0}{4}, \frac{M_0}{4}}(x_{0,j})$. Let us employ the partition of unity (Lemma 2.4.3). Set $V_j = \tilde{Q}_{\frac{r_0}{4}, \frac{M_0}{4}}(x_{0,j})$, $j = 1, \dots, N$, we have that there exist $\zeta_0, \zeta_1, \dots, \zeta_N \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \zeta_j(x) \leq 1, \quad j = 1, \dots, N, \quad \forall x \in \mathbb{R}^n$$

$$\text{supp } \zeta_j \subset V_j, \quad j = 1, \dots, N, \quad \text{supp } \zeta_0 \subset \mathbb{R}^n \setminus \partial\Omega,$$

$$\sum_{j=0}^N \zeta_j(x) = 1, \quad \forall x \in \mathbb{R}^n,$$

and

$$\sum_{j=1}^N \zeta_j(x) = 1, \quad \text{for every } x \text{ in a neighborhood of } \partial\Omega.$$

Be, also, $\eta \in C_0^\infty(\tilde{\Omega})$, such that $0 \leq \eta \leq 1$, $\eta(x) = 1$, for $x \in \Omega$. Set

$$\tilde{u} = \eta \left(\zeta_0 u + \sum_{j=1}^N \zeta_j \bar{u}_j \right),$$

By (3.4.12) and by the triangle inequality, we have

$$\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad (3.4.19)$$

where C depends on r_0 , n and M_0 only. Moreover, we have

$$\tilde{u}(x) = u(x), \quad \forall x \in \Omega, \quad (3.4.20)$$

$$\text{supp } \tilde{u} \subset \tilde{\Omega}. \quad (3.4.21)$$

Now, let us denote

$$Eu := \tilde{u}, \quad \forall u \in C^\infty(\bar{\Omega}).$$

E is a linear operator and satisfies the inequality

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}. \quad (3.4.22)$$

Now, let $u \in W^{1,p}(\Omega)$ and apply Theorem 3.3.3. Let therefore be $\{u_m\} \subset C^\infty(\bar{\Omega})$ such that

$$\{u_m\} \rightarrow u, \quad \text{in } W^{1,p}(\Omega).$$

We have, by (3.4.22),

$$\|Eu_m - Eu_{m'}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u_m - u_{m'}\|_{W^{1,p}(\Omega)}.$$

Hence $\{Eu_m\}$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ consequently it converges to a function which we continue to denote by $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ which satisfies trivially (3.4.19)–(3.4.21). ■

We merely state, with some comments, the Theorem of extension for $W^{k,p}(\Omega)$, where $k \geq 1$. We refer to [28] for a proof.

Theorem 3.4.5 (extension in $W^{k,p}$). *Let $k \geq 1$ and $p \in [1, +\infty)$. Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{k-1,1}$ with constants r_0, M_0 . Let d_0 be the diameter of Ω . Let $\tilde{\Omega}$ an open set of \mathbb{R}^n such that $\Omega \Subset \tilde{\Omega}$.*

Then there exists a bounded linear operator

$$E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n),$$

such that, for any $u \in W^{k,p}(\Omega)$ we have

$$Eu = u, \quad \text{on } \Omega,$$

$$\text{supp}(Eu) \subset \tilde{\Omega}.$$

Moreover, there exists a constant C depending on r_0, M_0, d_0, n, k and p only, such that

$$\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{k,p}(\Omega)}, \quad \forall u \in W^{k,p}(\Omega). \quad (3.4.23)$$

Exercise 1. (i) Let $u \in C^\infty(\overline{B_r^+})$, where $B_r^\pm = \{x \in \mathbb{R}^n : |x| < r, x_n \gtrless 0\}$. Let us define

$$\bar{u}(x) = \begin{cases} u(x), & \text{in } B_r^+, \\ v(x), & \text{in } B_r^-, \end{cases}$$

where

$$v(x) = -3u(x', -x_n) + 4u\left(x', -\frac{x_n}{2}\right).$$

Prove that, if $p \in [1, +\infty)$ then $\bar{u}(x) \in W^{2,p}(B_r)$ and the following inequality holds true

$$\|\bar{u}\|_{W^{2,p}(B_r)} \leq C \|u\|_{W^{2,p}(\Omega)}, \quad \forall u \in W^{2,p}(B_r^+).$$

(ii) Let us define

$$C^\infty(\overline{B_r^+}) \ni u \rightarrow Eu = \bar{u} \in W^{2,p}(B_r).$$

Prove that the operator E can be extended to $W^{2,p}(B_r^+)$ and that it satisfies $Eu = u$ in B_r^+ .

(iii) Let $k \in \mathbb{N}$ and let c_1, \dots, c_k be such that

$$\sum_{j=1}^k c_j \left(-\frac{1}{j}\right)^m = 1, \quad m = 0, 1, \dots, k-1,$$

(check that such c_1, \dots, c_k exist); let us define for any $u \in C^\infty(\overline{B_r^+})$

$$\bar{u}(x) = \begin{cases} u(x), & \text{in } B_r^+, \\ w(x), & \text{in } B_r^-, \end{cases}$$

where

$$w(x) = \sum_{j=1}^k c_j u\left(x', -\frac{x_n}{j}\right).$$

Prove that if $p \in [1, +\infty)$ then $\bar{u}(x) \in W^{k,p}(B_r)$ and the following inequality holds true

$$\|\bar{u}\|_{W^{k,p}(B_r)} \leq C \|u\|_{W^{k,p}(B_r^+)}, \quad \forall u \in W^{k,p}(B_r^+).$$

Moreover, deduce that the following operator

$$C^\infty(\overline{B_r^+}) \ni u \rightarrow E_k u = \bar{u} \in W^{k,p}(B_r)$$

can be extended to $W^{k,p}(B_r)$ and $E_k u = u$ in B_r^+ .

3.5 Traces in $W^{1,p}(\Omega)$

It is well-known that if $u \in C^0(\overline{\Omega})$, then we can define its trace on $\partial\Omega$, namely $u|_{\partial\Omega}$. If, on the other hand, $u \in L^p(\Omega)$, generally, it does not make sense to consider its trace on $\partial\Omega$. In the present Section we will see that we can define a notion of trace that extends the known one for the functions of $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$.

More precisely we have

Theorem 3.5.1 (trace Theorem). *Let Ω be a bounded open set of class $C^{0,1}$ with constants r_0 and M_0 . Let $p \in [1, +\infty)$. Let d_0 be the diameter of Ω . Then there exists a unique bounded linear operator*

$$T : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega),$$

which satisfies:

(i) $T(u) = u|_{\partial\Omega}$ for every $u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega)$;

(ii)

$$\|T(u)\|_{L^p(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),$$

where C depends by r_0 , M_0 , d_0 and p ;

(iii)

$$\int_{\Omega} u \operatorname{div} \Phi dx = - \int_{\Omega} \nabla u \cdot \Phi dx + \int_{\partial\Omega} (\Phi \cdot \nu) T u dS,$$

for every $u \in W^{1,p}(\Omega)$ and for every $\Phi \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

The function Tu is called the **trace** of u on $\partial\Omega$.

Proof. We first notice that, since $C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega)$ (Theorem 3.3.2) and Φ is arbitrary in $C^1(\overline{\Omega}, \mathbb{R}^n)$, if T there exists, then it is unique.

Let us prove the existence of T . First, let us consider the case where $u \in C^\infty(\overline{\Omega})$. Since $\partial\Omega$ is compact, we may consider a partition of unity subordinate to a finite covering $\{V_j\}_{1 \leq j \leq N}$, where $V_j = \tilde{Q}_{\frac{r_0}{2}, M_0}(x_{0,j})$, $j = 1, \dots, N$ ($\tilde{Q}_{\frac{r_0}{2}, M_0}(x_{0,j})$ is a cylinder isometric to $Q_{\frac{r_0}{2}, M_0}$)

$$\zeta_j \in C^\infty(\mathbb{R}^n), \quad 0 \leq \zeta_j \leq 1, \quad \operatorname{supp} \zeta_j \subset V_j, \quad \sum_{j=1}^n \zeta_j = 1 \text{ on } \partial\Omega.$$

Let $j \in \{1, \dots, N\}$ be fixed. Up to isometries we may assume $V_j = Q_{\frac{r_0}{2}, M_0}$ and

$$Q_{r_0, M_0} \cap \Omega = \{(x', x_n) \in Q_{r_0, M_0} : x_n > \varphi(x')\},$$

where $\varphi \in C^1(\overline{B'_{r_0}})$ satisfies $\varphi(0) = 0$ and

$$\|\varphi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla \varphi\|_{L^\infty(B'_{r_0})} \leq M_0 r_0.$$

Let $v = \zeta_j u$. For any $t \in [0, \frac{M_0 r_0}{2}]$ and any $x' \in B'_{r_0/2}$ we have

$$v(x', \varphi(x')) = v(x', \varphi(x') + t) - \int_{\varphi(x')}^{\varphi(x') + t} \partial_{x_n} v(x', x_n) dx_n.$$

Hölder inequality gives

$$|v(x', \varphi(x'))|^p \leq 2^{p-1} |v(x', \varphi(x') + t)|^p + 2^{p-1} t^{p-1} \int_{\varphi(x')}^{M_0 r_0} |\partial_{x_n} v(x', x_n)|^p dx_n$$

and if we integrate with respect to x_n over $[0, \frac{M_0 r_0}{2}]$ both the sides of the last inequality we get

$$\begin{aligned} \frac{M_0 r_0}{2} |v(x', \varphi(x'))|^p &\leq 2^{p-1} \int_{\varphi(x')}^{\varphi(x') + \frac{M_0 r_0}{2}} |v(x', x_n)|^p dx_n + \\ &+ \frac{2^{p-1}}{p} \left(\frac{M_0 r_0}{2} \right)^p \int_{\varphi(x')}^{M_0 r_0} |u \partial_{x_n} \zeta_j + \zeta_j \partial_{x_n} u|^p dx_n. \end{aligned}$$

Now, we multiply both the sides of the last inequality by $\sqrt{1 + |\nabla_{x'} \varphi(x')|^2}$, and we integrate over $B'_{r_0/2}$ obtaining

$$\int_{\partial\Omega} |u \zeta_j|^p dS \leq C \int_{\Omega} |u|^p dx + C \int_{\Omega} |\nabla u|^p dx,$$

where C depends by M_0 e r_0 . Therefore

$$\begin{aligned} \|u\|_{L^p(\partial\Omega)} &= \left\| \sum_{j=1}^N \zeta_j u \right\|_{L^p(\partial\Omega)} \leq \\ &\leq \sum_{j=1}^N \|\zeta_j u\|_{L^p(\partial\Omega)} \leq \\ &\leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in C^\infty(\overline{\Omega}). \end{aligned} \tag{3.5.1}$$

Set

$$Tu = u|_{\partial\Omega}, \quad \forall u \in C^\infty(\overline{\Omega}).$$

Inequality (3.5.1) implies

$$\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in C^\infty(\overline{\Omega}). \tag{3.5.2}$$

Let now $u \in W^{1,p}(\Omega)$. From Theorem 3.3.3 we have that there exists a sequence $\{u_m\}$ in $C^\infty(\overline{\Omega})$ such that

$$\{u_m\} \rightarrow u, \quad \text{in } W^{1,p}(\Omega).$$

In particular, (3.5.2) implies that $\{Tu_m\}$ is a Cauchy sequence in $L^p(\partial\Omega)$. Set

$$Tu = \lim_{m \rightarrow \infty} Tu_m.$$

Now, we observe that if $u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega)$, then the sequence $\{u_m\}$ constructed in the proof of Proposition 3.3.4 (with $\varepsilon = 1/m$) uniformly converges to u . Hence, for any $u \in C^0(\overline{\Omega}) \cap W^{1,p}(\Omega)$, we have

$$Tu = \lim_{m \rightarrow \infty} Tu_m = \lim_{m \rightarrow \infty} u_m = u, \quad \text{in } L^p(\partial\Omega).$$

Therefore, we have proved (i) and (ii), now let us prove (iii). Let $u \in W^{1,p}(\Omega)$ and let $\{u_m\}$ be a sequence in $C^\infty(\bar{\Omega})$ which converges to u in $W^{1,p}(\Omega)$. We have, for any $\Phi \in C^1(\bar{\Omega}, \mathbb{R}^n)$,

$$\begin{aligned} \int_{\Omega} u \operatorname{div} \Phi dx &= \lim_{m \rightarrow \infty} \int_{\Omega} u_m \operatorname{div} \Phi dx = \\ &= \lim_{m \rightarrow \infty} \left(- \int_{\Omega} \nabla u_m \cdot \Phi dx + \int_{\partial\Omega} (\Phi \cdot \nu) u_m dS \right) = \\ &= - \int_{\Omega} \nabla u \cdot \Phi dx + \int_{\partial\Omega} (\Phi \cdot \nu) T u dS. \end{aligned}$$

■

Remark 3.5.2. If $u \in W_0^{1,p}(\Omega)$ then $Tu = 0$. Actually, under the same assumption of Theorem 3.5.1 the conversely is also valid, but here we omit the proof and refer to Theorem 2 of Ch. 5 of [23]. ♦

The issue of traces will be taken up in Section 3.12.

3.6 The Sobolev spaces of function of one variable

In the present Section we will dwell briefly on the Sobolev spaces in the case where the space dimension is equal to 1. Let us observe that if $I \subset \mathbb{R}$ is a bounded open interval, then the theorems proved in the previous sections remain valid: it is certainly a useful exercise (left to the reader) to adapt the proofs of these theorems and observe that they turn out to be simplified with respect to the general case. In particular, by Theorem 3.3.3 we have that $C^\infty(\bar{I})$ is dense in $W^{k,p}(I)$, for every $p \in [1, +\infty)$ and, by the extension Theorem 3.4.4 it turns out that if $\tilde{I} \ni I$, where \tilde{I} is an open interval of \mathbb{R} , then there exists a bounded linear operator

$$E : W^{1,p}(I) \rightarrow W^{k,p}(\tilde{I})$$

which satisfies

$$\|Eu\|_{W^{1,p}(\tilde{I})} \leq \|u\|_{W^{1,p}(I)}, \quad \forall u \in W^{1,p}(I),$$

$$Eu|_I = u, \quad \text{supp}(Eu) \subset I.$$

Now we investigate the basic relations between the absolutely continuous functions and the functions of $W^{1,p}(I)$. By the above mentioned extension Theorem, we may consider the space $W^{1,p}(\mathbb{R})$, instead of $W^{1,p}(I)$.

Let us recall that if $u \in L^1_{loc}(\mathbb{R})$ then

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} |u(t) - u(x)| dt = 0, \quad \text{a.e. } x \in \mathbb{R}, \quad (3.6.1)$$

and

$$\lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} u(t) dt = u(x), \quad \text{a.e. } x \in \mathbb{R}. \quad (3.6.2)$$

When (3.6.2) holds true in x , we say that x is a **Lebesgue point** of u . For any $c \in \mathbb{R}$ we have

$$\mathbb{R} \ni x \rightarrow \int_c^x u(t) dt \in AC_{loc}(\mathbb{R}) \quad (3.6.3)$$

and

$$\left(\int_c^x u(t) dt \right)' = u(x), \quad \text{a.e. } x \in \mathbb{R}. \quad (3.6.4)$$

When $u \in L^1_{loc}(\mathbb{R})$, we set

$$u^*(x) = \begin{cases} \lim_{r \rightarrow 0} \frac{1}{2r} \int_{x-r}^{x+r} u(t) dt, & \text{provided the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

By (3.6.2) we have

$$u^*(x) = u(x), \quad \text{a.e. } x \in \mathbb{R}.$$

The function u^* is called the **precise representative** of u . In the sequel to this Section, if $f \in AC_{loc}(\mathbb{R})$ we will denote by f' its derivative, and if $g \in W^{1,p}_{loc}(\mathbb{R})$, we will denote by $\frac{d}{dx}g$ its weak derivative.

Theorem 3.6.1. *Let $p \in [1, +\infty)$. We have*

(i) *if $u \in W^{1,p}_{loc}(\mathbb{R})$, then $u^* \in AC_{loc}(\mathbb{R})$; moreover*

$$(u^*)' \in L^p_{loc}(\mathbb{R}), \quad \text{and} \quad \frac{d}{dx}u = (u^*)';$$

(ii) let $u \in L^p_{loc}(\mathbb{R})$. If $v \in AC_{loc}(\mathbb{R})$ satisfies

$$u = v, \quad \text{a.e. in } \mathbb{R}$$

and $v' \in L^p_{loc}(\mathbb{R})$ then $u \in W^{1,p}_{loc}(\mathbb{R})$ and $\frac{d}{dx}u = v$.

In order to prove the Theorem above we need the following Lemma

Lemma 3.6.2. Let $u \in L^1_{loc}(\mathbb{R})$ and let η_ε be a mollifier. If x is a Lebesgue point of u we have

$$u^\varepsilon(x) := (\eta_\varepsilon \star u)(x) \rightarrow u^*(x), \quad \text{as } \varepsilon \rightarrow 0^+.$$

Hence

$$u^\varepsilon(x) \rightarrow u(x), \quad \text{as } \varepsilon \rightarrow 0^+, \quad \text{a.e. } x \in \mathbb{R}.$$

Proof of Lemma. Let us recall

$$\eta_\varepsilon = \varepsilon^{-1} \eta(\varepsilon^{-1}x),$$

where $\text{supp } \eta \subset (-1, 1)$, $\eta \in C_0^\infty(\mathbb{R})$, $\eta \geq 0$ and

$$\int_{\mathbb{R}} \eta(x) dx = 1.$$

If $x \in \mathbb{R}$ is a Lebesgue point of u , then we have

$$\begin{aligned} |u^\varepsilon(x) - u(x)| &= \left| \frac{1}{\varepsilon} \int_{\mathbb{R}} \eta\left(\frac{x-t}{\varepsilon}\right) (u(t) - u(x)) dt \right| \leq \\ &\leq \|\eta\|_{L^\infty(\mathbb{R})} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |u(t) - u(x)| dt \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

■

Proof of Theorem 3.6.1.

1. Let $u \in W^{1,p}_{loc}(\mathbb{R})$. We have $u^\varepsilon \in C^\infty(\mathbb{R})$ and

$$u^\varepsilon(y) = u^\varepsilon(x) + \int_x^y (u^\varepsilon)'(t) dt, \quad \forall x, y \in \mathbb{R}. \quad (3.6.5)$$

Let $x_0 \in \mathbb{R}$ be a Lebesgue point of u (hence $u^*(x_0) = u(x_0)$). By (3.6.5) we have

$$u^\varepsilon(x) = u^\varepsilon(x_0) + \int_{x_0}^x (u^\varepsilon)'(t) dt \quad (3.6.6)$$

and, for any $\varepsilon, \delta > 0$,

$$|u^\varepsilon(x) - u^\delta(x)| \leq |u^\varepsilon(x_0) - u^\delta(x_0)| + \left| \int_{x_0}^x |(u^\varepsilon)' - (u^\delta)'| dt \right|. \quad (3.6.7)$$

Now, Lemma 3.6.2 yields

$$u^\varepsilon(x_0) \rightarrow u(x_0), \quad \text{as } \varepsilon \rightarrow 0. \quad (3.6.8)$$

Moreover

$$(u^\varepsilon)'(t) = \int_{\mathbb{R}} -\partial_y(\eta_\varepsilon(t-y)) u(y) dy = \int_{\mathbb{R}} \eta_\varepsilon(t-y) \frac{du(y)}{dy} dy,$$

which in turn implies

$$(u^\varepsilon)' \rightarrow \frac{du}{dt}, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } L^p_{loc}(\mathbb{R}). \quad (3.6.9)$$

Now, by (3.6.7)–(3.6.9) we have that $\{u^\varepsilon\}$ satisfies the Cauchy property on every compact set of \mathbb{R} . Therefore $\{u^\varepsilon\}$ uniformly converges to a continuous function v on every compact set of \mathbb{R} . Since we have

$$u^\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0, \quad \text{in } L^p_{loc}(\mathbb{R}),$$

we get

$$u = v, \quad \text{a.e. in } \mathbb{R}.$$

On the other hand, by (3.6.6) and by (3.6.9) (taking into account that $\{u^\varepsilon\} \rightarrow v$ on any compact), we have

$$v(x) = v(x_0) + \int_{x_0}^x \frac{du(t)}{dt} dt, \quad \forall x \in \mathbb{R}$$

which implies that $v \in AC_{loc}(\mathbb{R})$ e

$$v'(x) = \frac{du(x)}{dx}, \quad \text{a.e. in } \mathbb{R}. \quad (3.6.10)$$

Moreover,

$$\frac{1}{2r} \int_{x-r}^{x+r} u(t) dt = \frac{1}{2r} \int_{x-r}^{x+r} v(t) dt, \quad \forall x \in \mathbb{R}.$$

Hence, passing to the limit as $r \rightarrow 0$ we have

$$u^*(x) = v(x), \quad \forall x \in \mathbb{R}.$$

By the latter and by (3.6.10) we get

$$(u^*)'(x) = \frac{du(x)}{dx}, \quad \text{a.e. in } \mathbb{R}.$$

2. Let $u \in L^p_{loc}(\mathbb{R})$ satisfy

$$u = v, \quad \text{a.e. in } \mathbb{R},$$

where $v \in AC_{loc}(\mathbb{R})$ and $v' \in L^p_{loc}(\mathbb{R})$. We have

$$\int_{\mathbb{R}} u \Phi' dx = \int_{\mathbb{R}} v \Phi' dx = - \int_{\mathbb{R}} v' \Phi dx, \quad \forall \Phi \in C_0^\infty(\mathbb{R}).$$

Hence, as $v' \in L^p_{loc}(\mathbb{R})$, we get

$$\frac{du}{dx} = v', \quad \text{and} \quad u \in W^{1,p}_{loc}(\mathbb{R}).$$

■

Remark. The Extension Theorem implies that if $1 \leq p < +\infty$, $a, b \in \mathbb{R}$, $a < b$, denoting by E the extension operator, then we have

(i') if $u \in W^{1,p}(a, b)$, $(Eu)^* \in AC([a, b])$; in particular u is almost everywhere equal to an absolutely continuous function in $[a, b]$ and the weak derivative of u is equal to the classic derivative of $(Eu)^*$ in (a, b) ;

(ii') if $u \in L^p(a, b)$ and $v \in AC([a, b])$ satisfies

$$u = v, \quad \text{a.e. in } [a, b]$$

and $v' \in L^p(a, b)$ then $u \in W^{1,p}(a, b)$ and the weak derivative of u is equal to the classic derivative of v . ♦

Theorem 3.6.1 can be accomplished by the following

Proposition 3.6.3. *Let $p > 1$ and $u \in W^{1,p}_{loc}(a, b)$. We have that u is almost everywhere equal to a function $C^{0,\alpha}_{loc}(\mathbb{R})$, where $\alpha = 1 - 1/p$. Here $C^{0,\alpha}_{loc}(\mathbb{R})$ denotes the space of the functions u satisfying $u|_I \in C^{0,\alpha}(I)$ for every I compact interval of \mathbb{R} .*

Proof. Let I be a bounded interval. By (3.6.5), (by using Hölder inequality), we have, for any $x, y \in I$

$$\begin{aligned} |u^\varepsilon(x) - u^\varepsilon(y)| &= \left| \int_x^y (u^\varepsilon)'(t) dt \right| \leq \\ &\leq |x - y|^{1-1/p} \left(\int_I |(u^\varepsilon)'(t)|^p dt \right)^{1/p}. \end{aligned}$$

Hence, passing to the limit as $\varepsilon \rightarrow 0$, taking into account that $u^\varepsilon \rightarrow u^*$ and $u^*(x) = u(x)$ almost everywhere, we obtain

$$|u(x) - u(y)| \leq |x - y|^{1-1/p} \left(\int_I \left| \frac{du}{dt} \right|^p dt \right)^{1/p}, \quad \text{a.e. } x, y \in I.$$

■

3.7 The embedding theorems

In this Section we are interested in proving non trivial embedding theorems of $W^{k,p}(\Omega)$ in other function spaces. For instance we will be interested in establishing when it happens that $W^{k,p}(\Omega) \subset L^q(\Omega)$, for $q \neq p$ as well $W^{k,p}(\Omega) \subset C^{m,\gamma}(\Omega)$ for appropriate $m \in \mathbb{N}_0$, $0 < \gamma \leq 1$.

First we consider the space $W^{1,p}(\Omega)$ and we distinguish the following three cases

$$(a) \quad 1 \leq p < n, \quad (b) \quad n < p \leq +\infty, \quad (c) \quad p = n.$$

About case (c) we will just give brief hints.

3.7.1 Case $1 \leq p < n$. The Gagliardo – Nirenberg inequality

Let us assume

$$1 \leq p < n \tag{3.7.1}$$

and let us ask ourselves for what $q \in [1, +\infty]$ can be true an estimate like

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{3.7.2}$$

where C and q do not depend on u .

Let us assume that (3.7.2) is true and let us prove that, *necessarily*

$$q = \frac{np}{n-p}.$$

First, we examine the case $q \in [1, +\infty)$. Let $u \in C_0^\infty(\mathbb{R}^n)$ be **not identically equal to 0**, and, for any $\lambda > 0$, let

$$u_\lambda(x) = u(\lambda x), \quad \forall x \in \mathbb{R}^n.$$

Of course if (3.7.2) holds true, then

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n) \quad \forall \lambda > 0. \quad (3.7.3)$$

Now

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)}^q = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \lambda^{-n} \int_{\mathbb{R}^n} |u(x)|^q dx;$$

hence

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \quad (3.7.4)$$

and

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \lambda^p |(\nabla u)(\lambda x)|^p dx = \lambda^{p-n} \int_{\mathbb{R}^n} |\nabla u(x)|^p dx;$$

therefore

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}. \quad (3.7.5)$$

Since (3.7.4) and (3.7.5) hold true, we may write (3.7.3) as follows

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1+\frac{n}{q}-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall \lambda > 0. \quad (3.7.6)$$

Now, if

$$q > \frac{np}{n-p},$$

we get

$$1 + \frac{n}{q} - \frac{n}{p} < 1 + n \left(\frac{n-p}{np} \right) - \frac{n}{p} = 0.$$

Consequently

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \lim_{\lambda \rightarrow +\infty} C \lambda^{1+\frac{n}{q}-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)} = 0,$$

this is a contradiction because u does not vanish identically.

On the other hand, if

$$q < \frac{np}{n-p},$$

we have

$$1 + \frac{n}{q} - \frac{n}{p} > 0,$$

hence

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \lim_{\lambda \rightarrow 0^+} C \lambda^{1 + \frac{n}{q} - \frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)} = 0,$$

this is again a contradiction.

Finally, if $q = +\infty$, instead of (3.7.6) we have

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C \lambda^{1 - \frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall \lambda > 0.$$

and, by (3.7.1), passing to the limit as $\lambda \rightarrow +\infty$ we have a contradiction.

Here and in the sequel, if $1 \leq p < n$, we denote by p^* the number

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

and we call p^* the **Sobolev exponent** or the **Sobolev conjugate** of p . Let us notice

$$p^* = \frac{pn}{n-p} > p.$$

The Main Theorem of the present Subsection is the following one

Theorem 3.7.1 (The Gagliardo – Nirenberg inequality). *Let*

$$1 \leq p < n.$$

Then there exists C depending on p and n only such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_0^1(\mathbb{R}^n). \quad (3.7.7)$$

The most challenging part of the proof of Theorem 3.7.7 concerns the case $p = 1$ and this, in turn, is based on the following

Lemma 3.7.2. *Let $n \geq 2$ and*

$$g_j : \mathbb{R}^{n-1} \rightarrow [0, +\infty), \quad j = 1, \dots, n,$$

be measurable functions. Then

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{j=1}^n g_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) dx_1 \cdots dx_n &\leq \\ &\leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^{n-1}(y) dy \right)^{\frac{1}{n-1}}. \end{aligned} \quad (3.7.8)$$

Proof of Lemma 3.7.2. Let us proceed by induction on n . If $n = 2$, we have

$$\int_{\mathbb{R}^2} g_1(x_2)g_2(x_1)dx_1dx_2 = \int_{\mathbb{R}} g_1(x_2)dx_2 \int_{\mathbb{R}} g_2(x_1)dx_1.$$

Therefore, if $n = 2$, (3.7.8) holds true. Now, let us assume that (3.7.8) holds for n and let us prove it for $n + 1$. Hence, let us assume that **for any** nonnegative measurable functions $g_1, g_2 \cdots$, we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n g_j dx_1 \cdots dx_n \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^{n-1} dy \right)^{\frac{1}{n-1}},$$

let us notice that, to shorten the formula, we have omitted the variables. However, it is important to recall that g_j does not depend on x_j .

By the Hölder inequality we get

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} g_j dx_1 \cdots dx_{n+1} = \\ & = \int_{\mathbb{R}} dx_{n+1} \int_{\mathbb{R}^n} g_{n+1} \prod_{j=1}^n g_j dx_1 \cdots dx_n \leq \\ & \leq \int_{\mathbb{R}} dx_{n+1} \left(\int_{\mathbb{R}^n} g_{n+1}^n dx_1 \cdots dx_n \right)^{\frac{1}{n}} \times \\ & \times \left(\int_{\mathbb{R}^n} \prod_{j=1}^n g_j^{\frac{n}{n-1}} dx_1 \cdots dx_n \right)^{\frac{n-1}{n}} = \\ & = \left(\int_{\mathbb{R}^n} g_{n+1}^n dx_1 \cdots dx_n \right)^{\frac{1}{n}} \times \\ & \times \int_{\mathbb{R}} dx_{n+1} \left(\int_{\mathbb{R}^n} \prod_{j=1}^n g_j^{\frac{n}{n-1}} dx_1 \cdots dx_n \right)^{\frac{n-1}{n}}. \end{aligned} \tag{3.7.9}$$

Now let us apply the inductive assumption to the functions $g_j^{\frac{n}{n-1}}(\cdot, x_{n+1})$, $j = 1, \dots, n$. We get

$$\int_{\mathbb{R}^n} \prod_{j=1}^n g_j^{\frac{n}{n-1}} dx_1 \cdots dx_n \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n-1}}.$$

Hence, we have trivially

$$\left(\int_{\mathbb{R}^n} \prod_{j=1}^n g_j^{\frac{n}{n-1}} dx_1 \cdots dx_n \right)^{\frac{n-1}{n}} \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n}}.$$

The last inequality and (3.7.9) imply

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} g_j dx_1 \cdots dx_{n+1} &\leq \left(\int_{\mathbb{R}^n} g_{n+1}^n dx_1 \cdots dx_n \right)^{\frac{1}{n}} \times \\ &\times \int_{\mathbb{R}} dx_{n+1} \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n}}. \end{aligned} \quad (3.7.10)$$

Now, we set

$$h_j(x_{n+1}) = \left(\int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n}}$$

and we use the extended Hölder inequality:

$$\begin{aligned} \int_{\mathbb{R}} \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n}} dx_{n+1} &= \int_{\mathbb{R}} \prod_{j=1}^n h_j(x_{n+1}) dx_{n+1} \leq \\ &\leq \prod_{j=1}^n \left(\int_{\mathbb{R}} h_j^n(x_{n+1}) dx_{n+1} \right)^{\frac{1}{n}} = \\ &= \prod_{j=1}^n \left(\int_{\mathbb{R}} dx_{n+1} \int_{\mathbb{R}^{n-1}} g_j^n(y, x_{n+1}) dy \right)^{\frac{1}{n}} = \\ &= \prod_{j=1}^n \left(\int_{\mathbb{R}^n} g_j^n(y) dy \right)^{\frac{1}{n}}. \end{aligned}$$

The just obtained inequality and (3.7.10) yield

$$\int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} g_j dx_1 \cdots dx_{n+1} \leq \prod_{j=1}^{n+1} \left(\int_{\mathbb{R}^n} g_j^n dy \right)^{\frac{1}{n}}.$$

Proof of Lemma is concluded. ■

Proof of Theorem 3.7.1. Let $u \in C_0^1(\mathbb{R}^n)$. For any $j = 1, \dots, n$, we set

$$\begin{aligned}
f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) &= \\
&= \int_{\mathbb{R}} |\nabla u(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n)| dy_j.
\end{aligned} \tag{3.7.11}$$

We obtain

$$|u(x)| \leq f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \quad j = 1, \dots, n.$$

As a matter of fact, we have

$$u(x) = \int_{-\infty}^{x_j} \partial_{x_j} u(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n) dy_j, \quad j = 1, \dots, n,$$

from which, for any $j = 1, \dots, n$ we have

$$\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}} |\nabla u(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n)| dy_j = \\
&= f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).
\end{aligned} \tag{3.7.12}$$

Now by multiplying all (3.7.12) we get

$$|u(x)|^n \leq \prod_{j=1}^n f_j,$$

hence

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{j=1}^n f_j^{\frac{1}{n-1}}$$

and, by integrating over \mathbb{R}^n we have

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \int_{\mathbb{R}^n} \prod_{j=1}^n f_j^{\frac{1}{n-1}} dx.$$

At this stage let us exploit Lemma 3.7.2. Set

$$g_j = f_j^{\frac{1}{n-1}}, \quad j = 1, \dots, n$$

and we obtain

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} f_j(\eta) d\eta \right)^{\frac{1}{n-1}}. \tag{3.7.13}$$

Now, we notice that

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} f_j(\eta) d\eta = \\ &= \int_{\mathbb{R}^{n-1}} d\eta \int_{\mathbb{R}} |\nabla u(\eta_1, \dots, \eta_{j-1}, y_j, \eta_{j+1}, \dots, \eta_n)| dy_j = \\ &= \int_{\mathbb{R}^n} |\nabla u(x)| dx, \quad j = 1, \dots, n. \end{aligned}$$

By the just obtained equality and by (3.7.13) we get

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq \left(\int_{\mathbb{R}^n} |\nabla u(x)| dx \right)^{\frac{n}{n-1}},$$

which implies

$$\|u\|_{L^{1^*}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n)}. \quad (3.7.14)$$

Therefore (3.7.7) is proved for $p = 1$. Now, let $1 < p < n$ and $\alpha > 1$ to be chosen. By applying (3.7.14) to $|u|^\alpha$ we have

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{\alpha n}{n-1}} dx \right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |\nabla (|u(x)|^\alpha)| dx = \\ &= \int_{\mathbb{R}^n} \alpha |u(x)|^{\alpha-1} |\nabla u| dx \leq \\ &\leq \alpha \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{(\alpha-1)p}{p-1}} dx \right)^{\frac{p}{p-1}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, let us choose α satisfying

$$\frac{\alpha n}{n-1} = (\alpha-1) \frac{p}{p-1}$$

that is

$$\alpha = \frac{p(n-1)}{n-p},$$

notice that $\alpha > 1$, as $n > p > 1$. The above choice of α gives

$$\frac{\alpha n}{n-1} = (\alpha-1) \frac{p}{p-1} = \left(\frac{p(n-1)}{n-p} - 1 \right) \frac{p}{p-1} = \frac{pn}{n-p}.$$

Hence

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{pn}{n-p}} dx \right)^{\frac{n-1}{n}} \leq \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{pn}{n-p}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}$$

that is

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} dx \right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

On the other hand

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p} - \frac{1}{n} = \frac{1}{p^*}.$$

Therefore

$$\left(\int_{\mathbb{R}^n} |u(x)|^{p^*} dx \right)^{\frac{1}{p^*}} \leq \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

■

Theorem 3.7.3 (The Sobolev inequality). *Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ with constants M_0 and r_0 . Let d_0 be the diameter of Ω . Let us assume $1 \leq p < n$.*

Then there exists C depending on M_0 , r_0 , d_0 , p and n only such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \quad (3.7.15)$$

Proof. Since Ω is a bounded open set of \mathbb{R}^n of class $C^{0,1}$, we can apply the extension Theorem 3.4.4. Hence, there exists $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\tilde{u} = u, \text{ in } \Omega, \quad \text{supp } \tilde{u} \text{ compact of } \mathbb{R}^n. \quad (3.7.16)$$

Moreover, there exists C depending on M_0 , r_0 , d_0 , p and n only such that

$$\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \quad (3.7.17)$$

Proposition 3.4.1 implies that there exists a sequence $\{v_j\} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\{v_j\} \rightarrow \tilde{u}, \quad \text{in } W^{1,p}(\mathbb{R}^n).$$

Now, by Theorem 3.7.1 we have

$$\|v_j - v_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla v_j - \nabla v_m\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall j, m \in \mathbb{N},$$

from which it follows that $\{v_j\}$ is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$, hence there exists $v \in L^{p^*}(\Omega)$ such that

$$\{v_j\} \rightarrow v, \quad \text{in } L^{p^*}(\mathbb{R}^n).$$

Hence $\tilde{u} = v$; as a matter of fact, for any $R > 0$ we have $L^{p^*}(B_R) \subset L^p(B_R)$, consequently

$$\|v - \tilde{u}\|_{L^p(B_R)} \leq \|v - v_j\|_{L^p(B_R)} + \|v_j - \tilde{u}\|_{L^p(B_R)} \rightarrow 0, \quad \text{as } j \rightarrow \infty$$

and, as R is arbitrary, we get $\tilde{u} = v$.

Therefore $\tilde{u} \in L^{p^*}(\mathbb{R}^n)$ and passing to the limit in the following inequality

$$\|v_j\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla v_j\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall j \in \mathbb{N},$$

we obtain

$$\|\tilde{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla \tilde{u}\|_{W^{1,p}(\mathbb{R}^n)}. \quad (3.7.18)$$

On the other hand (3.7.16) implies

$$\|u\|_{L^{p^*}(\Omega)} \leq \|\tilde{u}\|_{L^{p^*}(\mathbb{R}^n)} \quad (3.7.19)$$

and (3.7.17) yields

$$\|\nabla \tilde{u}\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega).$$

so that, by the latter, by (3.7.18) and by (3.7.19) we get

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

■

Corollary 3.7.4. *Let Ω be a bounded open set of \mathbb{R}^n and let d_0 be its diameter. Let us assume $1 \leq p < n$. We have that, if $u \in W_0^{1,p}(\Omega)$, then $u \in L_0^{p^*}(\Omega)$ and there exists C depending on p , n and d_0 only, such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

Proof. Let $x_0 \in \Omega$ and $R = 2d_0$. Since $u \in W_0^{1,p}(\Omega)$ we have that the function

$$\tilde{u}(x) = \begin{cases} u(x), & \text{for } x \in \Omega \\ 0, & \text{for } x \in B_R \setminus \Omega, \end{cases}$$

belongs to $W_0^{1,p}(B_R(x_0))$. By the first Poincaré inequality (Theorem 3.4.2) and by Theorem 3.7.3 we obtain

$$\begin{aligned} \|u\|_{L^{p^*}(\Omega)} &= \|\tilde{u}\|_{L^{p^*}(B_R(x_0))} \leq \\ &\leq C \|\tilde{u}\|_{W^{1,p}(B_R(x_0))} \leq \\ &\leq C\tilde{C} \|\nabla\tilde{u}\|_{L^p(B_R(x_0))} = \\ &= C\tilde{C} \|\nabla u\|_{L^p(\Omega)}, \end{aligned}$$

where C is the same constant that occurs in the inequality (3.7.15), hence, it depends on p e da n only, and \tilde{C} is the same constant that occurs in the Poincaré inequality, hence, it depends on R , that is, on d_0 , only. ■

Counterexample. If $n > 1$ it **does not** happen that $W^{1,n}(\Omega) \subset L^\infty(\Omega)$. Let us prove what is claimed.

Let

$$u(x) = \log \log \left(1 + \frac{1}{|x|} \right), \quad \text{in } B_1.$$

We have $u \notin L^\infty(B_1)$ and

$$\nabla u(x) = \frac{x}{(|x|^3 + |x|^2) \log \left(1 + \frac{1}{|x|} \right)}.$$

Let us check that $u \in W^{1,n}(B_1)$. We have

$$\begin{aligned} \int_{B_1} |u|^n dx &= \int_{B_1} \left| \log \log \left(1 + \frac{1}{|x|} \right) \right|^n dx = \\ &= \omega_n \int_0^1 \rho^{n-1} \left| \log \log \left(1 + \frac{1}{\rho} \right) \right|^n d\rho < +\infty \end{aligned}$$

and (as $n > 1$)

$$\begin{aligned} \int_{B_1} |\nabla u|^n dx &= \omega_n \int_0^1 \frac{\rho^{n-1}}{(\rho^2 + \rho)^n \left(\log\left(1 + \frac{1}{\rho}\right)\right)^n} d\rho = \\ &= \omega_n \int_0^1 \frac{d\rho}{(\rho + 1)^n \rho \left(\log\left(1 + \frac{1}{\rho}\right)\right)^n} < +\infty. \end{aligned}$$

Therefore $u \in W^{1,n}(B_1)$, but $u \notin L^\infty(B_1)$.

Let us observe that if $n = 1$, for what was proved in Section 3.6, we have $W^{1,1}(I) \subset L^\infty(I)$, where I is a bounded open interval of \mathbb{R} . ♠

3.7.2 The Sobolev–Poincaré inequality

We begin by the following

Lemma 3.7.5. *If $p \in [1, +\infty)$, then there exists a constant C depending on p e da n only, such that*

$$\int_{B_r(x)} |u(y) - u(z)|^p dy \leq Cr^{n+p-1} \int_{B_r(x)} |\nabla u(y)|^p |y - z|^{1-n} dy, \quad (3.7.20)$$

for every $u \in C^1(\overline{B_r(x)})$, $r > 0$, $x \in \mathbb{R}^n$ and for every $z \in \overline{B_r(x)}$.

Proof. It is not restrictive to assume $x = 0$. By the Fundamental Theorem of Calculus we have, for any $y, z \in B_r$,

$$u(y) - u(z) = \int_0^1 \frac{d}{dt} u(z + t(y - z)) dt = \int_0^1 \nabla u(z + t(y - z)) dt \cdot (y - z);$$

which implies

$$|u(y) - u(z)|^p \leq |y - z|^p \int_0^1 |\nabla u(z + t(y - z))|^p dt.$$

Let $\rho > 0$ and let us integrate over $B_r \cap \partial B_\rho(z)$ both the sides of the last inequality. We get

$$\begin{aligned} \int_{B_r \cap \partial B_\rho(z)} |u(y) - u(z)|^p dS_y &\leq \int_{B_r \cap \partial B_\rho(z)} dS_y |y - z|^p \int_0^1 |\nabla u(z + t(y - z))|^p dt = \\ &= \rho^p \int_0^1 dt \int_{B_r \cap \partial B_\rho(z)} |\nabla u(z + t(y - z))|^p dS_y = \\ &= \rho^p \int_0^1 dt \int_{B_r \cap \partial B_{t\rho}(z)} |\nabla u(\xi)|^p t^{1-n} dS_\xi := (\star), \end{aligned}$$

where in the last inequality we set $\xi = z + t(y - z)$, so that $t^{n-1}dS_y = dS_\xi$. Now, if $\xi \in \partial B_{t\rho}(z)$, we have $\frac{|\xi - z|}{\rho} = t$; hence

$$\begin{aligned}
(\star) &= \rho^p \int_0^1 dt \int_{B_r \cap \partial B_{t\rho}(z)} |\nabla u(\xi)|^p \frac{|\xi - z|^{1-n}}{\rho^{1-n}} dS_\xi = \\
&= \rho^{n+p-1} \int_0^1 dt \int_{B_r \cap \partial B_{t\rho}(z)} |\nabla u(\xi)|^p |\xi - z|^{1-n} dS_\xi = \\
&= \rho^{n+p-2} \int_0^\rho d\tau \int_{B_r \cap \partial B_\tau(z)} |\nabla u(\xi)|^p |\xi - z|^{1-n} dS_\xi = \\
&= \rho^{n+p-2} \int_{B_r \cap B_\rho(z)} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi \leq \\
&\leq \rho^{n+p-2} \int_{B_r} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi.
\end{aligned}$$

Therefore

$$\int_{B_r \cap \partial B_\rho(z)} |u(y) - u(z)|^p dS_y \leq \rho^{n+p-2} \int_{B_r} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi.$$

Now, let us integrate w.r.t. ρ over $[0, r]$ both the sides of the previous inequality. We have

$$\begin{aligned}
\int_0^r d\rho \int_{B_r \cap \partial B_\rho(z)} |u(y) - u(z)|^p dS_y &\leq \int_0^r \rho^{n+p-2} d\rho \int_{B_r} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi = \\
&= \frac{r^{n+p-1}}{n+p-1} \int_{B_r} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi,
\end{aligned}$$

on the other hand, by using the polar coordinates, we have

$$\int_0^r d\rho \int_{B_r \cap \partial B_\rho(z)} |u(y) - u(z)|^p dS_y = \int_{B_r} |u(y) - u(z)|^p dy.$$

Therefore, for any $z \in B_r$, we have

$$\int_{B_r} |u(y) - u(z)|^p dy \leq \frac{r^{n+p-1}}{n+p-1} \int_{B_r} |\nabla u(\xi)|^p |\xi - z|^{1-n} d\xi$$

from which we obtain (3.7.20). ■

In what follows, for any $g \in L^1(B_r(x))$, we set

$$(g)_{x,r} = \int_{B_r(x)} g(y) dy = \frac{1}{|B_r(x)|} \int_{B_r(x)} g(y) dy.$$

Theorem 3.7.6 (The Sobolev–Poincaré inequality). *Let $1 \leq p < n$. Then there exists a constant C depending on p and n only such that*

$$\left(\int_{B_r(x)} |u(y) - (u)_{x,r}|^{p^*} dy \right)^{\frac{1}{p^*}} \leq Cr \left(\int_{B_r(x)} |\nabla u(y)|^p dy \right)^{\frac{1}{p}}, \quad (3.7.21)$$

for every $u \in W^{1,p}(B_r(x))$.

Proof. First, we assume $x = 0$ and $r = 1$. Let $u \in C^1(\overline{B_1})$. Lemma 3.7.5 and Hölder inequality give

$$\begin{aligned} \int_{B_1} \left| u(y) - \int_{B_1} u(z) dz \right|^p dy &= \int_{B_1} \left| \int_{B_1} (u(y) - u(z)) dz \right|^p dy \leq \\ &\leq \int_{B_1} dy \int_{B_1} |u(y) - u(z)|^p dz \leq \\ &\leq C \int_{B_1} dy \int_{B_1} |\nabla u(z)|^p |y - z|^{1-n} dz = \\ &= C \int_{B_1} dz |\nabla u(z)|^p \int_{B_1} |y - z|^{1-n} dy \leq \\ &\leq C_1 \int_{B_1} dz |\nabla u(z)|^p \int_{B_2(z)} |y - z|^{1-n} dy = \\ &= C_2 \int_{B_1} |\nabla u(z)|^p dz, \end{aligned}$$

where $C_1 = \frac{\omega_n}{n} C$, $C_2 = 2\omega_n C_1$. Hence, we have obtained the inequality

$$\int_{B_1} |u(y) - (u)_{0,1}|^p dy \leq C_2 \int_{B_1} |\nabla u(z)|^p dz.$$

Now, let us apply the Sobolev inequality (Theorem 3.7.3) to $u - (u)_{0,1}$. We get

$$\begin{aligned} \left(\int_{B_1} |u(y) - (u)_{0,1}|^{p^*} dy \right)^{\frac{1}{p^*}} &\leq C_3 \left[\int_{B_1} |u(y) - (u)_{0,1}|^{p^*} dy + \int_{B_1} |\nabla u(y)|^p dy \right]^{\frac{1}{p}} \leq \\ &\leq C_4 \left(\int_{B_1} |\nabla u(y)|^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, we have obtained

$$\left(\int_{B_1} |u(y) - (u)_{0,1}|^{p^*} dy \right)^{\frac{1}{p^*}} \leq C_4 \left(\int_{B_1} |\nabla u(y)|^p dy \right)^{\frac{1}{p}}, \quad (3.7.22)$$

where C_4 depend on n and p only.

Let now $u \in C^1(\overline{B_r}(0))$. Set

$$v(x) = u(rx), \quad x \in B_1$$

and apply (3.7.22) to function v . We have

$$(v)_{0,1} = (u)_{0,r}, \quad (3.7.23)$$

$$\int_{B_1} |\nabla v(y)|^p dy = r^p \int_{B_r} |\nabla u(x)|^p dx \quad (3.7.24)$$

and

$$\int_{B_1} |v(y) - (v)_{0,1}|^{p^*} dy = \int_{B_r} |u(x) - (u)_{0,r}|^{p^*} dx. \quad (3.7.25)$$

By applying inequality (3.7.22) to v and taking into account (3.7.23)–(3.7.25) we obtain (3.7.21) by density. ■

3.7.3 The Morrey inequality

Let E be any measurable of \mathbb{R}^n , here and in the sequel we say that $u^* : E \rightarrow \mathbb{R}$ is a *version* of a given function $u : E \rightarrow \mathbb{R}$ if

$$u = u^*, \quad \text{a.e. in } E.$$

Lemma 3.7.7. *If $n < p \leq +\infty$, then there exists a constant C depending on p and n only, such that we have*

$$|u(y) - u(z)| \leq Cr^{1-\frac{n}{p}} \int_{B_r(x)} |\nabla u(\xi)|^p |d\xi|, \quad (3.7.26)$$

for every $u \in C^1(\overline{B_r(x)})$, $r > 0$, $x \in \mathbb{R}^n$ and for every $y, z \in \overline{B_r(x)}$.

Proof. Let $u \in C^1(\overline{B_r(x)})$. Let us apply Lemma 3.7.5 for $p = 1$. We have, for any $y, z \in \overline{B_r(x)}$,

$$\begin{aligned}
|u(y) - u(z)| &= \int_{B_r} |u(y) - u(z)| d\xi \leq \\
&\leq \int_{B_r} |u(y) - u(\xi)| d\xi + \int_{B_r} |u(z) - u(\xi)| d\xi \leq \quad (3.7.27) \\
&\leq C \int_{B_r(x)} (|y - \xi|^{1-n} + |z - \xi|^{1-n}) |\nabla u(\xi)| d\xi,
\end{aligned}$$

where C depends on n only. Let $n < p < +\infty$, by the Hölder inequality we have

$$\begin{aligned}
&\int_{B_r(x)} (|y - \xi|^{1-n} + |z - \xi|^{1-n}) |\nabla u(\xi)| d\xi \leq \\
&\leq \left[\int_{B_r(x)} (|y - \xi|^{1-n} + |z - \xi|^{1-n}) d\xi \right]^{\frac{p-1}{p}} \left(\int_{B_r(x)} |\nabla u(\xi)|^p d\xi \right)^{\frac{1}{p}} \leq \\
&\leq 2^{\frac{1}{p-1}} \underbrace{\left[\int_{B_r(x)} (|y - \xi|^{-\frac{(n-1)p}{p-1}} + |z - \xi|^{-\frac{(n-1)p}{p-1}}) d\xi \right]^{\frac{p-1}{p}}}_{I} \left(\int_{B_r(x)} |\nabla u(\xi)|^p d\xi \right)^{\frac{1}{p}}. \quad (3.7.28)
\end{aligned}$$

Now, let us check that

$$I \leq C 2^{\frac{p-n}{p-1}} \left(\frac{p-1}{p-n} \right) r^{\frac{p-n}{p-1}}, \quad (3.7.29)$$

where C depends on n only. We have $B_r(x) \subset B_{2r}(y)$, for any $y \in B_r(x)$. Hence (taking into account that $p > n$ implies $\frac{(n-1)p}{p-1} < n$)

$$\begin{aligned}
\int_{B_r(x)} |y - \xi|^{-\frac{(n-1)p}{p-1}} d\xi &\leq \int_{B_{2r}(y)} |y - \xi|^{-\frac{(n-1)p}{p-1}} d\xi = \\
&= \int_{\partial B_1} dS \int_0^{2r} \rho^{-\frac{(n-1)p}{p-1}} \rho^{n-1} d\rho = \\
&= \omega_n 2^{\frac{p-n}{p-1}} \left(\frac{p-1}{p-n} \right) r^{\frac{p-n}{p-1}}.
\end{aligned}$$

Since a similar estimate holds true for the integral

$$\int_{B_r(x)} |z - \xi|^{-\frac{(n-1)p}{p-1}} d\xi$$

provided $z \in B_r(x)$, we get (3.7.29). By estimate (3.7.29) and by (3.7.27) we obtain

$$|u(y) - u(z)| \leq C_n 2^{\frac{1}{p-1}} 2^{1-\frac{n}{p}} r^{1-\frac{n}{p}} \left(\int_{B_r(x)} |\nabla u(\xi)|^p d\xi \right)^{\frac{1}{p}}. \quad (3.7.30)$$

The last obtained estimate implies (3.7.26) for every $n < p < +\infty$. If $p = +\infty$, we may pass to the limit as $p \rightarrow +\infty$ in (3.7.30). ■

Lemma 3.7.8. *If $n < p \leq +\infty$, then there exists a constant C depending on p and n such that*

$$|u(x)| \leq C \|\nabla u\|_{W^{1,p}(B_1(x))}, \quad (3.7.31)$$

for every $u \in C^1(\overline{B_1(x)})$, $r > 0$ and for every $x \in \mathbb{R}^n$.

Proof. By (3.7.26) ($y = x$, $r = 1$ and $z \in B_1(x)$) we get

$$|u(x)| \leq C \|\nabla u\|_{L^p(B_1(x))} + |u(z)|.$$

Now, by integrating both the side w.r.t. z on $B_1(x)$ and by applying the Hölder inequality, we have

$$\begin{aligned} |B_1(x)| |u(x)| &\leq C \|\nabla u\|_{L^p(B_1(x))} + \int_{B_1(x)} |u(z)| dz \leq \\ &\leq C' \left(\|\nabla u\|_{L^p(B_1(x))} + \|u\|_{L^p(B_1(x))} \right) \leq \\ &\leq C'' \|u\|_{W^{1,p}(B_1(x))}. \end{aligned}$$

Which gives (3.7.31). ■

Theorem 3.7.9 (the Morrey inequality). *Let $n < p < +\infty$ and let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ with constants M_0, r_0 . Then there exists a constant C , depending on p, n, M_0 and r_0 only, such that for every $u \in W^{1,p}(\Omega)$ there exists a version of u , $u^* \in C^{0,\gamma}(\overline{\Omega})$, where*

$$\gamma = 1 - \frac{n}{p}.$$

Moreover

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \|u\|_{W^{1,p}(\Omega)}. \quad (3.7.32)$$

Proof. Let us begin by proving that

$$\|v\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|v\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall v \in C_0^\infty(\mathbb{R}^n), \quad (3.7.33)$$

where C depends on p and n only. Indeed, by (3.7.31) we get trivially

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leq C \|v\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall v \in C_0^\infty(\mathbb{R}^n) \quad (3.7.34)$$

Now, we set

$$x = \frac{y+z}{2}, \quad \text{and} \quad r = |y-z|,$$

and by (3.7.26) we get, for any that $y, z \in \mathbb{R}^n$,

$$|v(y) - v(z)| \leq C |y-z|^\gamma \left(\int_{B_r(x)} |\nabla v(\xi)|^p d\xi \right)^{\frac{1}{p}} \leq C |y-z|^\gamma \|v\|_{W^{1,p}(\mathbb{R}^n)}. \quad (3.7.35)$$

Hence (3.7.34) and (3.7.35) give (3.7.33).

Now, as $\partial\Omega$ is of class $C^{0,1}$ with constants M_0, r_0 , by extension Theorem 3.4.4 we have that if $u \in W^{1,p}(\Omega)$ there exists $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \tilde{u}(x) = u(x), & \text{for } x \in \Omega, \\ \text{supp } \tilde{u}, & \text{is a compact of } \mathbb{R}^n, \\ \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} & \leq C \|u\|_{W^{1,p}(\Omega)}. \end{cases} \quad (3.7.36)$$

By Proposition 3.4.1 we derive that there exists a sequence $\{v_j\}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\{v_j\} \rightarrow \tilde{u}, \quad \text{in } W^{1,p}(\mathbb{R}^n) \quad (3.7.37)$$

and (3.7.33) implies

$$\|v_j - v_k\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|v_j - v_k\|_{W^{1,p}(\mathbb{R}^n)}$$

for every $j, k \in \mathbb{N}$. Hence $\{v_j\}$ is a Cauchy sequence in $C^{0,\gamma}(\mathbb{R}^n)$ and therefore there exists $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$v_j \rightarrow u^*, \quad \text{as } j \rightarrow \infty, \quad \text{in } C^{0,\gamma}(\mathbb{R}^n).$$

By the latter and by (3.7.36), (3.7.37) we obtain

$$u^*|_\Omega = \tilde{u}|_\Omega = u, \quad \text{a.e. in } \Omega.$$

Since (3.7.33) yields

$$\|v_j\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|v_j\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall j \in \mathbb{N},$$

passing to the limit, we have

$$\|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \leq C \|u^*\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C' C \|u\|_{W^{1,p}(\Omega)},$$

which concludes the proof. ■

3.7.4 The General Sobolev inequalities

By Theorems 3.7.3–3.7.9, proceeding by iteration we obtain the following general theorem, whose proof we leave to the reader.

Theorem 3.7.10 (Sobolev embedding). *Let Ω be a bounded open set of class $C^{0,1}$ with constants M_0, r_0 and let $u \in W^{k,p}(\Omega)$.*

(i) If

$$k < \frac{n}{p}, \quad (3.7.38)$$

then $u \in L^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}. \quad (3.7.39)$$

Moreover

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}, \quad (3.7.40)$$

where C depends on M_0, r_0, k and n only.

(ii) If

$$k > \frac{n}{p}, \quad (3.7.41)$$

then $u \in C^{m,\alpha}(\bar{\Omega})$, where $m = k - [\frac{n}{p}] - 1$ and

$$\alpha = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer number,} \\ \text{any positive number,} & \text{if } \alpha < 1 \text{ and } \frac{n}{p} \text{ is an integer number} \end{cases} \quad (3.7.42)$$

and

$$\|u\|_{C^{m,\alpha}(\bar{\Omega})} \leq C \|u\|_{W^{k,p}(\Omega)}, \quad (3.7.43)$$

where C depends on M_0, r_0, k and n only.

Examples.

If $n = 1$ and $u \in H^1(0, 1)$, then $u \in C^{0,1/2}([0, 1])$. If $n = 2$ and $u \in H^1(\Omega)$, then $u \in L^q(\Omega)$ for every $1 \leq q < \infty$ and, if $u \in H^2(\Omega)$ then $u \in C^{0,\alpha}(\overline{\Omega})$ for every $\alpha < 1$. Finally, if $u \in H^k(\Omega)$ for every $k \in \mathbb{N}$, then $u \in C^\infty(\overline{\Omega})$. ♠

3.8 The compactness theorems

In the previous Section we have proved that if Ω is a bounded open set of class $C^{0,1}$ and $1 \leq p < n$ then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$. Moreover the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

is continuous, as inequality (3.7.15) holds true. Similarly, (Theorem 3.7.9), for $n < p < +\infty$, the embedding

$$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p}}(\overline{\Omega}),$$

is continuous. In this Section we will prove compact embedding theorems, in particular, the Rellich – Kondrachov Theorem, which gives the compactness of the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega),$$

for $1 \leq p < n$ and $q < p^*$. This means that any bounded subset Y di $W^{1,p}(\Omega)$ is relatively compact in $L^q(\Omega)$ (namely, \overline{Y} is compact in $L^q(\Omega)$).

Theorem 3.8.1 (Rellich – Kondrachov). *Let Ω be a bounded open set of \mathbb{R}^n with boundary of class $C^{0,1}$ and let $1 \leq p < n$, $1 \leq q < p^* = \frac{np}{n-p}$. Then the embedding of $W^{1,p}(\Omega)$ in $L^q(\Omega)$ is compact.*

In order to prove Theorem 3.8.1 we need the following.

Lemma 3.8.2. *Let $1 \leq q < +\infty$, let Ω be a bounded open set of \mathbb{R}^n and let Λ be the subset of $L^q(\Omega)$ defined as follows*

$$\Lambda = \left\{ u \in L^q(\Omega) : \|u\|_{L^q(\Omega)} \leq 1 \right\}.$$

Let us assume

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{u \in \Lambda} \|u_\varepsilon - u\|_{L^q(\Omega)} \right) = 0, \quad (3.8.1)$$

where

$$u_\varepsilon(x) = \int_{\Omega} \eta_\varepsilon(x-y)u(y)dy$$

and $\eta_\varepsilon = \varepsilon^{-n}\eta(\varepsilon^{-1}x)$ where $\eta \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } \eta \subset B_1$, $\int_{\mathbb{R}^n} \eta(x)dx = 1$.

Then Λ is relatively compact in $L^q(\Omega)$.

Proof of Lemma 3.8.2. We prove that Λ is a totally bounded set in $L^q(\Omega)$.

Let $\delta > 0$. By (3.8.1) we have that there exists $\varepsilon_0 > 0$ so that

$$\|u_{\varepsilon_0} - u\|_{L^q(\Omega)} < \frac{\delta}{2}, \quad \forall u \in \Lambda. \quad (3.8.2)$$

Let

$$\Lambda_0 = \{u_{\varepsilon_0} : u \in \Lambda\}$$

Now we prove that Λ_0 is relatively compact in $C^0(\overline{\Omega})$.

Let us denote

$$M_0 = \sup_{\mathbb{R}^n} |\eta_{\varepsilon_0}|, \quad M_1 = \sup_{\mathbb{R}^n} |\nabla \eta_{\varepsilon_0}|.$$

We have, for any $u \in \Lambda$,

$$\begin{aligned} |u_{\varepsilon_0}(x)| &= \left| \int_{\Omega} \eta_{\varepsilon_0}(x-y)u(y)dy \right| \leq \\ &\leq M_0 |\Omega|^{1-\frac{1}{q}} \|u\|_{L^q(\Omega)} \leq \\ &\leq M_0 |\Omega|^{1-\frac{1}{q}} \end{aligned}$$

and, similarly,

$$|\nabla u_{\varepsilon_0}(x)| = \left| \int_{\Omega} \nabla \eta_{\varepsilon_0}(x-y)u(y)dy \right| \leq M_1 |\Omega|^{1-\frac{1}{q}}.$$

Therefore Λ_0 is equibounded and equicontinuous. Hence, the Arzelà–Ascoli Theorem implies that Λ_0 is relatively compact in $C^0(\overline{\Omega})$.

Now we prove that Λ_0 is relatively compact in $L^q(\Omega)$. The inequality

$$\|w\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}} \|w\|_{C^0(\overline{\Omega})}, \quad \forall w \in C^0(\overline{\Omega}),$$

implies that, for any $w \in C^0(\overline{\Omega}) \subset L^q(\Omega)$ and for any $r > 0$

$$B_{r'}^{C^0}(w) \subset B_r^{L^q}(w),$$

where $r' = r|\Omega|^{-\frac{1}{q}}$, $B_{r'}^{C^0}(w)$ is the open ball of $C^0(\overline{\Omega})$ centered at w with radius r' and $B_r^{L^q}(w)$ is the open ball of $L^q(\Omega)$ centered at w with radius r . Since Λ_0 is relatively compact in $C^0(\overline{\Omega})$, there exist $w_1, \dots, w_{N_r} \in C^0(\overline{\Omega})$ such that

$$\Lambda_0 \subset \bigcup_{j=1}^{N_r} B_{r'}^{C^0}(w_j) \subset \bigcup_{j=1}^{N_r} B_r^{L^q}(w_j).$$

All in all, Λ_0 is totally bounded set of $L^q(\Omega)$. Hence

$$\Lambda_0 \subset \bigcup_{j=1}^N B_{\delta/2}^{L^q}(w_j),$$

where N depends by $\delta > 0$. Consequently, if $u \in \Lambda$, then there exists $j_u \in \{1, \dots, N\}$ so that

$$\|u_{\varepsilon_0} - w_{j_u}\|_{L^q(\Omega)} < \frac{\delta}{2}.$$

By this inequality and by (3.8.2) we derive that, if $u \in \Lambda$ then

$$\|u - w_{j_u}\|_{L^q(\Omega)} < \delta.$$

Hence

$$\Lambda \subset \bigcup_{j=1}^N B_{\delta}^{L^q}(w_j),$$

which implies compactness of Λ . ■

Proof of Theorem 3.8.1. Let us apply Lemma 3.8.2. Set

$$\Lambda = \left\{ u \in W^{1,p}(\Omega) : \|u\|_{W^{1,p}(\Omega)} \leq 1 \right\}. \quad (3.8.3)$$

We begin by proving the Theorem for $q = 1$. Hence, let us prove that

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{u \in \Lambda} \|u_{\varepsilon} - u\|_{L^1(\Omega)} \right) = 0, \quad (3.8.4)$$

where

$$u_{\varepsilon} = (\bar{u} \star \eta_{\varepsilon}),$$

being \bar{u} the extension of u to 0 in $\mathbb{R}^n \setminus \Omega$.

Let $\delta > 0$ and let $\tilde{\Omega} \Subset \Omega$ satisfy

$$\left| \Omega \setminus \tilde{\Omega} \right| < \delta^{\frac{p^*}{p^*-1}}. \quad (3.8.5)$$

We have

$$\int_{\Omega} |u_{\varepsilon}(x) - u(x)| dx = \int_{\Omega \setminus \tilde{\Omega}} |u_{\varepsilon}(x) - u(x)| dx + \int_{\tilde{\Omega}} |u_{\varepsilon}(x) - u(x)| dx. \quad (3.8.6)$$

Now, by the Hölder inequality we derive

$$\begin{aligned} \int_{\Omega \setminus \tilde{\Omega}} |u_{\varepsilon}(x) - u(x)| dx &\leq \left| \Omega \setminus \tilde{\Omega} \right|^{1-\frac{1}{p^*}} \|u_{\varepsilon} - u\|_{L^{p^*}(\Omega)} \leq \\ &\leq \delta \left(\|u_{\varepsilon}\|_{L^{p^*}(\Omega)} + \|u\|_{L^{p^*}(\Omega)} \right), \end{aligned} \quad (3.8.7)$$

On the other hand by the Young inequality for convolutions, we have

$$\|u_{\varepsilon}\|_{L^{p^*}(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}. \quad (3.8.8)$$

Now, Theorem 3.7.3 gives

$$\|u\|_{L^{p^*}(\Omega)} \leq C_1 \|u\|_{W^{1,p}(\Omega)} \leq C_1, \quad \forall u \in \Lambda. \quad (3.8.9)$$

Therefore by (3.8.7) – (3.8.9) we obtain

$$\int_{\Omega \setminus \tilde{\Omega}} |u_{\varepsilon}(x) - u(x)| dx \leq 2C_1\delta, \quad \forall u \in \Lambda. \quad (3.8.10)$$

Now, let us consider second addend on the right-hand side of (3.8.6). Let $\bar{\varepsilon} = \text{dist}(\tilde{\Omega}, \partial\Omega)$. For any $x \in \tilde{\Omega}$ and for any $\varepsilon \in (0, \bar{\varepsilon})$, we have

$$u_{\varepsilon}(x) = \int_{\Omega} \eta_{\varepsilon}(x-y)u(y)dy = \int_{\Omega} \eta(\xi)u(x-\varepsilon\xi)d\xi.$$

Hence

$$\begin{aligned} \int_{\tilde{\Omega}} |u(x-\varepsilon\xi) - u(x)| dx &= \int_{\tilde{\Omega}} dx \left| \int_{\Omega} \eta(\xi)(u(x-\varepsilon\xi) - u(x))d\xi \right| \leq \\ &\leq \int_{\Omega} d\xi \int_{\tilde{\Omega}} \eta(\xi) |u(x-\varepsilon\xi) - u(x)| dx = \\ &= \int_{\Omega} \eta(\xi)d\xi \int_{\tilde{\Omega}} |u(x-\varepsilon\xi) - u(x)| dx. \end{aligned} \quad (3.8.11)$$

Let now $y \in B_1$ and $\varepsilon < \bar{\varepsilon}$, by applying Theorem 3.3.3 we have, for almost every $x \in \tilde{\Omega}$ and for every $\xi \in B_1$,

$$|u(x - \varepsilon\xi) - u(x)| = \left| \int_0^1 \nabla u(x - t\varepsilon\xi) \cdot \varepsilon\xi dt \right| \leq \varepsilon \int_0^1 |\nabla u(x - t\varepsilon\xi)| dt.$$

Hence, for any $u \in \Lambda$, $\xi \in B_1$ and $\varepsilon \in (0, \bar{\varepsilon})$ we have

$$\begin{aligned} \int_{\tilde{\Omega}} |u_\varepsilon(x) - u(x)| dx &\leq \varepsilon \int_0^1 dt \int_{\tilde{\Omega}} |\nabla u(x - t\varepsilon\xi)| dx = \\ &= \varepsilon \int_0^1 dt \int_{\tilde{\Omega} - t\varepsilon\xi} |\nabla u(z)| dz \leq \\ &\leq \varepsilon \int_{\Omega} |\nabla u(z)| dz \leq \varepsilon |\Omega|^{1-\frac{1}{p}} \|u\|_{W^{1,p}(\Omega)} \leq \\ &\leq \varepsilon |\Omega|^{1-\frac{1}{p}}. \end{aligned} \quad (3.8.12)$$

From what we obtained in (3.8.11) and by (3.8.12) we get

$$\int_{\tilde{\Omega}} |u_\varepsilon(x) - u(x)| dx \leq \varepsilon |\Omega|^{1-\frac{1}{p}}.$$

By the latter, by (3.8.6) and by (3.8.10) we have

$$\int_{\Omega} |u_\varepsilon(x) - u(x)| dx \leq 2C_1\delta + \varepsilon |\Omega|^{1-\frac{1}{p}}, \quad \forall u \in \Lambda, \forall \varepsilon \in (0, \bar{\varepsilon}), \quad (3.8.13)$$

which implies

$$\limsup_{\varepsilon \rightarrow 0} \left(\sup_{u \in \Lambda} \|u_\varepsilon - u\|_{L^1(\Omega)} \right) \leq 2C_1\delta$$

and, as δ is arbitrary, we get

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{u \in \Lambda} \|u_\varepsilon - u\|_{L^1(\Omega)} \right) = 0.$$

Hence, by Lemma 3.8.2, Λ is relatively compact in $L^1(\Omega)$.

Now we consider the case $1 < q < p^*$. Denoting $\theta = \frac{q-1}{p^*-1}$, we have $0 < \theta < 1$, $q = 1 - \theta + \theta p^*$, and

$$\begin{aligned} \int_{\Omega} |u_\varepsilon(x) - u(x)|^q dx &= \int_{\Omega} |u_\varepsilon(x) - u(x)|^{1-\theta+\theta p^*} dx \leq \\ &\leq \|u_\varepsilon - u\|_{L^1(\Omega)}^{1-\theta} \|u_\varepsilon - u\|_{L^{p^*}(\Omega)}^\theta. \end{aligned}$$

Hence

$$\|u_\varepsilon - u\|_{L^q(\Omega)} \leq \|u_\varepsilon - u\|_{L^1(\Omega)}^{\frac{1-\theta}{q}} \|u_\varepsilon - u\|_{L^{p^*}(\Omega)}^{\frac{\theta}{q}}. \quad (3.8.14)$$

On the other hand, by (3.8.9) we have

$$\|u\|_{L^{p^*}(\Omega)} \leq C_1, \quad \forall u \in \Lambda$$

and by the Young inequality we get

$$\|u_\varepsilon\|_{L^{p^*}(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)} \leq C_1, \quad \forall u \in \Lambda.$$

Hence, (3.8.13) and (3.8.14) give

$$\|u_\varepsilon - u\|_{L^q(\Omega)} \leq (2C_1)^{\frac{\theta}{q}} \left(2C_1\delta + \varepsilon|\Omega|^{1-\frac{1}{p}}\right)^{\frac{1-\theta}{q}}, \quad \forall u \in \Lambda.$$

Consequently

$$\lim_{\varepsilon \rightarrow 0} \left(\sup_{u \in \Lambda} \|u_\varepsilon - u\|_{L^q(\Omega)} \right) = 0$$

and by Lemma 3.8.2 we have that Λ is relatively compact in $L^q(\Omega)$. ■

Now we state and prove a compactness theorem in the case $p > n$.

Theorem 3.8.3. *Let Ω be a bounded open set of \mathbb{R}^n with boundary of class $C^{0,1}$ and let $p > n$, $\alpha \in (0, \gamma)$, where $\gamma = 1 - \frac{n}{p}$. Then the embedding*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}),$$

is compact.

Proof. Let $\{u_j\}$ be a sequence in $W^{1,p}(\Omega)$ satisfying

$$\|u_j\|_{W^{1,p}(\Omega)} \leq 1, \quad \forall j \in \mathbb{N}.$$

By Theorem 3.7.9 we have

$$\|u_j\|_{C^{0,\alpha}(\overline{\Omega})} \leq C_1, \quad \forall j \in \mathbb{N}, \quad (3.8.15)$$

where C_1 depends on p, n and Ω . The Arzelà–Ascoli Theorem yields that there exists a subsequence $\{u_{k_j}\}$ and $u \in C^0(\overline{\Omega})$ which satisfy

$$\{u_{k_j}\} \rightarrow u, \quad \text{uniformly.} \quad (3.8.16)$$

By (3.8.15) and (3.8.16) we have, for any $x, y \in \bar{\Omega}$, $x \neq y$,

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} = \lim_{j \rightarrow \infty} \frac{|u_{k_j}(x) - u_{k_j}(y)|}{|x - y|^\gamma} \leq C_1. \quad (3.8.17)$$

Hence $u \in C^{0,\gamma}(\bar{\Omega})$. Therefore $u \in C^{0,\alpha}(\bar{\Omega})$ for $0 < \alpha < \gamma$. Now, let us recall the following inequality (see Proposition 2.2.1):

$$\|f\|_{C^{0,\alpha}(\bar{\Omega})} \leq C \|f\|_{C^{0,\gamma}(\bar{\Omega})}^{\frac{\alpha}{\gamma}} \|f\|_{C^0(\bar{\Omega})}^{1-\frac{\alpha}{\gamma}},$$

For every $f \in C^{0,\gamma}(\bar{\Omega})$, where C depends by α , γ and Ω only. By applying such an inequality to $u_{k_j} - u$, taking into account (3.8.15)–(3.8.17), we easily obtain

$$\{u_{k_j}\} \rightarrow u, \quad \text{in } C^{0,\alpha}(\bar{\Omega}).$$

■

3.8.1 Counterexamples

1. The Rellich–Kondrachov Theorem does not hold for $q = p^*$. Indeed, we have the following counterexample. Let $u \in C_0^\infty(B_1 \setminus \{0\})$ be a **not identically vanishing** function and let

$$u_j(x) = j^{\frac{n}{p^*}} u(jx), \quad \forall j \in \mathbb{N}, \forall x \in B_1.$$

We have (see beginning of Section 3.7.1)

$$\|u_j\|_{L^{p^*}(B_1)} = \|u\|_{L^{p^*}(B_1)}, \quad \forall j \in \mathbb{N}, \quad (3.8.18)$$

$$\|u_j\|_{L^p(B_1)} = j^{-1} \|u\|_{L^p(B_1)}, \quad \forall j \in \mathbb{N}, \quad (3.8.19)$$

$$\|\nabla u_j\|_{L^p(B_1)} = \|\nabla u\|_{L^p(B_1)}, \quad \forall j \in \mathbb{N}. \quad (3.8.20)$$

Hence, by (3.8.19) and (3.8.20) we have

$$\|u_j\|_{W^{1,p}(B_1)} = C \|u\|_{W^{1,p}(B_1)} < +\infty, \quad \forall j \in \mathbb{N}. \quad (3.8.21)$$

Moreover

$$\lim_{j \rightarrow \infty} u_j(x) = 0, \quad \forall x \in B_1. \quad (3.8.22)$$

Now, if the embedding

$$W^{1,p}(B_1) \hookrightarrow L^{p^*}(B_1)$$

were compact, by (3.8.21) there should exist a subsequence $\{u_{k_j}\}$ and $v \in L^{p^*}(B_1)$ such that

$$\{u_{k_j}\} \rightarrow v, \quad \text{in } L^{p^*}(B_1). \quad (3.8.23)$$

Hence, by (3.8.18) we should have

$$\|v\|_{L^{p^*}(B_1)} = \|u\|_{L^{p^*}(B_1)}. \quad (3.8.24)$$

On the other hand, passing eventually to another subsequence, by (3.8.23) we should have

$$v(x) = \lim_{j \rightarrow \infty} u_{k_j}(x) \quad \text{a.e. } x \in B_1,$$

from the latter and from (3.8.22) we should have

$$v(x) = 0, \quad \text{a.e. } x \in B_1,$$

that would contradict (3.8.24).

2. Now, let us consider the case where $\Omega = \mathbb{R}^n$ and let us show that if $1 \leq p < n$ and $q \leq p^*$, then the embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n),$$

is **not** compact.

Let $u \in C_0^\infty(\mathbb{R}^n)$ be a not identically vanishing function such that $\text{supp } u \subset B_1$ and let

$$u_j(x) = u(x - 2je_1), \quad \forall j \in \mathbb{N}.$$

We obtain

$$\|u_j\|_{W^{1,p}(\mathbb{R}^n)} = \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall j \in \mathbb{N}$$

and

$$\|u_j - u_k\|_{L^q(\mathbb{R}^n)} = 2\|u\|_{L^q(\mathbb{R}^n)} > 0, \quad \forall j, k \in \mathbb{N}, j \neq k. \quad (3.8.25)$$

Hence $\{u_j\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ but, as (3.8.25) holds true, we cannot extract any subsequence that converges in $L^q(\mathbb{R}^n)$.

3. Let us prove that if Ω is a bounded open set and $p > n$, $\gamma = 1 - \frac{n}{p}$ then the embedding

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$$

is **not** compact.

Let $u \in C_0^\infty(B_1)$, not identically equal to 0. Denote by \bar{u} the extension of u to 0 in $\mathbb{R}^n \setminus B_1$. Let us denote

$$u_j(x) = \frac{1}{j^\gamma} \bar{u}(jx), \quad \forall j \in \mathbb{N}, \forall x \in \overline{B_1}.$$

Now, let us notice (the reader check as an exercise)

$$[\bar{u}]_{0,\gamma,\mathbb{R}^n} = [u]_{0,\gamma,B_1}$$

and

$$\begin{aligned} [u_j]_{0,\gamma,B_1} &= \sup_{x,y \in B_1, x \neq y} \frac{|u_j(x) - u_j(y)|}{|x - y|^\gamma} = \\ &= \sup_{x,y \in B_1, x \neq y} \frac{|\bar{u}(jx) - \bar{u}(jy)|}{|jx - jy|^\gamma} = \\ &= [\bar{u}]_{0,\gamma,\mathbb{R}^n}. \end{aligned}$$

In addition we have

$$\begin{aligned} \|u_j\|_{L^p(B_1)} &= \frac{1}{j^\gamma} \left(\int_{B_1} |u(jx)|^p dx \right)^{\frac{1}{p}} = \\ &= \frac{1}{j^{\gamma + \frac{n}{p}}} \left(\int_{B_{1/j}} |u(x)|^p dx \right)^{\frac{1}{p}} = \\ &= \frac{1}{j} \left(\int_{B_{1/j}} |u(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} \|\nabla u_j\|_{L^p(B_1)} &= \frac{1}{j^\gamma} \left(\int_{B_1} |(\nabla u)(jx)|^p j^p dx \right)^{\frac{1}{p}} = \\ &= \frac{j^{1 - \frac{n}{p}}}{j^\gamma} \left(\int_{B_{1/j}} |\nabla u(x)|^p dx \right)^{\frac{1}{p}} = \\ &= \left(\int_{B_{1/j}} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\{u_j\} \rightarrow 0, \quad \text{in } W^{1,p}(B_1),$$

in particular, $\{u_j\}$ is a bounded sequence in $W^{1,p}(B_1)$. On the other hand, if there was a subsequence $\{u_{k_j}\}$ of $\{u_j\}$ and $v \in C^{0,\gamma}(\overline{B_1})$ such that

$$\{u_{k_j}\} \rightarrow v, \quad \text{in } C^{0,\gamma}(\overline{B_1}),$$

we should necessarily have $v \equiv 0$ and

$$[u_j - v]_{0,\gamma,B_1} = [u_j]_{0,\gamma,B_1} = [\bar{u}]_{0,\gamma,\mathbb{R}^n} > 0$$

Which is a contradiction.

4. The case $p > n$, $\Omega = \mathbb{R}^n$, can be handle similarly to the case $p < n$. Let $u \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } u \subset B_1$, u not identically vanishing function; let $u_j(x) = u(x - 2je_1)$. We have

$$\|u_j\|_{W^{1,p}(\mathbb{R}^n)} = \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall j \in \mathbb{N}$$

and

$$[u_j - u_k]_{0,\alpha,\mathbb{R}^n} \geq [u_j]_{0,\alpha,\mathbb{R}^n} = [u]_{0,\alpha,\mathbb{R}^n} > 0, \quad \text{for } j \neq k,$$

where $\alpha \leq 1 - \frac{n}{p}$. Hence, no extracted sequence of $\{u_j\}$ can be a Cauchy sequence in $C^{0,\alpha}(\mathbb{R}^n)$.

3.9 The second Poincaré inequality

In Theorem 3.7.6 we have proved the Sobolev–Poincaré inequality that, in particular, holds in the following form (see the proof of the above mentioned Theorem)

$$\left(\int_{B_r(x)} |u(y) - (u)_{x,r}|^{p^*} dy \right)^{\frac{1}{p^*}} \leq Cr \left(\int_{B_r(x)} |\nabla u(y)|^p dy \right)^{\frac{1}{p}}, \quad (3.9.1)$$

for every $u \in W^{1,p}(B_r(x))$, where $p \in [1, +\infty)$ (actually it holds true for $p = +\infty$). We will now prove a more general version of (3.9.1).

Theorem 3.9.1 (The second Poincaré inequality). *Let Ω be a bounded connected open set of \mathbb{R}^n with $\partial\Omega$ of class $C^{0,1}$. Let $p \in [1, +\infty)$ and*

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u dx.$$

Then there exists a constant C depending on p, n and Ω only, such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \quad (3.9.2)$$

Proof. We argue by contradiction. Let us assume that (3.9.2) does not hold. Consequently for any $k \in \mathbb{N}$ there exists $u_k \in W^{1,p}(\Omega)$ such that

$$\|u_k - (u_k)_\Omega\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)}.$$

Let us denote

$$v_k = \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}, \quad \forall k \in \mathbb{N}.$$

We have

$$(v_k)_\Omega = 0,$$

$$\|v_k\|_{L^p(\Omega)} = 1$$

and

$$k \|\nabla v_k\|_{L^p(\Omega)} < 1. \quad (3.9.3)$$

Hence, there exists $M < +\infty$ such that

$$\|v_k\|_{W^{1,p}(\Omega)} \leq M.$$

Therefore, by Rellich–Kondrachov Theorem we have that there exists a subsequence $\{v_{k_j}\}$ of $\{v_k\}$, and $v \in L^p(\Omega)$ which satisfy

$$\{v_{k_j}\} \rightarrow v, \quad \text{in } L^p(\Omega).$$

Hence

$$v_\Omega = 0 \quad (3.9.4)$$

and

$$\|v\|_{L^p(\Omega)} = \lim_{j \rightarrow \infty} \|v_{k_j}\|_{L^p(\Omega)} = 1. \quad (3.9.5)$$

On the other hand by (3.9.3) we have

$$\int_{\Omega} v \partial_l \phi dx = \lim_{j \rightarrow \infty} \int_{\Omega} v_{k_j} \partial_l \phi dx = - \lim_{j \rightarrow \infty} \int_{\Omega} \partial_l v_{k_j} \phi dx = 0, \quad \forall \phi \in C_0^\infty(\Omega),$$

for $l = 1, \dots, n$. Consequently

$$\int_{\Omega} v \partial_l \phi dx = 0, \quad \forall \phi \in C_0^\infty(\Omega), \quad l = 1, \dots, n.$$

Therefore $\nabla v = 0$ in Ω (and, trivially, $v \in W^{1,p}(\Omega)$). Since Ω is a connected open set, Proposition 3.3.5 yields that there exists a constant $c_0 \in \mathbb{R}$ such that

$$v \equiv c_0,$$

and by (3.9.4) we have $c_0 = 0$ that contradicts (3.9.5). Therefore (3.9.2) holds true. ■

Remark. The proof of (3.9.2) that we have given before is not constructive and this does not allow us to further specify the dependence of the constant C on Ω . To fill this gap we refer the reader to [4]. ♦

3.10 The difference quotients

In this Section we provide the definition and the main properties of the difference quotients. These topics will turn out to be useful in the study of the regularity of the solutions of second order elliptic equations.

Definition 3.10.1. Let V and Ω be open sets \mathbb{R}^n such that $V \Subset \Omega$. Let $j \in \{1, \dots, n\}$ and let $u \in L^1_{loc}(\Omega)$. The following function

$$\delta_j^h u(x) = \frac{u(x + he_j) - u(x)}{h}, \quad \forall x \in V. \quad (3.10.1)$$

is called j -th partial quotient of u with increment $h \in \mathbb{R} \setminus \{0\}$, $|h| < \text{dist}(V, \partial\Omega)$. We denote

$$\delta^h u(x) = (\delta_1^h u(x), \dots, \delta_n^h u(x)), \quad \forall x \in V. \quad (3.10.2)$$

We have the following

Theorem 3.10.2. *Let Ω be an open set of \mathbb{R}^n .*

(i) *If $p \in [1, +\infty)$, $u \in W^{1,p}(\Omega)$, then*

$$\|\delta^h u\|_{L^p(V)} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \text{for } |h| < \frac{1}{2} \text{dist}(V, \partial\Omega), \quad h \neq 0 \quad (3.10.3)$$

where C depends on n only.

(ii) *Let us assume $p \in (1, +\infty)$, $u \in L^p(\Omega)$ and let us assume that there exists $C > 0$ satisfying*

$$\|\delta^h u\|_{L^p(V)} \leq C, \quad \text{for } |h| < \frac{1}{2} \text{dist}(V, \partial\Omega), \quad h \neq 0 \quad (3.10.4)$$

then

$$u \in W^{1,p}(V) \quad \text{and} \quad \|\nabla u\|_{L^p(V)} \leq C.$$

Proof. In order to prove (i) it suffices to assume $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ and to apply Theorem 3.3.2.

If $0 < |h| < \text{dist}(V, \partial\Omega)$, $j = 1, \dots, n$ and $x \in V$, we have

$$u(x + he_j) - u(x) = \int_0^1 \nabla u(x + the_j) \cdot (he_j) dt,$$

from which we have

$$|u(x + he_j) - u(x)| \leq |h| \int_0^1 |\nabla u(x + the_j)| dt.$$

By using Hölder inequality and by integrating both the sides of the last inequality over V , we get

$$\begin{aligned} \int_V |\delta_j^h u|^p dx &\leq \int_V dx \int_0^1 |\nabla u(x + the_j)|^p dt = \\ &= \int_0^1 dt \int_V |\nabla u(x + the_j)|^p dx \leq \\ &\leq \int_\Omega |\nabla u|^p dx. \end{aligned}$$

Now, let us prove (ii). Let us assume that for some $C > 0$ we have

$$\|\delta^h u\|_{L^p(V)} \leq C, \quad \text{for } 0 < |h| < \frac{1}{2} \text{dist}(V, \partial\Omega). \quad (3.10.5)$$

Claim

If $\phi \in C_0^\infty(V)$ and let us denote $K = \text{supp } \phi$, then for any $j \in \{1, \dots, n\}$ we have

$$\int_V u \delta_j^h \phi dx = - \int_V \delta_j^{-h} u \phi dx, \quad \text{for } 0 < |h| < \text{dist}(K, \partial V). \quad (3.10.6)$$

Proof of the Claim. Let us notice that

$$K - he_j \subset V \quad \text{for } 0 < |h| < \text{dist}(K, \partial V),$$

for $j = 1, \dots, n$. Hence we have

$$\begin{aligned}
\int_V u \delta_j^h \phi dx &= \frac{1}{h} \left\{ \int_V u(x) \phi(x + h e_j) dx - \int_V u(x) \phi(x) dx \right\} = \\
&= \frac{1}{h} \left\{ \int_{K - h e_j} u(x) \phi(x + h e_j) dx - \int_V u(x) \phi(x) dx \right\} = \\
&= \frac{1}{h} \left\{ \int_K u(x - h e_j) \phi(x) dx - \int_V u(x) \phi(x) dx \right\} = \\
&= \frac{1}{h} \left\{ \int_V u(x - h e_j) \phi(x) dx - \int_V u(x) \phi(x) dx \right\} = \\
&= - \int_V \delta_j^{-h} u \phi dx.
\end{aligned}$$

Claim is proved.

Let us fix $j \in \{1, \dots, n\}$. Since $L^p(V)$ is a reflexive Banach space for $1 < p < +\infty$, by

$$\sup \|\delta_j^{-h} u\|_{L^p(V)} \leq C$$

(recalling Theorems 2.3.31 and 2.3.32) there exists a sequence $\{h_k\}$ which goes to 0 and $v_j \in L^p(V)$, such that

$$\{\delta_j^{-h_k} u\} \rightharpoonup v_j, \quad \text{weakly in } L^p(V). \quad (3.10.7)$$

On the other hand, by the Dominated Convergence Theorem we have, for any $\phi \in C_0^\infty(\Omega)$ such that $\text{supp } \phi \subset V$,

$$\int_\Omega u \partial_j \phi dx = \lim_{k \rightarrow \infty} \int_\Omega u \delta_j^{h_k} \phi dx.$$

As a matter of fact

$$u(x) \delta_j^{h_k} \phi(x) \rightarrow u(x) \partial_j \phi(x), \quad \forall x \in \Omega \text{ as } k \rightarrow \infty$$

and

$$|u \delta_j^{h_k} \phi| \leq |u| \|\nabla \phi\|_{L^\infty(\Omega)} \chi_{\tilde{V}}, \quad \forall k \in \mathbb{N},$$

where

$$\tilde{V} = \left\{ x \in \Omega : \text{dist}(x, V) \leq \frac{1}{2} \text{dist}(V, \partial\Omega) \right\}.$$

Therefore

$$\begin{aligned}
\int_V u \partial_j \phi dx &= \int_\Omega u \partial_j \phi dx = \\
&= \lim_{k \rightarrow \infty} \int_\Omega u \left(\delta_j^{h_k} \phi \right) dx = \\
&= - \lim_{k \rightarrow \infty} \int_\Omega \left(\delta_j^{-h_k} u \right) \phi dx = \\
&= - \lim_{k \rightarrow \infty} \int_V \left(\delta_j^{-h_k} u \right) \phi dx = \\
&= - \int_V v_j \phi dx.
\end{aligned}$$

Consequently

$$\partial_j u = v_j, \quad \text{in the weak sense for } j = 1, \dots, n.$$

Hence $\nabla u \in L^p(V, \mathbb{R}^n)$, but $u \in L^p(V)$. Therefore $u \in W^{1,p}(V)$.

Finally, by (3.10.7) we have

$$\|\nabla u\|_{L^p(V)} \leq \liminf_{k \rightarrow \infty} \|\delta^{-h_k} u\|_{L^p(V)} \leq C,$$

(C is the same constant that occurs in (3.10.5)). ■

Remark. If $p = 1$, then (ii) of Theorem 3.10.2 does not hold. As a matter of fact, let $\Omega = (-2, 2)$ and

$$u(t) = \chi_{(-1,1)}.$$

We have $u \in L^1(-2, 2)$. Let $V = (-\frac{3}{2}, \frac{3}{2})$. Now, $\text{dist}(V, \partial\Omega) = \frac{1}{2}$ and for $0 < |h| < \frac{1}{4}$ we have (for $h > 0$)

$$\begin{aligned}
\delta^h u(t) &= \frac{\chi_{(-1,1)}(t+h) - \chi_{(-1,1)}(t)}{h} = \\
&= \frac{\chi_{(-1-h, 1-h)}(t) - \chi_{(-1,1)}(t)}{h} = \\
&= \frac{1}{h} \chi_{(-1, -1-h) \cup (1-h, 1)}.
\end{aligned}$$

Hence

$$\int_V |\delta^h u(t)| dt = 2, \quad \text{for } 0 < |h| < \frac{1}{4},$$

but (see Example 2 of Section 3.1)

$$u \notin W^{1,1}(V).$$



In the sequel we will use the following variant of Theorem 3.10.2.

Theorem 3.10.3. *Let $r > 0$. We have*

(i) *If $p \in [1, +\infty)$ and $u \in W^{1,p}(B_r^+)$, then for any $k \in \{1, \dots, n-1\}$ we have, for $0 < |h| < \frac{r}{2}$,*

$$\|\delta_k^h u\|_{L^p(B_{r/2}^+)} \leq C \|\partial_k u\|_{L^p(B_r^+)}, \quad (3.10.8)$$

where C depends on n only.

(ii) *Let $k \in \{1, \dots, n-1\}$. Let $p \in (1, +\infty)$, $u \in L^p(B_r^+)$ and let us suppose that there exists $C > 0$ such that*

$$\|\delta_k^h u\|_{L^p(B_{r/2}^+)} \leq C, \quad \text{for } 0 < |h| < \frac{r}{2}, \quad (3.10.9)$$

then

$$\partial_k u \in L^p(B_r^+) \quad \text{and} \quad \|\partial_k u\|_{L^p(B_r^+)} \leq C.$$

The proof of the above Theorem is completely analogous to the one of Theorem 3.10.2 and it is left to the reader as an exercise.

3.11 The dual space of $H_0^1(\Omega)$

Let Ω be an open set of \mathbb{R}^n . We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ (i.e. the space of the linear bounded form from $H_0^1(\Omega)$ to \mathbb{R}). If $F \in H^{-1}(\Omega)$, we write

$$\langle F, v \rangle := F(v), \quad v \in H_0^1(\Omega)$$

and

$$\|F\|_{H^{-1}(\Omega)} = \sup \left\{ \langle F, v \rangle : v \in H_0^1(\Omega), \|v\|_{H_0^1(\Omega)} \leq 1 \right\}.$$

The following Theorem holds true.

Theorem 3.11.1 (characterization of $H^{-1}(\Omega)$). *Let Ω be an open set of \mathbb{R}^n .*

(i) *$F \in H^{-1}(\Omega)$ if and only if there exist $f_0, f_1, \dots, f_n \in L^2(\Omega)$ satisfying*

$$\langle F, v \rangle = \int_{\Omega} f_0 v dx + \sum_{j=1}^n \int_{\Omega} f_j v_{x_j} dx, \quad \forall v \in H_0^1(\Omega). \quad (3.11.1)$$

(ii)

$$\begin{aligned} & \|F\|_{H^{-1}(\Omega)} = \\ & = \inf \left\{ \left(\sum_{j=0}^n \int_{\Omega} |f_j|^2 dx \right)^{1/2} : F \text{ satisfies (3.11.1) for } f_0, f_1, \dots, f_n \in L^2(\Omega) \right\}. \end{aligned}$$

We also write

$$F = f_0 - \sum_{j=1}^n \partial_j f_j.$$

If

$$f_1, \dots, f_n = 0,$$

we will identify the functional

$$\langle F, v \rangle = \int_{\Omega} f_0 v dx, \quad \forall v \in H_0^1(\Omega)$$

with f_0 and we will write $F \in L^2(\Omega)$. Similarly, if $f_0 \in H^k(\Omega)$, we will write $F \in H^k(\Omega)$.

Let us note that f_0, f_1, \dots, f_n are *not* uniquely determined. For instance, if Ω is bounded, then the functional

$$\langle F, v \rangle = \int_{\Omega} f_0 v dx,$$

where $f_0 \in L^2(\Omega)$, can also be represented by

$$\langle F, v \rangle = \int_{\Omega} (f_0 + 2x_1) v dx + \int_{\Omega} x_1^2 \partial_1 v dx, \quad \forall v \in H_0^1(\Omega)$$

and in infinite other ways.

Proof of Theorem 3.11.1. Let us equip $H_0^1(\Omega)$ with the scalar product

$$(u, v)_{H_0^1(\Omega)} = \int_{\Omega} (uv + \nabla u \cdot \nabla v) dx, \quad \forall u, v \in H_0^1(\Omega).$$

It is clear that if F is like (3.11.1), then $F \in H^{-1}(\Omega)$, as a matter of fact, by applying the Cauchy–Schwarz inequality we get

$$\begin{aligned} |\langle F, v \rangle| &= \left| \int_{\Omega} f_0 v dx + \sum_{j=1}^n \int_{\Omega} f_j \partial_j v dx \right| \leq \\ &\leq \left(\sum_{j=0}^n \int_{\Omega} |f_j|^2 dx \right)^{1/2} \|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{3.11.2}$$

Conversely, let us assume $F \in H^{-1}(\Omega)$. By the Riesz representation Theorem we have that there exists a unique $u \in H_0^1(\Omega)$ such that

$$\langle F, v \rangle = (u, v)_{H_0^1(\Omega)} = \int_{\Omega} (uv + \nabla u \cdot \nabla v) dx, \quad \forall v \in H_0^1(\Omega). \quad (3.11.3)$$

Hence, denoting

$$f_0 = u, \quad f_j = \partial_j u, \quad j = 1, \dots, n, \quad (3.11.4)$$

we have

$$\langle F, v \rangle = \int_{\Omega} f_0 v dx + \sum_{j=1}^n \int_{\Omega} f_j \partial_j v dx, \quad \forall v \in H_0^1(\Omega). \quad (3.11.5)$$

The proof of (i) is concluded.

Now, let us prove (ii). Let $u \in H_0^1(\Omega)$ and $f_j \in L^2(\Omega)$, $j = 0, 1, \dots, n$, be like in (3.11.4). Let $g_j \in L^2(\Omega)$, $j = 0, 1, \dots, n$ satisfy

$$\langle F, v \rangle = \int_{\Omega} \left(g_0 v + \sum_{j=1}^n g_j \partial_j v \right) dx, \quad \forall v \in H_0^1(\Omega).$$

Let us check that

$$\int_{\Omega} \sum_{j=0}^n |f_j|^2 dx \leq \int_{\Omega} \sum_{j=0}^n |g_j|^2 dx. \quad (3.11.6)$$

We have

$$\begin{aligned} \int_{\Omega} (|u|^2 + |\nabla u|^2) dx &= \langle F, u \rangle = \int_{\Omega} \left(g_0 u + \sum_{j=1}^n g_j \partial_j u \right) dx \leq \\ &\leq \left(\int_{\Omega} \sum_{j=0}^n |g_j|^2 dx \right)^{1/2} \left(\int_{\Omega} (|u|^2 + |\nabla u|^2) dx \right)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\Omega} \sum_{j=0}^n |f_j|^2 dx &= \int_{\Omega} (|u|^2 + |\nabla u|^2) dx \leq \\ &\leq \int_{\Omega} \sum_{j=0}^n |g_j|^2 dx, \end{aligned} \quad (3.11.7)$$

which proves (3.11.6).

In order to complete the proof, let us notice that by (3.11.2) we get

$$\|F\|_{H^{-1}(\Omega)} \leq \left(\sum_{j=0}^n \int_{\Omega} |f_j|^2 dx \right)^{1/2}. \quad (3.11.8)$$

On the other hand, setting

$$\tilde{v} = \frac{u}{\|u\|_{H_0^1(\Omega)}},$$

we obtain, by (3.11.5) (recall that u satisfies (3.11.4)),

$$\begin{aligned} \langle F, \tilde{v} \rangle &= \int_{\Omega} (u\tilde{v} + \nabla u \cdot \nabla \tilde{v}) dx = \\ &= \|u\|_{H_0^1(\Omega)} = \\ &= \left(\sum_{j=0}^n \int_{\Omega} |f_j|^2 dx \right)^{1/2}. \end{aligned} \quad (3.11.9)$$

Hence, by (3.11.8) and (3.11.9) we have

$$\|F\|_{H^{-1}(\Omega)} = \left(\sum_{j=0}^n \int_{\Omega} |f_j|^2 dx \right)^{1/2}.$$

By the just obtained equality and by (3.11.7) we obtain (ii). ■

Remark. Let us note that the greatest lower bound in (ii) is actually the minimum. ♦

Exercise. Let us denote by $H^{-m}(\Omega)$, $m \in \mathbb{N}$, the dual space of $H_0^m(\Omega)$. Prove that $F \in H^{-m}(\Omega)$ if and only if there exist $f_{\alpha} \in L^2(\Omega)$, $|\alpha| \leq m$ such that

$$\langle F, \tilde{v} \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} f_{\alpha} \partial^{\alpha} v dx, \quad \forall v \in H_0^m(\Omega)$$

and prove the analogue of the part (ii) of Theorem 3.11.1. ♣

3.12 The Sobolev spaces with noninteger exponents and traces

In the present Section we will provide a characterization of the traces of $H^k(\Omega)$ function, $k \in \mathbb{N}$. For this purpose we need to extend the notion of the Sobolev space that we have studied so far to the spaces with non integer exponents. First of all, we provide brief reminders of the Fourier transform.

3.12.1 Review of the Fourier transform

Let us denote by $D_j = \frac{1}{i}\partial_j$, $j = 1, \dots, n$.

Definition 3.12.1. We denote by \mathcal{S} the space of functions $f \in C^\infty(\mathbb{R}^n)$ which satisfy

$$p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f| < \infty, \quad \forall \alpha, \beta \in \mathbb{N}_0^n. \quad (3.12.1)$$

The topology on \mathcal{S} is defined by the seminorms $p_{\alpha,\beta}(f)$.

According to Definition 3.12.1, a sequence $\{f_k\} \subset \mathcal{S}$ converges to $f \in \mathcal{S}$ if and only if

$$\lim_{k \rightarrow \infty} p_{\alpha,\beta}(f_k - f) = 0, \quad \forall \alpha, \beta \in \mathbb{N}_0^n.$$

The space \mathcal{S} is known as **the Schwartz space** or, also, the space of rapidly decreasing functions; equipped with the family of seminorms $\{p_{\alpha,\beta}\}$, \mathcal{S} is a Fréchet space (for the definition of Fréchet space we refer to [69] and, in the present Notes, Section 9.2). We have

$$C_0^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^p(\mathbb{R}^n), \quad \forall p \in [1, +\infty].$$

It is simple to prove that $C_0^\infty(\mathbb{R}^n)$ is dense in \mathcal{S} . The function $f(x) = e^{-|x|^2}$ is an example of function that does not belong to $C_0^\infty(\mathbb{R}^n)$, but belongs to \mathcal{S} .

Definition 3.12.2. Let $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform by

$$\widehat{f}(\xi) := \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad \forall \xi \in \mathbb{R}^n.$$

We have

$$\left\| \widehat{f} \right\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}, \quad \forall f \in L^1(\mathbb{R}^n).$$

Actually, we have $\widehat{f} \in C^0(\mathbb{R}^n)$ and

$$\widehat{f}(\xi) \rightarrow 0, \text{ as } |\xi| \rightarrow +\infty. \quad (3.12.2)$$

Property (3.12.2) is known as **Riemann–Lebesgue Lemma**.

Let us recall that, if $f(x) = e^{-\frac{|x|^2}{2}}$ then

$$\widehat{f}(\xi) = (2\pi)^{n/2} e^{-\frac{|\xi|^2}{2}}.$$

Theorem 3.12.3. *Let $f \in \mathcal{S}$. Then we have $\widehat{f} \in \mathcal{S}$. Moreover the following properties hold.*

a) *The map*

$$\mathcal{S} \ni f \rightarrow \widehat{f} \in \mathcal{S}$$

is one-to-one, continuous, with continuous inverse and

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad \forall x \in \mathbb{R}^n; \quad (3.12.3)$$

b)

$$\widehat{D_x^\alpha f}(\xi) = \xi^\alpha \widehat{f}(\xi), \quad \forall f \in \mathcal{S};$$

c)

$$\widehat{(x^\alpha f)}(\xi) = (-1)^{|\alpha|} D_\xi^\alpha \widehat{f}(\xi) \quad \forall f \in \mathcal{S}.$$

Formula (3.12.3) is known as the **inversion formula for the Fourier transform**. If $f \in L^1(\mathbb{R}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix \cdot \xi} d\xi, \quad \text{a.e. in } \mathbb{R}^n. \quad (3.12.4)$$

Theorem 3.12.4. *Let $f, g \in \mathcal{S}$. We have*

$$\int_{\mathbb{R}^n} f(x) \widehat{g}(x) dx = \int_{\mathbb{R}^n} \widehat{f}(x) g(x) dx, \quad (3.12.5)$$

$$\int_{\mathbb{R}^n} f(x) \overline{g(x)} dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi, \quad (3.12.6)$$

$$\widehat{(f \star g)}(\xi) = \widehat{f}(\xi) \widehat{g}(\xi), \quad (3.12.7)$$

$$\widehat{(fg)}(\xi) = \frac{1}{(2\pi)^n} (\widehat{f} \star \widehat{g})(\xi). \quad (3.12.8)$$

Formula (3.12.6) is known as **Parseval formula** and it is equivalent to the following one

$$\|f\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}} \|\widehat{f}\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{S}. \quad (3.12.9)$$

Let us notice that the restriction of the linear operator \mathcal{F} over \mathcal{S} acts as follows

$$\mathcal{S} \ni f \rightarrow \mathcal{F}(f) := \widehat{f} \in \mathcal{S}.$$

Moreover \mathcal{F} is bijective and by (3.12.9) we have

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{S}. \quad (3.12.10)$$

Let us observe that, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and

$$C_0^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^2(\mathbb{R}^n),$$

then \mathcal{S} is dense in $L^2(\mathbb{R}^n)$. Hence (3.12.10) implies that the linear operator \mathcal{F} can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. We continue to denote by \mathcal{F} such an extension. Hence it is defined

$$\widehat{f} := \mathcal{F}(f), \quad \forall f \in L^2(\mathbb{R}^n).$$

It can be proved that the operator

$$\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

is bijective and Theorem 3.12.4 continue to holds. Moreover, denoting by

$$\mathcal{C} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n),$$

$$(\mathcal{C}(f))(x) = \frac{1}{(2\pi)^n} f(-x), \quad \forall f \in L^2(\mathbb{R}^n), \quad \forall x \in \mathbb{R}^n,$$

we have

$$f = \mathcal{C}\mathcal{F}(f), \quad \forall f \in L^2(\mathbb{R}^n). \quad (3.12.11)$$

If $f \in \mathcal{S}$ or $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then formula (3.12.11) is nothing but inversion formula (3.12.3) or (3.12.4) respectively. For the proofs and much more details we refer the reader to [36, Vol. I], [23], [69]

3.12.2 Fourier transform and $H^k(\mathbb{R}^n)$ spaces, $k \in \mathbb{N}_0$

Let us state and prove

Theorem 3.12.5. *Let $k \in \mathbb{N}_0$. The following properties hold*

(i) *Let $u \in L^2(\mathbb{R}^n)$. We have that $u \in H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |\xi|^2)^{k/2} \widehat{u}(\xi) \in L^2(\mathbb{R}^n). \quad (3.12.12)$$

(ii) *There exists a constant $C \geq 1$ depending on k and n only, such that*

$$C^{-1} \|u\|_{H^k(\mathbb{R}^n)} \leq \left\| (1 + |\xi|^2)^{k/2} \widehat{u}(\xi) \right\|_{L^2(\mathbb{R}^n)} \leq C \|u\|_{H^k(\mathbb{R}^n)}, \quad (3.12.13)$$

for every $u \in H^k(\mathbb{R}^n)$.

Proof. If $k = 0$, then (i) and (ii) are obvious. Let us assume $k \geq 1$ and let us begin to prove (i). Since $\mathcal{S} \subset H^k(\mathbb{R}^n)$, we have that if $u \in H^k(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} \partial^\alpha u \varphi dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u \partial^\alpha \varphi dx, \quad \text{for } |\alpha| \leq k, \quad \forall \varphi \in \mathcal{S}. \quad (3.12.14)$$

Claim I

$$u \in H^k(\mathbb{R}^n) \iff (i\xi)^\alpha \widehat{u}(\xi) \in L^2(\mathbb{R}^n), \quad \text{for } |\alpha| \leq k. \quad (3.12.15)$$

Proof of Claim I. Let us prove " \implies ". Let $u \in H^k(\mathbb{R}^n)$, R be an arbitrary positive number and $\psi \in C_0^\infty(B_R)$. Let us define

$$\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\psi(\xi)} e^{ix \cdot \xi} d\xi, \quad \forall x \in \mathbb{R}^n.$$

We have

$$\varphi \in \mathcal{S}, \quad \text{and} \quad \widehat{\varphi}(\xi) = \psi(\xi), \quad \forall \xi \in \mathbb{R}^n.$$

In addition, as $u \in L^2(\mathbb{R}^n)$,

$$(i\xi)^\alpha \widehat{u}(\xi)|_{B_R} \in L^2(B_R), \quad \text{for } |\alpha| \leq k,$$

hence $(i\xi)^\alpha \widehat{u}(\xi) \psi(\xi) \in L^2(\mathbb{R}^n)$ and by the Parseval identity we have, for $|\alpha| \leq k$,

$$\begin{aligned}
\int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{u}(\xi) \psi(\xi) d\xi &= \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{u}(\xi) \overline{\widehat{\varphi}(\xi)} d\xi = \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \widehat{u}(\xi) \overline{(i\xi)^\alpha \widehat{\varphi}(\xi)} d\xi = \\
&= (-1)^{|\alpha|} (2\pi)^n \int_{\mathbb{R}^n} u(x) \overline{\partial^\alpha \varphi(x)} dx = \\
&= (2\pi)^n \int_{\mathbb{R}^n} \partial^\alpha u(x) \overline{\varphi(x)} dx = \\
&= \int_{\mathbb{R}^n} \widehat{\partial^\alpha u}(\xi) \overline{\widehat{\varphi}(\xi)} d\xi = \\
&= \int_{\mathbb{R}^n} \widehat{\partial^\alpha u}(\xi) \psi(\xi) d\xi
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n} \left((i\xi)^\alpha \widehat{u} - \widehat{\partial^\alpha u} \right) \psi d\xi = 0, \quad \forall \psi \in C_0^\infty(B_R),$$

but $\widehat{\partial^\alpha u}|_{B_R} \in L^2(B_R)$ for $|\alpha| \leq k$, (because $u \in H^k(\mathbb{R}^n)$), therefore

$$(i\xi)^\alpha \widehat{u} = \widehat{\partial^\alpha u}, \quad \text{in } B_R$$

and since R is arbitrary,

$$(i\xi)^\alpha \widehat{u} = \widehat{\partial^\alpha u}, \quad \text{in } \mathbb{R}^n.$$

In particular

$$(i\xi)^\alpha \widehat{u} \in L^2(\mathbb{R}^n),$$

hence " \implies " is proved.

Now we prove " \impliedby ". Let us assume that

$$(i\xi)^\alpha \widehat{u}(\xi) \in L^2(\mathbb{R}^n), \quad \text{for } |\alpha| \leq k.$$

Let $u_\alpha \in L^2(\mathbb{R}^n)$, $|\alpha| \leq k$, be defined by

$$u_\alpha(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{u}(\xi) e^{ix \cdot \xi} d\xi.$$

For any $\phi \in C_0^\infty(\mathbb{R}^n)$ and any $|\alpha| \leq k$, we have (recall (3.12.6))

$$\begin{aligned}
\int_{\mathbb{R}^n} \partial^\alpha \phi(x) \overline{u}(x) dx &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\partial^\alpha \phi}(\xi) \overline{\widehat{u}(\xi)} d\xi = \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{\phi}(\xi) \overline{\widehat{u}(\xi)} d\xi = \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \overline{(i\xi)^\alpha \widehat{u}(\xi)} d\xi = \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\phi}(\xi) \overline{\widehat{u}_\alpha(\xi)} d\xi = \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi(x) \overline{u_\alpha(x)} dx.
\end{aligned}$$

Hence

$$\partial^\alpha u = u_\alpha \in L^2(\mathbb{R}^n), \quad \text{for } |\alpha| \leq k.$$

Therefore $u \in H^k(\mathbb{R}^n)$. Claim I is proved.

Claim II.

The following conditions are equivalent

- (a) $(i\xi)^\alpha \widehat{u} \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$,
- (b) $(1 + |\xi|^2)^{k/2} \widehat{u}(\xi) \in L^2(\mathbb{R}^n)$.

Proof of Claim II. First, let us note that (a) is equivalent to

$$\int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |\xi^\alpha|^2 |\widehat{u}(\xi)|^2 d\xi < +\infty,$$

hence, in order to prove that (a) and (b) are equivalent it suffices to prove that there exists $C \geq 1$ such that

$$C^{-1} \leq \frac{\sum_{|\alpha| \leq k} |\xi^\alpha|^2}{(1 + |\xi|^2)^k} \leq C, \quad \forall \xi \in \mathbb{R}^n. \quad (3.12.16)$$

To this purpose we notice that the function

$$g(\xi, \tau) = \frac{\sum_{|\alpha| \leq k} \tau^{2(k-|\alpha|)} |\xi^\alpha|^2}{(\tau^2 + |\xi|^2)^k},$$

is homogeneous of degree 0, it is continuous in $\mathbb{R}^{n+1} \setminus \{(0, 0)\}$, and

$$g(\xi, \tau) > 0, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}.$$

Hence there exists $C \geq 1$ such that

$$C^{-1} \leq g(\xi, \tau) \leq C$$

so that, if $\tau = 1$ we get (3.12.16), which, in turn implies the equivalence of (a) and (b). Claim II is proved.

By Claim I and Claim II we obtain (i).

Concerning (ii), it is enough to observe that by (3.12.11) we have

$$\partial^\alpha u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i\xi)^\alpha \widehat{u}(\xi) e^{ix \cdot \xi} d\xi, \text{ for } |\alpha| \leq k$$

and by the Parseval identity we have

$$\|u\|_{H^k(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |\xi^\alpha|^2 |\widehat{u}(\xi)|^2 d\xi.$$

Hence by (3.12.16) we derive (3.12.13). ■

3.12.3 The Sobolev spaces with noninteger exponents

Theorem 3.12.5 justifies the the following definition (instead of \mathbb{R}^n we will consider \mathbb{R}^m , with $m \in \mathbb{N}$ to avoid confusion later on, when we will need to set $m = n - 1$)

Definition 3.12.6. Let $m \in \mathbb{N}$ and let s be a real positive number, we say that $u \in H^s(\mathbb{R}^m)$ if

$$(1 + |\xi|^2)^{s/2} \widehat{u}(\xi) \in L^2(\mathbb{R}^m),$$

in this case we denote

$$\|u\|_{H^s(\mathbb{R}^m)} = \left(\frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi \right)^{1/2}. \quad (3.12.17)$$

It is evident that if $s \in \mathbb{N}$ we again obtain the Sobolev spaces with integer exponents that we have studied so far, nevertheless if $s \notin \mathbb{N}$ we obtain some new spaces, namely the **Sobolev spaces with noninteger exponents**, also known as "the Sobolev spaces with fractional exponent". It is simple to check that the norm $\|\cdot\|_{H^s(\mathbb{R}^m)}$ is induced by the scalar product

$$(u, v)_{H^s(\mathbb{R}^m)} = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} (1 + |\xi|^2)^s \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi.$$

We leave the reader to verify that $H^s(\mathbb{R}^m)$ is a Hilbert space.

Theorem 3.12.7. *If $0 < s < 1$ then the norm $\|u\|_{H^s(\mathbb{R}^m)}$ is equivalent to the norm*

$$\|u\| = \left(\|u\|_{L^2(\mathbb{R}^m)}^2 + |u|_{s, \mathbb{R}^m}^2 \right)^{1/2}, \quad (3.12.18)$$

where

$$|u|_{s, \mathbb{R}^m}^2 = \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{|u(x) - u(y)|^2}{|x - y|^{m+2s}} dy. \quad (3.12.19)$$

Proof. We need to prove that there exists $C \geq 1$ such that, for every $u \in H^s(\mathbb{R}^m)$ we have

$$C^{-1} \int_{\mathbb{R}^m} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi \leq |u|_{s, \mathbb{R}^m}^2 \leq C \int_{\mathbb{R}^m} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi. \quad (3.12.20)$$

Let us begin by observing that

$$\begin{aligned} |u|_{s, \mathbb{R}^m}^2 &= \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{|u(x+z) - u(x)|^2}{|z|^{m+2s}} dz = \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \frac{1}{|z|^{m+2s}} dz \int_{\mathbb{R}^m} \left| \widehat{u(\cdot+z)} - \widehat{u(\cdot)} \right|^2 d\xi = \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \frac{1}{|z|^{m+2s}} dz \int_{\mathbb{R}^m} |e^{iz \cdot \xi} - 1|^2 |\widehat{u}(\xi)|^2 d\xi = \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \phi(\xi) |\widehat{u}(\xi)|^2 d\xi, \end{aligned} \quad (3.12.21)$$

where

$$\phi(\xi) = \int_{\mathbb{R}^m} \frac{|e^{iz \cdot \xi} - 1|^2}{|z|^{m+2s}} dz.$$

Notice that ϕ is a homogeneous with degree $2s$. As a matter of fact, for any $t > 0$, we have

$$\begin{aligned} \phi(t\xi) &= \int_{\mathbb{R}^m} \frac{|e^{itz \cdot \xi} - 1|^2}{|z|^{m+2s}} dz = \\ &= \int_{\mathbb{R}^m} \frac{|e^{iy \cdot \xi} - 1|^2}{|t^{-1}y|^{m+2s}} \frac{dy}{t^m} = t^{2s} \int_{\mathbb{R}^m} \frac{|e^{iy \cdot \xi} - 1|^2}{|y|^{m+2s}} dy = \\ &= t^{2s} \phi(\xi), \quad \forall \xi \in \mathbb{R}^m. \end{aligned}$$

Moreover, ϕ is a continuous function in \mathbb{R}^m . In order to prove this, let $\xi_0 \in \mathbb{R}^m$ and let us check that

$$\lim_{\xi \rightarrow \xi_0} \int_{\mathbb{R}^m} \frac{|e^{iz \cdot \xi} - 1|^2}{|z|^{m+2s}} dz = \int_{\mathbb{R}^m} \frac{|e^{iz \cdot \xi_0} - 1|^2}{|z|^{m+2s}} dz. \quad (3.12.22)$$

We have

$$\lim_{\xi \rightarrow \xi_0} \frac{|e^{iz \cdot \xi} - 1|^2}{|z|^{m+2s}} = \frac{|e^{iz \cdot \xi_0} - 1|^2}{|z|^{m+2s}}, \quad \forall \xi \in \mathbb{R}^m$$

and, if $|\xi - \xi_0| < 1$, we have

$$\begin{aligned} \frac{|e^{itz \cdot \xi} - 1|^2}{|z|^{m+2s}} &= \frac{|e^{itz \cdot \xi} - 1|^2}{|z|^{m+2s}} \chi_{B_1}(z) + \frac{|e^{itz \cdot \xi} - 1|^2}{|z|^{m+2s}} \chi_{\mathbb{R}^m \setminus B_1}(z) \leq \\ &\leq \frac{C(1 + |\xi_0|)^2}{|z|^{m-2(1-s)}} \chi_{B_1}(z) + \frac{4}{|z|^{m+2s}} \chi_{\mathbb{R}^m \setminus B_1}(z) \in L^1(\mathbb{R}^m). \end{aligned}$$

Therefore by the Dominated Convergence Theorem we get (3.12.22).

Now, since ϕ is continuous in \mathbb{R}^m and $\phi(\xi) > 0$, for every $|\xi| = 1$, we have that there exists $C \geq 1$ such that

$$C^{-1}|\xi|^{2s} \leq \phi(\xi) \leq C|\xi|^{2s}, \quad \forall \xi \in \mathbb{R}^m.$$

By the last inequality and by (3.12.21) we obtain (3.12.20). ■

Similarly to the previous Theorem the following one can be proved

Theorem 3.12.8. *If $s > 0$, $s \notin \mathbb{N}$, then the norm $\|u\|_{H^s(\mathbb{R}^m)}$ is equivalent to the norm*

$$\|u\| = \left(\|u\|_{H^{[s]}(\mathbb{R}^m)}^2 + |u|_{s, \mathbb{R}^m}^2 \right)^{1/2}, \quad (3.12.23)$$

where

$$|u|_{s, \mathbb{R}^m}^2 = \sum_{|\alpha|=[s]} \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{m+2(s-[s])}} dy. \quad (3.12.24)$$

Theorems 3.12.7, 3.12.8 justify the following definition.

Definition 3.12.9. Let Θ be a bounded open set of \mathbb{R}^m of class $C^{0,1}$. Let $s \notin \mathbb{N}$ be a positive real number. We define $H^s(\Theta)$ as the space of functions $u \in H^{[s]}(\Theta)$ such that

$$|u|_{s, \Theta}^2 = \sum_{|\alpha|=[s]} \int_{\Theta} dx \int_{\Theta} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{m+2(s-[s])}} dy < +\infty,$$

equipped with the norm

$$\|u\|_{H^s(\Theta)} = \left(\|u\|_{H^{[s]}(\Theta)}^2 + |u|_{s, \Theta}^2 \right)^{1/2}.$$

It is not difficult to prove that the space $H^s(\Theta)$ is complete.

Now let us define $H^s(\partial\Omega)$, where Ω is a bounded open set. If $s \in \mathbb{N}$, we assume that $\partial\Omega$ is of class C^s . If $s \notin \mathbb{N}$ we assume that $\partial\Omega$ is of class $C^{[s],1}$.

We proceed basically as we did in Section 2.7 to define $L^p(\partial\Omega)$ (we will use the same notations as Section 2.7).

Let us begin by the case $s := k$, positive integer number. Thus, let us assume $\partial\Omega$ of class C^k with constants r_0, M_0 and let us cover $\partial\Omega$ by a finite number, N , of cylinders $\tilde{Q}_{r_0, 2M_0}(X_i)$, $i = 1 \cdots, N$, where $X_i \in \partial\Omega$ isometric to $Q_{r_0, 2M_0}$. Moreover let us assume that: $\Sigma_i := \tilde{Q}_{r_0, 2M_0}(X_i) \cap \partial\Omega$, for any $i = 1 \cdots, N$, up to isometry for which X_i is mapped in 0, is the graph of a function $\varphi_i \in C^k(\overline{B'_{r_0}})$ likewise Definition 2.7.1. We say that $f \in H^k(\partial\Omega)$ provided that the functions $f(x', \varphi_i(x'))$, $i = 1, \cdots, N$, belong to $H^k(B'_{r_0})$ and we denote

$$\|f\|_{H^k(\partial\Omega)} = \left(\sum_{i=1}^N \|f\|_{H^k(\Sigma_i)}^2 \right)^{1/2},$$

where

$$\|f\|_{H^k(\Sigma_i)} = \|f(\cdot, \varphi_i(\cdot))\|_{H^k(B_{r_0})}.$$

Of course, the norm $\|\cdot\|_{H^k(\partial\Omega)}$ depends on the particular family of cylinders that we use as a covering, but they are all equivalent norms. Moreover, $H^k(\partial\Omega)$ is a separable Hilbert space.

If $s \notin \mathbb{N}$, we say that $f \in H^s(\partial\Omega)$ provided that the functions $f(x', \varphi_i(x'))$ belong to $H^s(B'_r)$, for any $i = 1, \cdots, N$ and we denote

$$\|f\|_{H^s(\partial\Omega)} = \left(\sum_{i=1}^N \|f\|_{H^s(\Sigma_i)}^2 \right)^{1/2}.$$

The space $H^s(\partial\Omega)$ is complete and can be equipped of a Hilbert structure.

For an extended discussion of Sobolev spaces with noninteger exponent we refer the reader to [21, Cap. 6], [43, Cap. 6], [59, Cap. 2]. Here we limit ourselves to prove a Theorem that will be useful in the next Section.

Theorem 3.12.10 (density of $C^\infty(\mathbb{R}^m)$ in $H^s(\mathbb{R}^m)$). *If s is a positive real number then $C_0^\infty(\mathbb{R}^m)$ is dense in $H^s(\mathbb{R}^m)$.*

We premise the following

Lemma 3.12.11. *Let $\eta \in C_0^\infty(\mathbb{R}^m)$ satisfy*

(i) *supp $\eta \subset B_1$,*

(ii) *$\eta \geq 0$,*

(iii) *$\int_{\mathbb{R}^m} \eta(x) dx = 1$.*

Let us denote, for any $\varepsilon > 0$, $v \in H^s(\mathbb{R}^m)$, $s > 0$,

$$\eta_\varepsilon(x) = \varepsilon^{-m} \eta(\varepsilon^{-1}x)$$

and

$$v^\varepsilon = \eta_\varepsilon \star v, \quad \text{in } \mathbb{R}^m.$$

Then we have

$$\lim_{\varepsilon \rightarrow 0} \|\eta_\varepsilon \star v - v\|_{H^s(\mathbb{R}^m)} = 0. \quad (3.12.25)$$

Proof of the Lemma. We have

$$\widehat{\eta}_\varepsilon(\xi) = \int_{\mathbb{R}^m} \varepsilon^{-m} \eta(\varepsilon^{-1}x) e^{-ix \cdot \xi} dx = \int_{\mathbb{R}^m} \eta(y) e^{-i\varepsilon y \cdot \xi} dy = \widehat{\eta}(\varepsilon\xi).$$

Moreover

$$\lim_{\varepsilon \rightarrow 0} \widehat{\eta}(\varepsilon\xi) = \widehat{\eta}(0) = \int_{\mathbb{R}^m} \eta(x) dx = 1 \quad (3.12.26)$$

and

$$|\widehat{\eta}(\varepsilon\xi)| \leq \int_{\mathbb{R}^m} \eta(x) dx = 1, \quad \forall \xi \in \mathbb{R}^m, \forall \varepsilon > 0. \quad (3.12.27)$$

Hence, for any $v \in H^s(\mathbb{R}^m)$, we have

$$\|\eta_\varepsilon \star v - v\|_{H^s(\mathbb{R}^m)}^2 = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\widehat{\eta}(\varepsilon\xi) - 1|^2 |\widehat{v}(\xi)|^2 d\xi.$$

By the last equality, by the Dominated Convergence Theorem (take into account (3.12.26) and (3.12.27)) we obtain (3.12.25). ■

Proof of Theorem 3.12.10. The case where $s \in \mathbb{N}$ has been proved in Proposition 3.4.1. Let us consider the case $0 < s < 1$ (if $s > 1$, the proof proceeds in a similar way, and we leave the details to the reader).

Claim. Let us denote by \mathcal{H}_s the subspace of the functions of $H^s(\mathbb{R}^m)$ with compact support. Then \mathcal{H}_s is dense in $H^s(\mathbb{R}^m)$.

Proof of the Claim.

Let $\zeta \in C_0^\infty(\mathbb{R}^m)$ satisfy

$$0 \leq \zeta \leq 1, \quad \text{in } \mathbb{R}^m,$$

$$\zeta(x) = 1, \quad \forall x \in B_1, \quad \zeta(x) = 0, \quad \forall x \in \mathbb{R}^m \setminus B_2$$

and

$$|\nabla \zeta| \leq C_0, \quad \text{in } \mathbb{R}^m,$$

where C_0 is a constant. Let $R > 1$ and

$$\zeta_R(x) = \zeta(R^{-1}x).$$

We have

$$\|u - \zeta_R u\|_{L^2(\mathbb{R}^m)} \leq \|u\|_{L^2(\mathbb{R}^m \setminus B_R)} \rightarrow 0, \quad \text{as } R \rightarrow \infty.$$

Now, we prove

$$\lim_{R \rightarrow \infty} |u - u\zeta_R|_{s, \mathbb{R}^m} = 0. \quad (3.12.28)$$

In proving the latter, we will obtain as a by-product of the performed calculations that $u\zeta_R$ belongs to $H^s(\mathbb{R}^m)$.

We apply Theorem 3.12.7 and we write

$$|u - u\zeta_R|_{s, \mathbb{R}^m}^2 = \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{\Phi_R^2(x, y)}{|x - y|^{m+2s}} dy,$$

where

$$\Phi_R(x, y) = |(1 - \zeta_R(x))u(x) - (1 - \zeta_R(y))u(y)|.$$

We have

$$\begin{aligned} \Phi_R(x, y) &\leq |1 - \zeta_R(y)| |u(x) - u(y)| + |\zeta_R(x) - \zeta_R(y)| |u(x)| \leq \\ &\leq \chi_{\mathbb{R}^m \setminus B_R}(y) |u(x) - u(y)| + |\zeta_R(x) - \zeta_R(y)| |u(x)|. \end{aligned}$$

Hence

$$\begin{aligned} |u - u\zeta_R|_{s, \mathbb{R}^m}^2 &\leq 2 \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m \setminus B_R} \frac{|u(x) - u(y)|^2}{|x - y|^{m+2s}} dy + \\ &+ 2 \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x - y|^{m+2s}} dy. \end{aligned} \quad (3.12.29)$$

Set

$$I := \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x - y|^{m+2s}} dy.$$

Let us notice that

$$|\zeta_R(x) - \zeta_R(y)| = 0, \quad \text{for } |x| \geq 2R \text{ and } |y| \geq 2R$$

moreover,

$$|\zeta_R(x) - \zeta_R(y)| \leq \min \left\{ 2, \frac{C_0|x-y|}{R} \right\}, \quad \forall x \in \mathbb{R}^m, \forall y \in \mathbb{R}^m.$$

Hence

$$\begin{aligned} I &\leq \int_{|x| \leq 2R} dx \int_{\mathbb{R}^m} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x-y|^{m+2s}} dy + \\ &+ \int_{\mathbb{R}^m} dx \int_{|y| \leq 2R} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x-y|^{m+2s}} dy := I_1 + I_2. \end{aligned} \quad (3.12.30)$$

We have

$$\begin{aligned} I_1 &\leq \frac{C_0^2}{R^2} \int_{|x| \leq 2R} |u(x)|^2 dx \int_{|x-y| \leq \frac{2R}{C_0}} \frac{dy}{|x-y|^{m-2+2s}} + \\ &+ 4 \int_{|x| \leq 2R} |u(x)|^2 dx \int_{|x-y| > \frac{2R}{C_0}} \frac{dy}{|x-y|^{m+2s}} \leq \\ &\leq \frac{\omega_m}{2-2s} \frac{C_0^2}{R^2} \left(\frac{2R}{C_0} \right)^{2-2s} \int_{|x| \leq 2R} |u(x)|^2 dx + \\ &+ \frac{\omega_m}{2s} \left(\frac{2R}{C_0} \right)^{-2s} \int_{|x| \leq 2R} |u(x)|^2 dx \leq \\ &\leq CR^{-2s} \|u\|_{L^2(\mathbb{R}^m)}^2, \end{aligned} \quad (3.12.31)$$

where C depends by m and s only.

Concerning I_2 , we have

$$\begin{aligned}
I_2 &= \int_{|y| \leq 2R} dy \int_{\mathbb{R}^m} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x-y|^{m+2s}} dx = \\
&= \int_{|y| \leq 2R} dy \int_{|x-y| \leq \frac{2R}{C_0}} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x-y|^{m+2s}} dx + \\
&+ \int_{|y| \leq 2R} dy \int_{|x-y| > \frac{2R}{C_0}} \frac{|\zeta_R(x) - \zeta_R(y)|^2 |u(x)|^2}{|x-y|^{m+2s}} dx \leq \\
&\leq \frac{C_0^2}{R^2} \int_{|y| \leq 2R} dy \int_{|x| \leq 2R(1+1/C_0)} \frac{|u(x)|^2}{|x-y|^{m-2+2s}} dx + \\
&+ 4 \left(\frac{2R}{C_0} \right)^{-m-2s} \int_{|y| \leq 2R} dy \int_{|x-y| > \frac{2R}{C_0}} |u(x)|^2 dx.
\end{aligned}$$

By interchanging the order of integration in the second-to-last integral and trivially estimating from above the last integral, we obtain

$$\begin{aligned}
I_2 &\leq \frac{C_0^2}{R^2} \int_{|x| \leq 2R(1+1/C_0)} |u(x)|^2 dx \int_{|y| \leq 2R} \frac{dy}{|x-y|^{m-2+2s}} + \\
&+ 4 \frac{\omega_m}{m} \left(\frac{2R}{C_0} \right)^{-m-2s} (2R)^m \|u\|_{L^2(\mathbb{R}^m)}^2 \leq \\
&\leq \frac{C_0^2}{R^2} \int_{|x| \leq 2R(1+1/C_0)} |u(x)|^2 dx \int_{|y-x| \leq 2R(2+1/C_0)} \frac{dy}{|x-y|^{m-2+2s}} + \\
&+ CR^{-2s} \|u\|_{L^2(\mathbb{R}^m)}^2 \leq \\
&\leq C'R^{-2s} \|u\|_{L^2(\mathbb{R}^m)}^2,
\end{aligned} \tag{3.12.32}$$

where C e C' depend by m and s only.

By (3.12.30), (3.12.31) and (3.12.32) we get

$$I = I_1 + I_2 \leq CR^{-2s} \|u\|_{L^2(\mathbb{R}^m)}^2,$$

where C depends by s and m only.

By the just obtained inequality and by (3.12.29) we have

$$\|u - u\zeta_R\|_{s, \mathbb{R}^m}^2 \leq 2 \int_{\mathbb{R}^m} dx \int_{\mathbb{R}^m \setminus B_R} \frac{|u(x) - u(y)|^2}{|x-y|^{m+2s}} dy + CR^{-2s} \|u\|_{L^2(\mathbb{R}^m)}^2.$$

Since the following function belongs to $L^2(\mathbb{R}^m \times \mathbb{R}^m)$ (as $u \in H^s(\mathbb{R}^m)$)

$$\mathbb{R}^m \times \mathbb{R}^m \ni (x, y) \rightarrow \frac{|u(x) - u(y)|^2}{|x-y|^{m+2s}},$$

we obtain (3.12.28). The **Claim** is proved.

Now, let $\delta > 0$ and let R_0 be (recall (3.12.28)) such that

$$\|u - u\zeta_{R_0}\|_{s, \mathbb{R}^m} < \frac{\delta}{2}.$$

Lemma 3.12.11 implies that there exists $\varepsilon_0 > 0$ such that

$$\|u\zeta_{R_0} - (u\zeta_{R_0}) \star \eta_{\varepsilon_0}\|_{s, \mathbb{R}^m} < \frac{\delta}{2}.$$

Hence

$$\|u - (u\zeta_{R_0}) \star \eta_{\varepsilon_0}\|_{s, \mathbb{R}^m} < \delta$$

Since $(u\zeta_{R_0}) \star \eta_{\varepsilon_0} \in C_0^\infty(\mathbb{R}^m)$, the last inequality concludes the proof. ■

3.12.4 The Theorem of characterization of the traces.

We preliminarily examine the extension of the notion of trace of a function belonging to $H^1(\mathbb{R}_+^n)$, where $\mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$.

If $u \in H^1(\mathbb{R}_+^n)$, then the function

$$\tilde{u}(x', x_n) := u(x', |x_n|),$$

belongs to $H^1(\mathbb{R}^n)$ and, as $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$, there exists a sequence $\{v_j\}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\{v_j\} \rightarrow \tilde{u}, \quad \text{in } H^1(\mathbb{R}^n).$$

Hence

$$\{(v_j)|_{\mathbb{R}_+^n}\} \rightarrow u, \quad \text{in } H^1(\mathbb{R}_+^n).$$

Denoting

$$w = v_j - v_k, \quad j, k \in \mathbb{N},$$

we have

$$w(x', 0) = w(x', x_n) - \int_0^{x_n} \partial_{x_n} w(x', y) dy, \quad \forall x_n > 0,$$

from which we obtain

$$|w(x', 0)|^2 \leq 2|w(x', x_n)|^2 + 2x_n \int_0^{x_n} |\partial_{x_n} w(x', y)|^2 dy.$$

Now, integrating both the sides of the last inequality over $(0, \delta)$, $\delta > 0$, w.r.t. x_n , we have

$$\begin{aligned} \delta |w(x', 0)|^2 &\leq 2 \int_0^\delta |w(x', x_n)|^2 dx_n + 2\delta \int_0^\delta dx_n \int_0^{x_n} |\partial_{x_n} w(x', y)|^2 dy \leq \\ &\leq 2 \int_0^\delta |w(x', x_n)|^2 dx_n + 2\delta^2 \int_0^\delta |\partial_{x_n} w(x', y)|^2 dy. \end{aligned}$$

Integrating both the sides of the last inequality over \mathbb{R}^{n-1} w.r.t. x' , we have

$$\int_{\mathbb{R}^{n-1}} |w(x', 0)|^2 dx' \leq \frac{2}{\delta} \int_{\mathbb{R}^n} |w(x)|^2 dx + 2\delta \int_{\mathbb{R}^n} |\nabla w(x)|^2 dx.$$

Starting from this inequality we proceed as in the proof of Theorem 3.5.1 (inequality (3.5.2)) and we obtain the extension of the trace operator from the space

$$C_*^\infty(\mathbb{R}_+^n) := \left\{ \tilde{u}|_{\mathbb{R}_+^n} : \tilde{u} \in C_0^\infty(\mathbb{R}^n) \right\}$$

to the space $H^1(\mathbb{R}_+^n)$. In particular, we have

$$Tu \in L^2(\mathbb{R}^{n-1}), \quad \forall u \in H^1(\mathbb{R}_+^n)$$

and

$$\|Tu\|_{L^2(\mathbb{R}^{n-1})} \leq C \|u\|_{H^1(\mathbb{R}_+^n)}, \quad \forall u \in H^1(\mathbb{R}_+^n), \quad (3.12.33)$$

where C depends on n only.

The following Theorem provides a characterization of the image of $H^1(\Omega)$ by means the trace operator.

Theorem 3.12.12 (characterization of the trace). *Let Ω be either a bounded open set of class $C^{0,1}$ with constants M_0, r_0 , or $\Omega = \mathbb{R}_+^n$. Let*

$$T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$$

be the trace operator defined in Theorem 3.5.1 (in the case where $\Omega = \mathbb{R}_+^n$ the definition is given at the beginning of this Section).

Then we have

$$T(H^1(\Omega)) = H^{1/2}(\partial\Omega).$$

Moreover

(i)

$$\|T(u)\|_{H^{1/2}(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

where C depends on M_0 , r_0 and n only.

(ii) There exists a bounded, linear map

$$\mathcal{T} : H^{1/2}(\partial\Omega) \rightarrow H^1(\Omega)$$

such that

$$T(\mathcal{T}(h)) = h, \quad \forall h \in H^{1/2}(\partial\Omega).$$

In particular, denoting by $u = \mathcal{T}(h)$, we have

$$\|u\|_{H^1(\Omega)} \leq C \|h\|_{H^{1/2}(\partial\Omega)}, \quad \forall h \in H^{1/2}(\partial\Omega),$$

where C depends on M_0 , r_0 and n only.

In the general case the proof of Theorem 3.12.12, is quite technical. Here we limit ourselves to the case $\Omega = \mathbb{R}_+^n$. A complete treatment (including the traces of the functions belonging to $W^{k,p}(\Omega)$, $k \in \mathbb{N}$) can be founded in [43, Ch. 6] and in [59, Ch. 2, Secs. 2.3 – 2.5].

Proof of Theorem 3.12.12 in the case $\Omega = \mathbb{R}_+^n$.

We have proved, in (3.12.33), that $T \in L^2(\mathbb{R}^{n-1})$. Now we prove

$$Tu \in H^{1/2}(\mathbb{R}^{n-1}).$$

Claim

Let $v \in C_*^\infty(\mathbb{R}_+^n)$. Let us denote

$$h(x') = v(x', 0), \quad \forall x' \in \mathbb{R}^{n-1}.$$

We have

$$\widehat{h}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{v}(\xi', \xi_n) d\xi_n. \quad (3.12.34)$$

Proof of the Claim.

Since

$$v(x', x_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{v}(\xi) e^{ix \cdot \xi} d\xi,$$

we have

$$\begin{aligned} h(x') &= v(x', 0) = \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{v}(\xi) e^{ix' \cdot \xi} d\xi = \\ &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{v}(\xi', \xi_n) d\xi_n \right) e^{ix' \cdot \xi'} d\xi' \end{aligned}$$

and by proposition (a) of Theorem 3.12.3 we obtain (3.12.34). Claim is proved.

Now, by (3.12.34) we have

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |\widehat{h}(\xi')|^2 (1 + |\xi'|^2)^{1/2} d\xi' &\leq \frac{1}{4\pi^2} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |\widehat{v}(\xi', \xi_n)| d\xi_n \right)^2 (1 + |\xi'|^2)^{1/2} d\xi' = \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} d\xi' \left(\int_{\mathbb{R}} (1 + |\xi|^2)^{1/2} |\widehat{v}(\xi', \xi_n)| (1 + |\xi|^2)^{-1/2} d\xi_n \right)^2 \leq \\ &\leq \frac{1}{4\pi^2} \int_{\mathbb{R}^{n-1}} (1 + |\xi'|^2)^{1/2} d\xi' \left(\int_{\mathbb{R}} (1 + |\xi|^2) |\widehat{v}(\xi', \xi_n)|^2 d\xi_n \int_{\mathbb{R}} (1 + |\xi|^2)^{-1} d\xi_n \right). \end{aligned}$$

Moreover

$$\int_{\mathbb{R}} (1 + |\xi|^2)^{-1} d\xi_n = \int_{\mathbb{R}} \frac{d\xi_n}{1 + |\xi'|^2 + \xi_n^2} = \frac{\pi}{(1 + |\xi'|^2)^{1/2}}.$$

Therefore

$$\int_{\mathbb{R}^{n-1}} |\widehat{h}(\xi')|^2 (1 + |\xi'|^2)^{1/2} d\xi' \leq \frac{1}{4\pi} \int_{\mathbb{R}^n} (1 + |\xi|^2) |\widehat{v}(\xi)|^2 d\xi$$

that is

$$\|T(v)\|_{H^{1/2}(\mathbb{R}^{n-1})} \leq \frac{1}{2\sqrt{\pi}} \|v\|_{H^1(\mathbb{R}_+^n)}, \quad \forall v \in C_*^\infty(\mathbb{R}_+^n),$$

from which, by density we have

$$\|T(u)\|_{H^{1/2}(\mathbb{R}^{n-1})} \leq \frac{1}{2\sqrt{\pi}} \|u\|_{H^1(\mathbb{R}^n)}, \quad \forall u \in H^1(\mathbb{R}_+^n). \quad (3.12.35)$$

Now we prove (ii).

Let $h \in C_0^\infty(\mathbb{R}^{n-1})$ and, for any $\varepsilon > 0$, let

$$u_\varepsilon(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-(1+|\xi'|)(x_n+\varepsilon)} \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi', \quad \forall x \in \overline{\mathbb{R}_+^n}.$$

By applying Theorem 3.12.3 and by performing the derivative under the integral sign, it can be easily checked that

$$u_\varepsilon \in C^\infty(\overline{\mathbb{R}_+^n})$$

and

$$u_\varepsilon(x', 0) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-(1+|\xi'|)\varepsilon} \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi', \quad \forall x' \in \mathbb{R}^{n-1}.$$

Now we prove what follows

(a) $u_\varepsilon \in H^1(\mathbb{R}_+^n)$ and, denoting

$$u(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-(1+|\xi'|)x_n} \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi', \quad \forall x \in \mathbb{R}^n,$$

(let us notice that $u(\cdot, 0) = h$) we have,

$$u_\varepsilon \in H^1(\mathbb{R}_+^n) \quad \text{and} \quad u \in H^1(\mathbb{R}_+^n).$$

Moreover

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}, \quad (3.12.36)$$

and

$$\|u\|_{H^1(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}, \quad (3.12.37)$$

where C depends on n only.

(b)

$$u_\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0 \text{ in } H^1(\mathbb{R}_+^n).$$

Proof of (a).

The Parseval identity implies

$$\int_{\mathbb{R}^{n-1}} |u_\varepsilon(x', x_n)|^2 dx' = c_n \int_{\mathbb{R}^{n-1}} e^{-2(1+|\xi'|)(x_n+\varepsilon)} |\widehat{h}(\xi')|^2 d\xi',$$

where c_n depends on n only. Hence

$$\begin{aligned}
\int_{\mathbb{R}_+^n} |u_\varepsilon(x', x_n)|^2 dx' dx_n &= c_n \int_{\mathbb{R}^{n-1}} \left| \widehat{h}(\xi') \right|^2 d\xi' \left(\int_0^{+\infty} e^{-2(1+|\xi'|)(x_n+\varepsilon)} dx_n \right) = \\
&= c_n \int_{\mathbb{R}^{n-1}} \left| \widehat{h}(\xi') \right|^2 \frac{e^{-2\varepsilon(1+|\xi'|)}}{2(1+|\xi'|)} d\xi' \leq \\
&\leq \frac{(2\pi)^{n-1} c_n}{2} \|h\|_{L^2(\mathbb{R}^{n-1})}^2.
\end{aligned}$$

Therefore

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C \|h\|_{L^2(\mathbb{R}^{n-1})}, \quad (3.12.38)$$

where C depends on n only.

Now we estimate from above $\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^n)}$. We have

$$\partial_{x_n} u_\varepsilon(x', x_n) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} (1+|\xi'|) e^{-(1+|\xi'|)(x_n+\varepsilon)} \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi'.$$

Hence, arguing as above, we get

$$\begin{aligned}
&\int_{\mathbb{R}_+^n} |\partial_{x_n} u_\varepsilon(x', x_n)|^2 dx' dx_n = \\
&= c_n \int_{\mathbb{R}^{n-1}} \left| \widehat{h}(\xi') \right|^2 d\xi' \left(\int_0^{+\infty} (1+|\xi'|)^2 e^{-2(1+|\xi'|)(x_n+\varepsilon)} dx_n \right) = \\
&= \frac{c_n}{2} \int_{\mathbb{R}^{n-1}} (1+|\xi'|) \left| \widehat{h}(\xi') \right|^2 e^{-2\varepsilon(1+|\xi'|)} d\xi' \leq \\
&\leq \frac{(2\pi)^{n-1} c_n}{2} \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}^2.
\end{aligned} \quad (3.12.39)$$

Now, if $1 \leq j \leq n-1$, we have

$$\partial_{x_j} u_\varepsilon(x', x_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} i\xi_j e^{-2(1+|\xi'|)(x_n+\varepsilon)} \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi'$$

and arguing as in (3.12.39), we get

$$\int_{\mathbb{R}_+^n} |\partial_{x_j} u_\varepsilon(x', x_n)|^2 dx' dx_n \leq \frac{(2\pi)^{n-1} c_n}{2} \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}^2. \quad (3.12.40)$$

By (3.12.39) and (3.12.40) we get

$$\|\nabla u_\varepsilon\|_{L^2(\mathbb{R}^n)}^2 \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}^2,$$

where C depends on n only. By the just obtained inequality and by (3.12.38) we derive

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}$$

where C depends on n only.

As can be easily observed, the calculations performed above also apply to $\varepsilon = 0$ and similarly yield

$$\|u\|_{H^1(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})},$$

where C depends on n only. Hence we have proved (3.12.36) and (3.12.37). Proof of (a) is concluded.

Proof of (b).

Since

$$(u - u_\varepsilon)(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-(1+|\xi'|)x_n} \left(1 - e^{-\varepsilon(1+|\xi'|)}\right) \widehat{h}(\xi') e^{ix' \cdot \xi'} d\xi,$$

we easily obtain

$$\|u - u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^{n-1}} \frac{(1 - e^{-\varepsilon(1+|\xi'|)})^2}{1 + |\xi'|} \left|\widehat{h}(\xi')\right|^2 d\xi'$$

and

$$\|\nabla(u - u_\varepsilon)\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^{n-1}} (1 + |\xi'|) \left(1 - e^{-\varepsilon(1+|\xi'|)}\right)^2 \left|\widehat{h}(\xi')\right|^2 d\xi'.$$

Therefore, by the Dominated Convergence Theorem we get

$$\lim_{\varepsilon \rightarrow 0} \|u - u_\varepsilon\|_{H^1(\mathbb{R}^n)}^2 = 0.$$

By the previous limit and by the trace Theorem (that is, by (i)) we have

$$T(u) = \lim_{\varepsilon \rightarrow 0} T(u_\varepsilon), \quad \text{in } L^2(\mathbb{R}^{n-1}). \quad (3.12.41)$$

On the other hand

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|T(u_\varepsilon) - h\|_{L^2(\mathbb{R}^{n-1})} &= \\ &= \lim_{\varepsilon \rightarrow 0} c_n \int_{\mathbb{R}^{n-1}} \left(1 - e^{-\varepsilon(1+|\xi'|)}\right)^2 \left|\widehat{h}(\xi')\right|^2 d\xi' = 0 \end{aligned} \quad (3.12.42)$$

Therefore, by (3.12.41) and (3.12.42) we get

$$T(u) = h, \quad \forall h \in C_0^\infty(\mathbb{R}^{n-1}). \quad (3.12.43)$$

Now, we set

$$\mathcal{T}(h) = u.$$

By (3.12.37) we have

$$\|\mathcal{T}(h)\|_{H^1(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}, \quad \forall h \in C_0^\infty(\mathbb{R}^{n-1}) \quad (3.12.44)$$

and by (3.12.43) we have

$$T(\mathcal{T}(h)) = h, \quad \forall h \in C_0^\infty(\mathbb{R}^{n-1}). \quad (3.12.45)$$

Now, (3.12.35) and (3.12.44) give

$$\|T(\mathcal{T}(h))\|_{L^2(\mathbb{R}^n)} \leq C \|h\|_{H^{1/2}(\mathbb{R}^{n-1})}, \quad \forall h \in C_0^\infty(\mathbb{R}^{n-1}). \quad (3.12.46)$$

Finally, by density Theorem 3.12.10, by (3.12.44), (3.12.45) and by (3.12.46) the thesis follows. ■

3.13 Final comments and supplements

In this Section we will state, without proof, some theorems concerning the traces and Lipschitz continuous functions.

Concerning the traces, if $k \geq 1$ and if Ω is a bounded open set of \mathbb{R}^n of class $C^{k-1,1}$, it can be proved that (see [43], [59])

$$H^{k-1/2}(\partial\Omega) = T(H^k(\Omega)).$$

In the sequel we will be mainly interested in the cases $k = 1, 2$. In particular, if $u \in H^2(\Omega)$, then $\partial_k u \in H^1(\Omega)$ for $k = 1, \dots, n$, hence $\partial_k u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Moreover, we have

$$H^2(\partial\Omega) \subset H^{3/2}(\partial\Omega) \subset H^1(\partial\Omega).$$

If $u \in H^2(\Omega)$, we can define $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ on $\partial\Omega$ (ν unit outward normal vector) and we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{1/2}(\partial\Omega)} \leq C \|u\|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega), \quad (3.13.1)$$

where C depends by Ω only.

Similarly to what we saw in Section 3.5, the following Theorem can be proved

Theorem 3.13.1. *Let Ω be a bounded open set of \mathbb{R}^n of class $C^{1,1}$. If $u \in H^2(\Omega)$ then*

$$u \in H_0^2(\Omega) \quad \text{if and only if} \quad u|_{\partial\Omega} = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \quad (3.13.2)$$

Theorem 3.12.12 can be generalized as follows

Theorem 3.13.2. *Let Ω be a bounded open set of \mathbb{R}^n of class $C^{1,1}$. Then, for every*

$$(\psi_0, \psi_1) \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

there exists $u \in H^2(\Omega)$ such that

$$u|_{\partial\Omega} = \psi_0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \psi_1 \quad \text{on} \quad \partial\Omega$$

and

$$\|u\|_{H^2(\Omega)} \leq C \left(\|\psi_0\|_{H^{3/2}(\partial\Omega)} + \|\psi_1\|_{H^{1/2}(\partial\Omega)} \right),$$

where C depends on Ω only.

3.13.1 The space $H^{-1/2}(\partial\Omega)$

Let Ω be a bounded open set of \mathbb{R}^n of class $C^{0,1}$. We denote by $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega)$. Thus, $H^{-1/2}(\partial\Omega)$ is the space of the linear functionals

$$\Phi : H^{1/2}(\partial\Omega) \rightarrow \mathbb{R}$$

such that for a constant C we have

$$|\Phi(\varphi)| \leq C \|\varphi\|_{H^{1/2}(\partial\Omega)} \quad \forall \varphi \in H^{1/2}(\partial\Omega). \quad (3.13.3)$$

We define the norm, $\|\Phi\|_{H^{-1/2}(\partial\Omega)}$, of Φ in $H^{-1/2}(\partial\Omega)$ as the greatest lower bound of C satisfying (3.13.3).

3.13.2 The space $W_{loc}^{1,\infty}(\mathbb{R}^n)$ and $C_{loc}^{0,1}(\mathbb{R}^n)$

We say that $u \in C_{loc}^{0,1}(\mathbb{R}^n)$ provided that for any $x_0 \in \mathbb{R}^n$ there exists $r > 0$ such that

$$u|_{\overline{B_r(x_0)}} \in C^{0,1}(\overline{B_r(x_0)}),$$

Theorem 3.13.3. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$. We have that $u \in W_{loc}^{1,\infty}(\mathbb{R}^n)$ if and only if $u \in C_{loc}^{0,1}(\mathbb{R}^n)$.*

Proof. Let $u \in C_{loc}^{0,1}(\mathbb{R}^n)$. Theorem 2.6.1 implies that u is differentiable almost everywhere and $\nabla u = (\partial_1 u, \dots, \partial_n u) \in L_{loc}^\infty(\mathbb{R}^n)$. Hence, for any $1 \leq k \leq n$, we have

$$\int_{\mathbb{R}^n} u \partial_k \varphi dx = \int_{\mathbb{R}^n} (\partial_k (u\varphi) - \partial_k u \varphi) dx = - \int_{\mathbb{R}^n} \partial_k u \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Therefore $\partial_k u$ is the weak derivative of u , hence $u \in W_{loc}^{1,\infty}(\mathbb{R}^n)$.

Conversely, let $u \in W_{loc}^{1,\infty}(\mathbb{R}^n)$ and let $B_r(x_0)$ be a ball of \mathbb{R}^n . Let us denote

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) u(y) dy.$$

We have $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\nabla u_\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) \nabla u(y) dy.$$

Consequently, for every $\varepsilon \in (0, r]$, we get

$$|\nabla u_\varepsilon(x)| \leq \|\nabla u\|_{L^\infty(B_{2r}(x_0))} < +\infty, \quad \forall x \in B_r(x_0).$$

Hence

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \|\nabla u\|_{L^\infty(B_{2r}(x_0))} |x - y|, \quad \forall x, y \in B_r(x_0). \quad (3.13.4)$$

Moreover, Theorem 2.3.37 implies that

$$u_\varepsilon \rightarrow u, \quad \text{as } \varepsilon \rightarrow 0 \quad (\text{uniformly}).$$

By this and by (3.13.4), we have

$$|u(x) - u(y)| \leq \|\nabla u\|_{L^\infty(B_{2r}(x_0))} |x - y|, \quad \forall x, y \in B_r(x_0). \quad (3.13.5)$$

Therefore $u|_{\overline{B_r(x_0)}} \in C^{0,1}(\overline{B_r(x_0)})$. Since $B_r(x_0)$ is arbitrary, the proof is complete. ■

3.13.3 Almost everywhere differentiability of function belonging to $W_{loc}^{1,p}(\mathbb{R}^n)$ with $p > n$.

Theorem 3.13.4. *Let $n < p \leq +\infty$. If $u \in W_{loc}^{1,p}(\mathbb{R}^n)$, then u is almost everywhere differentiable in \mathbb{R}^n .*

Proof. Since $W_{loc}^{1,\infty}(\mathbb{R}^n) \subset W_{loc}^{1,p}(\mathbb{R}^n)$ for $p < +\infty$, we may assume that $n < p < +\infty$. Let $x \in \mathbb{R}^n$ be such that (Corollary 2.5.5)

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |\nabla u(\xi) - \nabla u(x)|^p d\xi = 0. \quad (3.13.6)$$

Let us denote

$$v(\xi) = u(\xi) - u(x) - \nabla u(x) \cdot (\xi - x), \quad \xi \in \mathbb{R}^n,$$

let $y \in \mathbb{R}^n$, $y \neq x$ and $r = |x - y|$. Theorem 3.3.3 and Lemma 3.7.7 give

$$\begin{aligned} |u(y) - u(x) - \nabla u(x) \cdot (y - x)| &= |v(y) - v(x)| \leq \\ &\leq Cr^{1-\frac{n}{p}} \left(\int_{B_r(x)} |\nabla v(\xi)|^p d\xi \right)^{1/p} = \\ &= C'|x - y| \left(\int_{B_r(x)} |\nabla u(\xi) - \nabla u(x)|^p d\xi \right)^{1/p}. \end{aligned}$$

Hence

$$u(y) - u(x) - \nabla u(x) \cdot (y - x) = o(|y - x|), \quad \text{as } y \rightarrow x.$$

Therefore u is differentiable in every point x which satisfies (3.13.6) and by Corollary 2.5.5 we conclude the proof. ■

Remark. From Theorems 3.13.3 and 3.13.4 one immediately obtains the Rademacher Theorem by a proof different from the one followed in Section 2.6. The reader is invited to make sure that following this new proof does not lead to "vicious circles". ♦

Chapter 4

The boundary value problems for second order elliptic equations and the Dirichlet to Neumann map

4.1 Introduction

Let Ω be a bounded open set of \mathbb{R}^n . Let us denote by $\mathbb{M}(n)$ the vector space of the matrices $n \times n$ whose entries are real numbers and let $A \in L^\infty(\Omega; \mathbb{M}(n))$, i.e. $A = \{a^{jk}\}_{j,k=1}^n$ is a matrix whose entries a^{jk} belong to $L^\infty(\Omega)$, for $j, k = 1, \dots, n$.

Throughout this Chapter we will assume that A satisfies the following condition of **uniform ellipticity**

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad (4.1.1)$$

where $\lambda \geq 1$ is a given number. We define

$$|A(x)|_{\mathbb{M}(n)} = \left(\sum_{j,k=1}^n (a^{jk}(x))^2 \right)^{\frac{1}{2}}, \quad \text{a.e. } x \in \Omega$$

and

$$\|A\|_{L^\infty(\Omega; \mathbb{M}(n))} = \||A(\cdot)|_{\mathbb{M}(n)}\|_{L^\infty(\Omega)}.$$

Let us notice that if A is symmetric, then the second inequality of (4.1.1) implies also

$$\sup_{|\xi|=1, |\eta|=1} |A(x)\xi \cdot \eta| \leq \lambda, \quad \forall x \in \Omega \quad (4.1.2)$$

and

$$\|a^{jk}\|_{L^\infty(\Omega)} \leq \lambda, \quad \text{for } j, k = 1, \dots, n. \quad (4.1.3)$$

Let us check (4.1.2) and (4.1.2). Let $\xi, \eta \in \mathbb{R}^n$ such that $|\xi| = 1$ and $|\eta| = 1$. By the symmetry of $A(x)$ we have

$$\begin{aligned} |A(x)\xi \cdot \eta| &= \frac{1}{4} |A(x)(\xi + \eta) \cdot (\xi + \eta) - A(x)(\xi - \eta) \cdot (\xi - \eta)| \leq \\ &\leq \frac{1}{4} (\lambda|\xi + \eta|^2 + \lambda|\xi - \eta|^2) = \\ &= \frac{\lambda}{4} (2|\xi|^2 + 2|\eta|^2) = \\ &= \lambda. \end{aligned}$$

Hence, (4.1.2) follows. Concerning (4.1.3), we have, for $j, k = 1, \dots, n$

$$\|a^{jk}(x)\|_{L^\infty(\Omega)} = |A(x)e_j \cdot e_k| \leq \lambda, \quad \text{a.e. } x \in \Omega.$$

In this Chapter we will tackle the Dirichlet problem for the operator $-\operatorname{div}(A\nabla u)$, which formally consists in determining $u \in H^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(A\nabla u) = F, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \quad (4.1.4)$$

where $\varphi \in H^{1/2}(\partial\Omega)$ and $F \in H^{-1}(\Omega)$. We will deal with the variational formulation of problem (4.1.4). Next we will deal with the existence and the uniqueness of the solutions in $H^1(\Omega)$ and subsequently we prove some regularity results for the same problem, i.e., in coarse terms, we will prove that if Ω , A , and F have greater regularity, then u also acquires more regularity.

The investigation on problem (4.1.4) will guide us to deal with the more general case in which, instead of $-\operatorname{div}(\nabla u)$, we will have the operator

$$-\sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu, \quad (4.1.5)$$

where $b^j, d^j, c \in L^\infty(\Omega)$, for $j = 1, \dots, n$.

4.2 The Lax–Milgram Theorem and the Fredholm Theorem

Let H be a real Hilbert space, let us denote by $\|\cdot\|$ and (\cdot, \cdot) the scalar product on H and the induced norm induced respectively. As usual we denote by H'

the dual space of H .

If A is a linear operator we denote by $\mathcal{R}(A)$ the range of A , that is

$$\mathcal{R}(A) := \{Au : u \in H\}$$

and by $\mathcal{N}(A)$ the kernel of A

$$\mathcal{N}(A) := \{u \in H : Au = 0 \in H\}.$$

Let

$$a : H \times H \rightarrow \mathbb{R}, \quad (4.2.1)$$

a bilinear form. We say that a is **continuous** if there exists $C > 0$ such that

$$|a(u, v)| \leq C \|u\| \|v\|, \quad \forall u, v \in H. \quad (4.2.2)$$

We say that the bilinear form a is **coercive** if there exists $\alpha > 0$ such that

$$\alpha \|u\|^2 \leq a(u, u), \quad \forall u \in H. \quad (4.2.3)$$

The following Theorem holds true.

Theorem 4.2.1 (Lax–Milgram). *Let a be a coercive bilinear form and let $F \in H'$. Then there exists a unique $u \in H$ such that*

$$a(u, v) = F(v), \quad \forall v \in H. \quad (4.2.4)$$

Moreover

$$\|u\| \leq \frac{1}{\alpha} \|F\|_{H'}, \quad (4.2.5)$$

where $\|\cdot\|_{H'}$ is the norm of H' .

Proof. The Riesz Representation Theorem implies that there exists a unique $f \in H$ such that

$$F(v) = (f, v), \quad \forall v \in H. \quad (4.2.6)$$

Moreover

$$\|F\|_{H'} = \|f\|_H.$$

Let $u \in H$ be fixed and observe that, as (4.2.2) holds, the map

$$H \ni v \rightarrow a(u, v) \in \mathbb{R},$$

is linear and bounded. The Riesz Representation Theorem implies that there exists a unique $Au \in H$ such that

$$a(u, v) = (Au, v), \quad \forall v \in H.$$

Hence, we have defined the map

$$A : H \rightarrow H,$$

such that

$$a(u, v) = (Au, v), \quad \forall u, v \in H. \quad (4.2.7)$$

By (4.2.6) and (4.2.7) we have that (4.2.4) is equivalent to

$$Au = f, \quad u \in H. \quad (4.2.8)$$

Now, we prove that the map A is **(i) linear**, **(ii) bounded** and **(iii) bijective**.

(i) Let $u_1, u_2 \in H$, $\lambda_1, \lambda_2 \in \mathbb{R}$. We have

$$\begin{aligned} (A(\lambda_1 u_1 + \lambda_2 u_2), v) &= a(\lambda_1 u_1 + \lambda_2 u_2, v) \\ &= \lambda_1 a(u_1, v) + \lambda_2 a(u_2, v) = \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) = \\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v), \quad \forall v \in H. \end{aligned}$$

Hence

$$A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda_2 Au_2, \quad \forall u_1, u_2 \in H.$$

(ii) By (4.2.2) we get

$$|(Au, v)| = |a(u, v)| \leq C \|u\| \|v\|, \quad \forall u, v \in H.$$

By the Cauchy–Schwarz inequality we get

$$\|Au\| \leq C \|u\|, \quad \forall u \in H.$$

Therefore A is bounded and

$$\|A\|_{\mathcal{L}(H)} \leq C,$$

where $\mathcal{L}(H)$ is the space of bounded linear map from H in itself.

(iii) Condition (4.2.3) implies

$$\alpha \|u\|^2 \leq a(u, u) = (Au, u) \leq \|Au\| \|u\|, \quad \forall u \in H.$$

Hence

$$\alpha \|u\| \leq \|Au\|, \quad \forall u \in H. \quad (4.2.9)$$

Since $\alpha > 0$, A is injective.

In order to prove that A is onto, we first prove that $\mathcal{R}(A)$ is closed. Let $\{w_k\}$ be a sequence in $\mathcal{R}(A)$ such that

$$\{w_k\} \rightarrow w, \quad (4.2.10)$$

and let us check that $w \in \mathcal{R}(A)$. Let $u_k \in H$, $k \in \mathbb{N}$, satisfy

$$Au_k = w_k, \quad \forall k \in \mathbb{N}.$$

By (4.2.9) we have

$$\|u_k - u_j\| \leq \frac{1}{\alpha} \|Au_k - Au_j\| = \frac{1}{\alpha} \|w_k - w_j\|, \quad \forall k, j \in \mathbb{N}.$$

Since $\{w_k\}$ converges, it is a Cauchy sequence. Consequently, $\{u_k\}$ is a Cauchy sequence too. Therefore there exists $u \in H$ such that

$$\{u_k\} \rightarrow u.$$

Now, by (4.2.10) and, as A is continuous, we obtain

$$w = \lim_{k \rightarrow \infty} w_k = \lim_{k \rightarrow \infty} Au_k = Au.$$

Therefore $w \in \mathcal{R}(A)$.

Now we prove

$$\mathcal{R}(A) = H. \quad (4.2.11)$$

We argue by contradiction. Let us assume that $\mathcal{R}(A) \subsetneq H$. Since $\mathcal{R}(A)$ is closed, there exists $w \in H \setminus \{0\}$ such that $w \perp \mathcal{R}(A)$, (by this we mean $(w, h) = 0$ for every $h \in \mathcal{R}(A)$). Hence

$$\alpha \|w\|^2 \leq a(w, w) = (Aw, w) = 0.$$

Consequently, we should have $w = 0$ that contradicts $w \neq 0$. Thus (4.2.11) is proved.

Now, since (4.2.4) and (4.2.8) are equivalent, there exists one and only one solution $u \in H$ of the problem

$$a(u, v) = F(v), \quad \forall v \in H. \quad (4.2.12)$$

Concerning estimate (4.2.5), it follows immediately by

$$\alpha \|u\|^2 \leq a(u, u) = F(u) \leq \|F\|_{H'} \|u\|. \quad (4.2.13)$$

■

Remark. We observe that, as a is a bilinear form, by (4.2.5) we obtain that if $F_j \in H'$, $j = 1, 2$ and $u_j \in H$ are solutions to

$$a(u_j, v) = F_j(v) \quad \forall v \in H,$$

then

$$\|u_1 - u_2\| \leq \frac{1}{\alpha} \|F_1 - F_2\|_{H'} \quad (4.2.14)$$

from which, in particular, we get again the uniqueness. ♦

We now recall the Fredholm Alternative Theorem [53]. Meanwhile, we recall that a linear operator \mathcal{K} from H in itself is said compact, provided that for every bounded set $M \subset H$, $\mathcal{K}(M)$ is relatively compact in H . We recall that a compact operator is necessarily bounded, nevertheless if H does not have finite dimension, the identity on H is a bounded operator, but it is not compact.

Theorem 4.2.2 (Fredholm Alternative). *Let*

$$\mathcal{K} : H \rightarrow H,$$

a linear compact operator. Then we have:

- (i) *the dimension of $\mathcal{N}(I - \mathcal{K})$ is finite;*
- (ii) *$\mathcal{R}(I - \mathcal{K})$ is a closed subspace;*
- (iii) *$\mathcal{R}(I - \mathcal{K}) = \mathcal{N}(I - \mathcal{K}^*)^\perp$,*
- (iv) *$\mathcal{N}(I - \mathcal{K}) = \{0\}$ if and only if $\mathcal{R}(I - \mathcal{K}) = H$;*
- (v) *$\dim \mathcal{N}(I - \mathcal{K}) = \dim \mathcal{N}(I - \mathcal{K}^*)$.*

Recall that \mathcal{K}^* is the adjoint operator \mathcal{K} defined by

$$(\mathcal{K}u, v) = (u, \mathcal{K}^*v), \quad \forall u, v \in H.$$

4.3 The variational formulation of the Dirichlet problem. Existence theorems

Let us begin by clarifying what we mean by **the variational formulation** of Dirichlet problem (4.1.4). We begin by the case in which the **condition at the boundary is homogeneous**. In this case (4.1.4) can be written (formally):

$$\begin{cases} -\operatorname{div}(A\nabla u) = F, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (4.3.1)$$

and the variational formulation of problem above is the following:

Determine u such that

$$\begin{cases} \int_{\Omega} A\nabla u \cdot \nabla v dx = F(v), & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \quad (4.3.2)$$

Let us observe that if Ω is an open set of class $C^{0,1}$, $a^{jk} \in C^1(\overline{\Omega})$, $j, k = 1, \dots, n$ and $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ and $F \in C^0(\overline{\Omega})$, then (4.3.1) is equivalent to (4.3.2). Indeed, under such assumptions, the divergence Theorem implies

$$\int_{\Omega} \operatorname{div}(A\nabla u)v dx = - \int_{\Omega} A\nabla u \cdot \nabla v dx, \quad \forall v \in C_0^\infty(\Omega). \quad (4.3.3)$$

Therefore, (4.3.1) implies (4.3.2) (taking into account Remark 3.5.2 and that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$). Conversely, if (4.3.2) holds true, then Theorem 3.5.1 implies $u|_{\partial\Omega} = 0$ and (4.3.2) gives

$$\int_{\Omega} (\operatorname{div}(A\nabla u) + F)v dx = 0, \quad \forall v \in C_0^\infty(\Omega), \quad (4.3.4)$$

that gives (4.3.1). Of course, under the general assumptions on Ω , A and F , the formulation (4.3.1) makes no sense, while the formulation (4.3.2) makes perfectly sense, and the Lax–Milgram Theorem will tell us easily that it is a well-posed problem. Actually, let $H = H_0^1(\Omega)$ and

$$a(u, v) = \int_{\Omega} A\nabla u \cdot \nabla v dx, \quad \forall u, v \in H_0^1(\Omega). \quad (4.3.5)$$

The form (4.3.5) is bilinear. Moreover by the Cauchy–Schwarz inequality we have, for any $u, v \in H_0^1(\Omega)$,

$$\begin{aligned} |a(u, v)| &\leq \|A\|_{L^\infty(\Omega; \mathbb{M}(n))} \int_{\Omega} |\nabla u| |\nabla v| dx \leq \\ &\leq \|A\|_{L^\infty(\Omega; \mathbb{M}(n))} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \end{aligned}$$

and by (4.1.1) we have

$$\lambda^{-1} \|\nabla u\|_{L^2(\Omega)}^2 \leq \int_{\Omega} A \nabla u \cdot \nabla u dx = a(u, u), \quad \forall u \in H_0^1(\Omega).$$

Now, recalling that $\|\nabla u\|_{L^2(\Omega)}$ and $\|u\|_{H^1(\Omega)}$ are two equivalent norms of $H_0^1(\Omega)$, we have only to apply the Lax–Milgram Theorem to conclude that problem (4.3.2) has an unique solution in $H_0^1(\Omega)$. Moreover the following inequality holds true

$$\|\nabla u\|_{L^2(\Omega)} \leq \lambda \|F\|_{H^{-1}(\Omega)}. \quad (4.3.6)$$

Inequality (4.3.6), together with the already existence and uniqueness results, implies that problem (4.3.2) is well-posed in $H_0^1(\Omega)$.

Now we consider the case where the boundary condition is **not homogeneous**, but the equation is still homogeneous.

Let Ω be an open set of \mathbb{R}^n of class $C^{0,1}$, let $\varphi \in H^{1/2}(\partial\Omega)$ and let us assume that (4.1.1) is satisfied. Formally, the Dirichlet problem can be written as

$$\begin{cases} -\operatorname{div}(A \nabla u) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \quad (4.3.7)$$

We wish to give the variational formulation of problem (4.3.7) and to prove the existence of the solutions in $H^1(\Omega)$ to this problem.

The variational formulation of (4.3.7) is

$$\begin{cases} \int_{\Omega} A \nabla u \cdot \nabla v dx = 0, & \forall v \in H_0^1(\Omega), \\ u = \varphi, & \text{on } \partial\Omega \quad (\text{in the sense of the traces}). \end{cases} \quad (4.3.8)$$

Notice that, in the case where A , u and φ are sufficiently regular, the first equation in (4.3.8) is equivalent to the first equation of (4.3.7).

In order to solve (4.3.7) ((4.3.8)), we proceed in the following way. Recalling Theorem 3.12.12, there exists $\Phi \in H^1(\Omega)$ such that

$$\Phi|_{\partial\Omega} = \varphi, \quad (\text{in the sense of the traces})$$

which in turn (by (ii) of Theorem 3.12.12) implies

$$\|\Phi\|_{H^1(\Omega)} \leq C \|\varphi\|_{H^{1/2}(\partial\Omega)}, \quad (4.3.9)$$

where C is a constant depending on Ω only. Set $w = u - \Phi$. Since u and Φ have the same trace on $\partial\Omega$ we have $w|_{\partial\Omega} = 0$, so that problem (4.3.7) can be written (formally),

$$\begin{cases} -\operatorname{div}(A\nabla w) = \operatorname{div}(A\nabla\Phi), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.3.10)$$

whose variational formulation is

$$\begin{cases} \int_{\Omega} A\nabla w \cdot \nabla v \, dx = \int_{\Omega} A\nabla\Phi \cdot \nabla v \, dx, & \forall v \in H_0^1(\Omega), \\ w \in H_0^1(\Omega). \end{cases} \quad (4.3.11)$$

Let us note that the bilinear form is still given by (4.3.5). The solution of problem (4.3.7) ((4.3.8)) is given by

$$u = w + \Phi \in H^1(\Omega). \quad (4.3.12)$$

Moreover, denoting

$$F(v) = \int_{\Omega} A\nabla\Phi \cdot \nabla v \, dx, \quad \forall v \in H_0^1(\Omega),$$

it turns out that $F \in H^{-1}(\Omega)$. As a matter of fact by the Cauchy–Schwarz inequality we have

$$|F(v)| \leq \lambda \|\nabla\Phi\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

By (4.3.9) and (4.3.10) we have

$$|F(v)| \leq C\lambda \|\varphi\|_{H^{1/2}(\partial\Omega)} \|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Therefore $F \in H^{-1}(\Omega)$ and the following inequality holds

$$\|F\|_{H^{-1}(\Omega)} \leq C' \|\varphi\|_{H^{1/2}(\partial\Omega)}, \quad (4.3.13)$$

where C' depends on Ω and λ only .

The Lax–Milgram Theorem implies that problem (4.3.11) has a unique solution $w \in H_0^1(\Omega)$, moreover by (4.2.5) we have

$$\|\nabla w\|_{L^2(\Omega)} \leq \lambda \|F\|_{H^{-1}(\Omega)} \leq \lambda C' \|\varphi\|_{H^{1/2}(\partial\Omega)}. \quad (4.3.14)$$

Now, by using (4.3.9), (4.3.12) and (4.3.14) we get

$$\|u\|_{H^1(\Omega)} \leq \|w\|_{H^1(\Omega)} + \|\Phi\|_{H^1(\Omega)} \leq C'' \|\varphi\|_{H^{1/2}(\partial\Omega)}, \quad (4.3.15)$$

where C'' depends by λ and Ω only. Notice that (4.3.15) implies, in particular, the uniqueness of solution of problem (4.3.7).

From what we have proved so far, we have the following

Theorem 4.3.1. *Let Ω be an open set of class $C^{0,1}$. Let us assume that $A \in L^\infty(\Omega; \mathbb{M}(n))$ and A satisfies (4.1.1). Let $F \in H^{-1}(\Omega)$ and $\varphi \in H^{1/2}(\partial\Omega)$. Then the following problem*

$$\begin{cases} -\operatorname{div}(A\nabla u) = F, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \quad (4.3.16)$$

whose variational formulation is

$$\begin{cases} \int_{\Omega} A\nabla u \cdot \nabla v dx = F(v), & \forall v \in H_0^1(\Omega), \\ u = \varphi, & \text{su } \partial\Omega \quad (\text{in the sense of traces}), \end{cases} \quad (4.3.17)$$

has a unique solution $u \in H^1(\Omega)$ and we have

$$\|u\|_{H^1(\Omega)} \leq C \left(\|F\|_{H^{-1}(\Omega)} + \|\varphi\|_{H^{1/2}(\partial\Omega)} \right), \quad (4.3.18)$$

where C depends on λ and Ω only.

Now let L be the following operator

$$Lu = - \sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu, \quad (4.3.19)$$

where $A \in L^\infty(\Omega; \mathbb{M}(n))$, $A = \{a^{jk}\}_{j,k=1}^n$, satisfies (4.1.1) and

$$b^j, d^j, c \in L^\infty(\Omega),$$

for $j = 1, \dots, n$.

Let us consider the Dirichlet problem

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{su } \partial\Omega, \end{cases} \quad (4.3.20)$$

where $f \in L^2(\Omega)$. The variational formulation of above problem is:

Determine u such that

$$\begin{cases} a(u, v) = (f, v), & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega), \end{cases} \quad (4.3.21)$$

where

$$a(u, v) = \int_{\Omega} (A \nabla u \cdot \nabla v + u d \cdot v - b \cdot \nabla uv - cuv) dx, \quad (4.3.22)$$

$d = (d^1, \dots, d^n)$ and $b = (b^1, \dots, b^n)$.

Problem (4.3.20) does not always have existence and uniqueness. To show this fact, let us consider the following simple example

$$\begin{cases} -u'' - u = f, & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \quad (4.3.23)$$

where $f \in L^2(0, \pi)$. The solutions to (4.3.23) have to be found among the functions of the type

$$C_1 \sin x + C_2 \cos x - \int_0^x \sin(x-t) f(t) dt.$$

By the boundary conditions we have $C_2 = 0$ and

$$\int_0^\pi f(t) \sin t dt = 0. \quad (4.3.24)$$

Therefore, if (4.3.24) is satisfied, then (4.3.23) has infinite solutions, given by

$$C \sin x - \int_0^x \sin(x-t) f(t) dt, \quad C \in \mathbb{R}$$

Whereas if (4.3.24) is not satisfied, then (4.3.23) has no solutions. So we cannot expect that the bilinear form (4.3.22) is always coercive. We can, however, prove the following Theorem that will be useful for establish some conditions of existence and uniqueness to problem (4.3.21)

Theorem 4.3.2. *Let Ω be a bounded open set of \mathbb{R}^n . Let us assume that $A \in L^\infty(\Omega; \mathbb{M}(n))$ and that A satisfies condition (4.1.1) and $b, d \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$, for $j = 1, \dots, n$. Moreover, let*

$$a(u, v) = \int_{\Omega} (A \nabla u \cdot \nabla v + u d \cdot \nabla v - b \cdot \nabla u v - c u v) dx.$$

Then a is a continuous bilinear form and there exist $\gamma_0 \geq 0$ and $\alpha_0 > 0$ such that for any $\gamma \geq \gamma_0$ we have

$$\alpha_0 \|u\|_{H^1(\Omega)}^2 \leq a(u, u) + \gamma \|u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega). \quad (4.3.25)$$

In particular,

$$a_\gamma(u, v) := a(u, v) + \gamma (u, v)_{L^2(\Omega)}, \quad (4.3.26)$$

is a continuous and coercive bilinear form for every $\gamma \geq \gamma_0$.

Proof. By the Cauchy–Schwarz inequality we have, for any $u, v \in H_0^1(\Omega)$

$$\begin{aligned} |a(u, v)| &\leq \int_{\Omega} \|A\|_{L^\infty(\Omega; \mathbb{M}(n))} |\nabla u| |\nabla v| dx + \\ &+ \int_{\Omega} \left(\|d\|_{L^\infty(\Omega; \mathbb{R}^n)} |u| |\nabla v| + \|b\|_{L^\infty(\Omega; \mathbb{R}^n)} |\nabla u| |v| + \|c\|_{L^\infty(\Omega)} |u| |v| \right) dx \leq \\ &\leq \|A\|_{L^\infty(\Omega; \mathbb{M}(n))} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|d\|_{L^\infty(\Omega; \mathbb{R}^n)} \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \\ &+ \|b\|_{L^\infty(\Omega; \mathbb{R}^n)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \\ &\leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \end{aligned}$$

where

$$C = \left(\|A\|_{L^\infty(\Omega; \mathbb{M}(n))} + \|d\|_{L^\infty(\Omega; \mathbb{R}^n)} + \|b\|_{L^\infty(\Omega; \mathbb{R}^n)} + \|c\|_{L^\infty(\Omega)} \right).$$

Therefore

$$|a_\gamma(u, v)| \leq (C + |\gamma|) \|v\|_{H^1(\Omega)} \|u\|_{H^1(\Omega)}, \quad \forall u, v \in H_0^1(\Omega)$$

which implies the continuity of a_γ for any $\gamma \in \mathbb{R}$.

Concerning coercivity of a_γ , we notice firstly that (4.1.1) gives

$$\int_{\Omega} A \nabla u \cdot \nabla u dx \geq \lambda^{-1} \|\nabla u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega). \quad (4.3.27)$$

Moreover, let ε be a positive number which we will choose later on. We get

$$\begin{aligned} \left| \int_{\Omega} u d \cdot \nabla u dx \right| &\leq \|d\|_{L^\infty(\Omega; \mathbb{R}^n)} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq \\ &\leq \frac{\varepsilon}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} \|d\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 \|u\|_{L^2(\Omega)}^2 \end{aligned} \quad (4.3.28)$$

similarly,

$$\left| \int_{\Omega} b \cdot \nabla u dx \right| \leq \frac{\varepsilon}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon} \|b\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 \|u\|_{L^2(\Omega)}^2. \quad (4.3.29)$$

Furthermore

$$\left| \int_{\Omega} c u^2 dx \right| \leq \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2. \quad (4.3.30)$$

Hence, by (4.3.27)–(4.3.30) we obtain

$$\begin{aligned} a_\gamma(u, u) &\geq (\lambda^{-1} - \varepsilon) \|\nabla u\|_{L^2(\Omega)}^2 + \\ &+ \left[\gamma - \left(\frac{1}{2\varepsilon} \|d\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 + \frac{1}{2\varepsilon} \|b\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 + \|c\|_{L^\infty(\Omega)} \right) \right] \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

Now, by choosing

$$\varepsilon = \frac{\lambda^{-1}}{2}$$

and denoting

$$\gamma_0 = \lambda \|d\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 + \lambda \|b\|_{L^\infty(\Omega; \mathbb{R}^n)}^2 + \|c\|_{L^\infty(\Omega)},$$

we have, for any $\gamma \geq \gamma_0$

$$a_\gamma(u, u) \geq \frac{\lambda^{-1}}{2} \|\nabla u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega).$$

Finally, the first Poincaré inequality (Theorem 3.4.2) implies that there exists α_0 , depending on the diameter of Ω , such that (4.3.25) is satisfied. ■

Exercise. Prove there that there exists $\delta > 0$ such that if $\text{diam}(\Omega) \leq \delta$ ($\text{diam}(\Omega)$ is the diameter of) then problem (4.3.20), where $f \in H^{-1}(\Omega)$, has a unique solution in $H_0^1(\Omega)$. [Hint: use the first Poincaré inequality].

The following operator is called **the (formal) adjoint** of the operator L

$$L^*v = - \sum_{j,k=1}^n \partial_k (a^{jk} \partial_j v + b^k v) + \sum_{j=1}^n d^j \partial_j v + cv. \quad (4.3.31)$$

To the operator L^* corresponds the bilinear form

$$a^*(v, u) = \int_{\Omega} (A^T \nabla v \cdot \nabla u - vb \cdot \nabla u + d \cdot \nabla v u - cvu) dx, \quad \forall u, v \in H_0^1(\Omega),$$

where A^T is the transposed of the matrix A . Let us notice that

$$a^*(v, u) = a(u, v), \quad \forall u, v \in H_0^1(\Omega).$$

Finally, let $f \in L^2(\Omega)$, we say that $v \in H_0^1(\Omega)$ is a weak solution of the adjoint problem

$$\begin{cases} L^*v = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.3.32)$$

provided that we have

$$\begin{cases} a^*(v, u) = (f, u)_{L^2(\Omega)}, & \forall u \in H_0^1(\Omega), \\ v \in H_0^1(\Omega). \end{cases} \quad (4.3.33)$$

The following Theorem holds true

Theorem 4.3.3. *Let L be operator (4.3.19) and L^* its formal adjoint.*

(i) The following alternative holds true.

either

(a) for any $f \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (4.3.34)$$

or

(b) there exists at least one **not identically vanishing solution** $u \in H_0^1(\Omega)$ to the homogeneous problem

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.3.35)$$

(ii) If (b) holds true, then, denoting by N the subspace $H_0^1(\Omega)$ of the solutions to (4.3.35) and by N^* the subspace of $H_0^1(\Omega)$ of the solutions to

$$\begin{cases} L^*v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.3.36)$$

we have that N and N^* have finite **dimension**, moreover

$$\text{dimension of } N = \text{dimension of } N^*.$$

(iii) Finally, problem (4.3.34) admits a solution in $H_0^1(\Omega)$ if and only if

$$(f, v)_{L^2(\Omega)} = 0, \quad \forall v \in N^*. \quad (4.3.37)$$

Proof. Let γ_0 be the same of Theorem 4.3.2. Let us fix $\gamma \geq \gamma_0$. Since a_γ , defined by (4.3.26), is a continuous and coercive bilinear form, we have that for any $g \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$L_\gamma u = g.$$

Set

$$L_\gamma^{-1}g = u.$$

Now, we notice that $u \in H_0^1(\Omega)$ solves the boundary value problem

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{su } \partial\Omega \end{cases} \quad (4.3.38)$$

if and only if

$$a_\gamma(u, v) = (\gamma u + f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

which, in turn, is equivalent to

$$u = L_\gamma^{-1}(\gamma u + f). \quad (4.3.39)$$

Let us denote

$$\mathcal{K}u = \gamma L_\gamma^{-1}u \quad (4.3.40)$$

and

$$h = L_\gamma^{-1}f. \quad (4.3.41)$$

Let us notice that \mathcal{K} is linear and it satisfies

$$a_\gamma(\mathcal{K}g, v) = \gamma(g, v)_{L^2(\Omega)}, \quad \forall g \in L^2(\Omega), \forall v \in H_0^1(\Omega).$$

Moreover, as (4.3.39) and (4.3.40) hold, (4.3.38) is equivalent to

$$u - \mathcal{K}u = h. \quad (4.3.42)$$

Also, we notice that we have $h \in H_0^1(\Omega)$ and by the definition of \mathcal{K} , we have $\mathcal{K}g \in H_0^1(\Omega)$, for any $g \in L^2(\Omega)$. Therefore, every function of $L^2(\Omega)$ which is a solution to (4.3.42) belongs to $H_0^1(\Omega)$.

Now we examine the solvability of (4.3.42) in $L^2(\Omega)$.

Let us begin to check that

$$\mathcal{K} : L^2(\Omega) \rightarrow L^2(\Omega),$$

is a well-defined compact operator. Let $g \in L^2(\Omega)$, set

$$w = \mathcal{K}g.$$

By (4.3.25) we have

$$\alpha_0 \|w\|_{H^1(\Omega)}^2 \leq a_\gamma(w, w) = \gamma(g, w)_{L^2(\Omega)} \leq \gamma \|g\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}.$$

Hence

$$\|\mathcal{K}g\|_{H^1(\Omega)} = \|w\|_{H^1(\Omega)} \leq \frac{\gamma}{\alpha_0} \|g\|_{L^2(\Omega)}. \quad (4.3.43)$$

Therefore the operator \mathcal{K} is well-defined from $L^2(\Omega)$ in itself. In addition \mathcal{K} is compact. As a matter of fact, let M be a bounded set of $L^2(\Omega)$, by (4.3.43), we have that $\mathcal{K}(M)$ is bounded in $H^1(\Omega)$ so that the Rellich–Kondrachov Theorem implies that $\mathcal{K}(M)$ is relatively compact in $L^2(\Omega)$.

Let us apply Theorem 4.2.2. By proposition (iv) of such a Theorem we have the following alternative:

either

(j) the equation

$$u - \mathcal{K}u = \tilde{h}. \quad (4.3.44)$$

has a unique solution in $L^2(\Omega)$, for every $\tilde{h} \in L^2(\Omega)$

or

(jj) there exists at least a not identically vanishing solution (in $L^2(\Omega)$) to the equation

$$u - \mathcal{K}u = 0. \quad (4.3.45)$$

If proposition (j) holds true, then, as (4.3.38) and (4.3.42) are equivalent, we have that there exists a unique solution to problem (4.3.34). Whereas, if proposition (jj) holds true, then, by (i) and (v) of Theorem 4.2.2, we have that the subspace $N \neq \{0\}$ of solutions of (4.3.45) (hence, of the solutions to (4.3.35)) has finite dimension. Let us observe that, in the latter case we have

$$\gamma \neq 0. \quad (4.3.46)$$

otherwise we should have $\mathcal{K} = 0$ and, by (4.3.45), $N = \{0\}$. Moreover the dimension of N is equal to the dimension of N^* , where N^* is the subspace of the solutions to

$$u - \mathcal{K}^*u = 0. \quad (4.3.47)$$

At this point, to conclude (b), let us examine what relationship holds true between \mathcal{K}^* and L^* .

Claim. Let us denote

$$L_\gamma^* = L^* + \gamma, \quad (4.3.48)$$

we have

$$\mathcal{K}^* = \gamma (L_\gamma^*)^{-1}. \quad (4.3.49)$$

Proof of the Claim. Firstly, recall that

$$\begin{aligned} \forall g \in L^2(\Omega) \quad \exists u \in H_0^1(\Omega) \text{ (unique) such that} \\ a_\gamma(u, v) = (g, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega) \end{aligned}$$

that is

$$u = L_\gamma^{-1}g. \quad (4.3.50)$$

By the Lax–Milgram Theorem we have

$$\begin{aligned} \forall \tilde{g} \in L^2(\Omega) \quad \exists \tilde{u} \in H_0^1(\Omega) \text{ (unique) such that} \\ a_\gamma^*(\tilde{u}, \tilde{v}) = (\tilde{g}, \tilde{v})_{L^2(\Omega)}, \quad \forall \tilde{v} \in H_0^1(\Omega) \end{aligned}$$

that is

$$\tilde{u} = (L_\gamma^*)^{-1} \tilde{g}. \quad (4.3.51)$$

Now, let us recall that

$$a_\gamma^*(\tilde{u}, \tilde{v}) = a_\gamma(\tilde{v}, \tilde{u}), \quad \forall \tilde{v} \in H_0^1(\Omega),$$

and let us choose $\tilde{v} = u$, where u is given by (4.3.50). We get

$$(\tilde{g}, u)_{L^2(\Omega)} = a_\gamma^*(\tilde{u}, u) = a_\gamma(u, \tilde{u}) = (g, \tilde{u})_{L^2(\Omega)}.$$

Hence, (4.3.50) and (4.3.51) give

$$(\tilde{g}, L_\gamma^{-1}g)_{L^2(\Omega)} = \left((L_\gamma^*)^{-1} \tilde{g}, g \right)_{L^2(\Omega)},$$

for every $\tilde{g} \in L^2(\Omega)$ and for every $g \in L^2(\Omega)$. Now, as $\mathcal{K}g = \gamma L_\gamma^{-1}g$ we obtain (4.3.49). The Claim is proved.

By (4.3.48) and (4.3.49) (taking into account that \mathcal{K}^* assumes its values in $H_0^1(\Omega)$) we have the equivalences

$$v - \mathcal{K}^*v = 0 \iff L_\gamma^*v - \gamma v = 0 \iff L^*v = 0.$$

Therefore, the solutions (4.3.47) to are all and only the solutions to (4.3.36) and by that also (b) is proved

Now, we prove (iii). Proposition (iii) of Theorem 4.2.2 implies that the boundary value problem (4.3.38) (which, we recall, is equivalent to equation (4.3.42)) admits a solution if and only if

$$(h, v)_{L^2(\Omega)} = 0, \quad \forall v \in N^*.$$

On the other hand, by $v = \mathcal{K}^*v$, (4.3.40) and by (4.3.41), we have

$$(f, v)_{L^2(\Omega)} = (f, \mathcal{K}^*v)_{L^2(\Omega)} = (\mathcal{K}f, v)_{L^2(\Omega)} = \gamma (h, v)_{L^2(\Omega)}$$

and, taking into account (4.3.46), we have that problem (4.3.34) has a solution in $H_0^1(\Omega)$ if and only if

$$(f, v)_{L^2(\Omega)} = 0, \quad \forall v \in N^*.$$

■

4.4 The Neumann problem

Let Ω be a connected open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$ and let A be a matrix whose entries are functions of $L^\infty(\Omega)$. Let us assume that A satisfies (4.1.1). Formally the Neumann problem for the equation

$$-\operatorname{div}(A\nabla u) = F,$$

may be written as follows

$$\begin{cases} -\operatorname{div}(A\nabla u) = F, & \text{in } \Omega, \\ A\nabla u \cdot \nu = g, & \text{on } \partial\Omega. \end{cases} \quad (4.4.1)$$

Concerning the variational formulation we first have to specify that

$$F \in (H^1(\Omega))' \quad \text{and} \quad g \in H^{-1/2}(\partial\Omega),$$

where $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$. Having done this, arguing similarly to the Dirichlet problem, we formulate the Neumann problem as follows:

Determine $u \in H^1(\Omega)$ such that

$$\begin{cases} \int_{\Omega} A\nabla u \cdot \nabla v dx = F(v) + \langle g, v \rangle_{H^{-1/2}, H^{1/2}}, & \forall v \in H^1(\Omega), \\ u \in H^1(\Omega), \end{cases} \quad (4.4.2)$$

where we mean

$$\langle g, v \rangle_{H^{-1/2}, H^{1/2}} = \langle g, T(v) \rangle_{H^{-1/2}, H^{1/2}},$$

here $T(v)$ is the trace of v on $\partial\Omega$.

Let us notice at once that, by setting $v = 1$, in (4.4.2) we have that a necessary condition (and, as we will see in Theorem 4.4.1, also sufficient) to ensure that the problem (4.4.2) admits solutions, is

$$F(1) + \langle g, 1 \rangle_{H^{-1/2}, H^{1/2}} = 0. \quad (4.4.3)$$

Also, we notice that if $u_0 \in H^1(\Omega)$ is a solution to problem (4.4.2), then all the solutions to (4.4.2) are given by

$$u_0 + C,$$

where C is any constant. Indeed, it is immediately checked that $u_0 + C$ is a solution to (4.4.2). Conversely, if $u \in H^1(\Omega)$ is a solution to (4.4.2), then

$$\int_{\Omega} A \nabla(u - u_0) \cdot \nabla v dx = 0, \quad \forall v \in H^1(\Omega).$$

Now, we choose $v = u - u_0$ and (4.1.1) gives

$$\lambda^{-1} \int_{\Omega} |\nabla(u - u_0)|^2 \leq \int_{\Omega} A \nabla(u - u_0) \cdot \nabla(u - u_0) dx = 0.$$

Since Ω is connected, we obtain that $u - u_0$ is a constant in Ω .

Thus, to ensure the uniqueness to the Neumann problem we may formulate it as follows

$$\begin{cases} \int_{\Omega} A \nabla u \cdot \nabla v dx = F(v) + \langle g, v \rangle_{H^{-1/2}, H^{1/2}}, & \forall v \in H^1(\Omega), \\ u \in \{w \in H^1(\Omega) : \int_{\Omega} w dx = 0\}. \end{cases} \quad (4.4.4)$$

Concerning the existence, we have

Theorem 4.4.1. *Let Ω be a connected open set of \mathbb{R}^n whose boundary is of class $C^{0,1}$. Let us assume that $A \in L^\infty(\Omega; \mathbb{M}(n))$ and that A satisfies (4.1.1). Let us assume that $F \in (H^1(\Omega))'$ and $g \in H^{-1/2}(\partial\Omega)$ satisfy (4.4.3).*

Then problem (4.4.4) has a unique solution and the following inequality holds true

$$\|u\|_{H^1(\Omega)} \leq C \left(\|F\|_{(H^1(\Omega))'} + \|g\|_{H^{-1/2}(\partial\Omega)} \right), \quad (4.4.5)$$

where C depends on λ and Ω only.

Proof. Set

$$\tilde{H} := \left\{ w \in H^1(\Omega) : \int_{\Omega} w dx = 0 \right\},$$

Theorem 3.9.1 implies that \tilde{H} is a Hilbert space equipped with the norm

$$\|w\|_{\tilde{H}} = \left(\int_{\Omega} |\nabla w|^2 dx \right)^{1/2}.$$

Moreover, the bilinear form on \tilde{H}

$$a(u, v) = \int_{\Omega} A \nabla u \cdot \nabla v dx, \quad (4.4.6)$$

is coercive and continuous. Now we check that the linear functional

$$\tilde{H} \ni v \rightarrow \tilde{F}(v) := F(v) + \langle g, v \rangle_{H^{-1/2}, H^{1/2}} \in \mathbb{R},$$

is well-defined and continuous on \tilde{H} . As a matter of fact, by Theorem 3.9.1 we have

$$|F(v)| \leq \|F\|_{(H^1(\Omega))'} \|v\|_{H^1(\Omega)} \leq C \|F\|_{H^{-1}(\Omega)} \|v\|_{\tilde{H}}, \quad \forall v \in \tilde{H}, \quad (4.4.7)$$

where C depends on Ω only. Moreover, recalling that $H^{-1/2}(\Omega)$ is the dual space of $H^{1/2}(\Omega)$ (compare Section 3.13.1), inequality (ii) of Theorem 3.5.1 gives

$$\begin{aligned} |\langle g, v \rangle_{H^{-1/2}, H^{1/2}}| &\leq \|g\|_{H^{-1/2}(\partial\Omega)} \|v\|_{H^{1/2}(\partial\Omega)} \leq \\ &\leq C \|g\|_{H^{-1/2}(\partial\Omega)} \|v\|_{H^1(\Omega)} \leq \\ &\leq C \|g\|_{H^{-1/2}(\partial\Omega)} \|v\|_{\tilde{H}}, \end{aligned} \quad (4.4.8)$$

where C depends by Ω only. Therefore, (4.4.7) and (4.4.8) give

$$|\tilde{F}(v)| \leq C \left(\|F\|_{H^{-1}(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right) \|v\|_{\tilde{H}}, \quad \forall v \in \tilde{H}. \quad (4.4.9)$$

Now, since bilinear form (4.4.6) is continuous and coercive on \tilde{H} and since (4.4.9) holds, \tilde{F} is a bounded linear functional on \tilde{H} . Therefore by the Lax-Milgram Theorem we have that there exists a unique $u \in \tilde{H}$ which satisfies

$$a(u, v) = \tilde{F}(v), \quad \forall v \in \tilde{H}. \quad (4.4.10)$$

Moreover

$$\|u\|_{H^1(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \leq C \left(\|F\|_{(H^1(\Omega))'} + \|g\|_{H^{-1/2}(\partial\Omega)} \right), \quad (4.4.11)$$

where C depends on λ and Ω only.

Now, let v be any function of $H^1(\Omega)$ and let us denote

$$v_\Omega = \frac{1}{|\Omega|} \int_\Omega v dx, \quad \tilde{v} = v - v_\Omega.$$

Since $\tilde{v} \in \tilde{H}$, by (4.4.3) and (4.4.10) we obtain

$$\begin{aligned} \int_\Omega A \nabla u \cdot \nabla v dx &= \int_\Omega A \nabla u \cdot \nabla \tilde{v} dx = \\ &= a(u, \tilde{v}) = F(\tilde{v}) + \langle g, \tilde{v} \rangle_{H^{-1/2}, H^{1/2}} = \\ &= F(v) + \langle g, v \rangle_{H^{-1/2}, H^{1/2}} - v_\Omega (F(1) + \langle g, 1 \rangle_{H^{-1/2}, H^{1/2}}) = \\ &= F(v) + \langle g, v \rangle_{H^{-1/2}, H^{1/2}}. \end{aligned} \quad (4.4.12)$$

Therefore u is a solution to problem (4.4.4). Estimate (4.4.4) follows by (4.4.11). ■

4.5 The Caccioppoli inequality

Theorem 4.5.1 (the Caccioppoli inequality). *Let $x_0 \in \mathbb{R}^n$ and $R > 0$. Let A be a symmetric matrix whose entries are measurable functions on $B_R(x_0)$. Let us assume A satisfies (4.1.1) (with $\Omega = B_R(x_0)$). Let $b \in L^\infty(B_R(x_0); \mathbb{R}^n)$ and $c \in L^\infty(B_R(x_0))$. Let $u \in H_{loc}^1(B_R(x_0))$ satisfy*

$$\int_{B_R(x_0)} A \nabla u \cdot \nabla v dx = \int_{B_R(x_0)} (b \nabla u + cu) v dx, \quad \forall v \in H_0^1(B_R(x_0)). \quad (4.5.1)$$

If $0 < r < \rho < R$, then we have

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho(x_0)} u^2 dx, \quad (4.5.2)$$

where C depends on λ , $R \|b\|_{L^\infty(B_R(x_0); \mathbb{R}^n)}$ and $R^2 \|c\|_{L^\infty(B_R(x_0); \mathbb{R}^n)}$ only.

Proof. It is not restrictive to assume $x_0 = 0$. Let $\eta \in C_0^\infty(B_\rho)$ satisfy

$$0 \leq \eta \leq 1; \quad \eta = 1, \quad \text{in } B_r \quad (4.5.3)$$

and

$$|\nabla \eta| \leq \frac{K}{\rho - r}, \quad (4.5.4)$$

where K is a positive constant. We choose in (4.5.1)

$$v = \eta^2 u$$

and we have

$$\int_{B_R} A \nabla u \cdot \nabla (\eta^2 u) dx = \int_{B_R} (b \nabla u + cu) \eta^2 u dx. \quad (4.5.5)$$

Hence

$$\begin{aligned}
& \int_{B_R} (A\nabla u \cdot \nabla u) \eta^2 dx = \int_{B_R} (b\nabla u + cu) \eta^2 u dx - \\
& - 2 \int_{B_R} (A\nabla u \cdot \nabla \eta) \eta u dx \leq \\
& \leq \int_{B_R} (|b| |\nabla u| |u| \eta^2 + |c| u^2 \eta^2) dx + \\
& + 2 \int_{B_R} (A\nabla u \cdot \nabla u)^{1/2} (A\nabla \eta \cdot \nabla \eta)^{1/2} |u| \eta dx \leq \\
& \leq \int_{B_R} (|b| |\nabla u| |u| \eta^2 + |c| u^2 \eta^2) dx + \\
& + \frac{1}{2} \int_{B_R} (A\nabla u \cdot \nabla u) \eta^2 dx + 2 \int_{B_R} (A\nabla \eta \cdot \nabla \eta) u^2 dx.
\end{aligned}$$

By moving to the left-hand side the second-to-last integral and, by estimating from above the last integral, we have

$$\begin{aligned}
& \frac{1}{2} \int_{B_R} (A\nabla u \cdot \nabla u) \eta^2 dx \leq \\
& \leq \int_{B_R} (|b| |\nabla u| |u| \eta^2 + |c| u^2 \eta^2) dx + \frac{2K^2 \lambda}{(\rho - r)^2} \int_{B_\rho} u^2 dx.
\end{aligned} \tag{4.5.6}$$

Now let us estimate from above the first integral on the right hand side of (4.5.6). We obtain, for $\varepsilon > 0$ to be chosen,

$$\begin{aligned}
\int_{B_R} (|b| |\nabla u| |u| \eta^2 + |c| u^2 \eta^2) dx & \leq \frac{\varepsilon}{2} \int_{B_R} |\nabla u|^2 \eta^2 dx + \\
& + \frac{1}{2\varepsilon} \|b\|_{L^\infty(B_R; \mathbb{R}^n)}^2 \int_{B_R} u^2 \eta^2 dx + \\
& + \|c\|_{L^\infty(B_R)} \int_{B_R} u^2 \eta^2 dx \leq \\
& \leq \frac{\varepsilon \lambda}{2} \int_{B_R} (A\nabla u \cdot \nabla u) \eta^2 dx + \\
& + C_\varepsilon \int_{B_R} u^2 \eta^2 dx,
\end{aligned} \tag{4.5.7}$$

where

$$C_\varepsilon = \frac{1}{2\varepsilon} \|b\|_{L^\infty(B_R; \mathbb{R}^n)}^2 + \|c\|_{L^\infty(B_R)}.$$

Using inequality (4.5.7) in (4.5.6), after a few easy calculations, we have

$$\begin{aligned} & \frac{1}{2} (1 - \varepsilon\lambda) \int_{B_R} (A\nabla u \cdot \nabla u) \eta dx \leq \\ & \leq C_\varepsilon \int_{B_R} u^2 \eta^2 dx + \frac{2K^2\lambda}{(\rho - r)^2} \int_{B_\rho} u^2 dx. \end{aligned} \quad (4.5.8)$$

Now we choose

$$\varepsilon = \varepsilon_0 := \frac{1}{2\lambda},$$

and we get

$$C_{\varepsilon_0} = \lambda \|b\|_{L^\infty(B_R; \mathbb{R}^n)}^2 + \|c\|_{L^\infty(B_R)}$$

and by (4.5.8) we have

$$\begin{aligned} \frac{\lambda^{-1}}{4} \int_{B_r} |\nabla u|^2 dx & \leq \frac{1}{4} \int_{B_R} (A\nabla u \cdot \nabla u) \eta^2 dx \leq \\ & \leq \frac{K_1}{(\rho - r)^2} \int_{B_\rho} u^2, \end{aligned} \quad (4.5.9)$$

where

$$K_1 = 2K^2\lambda + R^2 C_{\varepsilon_0}.$$

By (4.5.9) we obtain immediately

$$\int_{B_r} |\nabla u|^2 dx \leq \frac{4K_1\lambda}{(\rho - r)^2} \int_{B_\rho} u^2 dx,$$

so that (4.5.2) follows. ■

Exercise 1. Let $x_0 \in \mathbb{R}^n$ and $R > 0$. Let L be the operator

$$Lu = - \sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu, \quad (4.5.10)$$

where $A \in L^\infty(B_R(x_0); \mathbb{M}(n))$, $A = \{a^{jk}\}_{j,k=1}^n$, satisfies (4.1.1) and $b^j, d^j, c \in L^\infty(B_R(x_0))$, for $j = 1, \dots, n$. Let $f \in L^2(B_R(x_0))$ and let us assume that $u \in H^1(B_R(x_0))$ is a weak solution to

$$Lu = f, \quad \text{in } B_R(x_0).$$

Prove that, if $0 < r < \rho < R$ then we have

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho(x_0)} u^2 dx + C\rho^2 \int_{B_\rho(x_0)} f^2 dx, \quad (4.5.11)$$

where C depends on λ , $\|A\|_{L^\infty(B_R(x_0); \mathbb{M}(n))}$, $R\|d\|_{L^\infty(B_R(x_0); \mathbb{R}^n)}$, $R\|b\|_{L^\infty(B_R(x_0); \mathbb{R}^n)}$ and $R^2\|c\|_{L^\infty(B_R(x_0))}$ only. ♣

Exercise 2. Let $R > 0$ and $x_0 \in \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$. Let L be operator (4.5.10) and let $u \in H^1(B_R^+(x_0))$ satisfy

$$\begin{cases} Lu = f, & \text{in } B_R^+(x_0), \text{ in weak sense,} \\ u(x', 0) = 0, & x' \in B'_R(x_0) \text{ in the traces sense,} \end{cases}$$

then

$$\int_{B_r^+(x_0)} |\nabla u|^2 dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho^+(x_0)} u^2 dx + C\rho^2 \int_{B_\rho^+(x_0)} f^2 dx, \quad (4.5.12)$$

where C depends on λ , $\|A\|_{L^\infty(B_R^+(x_0); \mathbb{M}(n))}$, $R\|d\|_{L^\infty(B_R^+(x_0); \mathbb{R}^n)}$, $R\|b\|_{L^\infty(B_R^+(x_0); \mathbb{R}^n)}$ and $R^2\|c\|_{L^\infty(B_R^+(x_0))}$ only. ♣

Exercise 3. Let x_0 , R and A be like Exercise 2.

(a) Give the variational formulation of problem

$$\begin{cases} \operatorname{div}(A\nabla u) = 0, & \text{in } B_R^+(x_0), \\ (A\nabla u)(x', 0) \cdot e_n = 0, & \text{for } x' \in B'_R(x_0). \end{cases} \quad (4.5.13)$$

(b) Prove that

$$\int_{B_r^+(x_0)} |\nabla u|^2 dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho^+(x_0)} u^2 dx, \quad (4.5.14)$$

where C depends on λ and $\|A\|_{L^\infty(B_R^+(x_0); \mathbb{M}(n))}$ only. ♣

4.6 The regularity theorems

In Section 4 (Theorem 4.3.1) we have proved that if Ω is a bounded open set of \mathbb{R}^n , $A \in L^\infty(\Omega; \mathbb{M}(n))$ satisfies (4.1.1) and $F \in H^{-1}(\Omega)$, then the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = F, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4.6.1)$$

is well-posed in $H_0^1(\Omega)$.

It is natural to ask whether with more restrictive assumptions on the data Ω, A and F , u is more regular. More precisely, we ask whether there exists $k > 1$ such that $u \in H^k(\Omega)$. In carrying out this investigation it is convenient to distinguish between the **regularity in the interior** and **regularity at the boundary**. In the investigation of the regularity in the interior we are interested in whether for some $k > 1$ we have $u \in H_{loc}^k(\Omega)$, while in the investigation of regularity at the boundary we are interested in knowing whether for some $k > 1$ it happens that for every $x_0 \in \partial\Omega$ there exists a neighborhood of x_0 , \mathcal{U} , such that $u|_{\mathcal{U}} \in H^k(\Omega \cap \mathcal{U})$. As we should expect, in the study of regularity in the interior, the regularity of $\partial\Omega$ plays no role. In contrast, in the study of regularity at the boundary, the regularity of $\partial\Omega$ plays a crucial role.

Before going on to the rigorous treatment, let us illustrate in a rough manner the main idea that drives the study of the regularity in the interior; similar arguments can be made for the regularity at the boundary.

Let us consider the equation

$$-\Delta u = f, \quad \text{in } \Omega \quad (4.6.2)$$

where $f \in L^2(\Omega)$. Let $u \in H^1(\Omega)$ be a solution to (4.6.2); that is,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega). \quad (4.6.3)$$

Let us suppose that we know u be sufficiently regular (say $u \in H_{loc}^3(\Omega)$) so that the operations that we will make are allowed.

Let $x_0 \in \Omega$ and $R > 0$ satisfy $B_{2R}(x_0) \subset \Omega$. Let $\eta \in C_0^\infty(B_{2R}(x_0))$ such that

$$0 \leq \eta \leq 1; \quad \eta = 1, \quad \text{in } B_R(x_0) \quad (4.6.4)$$

and

$$|\nabla \eta| \leq \frac{K}{R}, \quad (4.6.5)$$

where K is a positive constant. Let $k \in \{1, \dots, n\}$. Multiply both the sides of (4.6.2) by $\partial_k(\eta^2 \partial_k u)$ and integrate over Ω or, equivalently, choose in (4.6.3)

$$v = \partial_k(\eta^2 \partial_k u) \quad (4.6.6)$$

obtaining

$$\int_{\Omega} \Delta u \partial_k(\eta^2 \partial_k u) \, dx = \int_{\Omega} f \partial_k(\eta^2 \partial_k u) \, dx. \quad (4.6.7)$$

Let us consider the left-hand side of (4.6.7), integration by per parts yields

$$\begin{aligned}
\int_{\Omega} \Delta u \partial_k (\eta^2 \partial_k u) dx &= \int_{\Omega} \sum_{j=1}^n \partial_j^2 u \partial_k (\eta^2 \partial_k u) dx = \\
&= - \int_{\Omega} \sum_{j=1}^n \partial_j (\partial_{jk}^2 u) \eta^2 \partial_k u dx = \\
&= \int_{\Omega} \sum_{j=1}^n \partial_{jk}^2 u \partial_j (\eta^2 \partial_k u) dx = \\
&= \int_{\Omega} \sum_{j=1}^n |\partial_{jk}^2 u|^2 \eta^2 dx + \\
&\quad + 2 \int_{\Omega} \sum_{j=1}^n (\partial_{jk}^2 u) \eta \partial_j \eta \partial_k u dx.
\end{aligned}$$

Concerning the right-hand side of (4.6.7), we get

$$\int_{\Omega} f \partial_k (\eta^2 \partial_k u) dx = \int_{\Omega} f \eta^2 \partial_k^2 u dx + 2 \int_{\Omega} f \eta \partial_k \eta \partial_k u dx.$$

Using in (4.6.7) the last two obtained equalities and summing up over k , we have

$$\begin{aligned}
\int_{\Omega} \sum_{j,k=1}^n |\partial_{jk}^2 u|^2 \eta^2 dx &= -2 \int_{\Omega} \sum_{j,k=1}^n (\partial_{jk}^2 u) \eta \partial_j \eta \partial_k u dx + \\
&\quad + \int_{\Omega} f \eta^2 \sum_{k=1}^n \partial_k^2 u dx + 2 \int_{\Omega} f \eta \sum_{k=1}^n \partial_k \eta \partial_k u dx := I.
\end{aligned} \tag{4.6.8}$$

Let $\varepsilon > 0$ to be chosen later on, let us denote by $\partial^2 u$ the Hessian matrix $\{\partial_{jk}^2 u\}_{j,k=1}^n$. We have

$$\begin{aligned}
I &\leq \varepsilon \int_{\Omega} |\partial^2 u|^2 \eta^2 dx + \frac{1}{\varepsilon} \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx + \\
&\quad + \frac{\varepsilon}{2} \int_{\Omega} |\partial^2 u|^2 \eta^2 dx + \frac{1}{2\varepsilon} \int_{\Omega} |f|^2 \eta^2 dx + \\
&\quad + \int_{\Omega} |f|^2 \eta^2 dx + \int_{\Omega} |\nabla \eta|^2 |\nabla u|^2 dx.
\end{aligned}$$

Now, in (4.6.8), we move to the left-hand side the terms that contain the second derivatives, and we get

$$\begin{aligned} \left(1 - \frac{3\varepsilon}{2}\right) \int_{\Omega} |\partial^2 u|^2 \eta^2 dx &\leq \left(1 + \frac{1}{2\varepsilon}\right) \int_{\Omega} |\nabla u|^2 |\nabla \eta|^2 dx + \\ &+ \left(1 + \frac{1}{2\varepsilon}\right) \int_{\Omega} |f|^2 \eta^2 dx. \end{aligned}$$

At this point, we choose $\varepsilon = \frac{1}{3}$ and by (4.6.4), (4.6.5) we get

$$\int_{B_R(x_0)} |\partial^2 u|^2 dx \leq \frac{5K^2}{R^2} \int_{B_{2R}(x_0)} |\nabla u|^2 dx + 5 \int_{B_{2R}(x_0)} |f|^2 dx. \quad (4.6.9)$$

We observe that (4.6.9) allows us to estimate the second derivatives of u in $L^2(B_R(x_0))$ by means of the finite quantity that occurs on the right. **It is evident that this estimate by itself do not provide a proof** that $u \in H^2(B_R(x_0))$, $j, k = 1, \dots, n$, since to obtain the estimate we exploited a regularity of u even greater than was proved (!). However, in the rigorous proofs that we will present soon in this Chapter, we will "retrace", in a sense, the previous steps by considering as test function, instead of the (4.6.6), the function

$$v = -\delta_k^{-h} (\eta^2 \delta_k^h u)$$

where δ_k^{-h} and δ_k^h , are the difference quotients studied in Section 3.10.

4.6.1 The regularity theorems in the interior

The Main Theorem of the present Subsection is the following.

Theorem 4.6.1 (regularity in the interior). *Let Ω be a bounded open set of \mathbb{R}^n . Let*

$$f \in L^2(\Omega). \quad (4.6.10)$$

Let A be a symmetric matrix. Let us assume that A satisfies (4.1.1), $A \in C^{0,1}(\Omega; \mathbb{M}(n))$ and it satisfies

$$|A(x) - A(y)| \leq E|x - y|, \quad \forall x, y \in \Omega, \quad (4.6.11)$$

where E is a positive number. Let us assume $u \in H^1(\Omega)$ is a solution to

$$- \operatorname{div}(A \nabla u) = f, \quad \text{in } \Omega. \quad (4.6.12)$$

Then we have

$$u \in H_{loc}^2(\Omega), \quad (4.6.13)$$

and, for any $B_{2R}(x_0) \subset \Omega$, the following estimate holds true

$$\begin{aligned} \sum_{|\alpha| \leq 2} R^{2|\alpha|} \int_{B_{2R}(x_0)} |\partial^\alpha u|^2 dx &\leq C (1 + E^2 R^2) \int_{B_{2R}(x_0)} u^2 dx + \\ &+ CR^4 \int_{B_{2R}(x_0)} f^2 dx, \end{aligned} \quad (4.6.14)$$

where C depends on λ only.

Proof. It is not restrictive to assume $0 \in \Omega$ and $R < \frac{1}{2} \text{dist}(0, \partial\Omega)$. Let $\eta \in C_0^\infty(B_{3R/2})$ satisfy

$$0 \leq \eta \leq 1; \quad \eta = 1, \quad \text{in } B_R \quad (4.6.15)$$

and

$$|\nabla \eta| \leq \frac{K}{R}, \quad (4.6.16)$$

where K is a positive constant.

Since $u \in H^1(\Omega)$ satisfies (4.6.12), we have

$$\int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx, \quad \forall v \in H_0^1(\Omega). \quad (4.6.17)$$

Let $h \in (-\frac{R}{8}, \frac{R}{8}) \setminus \{0\}$. Let us note that, if $w_1, w_2 \in H^1(\Omega)$ and $\text{supp } w_1 \subset B_{3R/2}$ (or $\text{supp } w_2 \subset B_{3R/2}$), then, for any $k \in \{1, \dots, n\}$, we have

$$\int_{\Omega} w_1 \delta_k^{-h} w_2 dx = - \int_{\Omega} w_2 \delta_k^h w_1 dx \quad (4.6.18)$$

and

$$\delta_k^h (w_1 w_2) = w_1^h \delta_k^h w_2 + w_2 \delta_k^h w_1, \quad (4.6.19)$$

where $w_1^h(x) = w_1(x + he_k)$. Concerning (4.6.18), just argue like in the the Claim of the proof of Theorem 3.10.2. While equality (4.6.18) follows easily by

$$\begin{aligned} h \delta_k^h (w_1 w_2) &= w_1(x + he_k) w_2(x + he_k) - w_1(x) w_2(x) = \\ &= w_1(x + he_k) w_2(x + he_k) - w_1(x + he_k) w_2(x) + \\ &+ w_1(x + he_k) w_2(x) - w_1(x) w_2(x) = \\ &= h (w_1^h \delta_k^h w_2 + w_2 \delta_k^h w_1). \end{aligned}$$

Now, let us choose as test function in (4.6.17)

$$v = -\delta_k^{-h} (\eta^2 \delta_k^h u). \quad (4.6.20)$$

We get

$$\begin{aligned}
\int_{\Omega} A \nabla u \cdot \nabla v dx &= - \int_{\Omega} A \nabla u \cdot [\delta_k^{-h} \nabla (\eta^2 \delta_k^h u)] dx = \\
&= \int_{\Omega} \delta_k^h (A \nabla u) \cdot \nabla (\eta^2 \delta_k^h u) dx = \\
&= \int_{\Omega} A^h (\delta_k^h \nabla u) \cdot \nabla (\eta^2 \delta_k^h u) dx + \\
&+ \int_{\Omega} (\delta_k^h A) \nabla u \cdot \nabla (\eta^2 \delta_k^h u) dx = \\
&= \int_{\Omega} [A^h (\delta_k^h \nabla u) \cdot (\delta_k^h \nabla u)] \eta^2 dx + \mathcal{R},
\end{aligned} \tag{4.6.21}$$

where

$$\begin{aligned}
\mathcal{R} &= \int_{\Omega} A^h (\delta_k^h \nabla u) \cdot (2\eta \nabla \eta \delta_k^h u) dx + \\
&+ \int_{\Omega} (\delta_k^h A) \nabla u \cdot (\delta_k^h \nabla u) \eta^2 dx + \\
&+ \int_{\Omega} (\delta_k^h A) \nabla u \cdot (2\eta \nabla \eta \delta_k^h u) dx.
\end{aligned}$$

Now, by (4.1.1) we get

$$\int_{\Omega} [A^h (\delta_k^h \nabla u) \cdot (\delta_k^h \nabla u)] \eta^2 dx \geq \lambda^{-1} \int_{\Omega} |\delta_k^h \nabla u|^2 \eta^2 dx. \tag{4.6.22}$$

Concerning \mathcal{R} , let ε be a positive number which will choose later on. By (4.6.11) we have

$$\begin{aligned}
|\mathcal{R}| &\leq \frac{c_n \lambda}{R} \int_{\Omega} |\delta_k^h \nabla u| |\delta_k^h u| \eta dx + E \int_{\Omega} |\delta_k^h \nabla u| |\nabla u| \eta^2 dx + \\
&+ \frac{E}{R} \int_{\Omega} |\nabla u| |\delta_k^h u| \eta^2 dx \leq \\
&\leq \varepsilon \int_{\Omega} |\delta_k^h \nabla u|^2 \eta^2 dx + \\
&+ \frac{C}{\varepsilon} (R^{-2} + E^2) \int_{B_{3R/2}} (|\delta_k^h u|^2 + |\nabla u|^2) dx,
\end{aligned} \tag{4.6.23}$$

where c_n depends on n only and C depends on λ and n only. Now let us choose

$$\varepsilon = \frac{\lambda^{-1}}{2}$$

so that, by (4.6.22) and (4.6.23), we get

$$\begin{aligned} \int_{\Omega} [A^h (\delta_k^h \nabla u) \cdot (\delta_k^h \nabla u)] \eta^2 dx + \mathcal{R} &\geq \frac{\lambda^{-1}}{2} \int_{\Omega} |\delta_k^h \nabla u|^2 \eta^2 dx - \\ &- 2C\lambda (R^{-2} + E^2) \int_{B_{3R/2}} (|\delta_k^h u|^2 + |\nabla u|^2) dx. \end{aligned} \quad (4.6.24)$$

Hence, (4.6.17), (4.6.20), (4.6.21) and (4.6.24) yield

$$\begin{aligned} \frac{\lambda^{-1}}{2} \int_{\Omega} |\delta_k^h \nabla u|^2 \eta^2 dx &\leq - \int_{\Omega} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx + \\ &+ C (R^{-2} + E^2) \int_{B_{3R/2}} (|\delta_k^h u|^2 + |\nabla u|^2) dx, \end{aligned} \quad (4.6.25)$$

where C depends on λ and n only.

Now, by Theorem 3.10.2–(i) (with $V = B_{3R/2}$ and $\Omega = B_{7R/4}$) we have

$$\int_{B_{3R/2}} |\delta_k^h u|^2 dx \leq \int_{B_{7R/4}} |\nabla u|^2 dx. \quad (4.6.26)$$

Moreover

$$\begin{aligned} \int_{\Omega} |\delta_k^{-h} (\eta^2 \delta_k^h u)|^2 dx &\leq C \int_{\Omega} |\nabla (\eta^2 \delta_k^h u)|^2 dx \leq \\ &\leq C' \int_{\Omega} |\eta \nabla \eta (\eta^2 \delta_k^h u)|^2 dx + C' \int_{\Omega} \eta^2 |\nabla \delta_k^h u|^2 dx \leq \\ &\leq \frac{C''}{R^2} \int_{B_{3R/2}} |\delta_k^h u|^2 dx + C' \int_{\Omega} \eta^2 |\delta_k^h \nabla u|^2 dx. \end{aligned} \quad (4.6.27)$$

Let now $\sigma > 0$ to be chosen, (4.6.27) implies

$$\begin{aligned} \left| \int_{\Omega} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx \right| &\leq \frac{1}{2\sigma} \int_{B_{2R}} f^2 dx + \frac{\sigma}{2} \int_{\Omega} |\delta_k^{-h} (\eta^2 \delta_k^h u)|^2 dx \leq \\ &\leq \frac{1}{2\sigma} \int_{B_{2R}} f^2 dx + \\ &+ C\sigma \left(R^{-2} \int_{B_{3R/2}} |\delta_k^h u|^2 dx + \int_{\Omega} \eta^2 |\delta_k^h \nabla u|^2 dx \right). \end{aligned} \quad (4.6.28)$$

Now we apply Theorem 3.10.2–(i) and inequality, (4.5.11), so that we have

$$R^{-2} \int_{B_{3R/2}} |\delta_k^h u|^2 dx \leq R^{-2} \int_{B_{7R/4}} |\nabla u|^2 dx \leq CR^{-4} \int_{B_{2R}} u^2 dx + C \int_{B_{2R}} f^2 dx.$$

By the last obtained estimate and by (4.6.28) we have

$$\left| \int_{\Omega} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx \right| \leq \frac{1}{2\sigma} \int_{B_{2R}} f^2 dx + C_* \sigma \left(R^{-4} \int_{B_{2R}} u^2 dx + \int_{\Omega} \eta^2 |\delta_k^h \nabla u|^2 dx \right),$$

where C_* is a constant depending on λ and n only. Inserting what we have just obtained into (4.6.25) we get

$$\begin{aligned} \left(\frac{\lambda^{-1}}{2} - C_* \sigma \right) \int_{\Omega} |\delta_k^h \nabla u|^2 \eta^2 dx &\leq \frac{1}{2\sigma} \int_{B_{2R}} f^2 dx + \\ &+ \frac{C(1 + \sigma + E^2 R^2)}{R^4} \int_{B_{2R}} u^2 dx. \end{aligned}$$

Now, choosing

$$\sigma = \frac{\lambda^{-1}}{4C_*}$$

and we have, for $k = 1, \dots, n$,

$$\int_{B_R} |\delta_k^h \nabla u|^2 \eta^2 dx \leq C \int_{B_{2R}} f^2 dx + \frac{C(1 + E^2 R^2)}{R^4} \int_{B_{2R}} u^2 dx.$$

By the last obtained inequality and by Theorem 3.10.2–(ii) we obtain

$$\partial_k u \in H^1(B_R),$$

for $k = 1, \dots, n$. Hence $u \in H^2(B_R)$ and

$$\sum_{|\alpha|=2} R^{2|\alpha|} \int_{B_R} |\partial^\alpha u|^2 dx \leq C(1 + E^2 R^2) \int_{B_{2R}} u^2 dx + CR^4 \int_{B_{2R}} f^2 dx.$$

Finally, by the latter and by (4.5.11) we get (4.6.14). ■

Exercise 1. Under the same assumptions of Theorem 4.6.1, prove that, if $0 < r < \rho$ and $B_\rho(x_0) \subset \Omega$ then

$$\begin{aligned} \sum_{|\alpha| \leq 2} (\rho - r)^{2|\alpha|} \int_{B_r(x_0)} |\partial^\alpha u|^2 dx &\leq C (1 + E^2 \rho^2) \int_{B_\rho(x_0)} u^2 dx + \\ &+ C(\rho - r)^4 \int_{B_\rho(x_0)} f^2 dx, \end{aligned} \quad (4.6.29)$$

where C depends on λ only. [Hint: consider a finite covering $\overline{B_r(x_0)}$ consisting of balls of the type $B_{\frac{\rho-r}{2}}(x)$, $x \in B_r(x_0)$, and apply (4.6.14)].

Exercise 2. Under the same assumption of Theorem 4.6.1, prove that, if

$$\Omega' \Subset \Omega$$

then

$$\begin{aligned} \sum_{|\alpha| \leq 2} \delta_0^{2|\alpha|} \int_{\Omega'} |\partial^\alpha u|^2 dx &\leq C (1 + E^2 d_0^2) \int_{\Omega} u^2 dx + \\ &+ C d_0^4 \int_{\Omega} f^2 dx, \end{aligned} \quad (4.6.30)$$

where d_0 is the diameter of Ω , $\delta_0 = \text{dist}(\Omega', \partial\Omega)$ and C depends on λ and $d_0 \delta_0^{-1}$ only. [Hint: use Exercise 1 and a partition of unity].

Exercise 3. (a) Generalize Theorem 4.6.1 to the equation

$$-\sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu = f$$

where $A = \{a^{jk}\}_{j,k=1}^n$ and f satisfy the same assumptions of Theorem 4.6.1, $d, b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$.

(b) Generalize (a) to the case where A is a nonsymmetric matrix. [Hint to (b): write the operator $\text{div}(A\nabla u)$ like

$$\text{div}(A^s \nabla u) + \text{terms of order less than 2},$$

where A^s is symmetric part of A . ♣

Theorem 4.6.2 (improved regularity in the interior). Let Ω be a bounded open set of \mathbb{R}^n with diameter d_0 . Let

$$f \in H^m(\Omega). \quad (4.6.31)$$

Let A be a symmetric matrix. Let us assume that A satisfies (4.1.1), let us assume that $A \in C^{m,1}(\bar{\Omega}; \mathbb{M}(n))$ and it satisfies

$$\|A\|_{C^{m,1}(\bar{\Omega}; \mathbb{M}(n))} \leq E_m, \quad (4.6.32)$$

where, E_m is a positive number and, recall,

$$\|A\|_{C^{m,1}(\bar{\Omega}; \mathbb{M}(n))} = \sum_{|\alpha| \leq m} d_0^{|\alpha|} \|\partial^\alpha A\|_{L^\infty(\Omega; \mathbb{M}(n))} + d_0^{m+1} \sum_{|\alpha|=m} [\partial^\alpha A]_{1,\Omega}.$$

Let us assume that $u \in H^1(\Omega)$ is a solution to

$$- \operatorname{div}(A \nabla u) = f, \quad \text{in } \Omega. \quad (4.6.33)$$

Then we have

$$u \in H_{loc}^{m+2}(\Omega), \quad (4.6.34)$$

moreover, if $B_{2R}(x_0) \subset \Omega$, then the following inequality holds true

$$\begin{aligned} \sum_{|\alpha| \leq m+2} R^{2|\alpha|} \int_{B_R(x_0)} |\partial^\alpha u|^2 dx &\leq C (1 + E_m^2) \int_{B_{2R}(x_0)} u^2 dx + \\ &+ C \sum_{|\alpha| \leq m} R^{2(|\alpha|+4)} \int_{B_{2R}(x_0)} |\partial^\alpha f|^2 dx, \end{aligned} \quad (4.6.35)$$

where C depends on λ only.

Proof. We simply consider the case $m = 1$, leaving the reader to complete the proof by induction. Let $l \in \{1, \dots, n\}$. Let \tilde{v} be any function belonging to $C_0^\infty(B_{3R/2}(x_0))$. Choose, as a test function,

$$v = -\partial_l \tilde{v}.$$

We get

$$- \int_{\Omega} A \nabla u \cdot \nabla \partial_l \tilde{v} dx = - \int_{\Omega} f \partial_l \tilde{v} dx. \quad (4.6.36)$$

Since $f \in H^1(\Omega)$, we have

$$- \int_{\Omega} f \partial_l \tilde{v} dx = \int_{\Omega} \partial_l f \tilde{v} dx. \quad (4.6.37)$$

Moreover, since $u \in H_{loc}^2(\Omega)$ (by Theorem 4.6.1), we have

$$\begin{aligned}
-\int_{\Omega} A \nabla u \cdot \nabla \partial_l \tilde{v} dx &= -\int_{\Omega} \sum_{j,k=1}^n a^{jk} \partial_k u \partial_l (\partial_j \tilde{v}) dx = \\
&= \int_{\Omega} \sum_{j,k=1}^n \partial_l (a^{jk} \partial_k u) \partial_j \tilde{v} dx = \\
&= \int_{\Omega} \sum_{j,k=1}^n a^{jk} \partial_k (\partial_l u) \partial_j \tilde{v} dx + \\
&+ \int_{\Omega} \sum_{j,k=1}^n \partial_l a^{jk} \partial_k u \partial_j \tilde{v} dx = \\
&= \int_{\Omega} \sum_{j,k=1}^n a^{jk} \partial_k (\partial_l u) \partial_j \tilde{v} dx - \\
&- \int_{\Omega} \sum_{j,k=1}^n \partial_j [(\partial_l a^{jk}) \partial_k u] \tilde{v} dx.
\end{aligned}$$

By the equality obtained above, by (4.6.36) and by (4.6.37) we have (recall that $C_0^\infty(B_{3R/2}(x_0))$ is dense in $H_0^1(B_{3R/2}(x_0))$)

$$\int_{B_{3R/2}(x_0)} A \nabla (\partial_l u) \cdot \nabla w dx = \int_{B_R(x_0)} \tilde{f} w dx, \quad \forall w \in H_0^1(B_{3R/2}(x_0)), \quad (4.6.38)$$

where

$$\tilde{f} = \partial_l f + \sum_{j,k=1}^n \partial_j [(\partial_l a^{jk}) \partial_k u].$$

Now $\tilde{f} \in L^2(B_{3R/2}(x_0))$. As a matter of fact

$$\begin{aligned}
&\int_{B_{3R/2}(x_0)} |\tilde{f}|^2 dx \leq \\
&\leq 2 \int_{B_{3R/2}(x_0)} |\partial_l f|^2 dx + cE_1^2 d_0^{-4} \int_{B_{3R/2}(x_0)} |\partial^2 u|^2 dx + \\
&+ cE_1^2 d_0^{-2} \int_{B_{3R/2}(x_0)} |\nabla u|^2 dx.
\end{aligned} \quad (4.6.39)$$

By the latter, by Theorem 4.6.1 (more precisely, by (4.6.29)) and by (4.6.38), we obtain

$$\begin{aligned}
R^4 \sum_{|\alpha|=3} \int_{B_R(x_0)} |\partial^\alpha u|^2 dx &\leq C(1 + E_1^2) \sum_{|\alpha|=2} \int_{B_{3R/2}(x_0)} |\partial^\alpha u|^2 dx + \\
&+ CR^4 \int_{B_{3R/2}(x_0)} |\tilde{f}|^2 dx.
\end{aligned} \tag{4.6.40}$$

At this point, we again apply (4.6.29) to estimate from above the derivatives of order less than or equal to 2 we obtain (4.6.35). ■

Exercise 4. Generalize Theorem 4.6.2 to the equation

$$-\sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j,k=1}^n b^j \partial_j u + cu = f$$

where A and f satisfy the same assumption of Theorem 4.6.2, $d, b \in C^{m-1,1}(\bar{\Omega}; \mathbb{R}^n)$, $c \in C^{m-1,1}(\bar{\Omega})$, $m \geq 1$. ♣

Corollary 4.6.3 (C^∞ regularity in the interior). *Let Ω an open set of \mathbb{R}^n . Let*

$$f \in C^\infty(\Omega).$$

Let $A \in C^\infty(\Omega; \mathbb{M}(n))$. Let us assume that A satisfies (4.1.1).

Let $u \in H^1(\Omega)$ be a solution to

$$-\operatorname{div}(A \nabla u) = f, \quad \text{in } \Omega.$$

Then we have

$$u \in C^\infty(\Omega).$$

Proof. Let $B_R(x_0) \Subset \Omega$. For any $m \geq 0$, we have $f \in H^m(B_R(x_0))$ and $A \in C^{m,1}(B_R(x_0); \mathbb{M}(n))$. Therefore Theorem 4.6.2 implies

$$u \in \bigcap_{m=0}^{\infty} H^m(B_R(x_0)) = C^\infty(B_R(x_0)),$$

Where the last equality is due to Theorem 3.7.10. Since $B_R(x_0)$ is arbitrary, the thesis follows. ■

Exercise 5. Prove Corollary 4.6.3 for the equation

$$-\sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu = f$$

where A satisfies the same assumption of Corollary 4.6.3 and $d, b \in C^\infty(\Omega; \mathbb{R}^n)$, $c \in C^\infty(\Omega)$. ♣

4.6.2 Regularity theorems at the boundary—global regularity

In this Section we will study the regularity at the boundary. The following Lemma is a crucial step in the proof of the forthcoming theorems

Lemma 4.6.4 (local regularity at the boundary). *Let $R > 0$ and*

$$f \in L^2(B_{2R}^+) \quad (4.6.41)$$

and let A be a symmetric matrix. Let us assume that A satisfies (4.1.1), $A \in C^{0,1}(\Omega; \mathbb{M}(n))$ and it satisfies

$$|A(x) - A(\bar{x})| \leq E|x - \bar{x}|, \quad \forall x, \bar{x} \in B_{2R}^+, \quad (4.6.42)$$

where E is a positive number. Let us assume that $u \in H^1(B_{2R}^+)$ satisfies

$$- \operatorname{div}(A\nabla u) = f, \quad \text{in } B_{2R}^+ \quad (4.6.43)$$

and

$$u(\cdot, 0) = 0, \quad \text{in the sense of the traces in } B'_{2R}. \quad (4.6.44)$$

Then we have

$$u \in H^2(B_R^+) \quad (4.6.45)$$

and the following estimate holds true

$$\begin{aligned} \sum_{|\alpha| \leq 2} R^{2|\alpha|} \int_{B_R^+} |\partial^\alpha u|^2 dx &\leq C(1 + E^2 R^2) \int_{B_{2R}^+} u^2 dx + \\ &+ CR^4 \int_{B_{2R}^+} f^2 dx, \end{aligned} \quad (4.6.46)$$

where C depends on λ only.

Proof. Let $\eta \in C_0^\infty(B_{3R/2})$ satisfy

$$0 \leq \eta \leq 1; \quad \eta = 1, \quad \text{in } B_R \quad (4.6.47)$$

and

$$|\nabla \eta| \leq \frac{K}{R}, \quad (4.6.48)$$

where K is a positive constant. Let $h \in (-\frac{R}{8}, \frac{R}{8}) \setminus \{0\}$ and let

$$k \in \{1, \dots, n-1\}.$$

Let us denote

$$v = -\delta_k^{-h} (\eta^2 \delta_k^h u). \quad (4.6.49)$$

By (4.6.44), taking into account that $\eta \in C_0^\infty(B_{3R/2})$, we have

$$v \in H_0^1(B_{2R}^+).$$

Therefore, by $u \in H^1(B_{2R}^+)$, by (4.6.41), (4.6.43) and by (4.6.44), we get

$$\int_{B_{2R}^+} A \nabla u \cdot \nabla v dx = \int_{B_{2R}^+} f v dx. \quad (4.6.50)$$

At this point we may argue likewise in the proof of Theorem 4.6.1. Actually, by using Theorem 3.10.3 instead of 3.10.2 we have

$$\begin{aligned} \frac{\lambda^{-1}}{2} \int_{B_{2R}^+} |\delta_k^h \nabla u|^2 \eta^2 dx &\leq - \int_{B_{2R}^+} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx + \\ &+ C (R^{-2} + E^2) \int_{B_{3R/2}^+} (|\delta_k^h u|^2 + |\nabla u|^2) dx, \end{aligned} \quad (4.6.51)$$

for any $k \in \{1, \dots, n-1\}$, where C depends on λ and n only. Let $\sigma > 0$ to be chosen, we get (compare with (4.6.28))

$$\begin{aligned} \left| \int_{B_{2R}^+} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx \right| &\leq \frac{1}{2\sigma} \int_{B_{2R}^+} f^2 dx + \\ &+ C\sigma \left(R^{-2} \int_{B_{3R/2}^+} |\delta_k^h u|^2 dx + \right. \\ &\left. + \int_{B_{2R}^+} \eta^2 |\nabla \delta_k^h u|^2 dx \right). \end{aligned} \quad (4.6.52)$$

By applying Theorem 3.10.3 and inequality (4.5.12), we get

$$\begin{aligned} R^{-2} \int_{B_{3R/2}^+} |\delta_k^h u|^2 dx &\leq R^{-2} \int_{B_{7R/4}^+} |\nabla u|^2 dx \leq \\ &\leq CR^{-4} \int_{B_{2R}^+} u^2 dx + C \int_{B_{2R}^+} f^2 dx. \end{aligned}$$

By the latter and by (4.6.52) we have

$$\left| \int_{\Omega} f \delta_k^{-h} (\eta^2 \delta_k^h u) dx \right| \leq \frac{1}{2\sigma} \int_{B_{2R}} f^2 dx + \\ + C_* \sigma \left(R^{-4} \int_{B_{2R}} u^2 dx + \int_{\Omega} \eta^2 |\nabla \delta_k^h u|^2 dx \right),$$

where C_* depends on λ and n only. By using in (4.6.51) the just obtained inequality, we obtain

$$\left(\frac{\lambda^{-1}}{2} - C_* \sigma \right) \int_{B_{2R}^+} |\delta_k^h \nabla u|^2 \eta^2 dx \leq \frac{1}{2\sigma} \int_{B_{2R}^+} f^2 dx + \\ + \frac{C(1 + \sigma + E^2 R^2)}{R^4} \int_{B_{2R}^+} u^2 dx.$$

Now let us choose

$$\sigma = \frac{\lambda^{-1}}{4C_*}$$

and we get, for $k = 1, \dots, n-1$,

$$\int_{B_R^+} |\delta_k^h \nabla u|^2 \eta^2 dx \leq C \int_{B_{2R}^+} f^2 dx + \frac{C(1 + E^2 R^2)}{R^4} \int_{B_{2R}^+} u^2 dx.$$

By this inequality and by Theorem 3.10.3-(ii) we get

$$\partial_k u \in H^1(B_R^+), \quad \text{for } k = 1, \dots, n-1$$

and applying again (4.5.12) we have

$$\sum_{\substack{k,j=1 \\ k+j < 2n}}^n \int_{B_R^+} |\partial_{jk}^2 u|^2 dx \leq \frac{C(1 + E^2 R^2)}{R^4} \int_{B_{2R}^+} u^2 dx + CR^4 \int_{B_{2R}^+} f^2 dx, \quad (4.6.53)$$

where C depends on λ n only.

Now, Theorem 4.6.1 implies that $u \in H_{loc}^2(B_R^+)$, this allows us to write the equation (4.6.42) in the form

$$\sum_{k,j=1}^n a^{jk} \partial_{jk}^2 u + \sum_{k,j=1}^n \partial_j a^{jk} \partial_k u = -f, \quad \text{a.e. in } B_R^+.$$

By the last equality we can find (taking into account that $a^{nn} \geq \lambda^{-1} > 0$)

$$\partial_n^2 u = -\frac{1}{a^{nn}} \left(\sum_{\substack{k,j=1 \\ k+j < 2n}}^n a^{jk} \partial_{jk}^2 u + \sum_{k,j=1}^n \partial_j a^{jk} \partial_k u + f \right). \quad (4.6.54)$$

By (4.6.54), (4.6.53), (4.6.42) we have $\partial_n^2 u \in H^2(B_R^+)$. Finally, by (4.5.12) we obtain (4.6.46). ■

Theorem 4.6.5 (global regularity). *Let Ω be a bounded open set of \mathbb{R}^n whose boundary is of class $C^{1,1}$ with constants M_0, r_0 . Let*

$$f \in L^2(\Omega) \quad (4.6.55)$$

and let A be a symmetric matrix. Let us assume that A satisfies (4.1.1), $A \in C^{0,1}(\Omega; \mathbb{M}(n))$ and it satisfies

$$|A(x) - A(\bar{x})| \leq E|x - \bar{x}|, \quad \forall x, \bar{x} \in \Omega, \quad (4.6.56)$$

where E is a positive number. Let us suppose that $u \in H_0^1(\Omega)$ is the solution to

$$- \operatorname{div}(A \nabla u) = f, \quad \text{in } \Omega. \quad (4.6.57)$$

Then we have

$$u \in H^2(\Omega), \quad (4.6.58)$$

and the following estimate holds true

$$\sum_{|\alpha| \leq 2} r_0^{2|\alpha|} \int_{\Omega} |\partial^{\alpha} u|^2 dx \leq C d_0^4 \int_{\Omega} f^2 dx, \quad (4.6.59)$$

where d_0 is the diameter of Ω and C depends by λ, E, M_0 and $\frac{d_0}{r_0}$ only.

Proof. Let $P \in \partial\Omega$. There exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in Q_{r_0, 2M_0} : x_n > g(x')\},$$

where $g \in C^{1,1}(\overline{B'_{r_0}})$,

$$g(0) = 0, \quad |\nabla_{x'} g(0)| = 0$$

and

$$\|g\|_{C^{1,1}(\overline{B'_{r_0}})} \leq M_0 r_0.$$

Let us consider the change of coordinates

$$\Phi : Q_{r_0, 2M_0} \rightarrow \mathbb{R}^n, \quad \Phi(x) = (x', x_n - g(x')).$$

Let us note that Φ "flattens the boundary" i.e.

$$Q_{r_0, 2M_0} \cap \Phi(\partial\Omega) = \{(y', 0) : y' \in B'_{r_0}\}.$$

$\Phi \in C^{1,1}(Q_{r_0, 2M_0} \cap \Omega)$ is injective and it is a local diffeomorphism. Let

$$W = Q_{r_0, 2M_0} \cap \Phi(\Omega)$$

and

$$\Psi : W \rightarrow Q_{r_0, 2M_0} \cap \Omega, \quad \Psi = \Phi^{-1}.$$

Let J be the jacobian matrix of Ψ . Then

$$\det J(y) = 1, \quad \forall y \in W.$$

Let $u \in H_0^1(\Omega)$ be the weak solutions to (4.6.57). Let \tilde{v} be any function belonging to $H_0^1(W)$ and set

$$v(x) = \tilde{v}(\Psi(x)).$$

We have

$$\int_{Q_{r_0, 2M_0} \cap \Omega} A \nabla u \cdot \nabla v dx = \int_{Q_{r_0, 2M_0} \cap \Omega} f v dx. \quad (4.6.60)$$

By the change of variables $x = \Psi(y)$ equation (4.6.60) becomes

$$\int_W \tilde{A}(y) \nabla w(y) \cdot \nabla \tilde{v}(y) dy = \int_W \tilde{f}(y) \tilde{v}(y) dy, \quad (4.6.61)$$

where

$$w(y) = u(\Psi(y)), \quad \forall y \in W,$$

$$\tilde{A}(y) = (J(y))^{-1} A(\Psi(y)) ((J(y))^{-1})^t, \quad \forall y \in W$$

and

$$\tilde{f}(y) = f(\Psi(y)), \quad \forall y \in W.$$

It is easy to check that

$$\tilde{\lambda}^{-1} |\xi|^2 \leq \tilde{A}(y) \xi \cdot \xi \leq \tilde{\lambda} |\xi|^2, \quad \forall y \in W, \forall \xi \in \mathbb{R}^n,$$

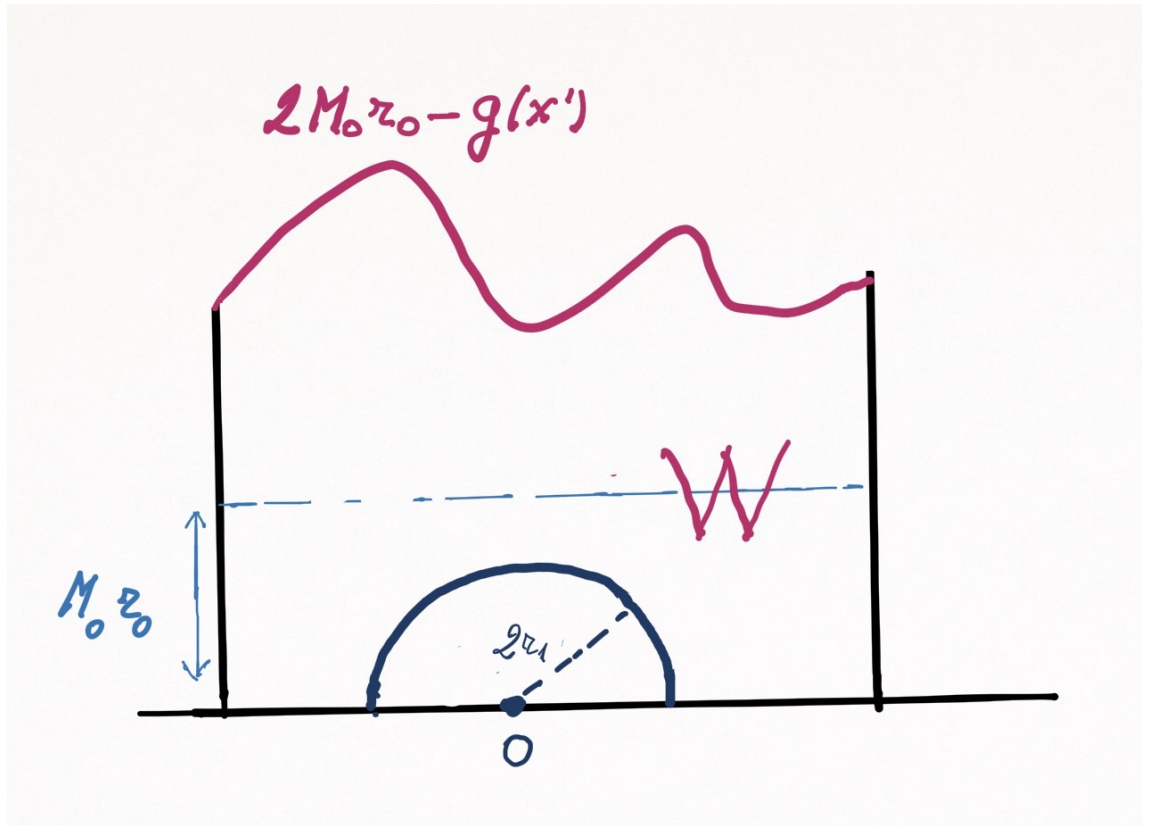


Figure 4.1:

where $\tilde{\lambda} \geq 1$ depends on λ and M_0 only. Moreover

$$w(y', 0) = 0, \quad y' \in B'_{r_0} \text{ in the sense of the traces.}$$

Also we have $\tilde{A} \in C^{0,1}(W)$ and

$$|\tilde{A}(y) - \tilde{A}(\bar{y})| \leq \tilde{E} |y - \bar{y}|, \quad \forall y, \bar{y} \in W,$$

where

$$\tilde{E} = C(E + M_0 r_0^{-1}),$$

and C depends on λ and M_0 only.

At this stage (compare Fig. 4.1) we introduce the quantity

$$r_1 = \frac{r_0}{2} \min\{1, M_0\}$$

in such a way that we have

$$B_{2r_1}^+ \subset W.$$

Applying Lemma 4.6.4 we get

$$\begin{aligned} \sum_{|\alpha| \leq 2} r_1^{2|\alpha|} \int_{B_{r_1}^+} |\partial^\alpha w|^2 dy &\leq C \left(1 + \tilde{E}^2 r_1^2\right) \int_{B_{2r_1}} w^2 dy + \\ &+ Cr_1^4 \int_{B_{2r_1}^+} \tilde{f}^2 dy, \end{aligned} \quad (4.6.62)$$

where C depends on $\tilde{\lambda}$ only. Coming back to the original variables, after some calculation and simple estimates, we have

$$\sum_{|\alpha| \leq 2} r_0^{2|\alpha|} \int_{\Psi(B_{r_1}^+)} |\partial^\alpha u|^2 dx \leq C \left(\int_{\Omega} u^2 dx + r_0^4 \int_{\Omega} f^2 dx \right), \quad (4.6.63)$$

where C depends on λ, M_0 and Er_0 only. On the other hand, as is easily checked, there is $\bar{C} \geq 1$, C depending on λ and M_0 only, such that, if $r_2 = \frac{r_0}{\bar{C}}$, we have

$$\Omega \cap B_{r_2}(P) \subset \Psi(B_{r_1}^+).$$

Therefore by (4.6.63) we get trivially

$$\sum_{|\alpha| \leq 2} r_0^{2|\alpha|} \int_{\Omega \cap B_{r_2}(P)} |\partial^\alpha u|^2 dx \leq C \left(\int_{\Omega} u^2 dx + r_0^4 \int_{\Omega} f^2 dx \right). \quad (4.6.64)$$

Now, by the compactness of $\partial\Omega$, we can extract a finite covering by $\{B_{r_2}(P)\}_{P \in \partial\Omega}$. Let $\{B_{r_2}(P_j)\}_{1 \leq j \leq N}$, such a finite covering, where $P_j \in \partial\Omega$, and let

$$\Lambda = \Omega \cap \bigcup_{j=1}^N B_{r_2}(P_j) \quad \text{e} \quad \Omega' = \Omega \setminus \Lambda.$$

We can make $\text{dist}(\Omega', \partial\Omega) \geq r_0/C$, where $C \geq 1$ depends on M_0 only, furthermore N depends on M_0 and $\frac{d_0}{r_0}$ only. Inequality (4.6.64) implies that there is a constant C depending on λ, E, M_0 and $\frac{d_0}{r_0}$ so that

$$\sum_{|\alpha| \leq 2} r_0^{2|\alpha|} \int_{\Lambda} |\partial^\alpha u|^2 dx \leq C \left(\int_{\Omega} u^2 dx + r_0^4 \int_{\Omega} f^2 dx \right).$$

By the just obtained inequality and by (4.6.30) we have

$$\sum_{|\alpha| \leq 2} r_0^{2|\alpha|} \int_{\Omega} |\partial^\alpha u|^2 dx \leq C \left(\int_{\Omega} u^2 dx + d_0^4 \int_{\Omega} f^2 dx \right), \quad (4.6.65)$$

By the first Poincaré inequality (Proposition (3.4.2)) and by inequality (4.3.6), we find

$$\int_{\Omega} u^2 dx \leq C d_0^4 \int_{\Omega} f^2 dx,$$

where C depends on λ only. By the last obtained inequality and by (4.6.65) we get (4.6.58). ■

Exercise 1. (a) Generalize Theorem 4.6.5 to the boundary value problem

$$\begin{cases} -\sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u + d^j u) + \sum_{j=1}^n b^j \partial_j u + cu = f, & \text{in } \Omega \\ u = 0, & \text{su } \partial\Omega. \end{cases}$$

where $A = \{a^{jk}\}_{j,k=1}^n$ and f satisfy the same assumptions of Theorem 4.6.1 and $d, b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$.

(b) Generalize the result obtained in (a) to the case where A is not a symmetric matrix. ♣

Theorem 4.6.6 (improved global regularity). *Let Ω be a bounded open set of \mathbb{R}^n of class $C^{m+1,1}$ with constants M_0, r_0 , let d_0 be the diameter of Ω . Let*

$$f \in H^m(\Omega) \quad (4.6.66)$$

and let A be a symmetric matrix. Let us assume that A satisfies (4.1.1), $A \in C^{m,1}(\bar{\Omega}; \mathbb{M}(n))$ and it satisfies

$$\|A\|_{C^{m,1}(\bar{\Omega}; \mathbb{M}(n))} \leq E_m,$$

where E_m is a positive number.

Let us suppose that $u \in H_0^1(\Omega)$ is the solution to

$$- \operatorname{div}(A \nabla u) = f, \quad \text{in } \Omega. \quad (4.6.67)$$

Then we have

$$u \in H^{m+2}(\Omega), \quad (4.6.68)$$

and the following estimate holds true

$$\sum_{|\alpha| \leq m+2} r_0^{2|\alpha|} \int_{\Omega} |\partial^\alpha u|^2 dx \leq C \sum_{|\alpha| \leq m} d_0^{2(|\alpha|+4)} \int_{\Omega} |\partial^\alpha f|^2 dx, \quad (4.6.69)$$

where C depends on λ, E_m, M_0 and $\frac{d_0}{r_0}$ only.

Proof. The proof is mostly similar to that of Theorem 4.6.2, so we focus here by considering, in the case $m = 1$, the steps in which the new proof differs from the proof of Theorem 4.6.2, leaving the details to the care of the reader. First of all, in analogy to Lemma 4.6.4, let us consider the following special situation. Let $R > 0$ and

$$f \in H^1(B_{2R}^+).$$

Let A be a symmetric matrix whose entries are measurable functions in B_{2R}^+ . Let us suppose that A satisfies (4.1.1) and that $A \in C^{1,1}(B_{2R}, \mathbb{M}(n))$. Let us suppose that $u \in H^1(B_{2R}^+)$ is a solution to

$$-\operatorname{div}(A\nabla u) = f, \quad \text{in } B_{2R}^+,$$

and

$$u(\cdot, 0) = 0, \quad \text{in } B'_{2R}, \text{ (in the sense of the traces)}. \quad (4.6.70)$$

Let us prove that

$$u \in H^3(B_r^+), \quad \forall r < R. \quad (4.6.71)$$

To this purpose, let us prove

Claim. Let $l = 1, \dots, n-1$. We have, for every $r \in (0, 2R)$,

$$\partial_l u(\cdot, 0) = 0, \quad \text{in } B'_r, \text{ (in the sense of the traces)}. \quad (4.6.72)$$

Proof of the Claim. First let us note that, since $u \in H^2(B_r^+)$, for every $r < R$, $\partial_l u(x', 0)$ is well-defined in the sense of traces. Now, let

$$v = \partial_l u$$

and let us denote by $T(v)$ the trace of v on $\{x_n = 0\}$. As a consequence of Theorem 3.5.1, $T(v)$ is characterised by the identity.

$$-\int_{B'_R} \Phi_n(x', 0) T(v) dx' = \int_{B_R^+} v \operatorname{div} \Phi dx + \int_{B_R^+} \nabla v \cdot \Phi dx, \quad (4.6.73)$$

for every $\Phi \in C_0^\infty(B_R; \mathbb{R}^n)$.

Let now Φ be any function belonging to $C_0^\infty(B_R; \mathbb{R}^n)$. By (4.6.71) and by Theorem 3.5.1 (applied to u), we get

$$\begin{aligned} \int_{B_R^+} v(\operatorname{div}\Phi)dx &= \int_{B_R^+} \partial_l u(\operatorname{div}\Phi)dx = - \int_{B_R^+} u(\partial_l \operatorname{div}\Phi)dx = \\ &= - \int_{B_R^+} u(\operatorname{div}\partial_l \Phi)dx = \\ &= \int_{B_R^+} \nabla u \cdot \partial_l \Phi dx + \int_{B_R^+} \partial_l \Phi_n(x', 0)T(u)dx' = \\ &= \int_{B_R^+} \nabla u \cdot \partial_l \Phi dx = - \int_{B_R^+} \partial_l \nabla u \cdot \Phi dx = \\ &= - \int_{B_R^+} \nabla v \cdot \Phi. \end{aligned}$$

Hence, (4.6.73) implies

$$T(v) = 0, \quad \text{in } B_r', \quad \forall r \in (0, 2R).$$

Claim is proved.

We now briefly and only formally show the most significant steps to complete the proof; we encourage the reader to treat the steps in a rigorous manner using appropriately the variational formulation in a similar way as we did in the proof of Theorem (4.6.2).

By calculating the derivative w.r.t. x_l , $l \in \{1, \dots, n-1\}$, of both the sides of the equation

$$- \sum_{j,k=1}^n \partial_j (a^{jk} \partial_k u) = f, \quad (4.6.74)$$

we obtain

$$- \sum_{j,k=1}^n \partial_j (a^{jk} \partial_k v) = \tilde{f}, \quad (4.6.75)$$

where $v = \partial_l u$ and

$$\tilde{f} = \partial_l f + \sum_{j,k=1}^n \partial_j ((\partial_l a^{jk}) \partial_k u).$$

Now, as $f \in H^1(B_{2R}^+)$, $a^{jk} \in C^{1,1}(B_{2R}^+)$, $u \in H^2(B_r^+)$ for every $r \in (0, 2R)$, we get

$$\tilde{f} \in L^2(B_r^+), \quad \forall r \in (0, 2R).$$

on the other hand, for every $r \in (0, 2R)$, we have $v \in H^1(B_r^+)$ and

$$v(\cdot, 0) = 0, \quad \text{in } x' \in B_r', \text{ (in the sense of the traces) .}$$

Therefore by Lemma 4.6.4 we have

$$\partial_l u = v \in H^2(B_r^+), \quad \forall r \in (0, 2R), \quad (4.6.76)$$

for $l \in \{1, \dots, n-1\}$. Moreover by (4.6.75) we get (likewise to (4.6.54))

$$\partial_n^3 u = \partial_n \left[-\frac{1}{a^{nn}} \left(\sum_{\substack{k,j=1 \\ k+j < 2n}}^n a^{jk} \partial_{jk}^2 u + \sum_{k,j}^n \partial_j a^{jk} \partial_k u + f \right) \right], \quad (4.6.77)$$

from which, taking into account (4.6.76), we get

$$u \in H^3(B_r^+), \quad \forall r \in (0, 2R).$$

Finally, inequality (4.6.69) (for $m = 1$) is obtained applying inequality (4.6.46) to (4.6.75). In addition, inequality (4.6.46) gives the estimates

$$\begin{aligned} R^6 \sum_{\substack{k,j,l=1 \\ k+j+l < 3n}}^n \|\partial_{jkl}^3 u\|_{L^2(B_R^+)}^2 &\leq C (1 + E_1^2) \int_{B_{2R}} u^2 dx + \\ &+ C \sum_{|\alpha| \leq 1} R^{2|\alpha|+4} \int_{B_{2R}^+} |\partial^\alpha|^2 dx. \end{aligned}$$

By the last inequality, by means of (4.6.77) (applying again inequality (4.6.46) to equation (4.6.74)), we obtain (4.6.69) (for $m = 1$).

In order to complete the proof, simply follow the proof of Theorem 4.6.5 taking into account that diffeomorphisms Φ e Ψ are, in this case, of class $C^{2,1}$ ($C^{m+1,1}$ in the general case). ■

Corollary 4.6.7 (C^∞ global regularity). *Let Ω be a bounded open set of \mathbb{R}^n with boundary of class C^∞ . Let*

$$f \in C^\infty(\bar{\Omega}).$$

Let $A \in C^\infty(\overline{\Omega}; \mathbb{M}(n))$. Let us assume that A satisfies (4.1.1).

Let us assume that $u \in H_0^1(\Omega)$ is a solution to

$$-\operatorname{div}(A\nabla u) = f, \quad \text{in } \Omega$$

Then we have

$$u \in C^\infty(\overline{\Omega}).$$

Proof. By Theorem 4.6.6 and by the Embedding Theorem 3.7.10 we have

$$u \in \bigcap_{m=0}^{\infty} H^m(\Omega) = C^\infty(\overline{\Omega}).$$

■

4.7 The Dirichlet to Neumann Map

Denote by $\mathbb{M}^S(n)$ the vector space of symmetric matrix $n \times n$ with real entries. Let Ω be a bounded open set of \mathbb{R}^n of class $C^{0,1}$. Let $A \in L^\infty(\Omega; \mathbb{M}^S(n))$ and let us suppose that (4.1.1) holds. Formally, the Dirichlet to Neumann Map can be constructed in the following way: let $\varphi \in H^{1/2}(\partial\Omega)$ and let $u \in H^1(\Omega)$ be the solution to the problem

$$\begin{cases} \operatorname{div}(A\nabla u) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \quad (4.7.1)$$

We have seen that u is uniquely determined by φ hence, *if it would make sense*, we could define the map

$$\varphi \rightarrow A\nabla u \cdot \nu, \quad (\text{conormal derivative of } u \text{ on } \partial\Omega). \quad (4.7.2)$$

Let us observe that if $u \in C^2(\overline{\Omega})$ and $A \in C^1(\overline{\Omega}; \mathbb{M}^S(n))$, by (4.7.1) we have

$$\int_{\partial\Omega} (A\nabla u \cdot \nu) v dS = \int_{\Omega} A\nabla u \cdot \nabla v dx, \quad \forall v \in C^1(\overline{\Omega}). \quad (4.7.3)$$

As a matter of fact, if $v \in C^1(\overline{\Omega})$, by the divergence Theorem and by (4.7.1) we get

$$\int_{\Omega} A\nabla u \cdot \nabla v dx = \int_{\Omega} (\operatorname{div}(vA\nabla u) - \operatorname{div}(A\nabla u)v) dx = \int_{\partial\Omega} (A\nabla u \cdot \nu) v dS.$$

Equality (4.7.3) allows us to "read" $A \frac{\partial u}{\partial \nu}$ by means integral on the right-hand side. Now, if $A \in L^\infty(\Omega; \mathbb{M}^S(n))$ and $u \in H^1(\Omega)$ then the integral on the right-hand side of (4.7.3) makes perfectly sense. Based on these insights we will define below $A \nabla u \cdot \nu$ as an element of $H^{-1/2}(\partial\Omega)$ (the dual space of $H^{1/2}(\partial\Omega)$).

First, notice that

$$\int_{\Omega} A \nabla u \cdot \nabla v dx = 0, \quad \forall v \in H_0^1(\Omega). \quad (4.7.4)$$

As a matter of fact, recalling (4.3.11) and (4.3.12), we have, for each $v \in H_0^1(\Omega)$,

$$\int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} \{A \nabla \Phi \cdot \nabla v + A \nabla w \cdot \nabla v\} dx = 0.$$

By (4.7.4), recalling Theorem 3.5.1, we have that if $\varphi \in H^1(\Omega)$ then the integral

$$\int_{\Omega} A \nabla u \cdot \nabla v dx \quad (4.7.5)$$

depends only by the trace of v on $\partial\Omega$. As a matter of fact, if $v_1, v_2 \in H^1(\Omega)$ have the same trace on $\partial\Omega$ then $v_1 - v_2 \in H_0^1(\Omega)$. Hence (4.7.4) implies

$$\int_{\Omega} A \nabla u \cdot \nabla v_1 dx = \int_{\Omega} A \nabla u \cdot \nabla v_2 dx.$$

Therefore, for every $\varphi \in H^{1/2}(\partial\Omega)$ it turns out to be well-defined the functional

$$L_\varphi : H^{1/2}(\partial\Omega) \rightarrow \mathbb{R},$$

which maps $\phi \in H^{1/2}(\partial\Omega)$ in the real number

$$L_\varphi(\phi) = \int_{\Omega} A \nabla u \cdot \nabla v dx,$$

where $v|_{\partial\Omega} = \phi$ (in the sense of traces).

We prove that L_φ is a linear and bounded functional.

The linearity of L_φ is trivial. Concerning the boundedness, let

$$\phi \in H^{1/2}(\partial\Omega),$$

by Theorem 3.12.12 we know that there exists $v \in H^1(\Omega)$ so that

$$v|_{\partial\Omega} = \phi, \quad (\text{in the sense of traces})$$

which satisfies

$$\|v\|_{H^1(\Omega)} \leq C \|\phi\|_{H^{1/2}(\partial\Omega)}, \quad (4.7.6)$$

where C is a constant depending on Ω only. Proceeding in a similar way to what we did to obtain (4.3.13) and taking into account (4.3.15), we get

$$\begin{aligned} |L_\varphi(\phi)| &= \left| \int_{\Omega} A \nabla u \cdot \nabla v dx \right| \leq \lambda \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \\ &\leq \bar{C} \|\varphi\|_{H^{1/2}(\partial\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)}, \end{aligned} \quad (4.7.7)$$

where \bar{C} is a constant depending on Ω and λ only. Therefore, the functional L_φ is bounded and it satisfies

$$\|L_\varphi\|_{H^{-1/2}(\partial\Omega)} \leq \bar{C} \|\varphi\|_{H^{1/2}(\partial\Omega)}, \quad \forall \varphi \in H^{1/2}(\partial\Omega). \quad (4.7.8)$$

Moreover, notice that (4.7.8) implies that the linear operator

$$H^{1/2}(\partial\Omega) \ni \varphi \rightarrow L_\varphi \in H^{-1/2}(\partial\Omega),$$

is bounded.

Now we set

$$A \nabla u \cdot \nu := L_\varphi$$

consequently, we write, for any $v \in H^1(\Omega)$ such that $v|_{\partial\Omega} = \phi$,

$$\langle A \nabla u \cdot \nu, \phi \rangle_{H^{-1/2}, H^{1/2}} = L_\varphi(\phi) = \int_{\Omega} A \nabla u \cdot \nabla v dx, \quad (4.7.9)$$

where, by $\langle \cdot, \cdot \rangle_{H^{-1/2}, H^{1/2}}$ we denote the scalar product in the duality. With the notations used so far we should have written $(A \nabla u \cdot \nu)(\phi)$ instead of $\langle A \nabla u \cdot \nu, \phi \rangle_{H^{-1/2}, H^{1/2}}$, but the latter notation it is certainly more handleable in the present context. .

Finally, we define the **Dirichlet to Neumann Map** as

$$\Lambda_A : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega), \quad \Lambda_A(\varphi) = A \nabla u \cdot \nu. \quad (4.7.10)$$

By (4.7.8) it follows that $\Lambda_A \in \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$, where, we recall, $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ denotes the space of the linear and bounded operators from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$. From the that construction we have performed so far we get

$$\langle \Lambda_A(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} A \nabla u \cdot \nabla v dx, \quad \forall \varphi, \phi \in H^{1/2}(\partial\Omega), \quad (4.7.11)$$

where $u \in H^1(\Omega)$ is the solution to (4.7.1) and v is any function of $H^1(\Omega)$ which satisfies $v|_{\partial\Omega} = \phi$. It is simple to check that

$$\begin{aligned} \|\Lambda_A\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} &= \\ &= \sup \left\{ \langle \Lambda_A(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} : \|\varphi\|_{H^{1/2}(\partial\Omega)} \leq 1, \|\phi\|_{H^{1/2}(\partial\Omega)} \leq 1 \right\}. \end{aligned}$$

In what follows we prove other simple but important properties of Λ_A .

We first observe that since the right-hand integral in (4.7.11) is independent of the choice of v (as long as it is a trace of ϕ) we can choose $v = w$ where $w \in H^1(\Omega)$ is the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(A \nabla w) = 0, & \text{in } \Omega, \\ w = \phi, & \text{on } \partial\Omega. \end{cases} \quad (4.7.12)$$

From this it follows that the bilinear form

$$H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \ni (\varphi, \phi) \rightarrow \langle \Lambda_A(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} \in \mathbb{R}$$

is **symmetric** that is

$$\langle \Lambda_A(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} = \langle \Lambda_A(\phi), \varphi \rangle_{H^{-1/2}, H^{1/2}}, \quad \forall \varphi, \phi \in H^{1/2}(\partial\Omega). \quad (4.7.13)$$

As a matter of fact

$$\begin{aligned} \langle \Lambda_A(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} &= \int_{\Omega} A \nabla u \cdot \nabla w dx = \\ &= \int_{\Omega} A \nabla w \cdot \nabla u dx = \\ &= \langle \Lambda_A(\phi), \varphi \rangle_{H^{-1/2}, H^{1/2}}. \end{aligned}$$

An important consequence of (4.7.13) is the following identity

Theorem 4.7.1 (the Alessandrini identity). *Let $A_1, A_2 \in L^\infty(\Omega; \mathbb{M}^S(n))$. Let us assume that A_1, A_2 satisfy (4.1.1). Let $\varphi, \phi \in H^{1/2}(\partial\Omega)$ and let $u_1, u_2 \in H^1(\Omega)$ be the solutions to*

$$\begin{cases} \operatorname{div}(A\nabla u_1) = 0, & \text{in } \Omega, \\ u_1 = \varphi, & \text{on } \partial\Omega \end{cases} \quad (4.7.14)$$

and

$$\begin{cases} \operatorname{div}(A\nabla u_2) = 0, & \text{in } \Omega, \\ u_2 = \phi, & \text{on } \partial\Omega, \end{cases} \quad (4.7.15)$$

then

$$\langle (\Lambda_{A_1} - \Lambda_{A_2})(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx. \quad (4.7.16)$$

Proof. By (4.7.13) we have (by setting for sake of brevity, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H^{-1/2}, H^{1/2}}$)

$$\begin{aligned} \langle (\Lambda_{A_1} - \Lambda_{A_2})(\varphi), \phi \rangle &= \langle \Lambda_{A_1}(\varphi), \phi \rangle - \langle \Lambda_{A_2}(\varphi), \phi \rangle = \\ &= \langle \Lambda_{A_1}(\varphi), \phi \rangle - \langle \Lambda_{A_2}(\phi), \varphi \rangle. \end{aligned} \quad (4.7.17)$$

Now, by (4.7.14) and (4.7.15) we have, respectively,

$$\langle \Lambda_{A_1}(\varphi), \phi \rangle = \int_{\Omega} A_1 \nabla u_1 \cdot \nabla u_2 dx$$

and

$$\langle \Lambda_{A_2}(\phi), \varphi \rangle = \int_{\Omega} A_2 \nabla u_2 \cdot \nabla u_1 dx = \int_{\Omega} A_2 \nabla u_1 \cdot \nabla u_2 dx$$

by inserting the latter in (4.7.17) we get

$$\langle (\Lambda_{A_1} - \Lambda_{A_2})(\varphi), \phi \rangle = \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx.$$

■

A simple consequence of the Alessandrini identity is the continuity of the map $A \rightarrow \Lambda_A$. Precisely we have the following

Proposition 4.7.2. *There exists a constant C depending on λ and Ω only so that, if $A_1, A_2 \in L^\infty(\Omega; \mathbb{M}^S(n))$ satisfy (4.1.1), then*

$$\|\Lambda_{A_1} - \Lambda_{A_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \leq C \|A_1 - A_2\|_{L^\infty(\Omega, \mathbb{M}^S(n))}. \quad (4.7.18)$$

Proof. By the Alessandrini identity and by the Cauchy–Schwarz inequality we have

$$\begin{aligned} \left| \langle (\Lambda_{A_1} - \Lambda_{A_2})(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} \right| &= \left| \int_{\Omega} (A_1 - A_2) \nabla u_1 \cdot \nabla u_2 dx \right| \leq \\ &\leq \|A_1 - A_2\|_{L^\infty(\Omega; \mathbb{M}^S(n))} \|\nabla u_1\|_{L^2(\Omega)} \|\nabla u_2\|_{L^2(\Omega)}. \end{aligned} \quad (4.7.19)$$

Now, (4.3.15) gives

$$\|\nabla u_1\|_{L^2(\Omega)} \leq C \|\varphi\|_{H^{1/2}(\partial\Omega)} \quad \text{and} \quad \|\nabla u_2\|_{L^2(\Omega)} \leq C \|\phi\|_{H^{1/2}(\partial\Omega)},$$

where C depends on λ and Ω only. Inserting the last obtained inequalities in (4.7.19) we have

$$\begin{aligned} \left| \langle (\Lambda_{A_1} - \Lambda_{A_2})(\varphi), \phi \rangle_{H^{-1/2}, H^{1/2}} \right| &\leq \\ &\leq C \|A_1 - A_2\|_{L^\infty(\Omega; \mathbb{M}^S(n))} \|\varphi\|_{H^{1/2}(\partial\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)} \end{aligned}$$

From which (4.7.18) follows. ■

Similarly, one can also define the **Neumann to Dirichlet Map**. Let $g \in H^{-1/2}(\partial\Omega)$ satisfy

$$\langle g, 1 \rangle = 0.$$

Let us consider the solution $u \in H^1(\Omega)$ to the Neumann problem

$$\begin{cases} \int_{\Omega} A \nabla u \cdot \nabla v dx = \langle g, \varphi \rangle_{H^{-1/2}, H^{1/2}}, & \forall v \in H^1(\Omega), \\ u \in \{w \in H^1(\Omega) : \int_{\Omega} w dx = 0\}, \end{cases} \quad (4.7.20)$$

we define the Neumann to Dirichlet Map as follows

$$\begin{aligned} \mathcal{N}_A : H^{-1/2}(\partial\Omega) &\rightarrow H^{1/2}(\partial\Omega), \\ \mathcal{N}_A(g) &= u|_{\partial\Omega}, \quad (\text{in the sense of the traces}). \end{aligned} \quad (4.7.21)$$

4.8 The inclusion inverse problem

Let $n = 2$ or $n = 3$ and let us assume that Ω represents an electrically conductor of constant conductivity, say, 1 and let us suppose that Ω contains an *inclusion* D of different conductivity, say k , with $k > 0$ and $k \neq 1$. We consider the problem of determining D from the knowledge of a density of prescribed current on $\partial\Omega$ and of the corresponding voltage u measured on $\partial\Omega$.

We provide a mathematical formulation of the problem. Let us assume that Ω is a bounded open set of \mathbb{R}^n ($n = 2, 3$) whose boundary is of class $C^{0,1}$, let $\phi \in H^{-1/2}(\partial\Omega)$ satisfy

$$\int_{\partial\Omega} \phi dS = 0, \quad (4.8.1)$$

Where we have denoted

$$\int_{\partial\Omega} \phi dS = \langle \phi, 1 \rangle_{H^{-1/2}, H^{1/2}}.$$

ϕ represents the density of prescribed current on $\partial\Omega$. If the inclusion is present, the electrostatic potential u is determined, up to an additive constant, as a solution of Neumann problem

$$\begin{cases} \operatorname{div}((1 + (k - 1)\chi_D) \nabla u) = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \phi, & \text{on } \partial\Omega, \end{cases} \quad (4.8.2)$$

where D is a measurable subset of Ω . In what follows we assume

$$\int_{\Omega} u(x) dx = 0, \quad (4.8.3)$$

which yields with the uniqueness of the boundary value problem (4.8.2).

The **inverse problem** consists in determining D , by assigning a non-trivial input ϕ and measuring the corresponding trace $u|_{\partial\Omega}$. The uniqueness of D is still an open question. We point out that if we dispose of the entire Neumann to Dirichlet Map (or the Dirichlet to Neumann Map), and ∂D is enough regular, the uniqueness can be proved (see [39]). Keep in mind that having the entire Dirichlet to Neumann Map is equivalent to being able to make infinite measurements on $\partial\Omega$. However, the ideas (developed detailed in [3]) that we will present here allow us to find **size estimates (of volume or area)** of the inclusion from certain integrals of the data, ϕ and $u|_{\partial\Omega}$ as we show below.

Let us consider the quantity

$$W = \int_{\partial\Omega} \phi u, \quad (4.8.4)$$

and compare it with

$$W_0 = \int_{\partial\Omega} \phi u_0, \quad (4.8.5)$$

where u_0 represents the solution to the Neumann problem

$$\begin{cases} \Delta u_0 = 0, & \text{in } \Omega, \\ \frac{\partial u_0}{\partial \nu} = \phi, & \text{on } \partial\Omega, \end{cases} \quad (4.8.6)$$

with

$$\int_{\Omega} u_0 = 0.$$

W and W_0 represent the power required to maintain the current ϕ , when the inclusion D is present and it is not present, respectively. Partially anticipating the results that **we will prove later in this Section, we have that if $k \neq 1$, $k > 0$, then the following inequality holds true**

$$C_1 \int_D |\nabla u_0|^2 \leq |W_0 - W| \leq C_2 \int_D |\nabla u_0|^2, \quad (4.8.7)$$

where C_1 and C_2 are positive constants depending on k only. Let us assume, for instance+, that D is a connected open set. Inequalities (4.8.7) implies that if $\phi \neq 0$ then $|D|$ is 0 if and only if $W_0 - W = 0$. Let us prove this claim. If $|D| = 0$ then by the second inequality we immediately have $W_0 - W = 0$. Conversely, if $W_0 - W = 0$, then by the first inequality we have that if $D \neq \emptyset$ then

$$u_0 = \text{constant in } D.$$

Since u_0 is an analytic function, we have

$$u_0 = \text{constant in } \Omega,$$

consequently $\phi = 0$, but we have assumed $\phi \neq 0$, therefore $D = \emptyset$.

Inequalities (4.8.7) can be proved as a consequence of general properties of the continuous symmetric coercive bilinear forms on a Hilbert space.

Introduce some notation. Let H be a real Hilbert space and H' its dual space.

Let $\lambda_0, \lambda_1 \in [1, +\infty)$ and let $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$ two symmetric bilinear forms on H which satisfy the conditions

$$\lambda_0^{-1}\|u\|^2 \leq a_0(u, u) \leq \lambda_0\|u\|^2, \quad \forall u \in H, \quad (4.8.8a)$$

$$\lambda_1^{-1}\|u\|^2 \leq a_1(u, u) \leq \lambda_1\|u\|^2, \quad \forall u \in H. \quad (4.8.8b)$$

Let us note that (4.8.8a) and (4.8.8b) imply, respectively, the continuity of a_0 and of a_1 . Just check this for $a_0(\cdot, \cdot)$. We have that (4.8.8a) implies

$$|a_0(u, v)| \leq \sqrt{a_0(u, u)}\sqrt{a_0(v, v)} \leq \lambda_0\|u\|\|v\|, \quad \forall u, v \in H.$$

Moreover, let

$$\alpha(u, v) = a_1(u, v) - a_0(u, v), \quad u, v \in H. \quad (4.8.9)$$

Let $F \in H'$. By the Lax–Milgram Theorem, there exist $u_1, u_0 \in H$ such that

$$a_j(u_j, v) = \langle F, v \rangle \quad \forall v \in H, \quad j = 0, 1. \quad (4.8.10)$$

Define

$$W_0 = \langle F, u_0 \rangle, \quad W_1 = \langle F, u_1 \rangle, \quad \delta W = W_0 - W_1. \quad (4.8.11)$$

Now we prove two simple lemmas.

Lemma 4.8.1. *The following equalities hold true.*

$$a_0(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0) = -\delta W, \quad (4.8.12a)$$

$$a_0(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) = \delta W, \quad (4.8.12b)$$

$$\alpha(u_1, u_0) = -\delta W. \quad (4.8.12c)$$

Proof. Let us check (4.8.12a).

$$\begin{aligned} & a_0(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0) = \\ & = a_0(u_1 - u_0, u_1 - u_0) - [a_1(u_0, u_0) - a_0(u_0, u_0)] = \\ & = a_1(u_1, u_1) - 2a_1(u_1, u_0) + a_1(u_0, u_0) - a_1(u_0, u_0) + a_0(u_0, u_0) = \\ & = a_1(u_1, u_1) - 2a_1(u_1, u_0) + a_0(u_0, u_0) = \\ & = \langle F, u_1 \rangle - 2\langle F, u_0 \rangle + \langle F, u_0 \rangle = \\ & = \langle F, u_1 - u_0 \rangle = -\delta W. \end{aligned}$$

Equality (4.8.12b) can be obtained similarly and (4.8.12c) is an immediate consequence of (4.8.9). ■

Lemma 4.8.2. *If one of the following conditions is satisfied*

$$\alpha(u, u) \geq 0, \quad \forall u \in H,$$

or

$$\alpha(u, u) \leq 0, \quad \forall u \in H,$$

then we have

$$|\alpha(u, v)| \leq |\alpha(u, u)|^{1/2} |\alpha(v, v)|^{1/2}, \quad \forall u, v \in H. \quad (4.8.13)$$

Proof. Let $u, v \in H$. If $\alpha(u, u) = 0$ and $\alpha(v, v) = 0$, then, assuming $\alpha(w, w) \geq 0$, for every $w \in H$, we have

$$0 \leq \alpha(u + tv, u + tv) = 2t\alpha(u, v), \quad \forall t \in \mathbb{R},$$

which implies $\alpha(u, v) = 0$ and (4.8.13) is proved.

If either $\alpha(u, u) \neq 0$ or $\alpha(v, v) \neq 0$, then, assuming, for instance, $\alpha(v, v) > 0$, we have

$$0 \leq \alpha(u + tv, u + tv) = t^2\alpha(v, v) + 2t\alpha(u, v) + \alpha(u, u), \quad \forall t \in \mathbb{R},$$

hence

$$(\alpha(u, v))^2 - \alpha(u, u)\alpha(v, v) \leq 0$$

which gives (4.8.13).

If $\alpha(w, w) \leq 0$, for every $w \in H$, the thesis follows easily by applying the previous procedure to $-\alpha(\cdot, \cdot)$. ■

Now we prove

Theorem 4.8.3. *Let $F \in H'$. Let us assume that the bilinear forms $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$ satisfy conditions (4.8.8) and let us assume that u_0, u_1 satisfy (4.8.10). If $\alpha(\cdot, \cdot)$ (defined by (4.8.9)), satisfies*

$$0 \leq \alpha(u, u) \leq C_0 a_0(u, u), \quad \forall u \in H, \quad (4.8.14)$$

where C_0 is a positive constant, then

$$\delta W \geq 0$$

and

$$\delta W \leq \alpha(u_0, u_0) \leq (1 + C_0)\delta W. \quad (4.8.15)$$

If $\alpha(\cdot, \cdot)$ satisfies the condition

$$\alpha(u, u) \leq 0, \quad \forall u \in H, \quad (4.8.16)$$

then

$$\delta W \leq 0$$

and

$$-C\delta W \leq -\alpha(u_0, u_0) \leq -\delta W, \quad (4.8.17)$$

where C is a positive constant depending on λ_0 and λ_1 only.

Proof. First we consider the case in which (4.8.14) holds. By (4.8.12b) we have $\delta W \geq 0$ and by (4.8.12a) we have $-\alpha(u_0, u_0) \leq -\delta W$. Therefore

$$\delta W \leq \alpha(u_0, u_0). \quad (4.8.18)$$

Now, let us estimate $\alpha(u_0, u_0)$ from above. Lemma 4.8.1 – b and Lemma 4.8.2 give

$$\begin{aligned} \alpha(u_0, u_0) &= \alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) + 2\alpha(u_0 - u_1, u_1) \leq \\ &\leq \alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) + 2|\alpha(u_0 - u_1, u_0 - u_1)|^{1/2}|\alpha(u_1, u_1)|^{1/2} \leq \\ &\leq \alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1) + \frac{1}{C_0}\alpha(u_0 - u_1, u_0 - u_1) + C_0\alpha(u_1, u_1) = \\ &= (1 + C_0)\left[\frac{1}{C_0}\alpha(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1)\right] \leq \\ &\leq (1 + C_0) \max\left\{1, \frac{1}{C_0}\right\} [a_0(u_0 - u_1, u_0 - u_1) + \alpha(u_1, u_1)] \leq \\ &\leq C_1\delta W, \end{aligned}$$

where, in the last inequality we have set

$$C_1 = (1 + C_0) \max\left\{1, \frac{1}{C_0}\right\}.$$

Hence we have

$$\alpha(u_0, u_0) \leq C_1\delta W.$$

By the just obtained inequality and by (4.8.18) we get (4.8.15).

Now we consider the case in which (4.8.16) holds. By (4.8.12a) we get $\delta W \leq 0$ and also

$$|\alpha(u_0, u_0)| \leq -\delta W. \quad (4.8.19)$$

Now we estimate $|\alpha(u_0, u_0)|$ from below. By (4.8.12c) we obtain, for $\varepsilon > 0$ to be chosen,

$$\begin{aligned} -\delta W &= \alpha(u_0, u_1) \leq (-\alpha(u_0, u_0))^{1/2} (-\alpha(u_1, u_1))^{1/2} \leq \\ &\leq \frac{\varepsilon}{2}(-\alpha(u_1, u_1)) + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)). \end{aligned} \quad (4.8.20)$$

By (4.8.12b) we have

$$-\alpha(u_1, u_1) = a_0(u_1 - u_0, u_1 - u_0) - \delta W. \quad (4.8.21)$$

Moreover, (4.8.8) gives

$$a_0(u_1 - u_0, u_1 - u_0) \leq \lambda_0 \|u_1 - u_0\|^2 \leq \lambda_0 \lambda_1 a_1(u_1 - u_0, u_1 - u_0).$$

By the just obtained inequality and by (4.8.21) we get

$$-\alpha(u_1, u_1) \leq \lambda_0 \lambda_1 a_1(u_1 - u_0, u_1 - u_0) - \delta W.$$

The last inequality together with (4.8.20) and (4.8.12a), give (we denote $A = \lambda_0 \lambda_1$)

$$\begin{aligned} -\delta W &\leq \frac{\varepsilon}{2} [A a_1(u_1 - u_0, u_1 - u_0) - \delta W] + \frac{1}{2\varepsilon} (-\alpha(u_0, u_0)) = \\ &= \frac{\varepsilon}{2} [A (a_1(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0)) + A \alpha(u_0, u_0) - \delta W] + \\ &+ \frac{1}{2\varepsilon} (-\alpha(u_0, u_0)) = \\ &= -\frac{\varepsilon}{2} (1 + A) \delta W + \left(\frac{1}{2\varepsilon} - A \frac{\varepsilon}{2} \right) (-\alpha(u_0, u_0)). \end{aligned}$$

Therefore

$$\left(1 - \frac{\varepsilon}{2}(1 + A)\right) |\delta W| \leq \frac{1 - A\varepsilon^2}{2\varepsilon} |\alpha(u_0, u_0)|.$$

If

$$\varepsilon = \min \left\{ \frac{1}{\sqrt{2A}}, \frac{1}{1 + A} \right\}$$

we have

$$\frac{2\varepsilon \left(1 - \frac{\varepsilon}{2}(1 + A)\right)}{1 - A\varepsilon^2} |\delta W| \leq |\alpha(u_0, u_0)|.$$

Ultimately, we have

$$C |\delta W| \leq |\alpha(u_0, u_0)|,$$

where C depends on λ_0, λ_1 only. ■

Remark 1. If (4.8.14) holds, condition (4.8.8) can be weakened by assuming that $a_0(\cdot, \cdot), a_1(\cdot, \cdot)$ are semidefinite positive. In turn, if case (4.8.16) occurs, it suffices to assume that $a_0(\cdot, \cdot), a_1(\cdot, \cdot)$ are semidefinite positive and satisfy

$$a_0(u, u) \leq C_1 a_1(u, u), \quad \forall u \in H,$$

where C_1 is a positive constant. \blacklozenge

Now we apply Theorem 4.8.3 to inclusion inverse problem. Let

$$H = \left\{ v \in H^1(\Omega) : \int_{\Omega} v dx = 0 \right\},$$

$$a_1(u, v) = \int_{\Omega} (1 + (k-1)\chi_D) \nabla u \cdot \nabla v, \quad u, v \in H,$$

$$a_0(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad u, v \in H.$$

Let us assume that $\phi \in H^{-1/2}(\partial\Omega)$ satisfies (4.8.1). Then Neumann problems (4.8.2), (4.8.6) can be formulated (see Section 4.4 and the proof of Theorem 4.4.1) as follows

$$a_1(u, v) = \langle g, v \rangle_{H^{-1/2}, H^{1/2}}, \quad \forall v \in H, \quad (4.8.22)$$

$$a_0(u, v) = \langle g, v \rangle_{H^{-1/2}, H^{1/2}}, \quad \forall v \in H. \quad (4.8.23)$$

Hence, in our case we have

$$H \ni v \rightarrow \langle F, v \rangle = \langle g, v \rangle_{H^{-1/2}, H^{1/2}} \in \mathbb{R},$$

$$\alpha(u, v) = (1-k) \int_D \nabla u \cdot \nabla v, \quad u, v \in H,$$

$$\delta W = \int_{\partial\Omega} \phi(u_0 - u_1),$$

where u_0 e u_1 are solutions to (4.8.23) and (4.8.22) respectively.

Case $k < 1$. In this case, if $C_0 = 1$ then (4.8.14) is satisfied. Hence by (4.8.15) we have

$$\frac{\delta W}{1-k} \leq \int_D |\nabla u_0|^2 \leq \frac{2\delta W}{1-k}. \quad (4.8.24)$$

Case $k > 1$. In this case (4.8.16) holds true. Hence, by (4.8.17) we get

$$-\frac{C\delta W}{k-1} \leq \int_D |\nabla u_0|^2 \leq -\frac{\delta W}{k-1}, \quad (4.8.25)$$

where C depends on k only. Inequalities (4.8.24) and (4.8.25) imply (4.8.7).

Let us observe that estimates (4.8.7) can be used to easily find a size estimate of D . For instance, if $u_0 = x_1 + c$ in Ω we have $\nabla u_0 = e_1$ and then by (4.8.7) we get.

$$C_2^{-1}|W_0 - W| \leq |D| \leq C_1^{-1}|W_0 - W|. \quad (4.8.26)$$

Obviously, to assign the value of u_0 on Ω is equivalent to assign some stringent conditions on the "input" current density. In the case in question, $\phi = e_1 \cdot \nu$ and it is not certain that, in practice, one one can make such a choice. For this reason it is useful to find the estimates (from above and below) of the measure of D for a generic nontrivial ϕ .

In order to examine this issue a little more deeply, let us begin by observing that to formulate Neumann problem (4.8.2) it is not necessary that D be an open, but it suffices that D be a **Lebesgue measurable set** of \mathbb{R}^n . If, for instance we know that

$$\text{dist}(D, \partial\Omega) \geq d > 0$$

it is not difficult to find an estimate from below of $|D|$ by exploiting the second inequality in (4.8.7), i.e. the inequality

$$|\delta W| \leq C_2 \int_D |\nabla u_0|^2. \quad (4.8.27)$$

Let us examine in which manner we can find an estimate from below of $|D|$.

We have

$$\int_D |\nabla u_0|^2 \leq |D| \max_D |\nabla u_0|^2. \quad (4.8.28)$$

Now, let $x_0 \in \bar{D}$ satisfy

$$|\nabla u_0(x_0)| = \max_D |\nabla u_0|. \quad (4.8.29)$$

By the Mean Property for harmonic functions, we have

$$\begin{aligned} \nabla u_0(x_0) &= \frac{1}{|B_{d/2}(x_0)|} \int_{B_{d/2}(x_0)} \nabla u_0(x) dx = \\ &= \frac{1}{|B_{d/2}(x_0)|} \int_{\partial B_{d/2}(x_0)} u_0(x) \nu dS, \end{aligned}$$

which implies

$$|\nabla u_0(x_0)| \leq \frac{2}{d} \max_{\partial B_{d/2}(x_0)} |u_0|. \quad (4.8.30)$$

Now, let us estimate $\max_{\partial B_{d/2}(x_0)} |u_0|$ from above. Let $\bar{x} \in \partial B_{d/2}(x_0)$ fulfill

$$|u_0(\bar{x})| = \max_{\partial B_{d/2}(x_0)} |u_0|,$$

by using again the Mean Property and the Cauchy–Schwarz inequality we get

$$\begin{aligned} |u_0(\bar{x})| &= \left| \frac{1}{|B_{d/4}(\bar{x})|} \int_{B_{d/4}(\bar{x})} u(y) dy \right| \leq \frac{|B_{d/4}(\bar{x})|^{1/2}}{|B_{d/4}(\bar{x})|} \left(\int_{B_{d/4}(\bar{x})} |u(y)|^2 dy \right)^{1/2} \leq \\ &\leq \frac{1}{|B_{d/4}(\bar{x})|^{1/2}} \|u_0\|_{L^2(\Omega)} \leq C \|\nabla u_0\|_{L^2(\Omega)}, \end{aligned} \quad (4.8.31)$$

where C depends by Ω and d only; in the last inequality of (4.8.31) we have applied Theorem 3.9.1. On the other hand, by (4.4.5) we have

$$\|\nabla u_0\|_{L^2(\Omega)} \leq C \|\phi\|_{H^{-1/2}(\partial\Omega)}.$$

The just obtained inequality and (4.8.31) yield

$$\max_{\partial B_{d/2}(x_0)} |u_0| = |u_0(\bar{x})| \leq C \|\phi\|_{H^{-1/2}(\partial\Omega)}.$$

Now, by this inequality and by (4.8.28)–(4.8.30) we get

$$\int_D |\nabla u_0|^2 \leq C_* |D| \|\phi\|_{H^{-1/2}(\partial\Omega)}^2, \quad (4.8.32)$$

where C_* is a constant depending on Ω and d only. Finally, by (4.8.27) and (4.8.32) we have

$$\frac{|\delta W|}{C_2 C_* \|\phi\|_{H^{-1/2}(\partial\Omega)}^2} \leq |D|.$$

To find an estimate from above of $|D|$ (of course, in terms of δW) is definitely more challenging and, as we have already mentioned in the case where D is an open set, such estimate from above has inevitably to do with the **unique continuation property** of solution to the Laplace equation.

When D is only a Lebesgue measurable set, even prove that

$$\delta W = 0 \implies |D| = 0, \quad (4.8.33)$$

is not trivial. To present here a proof of (4.8.33) we need the differentiation Lebesgue Theorem 2.5.1. In particular such a Theorem 2.5.1 implies that if D is a Lebesgue measurable set, then

$$\lim_{r \rightarrow 0} \frac{|D \cap B_r(x)|}{|B_r(x)|} = 1, \quad \text{a.e. } x \in D. \quad (4.8.34)$$

Let us set

$$\tilde{D} = \left\{ x \in D : \lim_{r \rightarrow 0} \frac{|D \cap B_r(x)|}{|B_r(x)|} = 1 \right\}. \quad (4.8.35)$$

The following Proposition holds true ([19]).

Proposition 4.8.4. *Let Ω be an open set of \mathbb{R}^n and let D be a Lebesgue measurable set such that $\overline{D} \subset \Omega$ and $|D| > 0$. Let $u \in H_{loc}^1(\Omega)$. Let us assume that u satisfies the condition*

$$u(x) = 0, \quad \forall x \in D. \quad (4.8.36)$$

Moreover, let us assume that, in a given point $x_0 \in \tilde{D}$ we have

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0)} u^2 dx, \quad (4.8.37)$$

for every $r > 0$ such that $\overline{B_{2r}(x_0)} \subset \Omega$, where C is independent of r .

Then we have

$$\int_{B_r(x_0)} u^2 dx \leq \mathcal{O}(r^k), \quad \text{as } r \rightarrow 0, \quad \forall k \in \mathbb{N}. \quad (4.8.38)$$

To prove Proposition 4.8.4 we need the following

Lemma 4.8.5. *If $R > 0$ and $u \in H^1(B_R)$ then*

$$\left(\int_{B_R} |u|^q dx \right)^{\frac{1}{q}} \leq \frac{C_{n,q}}{|B_R|^{\frac{1}{2} - \frac{1}{q}}} \left(\int_{B_R} [R^2 |\nabla u|^2 + u^2] dx \right)^{\frac{1}{2}}, \quad (4.8.39)$$

where q is an arbitrary number of $(1, +\infty)$ for $n = 2$, and it is equal to $\frac{2n}{n-2}$ for $n \geq 3$. Moreover, $C_{n,q}$ depends on q and n only.

Proof of Lemma 4.8.5. Set

$$v(y) = u(Ry), \quad \forall y \in B_1,$$

it turns out that $v \in H^1(B_1)$. Now, by the Embedding Sobolev Theorem (Theorem 3.7.10) and performing the change of variables $y = Rx$, we get

$$\begin{aligned}
\left(\int_{B_R} |u|^q dx\right)^{\frac{1}{q}} &= \left(R^n \int_{B_1} |u(Ry)|^q dy\right)^{\frac{1}{q}} = \\
&= R^{\frac{n}{q}} \left(\int_{B_1} |v(y)|^q dy\right)^{\frac{1}{q}} \leq \\
&\leq CR^{\frac{n}{q}} \left(\int_{B_1} [|\nabla v|^2 + |v|^2] dy\right)^{\frac{1}{2}} = \\
&= CR^{\frac{n}{q}} \left(\int_{B_1} [R^2 |(\nabla u)(Ry)|^2 + |u(Ry)|^2] dy\right)^{\frac{1}{2}} = \tag{4.8.40} \\
&= CR^{\frac{n}{q}} \left(R^{-n} \int_{B_R} [R^2 |\nabla u|^2 + |u|^2] dx\right)^{\frac{1}{2}} = \\
&= \frac{C\omega_n^{\frac{1}{2} - \frac{1}{q}}}{|B_R|^{\frac{1}{2} - \frac{1}{q}}} \left(\int_{B_R} [R^2 |\nabla u|^2 + |u|^2] dx\right)^{\frac{1}{2}}.
\end{aligned}$$

■

Proof of Proposition 4.8.4. Let us denote by $\rho = \text{dist}(x_0, \partial\Omega)$. Since $x_0 \in \tilde{D}$, we have that for any $\varepsilon > 0$ there exists $r_\varepsilon < 2\rho$ such that

$$\frac{|B_r(x_0) \setminus D|}{|B_r(x_0)|} < \varepsilon, \quad \forall r \in (0, r_\varepsilon].$$

By the above inequality and by Lemma 4.8.5 we have

$$\begin{aligned}
\int_{B_r(x_0)} u^2 dx &= \int_{B_r(x_0) \setminus D} u^2 dx \leq |B_r(x_0) \setminus D|^{1 - \frac{2}{q}} \left(\int_{B_r(x_0) \setminus D} |u|^q dx\right)^{\frac{2}{q}} \leq \\
&\leq |B_r(x_0) \setminus D|^{1 - \frac{2}{q}} \left(\int_{B_r(x_0)} |u|^q dx\right)^{\frac{2}{q}} \leq \\
&\leq C \left(\frac{|B_r(x_0) \setminus D|}{|B_r(x_0)|}\right)^{1 - \frac{2}{q}} \int_{B_r(x_0)} [r^2 |\nabla u|^2 + u^2] dx \leq \\
&\leq C\varepsilon^{1 - \frac{2}{q}} \int_{B_r(x_0)} [r^2 |\nabla u|^2 + u^2] dx,
\end{aligned}$$

where q is an arbitrary number of $(1, +\infty)$ for $n = 2$, and it is equal to $\frac{2n}{n-2}$ for $n \geq 3$. In addition, C depends on q and n only.

By the just obtained inequality and by (4.8.37) we get

$$\int_{B_r(x_0)} u^2 dx \leq C \varepsilon^{1-\frac{2}{q}} \int_{B_{2r}(x_0)} u^2 dx, \quad r \in (0, r_\varepsilon], \quad (4.8.41)$$

where C is independent on r and ε .

Let $k \in \mathbb{N}$ be arbitrary and let $\varepsilon > 0$ satisfy

$$C \varepsilon^{1-\frac{2}{q}} = 2^{-k}.$$

We further let us denote by r_k the value of r_ε that corresponds to this choice of ε . Let us introduce the following function

$$f(r) = \int_{B_r(x_0)} u^2 dx, \quad r \in (0, 2r_k].$$

Then (4.8.41) can be written as

$$f(r) \leq 2^{-k} f(2r), \quad r \in (0, r_k]. \quad (4.8.42)$$

Now, for any $0 < r < r_k$, let $m \in \mathbb{N}$ satisfy

$$2^{-m} r_k \leq r < 2^{1-m} r_k. \quad (4.8.43)$$

Iteration of (4.8.42) gives

$$f(r) \leq 2^{-k} f(2r) \leq \dots \leq 2^{-km} f(2^m r) \leq 2^{-km} f(2r_k).$$

On the other hand (4.8.43) implies

$$2^{-m} \leq \frac{r}{r_k},$$

hence

$$f(r) \leq \left(\frac{r}{r_k} \right)^k f(2r_k),$$

which gives (4.8.38). ■

Remark 2. By the proof of Proposition 4.8.4 it is clear that assumption (4.8.37) can be replaced by the assumption that there is $p > 2$ such that

$$\left(\int_{B_r} |u|^p dx \right)^{1/p} \leq C \left(\int_{B_{2r}} |u|^2 dx \right)^{1/2}. \quad (4.8.44)$$

See also Section 16.4. ♦

Theorem 4.8.6. *Let Ω be a connected bounded open set of \mathbb{R}^n and let*

$$D \subset \Omega$$

be a Lebesgue measurable set of positive measure. Let u be a harmonic function in Ω which vanishes on D . Then

$$u \equiv 0. \quad (4.8.45)$$

Remark 3. As will become clear from the proof, the boundedness assumption of Ω is not essential: we have introduced it solely for the purpose of easing the proof. We leave the simple extension to the reader.



Proof of Theorem 4.8.6. Since

$$D = \bigcup_{j=1}^{\infty} (D \cap \Omega_j), \quad (4.8.46)$$

where

$$\Omega_j = \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/j\},$$

we have

$$0 < |D| = \lim_{j \rightarrow \infty} |D \cap \Omega_j|,$$

hence $|D \cap \Omega_j| > 0$ for j large enough, in addition we have $\overline{D \cap \Omega_j} \subset \Omega$. Hence, provided to replace $D \cap \Omega_j$ to D , we may always assume $\overline{D} \subset \Omega$.

Let us apply Proposition 4.8.4. Inequality (4.8.37) is nothing more than the Caccioppoli inequality proved in Theorem 4.5.1. Hence we have, for $x_0 \in \tilde{D}$ (\tilde{D} is defined by (4.8.35)),

$$\int_{B_r(x_0)} u^2 dx = \mathcal{O}(r^k), \quad \text{as } r \rightarrow 0, \quad \forall k \in \mathbb{N}. \quad (4.8.47)$$

Let now $x \in \Omega$ satisfy $|x - x_0| < \frac{1}{2} \text{dist}(x_0, \partial\Omega)$. Set $r = |x - x_0|$, by the Mean Property and by the Cauchy–Schwarz inequality we have

$$\begin{aligned} |u(x)| &= \left| \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy \right| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |u(y)| dy \leq \\ &\leq \frac{1}{|B_r(x)|} \int_{B_{2r}(x_0)} |u(y)| dy \leq \frac{|B_{2r}(x_0)|}{|B_r(x)|} \left(\int_{B_{2r}(x_0)} |u(y)|^2 dy \right)^{1/2} \leq \\ &\leq c_n r^{n/2} \left(\int_{B_{2r}(x_0)} |u(y)|^2 dy \right)^{1/2}, \end{aligned}$$

where c_n depends on n only. From what has just been obtained and from (4.8.47), recalling that $r = |x - x_0|$, we have

$$u(x) = \mathcal{O}(|x - x_0|^k), \quad \text{as } x \rightarrow x_0, \quad \forall k \in \mathbb{N}.$$

Therefore, as u is an analytic function the thesis follows. ■

Part II

CAUCHY PROBLEM FOR PDEs AND STABILITY ESTIMATES

Chapter 5

The Cauchy problem for the first order PDEs

5.1 Review of ordinary differential equations

In this Section we give, without proof, some results on ordinary differential equations that we will need later on. For further discussion we refer to [63].

Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $x_0 = (x_{0,1}, \dots, x_{0,n})$, $a > 0$, $b > 0$. Set

$$Q = \{(t, x) \in \mathbb{R}^{n+1} : |t - t_0| \leq a, |x_i - x_{0,i}| \leq b\}$$

and let

$$f : Q \rightarrow \mathbb{R}^n,$$

a **continuous function** in Q which is **Lipschitz continuous with respect to the variable x** , that is

$$|f(t, x) - f(t, y)| \leq L|x - y|, \quad \forall (t, x), (t, y) \in Q. \quad (5.1.1)$$

Let us consider the following Cauchy problem: determine the function $x(t)$ differentiable in a neighborhood of t_0 and satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases} \quad (5.1.2)$$

here \dot{x} is the derivative of x w.r.t. t . The following Theorem holds true

Theorem 5.1.1. *Let $f \in C^0(Q)$ satisfy (5.1.1). Then there exists $\delta > 0$ and there exists a unique solution $x \in C^1([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n)$ to problem (5.1.2).*

Moreover, setting

$$M_i = \max_Q |f_i|, \quad M = \max_{1 \leq i \leq n} M_i,$$

we can choose $\delta = \min \left\{ a, \frac{b}{M} \right\}$.

The following Lemma will be very useful

Lemma 5.1.2 (Gronwall). *Let I be an interval of \mathbb{R} , $\alpha \in I$ and $c \geq 0$. Moreover, let $u, v \in C^0(I, \mathbb{R})$ where $v(t) \geq 0$ and $u(t) \geq 0$, for every $t \in I$.*

What follows holds true.

(i) If

$$v(t) \leq c + \int_{\alpha}^t u(s)v(s)ds, \quad \forall t \geq \alpha,$$

then

$$v(t) \leq ce^{\int_{\alpha}^t u(s)ds}, \quad \forall t \geq \alpha.$$

(ii) If

$$v(t) \leq c + \int_t^{\alpha} u(s)v(s)ds, \quad \forall t \leq \alpha$$

then

$$v(t) \leq ce^{\int_t^{\alpha} u(s)ds}, \quad \forall t \leq \alpha.$$

The Gronwall Lemma makes it simple to prove the continuous dependence result of the solution to (5.1.2) by the data t_0, x_0, f . More precisely, we have

Theorem 5.1.3 (Continuous dependence by the data). *Let $f, \tilde{f} \in C^0(Q)$ satisfy (5.1.1). Let $\sigma_1, \sigma_2, \varepsilon$ be positive numbers. Let us suppose that*

$$|t_0 - \tilde{t}_0| \leq \sigma_1, \quad |x_0 - \tilde{x}_0| \leq \sigma_2, \quad \max_Q |f - \tilde{f}| \leq \varepsilon.$$

Set

$$M = \max_Q |f|, \quad \tilde{M} = \max_Q |\tilde{f}|.$$

Then the following fact occurs:

There exists $\sigma_0 > 0$ depending on a, b, M, \tilde{M} such that if $\sigma_1, \sigma_2 < \sigma_0$, then there is $\delta > 0$ and $x, \tilde{x} \in C^1([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$ that satisfy what follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(t_0) = x_0, \end{cases}$$

$$\begin{cases} \dot{\tilde{x}}(t) = f(t, \tilde{x}(t)), \\ \tilde{x}(t_0) = \tilde{x}_0 \end{cases}$$

and

$$|x(t) - \tilde{x}(t)| \leq C(\sigma_1 + \sigma_2 + \varepsilon), \quad \forall t \in [t_0 - \delta, t_0 + \delta],$$

where C is a constant that depends on a, b, L, M, \tilde{M} only.

Theorem 5.1.4 (regularity). *Let $k \in \mathbb{N}$ and $f \in C^k(Q)$. Then the solution to Cauchy problem (5.1.2) belongs to $C^{k+1}([t_0 - \delta, t_0 + \delta]; \mathbb{R}^n)$, where $\delta = \min\{a, \frac{b}{M}\}$.*

We now describe the Theorem of **differentiability of the solution** of the Cauchy problem with respect to a parameter and with respect to the initial values. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 < \lambda_2$ and let

$$\tilde{Q} = \{(t, x; \lambda) \in \mathbb{R}^{n+2} : |t - t_0| \leq a, |x_i - x_{0,i}| \leq b, \lambda_1 \leq \lambda \leq \lambda_2\}.$$

Moreover, let $f \in C^1(\tilde{Q}, \mathbb{R}^n)$. If $(t_0, x_0, \bar{\lambda}) \in \tilde{Q}$ then there exists a unique solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x(t); \bar{\lambda}), \\ x(t_0) = x_0. \end{cases} \quad (5.1.3)$$

Let us denote by

$$\bar{x}(t, t_0, x_0; \bar{\lambda})$$

this solution. It can be proved (and for this we refer to [63, Ch. 1]) that \bar{x} is differentiable with respect to all the variables and the derivatives are continuous. In order to calculate the derivatives

$$\frac{\partial \bar{x}}{\partial t_0}, \quad \frac{\partial \bar{x}}{\partial x_{0,j}}, \quad \frac{\partial \bar{x}}{\partial \bar{\lambda}}, \quad j = 1, \dots, n,$$

we proceed in the following way: we write the system (5.1.3) in the form

$$\begin{cases} \frac{\partial}{\partial t} \bar{x}(t, t_0, x_0; \bar{\lambda}) = f(t, \bar{x}(t, t_0, x_0; \bar{\lambda}); \bar{\lambda}) \\ \bar{x}(t_0, t_0, x_0; \bar{\lambda}) = x_0, \end{cases} \quad (5.1.4)$$

next, we make the derivatives of both the sides of (5.1.4) obtaining a linear first-order system with Cauchy conditions in the "new unknowns"

$$\frac{\partial \bar{x}}{\partial t_0}, \quad \frac{\partial \bar{x}}{\partial x_{0,k}}, \quad \frac{\partial \bar{x}}{\partial \bar{\lambda}}.$$

For instance, in case $n = 1$, whether we are interested in calculating $\frac{\partial \bar{x}}{\partial t_0}$, we make the derivatives of both the sides of equation (5.1.4) with respect to t_0 and we get

$$\frac{\partial}{\partial t_0} \frac{\partial}{\partial t} \bar{x}(t, t_0, x_0; \bar{\lambda}) = \frac{\partial f}{\partial x}(t, \bar{x}(t, t_0, x_0; \bar{\lambda}); \bar{\lambda}) \frac{\partial}{\partial t_0} \bar{x}(t, t_0, x_0; \bar{\lambda})$$

and by the initial datum, making the derivative with respect to t_0 , we have

$$\left(\frac{\partial}{\partial t_0} \bar{x}(t, t_0, x_0; \bar{\lambda}) + \frac{\partial}{\partial t} \bar{x}(t, t_0, x_0; \bar{\lambda}) \right)_{|t=t_0} = 0.$$

From which, taking into account (5.1.4), we have

$$\frac{\partial}{\partial t_0} \bar{x}(t_0, t_0, x_0; \bar{\lambda}) = -f(t_0, x_0; \bar{\lambda}).$$

Now, set

$$U(t, t_0, x_0; \bar{\lambda}) = \frac{\partial}{\partial t_0} \bar{x}(t, t_0, x_0; \bar{\lambda}), \quad (5.1.5)$$

$$A(t, t_0, x_0; \bar{\lambda}) = \frac{\partial f}{\partial x}(t, \bar{x}(t, t_0, x_0; \bar{\lambda}); \bar{\lambda})$$

and interchanging the order of derivatives $\frac{\partial}{\partial t_0} \frac{\partial}{\partial t} \bar{x}$ (in the rigorous proof it is proved that this step is admissible, compare [63, Cap. 1]) we have

$$\begin{cases} \frac{\partial U}{\partial t} = A(t, t_0, x_0; \bar{\lambda}) U, \\ U(t_0, t_0, x_0; \bar{\lambda}) = -f(t_0, x_0; \bar{\lambda}), \end{cases}$$

that is a Cauchy problem for an ordinary differential equation in the new unknown U . When U is determined, also $\frac{\partial}{\partial t_0} \bar{x}$ turns out determined by (5.1.5). Similarly we proceed when $n > 1$ and for the others derivatives.

Likewise, if $f \in C^k(\tilde{Q}, \mathbb{R}^n)$, it can be proved that $\bar{x}(\cdot, t_0, x_0; \bar{\lambda})$ has continuous derivatives w.r.t. $t_0, x_{0,j}$ and $\bar{\lambda}$ up to order k .

5.2 First order linear PDEs

We begin by giving some definitions. Let Ω be a connected open set of \mathbb{R}^n and let

$$a : \Omega \rightarrow \mathbb{R}^n, \quad a(x) = (a_1(x), \dots, a_n(x)),$$

be a vector field on Ω . Let us denote by $P(x, \partial)$ the following linear differential operator

$$P(x, \partial) = a \cdot \nabla = \sum_{j=1}^n a_j(x) \partial_j, \quad x \in \Omega. \quad (5.2.1)$$

Throughout this Chapter we will call the **symbol of the operator** $P(x, \partial)$ the following homogeneous polynomial of first degree w.r.t. the variables ξ_1, \dots, ξ_n

$$P(x, \xi) = a(x) \cdot \xi = \sum_{j=1}^n a_j(x) \xi_j, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n. \quad (5.2.2)$$

We say that $\xi \in \mathbb{R}^n \setminus \{0\}$ is a **characteristic direction for $P(x, \partial)$ at the point $x_0 \in \Omega$** if

$$P(x_0, \xi) = 0. \quad (5.2.3)$$

A surface

$$\Gamma = \{x \in \Omega : \phi(x) = \phi(x_0)\},$$

where $\phi \in C^1(\Omega)$ and

$$\nabla \phi(x_0) \neq 0$$

is said a **characteristic surface at the point $x_0 \in \Omega$ for $P(x, \partial)$** if $\nabla \phi(x_0)$ is a characteristic direction for $P(x, \partial)$ at the point x_0 . That is

$$P(x_0, \nabla \phi(x_0)) = a \cdot \nabla \phi(x_0) = \sum_{j=1}^n a_j(x_0) \partial_j \phi(x_0) = 0, \quad (5.2.4)$$

let us note

$$P(x, \nabla \phi) = P(x, \partial) \phi.$$

We say that Γ is a characteristic surface for $P(x, \partial)$ if

$$P(x, \nabla \phi(x)) = P(x, \partial) \phi(x) = a(x) \cdot \nabla \phi(x) = 0, \quad \forall x \in \Gamma. \quad (5.2.5)$$

We say that the vectors

$$\nu(x_0) = -\frac{\nabla \phi(x_0)}{|\nabla \phi(x_0)|} \quad \text{and} \quad -\nu(x_0) = \frac{\nabla \phi(x_0)}{|\nabla \phi(x_0)|}$$

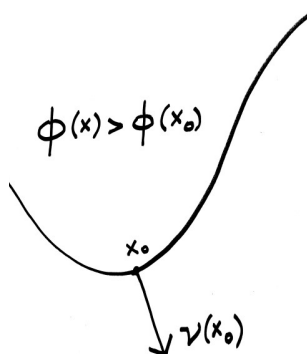


Figure 5.1:

Let us note that the versor $\nu(x_0)$ is directed toward the region $\{x \in \Omega : \phi(x) < 0\}$. We agree to say that $\nu(x_0)$ is the *unit outward normal* to Γ in x_0 and $-\nu(x_0)$ is the *inner outward normal* to Γ in x_0 , respectively.

Notice that Γ is a characteristic surface at x_0 for $P(x, \partial)$ if and only if

$$P(x_0, \nu(x_0)) = a(x_0) \cdot \nabla \nu(x_0) = 0, \quad (5.2.6)$$

In other words, Γ is a characteristic surface at x_0 if and only if $a(x_0)$ is a tangent vector to Γ at x_0 .

Of course, when $n = 2$, instead of the characteristic surfaces we will simply speak of the characteristic lines (or curves).

In this Section we will study the following **Cauchy problem** for the **first order linear differential equation**. Given the vector field $a(\cdot)$, and the function $h : \Gamma \rightarrow \mathbb{R}$ determine u such that

$$\begin{cases} P(x, \partial)u = c(x)u + f(x), \\ u|_{\Gamma} = h, \end{cases} \quad (5.2.7)$$

We will specify the assumptions on a, c, f, h in a while. Let us now premise

some simple example to the investigation of problem (5.2.7). We call Γ the **initial surface** and h the **initial datum** of Cauchy problem (5.2.7).

Example 1.

Let $\Omega = \mathbb{R}^2$ and let

$$P(x, y, \partial) = \partial_x + \partial_y.$$

Let us consider the equation

$$\partial_x u + \partial_y u = 0, \quad \text{in } \mathbb{R}^2. \quad (5.2.8)$$

We can easily determine all the $C^1(\mathbb{R}^2)$ solutions to equation (5.2.8). Actually, setting

$$\mu = (1, 1),$$

we can write (5.2.8)

$$\frac{\partial u}{\partial \mu} = 0, \quad \text{in } \mathbb{R}^2, \quad (5.2.9)$$

where $\frac{\partial u}{\partial \mu}$ denotes the derivatives of u w.r.t. direction μ (defined in Section 1.1).

It is clear that the functions $u \in C^1(\mathbb{R}^2)$ which satisfy (5.2.9) are all and only the functions constant on the lines parallel to the vector μ . Hence, for any fixed x_0 , we have

$$u(x_0 + t, t) = F(x_0), \quad t \in \mathbb{R}$$

from which by eliminating t , we have

$$u(x, y) = F(x - y). \quad (5.2.10)$$

Therefore it suffices to assume that $F \in C^1(\mathbb{R})$ for obtaining, by (5.2.10), all the solutions to (5.2.8).

Having (5.2.10) available, the study of the Cauchy problem for equation (5.2.8) is quite simple, and here we take the opportunity to highlight some important facts.

Let us consider the following Cauchy problem

$$\begin{cases} u_x + u_y = 0, \\ u(x, 0) = h(x), \end{cases} \quad (5.2.11)$$

where $h \in C^1(\mathbb{R})$. By (5.2.10), taking into account of the initial datum in (5.2.11), we have

$$h(x) = u(x, 0) = F(x), \quad x \in \mathbb{R}.$$

Therefore, the unique solution of problem (5.2.11) is given by

$$u(x, y) = h(x - y), \quad (x, y) \in \mathbb{R}^2. \quad (5.2.12)$$

Let us now consider a somewhat more general situation and let us assume that the Cauchy datum is assigned on a regular curve Γ of parametric equations

$$x = \bar{x}(\tau), \quad y = \bar{y}(\tau), \quad \tau \in I, \quad (5.2.13)$$

where I is an interval. Let us examine what happens when Γ is a **characteristic line**. We therefore consider the problem

$$\begin{cases} u_x + u_y = 0, \\ u(\tau, \tau) = h(\tau), \quad \tau \in \mathbb{R}. \end{cases} \quad (5.2.14)$$

We immediately realize that if h is not constant, the problem (5.2.14) has no solutions: we had, indeed, already observed that every $C^1(\mathbb{R}^2)$ which is a solution of the equation $u_x + u_y = 0$ must be constant on the lines parallel to the vector $\mu = (1, 1)$ and therefore, in particular, they must be constant on the line

$$\{(\tau, \tau) : \tau \in \mathbb{R}\}.$$

Moreover, if h is constant then Cauchy problem (5.2.14) has **infinite solutions** as we easily, if $h \equiv 0$ then every solutions to (5.2.14) is given by

$$u(x, y) = F(x - y), \quad \text{with } F(0) = 0.$$

More generally, if we have to face the Cauchy problem

$$\begin{cases} u_x + u_y = 0, \\ u(\bar{x}(\tau), \bar{y}(\tau)) = h(\tau), \quad \tau \in \mathbb{R} \end{cases} \quad (5.2.15)$$

and if a characteristic line intersects Γ at two distinct, say $P_0 = (\bar{x}(\tau_0), \bar{y}(\tau_0))$ and $P_1 = (\bar{x}(\tau_1), \bar{y}(\tau_1))$ where $\tau_0 \neq \tau_1$, then in order that problem (5.2.15) has solution, it is necessary that h satisfies the condition

$$h(\tau_0) = h(\tau_1)$$

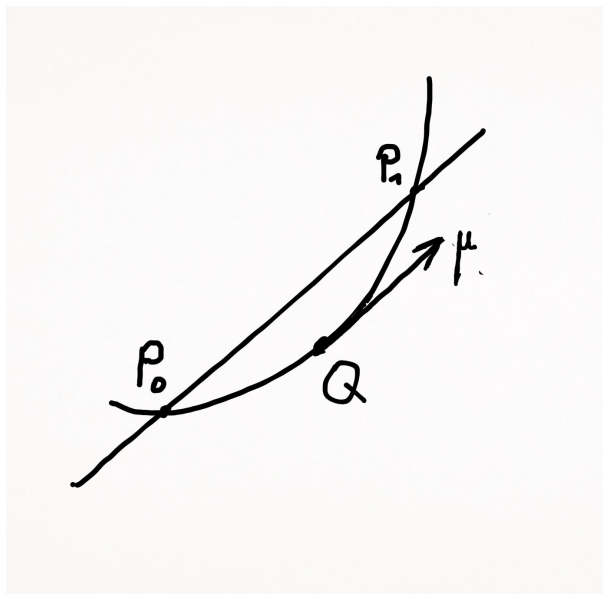


Figure 5.2:

and this imposes, in turn, some restrictions on the datum h itself. In other words, **not every initial value is admissible for Cauchy problem** (5.2.15). This is because the values of u on Γ are determined by the values of u on a portion smaller of Γ itself. Keep in mind that in the situation we have just considered, between the points P_0 and P_1 there must be a point $Q \in \Gamma$ at which the direction characteristic is tangent to Γ . That is, Γ characteristic line w.r.t. the operator $\partial_x + \partial_y$ at the point Q (Figure 5.2).

Let us further illustrate what has just been said. Let it be, then, (Figure 5.3)

$$\Gamma = \left\{ \left(\tau, \frac{\tau^2}{2} \right) : 0 < \tau < 2 \right\}$$

and let us consider the following Cauchy problem

$$\begin{cases} u_x + u_y = 0, \\ u\left(\tau, \frac{\tau^2}{2}\right) = h(\tau), \quad \tau \in (0, 2). \end{cases} \quad (5.2.16)$$

Notice that Γ is a characteristic line w.r.t. the operator $\partial_x + \partial_y$ at the point $Q = \left(1, \frac{1}{2}\right)$. Moreover, we check what follows.

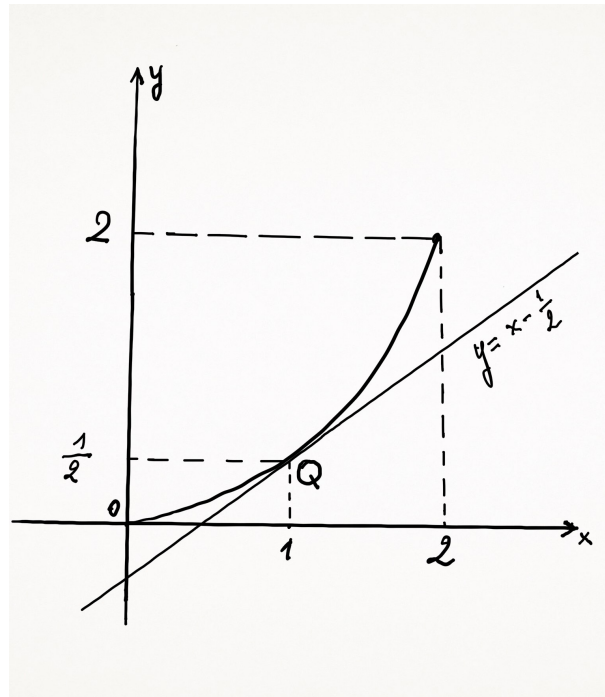


Figure 5.3:

(i) There exists a solution $u \in C^1(\mathbb{R}^2)$ to problem (5.2.16) if and only if $h \in C^1((0, 2))$ and

$$h(x) = h(2 - x), \quad \forall x \in (0, 2). \quad (5.2.17)$$

(ii) If $h \in C^1(0, 2)$ satisfies (5.2.17), then for any $0 < r < 1$, there exist infinite solutions to the Cauchy problem

$$\begin{cases} u_x + u_y = 0, & \text{in } B_r(Q), \\ u|_{\Gamma \cap B_r(Q)} = h. \end{cases} \quad (5.2.18)$$

Let us check (i). By (5.2.10) we have $u(x, y) = F(x - y)$. Consequently, in order to

$$u\left(x, \frac{x^2}{2}\right) = h(x),$$

we need to have

$$h(x) = F\left(x - \frac{x^2}{2}\right) = F\left(\frac{1}{2} - \frac{1}{2}(1 - x)^2\right), \quad \forall x \in (0, 2),$$

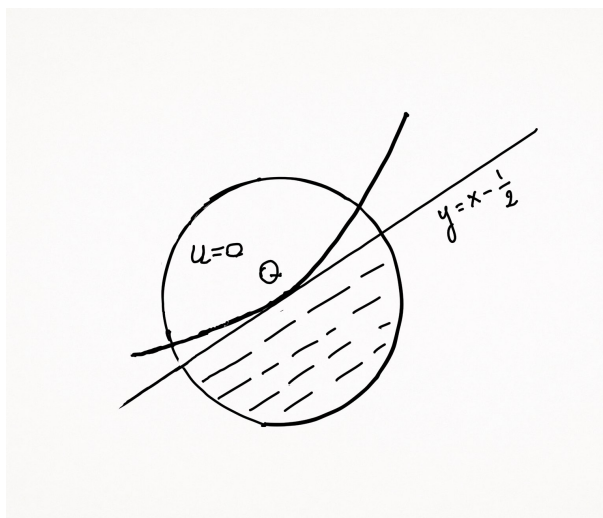


Figure 5.4:

from which (5.2.17) follows.

Let us check (ii). By the linearity of problem (5.2.18), we can choose $h \equiv 0$. Moreover, let $g \in C^1((0, 2))$ be an arbitrary function which satisfies

$$g\left(\frac{1}{2}\right) = g'\left(\frac{1}{2}\right) = 0.$$

Then it is easily checked that the functions

$$u(x, y) = \begin{cases} 0, & \text{in } B_r(Q) \cap \{y \geq x - \frac{1}{2}\}, \\ g(x - y), & \text{in } B_r(Q) \cap \{y < x - \frac{1}{2}\}, \end{cases} \quad (5.2.19)$$

are all solutions of Cauchy problem (Figure 5.4)

$$\begin{cases} u_x + u_y = 0, & \text{in } B_r(Q), \\ u|_{\Gamma \cap B_r(Q)} = 0. \end{cases} \quad (5.2.20)$$

5.3 The method of characteristics – the linear case

Let Ω be a connected open set of \mathbb{R}^n and let $a \in C^1(\Omega, \mathbb{R}^n)$,

$$a(x) = (a_1(x), \dots, a_n(x)).$$

Let us consider the operator

$$P(x, \partial) = \sum_{j=1}^n a_j(x) \partial_j, \quad x \in \Omega. \quad (5.3.1)$$

We will call **characteristic line** of $P(x, \partial)$ any solution to the system of ordinary differential equations – **characteristic equations**

$$\frac{dX(t)}{dt} = a(X(t)). \quad (5.3.2)$$

Let $u \in C^1(\Omega)$. Observe that, if we set

$$z(t) = u(X(t)),$$

then we have

$$\frac{dz(t)}{dt} = \frac{dX(t)}{dt} \cdot (\nabla u)(X(t)) = a(X(t)) \cdot (\nabla u)(X(t)). \quad (5.3.3)$$

This simple relationship is the starting point of the method of characteristics by which we will tackle and we will solve the Cauchy problem

$$\begin{cases} P(x, \partial)u = c(x)u + f(x), \\ u|_{\Gamma} = h, \end{cases} \quad (5.3.4)$$

where Γ is a portion of regular surface of parametric equations

$$x = \bar{x}(y), \quad y \in B'_1, \quad (5.3.5)$$

$\bar{x} \in C^1(B'_1)$ and

$$\text{Rank} \left(\frac{\partial \bar{x}}{\partial y}(y) \right) = \text{Rank} \begin{pmatrix} \partial_{y_1} \bar{x}_1 & \cdots & \partial_{y_{n-1}} \bar{x}_1 \\ \vdots & \cdots & \vdots \\ \partial_{y_1} \bar{x}_n & \cdots & \partial_{y_{n-1}} \bar{x}_n \end{pmatrix} = n - 1. \quad (5.3.6)$$

Moreover, we assume

$$c \in C^1(\Omega), \quad f \in C^1(\Omega). \quad (5.3.7)$$

The initial condition $u_\Gamma = h$ is expressed by

$$u(\bar{x}(y)) = h(\bar{x}(y)) := \bar{h}(y), \quad \forall y \in B'_1, \quad (5.3.8)$$

where

$$\bar{h} \in C^1(B'_1). \quad (5.3.9)$$

The **method of characteristics** consists of constructing a local change of coordinates of class C^1 ,

$$(-\delta, \delta) \times B'_r \ni (t, y) \rightarrow X(t, y) \in \mathbb{R}^n, \quad (5.3.10)$$

for suitable $\delta > 0$ and $r \in (0, 1)$. Where X has the following properties

$$X(0, y) = \bar{x}(y), \quad \forall y \in B'_r \quad (5.3.11)$$

and

$$\partial_t X(t, y) = a(X(t, y)), \quad \forall (t, y) \in (-\delta, \delta) \times B'_r. \quad (5.3.12)$$

In this way, setting

$$z(t, y) = u(X(t, y)), \quad C(t, y) = c(X(t, y)), \quad F(t, y) = f(X(t, y))$$

and taking into account (5.3.3), Cauchy problem (5.3.4) becomes

$$\begin{cases} \partial_t z(t, y) = C(t, y)z(t, y) + F(t, y), \\ z(0, y) = \bar{h}(y). \end{cases} \quad (5.3.13)$$

By the assumptions made on c , f , and h , see Section 5.1, it turns out that $z \in C^1((-\delta, \delta) \times B'_r)$ and, as we will see, that the function (under appropriate conditions)

$$u(x) = z(X^{-1}(x)) \quad (5.3.14)$$

is a solution to (5.3.4).

Let us begin to see under what conditions, the transformation defined by (5.3.10)–(5.3.12) is a diffeomorphism in a neighborhood of $0 \in \mathbb{R}^n$. Since,

$a \in C^1(\Omega)$ and $\bar{x} \in C^1(B'_1)$ we have (for the differentiability w. r. t. the parameters, see Section 5.1)

$$X \in C^1(J \times B'_1),$$

where J is a suitable neighborhood of 0.

Moreover

$$\begin{aligned} \frac{\partial X}{\partial(t, y)}(0, 0) &= \left(\underbrace{\partial_t X(0, 0), \partial_{y_1} X(0, 0), \dots, \partial_{y_{n-1}} X(0, 0)}_{\text{column vectors}} \right) = \\ &= (a(x_0), \partial_{y_1} \bar{x}(0), \dots, \partial_{y_{n-1}} \bar{x}(0)) \end{aligned}$$

and, since

$$\partial_t X(0, 0) = a(X(0, 0)) = a(x_0),$$

$$\text{Rank}(a(x_0), \partial_{y_1} \bar{x}(0), \dots, \partial_{y_{n-1}} \bar{x}(0)) = n - 1,$$

we have that the following conditions are equivalent

$$\text{Rank} \frac{\partial X}{\partial(t, y)}(0, 0) = n \quad (5.3.15)$$

and

$$a(x_0) \notin \langle \partial_{y_1} \bar{x}(0), \dots, \partial_{y_{n-1}} \bar{x}(0) \rangle, \quad (5.3.16)$$

where $\langle v_1, v_2, \dots, v_{n-1} \rangle$ is the vector space generated by v_1, v_2, \dots, v_{n-1} . Condition (5.3.16) is, in turn, equivalent to the condition that $a(x_0)$ is **not tangent to Γ in x_0** .

All in all, **if Γ is noncharacteristic in x_0 for operator (5.3.1)** then there exists $\delta > 0$ and $r \in (0, 1)$ such that X is a diffeomorphism in $(-\delta, \delta) \times B'_r$.

Now, we denote by $\mathcal{U}_{x_0} = X((-\delta, \delta) \times B'_r)$ and by

$$\Psi(x) = X^{-1}(x), \quad x \in \mathcal{U}_{x_0}. \quad (5.3.17)$$

Let us check that

$$u(x) = z(\Psi(x))$$

solves Cauchy problem (5.3.4).

Regarding the **initial condition**, by (5.3.11) we have immediately

$$\bar{x}(y) = X(0, y), \quad \forall y \in B'_r.$$

Hence, by (5.3.17) and recalling (5.3.13), we get

$$u(\bar{x}(y)) = z(\Psi(X(0, y))) = z(0, y) = \bar{h}(y), \quad \forall y \in B'_r. \quad (5.3.18)$$

Concerning the equation

$$\sum_{j=1}^n a_j(x) \partial_j u = c(x)u + f(x),$$

recall that by (5.3.17) we have

$$\left(\frac{\partial \Psi(x)}{\partial x} \right) \left(\frac{\partial X(\Psi(x))}{\partial(t, y)} \right) = I_n, \quad (5.3.19)$$

where I_n is the identity matrix $n \times n$. In particular, considering the first column on the right-hand side and the first column on the left-hand side of (5.3.19), we have

$$\begin{cases} \sum_{j=1}^n \partial_{x_j} \Psi_1 \partial_t X_j = 1, \\ \sum_{j=1}^n \partial_{x_j} \Psi_k \partial_t X_j = 0, \quad k = 2, \dots, n. \end{cases} \quad (5.3.20)$$

Now

$$\begin{aligned} \partial_{x_j} u(x) &= \partial_t z(\Psi(x)) \partial_{x_j} \Psi_1(x) + \partial_{y_1} z(\Psi(x)) \partial_{x_j} \Psi_2(x) + \\ &\quad \dots + \partial_{y_{n-1}} z(\Psi(x)) \partial_{x_j} \Psi_{n-1}(x). \end{aligned}$$

Hence (multiplying by $a_j(x)$ and summing up on j)

$$\begin{aligned} \sum_{j=1}^n a_j(x) \partial_{x_j} u &= \partial_t z(\Psi(x)) \sum_{j=1}^n \partial_{x_j} \Psi_1 a_j(x) + \\ &\quad + \partial_{y_1} z(\Psi(x)) \sum_{j=1}^n \partial_{x_j} \Psi_2 a_j(x) + \dots \\ &\quad + \partial_{y_{n-1}} z(\Psi(x)) \sum_{j=1}^n \partial_{x_j} \Psi_{n-1}(x) a_j(x). \end{aligned} \quad (5.3.21)$$

On the other hand by (5.3.12) we know

$$a_j(x) = (\partial_t X_j)(\Psi(x)).$$

By this equality, by (5.3.20) and by the equation in (5.3.13) we have

$$\sum_{j=1}^n a_j(x) \partial_{x_j} u(x) = \partial_t z(\Psi(x)) = c(x)u(x) + f(x).$$

Hence u is solution to Cauchy problem (5.3.4).

Finally, we observe that, by the hypotheses (5.3.7) (actually, it suffices $c \in C^0(\Omega)$), u is **the unique solution** of class C^1 to problem (5.3.4) in the neighborhood \mathcal{U}_{x_0} . Indeed, if $u_1, u_2 \in C^1(\mathcal{U}_{x_0})$ are two solutions then, setting

$$w = u_1 - u_2,$$

we have

$$\begin{cases} P(x, \partial)w = c(x)w, \\ w|_{\Gamma} = 0, \end{cases}$$

and setting

$$\tilde{z}(t, y) = w(X(t, y)),$$

by (5.3.13) we have

$$\begin{cases} \partial_t \tilde{z}(t, y) = C(t, y)\tilde{z}(t, y), \\ \tilde{z}(0, y) = 0. \end{cases}$$

From which we have $\tilde{z} = 0$ in $(-\delta, \delta) \times B'_r$, therefore $w = 0$ in \mathcal{U}_{x_0} .

The construction we have illustrated and the local uniqueness hold for any point of Γ .

Hence we have proved

Theorem 5.3.1. *Let $a \in C^1(\Omega, \mathbb{R}^n)$, $c \in C^1(\Omega)$ and $f \in C^1(\Omega)$. Let Γ be a non characteristic surface of parametric equations $x = \bar{x}(y)$, where $\bar{x} \in C^1(B'_1)$ and satisfying (5.3.6). Let h be a function C^1 on Γ (i.e. $h \circ \bar{x} \in C^1(B'_1)$).*

Then there exists a neighborhood \mathcal{U} of Γ such that there exists unique solution u in $C^1(\mathcal{U})$ to the Cauchy problem

$$\begin{cases} \sum_{j=1}^n a_j(x) \partial_j u = c(x)u + f(x), & x \in \mathcal{U}, \\ u|_{\Gamma} = h. \end{cases} \quad (5.3.22)$$

Given a surface Γ in \mathbb{R}^n we call **domain of dependence** of Γ with respect to the equation

$$\sum_{j=1}^n a_j(x) \partial_j u = c(x)u,$$

the largest closed set D_Γ for which we have

$$\begin{cases} \sum_{j=1}^n a_j(x) \partial_j u = c(x)u, & x \in D_\Gamma, \\ u|_\Gamma = 0, \end{cases} \implies u = 0 \quad \text{in } D_\Gamma.$$

Exercise 1. Let $0 < r < 1$ and

$$\Gamma = \left\{ (x', -\sqrt{1 - |x'|^2}) : |x'| < r \right\}.$$

Construct a vector field $a \in C^1(\overline{B'_1})$ such that the domain of dependence of Γ with respect to the equation

$$a(x) \cdot \nabla u = 0,$$

contains $\overline{B_1}$. ♣

Exercise 2. Apply the characteristic method to prove that the functions $u \in C^1(\mathbb{R}^n \setminus \{0\})$ which satisfy

$$\sum_{j=1}^n x_j \partial_j u = \alpha u,$$

are the homogeneous function of degree α . ♣

Exercise 3. Let b be a vector of \mathbb{R}^n and let $f \in C^0(\mathbb{R}^{n+1})$. Apply the characteristic method to solve the following Cauchy problem

$$\begin{cases} \partial_t u + b \cdot \nabla u = f(x, t), \\ u(x, 0) = 0. \end{cases}$$

The equation $\partial_t u + b \cdot \nabla u = f(x, t)$ is known as the **transport equation**.

♣

5.4 The method of characteristics – quasilinear case

Let J be an open interval of \mathbb{R} , let Ω be a connected open set of \mathbb{R}^n . Let $a \in C^1(J \times \Omega, \mathbb{R}^n)$ and $c \in C^1(J \times \Omega)$. The following equation

$$a(x, u) \cdot \nabla u = c(x, u), \quad (5.4.1)$$

is called a **first-order quasilinear equation**. Of course, a linear equations are special case of the quasilinear equations.

With minor modifications, the characteristics method studied in the Section 5.3 can be adapted to handle equation (5.4.1) and the related Cauchy problem. In the case of equation (5.4.1), the characteristic equation (5.4.1) is the following one

$$\begin{cases} \frac{dX}{dt}(t) = a(X(t), z(t)), \\ \frac{dz}{dt}(t) = c(X(t), z(t)). \end{cases} \quad (5.4.2)$$

Let us note that in the linear case, the equation

$$\frac{dz}{dt}(t) = c(X(t), z(t)),$$

is precisely the one satisfied by $z(t) = u(X(t))$ when u is a solution of the linear equation

$$a(x) \cdot \nabla u = c(x)u + f(x).$$

We continue to call **characteristic line**, the curve of parametric equations

$$(X, z) = (X(t), z(t)) \quad (5.4.3)$$

where $X(t), z(t)$ is a solution of the system (5.4.2). When there is no risk of ambiguity, we will call "characteristic line" also the projection on \mathbb{R}^n of the line (5.4.3).

Let us consider the Cauchy problem

$$\begin{cases} a(x, u) \cdot \nabla u = c(x, u), \\ u|_{\Gamma} = h, \end{cases} \quad (5.4.4)$$

where Γ is a portion of regular surface of parametric equations

$$x = \bar{x}(y), \quad \forall y \in B'_1.$$

To solve (5.4.4), we proceed similarly to what we did in the linear case. Namely, we consider $X(t, y)$ and $z(t, y)$ such that

$$\begin{cases} \partial_t X(t, y) = a(X(t, y), z(t, y)), \\ \partial_t z(t, y) = c(X(t, y), z(t, y)), \\ X(0, y) = \bar{x}(y), \\ z(0, y) = h(y) \end{cases} \quad (5.4.5)$$

and it can be checked, exactly as in Section 5.3 that if

$$(-\delta, \delta) \times B'_r \ni (t, y) \rightarrow X(t, y) \in \mathbb{R}^n,$$

for some $\delta > 0$ and $r \in (0, 1)$, is local change of coordinates of \mathbb{R}^n , then the function

$$u(x) := z(X^{-1}(x)), \quad (5.4.6)$$

is a solution to Cauchy problem (5.4.4). More precisely: setting $x_0 = \bar{x}(0)$ there exists a neighborhood \mathcal{U}_{x_0} such that the function u defined by (5.4.6) satisfies

$$\begin{cases} a(x, u) \cdot \nabla u = c(x, u), & \text{in } \mathcal{U}_{x_0}, \\ u|_{\Gamma \cap \mathcal{U}_{x_0}} = h. \end{cases} \quad (5.4.7)$$

In order to the map X be a local diffeomorphism, it suffices to have

$$\text{Rank} \frac{\partial X}{\partial(t, y)}(0, 0) = n \quad (5.4.8)$$

and since

$$\begin{aligned} \frac{\partial X}{\partial(t, y)}(0, 0) &= (\partial_t X(0, 0), \partial_{y_1} X(0, 0), \dots, \partial_{y_{n-1}} X(0, 0)) = \\ &= (a(x_0, h(x_0)), \partial_{y_1} \bar{x}(0), \dots, \partial_{y_{n-1}} \bar{x}(0)) \end{aligned}$$

and

$$\text{Rank} \left(\frac{\partial \bar{x}}{\partial y}(0) \right) = n - 1,$$

condition (5.4.8) is equivalent to

$$a(x_0, h(x_0)) \notin \langle \partial_{y_1} \bar{x}(0), \dots, \partial_{y_{n-1}} \bar{x}(0) \rangle,$$

(compare this condition with (5.3.16)).

Now we briefly consider the issue of continuous dependence by initial datum in problem (5.4.4).

Let u_k , $k = 1, 2$, satisfy

$$\begin{cases} a(x, u_k) \cdot \nabla u_k = c(x, u_k), \\ u_{k|\Gamma} = h_k. \end{cases} \quad (5.4.9)$$

We have

$$c(x, u_1) - c(x, u_2) = (u_1 - u_2) \int_0^1 \partial_u c(x, u_2(x) + t(u_1(x) - u_2(x))) dt$$

and, similarly

$$a(x, u_1) \cdot \nabla u_1 - a(x, u_2) \cdot \nabla u_2 = \bar{a}(x) \cdot \nabla (u_1 - u_2) + b(x),$$

where

$$\bar{a}(x) = a(x, u_1(x))$$

and

$$b(x) = -\nabla u_2(x) \cdot \int_0^1 \partial_u a(x, u_2(x) + t(u_1(x) - u_2(x))) dt.$$

Set

$$\bar{c}(x) = b(x) + \int_0^1 \partial_u c(x, u_2(x) + t(u_1(x) - u_2(x))) dt,$$

$$w = u_1 - u_2,$$

$$\bar{h} = h_1 - h_2.$$

By (5.4.9) we have

$$\begin{cases} \bar{a}(x) \cdot \nabla w = \bar{c}(x)w, \\ w|_{\Gamma} = \bar{h}. \end{cases} \quad (5.4.10)$$

Now, if $\bar{h} \equiv 0$ and if at a point $x_0 \in \Gamma$, then $a(x_0, h_1(x_0))$ is not tangent to Γ , then there exists a neighborhood \mathcal{U}_{x_0} of x_0 such that $w \equiv 0$ in \mathcal{U}_{x_0} , that is

$$u_1 \equiv u_2, \quad \text{in } \mathcal{U}_{x_0}.$$

As a matter of fact, the equation

$$\bar{a}(x) \cdot \nabla w = \bar{c}(x)w$$

is linear and Theorem 5.3.1 applies.

We leave as an exercise to the reader to prove that if $a(x_0, h_1(x_0))$ is not tangent to Γ in x_0 then there exists a neighborhood \mathcal{V}_{x_0} such that

$$\|w\|_{L^\infty(\mathcal{V}_{x_0})} \leq K \|\bar{h}\|_{L^\infty(\Gamma \cap \mathcal{V}_{x_0})}, \quad (5.4.11)$$

that is

$$\|u_1 - u_2\|_{L^\infty(\mathcal{V}_{x_0})} \leq K \|h_1 - h_2\|_{L^\infty(\Gamma \cap \mathcal{V}_{x_0})},$$

where K is a constant which depends on C^1 norm of a and c and on the (convex) angle between the vector $a(x_0, h_1(x_0))$ and the unit outward normal to Γ in x_0 .

We conclude this Section by the following

Example. Let us consider the following Cauchy problem

$$\begin{cases} u_y + uu_x = 0, \\ u(x, 0) = h(x), \quad x \in \mathbb{R}, \end{cases} \quad (5.4.12)$$

where $h \in C^1(\mathbb{R})$.

The characteristic equations are given by

$$\begin{cases} \frac{\partial x(t,s)}{\partial t} = z, \\ \frac{\partial y(t,s)}{\partial t} = 1, \\ \frac{\partial z(t,s)}{\partial t} = 0, \end{cases} \quad (5.4.13)$$

the initial conditions are

$$x(0, s) = s, \quad y(0, s) = 0 \quad z(0, s) = h(s). \quad (5.4.14)$$

By (5.4.13) and (5.4.14) we have easily

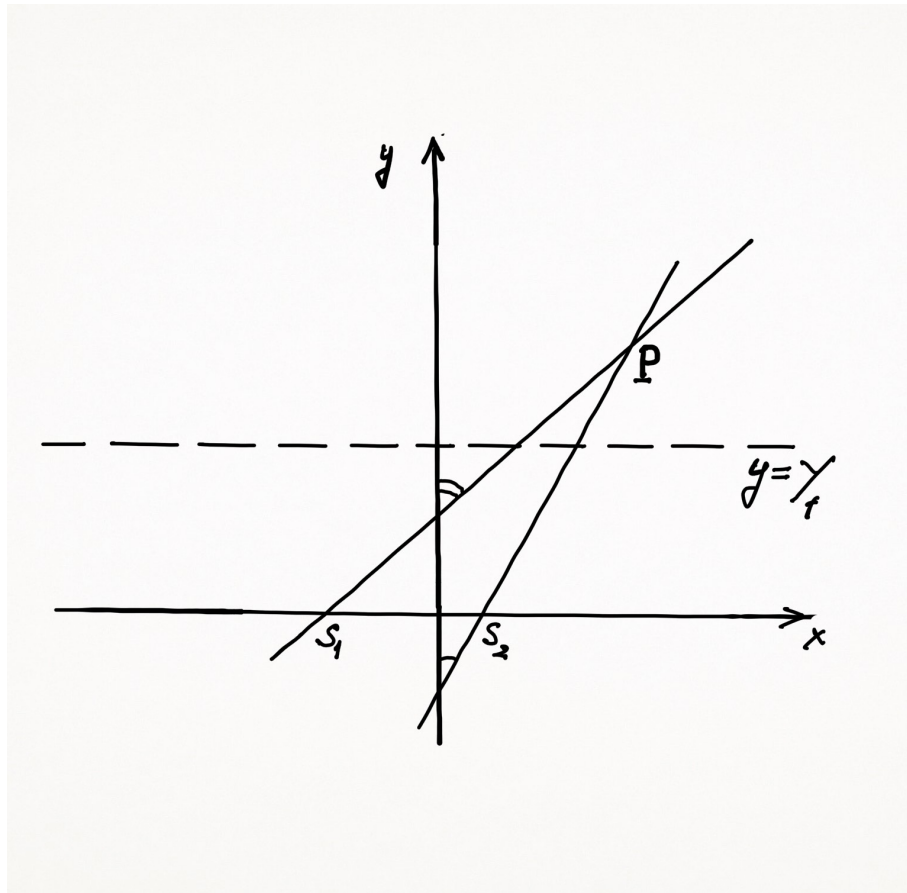


Figure 5.5:

$$\begin{cases} x(t, s) = s + th(s), \\ y(t, s) = t, \\ z(t, s) = h(s). \end{cases} \quad (5.4.15)$$

By the method shown in this Section, the solution to (5.4.12) is given by a function u such that

$$u(x(t, s), y(t, s)) = z(t, s) = h(s). \quad (5.4.16)$$

To express u in the variables x and y we eliminate s and t from the first two equations of (5.4.15). We have

$$\begin{cases} x = s + yh(s), \\ t = y. \end{cases}$$

To obtain s from the first equation it is necessary that $s \rightarrow s + yh(s)$ be injective, that is, it is necessary that

$$0 \neq \frac{d}{ds}(s + yh(s)) = 1 + yh'(s).$$

For instance, if $h' > 0$, let us assume

$$Y_0 := \sup_{s \in \mathbb{R}} -\frac{1}{h'(s)} < 0,$$

then we have that the solution to (5.4.12) is defined for all y such that

$$y > Y_0.$$

If $h'(s) < 0$, we assume

$$Y_1 := \inf_{s \in \mathbb{R}} -\frac{1}{h'(s)} > 0,$$

then we have the solution to (5.4.12) is defined for all y such that

$$y < Y_1.$$

Let us dwell for a while on the latter case and examine what happens above the line $y = Y_1$. Let us come back to (5.4.16); this relation tells us that u is constant on the projection of the characteristic passing through the point $(s, 0)$ and there it is equal to $h(s)$. Let now $s_1, s_2 \in \mathbb{R}$ satisfy $s_1 < s_2$, then the straight lines whose equations are given by

$$x = s_1 + yh(s_1)$$

and

$$x = s_2 + yh(s_2)$$

intersect at the point

$$P = \left(-\frac{s_2 h(s_1)}{h(s_2) - h(s_1)}, -\frac{s_2 - s_1}{h(s_2) - h(s_1)} \right),$$

that implies that the function u *cannot be continuous* in P . Let us observe that the point P is situated either on the line $y = Y_1$ or above it since, for an appropriate $\bar{s} \in (s_1, s_2)$, we have (Figure 5.5)

$$-\frac{s_2 - s_1}{h(s_2) - h(s_1)} = -\frac{1}{h'(\bar{s})} \geq Y_1.$$

It is, actually, of some physical interest to include (in an appropriate sense) the discontinuous solution among the solutions to problem (5.4.12) since they correspond to "shock waves". We refer for insights to [23, Ch. 3, Sec. 4].

5.5 Brief review on the fully nonlinear case

In this Section we wish briefly consider to the method of the characteristics to solve the Cauchy problem for the fully nonlinear equation

$$F(x, u(x), \nabla u(x)) = 0.$$

Namely, the Cauchy problem

$$\begin{cases} F(x, u(x), \nabla u(x)) = 0, & \text{in } \Omega, \\ u|_{\Gamma} = g, \end{cases} \quad (5.5.1)$$

where Ω is an open set \mathbb{R}^n , Γ is a regular surface of \mathbb{R}^n contained Ω ,

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (5.5.2)$$

is a function of class $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. The variables of F are $x \in \Omega$, $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$. Moreover $g : \Gamma \rightarrow \mathbb{R}$ is a function defined on Γ . The assumptions on Γ , F and g will be specified in more detail later on. We notice that in the linear and quasilinear cases investigated, respectively, in Sections 5.2 and 5.4, we have $F(x, z, p) = a(x) \cdot p - c(x)z - f(x)$, $F(x, z, p) = a(x, z) \cdot p - c(x, z)$. Notice that in the nonlinear case, generally, we cannot expect the uniqueness of the solutions to Cauchy (5.5.1). The following simple example will help us to understand this fact. Let us consider the Cauchy problem

$$\begin{cases} u_{x_1}^2 + u_{x_2}^2 = 1, & \text{in } \mathbb{R}^2, \\ u(x_1, 0) = g(x_1), & \text{for } x_1 \in \mathbb{R}. \end{cases} \quad (5.5.3)$$

We note that all we can say about $u_{x_2}(x_1, 0)$ is that it satisfies the condition

$$g'^2(x_1) + u_{x_2}^2(x_1, 0) = 1,$$

which leaves undetermined the sign of $u_{x_2}(x_1, 0)$. If, for instance, $g = 0$, then $u = x_2$ and $u = -x_2$ are both solutions of Cauchy problem (5.5.3).

We have already studied the method of characteristics for the linear and the quasilinear linear equations. In the nonlinear case we follow a procedure similar to the previous two cases, but in the nonlinear case it is less obvious which are the characteristic equations. For this purpose some geometrical considerations may be useful, which, nevertheless, we do not take up here, referring the interested reader to [41, Ch. 1]. We start by the following

Definition 5.5.1. Let F be the function (5.5.2). Let us assume that F is of class $C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. We call the characteristic equations related to the partial differential equation

$$F(x, u(x), \nabla u(x)) = 0, \text{ in } \Omega, \quad (5.5.4)$$

the following system of ordinary differential equations

$$\begin{cases} \frac{dX(t)}{dt} = \nabla_p F(X(t), z(t), p(t)), \\ \frac{dz(t)}{dt} = \nabla_p F(X(t), z(t), p(t)) \cdot p(t), \\ \frac{dp(t)}{dt} = -\partial_z F(X(t), z(t), p(t))p(t) - \nabla_x F(X(t), z(t), p(t)). \end{cases} \quad (5.5.5)$$

The function $X(\cdot)$, $z(\cdot)$, $p(\cdot)$ are called the **characteristic lines of the equation** (5.5.4). $X(\cdot)$ is called the **ray** or the **projected characteristic lines** on \mathbb{R}^n .

If F does not depend on z , system (5.5.5) is decoupled in z , while the first and the third equations constitute the **Hamilton–Jacobi system**:

$$\begin{cases} \frac{dX(t)}{dt} = \nabla_p F(X(t), p(t)), \\ \frac{dp(t)}{dt} = -\nabla_x F(X(t), p(t)). \end{cases} \quad (5.5.6)$$

Remarks.

1. Let us note that if $(X(\cdot), z(\cdot), p(\cdot))$ is a characteristic line for equation (5.5.4) then

$$F(X(t), z(t), p(t)) = \text{constant}. \quad (5.5.7)$$

As a matter of fact, exploiting (5.5.5), we have

$$\begin{aligned} \frac{d}{dt}F(X(t), z(t), p(t)) &= \nabla_x F(X(t), z(t), p(t)) \cdot \frac{dX}{dt} + \\ &+ F_z(X(t), z(t), p(t)) \frac{dz}{dt} + \nabla_p F(X(t), z(t), p(t)) \cdot \frac{dp}{dt} = \\ &= \nabla_x F \cdot \nabla_p F + F_z \nabla_p F \cdot p(t) + \\ &+ \nabla_p F \cdot (-F_z p(t) - \nabla_x F) = 0, \end{aligned}$$

In the last step, for the sake of brevity, we have omitted the arguments $X(t), z(t), p(t)$ in F .

2. Let us suppose that u is a solution of class C^2 to the equation

$$F(x, u(x), \nabla u(x)) = 0, \quad (5.5.8)$$

we wish to look for $X(t)$ (or, more precisely, for an equation for $X(t)$) such that, setting

$$z(t) = u(X(t)), \quad p(t) = \nabla u(X(t)),$$

it happens that $(X(t), z(t), p(t))$ solves system (5.5.5).

We have

$$\frac{dz(t)}{dt} = p(t) \cdot \frac{dX(t)}{dt}, \quad (5.5.9)$$

$$\frac{dp_i(t)}{dt} = \sum_{j=1}^n \partial_{ij}^2 u(X(t)) \frac{dX_j(t)}{dt}, \quad i = 1, \dots, n. \quad (5.5.10)$$

Now, calculating the derivatives of both the sides of equation (5.5.8) w.r.t. $x_i, i = 1, \dots, n$, we have

$$\sum_{j=1}^n \partial_{p_j} F \partial_{ij}^2 u(x) = -\partial_{x_i} F - \partial_z F \partial_{x_i} u(x), \quad (5.5.11)$$

where the argument of F in (5.5.11) is $(x, u(x), \nabla u(x))$. Now, let us observe what follows: if

$$\frac{dX_j(t)}{dt} = \partial_{p_j} F(X(t), z(t), p(t)), \quad j = 1, \dots, n, \quad (5.5.12)$$

then by (5.5.10) and by (5.5.11), calculated for $x = X(t)$, we have, for $i = 1, \dots, n$

$$\frac{dp_i(t)}{dt} = -\partial_{x_i} F(X(t), z(t), p(t)) - \partial_z F(X(t), z(t), p(t)) p_i(t) \quad (5.5.13)$$

and by (5.5.9) we have

$$\frac{dz(t)}{dt} = \sum_{j=1}^n \partial_{p_j} F(X(t), z(t), p(t)) p_j(t). \quad (5.5.14)$$

Equations (5.5.12), (5.5.13) and (5.5.14) are just the equations of the system (5.5.5). \blacklozenge

In order to solve Cauchy problem (5.5.1) we will follow an approach similar to that followed in the linear (and quasilinear) case by letting the projected characteristic lines, $X(t)$, play a similar role to that played, in the linear case, by the characteristic lines.

In what follows we will consider the case

$$\Gamma = \{x \in \Omega : x_n = 0\}. \quad (5.5.15)$$

We observe that we can always lead back to this situation, at least locally, even if Γ is given by

$$\Gamma = \{x \in \Omega : \phi(x) = 0\},$$

where $\phi \in C^3(\Omega)$ and $\phi(x_0) = 0$ for a given $x_0 \in \Omega$ and

$$\nabla \phi(x_0) \neq 0. \quad (5.5.16)$$

Indeed, thanks to (5.5.16), there exists a neighborhood, \mathcal{U} , of x_0 such that $\Gamma \cap \mathcal{U}$ is a graph of a function of $n - 1$ variables. If, for instance, let us suppose that $\phi_{x_n}(x_0) \neq 0$ then, up to a translation that moves x_0 to 0, we may assume that for an appropriate $\delta > 0$, we have

$$\Gamma \cap \mathcal{U} = \{(x', \varphi(x')) : x' \in B'_\delta\}, \quad (5.5.17)$$

where $\varphi \in C^3(B'_\delta)$, $\varphi(0) = |\nabla_{x'} \varphi(0)| = 0$. Now, let

$$\Lambda : B_\delta \subset \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n, \quad \Lambda(x) = (x', x_n - \varphi(x')),$$

$$\Lambda(\Gamma) = \{(y', 0) : y' \in B'_\delta\} = \{y \in B_\delta : -y_n = 0\}$$

and, setting

$$v(y) = u(\Lambda^{-1}(y)),$$

we easily obtain that the problem (5.5.1) takes the form

$$\begin{cases} \tilde{F}(y, v(y), \nabla_y v(y)) = 0, & \text{in } \mathcal{V}, \\ v(y) = \tilde{g}(y), & \text{for } y \in \Lambda(\Gamma) \cap \mathcal{V}, \end{cases} \quad (5.5.18)$$

where \mathcal{V} is a neighborhood of 0 and $\tilde{F} : \mathcal{V} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function of class $C^2(\mathcal{V} \times \mathbb{R} \times \mathbb{R}^n)$.

Theorem 5.5.2. *Let $R > 0$ and $F \in C^2(B_R \times \mathbb{R} \times \mathbb{R}^n)$. Let $g \in C^2(B'_R)$. Let $\eta \in \mathbb{R}$ satisfy*

$$F(0, g(0), \nabla_{x'} g(0), \eta) = 0 \quad (5.5.19)$$

and

$$F_{p_n}(0, g(0), \nabla_{x'} g(0), \eta) \neq 0, \quad (5.5.20)$$

then for some $r \in (0, R)$ there exists a unique solution $u \in C^2(B_r)$ to the initial-value problems

$$\begin{cases} F(x, u(x), \nabla u(x)) = 0, & \text{in } B_r, \\ u(x', 0) = g(x'), & \text{for } x' \in B'_r, \\ u_{x_n}(0, 0) = \eta. \end{cases} \quad (5.5.21)$$

Proof. Let us begin by proving **the uniqueness**. It will suffice to prove that if u is a solution to

$$\begin{cases} F(x, u(x), \nabla u(x)) = 0, & \text{in } B_R, \\ u(x', 0) = g(x'), & \text{for } x' \in B'_R, \\ u_{x_n}(0, 0) = \eta, \end{cases} \quad (5.5.22)$$

then there exists a neighborhood of 0 in which u is uniquely determined.

Set

$$z_0 = g(0), \quad p'_0 = \nabla_{x'} g(0).$$

By (5.5.19) and (5.5.20) we have

$$F(0, z_0, p'_0, \eta) = 0, \quad F_{p_n}(0, z_0, p'_0, \eta) \neq 0.$$

By applying the Implicit Function Theorem, we have that there exists $\delta \in (0, R]$ such that, setting

$$\mathcal{U} = B_\delta \times (z_0 - \delta, z_0 + \delta) \times B'_\delta(p'_0) \times (\eta - \delta, \eta + \delta),$$

we have that the set

$$\{(x, z, p) \in \mathcal{U} : F(x, z, p) = 0\},$$

is equal to the graph of the function

$$\psi : B_\delta \times (z_0 - \delta, z_0 + \delta) \times B'_\delta(p'_0) \rightarrow (\eta - \delta, \eta + \delta), \quad (5.5.23)$$

where ψ is of class C^2 and

$$\psi(0, z_0, p'_0) = \eta.$$

Now, since u satisfies (5.5.22), we have

$$F(x', 0, g(x'), \nabla_{x'} g(x'), u_{x_n}(x', 0)) = 0,$$

$$u_{x_n}(0, 0) = \eta,$$

and we have

$$u_{x_n}(x', 0) = \psi(x', 0, g(x'), \nabla_{x'} g(x')), \quad \forall x' \in B'_\delta.$$

Set

$$p_n(x') = \psi(x', 0, g(x'), \nabla_{x'} g(x'))$$

and

$$p^{(0)}(x') = (\nabla_{x'} g(x'), p_n(x')), \quad (5.5.24)$$

we have

$$\begin{cases} F(x, u(x), \nabla u(x)) = 0, & \text{in } B_\delta, \\ u(x', 0) = g(x'), & \text{for } x' \in B'_\delta, \\ \nabla u(x', 0) = p^{(0)}(x'), & \text{for } x' \in B'_\delta. \end{cases} \quad (5.5.25)$$

We denote by y an arbitrary point of B'_δ and recalling that by Remark 2 of the present Section, the function

$$t \rightarrow (X(t), z(t), p(t)) := (X(t, y), u(X(t, y)), \nabla u(X(t, y)))$$

is a solution to the characteristic equations (for each $y \in B'_\delta$)

$$\begin{cases} \frac{dX}{dt} = \nabla_p F(X, z, p), \\ \frac{dz}{dt} = \nabla_p F(X, z, p) \cdot p, \\ \frac{dp}{dt} = -\partial_z F(X, z, p)p - \nabla_x F(X, z, p) \end{cases} \quad (5.5.26)$$

and

$$\begin{cases} X(0, y) = (y, 0), \\ z(0, y) = u(X(0, y)) = g(y), \\ p(0, y) = \nabla u(X(0, y)) = p^{(0)}(y). \end{cases} \quad (5.5.27)$$

Therefore, due to the uniqueness of the solution to Cauchy problem (5.5.26)–(5.5.27) it turns out that $u(X(t, y))$ is determined for every $y \in B'_\delta$ and for every t in a neighborhood of 0 (this neighborhood depends on y). To conclude the proof, it suffices, therefore, to prove that the map

$$(t, y) \rightarrow X(t, y), \quad (5.5.28)$$

is a diffeomorphism in a neighborhood of 0. From what we said in Section 5.1 (final part), map (5.5.28) is of class C^2 . To establish that it is a local diffeomorphism, it suffices to check that the Jacobian matrix of $(t, y) \rightarrow X(t, y)$ is nonsingular in 0. Now from (5.5.27) we have

$$\frac{\partial X(0, 0)}{\partial(t, y)} = \begin{pmatrix} \partial_t X_1(0, 0) & \partial_{y_1} X_1(0, 0) & \cdots & \partial_{y_{n-1}} X_1(0, 0) \\ & \vdots & \cdots & \vdots \\ \partial_t X_n(0, 0) & \partial_{y_1} X_n(0, 0) & \cdots & \partial_{y_{n-1}} X_n(0, 0) \end{pmatrix}. \quad (5.5.29)$$

On the other hand

$$\begin{aligned} \partial_{y_i} X_j(0, 0) &= \delta_{ij}, \text{ for } 1 \leq i \leq n-1, 1 \leq j \leq n-1, \\ \partial_{y_i} X_n(0, 0) &= 0, \text{ for } 1 \leq i \leq n-1 \end{aligned}$$

and, for $1 \leq j \leq n-1$,

$$\partial_t X_j(0, 0) = \partial_{p_j} F(X(0, 0), z(0, 0), p(0, 0)) = \partial_{p_j} F(0, g(0), \nabla_{x'} g(0), \eta).$$

Hence

$$\det \left(\frac{\partial X(0, 0)}{\partial(t, y)} \right) = (-1)^n \partial_{p_n} F(0, g(0), \nabla_{x'} g(0), \eta) \neq 0, \quad (5.5.30)$$

from which it follows that map (5.5.28) is a local diffeomorphism. The proof of uniqueness is complete.

Now, let us prove **the existence** of the solution to problem (5.5.21). Let $p^{(0)}(x')$ be defined by (5.5.24) and let $(X(t, y), z(t, y), p(t, y))$ be the solution of the Cauchy problem comprising the system (5.5.26) and the initial conditions

$$\begin{cases} X(0, y) = (y, 0), \\ z(0, y) = g(y), \\ p(0, y) = p^{(0)}(y). \end{cases} \quad (5.5.31)$$

Set

$$f(t, y) = F(X(t, y), z(t, y), p(t, y)),$$

we have

$$\begin{aligned} f(0, y) &= F((y, 0), g(y), p^{(0)}(y)) = \\ &= F((y, 0), g(y), \nabla_y g(y), \psi((y, 0), g(y), \nabla_y g(y))) = 0, \end{aligned}$$

where ψ is given by (5.5.23). Hence, by (5.5.7), we have

$$f(t, y) = F(X(t, y), z(t, y), p(t, y)) = 0. \quad (5.5.32)$$

Moreover, in a completely similar way to what has been done above for the uniqueness we have that there exists $\delta_1 > 0$ and a neighbourhood of 0, \mathcal{U}_0 , such that

$$B_{\delta_1} \ni (t, y) \rightarrow X(t, y) \in \mathcal{U}_0,$$

is a diffeomorphism of class $C^2(B_{\delta_1}(0))$. We denote the inverse of $X(\cdot, \cdot)$ by

$$X^{-1}(x) = (t(x), y(x))$$

and set

$$u(x) = z(t(x), y(x)), \quad p(x) = p(t(x), y(x)).$$

The remaining part of the proof consists of proving that u satisfies (5.5.21). First of all, we check that

$$u(x', 0) = g(x'). \quad (5.5.33)$$

To this purpose we note that

$$t(x', 0) = 0, \quad y(x', 0) = x'.$$

Hence

$$u(x', 0) = z(t(x', 0), y(x', 0)) = z(0, x') = g(x').$$

Therefore, we have (5.5.33). We will check the condition $\partial_{x_n} u(0, 0) = \eta$ later on, now we check that u satisfies the equation

$$F(x, u(x), \nabla u(x)) = 0. \quad (5.5.34)$$

Firstly, we observe that from (5.5.32) we have

$$F(x, u(x), p(x)) = f(t(x), y(x)) = 0, \quad \forall x \in \mathcal{U}_0. \quad (5.5.35)$$

Therefore, to prove (5.5.34) it suffices to prove that

$$p(x) = \nabla u(x), \quad \forall x \in \mathcal{U}_0. \quad (5.5.36)$$

To this purpose we prove the following claims:

Claim I

$$\partial_t z(t, y) = \sum_{j=1}^n p_j(t, y) \partial_t X_j(t, y), \quad \forall (t, y) \in B_{\delta_1}. \quad (5.5.37)$$

Claim II

$$\partial_{y_i} z(t, y) = \sum_{j=1}^n p_j(t, y) \partial_{y_i} X_j(t, y), \quad \forall (t, y) \in B_{\delta_1}. \quad (5.5.38)$$

Claim I follows by the first and the second equation of (5.5.26). As a matter of fact, we have

$$\partial_t z(t, y) = \nabla_p F(X(t, y), z(t, y), p(t, y)) \cdot p = \partial_t X(t, y) \cdot p(t, y).$$

The proof of **Claim II** is less immediate than Claim I. Set

$$h_i(t, y) = \partial_{y_i} z(t, y) - \sum_{j=1}^n p_j(t, y) \partial_{y_i} X_j(t, y). \quad (5.5.39)$$

By (5.5.24) and recalling that

$$\begin{aligned} \partial_{y_i} X_j(0, y) &= \delta_{ij}, \text{ for } 1 \leq i, j \leq n-1, \\ \partial_{y_i} X_n(0, y) &= 0, \text{ for } 1 \leq i \leq n-1, \end{aligned} \quad (5.5.40)$$

we have, for $i = 1, \dots, n-1$,

$$h_i(0, y) = \partial_{y_i} z(0, y) - \sum_{j=1}^n p_j(0, y) \partial_{y_i} X_j(0, y) = \partial_{y_i} g(y) - p_i^{(0)}(y) = 0. \quad (5.5.41)$$

Now, we prove that $h_i(\cdot, y)$ satisfies

$$\partial_t h_i(t, y) = -\partial_z F(X(t, y), z(t, y), p(t, y)) h_i(t, y). \quad (5.5.42)$$

By (5.5.37) we have

$$\partial_{t y_i}^2 z = \sum_{j=1}^n (\partial_{y_i} p_j \partial_t X_j + p_j \partial_{t y_i}^2 X_j). \quad (5.5.43)$$

Now making the derivative w.r.t. t of both the sides of (5.5.39) we have

$$\partial_t h_i = \partial_{t y_i}^2 z - \sum_{j=1}^n (\partial_t p_j \partial_{y_i} X_j + p_j \partial_{t y_i}^2 X_j).$$

By this equality, by (5.5.43) and by (5.5.26) we have

$$\begin{aligned} \partial_t h_i &= \sum_{j=1}^n (\partial_{y_i} p_j \partial_t X_j - \partial_t p_j \partial_{y_i} X_j) = \\ &= \sum_{j=1}^n (\partial_{y_i} p_j \partial_{p_j} F - (-\partial_{x_j} F - \partial_z F p_j) \partial_{y_i} X_j) = \\ &= \sum_{j=1}^n (\partial_{y_i} p_j \partial_{p_j} F + \partial_{x_j} F \partial_{y_i} X_j + \partial_z F p_j \partial_{y_i} X_j). \end{aligned} \quad (5.5.44)$$

Now by (5.5.32), making the derivative w.r.t. y_i of both the sides, we get

$$\sum_{j=1}^n (\partial_{y_i} p_j \partial_{p_j} F + \partial_{x_j} F \partial_{y_i} X_j) = -\partial_z F \partial_{y_i} z$$

and inserting the latter into (5.5.44) we get

$$\begin{aligned} \partial_t h_i &= -\partial_z F \partial_{y_j} z + \sum_{j=1}^n \partial_z F p_j \partial_{y_i} X_j = \\ &= -\partial_z F \left(\partial_{y_i} z - \sum_{j=1}^n p_j \partial_{y_i} X_j \right) = \\ &= -\partial_z F h_i. \end{aligned}$$

All in all, by the latter and by (5.5.41) we get, for $i = 1, \dots, n-1$,

$$\begin{cases} \partial_t h_i = -\partial_z F h_i, \\ h_i(0, y) = 0, \end{cases}$$

from which we have

$$h_i(t, y) = 0, \quad \text{for } i = 1, \dots, n-1.$$

Claim II is proved.

Now, let us prove (5.5.36). First, we recall that $u(x) = z(X^{-1}(x)) = z(t(x), y(x))$. We have by (5.5.37) and (5.5.38),

$$\begin{aligned} \partial_{x_i} u &= \partial_t z \partial_{x_i} t + \sum_{j=1}^{n-1} \partial_{y_j} z \partial_{x_i} y_j = \\ &= \left(\sum_{k=1}^n p_k \partial_t X_k \right) \partial_{x_i} t + \sum_{j=1}^{n-1} \sum_{k=1}^n p_k \partial_{y_j} X_k \partial_{x_i} y_j = \\ &= \sum_{k=1}^n p_k \left(\partial_t X_k \partial_{x_i} t + \sum_{j=1}^{n-1} \partial_{y_j} X_k \partial_{x_i} y_j \right) = \\ &= \sum_{k=1}^n p_k \partial_{x_i} (X_k (X^{-1}(x))) = \\ &= \sum_{k=1}^n p_k \delta_{ik} = p_i \end{aligned}$$

for $i = 1, \dots, n$. From which we have (5.5.36) and, in particular,

$$\partial_{x_n} u(0) = p_n(0) = \eta.$$

Which concludes the proof. ■

5.6 Appendix: geodesics and Hamilton–Jacobi equations

We warn that throughout this Appendix we will adopt the convention of repeated indices. In addition, we will strictly adhere to the notation on indices (upper or lower) for the components of a tensor.

In the first part of this Appendix we will present the rudiments of the theory of Hamilton–Jacobi equations, these topics can be carried out in a more general way, for more details we refer to [23].

Let Ω be an open set of \mathbb{R}^n , we say that a real–valued function,

$$L \in C^\infty(\Omega \times \mathbb{R}^n)$$

is a **Lagrangian** on Ω . An example of Lagrangian that we are interested in is given by

$$L(x, q) = \frac{1}{2} g_{ij}(x) q^i q^j, \quad \forall x \in \Omega, \quad \forall q \in \mathbb{R}^n, \quad (5.6.1)$$

where $\{g_{ij}(x)\}_{i,j=1}^n$ is a real symmetric nonsingular matrix, $n \times n$, whose entries belong to $C^\infty(\Omega)$.

Given a Lagrangian L we will call **equation of Euler–Lagrange** the differential equation in the unknown $x = x(t)$

$$\frac{d}{dt} \left(\nabla_q L(x(t), \dot{x}(t)) \right) - \nabla_x L(x(t), \dot{x}(t)) = 0. \quad (5.6.2)$$

Here and in the sequel we will indistinctly let us denote by $\frac{df}{dt}$ or by $\dot{f}(t)$ the derivative with respect to t of a differentiable function f . The solutions $x : [t_0, t_1] \rightarrow \mathbb{R}^n$ of (5.6.2) are also called the **extremal** of the functional

$$\int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt. \quad (5.6.3)$$

Assumption I. In what follows we suppose that, for every $p \in \mathbb{R}^n$, the equation

$$\nabla_q L(x, q) = p, \quad (5.6.4)$$

has a unique solution of class $C^\infty(\Omega \times \mathbb{R}^n)$. We denote such a solution by $q(x, p)$.

The function

$$H(x, p) = p \cdot q(x, p) - L(x, q(x, p)). \quad (5.6.5)$$

is called the **Hamiltonian** associated to L

We have

Theorem 5.6.1. *Let $x = x(t)$ be a solution to Euler–Lagrange equation*

$$\frac{d}{dt} \left(\nabla_q L(x(t), \dot{x}(t)) \right) - \nabla_x L(x(t), \dot{x}(t)) = 0. \quad (5.6.6)$$

Then, setting

$$p(t) = \nabla_q L(x(t), \dot{x}(t)),$$

it turns out that $(x(t), p(t))$ is a solution of the Hamilton–Jacobi system

$$\begin{cases} \frac{dx}{dt} = \nabla_p H(x(t), p(t)), \\ \frac{dp}{dt} = -\nabla_x H(x(t), p(t)). \end{cases} \quad (5.6.7)$$

Moreover

$$H(x(t), p(t)) = \text{constant}. \quad (5.6.8)$$

Proof. Let $x(t)$ be a solution to equation (5.6.6). Then, since equation (5.6.4) has a unique solution, $q(x, p)$, and since

$$p(t) = \nabla_q L(x(t), \dot{x}(t)),$$

we have

$$\dot{x}(t) = q(x(t), p(t)). \quad (5.6.9)$$

Now we have, for $i = 1, \dots, n$,

$$\begin{aligned} \partial_{p_i} H(x, p) &= \partial_{p_i} (p \cdot q(x, p) - L(x, q(x, p))) = \\ &= p_k \partial_{p_i} q^k(x, p) + q^i(x, p) - \partial_{q^k} L(x, q(x, p)) \partial_{p_i} q^k(x, p) = \\ &= \partial_{p_i} q^k(x, p) (p_k - \partial_{q^k} L(x, q(x, p))) + q^i(x, p) = \\ &= q^i(x, p). \end{aligned}$$

Hence, recalling (5.6.9), we have, for $i = 1, \dots, n$,

$$\frac{dx^i(t)}{dt} = \partial_{p_i} H(x(t), p(t)),$$

which is the first equation of system (5.6.7). Concerning the second equation, for $i = 1, \dots, n$, we have

$$\begin{aligned} \partial_{x^i} H(x, p) &= \partial_{x^i} (p \cdot q(x, p) - L(x, q(x, p))) = \\ &= p_k \partial_{x^i} q^k(x, p) - \partial_{x^i} L(x, q(x, p)) - \partial_{q^k} L(x, q(x, p)) \partial_{x^i} q^k(x, p) = \\ &= \partial_{x^i} q^k(x, p) (p_k - \partial_{q^k} L(x, q(x, p))) - \partial_{x^i} L(x, q(x, p)) = \\ &= -\partial_{x^i} L(x, q(x, p)). \end{aligned} \quad (5.6.10)$$

On the other hand, by (5.6.6) and (5.6.9), we have, for $i = 1, \dots, n$,

$$\begin{aligned} \partial_{x^i} L(x(t), q(x(t), p(t))) &= \partial_{x^i} L(x(t), \dot{x}(t)) = \\ &= \frac{d}{dt} \partial_{q^i} L(x(t), \dot{x}(t)) = \\ &= \frac{dp_i(t)}{dt}. \end{aligned}$$

By the just obtained equality and by (5.6.10) we get

$$\frac{dp_i(t)}{dt} = -\partial_{x^i} H(x(t), p(t)), \quad \text{for } i = 1, \dots, n. \quad (5.6.11)$$

which is the second equation of system (5.6.7).

Finally, (5.6.8) follows by (5.6.7) and by

$$\begin{aligned} \frac{d}{dt} H(x(t), p(t)) &= \partial_{x^i} H(x(t), p(t)) \frac{dx^i(t)}{dt} + \partial_{p_i} H(x(t), p(t)) \frac{dp_i(t)}{dt} = \\ &= \partial_{x^i} H(x(t), p(t)) \partial_{p_i} H(x(t), p(t)) - \partial_{p_i} H(x(t), p(t)) \partial_{x^i} H(x(t), p(t)) = 0. \end{aligned}$$

■

Assumption II. Let H be the Hamiltonian associated to the Lagrangian L which satisfies Assumption I. Let us suppose that, for every $q \in \mathbb{R}^n$, the equation

$$\nabla_p H(x, p) = q$$

has a unique solution of class $C^\infty(\Omega \times \mathbb{R}^n)$. We denote by $p(x, q)$ such solution. Let us notice that (5.6.5) implies trivially

$$L(x, q) = p(x, q) \cdot q - H(x, p(x, q)). \quad (5.6.12)$$

We now prove the converse of Theorem 5.6.1.

Theorem 5.6.2. *Let H the Hamiltonian associated to L and let us suppose that Assumption I and II hold true. Moreover, let us suppose $(x(t), p(t))$ that is a solution to the Hamilton–Jacobi equation*

$$\begin{cases} \frac{dx(t)}{dt} = \nabla_p H(x(t), p(t)), \\ \frac{dp(t)}{dt} = -\nabla_x H(x(t), p(t)). \end{cases} \quad (5.6.13)$$

Then $x(t)$ is a solution to Euler–Lagrange equation

$$\frac{d}{dt} \left(\nabla_q L(x(t), \dot{x}(t)) \right) - \nabla_x L(x(t), \dot{x}(t)) = 0. \quad (5.6.14)$$

Proof. By Assumption II and by (5.6.13), in particular by

$$\nabla_p H(x(t), p(t)) = \dot{x}(t),$$

we have

$$p(t) = p(x(t), \dot{x}(t)).$$

Now, by (5.6.12) we have

$$L(x, \dot{x}(t)) = p(t) \cdot \dot{x}(t) - H(x(t), p(t)). \quad (5.6.15)$$

On the other hand

$$\nabla_q L(x, q) = p(x, q),$$

hence

$$\nabla_q L(x(t), \dot{x}(t)) = p(x(t), \dot{x}(t)) = p(t).$$

Therefore, taking into account (5.6.13), we have

$$\frac{d}{dt} \nabla_q L(x(t), \dot{x}(t)) = \dot{p}(t) = -\nabla_x H(x(t), p(t)). \quad (5.6.16)$$

Now, let us make the derivatives w.r.t. x^i of both the sides of (5.6.12)

$$\begin{aligned} \partial_{x^i} L(x, q) &= q^k \partial_{x^i} p_k(x, q) - \partial_{x^i} H(x, p(x, q)) - \\ &\quad - \partial_{p_k} H(x, p(x, q)) \partial_{x^i} p_k(x, q) = \\ &= (q^k - \partial_{p_k} H(x, p(x, q))) \partial_{x^i} p_k(x, q) - \partial_{x^i} H(x, p(x, q)), \end{aligned}$$

from which we have

$$\begin{aligned}
\partial_{x^i} L(x(t), \dot{x}(t)) &= \\
&= \left(\frac{dx^k}{dt} - \partial_{p_k} H(x(t), p(t)) \right) \partial_{x^i} p_k(x(t), p(t)) - \\
&\quad - \partial_{x^i} H(x(t), p(t)) = \\
&= -\partial_{x^i} H(x(t), p(t)).
\end{aligned}$$

By the just obtained inequality and by (5.6.16) we have

$$\frac{d}{dt} \nabla_q L(x(t), \dot{x}(t)) = \nabla_x L(x(t), \dot{x}(t)).$$

■

Now let us consider the **geodesic lines** with respect to the Riemannian metric

$$g_{ij}(x) dx^i \otimes dx^j,$$

where $\{g_{ij}(x)\}_{i,j=1}^n$ is a symmetric real matrix $n \times n$ whose entries belong to $C^\infty(\Omega)$. Let us suppose

$$\lambda^{-1} |\xi|^2 \leq g_{ij}(x) \xi^i \xi^j \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega, \quad (5.6.17)$$

where $\lambda \geq 1$. Let us denote by $\{g^{ij}(x)\}_{i,j=1}^n$ the inverse matrix of $\{g_{ij}(x)\}_{i,j=1}^n$.

Definition 5.6.3. We say that the path

$$\gamma : [t_0, t_1] \rightarrow \Omega,$$

is a **geodesic line** with respect to the Riemannian metric $g_{ij}(x) dx^i \otimes dx^j$, if $\gamma \in C^\infty([t_0, t_1], \Omega)$ and it solves the equations

$$\frac{d^2 \gamma^h(t)}{dt^2} + \Gamma_{ij}^h(\gamma(t)) \frac{d\gamma^i(t)}{dt} \frac{d\gamma^j(t)}{dt} = 0, \quad h = 1, \dots, n, \quad (5.6.18)$$

where (Christoffel symbols), for $i, j, h = 1, \dots, n$,

$$\Gamma_{ij}^h(x) = \frac{1}{2} g^{hk}(x) [\partial_i g_{kj}(x) + \partial_j g_{ki}(x) - \partial_k g_{ij}(x)]. \quad (5.6.19)$$

The following Proposition holds true.

Proposition 5.6.4. *The path $\gamma : [t_0, t_1] \rightarrow \Omega$ is a geodesic line w.r.t. the Riemannian metric $g_{ij}(x) dx^i \otimes dx^j$ if and only if γ is an extremal of the Lagrangian*

$$L(x, q) = g_{ij}(x) q^i q^j, \quad x \in \Omega, \quad q \in \mathbb{R}. \quad (5.6.20)$$

Proof. Let us write the Euler–Lagrange equation

$$\frac{d}{dt} \left(\nabla_q L \left(x(t), \dot{x}(t) \right) \right) - \nabla_x L \left(x(t), \dot{x}(t) \right) = 0. \quad (5.6.21)$$

We have

$$\begin{aligned} \partial_{q_k} L \left(x(t), \dot{x}(t) \right) &= 2g_{kj}(x(t)) \frac{dx^j(t)}{dt}, \\ \frac{d}{dt} \left(\partial_{q_k} L \left(x(t), \dot{x}(t) \right) \right) &= 2g_{kj}(x(t)) \frac{d^2x^j(t)}{dt^2} + \\ &\quad + 2\partial_{x_i} g_{kj}(x(t)) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} \end{aligned}$$

and

$$\partial_{x_k} L \left(x(t), \dot{x}(t) \right) = \partial_{x_k} g_{ij}(x(t)) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt}.$$

Therefore the Euler–Lagrange equation can be written (we omit the variables, for the sake of brevity)

$$g_{kj} \frac{d^2x^j}{dt^2} = \left(\frac{1}{2} \partial_{x_k} g_{ij} - \partial_i g_{kj} \right) \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

From which we have

$$\begin{aligned} \frac{d^2x^h}{dt^2} &= g^{hk} \left(\frac{1}{2} \partial_{x_k} g_{ij} - \partial_i g_{kj} \right) \frac{dx^i}{dt} \frac{dx^j}{dt} = \\ &= \frac{1}{2} g^{hk} \left(\partial_{x_k} g_{ij} - \partial_{x_i} g_{kj} - \partial_{x_j} g_{ki} \right) \frac{dx^i}{dt} \frac{dx^j}{dt} = \\ &= -\Gamma_{ij}^h \frac{dx^i}{dt} \frac{dx^j}{dt}. \end{aligned}$$

Hence the equation

$$\frac{d^2x^h}{dt^2} + \Gamma_{ij}^h \frac{dx^i}{dt} \frac{dx^j}{dt} = 0,$$

is equivalent to the Euler–Lagrange equation related to L and by (5.6.18) the thesis follows. ■

Remark. Let $\{g_{ij}(x)\}_{i,j=1}^n$ be a matrix like in Proposition 5.6.4, set

$$L(x, q) = \frac{1}{2} g_{ij}(x) q^i q^j, \quad \forall x \in \Omega, \quad \forall q \in \mathbb{R}^n.$$

Notice that, by (5.6.1), we can write (5.6.4) as

$$g_{ij}q^j = p_i, \quad \text{for } i = 1, \dots, n.$$

Hence

$$H(x, p) = g^{ij}p_jp_i - \frac{1}{2}g_{ij}g^{ih}p_hg^{ik}p_k = \frac{1}{2}g^{ij}p_jp_i.$$

◆

The following Theorem holds true

Theorem 5.6.5. *Let $\{g_{ij}(x)\}_{i,j=1}^n$ be a real symmetric matrix $n \times n$ whose entries belong to $C^\infty(\Omega)$ and let us assume that it satisfies (5.6.17). Let $u \in C^\infty(\Omega)$ be a solution to the eikonal equation*

$$g^{ij}(x) \partial_{x^i}u \partial_{x^j}u = 1 \tag{5.6.22}$$

and let $x = \gamma(t)$ be a solution to the system

$$\frac{dx^i(t)}{dt} = g^{ij}(x(t)) \partial_{x^j}u(x(t)), \quad i = 1, \dots, n.$$

Then $x = \gamma(t)$ is a geodesic line w.r.t. the Riemannian metric

$$g_{ij}(x)dx^i \otimes dx^j.$$

Proof. Let

$$L(x, q) = \frac{1}{2}g_{ij}(x)q^iq^j, \quad \forall x \in \Omega, \forall q \in \mathbb{R}^n.$$

Let H be the Hamiltonian of L , that is

$$H(x, p) = \frac{1}{2}g^{ij}(x)p_jp_i \quad \forall x \in \Omega, \forall p \in \mathbb{R}^n.$$

Set

$$p(t) = \nabla u(\gamma(t)), \tag{5.6.23}$$

where u is a solution to equation (5.6.22) and γ is a solution to the equations

$$\frac{d\gamma^i(t)}{dt} = g^{ij}(\gamma(t)) \partial_{x^j}u(\gamma(t)) (= \partial_{p_i}H(\gamma(t), p(t))), \tag{5.6.24}$$

for $i = 1, \dots, n$.

Now, we make the derivative w.r.t. x^k of both the sides of equation (5.6.22) and we get

$$2(g^{ij}(x)\partial_{x^i x^k}^2 u)\partial_{x^j} u + (\partial_{x^k} g^{ij}(x))\partial_{x^i} u \partial_{x^j} u = 0. \quad (5.6.25)$$

By (5.6.23), (5.6.24) and (5.6.25) we have, for $i = 1, \dots, n$,

$$\begin{aligned} \frac{dp_i(t)}{dt} &= \partial_{x^i x^k}^2 u(\gamma(t)) \frac{d\gamma^k(t)}{dt} = \\ &= \partial_{x^i x^k}^2 u(\gamma(t)) g^{kj}(\gamma(t)) \partial_{x^j} u(\gamma(t)) = \\ &= (g^{kj}(\gamma(t)) \partial_{x^i x^k}^2 u(\gamma(t))) \partial_{x^j} u(\gamma(t)) = \\ &= -\frac{1}{2} \partial_{x^i} g^{jk}(\gamma(t)) \partial_{x^j} u(\gamma(t)) \partial_{x^k} u(\gamma(t)) = \\ &= -\partial_{x^i} H(\gamma(t), p(t)). \end{aligned}$$

The just obtained equality and (5.6.24) implies that $(\gamma(t), p(t))$ is a solution to the system

$$\begin{cases} \frac{d\gamma(t)}{dt} = \nabla_p H(x(t), \gamma(t)), \\ \frac{dp(t)}{dt} = -\nabla_x H(\gamma(t), p(t)). \end{cases} \quad (5.6.26)$$

Therefore, by Theorem 5.6.2 and by Proposition (5.6.4) the thesis follows. ■

Remark. Let us observe that if u is a solution to the equation

$$g^{ij}(x)\partial_{x^i} u \partial_{x^j} u = 1$$

and $\gamma(t)$ is a solution to the system

$$\frac{d\gamma^i(t)}{dt} = g^{ij}(\gamma(t)) \partial_{x^j} u(\gamma(t)), \quad i = 1, \dots, n,$$

then t is the natural parameter (in the Riemannian metric) of the path $x = \gamma(t)$. As a matter of fact we have

$$\begin{aligned} g_{ij}(\gamma(t)) \frac{d\gamma^i(t)}{dt} \frac{d\gamma^j(t)}{dt} &= g_{ij}(\gamma(t)) (g^{ik}(\gamma(t)) \partial_{x^k} u(\gamma(t))) (g^{jl}(\gamma(t)) \partial_{x^l} u(\gamma(t))) = \\ &= \delta_j^k \partial_{x^k} u(\gamma(t)) g^{jl}(\gamma(t)) \partial_{x^l} u(\gamma(t)) = \\ &= g^{jl}(\gamma(t)) \partial_{x^l} u(\gamma(t)) \partial_{x^j} u(\gamma(t)) = 1. \end{aligned}$$

Moreover, for fixed $\bar{x} \in \mathbb{R}^n$, set

$$u(\bar{x}) = R_0.$$

If γ is the solution to the Cauchy problem

$$\begin{cases} \frac{d\gamma^i(t)}{dt} = g^{ij}(\gamma(t)) \partial_{x^j} u(\gamma(t)), & i = 1, \dots, n, \\ \gamma(R_0) = \bar{x}, \end{cases}$$

then

$$u(\gamma(t)) = t, \quad \forall t \in I \quad (5.6.27)$$

(I is the maximal interval of the solution γ). As a matter of fact we have

$$\begin{aligned} \frac{d}{dt} u(\gamma(t)) &= \partial_{x^i} u(\gamma(t)) \frac{d\gamma^i(t)}{dt} = \\ &= \partial_{x^i} u(\gamma(t)) g^{ij}(\gamma(t)) \partial_{x^j} u(\gamma(t)) = 1, \end{aligned}$$

hence

$$u(\gamma(t)) = t + C,$$

where C is a constant which can be determined easily in the following way

$$R_0 = u(\bar{x}) = u(\gamma(R_0)) = R_0 + C,$$

hence $C = 0$ and (5.6.27) is proved. These comments will be used in Ch. 15.

◆

Let us conclude this Appendix by some propositions on the extremal and other comments on the geodesics lines.

Proposition 5.6.6. *Let $L \in C^\infty(\Omega \times \mathbb{R}^n)$. We have what follows.*

(i) *if $\bar{x} \in C^\infty([t_0, t_1], \mathbb{R}^n)$ is an extremal of the functional*

$$\int_{t_0}^{t_1} L\left(x(t), \frac{dx(t)}{dt}\right) dt, \quad (5.6.28)$$

we have

$$\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d\bar{x}(t)}{dt} - L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) = \text{constant}. \quad (5.6.29)$$

(ii) If $L(x, q)$ is an homogeneous function w.r.t. q of degree $\alpha \neq 1$ and $\bar{x}(t)$ is an extremal of functional (5.6.28) then

$$L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) = \text{constant}. \quad (5.6.30)$$

Proof.

(i) Set

$$F(t) = \nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d\bar{x}(t)}{dt} - L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right).$$

We have

$$\begin{aligned} \frac{dF(t)}{dt} &= \frac{d}{dt} \left(\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \right) \cdot \frac{d\bar{x}(t)}{dt} + \nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d^2\bar{x}(t)}{dt^2} - \\ &- \nabla_x L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d\bar{x}(t)}{dt} - \nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d^2\bar{x}(t)}{dt^2} = \\ &= \left[\frac{d}{dt} \left(\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \right) - \nabla_x L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \right] \cdot \frac{d\bar{x}(t)}{dt} = 0. \end{aligned}$$

From which the thesis follows.

(ii) By point (i) we have

$$\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d\bar{x}(t)}{dt} - L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) = \text{constant}.$$

On the other hand by the homogeneity of $L(x, \cdot)$ we get

$$\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \cdot \frac{d\bar{x}(t)}{dt} = \alpha L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right).$$

Therefore

$$(\alpha - 1)L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) = \text{constant}$$

and recalling that $\alpha \neq 1$, the thesis follows. ■

Proposition 5.6.7. Let $L \in C^\infty(\Omega \times \mathbb{R}^n)$ be an homogeneous function w.r.t. q of degree 2. If $\bar{x} \in C^\infty([t_0, t_1], \mathbb{R}^n)$ is an extremal of the functional

$$\int_{t_0}^{t_1} L\left(x(t), \frac{dx(t)}{dt}\right) dt, \quad (5.6.31)$$

and

$$L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) > 0, \quad \forall t \in [t_0, t_1], \quad (5.6.32)$$

then \bar{x} is an extremal of the functional

$$\int_{t_0}^{t_1} \sqrt{L\left(x(t), \frac{dx(t)}{dt}\right)} dt. \quad (5.6.33)$$

Proof. By Proposition (5.6.6) and by (5.6.32) we may set

$$c_0^2 = L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) > 0,$$

where c_0 is a positive constant. Since \bar{x} is an extremal of the functional (5.6.31), we have

$$\begin{aligned} \frac{d}{dt} \left(\nabla_q \sqrt{L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right)} \right) &= \frac{1}{2c_0} \frac{d}{dt} \left(\nabla_q L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) \right) = \\ &= \frac{1}{2c_0} \nabla_x L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right) = \\ &= \nabla_x \sqrt{L\left(\bar{x}(t), \frac{d\bar{x}(t)}{dt}\right)}. \end{aligned}$$

Hence \bar{x} is an extremal of functional (5.6.33). ■

Proposition 5.6.8. *Let us suppose that L satisfies the same assumptions of Proposition 5.6.7. Let φ an extremal of functional*

$$\int_{t_0}^{t_1} \sqrt{L\left(x(t), \frac{dx(t)}{dt}\right)} dt. \quad (5.6.34)$$

Let us suppose

$$L\left(\varphi(t), \frac{d\varphi(t)}{dt}\right) > 0, \quad \forall t \in [t_0, t_1],$$

then there exists a unique parametrization $t(\tau)$, $t'(\tau) > 0$ in $[\tau_0, \tau_1]$ ($t(\tau_0) = t_0$ and $t(\tau_1) = t_1$), such that, setting $\psi(\tau) = \varphi(t(\tau))$, we have

$$L\left(\psi(\tau), \frac{d\psi(\tau)}{d\tau}\right) = \text{constant}. \quad (5.6.35)$$

Moreover the path $x = \psi(\tau)$ is an extremal of the functional

$$\int_{\tau_0}^{\tau_1} L\left(x(\tau), \frac{dx(\tau)}{d\tau}\right) d\tau. \quad (5.6.36)$$

Proof. Let

$$f(t) = \sqrt{L\left(\varphi(t), \frac{d\varphi(t)}{dt}\right)}, \quad \forall t \in [t_0, t_1].$$

Let $c_0 > 0$ a be constant and let $t(\tau)$ satisfy

$$\int_{t_0}^{t(\tau)} f(t)dt = c_0\tau, \quad \forall \tau \in [\tau_0, \tau_1].$$

Set $\psi(\tau) = \varphi(t(\tau))$; we get, by the homogeneity of $L(x, \cdot)$,

$$\begin{aligned} \sqrt{L\left(\psi(\tau), \frac{d\psi(\tau)}{d\tau}\right)} &= \sqrt{L\left(\varphi(t(\tau)), \frac{d\varphi}{dt}(t(\tau))\right)} t'(\tau) = \\ &= t'(\tau) \sqrt{L\left(\varphi(t(\tau)), \frac{d\varphi}{dt}(t(\tau))\right)} = c_0. \end{aligned}$$

Hence (5.6.35) is proved.

Now, let us prove that $x = \psi(\tau)$ is an extremal of the functional (5.6.36). Set

$$F(t) = \frac{1}{f(t)} \nabla_q L\left(\varphi(t), \frac{d\varphi(t)}{dt}\right)$$

and

$$G(t) = \frac{1}{f(t)} \nabla_x L\left(\varphi(t), \frac{d\varphi(t)}{dt}\right).$$

Since φ is an extremal of functional (5.6.34), we have

$$\frac{dF(t)}{dt} = G(t). \quad (5.6.37)$$

Now, recalling

$$\sqrt{L\left(\psi(\tau), \frac{d\psi(\tau)}{d\tau}\right)} = c_0,$$

we have (by the homogeneity of L w.r.t. q)

$$\begin{aligned} \nabla_q L\left(\psi(\tau), \frac{d\psi(\tau)}{d\tau}\right) &= t'(\tau) \nabla_q L\left(\varphi(t(\tau)), \frac{d\varphi}{dt}(t(\tau))\right) = \\ &= t'(\tau) \sqrt{L\left(\varphi(t(\tau)), \frac{d\varphi}{dt}(t(\tau))\right)} F(t(\tau)) = \\ &= c_0 F(t(\tau)). \end{aligned}$$

Hence, recalling (5.6.37) (and the homogeneity of L w.r.t. q), we have

$$\begin{aligned}
\frac{d}{d\tau} \left(\nabla_q L \left(\psi(\tau), \frac{d\psi(\tau)}{d\tau} \right) \right) &= c_0 \frac{d}{d\tau} (F(t(\tau))) = \\
&= c_0 t'(\tau) \frac{dF}{dt}(t(\tau)) = c_0 t'(\tau) G(t(\tau)) = \\
&= \frac{c_0 t'(\tau)}{f(t(\tau))} \nabla_x L \left(\varphi(t(\tau)), \frac{d\varphi}{dt}(t(\tau)) \right) = \\
&= \frac{c_0}{t'(\tau) f(t(\tau))} \nabla_x L \left(\psi(\tau), \frac{d\psi(\tau)}{d\tau} \right) = \\
&= \nabla_x L \left(\psi(\tau), \frac{d\psi(\tau)}{d\tau} \right).
\end{aligned}$$

Hence ψ is an extremal of functional (5.6.36). ■

Remark. Let $\{g_{ij}(x)\}_{i,j=1}^n$ be a real symmetric matrix $n \times n$ whose entries belong to $C^\infty(\Omega)$ and let us assume that it satisfies (5.6.17). Let

$$L(x, q) = g_{ij}(x) q^i q^j, \quad \forall x \in \Omega, \quad \forall q \in \mathbb{R}^n.$$

By Proposition 5.6.7 we have that, if $x = \gamma(t)$ is a geodesic line, i.e. it is an extremal of the functional

$$\int_{t_0}^{t_1} L \left(x(t), \frac{dx(t)}{dt} \right) dt = \int_{t_0}^{t_1} g_{ij}(x(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} dt, \quad (5.6.38)$$

then $x = \gamma(t)$ is also an extremal of the functional

$$\int_{t_0}^{t_1} L \left(x(t), \frac{dx(t)}{dt} \right) dt = \int_{t_0}^{t_1} \sqrt{g_{ij}(x(t))} \frac{dx^i}{dt} \frac{dx^j}{dt} dt. \quad (5.6.39)$$

On the other hand, by Proposition 5.6.8, we have that if $x = \gamma(t)$ is an extremal of functional (5.6.39) and if $t(\tau)$ is strictly increasing and it satisfies

$$t'(\tau) \sqrt{g_{ij}(\gamma(t(\tau)))} \frac{d\gamma^i}{dt}(t(\tau)) \frac{d\gamma^j}{dt}(t(\tau)) = c, \quad (5.6.40)$$

where $c > 0$ is a positive constant, then $x = \gamma(t(\tau))$ is an extremal of

$$\int_{t_0}^{t_1} L \left(x(t), \frac{dx(t)}{dt} \right) dt = \int_{\tau_0}^{\tau_1} g_{ij}(x(\tau)) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau, \quad (5.6.41)$$

where τ_0 and τ_1 satisfy $t(\tau_0) = t_0$ and $t(\tau_1) = t_1$. Let us notice that if $c = 1$, then condition (5.6.40) means that τ is the natural parameter of the path $x = \gamma(t)$ (extremal of (5.6.39)) in the riemannian metric $g_{ij}(x) dx^i \otimes dx^j$. ♦

Chapter 6

Real analytic functions

6.1 Power series

In this chapter we will consider the multiple series

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha, \quad (6.1.1)$$

where $c_\alpha \in \mathbb{R}$ (or $c_\alpha \in \mathbb{C}$).

When we say that the series (6.1.1) converges, we will mean *always* that it is **absolutely convergent**. That is

$$\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha| < +\infty.$$

Therefore, if the series (6.1.1) converges, the value of the sum in (6.1.1) does not depend on the order of the terms c_α . If $c_\alpha(x)$ are functions, we will naturally extend the notions of uniform, total convergence, $C^k(\overline{\Omega})$ convergence and so on. For instance, we will say that

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha(x), \quad (6.1.2)$$

uniformly converges to a function f in a set $K \subset \mathbb{R}^n$ provided that:

- (i) for every $x \in K$, $\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha(x)|$ converges,
- (ii) we have

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha(x), \quad \forall x \in K$$

and

(iii)

$$\lim_{N \rightarrow +\infty} \sup_{x \in K} \left| f(x) - \sum_{|\alpha| \leq N} c_\alpha(x) \right| = 0.$$

Let $c_\alpha \in \mathbb{R}$ (or $c_\alpha \in \mathbb{C}$) we call **power series** a series like

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha. \quad (6.1.3)$$

For any $y \in \mathbb{R}^n$ set

$$Q_y = \{x \in \mathbb{R}^n : |x_j| \leq |y_j|, j = 1, \dots, n\}.$$

We have

Proposition 6.1.1. *If series (6.1.3) converges at a point $y \in \mathbb{R}^n$ then the series uniformly converges in Q_y .*

Proof. The convergence at y of (6.1.3) is equivalent to

$$\sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha| |y^\alpha| < +\infty.$$

Hence, we have

$$\sum_{\alpha \in \mathbb{N}_0^n} \sup_{Q_y} |c_\alpha x^\alpha| \leq \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha| |y^\alpha| < +\infty.$$

From which we get the total convergence and, consequently, the uniform convergence, of series (6.1.3). ■

Proposition 6.1.1 implies that the sum of series (6.1.3) is continuous in Q_y .

The differentiability will be proved in Proposition 6.1.3 to prove such a Proposition we need

Lemma 6.1.2. *Let us denote by $v = (1, 1, \dots, 1)$. If $|x_j| < 1$ $j = 1, \dots, n$, we have*

$$\sum_{\alpha \in \mathbb{N}_0^n} x^\alpha = \frac{1}{(v-x)^v}, \quad (6.1.4)$$

and

$$\sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} = \partial^\beta \left(\frac{1}{(v-x)^v} \right) = \frac{\beta!}{(v-x)^{v+\beta}}. \quad (6.1.5)$$

Proof. Concerning the convergence of series (6.1.4), we have, for $|x_j| < 1$, $j = 1, \dots, n$,

$$\begin{aligned} \sum_{\alpha \leq vN} |x^\alpha| &= \sum_{\alpha \leq vN} |x_j^{\alpha_j}| = \\ &= \prod_{j=1}^n \sum_{\alpha_j \leq N} |x_j^{\alpha_j}| = \\ &= \prod_{j=1}^n \frac{1 - |x_j|^{N+1}}{1 - |x_j|} \rightarrow \prod_{j=1}^n \frac{1}{1 - |x_j|}, \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Concerning the sum of the series we have, similarly,

$$\sum_{\alpha \leq vN} x^\alpha = \lim_{N \rightarrow \infty} \prod_{j=1}^n \sum_{\alpha_j \leq N} x_j^{\alpha_j} = \frac{1}{(1 - x_1) \cdots (1 - x_n)} = \frac{1}{(v - x)^v}.$$

Now, let us prove (6.1.5). Recalling (1.2.2)

$$\partial^\beta x^\alpha = \begin{cases} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta}, & \text{for } \alpha \geq \beta, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta} &= \sum_{\alpha \in \mathbb{N}_0^n} \partial^\beta x^\alpha = \\ &= \prod_{j=1}^n \sum_{\alpha_j \in \mathbb{N}_0} \partial^{\beta_j} x_j^{\alpha_j} = \\ &= \prod_{j=1}^n \partial_j^{\beta_j} \frac{1}{1 - x_j} = \partial^\beta \left(\frac{1}{(v - x)^v} \right) = \\ &= \frac{\beta!}{(v - x)^{v + \beta}}. \end{aligned}$$

■

Proposition 6.1.3. *If series (6.1.3) converges at the point $y \in \mathbb{R}^n$ and $y_j > 0$ for every $j = 1, \dots, n$ then, denoted by f the sum of such a series, we have $f \in C^\infty(\text{Int}(Q_y))$, where $\text{Int}(Q_y)$ is the interior part of Q_y .*

Moreover

$$\partial^\alpha f(0) = \frac{1}{\alpha!} c_\alpha. \quad (6.1.6)$$

Proof. In order to prove that $f \in C^\infty(\text{Int}(Q_y))$ it suffices to prove that for every $q \in (0, 1)$ and for every $\beta \in \mathbb{N}_0^n$ we have

$$\sum_{\alpha \in \mathbb{N}_0^n} \sup_{Q_{qy}} |\partial^\beta (c_\alpha x^\alpha)| < +\infty.$$

By (1.2.2) we have

$$\partial^\beta (c_\alpha x^\alpha) = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!} c_\alpha x^{\alpha-\beta}, & \text{for } \alpha \geq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^n} \sup_{Q_{qy}} |\partial^\beta (c_\alpha x^\alpha)| &\leq \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} |c_\alpha| |(qy)^{\alpha-\beta}| = \\ &= \frac{1}{|y^\beta|} \sum_{\alpha \geq \beta} \frac{\alpha! |c_\alpha y^\alpha|}{(\alpha-\beta)!} q^{|\alpha-\beta|}. \end{aligned} \quad (6.1.7)$$

Now, since (6.1.3) converges in $y \in \mathbb{R}^n$, we get

$$|c_\alpha y^\alpha| \leq \mu_y := \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha y^\alpha| < +\infty. \quad (6.1.8)$$

By the above obtained inequality and by (6.1.7) we have (for $y \neq 0$)

$$\sum_{\alpha \in \mathbb{N}_0^n} \sup_{Q_{qy}} |\partial^\beta (c_\alpha x^\alpha)| \leq \frac{\mu_y}{|y^\beta|} \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} q^{|\alpha-\beta|}. \quad (6.1.9)$$

Applying (6.1.5) with $x = qv = q(1, 1 \dots, 1)$, we have

$$\begin{aligned} \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} q^{|\alpha-\beta|} &= \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha-\beta)!} (qv)^{\alpha-\beta} = \\ &= \frac{\beta!}{(v-vq)^{v+\beta}} = \\ &= \frac{\beta!}{(1-q)^{n+|\beta|}}. \end{aligned} \quad (6.1.10)$$

By the just obtained equality and by (6.1.9) we obtain

$$\sum_{\alpha \in \mathbb{N}_0^n} \sup_{Q_{yq}} |\partial^\beta (c_\alpha x^\alpha)| \leq \frac{\mu_y}{|y^\beta|} \frac{\beta!}{(1-q)^{n+|\beta|}} < +\infty. \quad (6.1.11)$$

We have proved so far that for each $q \in (0, 1)$ we have $f \in C^\infty(Q_{yq})$. Therefore $f \in C^\infty(\text{Int}(Q_y))$.

Concerning (6.1.6) we have

$$\partial^\beta f(x) = \sum_{\alpha \in \mathbb{N}_0^n} \partial^\beta (c_\alpha x^\alpha) = \sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha - \beta)!} c_\alpha x^{\alpha - \beta}.$$

Therefore

$$\partial^\beta f(0) = \frac{1}{\beta!} c_\beta.$$

■

Exercise 1. Let $f(x)$ be the sum of the series

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha,$$

in Q_ϱ where $\varrho = (\varrho_1, \dots, \varrho_n)$, with $\varrho_j > 0$, $j = 1, \dots, n$. Let $q \in (0, 1)$ and set

$$r = (1-q) \min_{1 \leq j \leq n} \varrho_j, \quad \text{and} \quad \mu_\varrho = \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha \varrho^\alpha|. \quad (6.1.12)$$

Then we have

$$\sup_{Q_{q\varrho}} |\partial^\beta f| \leq (1-q)^{-n} \mu_\varrho r^{-|\beta|} \beta!. \quad (6.1.13)$$

Solving Exercise 1.

Is an immediate consequence of (6.1.11). ♣

Exercise 2. Prove that

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{|\alpha!|}{\alpha!} x^\alpha = \frac{1}{1 - \sum_{j=1}^n x_j}, \quad \text{for} \quad \sum_{j=1}^n |x_j| < 1, \quad (6.1.14)$$

$$\sum_{\alpha \geq \beta} \frac{|\alpha!|}{(\alpha - \beta)!} x^{\alpha - \beta} = \frac{|\beta!|}{\left(1 - \sum_{j=1}^n x_j\right)^{1+|\beta|}}, \quad \text{for} \quad \sum_{j=1}^n |x_j| < 1. \quad (6.1.15)$$

Solving Exercise 2.

By (1.2.3) we have

$$(x_1 + x_2 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha, \quad m \in \mathbb{N}_0.$$

Hence, for $\sum_{j=1}^n |x_j| < 1$, we have

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^n} \frac{|\alpha!|}{\alpha!} x^\alpha &= \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{|\alpha!|}{\alpha!} x^\alpha = \\ &= \sum_{m=0}^{\infty} (x_1 + x_2 + \cdots + x_n)^m = \\ &= \frac{1}{1 - (x_1 + x_2 + \cdots + x_n)}, \end{aligned}$$

from which we get (6.1.14).

Concerning (6.1.15), it suffices to note that by (6.1.14), we have, for $\sum_{j=1}^n |x_j| < 1$,

$$\begin{aligned} \sum_{\alpha \geq \beta} \frac{|\alpha!|}{(\alpha - \beta)!} x^{\alpha - \beta} &= \sum_{\alpha \in \mathbb{N}_0^n} \frac{|\alpha!|}{\alpha!} \partial^\beta (x^\alpha) = \\ &= \partial^\beta \left(\frac{1}{1 - (x_1 + x_2 + \cdots + x_n)} \right) = \\ &= \frac{|\beta!|}{(1 - (x_1 + x_2 + \cdots + x_n))^{1+|\beta|}}. \end{aligned}$$

6.2 Analytic functions in an open set of \mathbb{R}^n

Definition 6.2.1. Let Ω be an open set of \mathbb{R}^n and $f : \Omega \rightarrow \mathbb{R}$ (or \mathbb{C}) a function. We say that f is an **analytic function of real variables** in $x_0 \in \Omega$ if there exist a neighborhood \mathcal{U}_{x_0} of x_0 , $c_\alpha \in \mathbb{R}$ (\mathbb{C}), $\alpha \in \mathbb{N}_0^n$, such that

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (x - x_0)^\alpha, \quad \forall x \in \mathcal{U}_{x_0}.$$

We say that f is a **analytic function of real variables** in Ω if it is analytic function of real variables in every $x_0 \in \Omega$.

We say that $f : \Omega \rightarrow \mathbb{R}^m$ (or \mathbb{C}^m) where $m \in \mathbb{N}$, $f = (f_1, \cdots, f_m)$ is an **analytic function of real variables** in $x_0 \in \Omega$ (or in Ω) provided f_j are analytic functions of real variables in $x_0 \in \Omega$ (or in Ω).

In what follows, if there is no ambiguity, we will simply say "**analytic functions**", omitting the expression "of real variables".

We will denote by $C^\omega(\Omega; \mathbb{R}^m)$ ($C^\omega(\Omega; \mathbb{C}^m)$) the class of analytic function defined on Ω with values in \mathbb{R}^m (\mathbb{C}^m), if $m = 1$ we write, if there is no ambiguity, simply $C^\omega(\Omega)$ to denote $C^\omega(\Omega; \mathbb{R})$ ($C^\omega(\Omega; \mathbb{C})$).

Proposition 6.1.3 immediately gives:

$$C^\omega(\Omega) \subset C^\infty(\Omega). \quad (6.2.1)$$

Moreover, for every $x_0 \in \Omega$ there exists a neighborhood \mathcal{U}_{x_0} of x_0 such that

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial^\alpha f(x_0) (x - x_0)^\alpha, \quad \forall x \in \mathcal{U}_{x_0} \quad (6.2.2)$$

and there exist $M > 0$, $r > 0$ (depending on x_0) and $\tilde{\mathcal{U}}_{x_0}$, neighborhood of x_0 , with $\tilde{\mathcal{U}}_{x_0} \Subset \mathcal{U}_{x_0}$, such that

$$|\partial^\alpha f(x)| \leq M |\alpha|! r^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n, \quad \forall x \in \tilde{\mathcal{U}}_{x_0}, \quad (6.2.3)$$

((6.2.3) follows by (6.1.13)).

We recall that the inclusion (6.2.1) is proper. As a matter of fact, the function

$$f(t) = \begin{cases} e^{-1/t^2}, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases} \quad (6.2.4)$$

belongs to $C^\infty(\mathbb{R})$, but it is not analytic. As a matter of fact, since we have $f^{(k)}(0) = 0$, for every $k \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) t^k = 0 \neq f(t), \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

Therefore, for every $t \neq 0$, $f(t)$ is different from the sum of its Taylor series.

The analytic functions enjoy the **unique continuation property**. Indeed, the following holds true.

Theorem 6.2.2. *Let Ω be a connected open set of \mathbb{R}^n and $f \in C^\omega(\Omega)$. Let $x_0 \in \Omega$. Then we have*

$$\partial^\alpha f(x_0) = 0, \quad \forall \alpha \in \mathbb{N}_0^n \quad \implies \quad f \equiv 0, \text{ in } \Omega.$$

In particular, if f vanishes identically in an open set (not empty) of Ω then f vanishes identically in Ω .

Proof. Let

$$\tilde{\Omega} = \{x \in \Omega : \partial^\alpha f(x) = 0, \quad \forall \alpha \in \mathbb{N}_0^n\}.$$

It is clear that $\tilde{\Omega} \neq \emptyset$ because $x_0 \in \tilde{\Omega}$. Therefore, whether we prove that $\tilde{\Omega}$ is at the same time an open and a closed set in Ω (in the topology induced by \mathbb{R}^n), as Ω is connected, we have $\Omega = \tilde{\Omega}$ and the thesis follows.

Since f is continuous, the set $\tilde{\Omega}$ is closed in Ω . Now we prove that $\tilde{\Omega}$ is an open set of Ω .

Let $\tilde{x} \in \tilde{\Omega}$. By the analyticity of f we have that there exists a neighborhood $\mathcal{U}_{\tilde{x}}$ of \tilde{x} such that

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial^\alpha f(\tilde{x})(x - \tilde{x})^\alpha, \quad \forall x \in \mathcal{U}_{\tilde{x}}.$$

On the other hand, $\partial^\alpha f(\tilde{x}) = 0$ for every $\alpha \in \mathbb{N}_0^n$, hence $f(x) = 0$ for every $x \in \mathcal{U}_{\tilde{x}}$. Hence

$$\partial^\alpha f(x) = 0, \quad \forall \alpha \in \mathbb{N}_0^n, \quad \forall x \in \mathcal{U}_{\tilde{x}},$$

from which $\mathcal{U}_{\tilde{x}} \subset \tilde{\Omega}$. Therefore $\Omega = \tilde{\Omega}$. ■

In Theorem 6.2.4 we will prove that the analytic functions can be characterized by the growth of their derivatives. We premise the following

Definition 6.2.3. Let $m \in \mathbb{N}$ and let Ω be an open set of \mathbb{R}^n . Let $f : \Omega \rightarrow \mathbb{R}^m$ (or \mathbb{C}^m), $f = (f_1, \dots, f_m)$. Let $x_0 \in \Omega$ and $M, r > 0$, $j = 1, \dots, m$. We write

$$f \in \mathcal{C}_{M,r}(x_0),$$

provided that $f \in C^\infty$ in a neighborhood of x_0 and we have

$$|\partial^\alpha f_j(x_0)| \leq M |\alpha|! r^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n, \quad j = 1, \dots, m.$$

Theorem 6.2.4. Let Ω be an open set of \mathbb{R}^n and $f : \Omega \rightarrow \mathbb{R}^m$ (or \mathbb{C}^m). The following conditions are equivalent:

- (i) f is analytic in Ω ,
- (ii) for any compact K , $K \subset \Omega$, there exist $M, r > 0$ (depending on K) such that

$$f \in \mathcal{C}_{M,r}(x_0), \quad \forall x_0 \in K.$$

Proof. It suffices to consider the case $m = 1$.

We prove that (i) \Rightarrow (ii).

By (6.2.3) we know that if $y \in \Omega$, there exist $M_y, r_y > 0$ and a neighborhood \mathcal{U}_y of y such that

$$\sup_{\mathcal{U}_y} |\partial^\alpha f| \leq M_y |\alpha|! r_y^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n. \quad (6.2.5)$$

Let $K \subset \Omega$ be a compact, then $\{\mathcal{U}_y\}_{y \in K}$ is an open covering of K , hence there exist $\mathcal{U}_{y_1}, \dots, \mathcal{U}_{y_N}$ such that

$$K \subset \bigcup_{j=1}^N \mathcal{U}_{y_j}.$$

Set

$$M = \max_{1 \leq j \leq N} M_{y_j}, \quad r = \min_{1 \leq j \leq N} r_{y_j}.$$

By (6.2.5) we have

$$\sup_K |\partial^\alpha f| \leq \sup_{\bigcup_{j=1}^N \mathcal{U}_{y_j}} |\partial^\alpha f| \leq M |\alpha|! r^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n. \quad (6.2.6)$$

We prove that (ii) \Rightarrow (i).

Let us suppose that (ii) holds. Let $x_0 \in \Omega$ and $\rho > 0$ satisfy $\overline{B_\rho(x_0)} \subset \Omega$. It is not restrictive to assume that $x_0 = 0 \in \Omega$. Let us choose $K = \overline{B_\rho}$ and let $M, r > 0$ satisfy

$$|\partial^\alpha f(x)| \leq M |\alpha|! r^{-|\alpha|}, \quad \forall x \in \overline{B_\rho}, \quad \forall \alpha \in \mathbb{N}_0^n. \quad (6.2.7)$$

Set $d, 0 < d < \min\{r, \rho\}$, we now prove

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha, \quad \text{for every } x \text{ such that } \sum_{j=1}^n |x_j| \leq d.$$

Let x satisfy $\sum_{j=1}^n |x_j| \leq d$ and set

$$\phi(t) = f(tx), \quad t \in [0, 1].$$

We have, for any $m \in \mathbb{N}$,

$$f(x) = \phi(1) = \sum_{k=0}^{m-1} \frac{1}{k!} \phi^{(k)}(0) + \mathcal{R}_m = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha + \mathcal{R}_m. \quad (6.2.8)$$

where

$$\mathcal{R}_m = \frac{1}{(m-1)!} \int_0^1 (1-t)^{(m-1)} \phi^{(m)}(t) dt.$$

By (1.2.3), (6.2.7) and by $\sum_{j=1}^n |x_j| \leq d$ we have

$$\begin{aligned} \left| \frac{1}{m!} \phi^{(m)}(t) \right| &= \left| \sum_{|\alpha|=m} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha \right| \leq \\ &\leq \sum_{|\alpha|=m} \frac{|\alpha!|}{\alpha!} M r^{-|\alpha|} |x^\alpha| = \\ &= M r^{-m} \sum_{|\alpha|=m} \frac{|\alpha!|}{\alpha!} |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} = \\ &= M r^{-m} \left(\sum_{j=1}^n |x_j| \right)^m \leq \\ &\leq M \left(\frac{d}{r} \right)^m. \end{aligned}$$

Therefore

$$\begin{aligned} |\mathcal{R}_m| &\leq \frac{1}{(m-1)!} \int_0^1 (1-t)^{(m-1)} |\phi^{(m)}(t)| dt \leq \\ &\leq \frac{1}{(m-1)!} \int_0^1 (1-t)^{(m-1)} m! M \left(\frac{d}{r} \right)^m dt = \quad (6.2.9) \\ &= M \left(\frac{d}{r} \right)^m. \end{aligned}$$

All in all, by (6.2.8) and (6.2.9) we get, if $\sum_{j=1}^n |x_j| \leq d$ (recall $d < r$),

$$\left| f(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha \right| \leq M \left(\frac{d}{r} \right)^m \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

■

6.3 Majorant functions

In the proof of the Cauchy–Kowalevskaya Theorem we will make use the **method of majorant series**. We give the following

Definition 6.3.1. Let $m \in \mathbb{N}$. Let \mathcal{U}_{x_0} be a neighborhood of $x_0 \in \mathbb{R}^n$ and let $f : \mathcal{U}_{x_0} \rightarrow \mathbb{R}^m$ (or \mathbb{C}^m), $F : \mathcal{U}_{x_0} \rightarrow \mathbb{R}^m$, $f = (f_1, \dots, f_m)$, $F = (F_1, \dots, F_m)$. Let us suppose that $f_j, F_j \in C^\infty(\mathcal{U}_{x_0})$, $j = 1, \dots, m$. We say that F is a **majorant** of f or F **majorizes** f in x_0 and we write

$$f \preceq F, \quad \text{in } x_0,$$

provided

$$|\partial^\alpha f_j(x_0)| \leq \partial^\alpha F_j(x_0), \quad \forall \alpha \in \mathbb{N}_0^n, \quad j = 1, \dots, m.$$

Remark. Notice that if $f \preceq F$ in x_0 , then, we have $f \preceq \tilde{F}$ in x_0 , where

$$\tilde{F}(x) = F(a_1^{-1}x_1, \dots, a_n^{-1}x_n)$$

for every $a_j \in (0, 1]$, $j = 1, \dots, n$. \blacklozenge

In what follows we will assume, without any restriction that $x_0 = 0$ and we will write simply (if there is no ambiguity) $f \preceq F$ instead of $f \preceq F$ in 0.

Proposition 6.3.2. Let $f : \mathcal{U}_0 \rightarrow \mathbb{C}^m$, where \mathcal{U}_0 is a neighborhood of $0 \in \mathbb{R}^n$ and, for given $M, r > 0$, let

$$\phi_{M,r}(x) = \frac{Mr}{r - (x_1 + \dots + x_n)}.$$

Then we have

- (i) $f \in \mathcal{C}_{M,r}(0)$ if and only if $f \preceq v_m \phi_{M,r}$ (here $v_m = \underbrace{(1, \dots, 1)}_m$),
(ii) $f \in \mathcal{C}_{M,r}(0)$ and $f(0) = 0$ if and only if $f \preceq v_m (\phi_{M,r} - M)$.

Proof

(i) is an immediate consequence of (6.1.14). We have, indeed,

$$\phi_{M,r}(x) = \frac{M}{1 - \frac{x_1 + \dots + x_n}{r}} = \sum_{\alpha \in \mathbb{N}_0^n} \frac{M|\alpha|!r^{-|\alpha|}}{\alpha!} x^\alpha.$$

Hence $\partial^\alpha \phi_{M,r}(0) = M|\alpha|!r^{-|\alpha|}$, from which we get (i). (ii) is a trivial consequence of (i). \blacksquare

From the derivation rules we have

Proposition 6.3.3. *Let $f, g : \mathcal{U}_0 \rightarrow \mathbb{C}$; $F, G : \mathcal{U}_0 \rightarrow \mathbb{R}$ where \mathcal{U}_0 is a neighborhood of $0 \in \mathbb{R}^n$. If,*

$$f \preceq F, \quad \text{and} \quad g \preceq G$$

then

$$f + g \preceq F + G, \quad fg \preceq FG.$$

Proof. The proof is left as an exercise to the reader. ■

Proposition 6.3.4. *Let $n, m, p \in \mathbb{N}$ and*

$$f, F : \mathcal{U}_0 \subset \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

where \mathcal{U}_0 is a neighborhood of 0. Let us suppose

$$f(0) = F(0) = 0,$$

and let \mathcal{V} be an open set of \mathbb{R}^m such that $\mathcal{U}_0 \subset \mathcal{V}$. Let

$$g, G : \mathcal{V} \rightarrow \mathbb{R}^p.$$

Let us assume that f, F, g, G be of class C^∞ and

$$f \preceq F, \quad g \preceq G.$$

Then we have

$$g \circ f \preceq G \circ F.$$

Proof. Set $h = g \circ f$ e $H = G \circ F$. By the chain rule we have that for every $\alpha \in \mathbb{N}_0^n$ there exists a polynomial P_α with positive coefficients independent of f, g, F, G such that, for $j = 1, \dots, p$, we have

$$\partial^\alpha h_j(0) = P_\alpha (\partial^\beta g_l(0), \dots, \partial^\gamma f_k(0)),$$

$$\partial^\alpha H_j(0) = P_\alpha (\partial^\beta G_l(0), \dots, \partial^\gamma F_k(0)).$$

Since

$$|\partial^\gamma f_k(0)| \leq \partial^\gamma F_k(0), \quad \forall \beta \in \mathbb{N}_0^n, \quad k = 1, \dots, m,$$

$$|\partial^\beta g_l(0)| \leq \partial^\beta G_l(0), \quad \forall \beta \in \mathbb{N}_0^m, \quad l = 1, \dots, p,$$

and taking into account that the coefficients of P_α are positive, we get

$$\begin{aligned} |\partial^\alpha h_j(0)| &\leq P_\alpha (|\partial^\beta g_l(0)|, \dots, |\partial^\gamma f_k(0)|) \leq \\ &\leq P_\alpha (|\partial^\beta G_l(0)|, \dots, |\partial^\gamma F_k(0)|) = \\ &= \partial^\alpha H_j(0). \end{aligned}$$

■

Proposition 6.3.5. *Let $n, m, p \in \mathbb{N}$. Let M, r, μ, ρ be positive numbers and $\tilde{x} \in \mathbb{R}^n, \tilde{y} \in \mathbb{R}^m$. Let $f \in \mathcal{C}_{M,r}(\tilde{x})$ be a function with values in \mathbb{R}^m , $g \in \mathcal{C}_{\mu,\rho}(\tilde{y})$ be a function with values in \mathbb{R}^p , $\tilde{y} = f(\tilde{x})$. Then*

$$h := g \circ f \in \mathcal{C}_{\mu, \frac{\rho r}{Mm+\rho}}(\tilde{x}). \quad (6.3.1)$$

Proof. Set

$$g^*(y) = g(y + \tilde{y}), \quad f^*(x) = f(x + \tilde{x}) - f(\tilde{x}).$$

We have

$$h(x + \tilde{x}) = g(\tilde{y} + f(x + \tilde{x}) - f(\tilde{x})) = g^*(f^*(x))$$

and

$$f^* \in \mathcal{C}_{M,r}(0), \quad g^* \in \mathcal{C}_{\mu,\rho}(0).$$

Proposition 6.3.2 implies

$$f^* \preceq v_m (\phi_{M,r} - M). \quad (6.3.2)$$

$$g^* \preceq v_p \phi_{\mu,\rho}. \quad (6.3.3)$$

Set

$$\chi(x) = \phi_{\mu,\rho} ((\phi_{M,r}(x) - M)).$$

By Proposition 6.3.4, by (6.3.2) and (6.3.3) we have

$$h(x + \tilde{x}) = (g^* \circ f^*)(0) \preceq v_p \chi(x). \quad (6.3.4)$$

Moreover it is simple to obtain

$$\begin{aligned} \chi(x) &= \phi_{\mu,\rho} (\phi_{M,r}(x) - M) = \\ &= \frac{\mu\rho}{\rho - m (\phi_{M,r}(x) - M)} = \\ &= \frac{\mu\rho (r - (x_1 + \dots + x_n))}{\rho r - (\rho + mM)(x_1 + \dots + x_n)}. \end{aligned}$$

Now we check that

$$\chi(x) \preccurlyeq \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \cdots + x_n)}. \quad (6.3.5)$$

To this purpose, set $A = \rho r$, $B = \rho + mM$, $t = x_1 + \cdots + x_n$,

$$\tilde{\chi}(t) = \mu\rho \frac{r - t}{A - Bt}.$$

We have

$$\chi(x) = \tilde{\chi}(x_1 + \cdots + x_n). \quad (6.3.6)$$

and

$$\begin{aligned} \frac{r - t}{A - Bt} &= \frac{r/A}{1 - Bt/A} - \frac{t}{A} \frac{1}{(1 - \frac{B}{A}t)} = \\ &= \frac{r}{A} \sum_{k=0}^{\infty} \left(\frac{Bt}{A}\right)^k - \frac{t}{A} \sum_{k=0}^{\infty} \left(\frac{Bt}{A}\right)^k = \\ &= \frac{r}{A} + \frac{1}{A} \sum_{k=1}^{\infty} \left[\left(\frac{B}{A}\right)^k r - \left(\frac{B}{A}\right)^{k-1} t \right] t^k \preccurlyeq \\ &\preccurlyeq \frac{r}{A} + \frac{r}{A} \sum_{k=1}^{\infty} \left(\frac{B}{A}\right)^k t^k = \frac{r}{A - Bt}. \end{aligned} \quad (6.3.7)$$

By (6.3.6) and (6.3.7) we obtain (6.3.5). Finally, by (6.3.4), (6.3.5) and by Proposition 6.3.2 we get (6.3.1). ■

Theorem 6.3.6. *Let $n, m, p \in \mathbb{N}$. Let Ω_1 be an open set of \mathbb{R}^n , and Ω_2 be an open set of \mathbb{R}^m . Let $f \in C^\omega(\Omega_1, \mathbb{R}^m)$ satisfy $f(\Omega_1) \subset \Omega_2$. Let $g \in C^\omega(\Omega_2, \mathbb{R}^p)$. Then $g \circ f \in C^\omega(\Omega_1, \mathbb{R}^p)$.*

Proof. Is an immediate consequence of Proposition 6.3.5 and of Theorem 6.2.4. ■

In what follows we will use the **Inverse Function Theorem** and the **Implicit Function Theorem for analytic functions**.

For instance, we will exploit the following fact. If $\phi : \Omega \rightarrow \mathbb{R}$ is an analytic function in Ω and

$$\nabla\phi(x_0) \neq 0,$$

$x_0 \in \Omega$, then there exist $r, \delta > 0$ and an isometry

$$\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

such that

$$\Psi(0) = x_0,$$

$$\Psi^{-1}(\{x \in \Omega : \phi(x) = \phi(x_0)\}) \cap Q_{r,2M} = \{(x', \varphi(x')) : x' \in B'_r\}$$

where $\varphi \in C^\omega(B'_r; \mathbb{R})$ and it satisfies

$$\varphi(0) = 0, \quad |\nabla \varphi(0)| = 0,$$

and

$$\|\varphi\|_{C^1(\overline{B'_{r_0}})} \leq Mr.$$

We will not prove this Theorem which can be proved by the method of the majorant functions that we will learn to use in the next Chapter.

Chapter 7

The Cauchy problem for PDEs with analytic coefficients

7.1 Formulation of the Cauchy problem

In this Section we will give a fairly general formulation of the Cauchy problem. Although we are mainly interested in the linear operators, the formulation that we will give also applies to the fully nonlinear operators.

Let Ω be a connected open set of \mathbb{R}^n . Let $x_0 \in \Omega$, $\phi \in C^m(\Omega; \mathbb{R})$, where $m \in \mathbb{N}$. Let us suppose that

$$\phi(x_0) = 0. \quad (7.1.1)$$

Set

$$\Gamma = \{x \in \Omega : \phi(x) = 0\} \quad (7.1.2)$$

and let us assume

$$\nabla\phi(x) \neq 0, \quad \forall x \in \Gamma. \quad (7.1.3)$$

Let us denote by

$$\nu(x) = -\frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \quad \forall x \in \Gamma. \quad (7.1.4)$$

Let be given the function g_0, g_1, \dots, g_{m-1} , defined on Γ , and let $F(x, (p_\alpha)_{|\alpha| \leq m})$ be a function defined on $\Omega \times \mathbb{R}^{N_m}$, where $N_m \in \mathbb{N}$ depends on m only. The **Cauchy problem** is formulated as follows.

Determine u of class C^m in a neighborhood \mathcal{U} of x_0 such that

$$\begin{cases} F(x, (\partial^\alpha u)_{|\alpha| \leq m}) = 0, & \forall x \in \mathcal{U}, \\ \frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), & j = 0, 1, \dots, m-1, \forall x \in \Gamma \cap \mathcal{U}. \end{cases} \quad (7.1.5)$$

The functions g_0, g_1, \dots, g_{m-1} and Γ are called, respectively, the **initial data** or the **initial values** and the **initial surface** of Cauchy problem (7.1.5). The equations

$$\frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), \quad j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma \cap \mathcal{U}, \quad (7.1.6)$$

are called the **initial conditions of the Cauchy problem**. Of course, it makes sense and interest to set more general initial conditions. For instance, instead of the vector field $\nu(x)$, we may consider a vector field $\ell(x)$ in a neighborhood of Γ , of class C^{m-1} , requiring that

$$\frac{\partial^j u(x)}{\partial \ell^j} = g_j(x), \quad j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma \cap \mathcal{U}. \quad (7.1.7)$$

We can easily check that if the vector field $\ell(x)$ and the functions g_j are smooth enough and if $\ell(x) \cdot \nu(x) \neq 0$, for any $x \in \Gamma$, then conditions (7.1.6) and (7.1.7) are equivalent. To realize this, let us consider the simple case where $m = 2$, $\Omega = \mathbb{R}^n$, $\phi(x) = x_n$, hence $\nu(x) = -e_n$ for every $x \in \Gamma$ and $\ell = (\ell', \ell_n)$, where $\ell' = (\ell_1, \dots, \ell_{n-1})$, is a vector field such that $\ell_n(x') \neq 0$ for every $x' \in \mathbb{R}^{n-1}$. Let us assume $u \in C^2(\mathbb{R}^n)$ and

$$u(x', 0) = g_0(x'), \quad \partial_n u(x', 0) = -g_1(x'), \quad \forall x' \in \mathbb{R}^{n-1}, \quad (7.1.8)$$

where $g_0 \in C^1(\mathbb{R}^{n-1})$ and $g_1 \in C^0(\mathbb{R}^{n-1})$. By the first equation in (7.1.8) we have

$$\nabla_{x'} u(x', 0) = \nabla_{x'} g_0(x'), \quad \forall x' \in \mathbb{R}^{n-1}$$

that, together with $\partial_n u(x', 0) = -g_1(x')$, gives

$$\frac{\partial u}{\partial \ell}(x', 0) = \ell' \cdot \nabla_{x'} g_0(x') - \ell_n g_1(x'), \quad \forall x' \in \mathbb{R}^{n-1}. \quad (7.1.9)$$

Conversely, let us suppose that

$$u(x', 0) = \tilde{g}_0(x'), \quad \frac{\partial}{\partial \ell} u(x', 0) = \tilde{g}_1(x'), \quad \forall x' \in \mathbb{R}^{n-1}, \quad (7.1.10)$$

where $\tilde{g}_0 \in C^1(\mathbb{R}^{n-1})$ and $\tilde{g}_1 \in C^0(\mathbb{R}^{n-1})$. We have, by the first equation in (7.1.10) and taking into account that $\ell_n \neq 0$,

$$\partial_n u(x', 0) = \frac{1}{\ell_n(x')} (-\ell'(x') \cdot \nabla_{x'} \tilde{g}_0(x') + \tilde{g}_1(x')).$$

We also notice that if $\ell_n = 0$ at some point $x'_0 \in \mathbb{R}^{n-1}$ then between (7.1.8) and (7.1.10) there is no equivalence. Actually, if (7.1.8) holds, then we can equally get (7.1.9), but by (7.1.10) we see that between \tilde{g}_0 and \tilde{g}_1 the following condition of compatibility needs to be fulfilled

$$-\ell'(x'_0) \cdot \nabla_{x'} \tilde{g}_0(x'_0) + \tilde{g}_1(x'_0) = 0.$$

On the other hand, if this condition is satisfied, it is undetermined the value of $\partial_n u(x'_0, 0)$.

7.2 The characteristic surfaces

The notion of the **characteristic surface** has a fundamental importance in the investigation of the Cauchy problem. Roughly speaking, we say that the **surface** Γ is noncharacteristic if, assuming $\phi, u, F, g_j \in C^\infty$, for $j = 0, 1, \dots, m-1$, all the derivatives of u on Γ can be determined from by (7.1.5).

Of course, we need to specify this notion and arrive at a formal definition, however in what follows we will not tackle problem (7.1.5) in its full generality, but we will limit ourselves to the linear case, i.e.

$$F(x, (\partial^\alpha u)_{|\alpha| \leq m}) = P(x, \partial)u - f(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u - f(x),$$

Where we will assume, unless explicitly otherwise stated,

$$\phi, a_\alpha, f \in C^\infty(\Omega), \quad |\alpha| \leq m \quad (7.2.1)$$

and that (7.1.3) holds.

To motivate the definition of a characteristic surface that we will give, we begin by considering the following Cauchy problem

$$\begin{cases} P(x, \partial)u = f(x), & \forall x \in B_R, \\ \partial_n^j u(x', 0) = g_j(x'), & j = 0, 1, \dots, m-1, \forall x' \in B'_R. \end{cases} \quad (7.2.2)$$

Here $\phi(x) = -x_n$, $\Gamma = \{x \in B_R : x_n = 0\}$.

Now, if g_j , $j = 0, 1, \dots, m-1$, are of class C^∞ , and if there exists a solution $u \in C^\infty(B_R)$ to problem (7.2.2), we have, by the initial condizions,

$$\partial^{\alpha'} \partial_n^j u(x', 0) = \partial^{\alpha'} g_j(x'), \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \quad j = 0, 1, \dots, m-1. \quad (7.2.3)$$

Let us notice that by (7.2.3) we *cannot* determine the derivatives

$$\partial_n^j u(x', 0), \quad j \geq m \quad (7.2.4)$$

and a fortiori, we cannot determine the derivatives $\partial^{\alpha'} \partial_n^j u(x', 0)$ for $j \geq m$, $\alpha' \in \mathbb{N}_0^{n-1}$. To gain such derivatives we should exploit the equation

$$P(x, \partial)u = f(x)$$

which we write in the form

$$a_{(0', m)}(x', 0) \partial_n^m u(x', 0) = - \sum_{|\alpha'| \leq m, \alpha_n < m} a_\alpha(x', 0) \partial^\alpha u(x', 0) + f(x', 0). \quad (7.2.5)$$

Let us note that to the right-hand side of (7.2.5), by (7.2.3), all the derivatives that appear can be expressed in terms of the initial data and their derivatives. Therefore, if

$$a_{(0', m)}(x', 0) \neq 0, \quad \forall x' \in B'_R, \quad (7.2.6)$$

by (7.2.5) we determine $\partial_n^m u(x', 0)$, hence we can calculate

$$\partial^{\alpha'} \partial_n^m u(x', 0), \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \forall x' \in B'_R. \quad (7.2.7)$$

Actually, condition (7.2.6) allows us to deduce all the derivatives of $u(x', 0)$. As a matter of fact, by calculating the derivatives w.r.t. x_n of both the sides of (7.2.5), we have

$$\begin{aligned} a_{(0', m)}(x) \partial_n^{m+1} u(x) &= \underbrace{-\partial_n a_{(0', m)}(x) \partial_n^m u(x)}_{\text{known for } x_n=0 \text{ (by (7.2.7))}} - \\ &\quad - \underbrace{\partial_n \left(\sum_{|\alpha'| \leq m, \alpha_n < m} a_\alpha(x) \partial^\alpha u(x) \right)}_{\text{known for } x_n=0 \text{ (by (7.2.3))}} + \\ &\quad + \underbrace{\partial_n f(x)}_{\text{known for } x_n=0}. \end{aligned} \quad (7.2.8)$$

Again, condition (7.2.6) allows us to derive from (7.2.8) the derivative

$$\partial_n^{m+1}u(x', 0), \quad \forall x' \in B'_R.$$

Of course, we can further make the derivatives of both the sides of (7.2.8) and we determine $\partial_n^{m+2}u(x', 0)$. Iterating the procedure we obtain the derivatives

$$\partial_n^k u(x', 0), \quad \forall k \in \mathbb{N}_0, \quad \forall x' \in B'_R$$

and then calculating the derivatives w.r.t. x_1, x_2, \dots, x_{n-1} we can calculate all the derivatives of u at the points $(x', 0) \in B'_R$.

At this point we note that condition (7.2.6) can be written

$$P_m((x', 0), \nu) = i^m \sum_{|\alpha|=m} a_\alpha(x', 0) \nu^\alpha \neq 0, \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \quad \forall x' \in B'_R \quad (7.2.9)$$

being, in this case, $\nu = e_n$ and recalling that $P_m(x, \xi) = i^m \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$.

Now we give the following general definition

Definition 7.2.1. Let Ω be an open set of \mathbb{R}^n and $x_0 \in \Omega$. Let $a_\alpha \in C^0(\Omega)$, for any $|\alpha| \leq m$. Let

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

be a linear differential operator of order m .

We say that $\ell \in \mathbb{R}^n$ is a **characteristic direction** for the operator $P(x, \partial)$ in x_0 if

$$P_m(x_0, \ell) = 0. \quad (7.2.10)$$

Let $\phi \in C^1(\Omega)$ satisfy

$$\nabla \phi(x_0) \neq 0.$$

We say that $\Gamma = \{x \in \Omega : \phi(x) = \phi(x_0)\}$ is a **characteristic surface in x_0** for the operator $P(x, \partial)$ provided $\nu(x_0) = -\frac{\nabla \phi(x_0)}{|\nabla \phi(x_0)|}$ is a characteristic direction for $P(x, \partial)$ in x_0 that is, if

$$P_m(x_0, \nu(x_0)) = i^m \sum_{|\alpha|=m} a_\alpha(x_0) \nu^\alpha(x_0) = 0,$$

or, equivalently,

$$P_m(x_0, \nabla \phi(x_0)) = 0.$$

In the sequel, we will say that Γ is a **noncharacteristic surface in x_0** for the operator $P(x, \partial)$, provided that

$$P_m(x_0, \nabla\phi(x_0)) \neq 0.$$

We will say that Γ is a **characteristic surface** for the operator $P(x, \partial)$ if it is characteristic **at each point of Γ** . Finally, we say that Γ is a **noncharacteristic surface** for the operator $P(x, \partial)$ as long as it is noncharacteristic **at every point of Γ** . Let us notice that being a "non characteristic" is more restrictive than the negation of "characteristic surface." This little abuse will simplify the form of expression later on.

We notice that the previous definition **involves only the principal part of the operator $P(x, \partial)$** .

For completeness, we also give a definition of a non characteristic surface for the quasilinear operator of order m

$$\mathcal{P}(u) = \sum_{|\alpha|=m} a_\alpha(x, (\partial^\beta u)_{\|\beta\| \leq m-1}) \partial^\alpha u + a_0(x, (\partial^\beta u)_{\|\beta\| \leq m-1}), \quad (7.2.11)$$

where $a_\alpha(x, (p^\beta)_{\|\beta\| \leq m-1})$, a_0 are given functions.

Definition 7.2.2. Let Ω , x_0 , ϕ , Γ be like in Definition 7.2.1 and let \mathcal{P} be like in (7.2.11). We say that Γ is a **noncharacteristic surface in x_0** for the operator \mathcal{P} if Γ is a noncharacteristic surface in x_0 for to the operator

$$\sum_{|\alpha|=m} a_\alpha(x, (p^\beta)_{\|\beta\| \leq m-1}) \partial^\alpha$$

for each value of p^β , for $|\beta| \leq m-1$. We say that Γ is a **noncharacteristic surface** for the operator \mathcal{P} if it is a noncharacteristic in each point of Γ .

Remark. Definition 7.2.2 is actually more restrictive than that would be needed to determine $\partial^\gamma u(x', 0)$ for all $\gamma \in \mathbb{N}_0^n$. Let us consider, for instance, the Cauchy problem

$$\begin{cases} \mathcal{P}(u) = 0, & \forall x \in B_R, \\ \partial_n^j u(x', 0) = g_j(x'), & j = 0, 1, \dots, m-1, \forall x' \in B'_R, \end{cases} \quad (7.2.12)$$

Let us suppose that a_α, a_0, g_j are functions C^∞ . We have proved before that the derivatives $\partial^\gamma u(x', 0)$, for $|\gamma| \leq m - 1$, depend on the initial data only. Hence the value of $a_\alpha((x', 0), (\partial^\beta u(x', 0))_{|\beta| \leq m-1})$ are determined by the Cauchy data only. Therefore, it should be more natural to say that Γ is a noncharacteristic surface for the operator \mathcal{P} provided that Γ is a noncharacteristic surface in x_0 for to the operator

$$\sum_{|\alpha|=m} a_\alpha((x', 0), (\partial^\beta u(x', 0))_{|\beta| \leq m-1}) \partial^\alpha.$$

◆

7.3 Transformation of a linear differential operator.

We wish to examine the transformation of the principal part of the linear differential operator

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha,$$

under the action of

$$\Lambda \in C^m(\Omega, \mathbb{R}^n),$$

where Ω is an open set of \mathbb{R}^n and $\Lambda = (\Lambda_1, \dots, \Lambda_n)$ is a diffeomorphism of class C^m . By this we mean that Λ is injective and it satisfies

$$\det(\partial_x \Lambda(x)) \neq 0, \quad \forall x \in \Omega,$$

where $\partial_x \Lambda(x)$ is the jacobian matrix of Λ . Let $u \in C^m(\Omega)$ and set

$$v(y) = u(\Lambda^{-1}(y)), \quad \forall y \in \tilde{\Omega} := \Lambda(\Omega),$$

we have $v \in C^m(\tilde{\Omega})$.

Moreover by

$$u(x) = v(\Lambda(x)), \quad \forall x \in \Omega,$$

we have

$$\partial_{x_j} u(x) = \sum_{k=1}^n (\partial_{y_k} v)(\Lambda(x)) \partial_{x_j} \Lambda_k(x), \quad j = 1, \dots, n, \quad \forall x \in \Omega$$

$$\partial_{x_j x_i}^2 u(x) = \sum_{h,k=1}^n (\partial_{y_h y_k} v)(\Lambda(x)) \partial_{x_i} \Lambda_h(x) \partial_{x_j} \Lambda_k(x) +$$

$$\begin{aligned}
 \partial_{x_j x_i}^2 u(x) &= \sum_{h,k=1}^n (\partial_{y_h y_k} v)(\Lambda(x)) \partial_{x_i} \Lambda_h(x) \partial_{x_j} \Lambda_k(x) + \\
 &+ \sum_{k=1}^n (\partial_{y_k} v)(\Lambda(x)) \partial_{x_j x_i} \Lambda_k(x) = \\
 &= \left(((\partial_x \Lambda(x))^t \partial_y)_i ((\partial_x \Lambda(x))^t \partial_y)_j \right) v(\Lambda(x)) + \\
 &+ (\text{first order terms}).
 \end{aligned}$$

In general we have

$$\partial_x^\alpha u(x) = \left(((\partial_x \Lambda(x))^t \partial_y)^\alpha \right) v(\Lambda(x)) + (\text{terms of order less than } |\alpha|). \quad (7.3.1)$$

Now, let us denote by $\tilde{P}(y, \partial_y)$ the transformed operator of $P(x, \partial_x)$ through Λ , that is the operator satisfying

$$\left(\tilde{P}(y, \partial_y) v(y) \right)_{|y=\Lambda(x)} = P(x, \partial_x) u(x). \quad (7.3.2)$$

By (7.3.2) we have that the principal part of $\tilde{P}(y, \partial_y)$, $\tilde{P}_m(y, \partial_y)$, is given by

$$\begin{aligned}
 \tilde{P}_m(y, \partial_y) &= \sum_{|\alpha|=m} a_\alpha(\Lambda^{-1}(y)) \left((\partial_x \Lambda(x))^t \partial_y \right)_{|x=\Lambda^{-1}(y)}^\alpha = \\
 &= P(x, \partial_x (\Lambda(x))^t \partial_y)_{|x=\Lambda^{-1}(y)}
 \end{aligned} \quad (7.3.3)$$

and its symbol $\tilde{P}_m(y, \eta)$ is given (up to the multiplicative constant i^m) by

$$\tilde{P}_m(y, \eta) = \sum_{|\alpha|=m} a_\alpha(\Lambda^{-1}(y)) \left((\partial_x \Lambda(x))^t \eta \right)_{|x=\Lambda^{-1}(y)}^\alpha. \quad (7.3.4)$$

From what we have so far established, we easily obtain

Theorem 7.3.1 (invariant property of the characteristic surfaces).

Let Ω be an open set of \mathbb{R}^n , and let $x_0 \in \Omega$, $\phi \in C^m(\Omega)$,

$$\Gamma = \{x \in \Omega : \phi(x) = \phi(x_0)\}.$$

Let us suppose that

$$\nabla \phi(x) \neq 0, \quad \forall x \in \Gamma.$$

Moreover, let $P(x, \partial)$ be a linear differential operator of order m and $\Lambda \in C^m(\Omega, \mathbb{R}^n)$ be a diffeomorphism of class C^m .

Then Γ is a noncharacteristic surface for $P(x, \partial)$ if and only if $\Lambda(\Gamma)$ is a noncharacteristic surface for the operator $\tilde{P}(y, \partial_y)$.

Proof. It is not restrictive to assume $x_0 = 0$ and $\Lambda(x_0) = 0$. Now, since $\Gamma = \{x \in \Omega : \phi(x) = 0\}$, we have

$$\Lambda(\Gamma) = \left\{ y \in \Lambda(\Omega) : \tilde{\phi}(y) = 0 \right\},$$

where $\tilde{\phi} = \phi \circ \Lambda$. On the other hand, Γ is a noncharacteristic surface for $P(x, \partial)$ if and only if

$$P_m(x, \nabla \phi(x)) \neq 0, \quad \forall x \in \Gamma,$$

but we have

$$\nabla_y \tilde{\phi}(y) = (\partial_x \Lambda(x))|_{x=\Lambda^{-1}(y)}^t (\nabla_x \phi) (\Lambda^{-1}(y)).$$

Therefore by (7.3.4) we have

$$\tilde{P}_m(y, \nabla_y \tilde{\phi}(y)) = i^m \sum_{|\alpha|=m} a_\alpha(x) (\nabla_x \phi)^\alpha = P_m(x, \nabla_x \phi(x))$$

from which the thesis follows. ■

The following Proposition holds true.

Proposition 7.3.2. *Let Ω be an open set of \mathbb{R}^n and let $x_0 \in \Omega$ and $\phi \in C^\infty(\Omega)$ satisfy*

$$\nabla \phi(x_0) \neq 0.$$

Let $f, a_\alpha \in C^\infty(\Omega)$ for $|\alpha| \leq m$. Let us assume that

$$\Gamma = \{x \in \Omega : \phi(x) = \phi(x_0)\}$$

is a noncharacteristic surface in x_0 for the operator $P(x, \partial)$. Let $g_j \in C^\infty(\Omega)$, for $j = 0, 1, \dots, m-1$.

If u is a C^∞ solution in a neighborhood \mathcal{U} of x_0 of the Cauchy problem

$$\begin{cases} P(x, \partial)u = f(x), & \forall x \in \mathcal{U}, \\ \frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), & j = 0, 1, \dots, m-1, \forall x \in \Gamma \cap \mathcal{U}, \end{cases} \quad (7.3.5)$$

then the derivatives $\partial^\alpha u(x_0)$ are uniquely determined for every $\alpha \in \mathbb{N}_0^n$.

Proof. Although the proof follows the line indicated in the particular case $\phi(x) = -x_n$ we want to dwell on some details that may be useful later.

We split the proof into two steps. In **Step I** we determine the derivatives of u of order less than m on Γ in, **Step II** we determine the higher order derivatives of u .

Step I. We begin by proving that if $\phi \in C^k(\Omega)$, $k \geq 1$, $\nabla\phi(x_0) \neq 0$ and v is any C^k function in a neighborhood of x_0 then there exists a neighborhood \mathcal{V} of x_0 such that the derivatives $\frac{\partial^j v(x)}{\partial \nu^j}$ on $\Gamma \cap \mathcal{V}$, for $j = 0, 1, \dots, k$ determine the derivatives $\partial^\alpha v$ on $\Gamma \cap \mathcal{V}$ for each $|\alpha| \leq k$.

Since $\nabla\phi(x_0) \neq 0$, we may limit ourselves, up to isometries, to consider the case where, we have, for a suitable $\delta > 0$,

$$\Gamma = \{(x', \varphi(x')) : x' \in B'_\delta\}, \quad (7.3.6)$$

where $\varphi \in C^k(\overline{B'_\delta})$,

$$\varphi(0) = |\nabla\varphi(0)| = 0.$$

We have

$$\nu((x', \varphi(x'))) = \frac{(\nabla_{x'}\varphi(x'), -1)}{\sqrt{1 + |\nabla_{x'}\varphi|^2}}. \quad (7.3.7)$$

Also set

$$\mu(x') := \nu((x', \varphi(x'))) \sqrt{1 + |\nabla_{x'}\varphi|^2} = (\nabla_{x'}\varphi(x'), -1). \quad (7.3.8)$$

We proceed by induction on the order s of derivatives. If $s = 0$, then v is known on Γ , but to better understand the procedure, we also consider the case $k = 1$. In such a case we have, for $x' \in B'_\delta$,

$$v(x', \varphi(x')) = g_0(x'), \quad (7.3.9)$$

$$\begin{aligned} \sum_{j=1}^{n-1} (\partial_j v)(x', \varphi(x')) \partial_j \varphi(x') - (\partial_n v)(x', \varphi(x')) &= \\ = g_1(x') \sqrt{1 + |\nabla_{x'}\varphi(x')|^2}. \end{aligned} \quad (7.3.10)$$

Making the derivatives w.r.t. x_i of both the sides of (7.3.9), for $i = 1, \dots, n-1$, we get, (for the sake of brevity, omit the variables)

$$\partial_i v + \partial_n v \partial_i \varphi = \partial_i g_0.$$

Hence

$$\partial_i v = \partial_i g_0 - \partial_n v \partial_i \varphi$$

and, inserting these derivatives in (7.3.10), we have

$$\sum_{j=1}^{n-1} \partial_j g_0 \partial_j \varphi - (1 + |\nabla_{x'} \varphi|^2) \partial_n v = g_1 \sqrt{1 + |\nabla_{x'} \varphi|^2}$$

from which we get

$$\partial_n v = \frac{1}{1 + |\nabla_{x'} \varphi|^2} \left(\nabla_{x'} g_0 \cdot \nabla_{x'} \varphi - g_1 \sqrt{1 + |\nabla_{x'} \varphi|^2} \right).$$

Now let us prove that if $\partial^\alpha v$ are determined by $|\alpha| \leq s$ on Γ (with $s \leq k-1$) then $\partial^\alpha v$ are determined on Γ for $|\alpha| \leq s+1$.

Let

$$\partial^\alpha v(x', \varphi(x')) = h_\alpha(x'), \quad \text{for } |\alpha| = s, \quad \text{on } \Gamma \quad (7.3.11)$$

and set, for $j = 0, 1, \dots, k$,

$$\tilde{g}_j(x') = (1 + |\nabla_{x'} \varphi|^2)^{j/2} \frac{\partial^j v}{\partial \nu^j} = \sum_{|\alpha|=j} \mu^\alpha(x') \partial^\alpha v(x', \varphi(x')).$$

Then, besides (7.3.11), we know that

$$\sum_{|\alpha|=s+1} \mu^\alpha \partial^\alpha v = \tilde{g}_{s+1} \quad \text{on } \Gamma, \quad (7.3.12)$$

which we write

$$\sum_{j=0}^{s+1} \sum_{|\alpha'|=s+1-j} \mu'^{\alpha'} \mu^j \partial_n^j \partial^{\alpha'} v = \tilde{g}_{s+1} \quad \text{on } \Gamma, \quad (7.3.13)$$

where $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ and $\mu' = (\mu_1, \dots, \mu_{n-1})$.

At this point we express $\partial_n^j \partial^{\alpha'} v$, for $|\alpha'| = s+1-j$, through the functions $\partial_n^{s+1} v$ and h_γ for $|\gamma| \leq s$.

If $j < s+1$, there exists $i \in \{1, \dots, n-1\}$ such that $\alpha' - e_i \geq 0$. Set $\beta' = \alpha' - e_i$ and let us recall that by (7.3.11) we have

$$\partial_n^j \partial^{\beta'} v = h_{(\beta', j)},$$

from which, making the derivative w.r.t. x_i we have

$$\underbrace{\partial_n^j \partial_i \partial^{\beta'} v}_{\partial^{\alpha'} v} + \mu_i \partial_n^{j+1} \partial^{\beta'} v = \partial_i h_{(\beta', j)},$$

hence

$$\partial_n^j \partial^{\alpha'} v = \partial_i h_{(\beta', j)} - \mu_i \partial_n^{j+1} \partial^{\beta'} v.$$

Now, if $\beta' \neq 0$, we proceed in a similar manner for $\partial_n^j \partial^{\alpha'} v$ and then we iterate. Denoting by $H_{(\alpha', j)}$ the functions of the type

$$\sum_{k=0}^j \sum_{|\beta'|=s-j} c_{(\beta', k)} h_{(\beta', k)},$$

where $c_{(\beta', k)}$ are known functions expressible by means of μ_i , $i = 1, \dots, n-1$, we obtain

$$\partial_n^j \partial^{\alpha'} v = H_{(\alpha', j)} + (-1)^{s+1-j} \mu^{\alpha'} \partial_n^{s+1} v \quad \text{for } |\alpha| = s+1-j. \quad (7.3.14)$$

Inserting (7.3.14) in (7.3.13) we get (recall $\mu_n = -1$)

$$\begin{aligned} \tilde{g}_{s+1} &= \sum_{j=0}^{s+1} \sum_{|\alpha'|=s+1-j} (-1)^j \mu^{\alpha'} \left(H_{(\alpha', j)} + (-1)^{s+1-j} \mu^{\alpha'} \partial_n^{s+1} v \right) = \\ &= \sum_{j=0}^{s+1} \sum_{|\alpha'|=s+1-j} (-1)^j \mu^{\alpha'} H_{(\alpha', j)} + \left(\sum_{j=0}^{s+1} \sum_{|\alpha'|=s+1-j} (-1)^j \mu^{2\alpha'} \right) \partial_n^{s+1} v. \end{aligned}$$

From which we have

$$\begin{aligned} (-1)^{s+1} \left(1 + \sum_{1 \leq |\alpha'| \leq s+1} \mu^{2\alpha'} \right) \partial_n^{s+1} v &= \\ = \tilde{g}_{s+1} - \sum_{j=0}^{s+1} \sum_{|\alpha'|=s+1-j} (-1)^j \mu^{\alpha'} H_{(\alpha', j)}. \end{aligned} \quad (7.3.15)$$

Since

$$\sum_{1 \leq |\alpha'| \leq s+1} \mu^{2\alpha'} \geq 0,$$

by (7.3.15) we determine $\partial_n^{s+1} v$.

Step II. Let us consider the Cauchy problem

$$\begin{cases} P(x, \partial)u = f(x), & \forall x \in \mathcal{U}, \\ \frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), & j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma \cap \mathcal{U}. \end{cases} \quad (7.3.16)$$

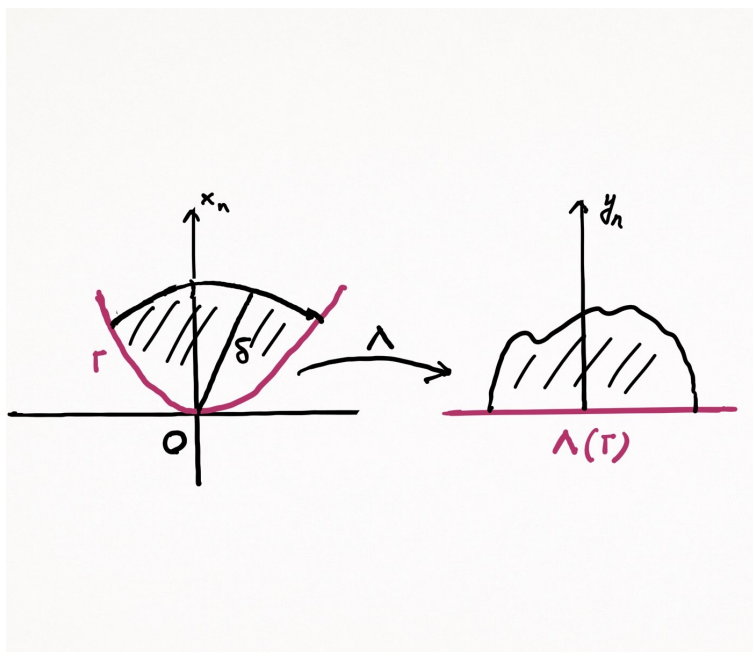


Figure 7.1:

Like Step I we assume that Γ is the graph (7.3.6), where $\varphi \in C^\infty(\overline{B'_\delta})$, $\varphi(0) = |\nabla_{x'}\varphi(0)| = 0$. By what was proved in Step I we can determine, from the functions g_0, g_1, \dots, g_{m-1} only, the derivatives

$$\partial^\alpha u, \quad \text{for } |\alpha| \leq m-1, \quad \text{on } \Gamma. \quad (7.3.17)$$

Let now us show in which a way we determine the other derivatives of u on Γ . Let

$$\Lambda : B_\delta \subset \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n, \quad \Lambda(x) = (x', x_n - \varphi(x')),$$

we have

$$\Lambda(\Gamma) = \{(y', 0) : y' \in B'_\delta\} = \{y \in B_\delta : -y_n = 0\}$$

and, set

$$v(y) = u(\Lambda^{-1}(y)).$$

Equality (7.3.1) implies that the derivatives

$$\partial^\alpha v(y', 0), \quad \text{for } |\alpha| \leq m-1, \quad \forall y' \in B'_\delta,$$

are all uniquely determined. In particular, the following derivatives are determined

$$\partial_n^j v(y', 0), \quad \text{for } j = 0, 1, \dots, m-1, \quad \forall y' \in B'_\delta. \quad (7.3.18)$$

On the other hand, by (7.3.2) we have that v , solves the following equation in a neighborhood of 0

$$\tilde{P}(y, \partial_y)v(y) := \sum_{|\alpha| \leq m} b_\alpha \partial_y^\alpha v(y) = \tilde{f}(y),$$

where $\tilde{f}(y) = f(\Lambda^{-1}(y))$. Now, by Proposition 7.3.1 we have that $\Lambda(\Gamma)$ is a noncharacteristic surface for $\tilde{P}(y, \partial_y)$. Since $\Lambda(\Gamma) = \{y \in B_\delta : -y_n = 0\}$ we have

$$b_{(0,m)}(y', 0) = \sum_{|\alpha|=m} b_\alpha(y', 0) e_n^\alpha \neq 0 \quad \forall y' \in B'_\delta.$$

We are therefore reduced to the same situation examined at the beginning of this Section 7.2 and we then calculate the derivatives $\partial_y^\alpha v(y', 0)$ for $|\alpha| \geq m$ from the derivatives of g_j , $j = 0, 1, \dots, m-1$ and the coefficients (and their derivatives) of $\tilde{P}(y, \partial_y)$. Finally, by exploiting formula (7.3.1) we obtain the derivatives $\partial_x^\alpha u$ on Γ for $|\alpha| \geq m$. ■

We conclude this Section with some examples and remarks.

Remarks. Let $P(x, \partial)$, $x \in \Omega$, be a linear differential operator whose principal part is $P_m(x, \partial)$.

1. We call **characteristic equation** the non linear first order equation in the unknown ϕ

$$P_m(x, \nabla \phi(x)) = 0. \quad (7.3.19)$$

2. We say that $P(x, \partial)$ is **elliptic in the point** $x_0 \in \Omega$ if

$$P_m(x_0, \xi) \neq 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

It is evident that the elliptic operators have not characteric surface. If $P(x, \partial)$ is elliptic in each point of Ω we say that $P(x, \partial)$ is **elliptic in Ω** . Each linear differential operators of one variable are elliptic

$$P(t, \frac{d}{dt}) = a_m(t) \frac{d^m}{dt^m} + \dots + a_0(t), \quad t \in I,$$

where $a_m(t) \neq 0$ for $t \in I$, where I is an interval of \mathbb{R} . As a matter of fact

$$P(t, \xi) = a_m(t) \xi^m \neq 0, \quad t \in I, \quad \forall \xi \in \mathbb{R} \setminus \{0\}.$$

A remarkable example of an elliptic operator is the operator of Cauchy-Riemann

$$P((x, y), \partial_x, \partial_y) = \partial_x + i\partial_y.$$

If $m = 2$ and a_{jk} , where $a_{jk}(x) = a_{kj}(x)$ for $x \in \Omega$, $j, k = 1, \dots, n$ are real-valued functions in \mathbb{R} , we define a **uniformly elliptic** operator with bounded coefficients an operator of the type

$$P(x, \partial) = \sum_{j,k=1}^n a_{jk}(x) \partial_{jk}^2 + \sum_{j=1}^n b_j(x) \partial_j + c(x)$$

such that there exists a constant $\lambda \geq 1$ satisfying

$$\lambda^{-1} |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \leq \lambda |\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

◆

Exercise. Let us consider the following operator with constant coefficients

$$P(\partial) = \sum_{j,k=1}^n a_{jk} \partial_{jk}^2, \quad (7.3.20)$$

where $\{a_{jk}\}$ is a real symmetric matrix. If $\det\{a_{jk}\} \neq 0$ then there exists a nonsingular matrix C such that, setting $y = Cx$, operator (7.3.20), is transformed in

$$P(\partial_y) = \sum_{j=1}^n \varkappa_j \partial_{y_j}^2,$$

where \varkappa_j , $j = 1, \dots, n$, is equal either to 1 or to -1 . ♣

7.4 The Cauchy-Kovalevskaya Theorem

In what follows we denote by Ω an open set of \mathbb{R}^n , $x_0 \in \Omega$ and by $\phi \in C^\omega(\Omega, \mathbb{R})$ a function such that

$$\nabla \phi(x) \neq 0, \quad \forall x \in \Gamma := \{x \in \Omega : \phi(x) = \phi(x_0)\}. \quad (7.4.1)$$

By (7.4.1) we have that for every $\hat{x} \in \Gamma$ there exist $r, M > 0$ and an isometry Ψ under which we have $\Psi(0) = \hat{x}$, and

$$\Psi^{-1}(\Gamma) \cap Q_{r,2M} = \{(x', \varphi(x')) : x' \in B'_r\}$$

where $\varphi \in C^\omega(B'_r; \mathbb{R})$ and it satisfies

$$\varphi(0) = 0, \quad |\nabla\varphi(0)| = 0,$$

$$\|\varphi\|_{C^1(\overline{B'_r})} \leq Mr.$$

We say that a function

$$h : \Gamma \rightarrow \mathbb{C}$$

is analytic on Γ , provided that

$$(h \circ \Psi)(\cdot, \varphi(\cdot)) \in C^\omega(B'_r).$$

In this Section we will prove

Theorem 7.4.1 (Cauchy–Kovalevskaya). *Let $m \in \mathbb{N}$ and let Ω be an open set of \mathbb{R}^n , $x_0 \in \Omega$, $\phi \in C^\omega(\Omega, \mathbb{R})$ which satisfies (7.4.1). Moreover, let $P(x, \partial)$ be the linear differential operator*

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha, \quad (7.4.2)$$

where $a_\alpha \in C^\omega(\Omega)$, for $|\alpha| \leq m$. Let g_0, g_1, \dots, g_{m-1} be **analytic functions on** Γ . Let us assume that Γ is a noncharacteristic surface for the operator $P(x, \partial)$. Let $f \in C^\omega(\Omega)$.

Then for every $\tilde{x} \in \Gamma$ there exists a neighborhood $\mathcal{U}_{\tilde{x}}$ such that the Cauchy problem

$$\begin{cases} P(x, \partial)u(x) = f(x), & \forall x \in \mathcal{U}_{\tilde{x}}, \\ \frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), & j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma \cap \mathcal{U}_{\tilde{x}} \end{cases} \quad (7.4.3)$$

has a unique analytic solution in $\mathcal{U}_{\tilde{x}}$.

In order to prove Cauchy-Kovalevskaya Theorem we need two preliminary steps

- (i) local flatness of initial surface;
- (ii) transformation of problem (7.4.3) to a Cauchy problem for a first order system.

We have already considered **point (i)** in the context of the proof of Step II of Proposition 7.3.2. Here it suffices to add that, referring to the

notations used in the above proof, the function φ introduced there is, not only C^∞ , but also analytic in B'_δ and, consequently, the map

$$\Lambda : B_\delta \subset \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n, \quad \Lambda(x) = (x', x_n - \varphi(x')), \quad (7.4.4)$$

which, we recall, flattens Γ in the sense that

$$\Lambda(\Gamma) = \{(y', 0) : y' \in B'_\delta\} = \{y \in B_\delta : -y_n = 0\}.$$

Moreover, setting

$$v(y) = u(\Lambda^{-1}(y)), \quad (7.4.5)$$

the operator $\tilde{P}(y, \partial_y)$, defined by

$$\left(\tilde{P}(y, \partial_y)v(y) \right)_{|y=\Lambda(x)} = P(x, \partial_x)u(x), \quad (7.4.6)$$

has its principal part $\tilde{P}_m(y, \partial_y)$, given by

$$\tilde{P}_m(y, \partial_y) = \sum_{|\alpha|=m} b_\alpha(y) \partial^\alpha, \quad (7.4.7)$$

where

$$b_\alpha(y) = a_\alpha(\Lambda^{-1}(y)), \quad \text{for } |\alpha| \leq m, \quad (7.4.8)$$

are analytic functions on B_δ .

Again by Proposition 7.3.2 we know that the new initial data $\tilde{g}_j(y') = \partial_n^j v(y', 0)$, $j = 0, 1, \dots, m-1$, are determined by the initial data g_j and that \tilde{g}_j are analytic in a neighborhood of $0 \in \mathbb{R}^{n-1}$. Moreover, we know that the surface

$$\{y \in B_\delta : -y_n = 0\}$$

is noncharacteristic. Therefore

$$b_{(0,m)}(y', 0) \neq 0, \quad \forall y' \in B'_\delta(0).$$

Hence, setting

$$\tilde{b}_\alpha(y) = -\frac{b_\alpha(y)}{b_{(0,m)}(y)}, \quad \tilde{f}(y) = \frac{f(\Lambda^{-1}(y))}{b_{(0,m)}(y)},$$

we write problem (7.4.3) as

$$\begin{cases} \partial_n^m v = \sum_{|\alpha| \leq m, \alpha_n \leq m-1} \tilde{b}_\alpha(y) \partial^\alpha v + \tilde{f}(y), \\ \partial_n^j v(y', 0) = \tilde{g}_j(y'), \quad j = 0, 1, \dots, m-1. \end{cases} \quad (7.4.9)$$

We may easily transform problem (7.4.9) into a Cauchy problem with homogeneous initial conditions. To this purpose it suffices to define the function

$$H(y) = \sum_{j=0}^{m-1} \frac{y_n^j}{j!} \tilde{g}_j(y')$$

and set

$$w = v - H,$$

obtaining

$$\begin{cases} \partial_n^m w = \sum_{|\alpha| \leq m, \alpha_n \leq m-1} \tilde{b}_\alpha(y) \partial^\alpha w + F(y), \\ \partial_n^j w(y', 0) = 0, \quad j = 0, 1, \dots, m-1. \end{cases} \quad (7.4.10)$$

where

$$F(y) = \tilde{f}(y) - \partial_n^m H(y) + \sum_{|\alpha| \leq m, \alpha_n \leq m-1} \tilde{b}_\alpha(y) \partial^\alpha H(y). \quad (7.4.11)$$

(ii). The idea of the transformation is simple and it partly replicates the one usually followed to reduce a Cauchy problem for ordinary differential equations of order m to a Cauchy problem for a first-order system. However, in our case the unknown depends on $n > 1$ variables, and this requires further arrangements.

In order to highlight the main steps we illustrate the procedure in the case where $n = 2$ and the operator is equal to its principal part only. Next we will outline how to proceed in the general case.

Let us consider the Cauchy problem

$$\begin{cases} \partial_t^m u = \sum_{j \leq m-1} \sum_{i+j=m} a_{i,j}(x, t) \partial_x^i \partial_t^j u + f(x, t), \\ \partial_t^j u(x, 0) = 0, \quad j = 0, 1, \dots, m-1, \quad \forall x \in \mathbb{R}. \end{cases} \quad (7.4.12)$$

Let us assume that $m \geq 2$ and let $u(x, t)$ be a C^∞ solution of (7.4.12). We set

$$V_{i,j} = \partial_x^i \partial_t^j u, \quad \text{for } i + j \leq m - 1.$$

It is simple to check what follows

$$\partial_t V_{i,j} = V_{i,j+1}, \quad \text{for } i + j < m - 1, \quad (7.4.13a)$$

$$\partial_t V_{i,j} = \partial_x V_{i-1,j+1}, \quad \text{for } i + j = m - 1, \quad i > 0, \quad (7.4.13b)$$

$$\partial_t V_{0,m-1} = \sum_{i+j=m-1} a_{i+1,j}(x,t) \partial_x V_{i,j} + f(x,t), \quad (7.4.13c)$$

$$V_{i,j}(x,0) = 0 \quad \text{for } i + j \leq m - 1. \quad (7.4.13d)$$

Hence, if $V_{0,0}$ is a solution to Cauchy problem (7.4.12), then it is a solution to Cauchy problem (7.4.13a)–(7.4.13d). Thus, in order to prove the equivalence of problem (7.4.12) and problem (7.4.13a)–(7.4.13d) it suffices to prove the converse. Let us suppose, therefore, that $V = (V_{i,j})_{i+j \leq m-1}$ is a C^∞ a solution to problem (7.4.13a)–(7.4.13d) and let us prove that $V_{0,0}$ is a solution to problem (7.4.12).

1. We prove that

$$\text{If } i + j = m - 1, i > 0 \quad \text{then} \quad V_{i,j} = \partial_x V_{i-1,j}. \quad (7.4.14)$$

Proof. Let $i + j \leq m - 1$ and $i > 0$. By (7.4.13b) we have

$$\partial_t V_{i,j} = \partial_x V_{i-1,j+1}, \quad \text{for } i + j = m - 1, \quad i > 0. \quad (7.4.15)$$

Now, since $(i - 1) + j = m - 2$, by (7.4.13a) we have

$$V_{i-1,j+1} = \partial_t V_{i-1,j}. \quad (7.4.16)$$

Hence, by (7.4.15) e (7.4.16) we have

$$\partial_t (V_{i,j} - \partial_x V_{i-1,j}) = 0. \quad (7.4.17)$$

On the other hand, by (7.4.13d) we have

$$(V_{i,j} - \partial_x V_{i-1,j})(x,0) = 0$$

this equality and (7.4.17) implies

$$V_{i,j} = \partial_x V_{i-1,j}.$$

2. We prove that

$$\text{If } i + j \leq m - 1, i > 0 \quad \text{then} \quad V_{i,j} = \partial_x V_{i-1,j}. \quad (7.4.18)$$

Proof. Set $l = (m - 1) - (i + j)$ and let us proceed by induction on l . If $l = 0$, then (7.4.18) holds true, because it is nothing but (7.4.14). Let now let us suppose that (7.4.18) holds true for l and prove it for $l + 1$. Hence, let us suppose that

$$\text{if } i + j = (m - 1) - l, i > 0 \quad \text{then} \quad V_{i,j} = \partial_x V_{i-1,j}. \quad (7.4.19)$$

Let i, j satisfy $i + j = (m - 1) - (l + 1)$ and $i > 0$. Since $i + j < m - 1$ by (7.4.13a) we have

$$\partial_t V_{i,j} = V_{i,j+1}. \quad (7.4.20)$$

Since we have $i + (j + 1) = (m - 1) - l$, (7.4.19) gives

$$V_{i,j+1} = \partial_x V_{i-1,j+1}. \quad (7.4.21)$$

Since $(i - 1) + j = (m - 1) - l - 2 < m - 1$, (7.4.13a) gives

$$V_{i-1,j+1} = \partial_t V_{i-1,j}, \quad \text{for } i + j < m - 1. \quad (7.4.22)$$

Hence, by (7.4.21) e (7.4.22) we have

$$V_{i,j+1} = \partial_x V_{i-1,j+1} = \partial_t \partial_x V_{i-1,j},$$

by the latter and by (7.4.20) we have

$$\partial_t (V_{i,j} - \partial_x V_{i-1,j}) = 0 \quad (7.4.23)$$

so, by (7.4.13d) we have that, if $i + j = (m - 1) - (l + 1)$, $i > 0$, then

$$V_{i,j} = \partial_x V_{i-1,j}.$$

(7.4.18) is proved.

Conclusions. Iteration of (7.4.18) implies what follows:

$$\text{if } i + j \leq m - 1, i > 0 \quad \text{then} \quad V_{i,j} = \partial_x^i V_{0,j}. \quad (7.4.24)$$

On the other hand, (7.4.13a) gives

$$V_{0,j} = \partial_t V_{0,j-1} = \cdots = \partial_t^j V_{0,0}. \quad (7.4.25)$$

All in all, by (7.4.24) and (7.4.25) we get

$$V_{i,j} = \partial_x^i \partial_t^j V_{0,0}$$

and using this equality into (7.4.13c)–(7.4.13d) we have that $V_{0,0}$ solves (7.4.12).

We may rewrite problem (7.4.13a)–(7.4.13d) in a more concentrated form as follows

$$\begin{cases} \partial_t V(x, t) = B(x, t) \partial_x V + F(x, t), \\ V(x, 0) = 0. \end{cases} \quad (7.4.26)$$

where, for an appropriate $N \in \mathbb{N}$, V is a function with values in \mathbb{R}^N , B is an $N \times N$ matrix whose entries are analytic and $F = fe_N$.

In the case $n > 2$ one may similarly reduce Cauchy problem (7.4.3) to a Cauchy problem for a first order system. We outline the procedure (the details of which we leave to the reader). First, it is convenient to introduce the following notation. If $\alpha \in \mathbb{N}_0^{n-1} \setminus \{0\}$ is a multi-index, we set

$$i(\alpha) = \min\{i : \alpha_i > 0\}.$$

In addition, we set $t = x_n$ and

$$V_{\alpha,j} = \partial_t^j \partial_{x'}^\alpha w, \quad \text{for } |\alpha| + j \leq m - 1.$$

By (7.4.10) we have

$$\partial_t V_{\alpha,j} = V_{\alpha,j+1}, \quad \text{for } |\alpha| + j < m - 1, \quad (7.4.27a)$$

$$\partial_t V_{\alpha,j} = \partial_{x_{i(\alpha)}} V_{\alpha - e_{i(\alpha)}, j+1}, \quad \text{for } |\alpha| + j = m - 1, |\alpha| > 0, \quad (7.4.27b)$$

$$\partial_t V_{0,m-1} = \sum_{|\alpha|+j=m, j < m} c_{\alpha,j} \partial_{x_{i(\alpha)}} V_{\alpha - e_{i(\alpha)}, j} + \sum_{|\alpha|+j \leq m} d_{\alpha,j} V_{\beta,j} + F, \quad (7.4.27c)$$

$$V_{\alpha,j}(x', 0) = 0 \quad \text{for } |\alpha| + j \leq m - 1, \quad (7.4.27d)$$

where $c_{\alpha,j}$, $d_{\alpha,j}$, F are analytic functions in the variables x' and t .

Proof of the Cauchy–Kovalevskaya Theorem.

Taking into account what has been done in (i) and (ii) and changing the notations a little, we may reformulate Cauchy problem (7.4.10) as follows.

$$\begin{cases} \partial_t U_k(x, t) = \sum_{j=1}^{n-1} \sum_{l=1}^N B_j^{lk} \partial_{x_j} U_l + \sum_{l=1}^N C^{lk} U_l + F_k, & k = 1, \dots, N, \\ U_k(x, 0) = 0, & k = 1, \dots, N, \end{cases} \quad (7.4.28)$$

where B_j^{lk} , C^{lk} , F_k , $j = 1, \dots, n-1$, $l, k = 1, \dots, N$ are analytic functions in a neighborhood of 0.

In order to prove the existence and the uniqueness for Cauchy problem (7.4.28) we proceed as follows:

Step I. For any function $U \in C^\infty$ that we suppose to satisfy (7.4.28), we will calculate the derivatives

$$\partial^\alpha U_k(0,0) := U_{k,\alpha}, \quad \forall \alpha \in \mathbb{N}_0^n, \quad k = 1, \dots, N.$$

Setting, $\partial^\alpha = \partial_x^{\alpha'} \partial_t^{\alpha_n}$, for $\alpha = (\alpha', \alpha_n)$, we will have

$$U_{k,(\alpha',0)} = 0, \quad \forall \alpha' \in \mathbb{N}_0^{n-1}, \quad k = 1, \dots, N. \quad (7.4.29)$$

Step II. We will show what follows. Let us assume that the functions \tilde{B}_j^{lk} , \tilde{C}^{lk} , \tilde{F} and $\tilde{\varphi}$ (the latter is independent of t) satisfy the following conditions

$$(a) \quad B_j^{lk} \preceq \tilde{B}_j^{lk}, \quad C^{lk} \preceq \tilde{C}^{lk}, \quad \text{for } j = 1, \dots, n-1, \quad l, k = 1, \dots, N, \quad F \preceq \tilde{F}, \\ 0 \preceq \tilde{\varphi}$$

and let us assume that

(b) it occurs that for any C^∞ solution \tilde{U} to the Cauchy problem

$$\begin{cases} \partial_t \tilde{U}_k = \sum_{j=1}^{n-1} \sum_{l=1}^N \tilde{B}_j^{lk} \partial_{x_j} \tilde{U}_l + \sum_{l=1}^N \tilde{C}^{lk} \tilde{U}_l + \tilde{F}_k, \\ \tilde{U}_k(x, 0) = \tilde{\varphi}_k(x), \end{cases} \quad (7.4.30)$$

we will have

$$|U_{k,\alpha}| \leq \partial^\alpha \tilde{U}_k(0,0), \quad \forall \alpha \in \mathbb{N}_0^n, \quad k = 1, \dots, N. \quad (7.4.31)$$

Step III. We will construct some majorants \tilde{B}_j^{lk} , \tilde{C}^{lk} , \tilde{F} and $\tilde{\varphi}$ for which Cauchy problem (7.4.30) **does indeed have an analytic solution**. Let us denote again by \tilde{U} such a solution. By Step II and, in particular, by (7.4.31) it will follow that the power series

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} U_\alpha x^{\alpha'} t^{\alpha_n},$$

will converge in a neighborhood of 0 and its sum, which we denote by V , will satisfy, the system

$$\partial_t V_k(x, t) = \sum_{j=1}^{n-1} \sum_{l=1}^N B_j^{lk}(x, t) \partial_{x_j} V_l + \sum_{l=1}^N C^{lk}(x, t) V_l + F_k(x, t). \quad (7.4.32)$$

Indeed, for $k = 1, \dots, N$, the analytic functions

$$\partial_t V_k$$

and

$$\sum_{l=1}^N B_j^{lk}(x, t) \partial_{x_j} V_l + \sum_{l=1}^N C^{lk}(x, t) V_l + F_k(x, t),$$

will have (by construction) all the derivatives equal at 0. Furthermore, by (7.4.29), we will have

$$V_k(x, 0) = 0, \quad k = 1, \dots, N \quad (7.4.33)$$

and we will then have proved the existence of a solution to Cauchy problem (7.4.28).

Step IV. The uniqueness in the class of analytic functions in a connected neighborhood of 0 for Cauchy problem (7.4.28) will be a consequence of **Step I** and of the unique continuation property for the analytic functions (Theorem 6.2.2).

Step I. By the initial conditions $U(x, 0) = 0$ we have

$$\partial_x^{\alpha'} U_k(x, 0) = 0, \quad k = 1, \dots, N, \quad (7.4.34)$$

which implies (7.4.29). Now, for every $\alpha \in \mathbb{N}_0^n$, where $\alpha_n > 0$, we have, for $k = 1, \dots, N$,

$$\begin{aligned} \partial^\alpha U_k(0, 0) &= P_{k, \alpha} (\partial^\gamma B_j^{kl}, \dots, \partial^\delta C^{kl}, \dots, \partial^\beta U_h)_{x=0, t=0} + \\ &+ \partial_x^{\alpha'} \partial_t^{\alpha_n - 1} F_k(0, 0), \end{aligned} \quad (7.4.35)$$

where $P_{k, \alpha}$ is a polynomial with positive coefficients and the multi-indices β in the derivatives $\partial^\beta U_h$ satisfy

$$|\beta| \leq |\alpha|, \quad \text{e} \quad \beta_n \leq \alpha_n - 1.$$

In particular, (7.4.35) is a recursive relation on the derivatives of U . To prove (7.4.35) it suffices to make the derivatives of both the sides of the equations

in (7.4.28). Concerning the positivity of the coefficients of $P_{k,\alpha}$ it suffices to keep in mind that we only use the rules of derivation of a product and a sum of functions.

For instance, if $i = 1, \dots, n-1$, we have

$$\begin{aligned} \partial_t \partial_{x_i} U_k &= \sum_{j=1}^{n-1} \sum_{l=1}^N \left(B_j^{lk} \partial_{x_i x_j}^2 U_l + \partial_{x_i} B_j^{lk} \partial_{x_j} U_l \right) + \\ &+ \sum_{l=1}^N \left(C^{lk} \partial_{x_i} U_l + \partial_{x_i} C^{lk} U_l \right) + \partial_{x_i} F_k \end{aligned} \quad (7.4.36)$$

and, taking into account (7.4.34), we have

$$\partial_t \partial_{x_i} U_k(0, 0) = \partial_{x_i} F_k(0, 0).$$

Analogously,

$$\partial_t \partial_x^{\alpha'} U_k(0, 0) = \partial_x^{\alpha'} F_k(0, 0)$$

and

$$\begin{aligned} \partial_t^2 U_k &= \sum_{j=1}^{n-1} \sum_{l=1}^N \left(B_j^{lk} \partial_t \partial_{x_j} U_l + \partial_t B_j^{lk} \partial_{x_j} U_l \right) + \\ &+ \sum_{l=1}^N \left(C^{lk} \partial_t U_l + \partial_t C^{lk} U_l \right) + \partial_t F_k. \end{aligned} \quad (7.4.37)$$

Let us observe that all the derivatives of U in $(0, 0)$ that occur in (7.4.37) can be obtained by (7.4.36) and by (7.4.34). A similar argument applies to $\partial_t^2 \partial_x^{\alpha'} U(0, 0), \dots, \partial_t^j \partial_x^{\alpha'} U(0, 0), j = 1, \dots, \alpha' \in \mathbb{N}_0^{n-1}$.

Step II. Let \tilde{U} be a solution to problem (7.4.30). In a similar way to what we have done in **Step I** we obtain, for $\alpha \in \mathbb{N}_0^n$ with $\alpha_n > 0$, that for each $k = 1, \dots, N$,

$$\begin{aligned} \partial^\alpha \tilde{U}_k(0, 0) &= P_{k,\alpha} \left(\partial^\gamma \tilde{B}_j^{kl}, \dots, \partial^\delta \tilde{C}^{kl}, \dots, \partial^\beta \tilde{U}_h \right)_{x=0, t=0} + \\ &+ \partial_x^{\alpha'} \partial_t^{\alpha_n - 1} \tilde{F}_k(0, 0), \end{aligned} \quad (7.4.38)$$

where $P_{k,\alpha}$ is the same polynomial with positive coefficients that occurs in (7.4.35) and (as in (7.4.35)) the multi-indices β in the derivatives $\partial^\beta U_h$ satisfy $|\beta| \leq |\alpha|$ and $\beta_n \leq \alpha_n - 1$. Furthermore, we have that

$$\partial_x^{\alpha'} \tilde{U}_k(0, 0) = \partial_x^{\alpha'} \tilde{\varphi}_k(0) \quad (7.4.39)$$

and by $0 \preceq \tilde{\varphi}$ we have

$$0 \leq \partial_x^{\alpha'} \tilde{\varphi}_k(0). \quad (7.4.40)$$

In order to prove (7.4.31) one can proceed by induction on the order α_n of the derivative with respect to t . If $\alpha_n = 0$ then we have

$$|U_{k,(\alpha',0)}| \leq \partial_x^{\alpha'} \tilde{\varphi}_k(0) = \partial_x^{\alpha'} \tilde{U}_k(0, 0), \quad \forall \alpha \in \mathbb{N}_0^{n-1}, \quad k = 1, \dots, N. \quad (7.4.41)$$

Now, let us suppose that for a given α_n we have

$$|U_{k,(\alpha',\alpha_n)}| \leq \partial_t^{\alpha_n} \partial_x^{\alpha'} \tilde{U}_k(0, 0), \quad \forall \alpha \in \mathbb{N}_0^{n-1} \quad k = 1, \dots, N. \quad (7.4.42)$$

We have, for $k = 1, \dots, N$,

$$\begin{aligned} U_{k,(\alpha',\alpha_n+1)} &= \partial_t^{\alpha_n+1} \partial_x^{\alpha'} U_k(0, 0) = \partial_x^{\alpha'} \partial_t^{\alpha_n} F_k(0, 0) + \\ &+ P_{k,(\alpha',\alpha_n+1)} (\partial^\gamma B_j^{kl}, \dots, \partial^\delta C^{kl}, \dots, \partial^\beta U_h)_{x=0,t=0}, \end{aligned}$$

where

$$|\beta| \leq |\alpha|, \quad \text{and} \quad \beta_n \leq \alpha_n.$$

Then, since the coefficients of $P_{k,(\alpha_n+1)}$ are positive, using **(a)** of **Step II** and the inductive hypothesis (7.4.42), we have

$$\begin{aligned} |U_{k,(\alpha',\alpha_n+1)}| &\leq \left| \partial_x^{\alpha'} \partial_t^{\alpha_n} F_k(0, 0) \right| + \\ &+ P_{k,(\alpha',\alpha_n+1)} (|\partial^\gamma B_j^{kl}|, \dots, |\partial^\delta C^{kl}|, \dots, |U_{h,\beta}|)_{x=0,t=0} \leq \\ &\leq \partial_x^{\alpha'} \partial_t^{\alpha_n} \tilde{F}_k(0, 0) + \\ &+ P_{k,(\alpha',\alpha_n+1)} (\partial^\gamma \tilde{B}_j^{kl}, \dots, \partial^\delta \tilde{C}^{kl}, \dots, \partial^\beta \tilde{U}_h)_{x=0,t=0} = \\ &= \partial_t^{\alpha_n+1} \partial_x^{\alpha'} \tilde{U}_k(0, 0). \end{aligned} \quad (7.4.43)$$

Step III. We may assume that for appropriate positive numbers M_1, M_2 and ρ_1, ρ_2 with $M_1 \geq 1$, we have

$$B_j^{lk}, C^{lk} \in \mathcal{C}_{M_1, \rho_1}(0), \quad j = 1, \dots, n-1, \quad l, k = 1, \dots, N, \quad (7.4.44)$$

$$F_k \in \mathcal{C}_{M_2, \rho_2} \quad k = 1, \dots, N. \quad (7.4.45)$$

We set

$$M = \frac{M_2}{M_1}, \quad \rho = \min\{\rho_1, \rho_2\}.$$

By Proposition 6.3.2, we may choose, for $j = 1, \dots, n-1, l, k = 1, \dots, N$,

$$\tilde{B}_j^{lk} = \tilde{C}^{lk} = \frac{M_1 \rho}{\rho - (\sigma^{-1}t + x_1 + \dots + x_{n-1})}$$

and

$$\tilde{F}_k = \frac{M_2 \rho}{\rho - (\sigma^{-1}t + x_1 + \dots + x_{n-1})},$$

where $\sigma \in (0, 1]$ is to be chosen. System (7.4.30) becomes, for $k = 1, \dots, N$,

$$\begin{aligned} \partial_t \tilde{U}_k(x, t) &= \\ &= \frac{M_1 \rho}{\rho - (\sigma^{-1}t + x_1 + \dots + x_{n-1})} \left(\sum_{j=1}^{n-1} \sum_{l=1}^N \partial_{x_j} \tilde{U}_l + \sum_{l=1}^N \tilde{U}_l + M \right). \end{aligned} \quad (7.4.46)$$

At this point we search for a solution to equation (7.4.46) of the form

$$\tilde{U}_k(x, t) = w(\sigma^{-1}t + x_1 + \dots + x_{n-1}). \quad (7.4.47)$$

Set $s = \sigma^{-1}t + x_1 + \dots + x_{n-1}$ and

$$\phi(s) = \frac{M_1}{1 - \rho^{-1}s} \quad (7.4.48)$$

we have

$$(\sigma^{-1} - \phi(s)N(n-1)) \frac{dw}{ds} = N\phi(s)w + M\phi(s). \quad (7.4.49)$$

Now we choose $\sigma > 0$ so that

$$\sigma^{-1} - \phi(s)N(n-1) > 0$$

in a neighborhood of 0. For instance, we choose

$$\sigma = \sigma_0 := \frac{1}{2NM_1(n-1)}. \quad (7.4.50)$$

We get

$$\sigma_0^{-1} - \phi(s)N(n-1) = NM_1(n-1)(1 - 2s/\rho) > 0, \quad \text{for } |s| < \frac{\rho}{2}.$$

Setting

$$h(s) = \frac{\phi}{\sigma_0^{-1} - N(n-1)\phi} = \frac{1}{N(n-1)} \frac{\rho}{\rho - 2s},$$

equation (7.4.49) becomes

$$\frac{dw}{ds} = Nh(s)w + Mh(s). \quad (7.4.51)$$

Let w_0 the solution to (7.4.51) such that

$$w_0(0) = 0. \quad (7.4.52)$$

We have

$$w_0 = \frac{M}{N} \left[\exp \left(N \int_0^s h(\eta) d\eta \right) - 1 \right] = \frac{M}{N} \left[\left(\frac{\rho}{\rho - 2s} \right)^{\frac{\rho}{2(n-1)}} - 1 \right]. \quad (7.4.53)$$

In particular, w_0 is analytic in $(-\frac{\rho}{2}, \frac{\rho}{2})$ and

$$0 \preccurlyeq w_0. \quad (7.4.54)$$

The latter relationship can be easily checked by using formula (7.4.53) or can be also easily derived from (7.4.51) and (7.4.53), by expressing the derivatives of w_0 in 0 by means of those of lower order and noticing that they are all nonnegative. Now, for $k = 1, \dots, N$, let us consider the following functions

$$\tilde{U}_k(x, t) = w_0(\sigma_0^{-1}t + x_1 + \dots + x_{n-1}), \quad (7.4.55)$$

we have that \tilde{U}_k are solutions to equations (7.4.46) (when $\sigma = \sigma_0$) and by Proposition 6.3.4, they are analytic. Moreover by (1.2.3) we have

$$\begin{aligned} \tilde{\varphi}_k(x) &= \tilde{U}_k(x, 0) = w_0(x_1 + \dots + x_{n-1}) = \\ &= \sum_{m=0}^{\infty} w_0^{(m)}(0) \sum_{|\alpha'|=m} \frac{1}{\alpha'!} x^{\alpha'}. \end{aligned} \quad (7.4.56)$$

From which, taking into account (7.4.54), it is obvious that

$$0 \preccurlyeq \tilde{\varphi}_k, \quad \text{per } k = 1, \dots, N. \quad (7.4.57)$$

All in all, \tilde{U} is an analytic solution in a neighborhood of $(0, 0)$, of the Cauchy problem

$$\begin{cases} \partial_t \tilde{U}_k = H(x, t) \left(\sum_{j=1}^{n-1} \sum_{l=1}^N \partial_{x_j} \tilde{U}_l + \sum_{l=1}^N \tilde{U}_l + M \right), \\ \tilde{U}_k(x, 0) = \tilde{\varphi}_k(x), \end{cases} \quad (7.4.58)$$

where

$$H(x, t) = \frac{M_1 \rho}{\rho - (\sigma_0^{-1} t + x_1 + \cdots + x_{n-1})}.$$

Since \tilde{U} is analytic, (7.4.43) implies that the following power series converges in a neighborhood \mathcal{U} of $(0, 0)$

$$\sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} U_\alpha x^{\alpha'} t^{\alpha_n},$$

in addition its sum, U , solves Cauchy problem (7.4.28) in \mathcal{U} .

Step IV. The uniqueness to Cauchy problem (7.4.28) in the class of analytic functions in a connected neighborhood of 0 is a consequence of **Step I** and of the unique continuation property for analytic functions. As a matter of fact, if V' V'' are analytic solutions of (7.4.28),+ then

$$\partial^\alpha V'_k(0, 0) = U_{k, \alpha} = \partial^\alpha V''_k(0, 0), \quad k = 1, \dots, N$$

so that, by Theorem 6.2.2, we have $V' = V''$ in a neighborhood of 0 .

■

Remark on the neighborhood in which there exist solutions of the Cauchy problem

In what follows we will be interested in having some detailed information about the dependence of the neighborhood \mathcal{U} by the known term f and, consequently, by the initial data of Cauchy problem (7.4.3). From Step III of the previous proof we can say that the neighborhood \mathcal{U} does not depend on the constant M_2 . To clarify what we have just claimed, it suffices to apply Proposition 6.3.5 to the composite function

$$w_0(\sigma_0^{-1} t + x_1 + \cdots + x_{n-1}).$$

Let us observe that by (7.4.53), setting $\kappa = \frac{\rho}{2(n-1)}$, we have there exists a constant $c_\kappa \geq 1$ such that

$$0 \leq w_0^{(m)}(0) \leq \frac{c_\kappa M}{N} (c_\kappa \rho^{-1})^m m!, \quad \forall m \in \mathbb{N}_0.$$

Hence

$$w_0 \in \mathcal{C}_{\frac{c_\kappa M}{N}, \frac{\rho}{c_\kappa}}.$$

On the other hand we get trivially

$$(\sigma_0^{-1}t + x_1 + \cdots + x_{n-1}) \in \mathcal{C}_{(\sigma_0^{-1}+n-1), 1}.$$

Hence

$$w_0 (\sigma_0^{-1}t + x_1 + \cdots + x_{n-1}) \in \mathcal{C}_{\frac{c_\kappa M}{N}, R},$$

where

$$R = \frac{\rho}{c_\kappa \sigma_0^{-1} + n - 1}$$

and by (7.4.50) it turns out that σ_0 does not depend by M_2 . Therefore R depends on M_1 , ρ , n and N only. Moreover, we can choose $\mathcal{U} = \{(x, t) \in \mathbb{R}^n : |t| + |x_1| + \cdots + |x_{n-1}| < R\}$ as the neighborhood in which the Cauchy problem (7.4.28) admits a solution.

In preparation for what we will do later, let us go back to consider the case of a linear differential operator of order m given by

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha \quad (7.4.59)$$

and let us consider the following Cauchy problem

$$\begin{cases} P(x, \partial)u(x) = f(x), \\ \partial_n^j u(x', 0) = 0, \quad j = 0, 1, \dots, m-1, \forall x' \in B'_1. \end{cases} \quad (7.4.60)$$

Let us suppose that, for given M_0, ρ_0 , we have

$$P_m((x', 0), e_n) \neq 0, \quad \forall x' \in \overline{B'_1}, \quad (7.4.61)$$

$$a_\alpha \in \mathcal{C}_{M_0, \rho_0}(z), \quad |\alpha| \leq m, \quad \forall z \in \overline{B'_1} \times [-\delta_0, \delta_0], \quad (7.4.62)$$

$$f \text{ be a polynomial.} \quad (7.4.63)$$

Then the solution to problem (7.4.60) there exists in $\overline{B'_1} \times [-\delta, \delta]$ (actually, in a neighborhood of $\overline{B'_1} \times \{0\}$) where $\delta > 0$ depends on M_0, ρ_0, δ_0 and $\min\{|P_m((x', 0), e_n)| : x' \in \overline{B'_1}\}$, **but does not depend by the polynomial f** . In order to check this assertion, let h be the degree of the polynomial f . Set

$$K = 1 + \sum_{|\beta| \leq h} \max_{\overline{B'_1} \times [-\delta_0, \delta_0]} |\partial^\beta f|.$$

It is evident that, setting.

$$\tilde{u} = \frac{u}{K}, \quad \tilde{f} = \frac{f}{K},$$

u solves Cauchy problem (7.4.60) if and only if v solves the following Cauchy problem

$$\begin{cases} P(x, \partial)\tilde{u}(x) = \tilde{f}(x), \\ \partial_n^j \tilde{u}(x', 0) = 0, \quad j = 0, 1, \dots, m-1, \quad \forall x' \in \overline{B'_1}. \end{cases} \quad (7.4.64)$$

On the other hand, because of the way we defined \tilde{f} we can certainly state that

$$\tilde{f} \in \mathcal{C}_{1,1}(z), \quad \forall z \in \overline{B'_1} \times [-\delta_0, \delta_0]. \quad (7.4.65)$$

We can then return to problem (7.4.28). Hence by applying the Cauchy–Kovalevskaya Theorem and taking into account that a_α , $|\alpha| \leq m$, are analytic functions in a neighborhood of $\overline{B'_1}(0) \times \{0\}$, we conclude that the solution of Cauchy problem (7.4.60) exists and it is analytic in $\overline{B'_1} \times [-\delta, \delta]$ where $\delta > 0$ depends on M_0, ρ_0, δ_0 and on

$$\min\{|P_m((x', 0), e_n)| : x' \in \overline{B'_1}\}$$

(but does not depend on f).

Obviously, if (7.4.61), (7.4.62), (7.4.63) hold, similar conclusions are valid to the Cauchy problem

$$\begin{cases} P(x, \partial)u(x) = f(x), \quad \text{se } x \in \mathcal{U}_{\tilde{x}}, \\ \partial_n^j u(x', 0) = g_j, \quad j = 0, 1, \dots, m-1, \quad \forall x' \in \overline{B'_1}. \end{cases} \quad (7.4.66)$$

provided that g_j are polynomials for $j = 0, 1, \dots, m-1$. ♦

7.5 Further comments on the Cauchy–Kovalevskaya Theorem. Examples

7.5.1 A few brief note on the qualislinear and the non-linear case

It is not difficult to adapt the proof of the Cauchy–Kovalevskaya Theorem to the case of a quasilinear operator

$$\sum_{|\alpha|=m} a_\alpha(x, (\partial^\beta u)_{\|\beta\|\leq m-1}) \partial^\alpha u + a_0(x, (\partial^\beta u)_{\|\beta\|\leq m-1}).$$

In this case, we recall, the Cauchy problem is

$$\begin{cases} \mathcal{P}(u) = 0, \\ \partial_n^j u(x) = g_j(x), \quad j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma. \end{cases} \quad (7.5.1)$$

One can prove that if a_α, a_0, g_j are analytic functions, Γ is analytic and noncharacteristic (Definition 7.2.2) then for every $\tilde{x} \in \Gamma$ there exists a neighborhood $\mathcal{U}_{\tilde{x}}$ in which Cauchy problem (7.5.1) has analytic solution. For the proof we refer to [23].

In (7.1.5) we have formulated the Cauchy problem for the fully nonlinear equation

$$\begin{cases} F(x, (\partial^\alpha u)_{|\alpha|\leq m}) = 0, \\ \frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), \quad j = 0, 1, \dots, m-1, \quad \forall x \in \Gamma. \end{cases} \quad (7.5.2)$$

Here we only outline the proof of the existence of the solutions to problem (7.5.2) referring for more details to [18, Chapter 1], [21, Chapter 1].

Let us consider the case where $\Gamma = \{x_n = 0\}$. We know that we may always reduce to this case by "flattening" Γ (by map (7.4.4)). So, instead of the conditions $\frac{\partial^j u(x)}{\partial \nu^j} = g_j(x)$, $j = 0, 1, \dots, m-1$, for $x \in \Gamma$, we may consider

$$\partial_n^j u(x', 0) = g_j(x') \quad j = 0, 1, \dots, m-1, \quad \forall x' \in B'_r, \quad (7.5.3)$$

where $r > 0$. In the first part of the proof of Proposition 7.3.2 we have already seen that conditions (7.5.3) allow us to determine the derivatives

$$\partial_{x'}^{\alpha'} \partial_n^j u(x', 0) = \partial_{x'}^{\alpha'} g_j(x') \quad j = 0, 1, \dots, m-1, \quad \alpha' \in \mathbb{N}_0^{n-1} \quad x' \in B'_r. \quad (7.5.4)$$

without involving the equation $F(x, (\partial^\alpha u)_{|\alpha| \leq m}) = 0$. Let us recall that in order to calculate the derivative $\partial_n^m u(x', 0)$ we need to use the equation. More precisely, we have

$$F(x', 0, (\partial_{x'}^{\alpha'} g_j)_{|\alpha'|+j \leq m, j \leq m-1}(x'), \partial_n^m u(x', 0)) = 0. \quad (7.5.5)$$

To find $z = \partial_n^m u(x', 0)$ from equation (7.5.5) we need that the equation

$$F(x', 0, (\partial_{x'}^{\alpha'} g_j)_{|\alpha'|+j \leq m, j \leq m-1}(x'), z) = 0, \quad (7.5.6)$$

admits a solution. If, for instance, we require

$$(\partial_z F)(x', 0, (\partial_{x'}^{\alpha'} g_j)_{|\alpha'|+j \leq m, j \leq m-1}(x'), z) \neq 0 \quad (7.5.7)$$

then we may express z as a function of variable x' of class C^1 (provided F is of class C^1) or as an analytic function, provided F is analytic. Let us observe that condition (7.5.7) makes it possible to write, in a neighborhood of Γ , the equation

$$F(x, (\partial^\alpha u)_{|\alpha| \leq m}) = 0$$

like

$$\partial_n^m u = G(x, (\partial^\alpha u)_{|\alpha| \leq m, \alpha_n < m}), \quad (7.5.8)$$

where G is analytic (provided F is analytic). Hence, in the fully nonlinear case, condition (7.5.7) may replace the condition that Γ is noncharacteristic surface for a linear (or quasilinear) operator. Furthermore, since we can find $\partial_n^m u(x', 0)$ from (7.5.5), we set $g_m(x') = \partial_n^m u(x', 0)$, by making the derivatives of both the sides of (7.5.8) w.r.t. x_n we have

$$\begin{cases} \partial_n^{m+1} u = \sum_{|\alpha| \leq m, j < m} a_{\alpha', j} \partial_{x'}^{\alpha'} \partial_n^{j+1} u + (\partial_n G)(x, (\partial^\alpha u)_{|\alpha| \leq m-1, \alpha_n}), \\ \frac{\partial^j u(x', 0)}{\partial \nu^j} = g_j(x'), \quad j = 0, 1, \dots, m, \quad \forall x \in \Gamma. \end{cases}$$

where

$$a_{\alpha', j} = \left(\partial_{p_{\alpha', j}} G \right) \left(x, (\partial_{x'}^{\alpha'} \partial_n^j u)_{|\alpha'|+j \leq m, j < m} \right),$$

for $|\alpha'| + j \leq m, j < m$.

In other words, if conditions (7.5.6) and (7.5.7) hold we may reformulate Cauchy problem (7.5.2) as a Cauchy problem for a quasilinear equation of order $m + 1$ and we may use the existence results that obtained in the quasilinear case.

7.5.2 Comments about the existence and the uniqueness of solutions. Examples and counterexamples

In this Subsection we return to the **linear case**. In general, if Γ is a characteristic surface for the operator $P(x, \partial)$, neither existence nor uniqueness for Cauchy problem (7.4.3) can be expected. Let us look at some example.

(a) Let $P_m(\xi)$ be a homogeneous polynomial of degree m and let us assume

$$P_m(N) = 0,$$

where $N \in \mathbb{R}^n \setminus \{0\}$. Then the hyperplane

$$\pi = \{x \cdot N = 0\}$$

is a characteristic surface for the operator $P_m(\partial)$. Now, let us consider the functions

$$u^{(t)}(x) = e^{tx \cdot N} - \sum_{k=0}^{m-1} \frac{t^k (x \cdot N)^k}{k!}, \quad t \in \mathbb{R}$$

where $t \neq 0$. It is easy to check that, for every $t \in \mathbb{R}$, $u^{(t)}$ solves the Cauchy problem

$$\begin{cases} P_m(\partial)u(x) = 0, \\ \frac{\partial^j u}{\partial N^j} = 0, \quad j = 0, 1, \dots, m-1, \text{ on } \pi, \end{cases} \quad (7.5.9)$$

Hence, does not hold the uniqueness for problem (7.5.9).

(b) The Cauchy–Kovalevskaya Theorem gives the existence and uniqueness of solutions to the Cauchy problem having initial data on a noncharacteristic surface *under the assumption of analyticity of all the data of the problem*. Regarding the existence of the solutions, if we desire to preserve the same generality of the Theorem, the assumptions of analyticity cannot be reduced. To prove this, it suffices to consider the following Cauchy problem

$$\begin{cases} \partial_y^2 u + \partial_x^2 u = 0, \\ u(x, 0) = g_0(x), \text{ for } x \in (-r, r), \\ \partial_y u(x, 0) = g_1(x), \text{ for } x \in (-r, r). \end{cases} \quad (7.5.10)$$

Since u satisfies the Laplace equation $\partial_y^2 u + \partial_x^2 u = 0$ in a neighborhood of $(0, 0)$ it is analytic in such a neighborhood, therefore the initial data, $u(x, 0) = g_0(x)$ and $\partial_y u(x, 0) = g_1(x)$ also need to be analytic.

Incidentally, even though we consider the "one-sided" Cauchy problem the situation do not change in a significant way. Let us consider, indeed, the problem

$$\begin{cases} \partial_y^2 u + \partial_x^2 u = 0, & \text{for } (x, y) \in B_r^+, \\ u(x, 0) = g_0(x), & \text{for } x \in (-r, r), \\ \partial_y u(x, 0) = g_1(x), & \text{for } x \in (-r, r), \end{cases}$$

where $u \in C^2(B_r^+) \cap C^0(\overline{B_r^+})$. Let us suppose for simplicity that $g_1 \equiv 0$ and let us consider the even reflection w.r.t. x -axis of u

$$v(x, y) = u(x, |y|)$$

then, by the Schwarz reflection principle, we have

$$\Delta v = 0, \text{ in } B_r$$

and again we have that v is an analytic function and therefore g_0 is an analytic function.

Exercise 1. Let us suppose $g_0, g_1 \in C^\omega(-r, r)$ in (7.5.10). Prove that the solution to Cauchy problem (7.5.10) is given by

$$u(x, y) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{g_0^{(2n)}(x) y^{2n}}{(2n)!} + \frac{g_1^{(2n+1)}(x) y^{2n+1}}{(2n+1)!} \right). \quad (7.5.11)$$

♣

(c) Let us consider the Cauchy problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, \\ u(x, 0) = g_0(x), & \text{for } x \in \mathbb{R}, \\ \partial_t u(x, 0) = g_1(x), & \text{for } x \in \mathbb{R}. \end{cases} \quad (7.5.12)$$

In this case the straight line $\{t = 0\}$ is a characteristic line for the operator $\partial_t - \partial_x^2$. It is evident that if we do not require

$$g_1(x) = g_0''(x), \quad \forall x \in \mathbb{R}, \quad (7.5.13)$$

then Cauchy problem (7.5.12) has no solutions. As a matter of fact, if problem (7.5.12) admits solutions, even just of class C^2 , then

$$g_0''(x) = \partial_x^2 u(x, 0) = \partial_t u(x, 0) = g_1(x), \quad \text{for } |x| < 1$$

and therefore (7.5.13) hold.

We now check that even condition (7.5.13) is satisfied we can exhibit a $g_0 \in C^\omega$ such that problem (7.5.12) has no solutions. First, it is evident that if (7.5.13) is satisfied, then problem (7.5.12) can be formulated as

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, \\ u(x, 0) = g_0(x), \quad \text{for } x \in \mathbb{R}. \end{cases} \quad (7.5.14)$$

Let

$$g_0(x) = \frac{1}{1+x^2}.$$

We have that $g_0 \in C^\omega(\mathbb{R})$ and

$$g_0^{(2k)}(0) = (-1)^k (2k)!, \quad \text{for } k \in \mathbb{N}_0.$$

Now, if there exists a solution u **analytic** in a neighborhood of $(0, 0)$ of problem 7.5.14, then $u(0, t)$ should be expanded in Taylor series in $t = 0$

$$\partial_t^j u(0, 0) = \partial_x^{2j} u(0, 0) = g_0^{(2j)}(0) = (-1)^j (2j)!,$$

on the other side, the power series

$$\sum_{j=0}^{\infty} \frac{(-1)^j (2j)!}{j!} t^j$$

has the radius of convergence equal to zero. Hence $u(0, t)$ is not analytic.

Also, we note that the Cauchy problem

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, \\ u(x, 0) = 0, \quad \text{for } x \in \mathbb{R}. \end{cases} \quad (7.5.15)$$

admits only one analytic solution, namely the null solution. As a matter of fact, the null function is trivially solution of problem (7.5.15) and if u is an analytic solution of (7.5.15) then we have, for each $i, j \in \mathbb{N}_0$

$$\partial_x^i \partial_t^j u(0, 0) = \partial_x^{i+2j} u(0, 0) = 0$$

hence $u \equiv 0$.

Of course the one-sided Cauchy problems are also of interest

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & \text{for } x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = g_0(x), & \text{for } x \in \mathbb{R}. \end{cases} \quad (7.5.16)$$

For this problem it turns out that (if g_0 is regular enough)

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} g_0(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi$$

is a solution to (7.5.16). In particular if $g_0 = \frac{1}{1+x^2}$, then Cauchy problem (7.5.16) admits solutions (of course, nonanalytic w.r.t. t).

Keep in mind that the problems that we have considered in the point (c) are not written in the form (7.4.12). As a matter of fact, the term on the right-hand side of equation $\partial_t u = \partial_x^2 u$ has order 2 greater than the order of derivative $\partial_t u$, on the left-hand side.

(d) In a strong contrast with the example considered in (b) we present now the following example for the vibrating string equation. Let us consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \\ u(x, 0) = g_0(x), & \text{for } x \in (-1, 1), \\ \partial_t u(x, 0) = g_1(x), & \text{for } x \in (-1, 1). \end{cases} \quad (7.5.17)$$

Let us first prove problem (7.5.17) has at most one solution $u \in C^2(\overline{Q})$ where $Q = \{(x, t) \in \mathbb{R}^2 : |x| + |t| \leq 1\}$.

To this purpose we observe that

$$0 = (\partial_t^2 u - \partial_x^2 u) \partial_t u = \frac{1}{2} (\partial_t (\partial_t u)^2 - 2\partial_x (\partial_x u \partial_t u) + \partial_t (\partial_x u)^2). \quad (7.5.18)$$

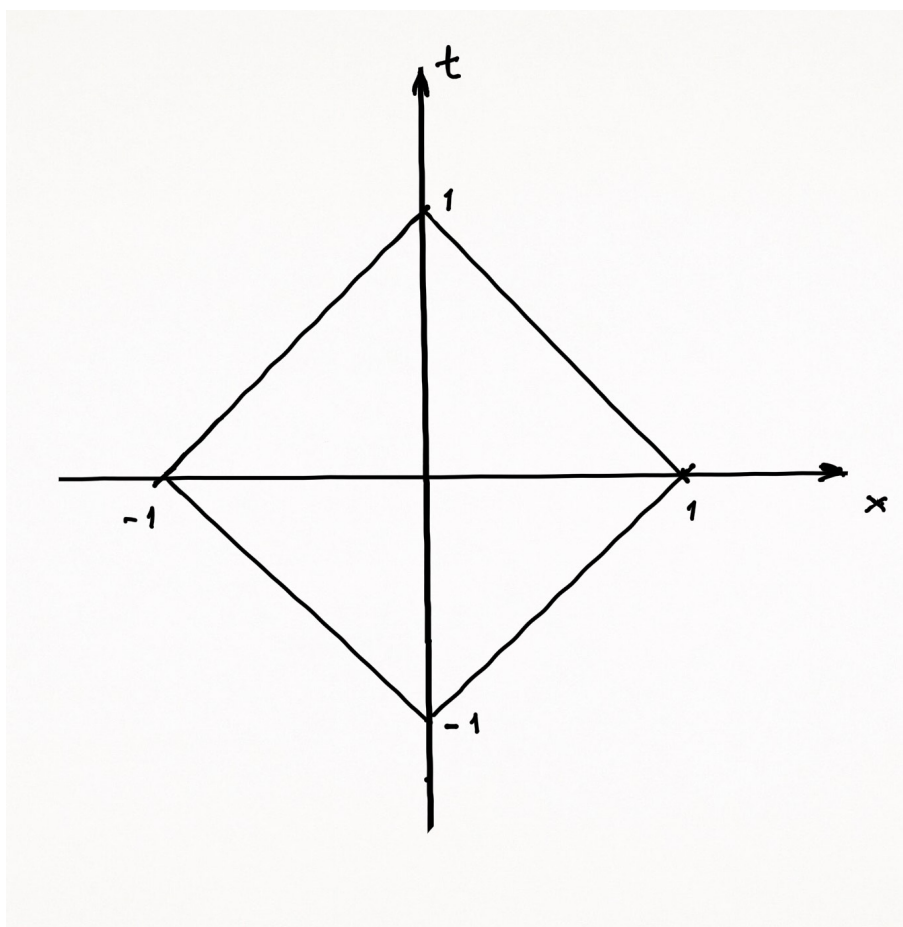


Figure 7.2: $Q = \{(x, t) \in \mathbb{R}^2 : |x| + |t| \leq 1\}$

Moreover, let us suppose that $g_0 = 0$ and $g_1 = 0$ in (7.5.17). We integrate both the sides of (7.5.18) over

$$\mathcal{T}_\delta = Q \cap \{(x, t) \in \mathbb{R}^2 : 0 < t < 1 - \delta\},$$

where $\delta \in (0, 1)$ is arbitrary. We obtain, by the divergence Theorem,

$$\begin{aligned} 0 &= \iint_{\mathcal{T}_\delta} (\partial_t^2 u - \partial_x^2 u) \partial_t u dx dt = \\ &= \frac{1}{2} \int_{\partial \mathcal{T}_\delta} ((\partial_t u)^2 \nu_t - 2(\partial_x u \partial_t u) \nu_x + (\partial_x u)^2 \nu_x) dS = \\ &= -\frac{1}{2} \int_{-1}^1 ((\partial_t u)^2(x, 0) + (\partial_x u)^2(x, 0)) dx + \\ &+ \frac{1}{2\sqrt{2}} \int_{1-\delta}^1 (\partial_t u(x, 1-x) - \partial_x u(x, 1-x))^2 dx + \\ &+ \frac{1}{2} \int_{-1+\delta}^{1-\delta} ((\partial_t u)^2(x, \delta) + (\partial_x u)^2(x, \delta)) dx + \\ &+ \frac{1}{2\sqrt{2}} \int_{-1}^{-1+\delta} (\partial_t u(x, 1+x) + \partial_x u(x, 1+x))^2 dx \geq \\ &\geq \frac{1}{2} \int_{-1+\delta}^{1-\delta} ((\partial_t u)^2(x, \delta) + (\partial_x u)^2(x, \delta)) dx. \end{aligned} \quad (7.5.19)$$

Hence

$$\int_{-1+\delta}^{1-\delta} ((\partial_t u)^2(x, \delta) + (\partial_x u)^2(x, \delta)) dx = 0$$

from which we have $(\partial_t u)^2(x, \delta) + (\partial_x u)^2(x, \delta) = 0$ and, since δ is arbitrary, we have $\partial_x u = \partial_t u = 0$ in $Q \cap \{(x, t) \in \mathbb{R}^2 : 0 \leq t\}$. Finally, since $u(x, 0) = \partial_x u(x, 0) = 0$, we have $u = 0$ in $Q \cap \{(x, t) \in \mathbb{R}^2 : 0 \leq t\}$. Similarly, we obtain $u = 0$ in $Q \cap \{(x, t) \in \mathbb{R}^2 : 0 \geq t\}$. Therefore $u = 0$ in Q .

The existence of solutions also does not require that the Cauchy data g_0 and g_1 are analytic. As a matter of fact it is checked straightforwardly that, if $g_0 \in C^2([-1, 1])$ and $g_1 \in C^1([-1, 1])$, then the solution to Cauchy problem (7.5.17) is given by

$$u(x, t) = \frac{g_0(x+t) + g_0(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g_1(\eta) d\eta. \quad (7.5.20)$$

Conclusion. From the short discussion of this Section we can say that, in the context of partial differential equations, the Cauchy–Kovalevskaya Theorem represents for us more a starting point than an ending point. Starting

with the next Section we will focus more on the issue of the uniqueness, under assumptions which are weaker than the analyticity of all the data.

7.6 The Holmgren Theorem

Let us start by recalling the **divergence Theorem**. Let D a bounded open \mathbb{R}^n such that its boundary ∂D is of class $C^{0,1}$. Then we have

$$\int_D \partial_j u dx = \int_{\partial D} u \nu_j dS, \quad j = 1, \dots, n \quad \forall u \in C^1(\overline{D}), \quad (7.6.1)$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward normal to ∂D and dS is the $(n-1)$ -element of surface.

Let $m \in \mathbb{N}$, $a_\alpha \in C^m(\overline{D})$, $|\alpha| \leq m$ and

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha. \quad (7.6.2)$$

We call the **(formal) adjoint operator** of $P(x, \partial)$ the following operator

$$C^m(\overline{D}) \ni u \rightarrow P^*(x, \partial)u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha(x)u). \quad (7.6.3)$$

Let us note that, up to the sign, the principal part of $P^*(x, \partial)$ is equal to the principal part of $P(x, \partial)$.

The following **Green identity** holds true, for any $u, v \in C^m(\overline{D})$,

$$\int_D (vP(x, \partial)u - uP^*(x, \partial)v) dx = \int_{\partial D} \mathcal{M}(u, v; \nu) dS, \quad (7.6.4)$$

where $\mathcal{M}(u, v; \nu)$ is linear w.r.t. u, v and ν . Moreover

$$\mathcal{M}(u, v; \nu) = \sum_{|\beta|+|\gamma| \leq m-1} c_{\beta\gamma}(x) \partial^\beta u \partial^\gamma v, \quad (7.6.5)$$

where $c_{\beta\gamma}$, $|\beta| + |\gamma| \leq m-1$, belong (at least) to $C^0(\partial D)$.

Identity (7.6.4) can be obtained by applying repeatedly the following simple identity

$$v(x)a(x)\partial_j u(x) = \partial_j (v(x)a(x)u(x)) - \partial_j (v(x)a(x)) u(x)$$

obtaining

$$\begin{aligned}
 v(x)a_\alpha(x)\partial^\alpha u(x) &= v(x)a_\alpha(x)\partial_j\partial^{\alpha-e_j}u(x) = \\
 &= \partial_j(v(x)a_\alpha(x)\partial^{\alpha-e_j}u(x)) - \partial_j(v(x)a_\alpha(x))\partial^{\alpha-e_j}u(x) = \\
 &= \dots = \\
 &= \operatorname{div}(F_\alpha) + (-1)^{|\alpha|}\partial^\alpha(a_\alpha(x)v(x))u(x),
 \end{aligned} \tag{7.6.6}$$

where F_α is a suitable vector field. Next we add up the identities obtained in (7.6.6), we integrate over D the obtained new identity, and by the divergence Theorem we get (7.6.4).

Before stating the Holmgren Theorem, let us introduce some notation.

Let Ω be an open set of \mathbb{R}^n , $x_0 \in \Omega$ and let $\phi \in C^2(\Omega)$. Let us denote by

$$\Gamma = \{x \in \Omega : \phi(x) = \phi(x_0)\}.$$

We will assume

$$\nabla\phi(x) \neq 0, \quad \forall x \in \Gamma. \tag{7.6.7}$$

If \mathcal{U} is a neighborhood of x_0 we denote by \mathcal{U}_+ the set

$$\mathcal{U}_+ = \mathcal{U} \cap \{x \in \Omega : \phi(x) \geq \phi(x_0)\}. \tag{7.6.8}$$

Theorem 7.6.1 (Holmgren). *Let $a_\alpha \in C^\omega(\Omega)$, $|\alpha| \leq m$. Let us suppose that Γ is a noncharacteristic surface in x_0 for the operator*

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha. \tag{7.6.9}$$

Then there exists a neighborhood \mathcal{U} of x_0 such that we have:

if $u \in C^m(\overline{\mathcal{U}_+})$ satisfies

$$\begin{cases} P(x, \partial)u = 0, & \text{in } \mathcal{U}_+ \\ \partial^\alpha u = 0, & \text{for } |\alpha| \leq m-1, \quad x \in \Gamma \cap \mathcal{U}_+. \end{cases} \tag{7.6.10}$$

Then we have

$$u \equiv 0 \quad \text{in } \mathcal{U}_+. \tag{7.6.11}$$

Remarks. Before starting with the proof of Theorem 7.6.1 we observe what follows.

(i) In (7.6.10), u is required to be a solution to the equation $P(x, \partial)u = 0$ in \mathcal{U}_+ , unlike the Cauchy-Kovalevskaya Theorem in which u is required to be

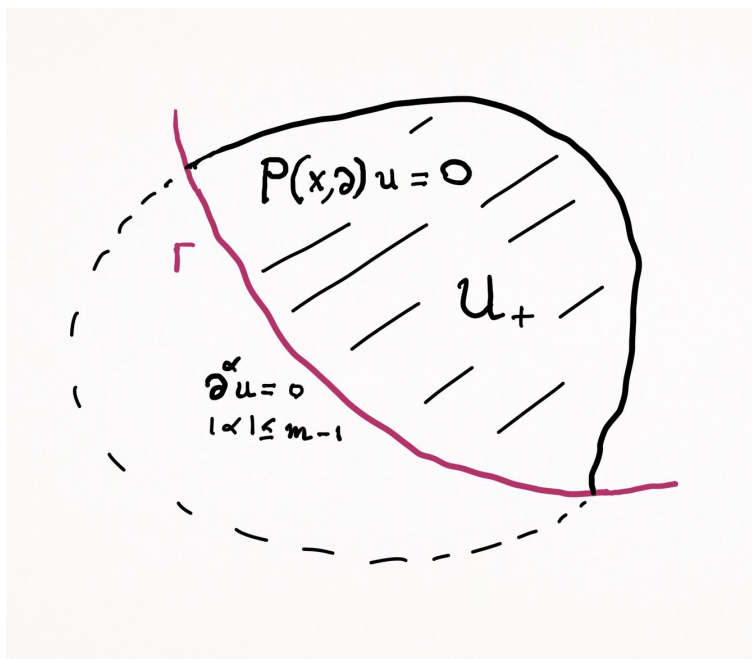


Figure 7.3:

a solution in a full neighborhood of x_0 . Furthermore, it is only required that $u \in C^m(\overline{U_+})$.

(ii) The initial surface Γ is assumed to be of class C^2 , thus, not analytic like in the Cauchy-Kovalevskaya Theorem. Also, let us note that in (7.6.10) we require $\partial^\alpha u = 0$ for $|\alpha| \leq m - 1$, on $\Gamma \cap U_+$ and not just that

$$\frac{\partial^j u}{\partial \nu^j} = 0, \quad \text{for } j = 0, 1, \dots, m - 1.$$

Of course, if we want to assume the latter conditions we should require $\phi \in C^{m-1}(\Omega)$ (compare with the proof of the first part of Proposition 7.3.2).

◆

Proof of the Holmgren Theorem.

We may assume Γ be a graph of a function. More precisely we may assume that.

$$\Gamma = \{(x', \psi(x')) : x' \in B'_{r_0}\}, \quad (7.6.12)$$

where $\psi \in C^2(\overline{B'_{r_0}})$ satisfies

$$\psi(0) = |\nabla_{x'} \psi(0)| = 0. \quad (7.6.13)$$

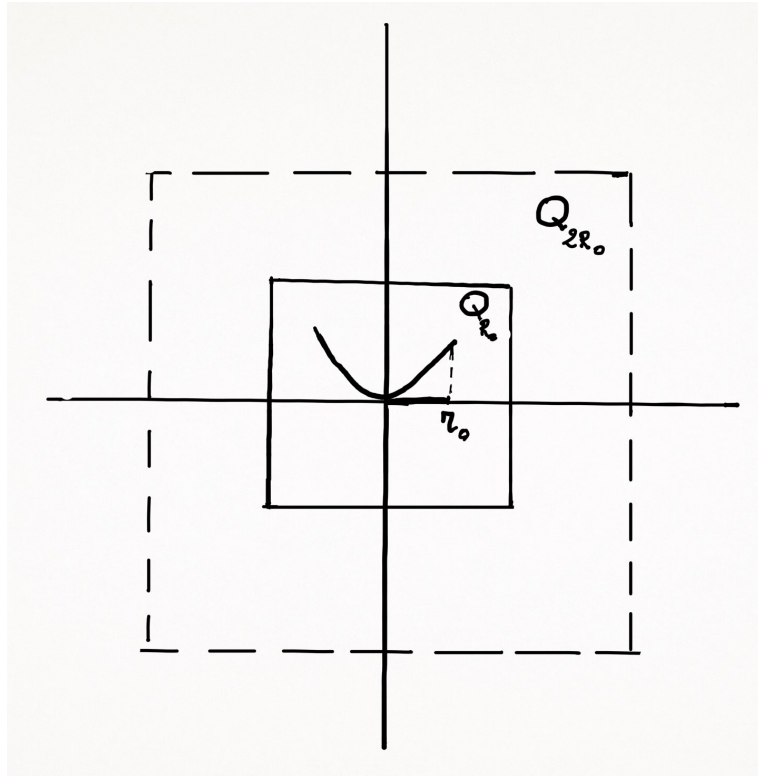


Figure 7.4:

We will divide the proof of the Theorem into two steps. In the first step we will assume that ψ is a **strictly convex function**. In the second step we will reduce to the first part by means of the so-called Holmgren transformation.

Step I. Let ψ strictly convex, let R_0 satisfy

$$\Gamma \subset Q_{R_0} := B'_{R_0} \times (-R_0, R_0) \quad (7.6.14)$$

and, by assumptions,

$$a_\alpha \in C^\omega(Q_{2R_0}). \quad (7.6.15)$$

Since Γ is a noncharacteristic surface in 0 we may assume, recalling (7.6.13),

$$|P_m(0, e_n)| = c_0 > 0. \quad (7.6.16)$$

By the continuity of the coefficients of $P_m(x, \partial)$ and by (7.6.16) we have that there exists $\rho_1 > 0$ such that

$$|P_m((x', h), e_n)| \geq \frac{c_0}{2}, \quad \forall x' \in B'_{\rho_1}(0), |h| \leq \rho_1. \quad (7.6.17)$$

This implies that for every $h \in [-\rho_1, \rho_1]$ the flat surface

$$\{(x', h) : x' \in B'_{\rho_1}(0)\}$$

is noncharacteristic.

Let now f be a polynomial. By the Cauchy–Kovalevskaya Theorem and by the Remark which follows such a Theorem, there exists ρ_2 , $0 < \rho_2 < \rho_1$, ρ_2 independent of f such that, there exists an analytic solution in $\overline{B'_{\rho_2}(0)} \times [h - \rho_2, h + \rho_2]$ to the following Cauchy problem

$$\begin{cases} P^*(x, \partial)w = f, \\ \partial^\alpha w(x', h) = 0, \quad |\alpha| \leq m - 1. \end{cases} \quad (7.6.18)$$

Let

$$h_0 = \min_{\partial B'_{\rho_2}(0)} \psi,$$

$$h_1 = \min \left\{ h_0, \frac{\rho_2}{2} \right\}.$$

Let us notice that by the strict convexity of ψ , h_1 is positive. Let us choose in (7.6.18)

$$h = h_1. \quad (7.6.19)$$

Let us consider the set

$$D = \{(x', x_n) : x' \in B'_{\rho_2}(0), \psi(x') < x_n < h_1\}.$$

We have that D has a "lens" shape in particular on the boundary of D there are no vertical segments. Let us note that, because of the way we choose h_1 , we have $w \in C^\omega(\overline{D})$.

Now, by the assumptions u belongs to $C^m(\overline{D})$ and it is a solution to the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } D, \\ \partial^\alpha u = 0, & \text{for } |\alpha| \leq m - 1, \quad x \in \Gamma \cap D. \end{cases} \quad (7.6.20)$$

By the Green identity (7.6.4), we have

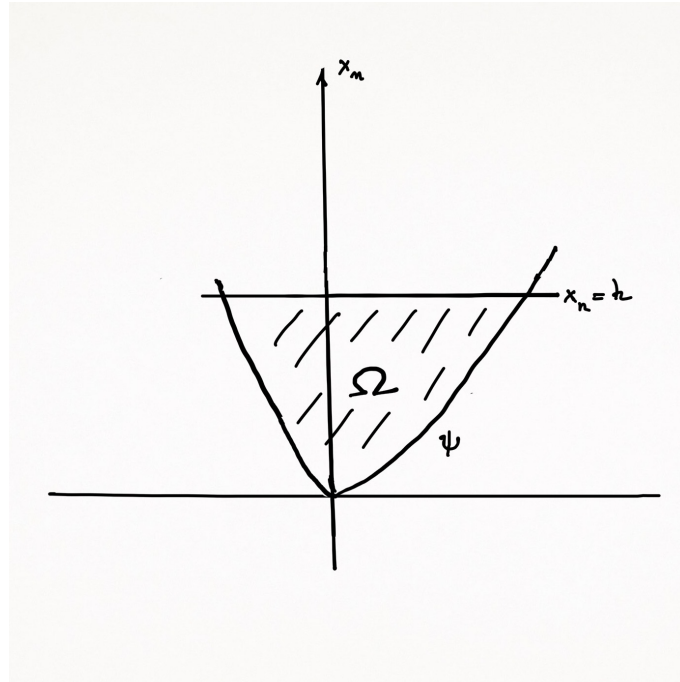


Figure 7.5:

$$\begin{aligned}
 \int_D f u dx &= \int_D u P^*(x, \partial) w dx = \\
 &= \int_D (u P^*(x, \partial) w - w P(x, \partial) u) dx = \\
 &= \int_{\partial D} \mathcal{M}(u, w; \nu) dS = 0.
 \end{aligned} \tag{7.6.21}$$

To prove that

$$\int_{\partial D} \mathcal{M}(u, w; \nu) dS = 0,$$

it suffices to write the integral over ∂D as a sum of two integrals, say I_1 and I_2 , with the same integrand $\mathcal{M}(u, w; \nu)$, where I_1 is the integral over a portion of the graph of ψ , on which $\partial^\alpha u = 0$, for $|\alpha| \leq m - 1$, and I_2 is the integral over a portion of hyperplane $\{x_n = h_1\}$ on which, by (7.6.18) and (7.6.19), we have $\partial^\alpha w = 0$, for $|\alpha| \leq m - 1$. Hence, by (7.6.5), both I_1 and I_2 is equal to zero. Therefore, by (7.6.21) we get

$$\int_D f u dx = 0, \quad \text{for every polynomial } f$$

and since the set of polynomials is dense in $C^0(\overline{D})$ (Theorem 2.1.2) we have

$$u \equiv 0, \quad \text{in } D.$$

The first part of proof is concluded.

Step II. Now, we suppose that Γ satisfies (7.6.13) e (7.6.12), but we *do not suppose* that ψ is strictly convex. We may reduce to the case discussed in Step t I using the following **Holmgren transformation**

$$\Lambda : \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n, \quad x \rightarrow y = \Lambda(x', x_n) = \left(x', x_n + \frac{A}{2}|x'|^2 \right), \quad (7.6.22)$$

where $A > 0$ is to be chosen. Let us note that Λ is a diffeomorphism. Let it be further

$$\tilde{\Gamma} := \Lambda(\Gamma) = \left\{ (x', \tilde{\psi}(x')) : x' \in B'_{r_0} \right\}, \quad (7.6.23)$$

where

$$\tilde{\psi} := \psi(x') + \frac{A}{2}|x'|^2. \quad (7.6.24)$$

Let us choose A in such a way that $\tilde{\psi}$ is strictly convex. For this purpose it suffices to have

$$A > \left\| \partial^2 \psi \right\|_{L^\infty(B'_{r_0})},$$

where $\partial^2 \psi$ is the Hessian matrix of ψ . Fix such a number A . Let us denote by $\tilde{P}(y, \partial_y)$ the transformed operator by means of Λ

$$\tilde{P}(y, \partial_y)v(y)|_{y=\Lambda(x)} = P(x, \partial_x)u(x),$$

where v is defined by

$$v(\Lambda(x)) = u(x).$$

Let us notice that the coefficients of \tilde{P} are analytic functions. Setting $\tilde{\mathcal{U}}_+ = \Lambda(\mathcal{U}_+)$, we have that $v \in C^m(\tilde{\mathcal{U}}_+)$ is a solution to the Cauchy problem

$$\begin{cases} \tilde{P}(y, \partial_y)v = 0, & \text{in } \mathcal{U}_+, \\ \partial^\alpha v(y) = 0, & \text{for } |\alpha| \leq m-1, \quad y \in \tilde{\Gamma} \cap \mathcal{U}_+. \end{cases}$$

Let us recall (compare with (7.3.4)) that the symbol of the principal part of $\tilde{P}_m(y, \eta)$ is given by

$$\tilde{P}_m(y, \eta) = P(x, \partial_x(\Lambda(x))^t \eta)|_{x=\Lambda^{-1}(y)}.$$

Since

$$(\Lambda(0))^t e_n = e_n$$

we have

$$\tilde{P}_m(0, e_n) = P_m(0, e_n) \neq 0.$$

In short, we come back to the case already treated in Step I. Therefore, for a suitable neighborhood \mathcal{U} of 0, we have $v \equiv 0$ in \mathcal{U}_+ which implies $u \equiv 0$ in U_+ . ■

Remarks about the Holmgren Theorem.

1. If Γ is a noncharacteristic surface, Theorem 7.6.1 allows us to say that there exists an open set S such that $\Gamma \subset S$ and such that, denoting by $S_+ = S \cap \{x \in \Omega : \phi(x) \geq \phi(x_0)\}$, it occurs that if $v \in C^m(\overline{S_+})$ is a solution of the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } S_+, \\ \partial^\alpha u(x) = 0, & \text{for } |\alpha| \leq m-1, \quad x \in \Gamma, \end{cases} \quad (7.6.25)$$

then $u \equiv 0$ in S_+ .

Nevertheless, the statement of Holmgren Theorem does not clarify sufficiently how large the set S (or S_+) can be. Actually, one would expect that $\partial S \setminus \Gamma$ should consist of characteristic surfaces or, in other words, that the uniqueness for the Cauchy problem would hold until a characteristic surface is encountered.

If $P(x, \partial)$ is an elliptic operator with analytic coefficients in an open connected set Ω of \mathbb{R}^n (Section 7.3) and if Γ is a portion of a regular surface, then, since the ellipticity of $P(x, \partial)$ guarantees us that Γ is not characteristic, we would expect the same ellipticity of $P(x, \partial)$ guarantees that a solution of $P(x, \partial)u = 0$ in Ω , with null Cauchy data on Γ , is identically null on Ω . For instance, if $P(x, \partial) = \Delta$, the above occurs. As a matter of fact, it is enough to keep in mind that the solutions of $\Delta u = 0$ are analytic in Ω to obtain that $u \equiv 0$ in Ω .

If we have the vibrating string operator $\partial_t^2 - \partial_x^2$ we know that if u is a solution of $\partial_t^2 u - \partial_x^2 u = 0$ with zero initial conditions on $\Gamma = (-R, R)$ then u vanishes in the square $\{|x| + |t| < R\}$ that is, u vanishes in a region bounded by characteristic lines parallel to $\{x + t = 0\}$, $\{x - t = 0\}$.

Neither the situation described for the Laplace equation nor the one described for the vibrating string operator are a direct consequence of the statement of Theorem 7.6.1. A general answer to the problems is given by the Global Uniqueness Theorem proved by F. John, of which we will here provide the statement and examine some of its consequences.

2. The assumptions of Theorem 7.6.1 can be weakened. Here we merely give a few hints and refer to [34, Theorem 5.3.1] the interested reading in learning more about the topic. We point out, in particular, that:

(i) one may assume $\psi \in C^1(\Omega)$.

(ii) one may give a distributional formulation of Cauchy problem (7.6.10) and in this framework prove the uniqueness of the solution.

3. It is worth mentioning that the Holmgren uniqueness Theorem cannot be extended to the nonlinear case. For further information we refer the reader to [26], [55]. ♦

7.6.1 Statement of the Holmgren–John Theorem. Examples

Here we only state the Holmgren–John global uniqueness Theorem, for the proof we refer to [41]) or, in these notes, to the Chapter 11 in which we will prove the Stability Theorem due to F. John himself and from which we can trivially deduce the uniqueness.

In order to state the Global Uniqueness Theorem, we need the following definition (in it we follow the terminology introduced in [41]).

Definition 7.6.2. Let

$$F : B'_1 \times (0, 1) \rightarrow \mathbb{R}^n, \quad (7.6.26)$$

a function satisfying the following properties (let us denote by $y' \in B'_1$ and $\lambda \in (0, 1)$ the independent variables):

(i) F is injective,

(ii) F is analytic in $B'_1 \times (0, 1)$,

(iii) for every $(y', \lambda) \in B'_1 \times (0, 1)$ the jacobian matrix $\partial_{y', \lambda} F(y', \lambda)$ is nonsingular.

We call **analytic field** in \mathbb{R}^n the family of sets $\{S_\lambda\}_{\lambda \in (0, 1)}$, where

$$S_\lambda = \{F(y', \lambda) : y' \in (0, 1)\}, \quad \text{for } \lambda \in (0, 1). \quad (7.6.27)$$

We call **support of the analytic field** the open set (see Figure 7.6)

$$\Sigma := \bigcup_{\lambda \in (0, 1)} S_\lambda.$$

We denote, for any $\mu \in (0, 1)$,

$$\Sigma_\mu := \bigcup_{\lambda \in (0, \mu]} S_\lambda.$$

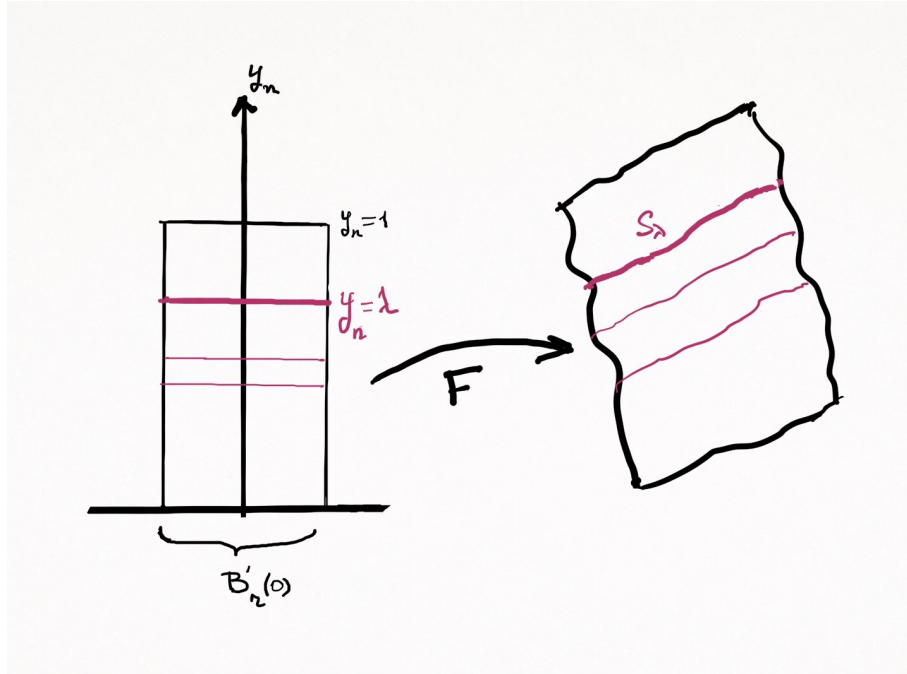


Figure 7.6:

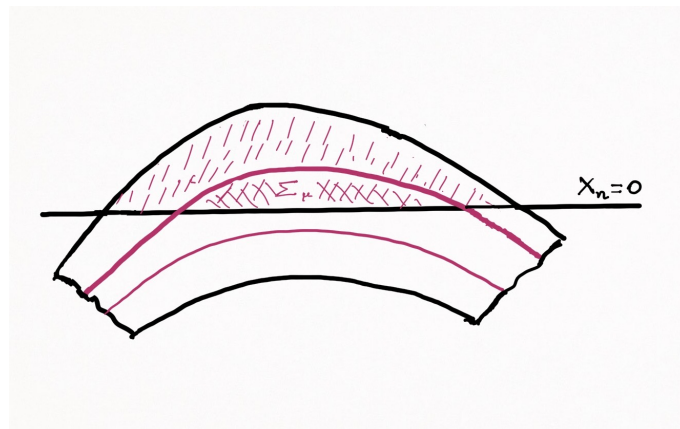


Figure 7.7:

Theorem 7.6.3 (Holmgren–John). *Let S_λ be an analytic field in \mathbb{R}^n and let Σ be its support. Let*

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

be a linear differential operator of order m , where $a_\alpha \in C^\omega(\Sigma)$. Let us define the sets (see Figure 7.7)

$$\mathcal{R} = \{x \in \mathbb{R}^n : x \in \Sigma, \quad x_n \geq 0\}, \quad (7.6.28)$$

$$\mathcal{Z} = \{(x', 0) : (x', 0) \in \Sigma\}. \quad (7.6.29)$$

Let us suppose

- (a) \mathcal{Z} and S_λ , $\lambda \in (0, 1)$, are noncharacteristic for $P(x, \partial)$,
- (b) for every $\mu \in (0, 1)$, $\mathcal{R} \cap \Sigma_\mu$, is a **closed set** of \mathbb{R}^n .

Then we have that if $u \in C^m(\mathcal{R})$ is a solution to the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } \mathcal{R}, \\ \partial^\alpha u = 0, & \text{for } |\alpha| \leq m - 1, \quad x \in \mathcal{Z}, \end{cases} \quad (7.6.30)$$

we have

$$u \equiv 0.$$

Let us illustrate somewhat the assumptions of Theorem 7.6.3. We observe that \mathcal{Z} is a portion of the hyperplane $\{x_n = 0\}$: this is exclusively an expository choice, actually \mathcal{Z} can be any C^2 noncharacteristic surface for $P(x, \partial)$. Furthermore, hypothesis (b) assures us that in the boundary of $\mathcal{R} \cap \Sigma_\mu$, for $\mu \in (0, 1)$, there are no vertical segments, in other words, $\mathcal{R} \cap \Sigma_\mu$ has a "lens" shape which we have already encountered in the proof of the Theorem 7.6.1.

Example 1 – Wave equation.

Let us denote by $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ the independent variables, let

$$K = \{(x, t) \in \mathbb{R}^{n+1} : |x| < |1 - t|\}. \quad (7.6.31)$$

Let us prove that if $u \in C^2(K)$ is a solution to the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, & \text{in } K, \\ u(x, 0) = 0, & \text{for } |x| < 1, \\ \partial_t u(x, 0) = 0, & \text{for } |x| < 1, \end{cases} \quad (7.6.32)$$

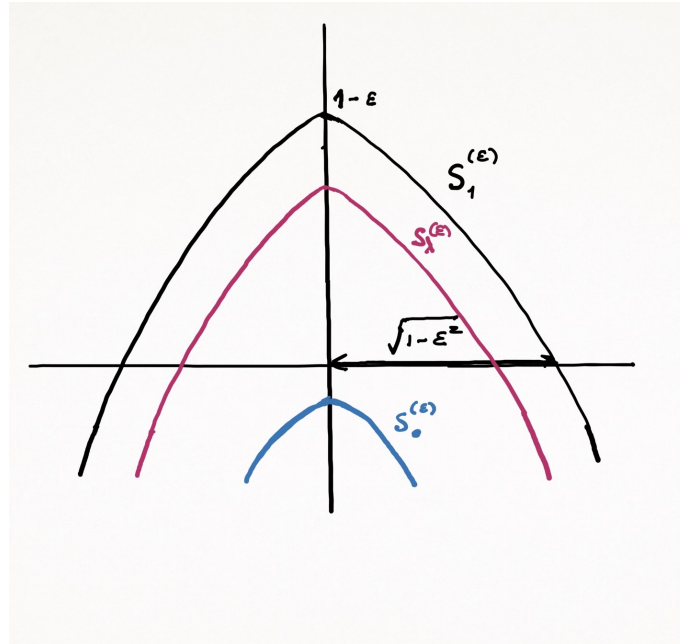


Figure 7.8:

then $u = 0$ in K .

We apply Theorem 7.6.3

In this case we have

$$P(\partial_t, \partial_x) = \partial_t^2 u - \Delta_x.$$

Let $\varepsilon \in (0, 1)$ be fixed and let, for $\lambda \in (0, 1)$ (Figure 7.8)

$$S_\lambda^\varepsilon = \left\{ (x, t) \in \mathbb{R}^{n+1} : t = 1 - \sqrt{(1 - \lambda + \varepsilon)^2 + |x|^2}, \quad |x| < \sqrt{1 - \varepsilon^2} \right\}.$$

Let

$$F : B_{\sqrt{1-\varepsilon^2}}(0) \times (0, 1) \rightarrow \mathbb{R}^{n+1},$$

$$F(y, \lambda) = \left(y, 1 - \sqrt{(1 - \lambda + \varepsilon)^2 + |y|^2} \right).$$

Moreover, set

$$\begin{aligned} \Sigma^\varepsilon &= \bigcup_{\lambda \in (0, 1)} S_\lambda^\varepsilon = \\ &= \left\{ 1 - \sqrt{(1 - \lambda + \varepsilon)^2 + |x|^2} < t < 1 - \sqrt{\varepsilon^2 + |x|^2}, \quad |x| < \sqrt{1 - \varepsilon^2} \right\}, \end{aligned}$$

$$\mathcal{R}^\varepsilon = \left\{ (x, t) \in \mathbb{R}^{n+1} : 0 \leq t < 1 - \sqrt{\varepsilon^2 + |x|^2} \right\},$$

$$\mathcal{Z}^\varepsilon = \left\{ (x, 0) : |x| < \sqrt{1 - \varepsilon^2} \right\}$$

and

$$\Sigma_\mu^\varepsilon = \left\{ (x, t) \in \mathbb{R}^{n+1} : 0 \leq t \leq 1 - \sqrt{(1 - \mu + \varepsilon)^2 + |x|^2} \right\}.$$

For each $0 < \mu < \varepsilon$ we have $\mathcal{R}^\varepsilon \cap \Sigma_\mu^\varepsilon = \emptyset$ and, for each $\varepsilon \leq \mu < 1$, we have

$$\Sigma_\mu^\varepsilon = \mathcal{R}^\varepsilon \cap \Sigma_\mu^\varepsilon.$$

Finally, let us check that \mathcal{Z} and S_λ^ε are noncharacteristic. We have trivially that \mathcal{Z} is a noncharacteristic surface. Concerning S_λ^ε , we notice that

$$S_\lambda^\varepsilon = \left\{ (x, t) \in \mathbb{R}^{n+1} : \phi(x, t) = (1 - \lambda + \varepsilon)^2, \quad t < 1 \right\},$$

where

$$\phi(x, t) = (1 - t)^2 - |x|^2.$$

Therefore we have

$$\nabla_{x,t}\phi(x, t) = (-2x, 2(t - 1)), \quad P(\nabla_{x,t}\phi(x, t)) = 4((1 - t)^2 - |x|^2).$$

Hence, if $(x, t) \in S_\lambda^\varepsilon$, then

$$P(\nabla_{x,t}\phi(x, t)) = 4(1 - \lambda + \varepsilon)^2 > 0.$$

Therefore S_λ^ε is noncharacteristic.

By Theorem 7.6.3 we get

$$u = 0, \quad \text{in } \mathcal{R}^\varepsilon$$

and, since ε is arbitrary, we have

$$u = 0, \quad \text{for } 0 \leq t < 1 - |x|.$$

Similarly we can check that $u = 0$ for $0 \leq |x| - 1 < t \leq 0$. Therefore $u = 0$ in K . ♠

Example 2 – Elliptic equations with analytic coefficients.

Let

$$P(x, \partial)$$

be a **linear elliptic operator** of order m whose coefficients are analytic functions. Let us begin by proving the following unique continuation property

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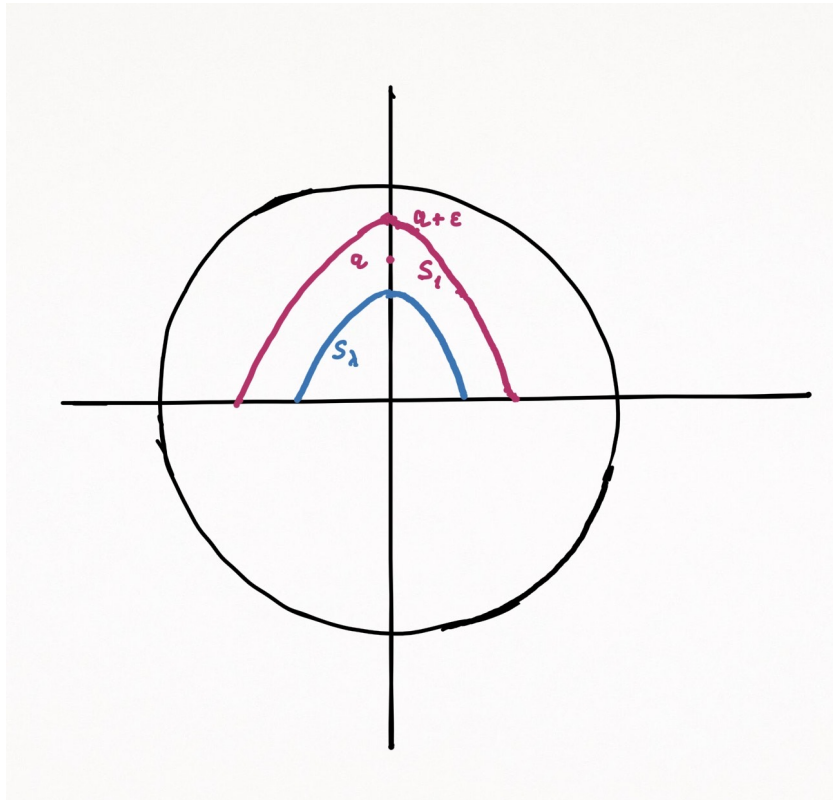


Figure 7.9:

Proposition 7.6.4. *Let ρ, R be such that $0 < \rho < R$. Let $u \in C^m(B_R)$ and let us suppose that*

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_R, \\ u = 0, & \text{in } B_\rho, \end{cases} \quad (7.6.33)$$

then

$$u \equiv 0, \quad \text{in } B_R.$$

Proof. Let $x_0 \in B_R \setminus \overline{B_\rho}$ and let us prove that $u(x_0) = 0$. We may assume that x_0 lies on the x_n -axis, because, by means of a rotation of \mathbb{R}^n , we may always reduce to this case (Figure 7.9). Hence let us suppose that

$$x_0 = ae_n,$$

where $\rho < a < R$.

Trivially we have

$$\partial^\alpha u(x', 0) = 0, \quad |\alpha| \leq m, \quad x' \in B'_\rho. \quad (7.6.34)$$

Let ε be such that $0 < \varepsilon < \min\{R - a, \rho\}$ and let S_λ be defined by

$$x_n = (a + \varepsilon)^2 \left[\lambda - \frac{|x'|^2}{(\rho - \varepsilon)^2} \right], \quad x' \in B'_{\rho - \varepsilon}, \quad 0 < \lambda < 1.$$

Recalling that an elliptic operator has no characteristics, by Theorem 7.6.3 we have $u(ae_n) = 0$. ■

It is evident from the proof of the above Proposition that if $u \in C^m(B_R)$, is a solution to $P(x, \partial)u = 0$ in B_R and it is null in $B_\rho(\tilde{x}) \subset B_R$ we have that $u \equiv 0$ in B_R .

Theorem 7.6.5. *Let Ω be a connected open set of \mathbb{R}^n and let*

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha,$$

be a linear differential operator of order m , elliptic in Ω , where

$$a_\alpha \in C^\omega(\Omega \cup \Gamma),$$

and $\Gamma \subset \partial\Omega$. Let us assume that Γ is a local graph of class C^2 . If $u \in C^m(\Omega \cup \Gamma)$ is a solution to the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } \Omega, \\ \partial^\alpha u = 0, & |\alpha| \leq m - 1 \text{ on } \Gamma, \end{cases} \quad (7.6.35)$$

then

$$u = 0, \quad \text{in } \Omega.$$

Proof. By Theorem 7.6.1 (or 7.6.3) we have there exists an open set (in the topology induced on $\Omega \cup \Gamma$) $\mathcal{U}_+ \subset \Omega$ such that $\partial\mathcal{U}_+ \cap \Gamma \neq \emptyset$ and such that

$$u = 0, \quad \text{in } \mathcal{U}_+. \quad (7.6.36)$$

Set

$$A = \{x \in \Omega : \exists \rho_x > 0 \text{ such that } u = 0, \text{ in } B_{\rho_x}(x)\}. \quad (7.6.37)$$

By (7.6.36) we have $A \neq \emptyset$ and, trivially, we have that A is an open set in Ω . In order to prove the assertion, it suffices to prove that A is also closed in Ω and, since Ω is connected, we have $A = \Omega$ from which we will obtain the thesis. In order to prove that A is closed in Ω it suffices to prove that if $\{x_j\}_{j \in \mathbb{N}}$ is a sequence of A such that

$$\lim_{j \rightarrow \infty} x_j = x_0, \quad (7.6.38)$$

where $x_0 \in \Omega$, then $x_0 \in A$.

Let $\varepsilon > 0$ satisfy $B_\varepsilon(x_0) \subset \Omega$. By (7.6.38), there exists j_0 such that

$$|x_0 - x_{j_0}| < \frac{\varepsilon}{4}.$$

Since $x_{j_0} \in A$ there exists $\rho_{x_{j_0}} > 0$ such that

$$u = 0, \quad \text{in } B_{\rho_{x_{j_0}}}(x_{j_0}) \subset \Omega. \quad (7.6.39)$$

Set $\bar{\rho} = \min\{\frac{\varepsilon}{4}, \rho_{x_{j_0}}\}$, we have $B_{\bar{\rho}}(x_{j_0}) \subset B_\varepsilon(x_0)$ and, by (7.6.39), we have $u = 0$ in $B_{\bar{\rho}}(x_{j_0})$, then, by Proposition 7.6.4 and subsequent remarks, we have $u = 0$ in $B_\varepsilon(x_0)$. Therefore $x_0 \in A$ and the thesis follows. ■

Remarks on Theorem 7.6.5.

1. At this point we report, for information only, that if $P(x, \partial)$ is an elliptic operator in Ω , $f \in C^\omega(\Omega)$ and $u \in C^m(\Omega)$ is a solution of the equation $P(x, \partial)u = f$ in Ω , then $u \in C^\omega(\Omega)$, see [57] for a proof. It is evident that if we had used this regularity property of the solutions, Theorem 7.6.5 would be a consequence of the unique continuation property for analytic functions (and the Holmgren Theorem). However, to prove Theorem 7.6.5 we *did not* need the above regularity result.

2. Let us consider some interesting consequences of Theorem 7.6.5. Let Ω , Γ and $P(x, \partial)$ be as in Theorem 7.6.5, further let us suppose that Γ is of class C^m . We denote by \mathcal{X}_Γ the class of functions

$$g : \Gamma \rightarrow \mathbb{R}^m, \quad g(x) = (g_0(x), g_1(x), \dots, g_{m-1}(x)), \quad \forall x \in \Gamma$$

such that there exists $u \in C^m(\Omega \cup \Gamma)$ solution to the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } \Omega, \\ \frac{\partial^j u}{\partial \nu^j} = g_j, & j = 0, 1, \dots, m-1 \text{ on } \Gamma. \end{cases} \quad (7.6.40)$$

The class \mathcal{X}_Γ enjoys the following property.

For every $\Gamma_0 \subset \Gamma$, Γ_0 open in Γ in the induced topology, it occurs that

$$g \in \mathcal{X}_\Gamma, \text{ and } g = 0 \text{ on } \Gamma_0 \implies g = 0 \text{ on } \Gamma \quad (7.6.41)$$

and, therefore, by linearity, if $g, \tilde{g} \in \mathcal{X}_\Gamma$ and $g = \tilde{g}$ on Γ_0 then $g = \tilde{g}$ on Γ . In other words, the class \mathcal{X}_Γ must enjoy the unique continuation property (7.6.41). By this same fact we deduce that if the initial data of a Cauchy problem for $P(x, \partial)$ belong to the class of functions $C^k(\Gamma, \mathbb{R}^m)$, for any $k \geq m$, such a Cauchy problem cannot, in general, admit solutions. As a matter of fact, the class $C^k(\Gamma, \mathbb{R}^m)$ does not enjoy property (7.6.41).

The proof of the assertion above is very simple. As a matter of fact, let $g \in \mathcal{X}_\Gamma$ and let $u \in C^k(\Omega \cup \Gamma)$, $k \geq m$ be a solution of problem (7.6.41). Let us suppose

$$g = 0, \quad \text{on } \Gamma_0.$$

Then applying Theorem 7.6.5 (with Γ_0 in the place of Γ) we have that $u = 0$ in Ω and being $u \in C^k(\Omega \cup \Gamma)$, we have $g = u|_\Gamma = 0$. \blacklozenge

Example 3 – One-dimensional heat equation.

We consider the following Cauchy problem

$$\begin{cases} \partial_x^2 u - \partial_t u = 0, & \text{in } D := (0, 1) \times (0, 1), \\ u(0, t) = 0, & \text{for } t \in (a, b), \\ \partial_x u(0, t) = 0, & \text{for } t \in (a, b), \end{cases} \quad (7.6.42)$$

where a, b are given numbers and such that $0 < a < b < 1$. Let us notice that problem (7.6.42) is a **noncharacteristic Cauchy problem**, since the initial line is $\{x = 0\}$.

By using the same arguments exploited in **Example 1** and in **Example 2** (Proposition 7.6.4) it can be proved easily that if $u \in C^2(\overline{D})$ then $u = 0$ in $[0, 1] \times [a, b]$. The details are left to the reader (it is useful to keep in mind that the only characteristics of the operator $\partial_x^2 - \partial_t$ are the straight lines $t = t_0$, for any $t_0 \in \mathbb{R}$).

It is quite natural to wonder whether a solution of (7.6.42) is null in D . The answer to this question is negative as proved in an example due to **Tychonoff**, [74], which we will discuss below.

First, let us consider the following Cauchy problem.

$$\begin{cases} \partial_x^2 u - \partial_t u = 0, \\ u(0, t) = \varphi(t), \quad \text{for } t \in \mathbb{R}, \\ \partial_x u(0, t) = 0, \quad \text{for } t \in \mathbb{R}, \end{cases} \quad (7.6.43)$$

For the time being, let us just assume that $\varphi \in C^\infty(\mathbb{R})$ and let us search, at first just formally, a solution of the type

$$u(x, t) = \sum_{j=0}^{\infty} a_j(t) x^j, \quad (7.6.44)$$

where a_j are functions to be found.

Obviously we have to require that

$$a_0(t) = \varphi(t), \quad \text{and} \quad a_1(t) = 0$$

and, by requiring that (7.6.44) is a solution to the equation $\partial_x^2 u - \partial_t u = 0$, we need to require

$$\sum_{j=0}^{\infty} a'_j(t) x^j - \sum_{j=2}^{\infty} j(j-1) a_j(t) x^{j-2} = 0,$$

from which we have

$$a_{j+2}(t) = \frac{1}{(j+2)(j+1)} a'_j(t), \quad \forall j \in \mathbb{N}_0.$$

Hence

$$a_{2k}(t) = \frac{\varphi^{(k)}(t)}{(2k)!}, \quad a_{2k+1}(t) = 0 \quad \forall k \in \mathbb{N}_0.$$

Therefore

$$u(x, t) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(t)}{(2k)!} x^{2k}. \quad (7.6.45)$$

In order that (7.6.45) is actually a solution to Cauchy problem (7.6.43) (in a neighborhood of $\{0\} \times \mathbb{R}$) it suffices to require that there exist two positive numbers c and M such that

$$|\varphi^{(k)}(t)| \leq cM^k(2k!), \quad \forall k \in \mathbb{N}_0. \quad (7.6.46)$$

Let now

$$\varphi(t) = \begin{cases} e^{-\frac{1}{t^2}}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases} \quad (7.6.47)$$

We show now that (7.6.46) is satisfied. We will prove, indeed,

$$|\varphi^{(k)}(t)| \leq \left(\frac{9k}{2e}\right)^{\frac{k}{2}} k!, \quad \forall k \in \mathbb{N}_0. \quad (7.6.48)$$

which (by the Stirling formula, (1.2.5)) implies (7.6.46).

To prove (7.6.48) we use the Cauchy formula for the holomorphic functions. Let therefore $t > 0$ and let S be the circumference centered at $t + i0$ and with radius $\frac{t}{2}$ in the complex plane, i.e.

$$S = \left\{ t \left(1 + \frac{1}{2} e^{i\vartheta} \right) : \vartheta \in [0, 2\pi) \right\}.$$

We have, for any $k \in \mathbb{N}_0$,

$$\varphi^{(k)}(t) = \frac{k!}{2\pi i} \int_S \frac{e^{-\frac{1}{z^2}}}{(z-t)^{k+1}} dz.$$

Now, when $z = t(1 + \frac{1}{2}e^{i\vartheta}) \in S$, it is easily checked that

$$\frac{1}{z} = \frac{4}{3t} + \frac{2}{3t} e^{i\vartheta}.$$

Therefore

$$\begin{aligned} \Re \left(\frac{1}{z^2} \right) &= \left(\frac{4}{3t} \right)^2 \left[\left(1 + \frac{1}{2} \cos \vartheta \right)^2 - \left(\frac{1}{2} \sin \vartheta \right)^2 \right] = \\ &= \left(\frac{4}{3t} \right)^2 \left[\frac{1}{4} + \frac{1}{2} (1 + \cos \vartheta)^2 \right] \geq \frac{4}{9t^2}. \end{aligned}$$

Therefore we have

$$\begin{aligned}
|\varphi^{(k)}(t)| &\leq \frac{k!}{2\pi} \int_{|z-t|=\frac{t}{2}} \left| \frac{e^{-\frac{1}{z^2}}}{(z-t)^{k+1}} \right| ds \leq \\
&\leq \frac{k!}{2\pi} \int_{|z-t|=\frac{t}{2}} \frac{e^{-\frac{4}{9t^2}}}{|z-t|^{k+1}} ds = \\
&= \frac{k!}{2\pi} \left(2\pi \frac{t}{2} \right) \frac{1}{(t/2)^{k+1}} e^{-\frac{4}{9t^2}} = \\
&= \frac{2^k k!}{t^k} e^{-\frac{4}{9t^2}}.
\end{aligned} \tag{7.6.49}$$

Since

$$\sup \frac{1}{t^k} e^{-\frac{4}{9t^2}} = \left(\frac{9k}{8e} \right)^{k/2},$$

by (7.6.49) we get (7.6.48) which in turn implies that the series in (7.6.45) converges for every $x \in \mathbb{R}$ and its sum, u , is indeed the solution to Cauchy problem (7.6.43).

From what we have just established, it turns out that, denoting by ψ the following function

$$\psi(t) = \begin{cases} e^{-\frac{1}{(t-b)^2}}, & \text{for } t > b, \\ 0, & \text{per } a \leq t \leq b, \\ e^{-\frac{1}{(a-t)^2}}, & \text{for } t < a, \end{cases}$$

we have that

$$\tilde{u} = \sum_{k=0}^{\infty} \frac{\psi^{(k)}(t)}{(2k)!} x^{2k},$$

is the solution to the Cauchy problem

$$\begin{cases} \partial_x^2 \tilde{u} - \partial_t \tilde{u} = 0, & \text{in } \mathbb{R}^2, \\ \tilde{u}(0, t) = \psi(t), & \text{for } t \in \mathbb{R}, \\ \partial_x \tilde{u}(0, t) = 0, & \text{for } t \in \mathbb{R} \end{cases}$$

and $\tilde{u} = 0$ in $\mathbb{R} \times [a, b]$, but if $t_0 \notin [a, b]$ then $\tilde{u}(\cdot, t_0)$ does not identically vanish, more precisely $\tilde{u}(\cdot, t_0)$ does not vanish in any open set of \mathbb{R} because,

as $\tilde{u}(\cdot, t_0)$ is analytic (as it is the sum of a series of powers in the variable x) one would have that $\psi^{(k)}(t_0) = 0$ for every $k \in \mathbb{N}_0$ that is false.

We conclude this discussion about the heat equation by observing that does not hold the uniqueness in $C^2(\mathbb{R} \times [0, +\infty))$ to the following Cauchy problem **characteristic**

$$\begin{cases} \partial_x^2 u - \partial_t u = 0, & \text{in } \mathbb{R}^2, \\ u(x, 0) = 0, & \text{for } x \in \mathbb{R}. \end{cases} \quad (7.6.50)$$

It suffices to consider the function u defined by (7.6.45) where φ defined by (7.6.47) and we have that u is the solution of problem (7.6.50), but it does not vanish identically. ♠

Exersice. Let $T, r > 0$ and denote

$$K = \{(x, t) \in \mathbb{R}^{n+1} : |x| + |t| < T + r\}.$$

Prove that if $u \in C^2(K)$ satisfies

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, & \text{in } K \\ u = 0, & \text{in } B_r \times (-T, T), \end{cases}$$

then

$$u = 0, \quad \text{in } K.$$

♣

Chapter 8

Uniqueness for an inverse problem

8.1 Introduction

In this short chapter we present an inverse problem for the Laplace equation. The direct problem is nothing but the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \quad (8.1.1)$$

For simplicity, we assume that Ω is a bounded and connected open of \mathbb{R}^n whose boundary is of class C^∞ and $\varphi \in C^\infty(\partial\Omega)$. In Chapter 4 we saw (see in particular, Corollary 4.6.7) that, under these assumptions, there exists a unique $u \in C^\infty(\overline{\Omega})$, which is the solution to (8.1.1).

Now, let us suppose that a portion of $\partial\Omega$, which we will call $\Gamma^{(i)}$, is unknown and that we have

$$\varphi(x) = 0, \quad \forall x \in \Gamma^{(i)} \quad (8.1.2)$$

and let us suppose that we know

$$\frac{\partial u}{\partial \nu} = \psi, \quad \text{su } \Sigma, \quad (8.1.3)$$

where $\Sigma \subset \partial\Omega \setminus \Gamma^{(i)}$. We are interested in determining $\Gamma^{(i)}$. This is our **inverse problem**. Let us note that if we consider all the data of the problem we should write

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \psi, & \text{on } \Sigma, \end{cases} \quad (8.1.4)$$

which evidently is an overdetermined problem from which there is to be expected a compatibility relation between φ, ψ and $\partial\Omega$ so that, if we assume to know φ, ψ and $\partial\Omega \setminus \Gamma^{(i)}$, we can reasonably hope to obtain some information about $\Gamma^{(i)}$ itself. It is evident that if $\varphi \equiv 0$ on $\partial\Omega$ then we have no information on $\Gamma^{(i)}$. We will see that this trivial case is (under precise assumptions) the only case in which $\Gamma^{(i)}$ is not uniquely determined.

Instead of considering the inverse problem as an overdetermined problem, it turns out to be more efficient to consider the inverse problem from the point of view which we now illustrate. Since the direct problem has a unique solution $u \in C^\infty(\overline{\Omega})$ for every bounded open set of class C^∞ , and for any $\varphi \in C^\infty(\partial\Omega)$ satisfying (8.1.2), it turns out that the derivative

$$\frac{\partial u}{\partial \nu}|_\Sigma$$

is a function of $\Gamma^{(i)}$. Let us let us denote such a function by

$$\mathbf{U}(\Gamma^{(i)}),$$

the inverse problem we are interested in may be formulated as follows

Determine the solution to the equation $\mathbf{U}(\Gamma^{(i)}) = \psi$.

In the next Section we will specify the assumptions and we will formulate the uniqueness theorem for the inverse problem above.

8.2 Statement of the uniqueness theorem for the inverse problem

Let us suppose that Ω is as above and let us suppose that $\partial\Omega$ is the union of two internally disjoint portions, $\Gamma^{(a)}$ ("accessible" portion) and $\Gamma^{(i)}$ ("inaccessible" portion). More precisely, let us suppose that:

$$\partial\Omega = \Gamma^{(a)} \cup \Gamma^{(i)} \quad (8.2.1)$$

$$\Gamma^{(a)} \text{ and } \Gamma^{(i)} \text{ closed in } \partial\Omega \quad \text{and} \quad \overline{\Gamma^{(a)}} = \Gamma^{(a)}, \quad \overline{\Gamma^{(i)}} = \Gamma^{(i)}, \quad (8.2.2)$$

we equip $\partial\Omega$ with the topology induced by the Euclidean topology of \mathbb{R}^n ,

$$\Gamma^{(a)\circ} \cap \Gamma^{(i)\circ} = \emptyset. \quad (8.2.3)$$

Hence

$$\Gamma^{(i)} = \partial\Omega \setminus \Gamma^{(a)\circ} \quad \text{and} \quad \Gamma^{(a)} = \partial\Omega \setminus \Gamma^{(i)\circ} \quad (8.2.4)$$

and

$$\Gamma^{(a)\circ} \quad \text{is connected}, \quad (8.2.5)$$

therefore, (8.2.2) implies $\Gamma^{(a)}$ is connected.

Let Σ be a compact subset of $\partial\Omega$ such that

$$\Sigma \Subset \Gamma^{(a)\circ}, \quad \Sigma^\circ \neq \emptyset. \quad (8.2.6)$$

Let $\varphi \in C^\infty(\partial\Omega)$ satisfy

$$\text{supp } \varphi \subset \Gamma^{(a)\circ}. \quad (8.2.7)$$

We now state the Theorem

Theorem 8.2.1 (uniqueness). *Let Ω_k , $k = 1, 2$, be two bounded connected open sets of \mathbb{R}^n whose boundary is of class C^∞ . Let us assume that*

$$\partial\Omega_1 = \Gamma^{(a)} \cup \Gamma_1^{(i)} \quad \text{and} \quad \partial\Omega_2 = \Gamma^{(a)} \cup \Gamma_2^{(i)}, \quad (8.2.8)$$

where $\Gamma^{(a)}, \Gamma_k^{(i)}$, $k = 1, 2$, satisfies (8.2.2), (8.2.3) and (8.2.5). Let us assume that $\varphi_k \in C^\infty(\partial\Omega_k)$ do not vanish identically and satisfying (8.2.7). Moreover, let us assume

$$\varphi_1 = \varphi_2, \quad \text{on } \Gamma^{(a)}. \quad (8.2.9)$$

Let $u_k \in C^\infty(\overline{\Omega_k})$, $k = 1, 2$, be the solutions to

$$\begin{cases} \Delta u_k = 0, & \text{in } \Omega_k, \\ u_k = \varphi, & \text{on } \partial\Omega_k. \end{cases} \quad (8.2.10)$$

Let us assume that

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}, \quad \text{on } \Sigma, \quad (8.2.11)$$

where Σ is a compact subset of $\partial\Omega$ which satisfies (8.2.6).

Then we have

$$\Omega_1 = \Omega_2, \quad (8.2.12)$$

and, consequently,

$$\Gamma_1^{(i)} = \Gamma_2^{(i)}.$$

The proof will be given in the next Section.

8.3 Proof of the uniqueness

The idea of the proof of uniqueness Theorem is quite simple, but it requires propositions of general topology that we will prove separately for the purpose of not breaking the main argument of the proof.

We argue by contradiction. We assume that one of the two sets $\Omega_1 \setminus \overline{\Omega}_2$, $\Omega_2 \setminus \overline{\Omega}_1$, is not empty. For instance, let us assume

$$\Omega_1 \setminus \overline{\Omega}_2 \neq \emptyset. \quad (8.3.1)$$

Define G as

$$G = \bigcup_{A \in \mathcal{A}} A, \quad (8.3.2)$$

where

$$\mathcal{A} = \{A \text{ open set of } \mathbb{R}^n : A \subset \Omega_1 \cap \Omega_2, \Gamma^{(a)} \subset \overline{A}, A \text{ connected}\}. \quad (8.3.3)$$

As we will prove in Proposition 8.3.1 (and as can be expected), G is a connected open set, moreover $\Gamma^{(a)} \subset \overline{G}$ and $G \subset \Omega_1 \cap \Omega_2$. From the latter and from (8.3.1) we have

$$\Omega_1 \setminus \overline{G} \neq \emptyset. \quad (8.3.4)$$

Set

$$u := u_1 - u_2, \quad \text{in } G.$$

We have

$$\begin{cases} \Delta u = 0, & \text{in } G, \\ u = 0, & \text{on } \Sigma, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma. \end{cases}$$

Theorem 7.6.5 implies

$$u \equiv 0, \quad \text{in } G \quad (8.3.5)$$

Claim.

$$u_1 = 0, \quad \text{su } \partial(\Omega_1 \setminus \overline{G}). \quad (8.3.6)$$

Proof of Claim. We exploit the following relationship (proved in Proposition 8.3.3)

$$\partial(\Omega_1 \setminus \overline{G}) \subset \Gamma_1^{(i)} \cup (\Gamma_2^{(i)} \cap \partial G). \quad (8.3.7)$$

Let $x \in \partial(\Omega_1 \setminus \overline{G})$, then by (8.3.7) we distinguish two cases:

(a) $x \in \Gamma_1^{(i)}$,

(b) $x \in \Gamma_2^{(i)} \cap \partial G$.

In case (a) $u_1(x) = 0$ because $u = \varphi_1$ on $\partial\Omega_1$ and $\varphi_1 = 0$ on $\Gamma_1^{(i)}$.

In case (b), we have $u_2(x) = 0$ because $u = \varphi_2$ on $\partial\Omega_2$ and $\varphi_2 = 0$ on $\Gamma_2^{(i)}$.

On the other hand, by (8.3.5) and taking into account that u is continuous in \overline{G} , we have

$$u_1(x) = u_1(x) - u_2(x) = u(x) = 0.$$

The Claim is proved.

Therefore u_1 solves the Dirichlet problem

$$\begin{cases} \Delta u_1 = 0, & \text{in } \Omega_1 \setminus \overline{G}, \\ u_1 = 0, & \text{on } \partial(\Omega_1 \setminus \overline{G}), \end{cases}$$

and the maximum principle implies

$$u_1 = 0, \quad \text{in } \Omega_1 \setminus \overline{G}.$$

Now, taking into account (8.3.4), the unique continuation property gives

$$u_1 = 0, \quad \text{in } \Omega_1.$$

From which we have

$$\varphi_1 = 0, \quad \text{in } \partial\Omega_1,$$

But this contradicts the assumption that φ_1 does not vanish identically. Therefore $\Omega_1 \setminus \overline{\Omega}_2 \neq \emptyset$. Hence $\Omega_1 \subset \overline{\Omega}_2$. Similarly we have $\Omega_2 \subset \overline{\Omega}_1$. Therefore, $\overline{\Omega}_1 = \overline{\Omega}_2$. On the other hand (see Exercise of Section 2.7)

$$\Omega_1 = \overset{\circ}{\overline{\Omega}}_1 = \overset{\circ}{\overline{\Omega}}_2 = \Omega_2.$$

■

Proposition 8.3.1. *G , defined by (8.3.2) is an open nonempty set and it enjoys the following properties*

- (a) $G \subset \Omega_1 \cap \Omega_2$,
- (b) $\Gamma^{(a)} \subset \overline{G}$,
- (c) G is connected.

Proof. Since Ω_k , $k = 1, 2$, are of class C^∞ , there exist r_0, M_0 , positive numbers, such that Ω_k , $k = 1, 2$, are of class $C^{1,1}$ with constant r_0, M_0 . Proposition 2.11.8 implies that there exists $\mu_1 > 0$ such that, the following map is continuous

$$\Phi : \partial\Omega \times (0, \mu_1 r_0) \rightarrow \mathbb{R}^n,$$

$$\Phi(y, t) = y - t\nu(y), \quad \forall (y, t) \in \partial\Omega \times (0, \mu_1 r_0).$$

For any $\delta \in (0, \mu_1 r_0)$ we denote

$$\Lambda_\delta = \Phi \left(\overset{\circ}{\Gamma}^{(a)} \times (0, \delta) \right).$$

Let us check

$$\Lambda_\delta \in \mathcal{A}, \tag{8.3.8}$$

where \mathcal{A} is defined in (8.3.3). It evident that $\Lambda_\delta \neq \emptyset$. Moreover, by the continuity of Φ^{-1} (see (c) of Proposition 2.11.8), Λ_δ is an open set (see (a) of Proposition 2.11.8)

$$\Lambda_\delta \subset \Omega_1, \quad \text{and} \quad \Lambda_\delta \subset \Omega_2.$$

Also, Λ_δ is connected, as the image of the connected set $\overset{\circ}{\Gamma}^{(a)} \times (0, \delta)$ by means of the continuous map Φ . To complete the proof of (8.3.8), it suffices to check

$$\overset{\circ}{\Gamma}^{(a)} \subset \overline{\Lambda}_\delta. \tag{8.3.9}$$

Let $x \in \Gamma^{(a)}$ and let $r > 0$ arbitrary. Since, by (8.2.2),

$$\overline{\Gamma^{(a)}} = \Gamma^{(a)},$$

we have $B_r(x) \cap \overset{\circ}{\Gamma^{(a)}} \neq \emptyset$. Let $y \in B_r(x) \cap \overset{\circ}{\Gamma^{(a)}}$, if t is a positive number small enough, we have

$$y - t\nu(y) \in B_r(x) \cap \Lambda_\delta.$$

Therefore

$$B_r(x) \cap \Lambda_\delta \neq \emptyset, \quad \forall r > 0.$$

Hence $x \in \overline{\Lambda}_\delta$ and (8.3.9) is proved. Now, let us notice that

$$\Lambda_\delta \cap A \neq \emptyset, \quad \forall A \in \mathcal{A}. \quad (8.3.10)$$

Let us fix $A \in \mathcal{A}$ and let $x \in \overset{\circ}{\Gamma^{(a)}}$. Since x is an interior point of $\Gamma^{(a)}$, $\Gamma^{(a)} \subset \partial\Omega_1 \cap \partial\Omega_2$ and since Ω_1 and Ω_2 are of class $C^{1,1}$, there exists $\bar{r} > 0$ such that

$$B_{\bar{r}}(x) \cap \Omega_1 = B_{\bar{r}}(x) \cap \Omega_2 \subset \Lambda_\delta.$$

Hence

$$B_{\bar{r}}(x) \cap A \subset \Lambda_\delta.$$

On the other hand, since $\overset{\circ}{\Gamma^{(a)}} \subset \overline{A}$, we have $x \in \overline{A}$. Hence

$$\emptyset \neq B_{\bar{r}}(x) \cap A \subset \Lambda_\delta \cap A,$$

which implies (8.3.10).

Now, since

$$G = \bigcup_{A \in \mathcal{A}} A \quad (8.3.11)$$

and since $\Lambda_\delta \in \mathcal{A}$, we have that $G \neq \emptyset$ and (trivially)

$$G \subset \Omega_1 \cap \Omega_2, \quad \Gamma^{(a)} \subset \overline{\Lambda}_\delta \subset \overline{G}.$$

Hence (a) and (b) are proved. It remains to prove (c). Let $x, y \in G$ and let $A, B \in \mathcal{A}$ satisfy $x \in A$ e $y \in B$. By (8.3.10), we have $\Lambda_\delta \cap A \neq \emptyset$ and $\Lambda_\delta \cap B \neq \emptyset$. Let

$$z \in \Lambda_\delta \cap A \quad \text{and} \quad w \in \Lambda_\delta \cap B$$

and let γ_1 be a continuous path that joins x and z in A , γ_2 be a continuous path that joins z and w in Λ_δ and γ_3 be a continuous path that joins w and y in B . Set

$$\gamma = \gamma_1 \vee \gamma_2 \vee \gamma_3.$$

γ is a continuous path that joins x and y in $A \cup \Lambda_\delta \cup B$. Hence $A \cup \Lambda_\delta \cup B \in \mathcal{A}$ and we have $A \cup \Lambda_\delta \cup B \subset G$. Consequently γ joins x and y in G . All in all, G is connected. ■

We have

Proposition 8.3.2. *Let C a nonempty set of \mathbb{R}^n and A an open set of \mathbb{R}^n . then*

$$A \cap \overline{C} \neq \emptyset \iff A \cap C \neq \emptyset \quad (8.3.12)$$

Proof. The implication " \Leftarrow " is trivial. Concerning the implication " \Rightarrow ", let $z \in A \cap \overline{C}$ and $B_r(z) \subset A$. Since $z \in \overline{C}$, we have $\emptyset \subsetneq B_r(z) \cap C \subset A \cap C$. Therefore $A \cap C \neq \emptyset$. ■

Proposition 8.3.3. *Let G defined by (8.3.2). Let us suppose*

$$\Omega_1 \setminus \overline{\Omega}_2 \neq \emptyset. \quad (8.3.13)$$

Then we have

$$\partial(\Omega_1 \setminus \overline{G}) \subset \Gamma_1^{(i)} \cup (\Gamma_2^{(i)} \cap \partial G). \quad (8.3.14)$$

Proof.

Step I. First, let us notice that, since $\Omega_1 \setminus G \supset \Omega_1 \setminus (\overline{\Omega_1 \cap \Omega_2}) \not\supseteq \emptyset$ and $\Omega_1 \setminus \overline{G} \neq \mathbb{R}^n$, we have $\partial(\Omega_1 \setminus \overline{G}) \neq \emptyset$.

Now we prove that

$$\partial(\Omega_1 \setminus \overline{G}) \subset \partial\Omega_1 \cup \partial\Omega_2. \quad (8.3.15)$$

We argue by contradiction. Let $x_0 \in \partial(\Omega_1 \setminus \overline{G})$ and let us suppose that

$$x_0 \notin \partial\Omega_1 \cup \partial\Omega_2. \quad (8.3.16)$$

It cannot be the case that $x_0 \notin \overline{\Omega}_1$ because if it were, it would exist $\rho > 0$ such that $B_\rho(x_0) \cap (\Omega_1 \setminus \overline{G}) \subset B_\rho(x_0) \cap \Omega_1 = \emptyset$, that contradicts $\partial(\Omega_1 \setminus \overline{G}) \neq \emptyset$. Let now examine the following two cases:

- (a) $x_0 \in \Omega_1 \cap \Omega_2$
- (b) $x_0 \in \Omega_1 \cap (\mathbb{R}^n \setminus \overline{\Omega}_2)$.

Case (a). Let $\rho > 0$ such that

$$B_\rho(x_0) \subset \Omega_1 \cap \Omega_2. \quad (8.3.17)$$

On the other hand, $x_0 \in \partial(\Omega_1 \setminus \overline{G})$, hence

$$\begin{cases} B_\rho(x_0) \cap (\Omega_1 \setminus \overline{G}) \neq \emptyset, \\ B_\rho(x_0) \cap (\mathbb{R}^n \setminus (\Omega_1 \setminus \overline{G})) \neq \emptyset. \end{cases} \quad (8.3.18)$$

By (8.3.17) and by the second of (8.3.18), we have

$$\begin{aligned} \emptyset \subsetneq B_\rho(x_0) \cap (\mathbb{R}^n \setminus (\Omega_1 \setminus \overline{G})) &= B_\rho(x_0) \cap [(\mathbb{R}^n \setminus \Omega_1) \cup (\overline{G} \cap \Omega_1)] = \\ &= B_\rho(x_0) \cap (\overline{G} \cap \Omega_1) \subset \\ &\subset B_\rho(x_0) \cap \overline{G}. \end{aligned}$$

All in all, $B_\rho(x_0) \cap \overline{G} \neq \emptyset$ and, by Proposition 8.3.2 we get

$$B_\rho(x_0) \cap G \neq \emptyset. \quad (8.3.19)$$

Moreover, by the first relationship of (8.3.18), we obtain

$$B_\rho(x_0) \cap (\mathbb{R}^n \setminus \overline{G}) \supset B_\rho(x_0) \cap (\Omega_1 \setminus \overline{G}) \supsetneq \emptyset. \quad (8.3.20)$$

Therefore (recalling (8.3.17))

$$G \subsetneq B_\rho(x_0) \cup G \subset \Omega_1 \cap \Omega_2. \quad (8.3.21)$$

Claim. $B_\rho(x_0) \cup G$ is a connected set.

Proof of the Claim. Let $x, y \in B_\rho(x_0) \cup G$. Let us prove that there exists a continuous path γ that joins x and y in $B_\rho(x_0) \cup G$. If $x, y \in G$ we have nothing to prove because (by Proposition 8.3.1) G is connected. If $x, y \in B_\rho(x_0)$, of course we have nothing to prove. Hence, let us suppose that $x \in G$ and $y \in B_\rho(x_0)$. By (8.3.19) there exists $z \in B_\rho(x_0) \cap G$. Let γ_1 be a continuous path that joins x and z in G and γ_2 be a continuous path that joins z and y in $B_\rho(x_0)$. Let

$$\gamma = \gamma_1 \vee \gamma_2.$$

γ is a continuous path that joins x and y in $B_\rho(x_0) \cup G$. Claim is proved.

Now, from (b) of Proposition 8.3.1 we have $\overline{B_\rho(x_0) \cup G} \supset \Gamma^{(a)}$ and recalling the definition of G we would have

$$G = B_\rho(x_0) \cup G,$$

which contradicts (8.3.21).

Case (b). Let $\bar{\rho} > 0$ satisfy

$$B_{\bar{\rho}}(x_0) \subset \Omega_1 \cap (\mathbb{R}^n \setminus \bar{\Omega}_2). \quad (8.3.22)$$

Since $x_0 \in \partial(\Omega_1 \setminus G)$, we again obtain (8.3.19). Therefore

$$\emptyset \subsetneq B_{\bar{\rho}}(x_0) \cap G \subset \Omega_1 \cap \Omega_2 \subset \Omega_2,$$

hence $B_{\bar{\rho}}(x_0) \cap \Omega_2 \neq \emptyset$, that contradicts (8.3.21). Hence (8.3.22) is proved.

Step II. Now we prove

$$\partial(\Omega_1 \setminus G) \subset \Gamma_1^{(i)} \cup \Gamma_2^{(i)}. \quad (8.3.23)$$

To this aim, let us prove

$$\Gamma^{(a)} \cap \partial(\Omega_1 \setminus \bar{G}) = \emptyset. \quad (8.3.24)$$

Let us suppose that (8.3.24) does not hold. Hence, there exists \bar{x} such that

$$\bar{x} \in \Gamma^{(a)} \cap \partial(\Omega_1 \setminus \bar{G}). \quad (8.3.25)$$

Since $\bar{x} \in \Gamma^{(a)}$ and $\Gamma^{(a)} \subset (\partial\Omega_1) \cap (\partial\Omega_2)$, taking into account that $\partial\Omega_k$, $k = 1, 2$, are of class $C^{1,1}$, there exist \bar{r}, \bar{M} positive numbers, such that

- (i) $Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \partial\Omega_1 = Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \partial\Omega_2 \subset \Gamma^{(a)}$,
- (ii) $Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1 = Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_2$ and they are connected.

Now we prove that

$$G \subsetneq G \cup (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \subset \Omega_1 \cap \Omega_2 \quad (8.3.26)$$

and

$$G \cup (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \quad \text{is connected.} \quad (8.3.27)$$

From these we will arrive to a contradiction.

First, by $G \subset \Omega_1 \cap \Omega_2$ we have

$$G \cap (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \subset \Omega_1 \cap \Omega_2. \quad (8.3.28)$$

On the other hand, since, by (8.3.25) and $\bar{x} \in \partial(\Omega_1 \setminus \bar{G})$, we obtain

$$\emptyset \subsetneq Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap (\Omega_1 \setminus \bar{G}) \subset (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \setminus G.$$

Hence

$$G \subsetneq G \cup (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1)$$

and (8.3.26) is proved.

In order to prove that the set $G \cup (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1)$ is connected, we first of all check that

$$(Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cap G \neq \emptyset, \quad (8.3.29)$$

As a matter of fact, by $\bar{x} \in \partial(\Omega_1 \setminus \bar{G})$ we have

$$\begin{aligned} \emptyset &\subsetneq (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cap [\mathbb{R}^n \setminus (\Omega_1 \setminus \bar{G})] = \\ &= (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cap [(\mathbb{R}^n \setminus \Omega_1) \cup (\Omega_1 \cap \bar{G})] = \\ &= (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cap \bar{G}. \end{aligned}$$

Hence

$$(Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cap \bar{G} \neq \emptyset$$

and by Proposition 8.3.2 we get (8.3.29).

At this point, in order to prove that $(Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1) \cup G$ is connected, we need only recall (ii) and to proceed as in the proof of the Claim in Step I. Therefore (8.3.27) is proved. Now, since

$$\overline{G \cup (Q_{\bar{r}, 2\bar{M}}(\bar{x}) \cap \Omega_1)} \supset \Gamma^{(a)},$$

by (8.3.26), (8.3.27) and by the definition of G , we arrive to a contradiction. Hence (8.3.24) holds, which combined with (8.3.15) implies

$$\begin{aligned} \partial(\Omega_1 \setminus G) &\subset (\partial\Omega_1 \cup \partial\Omega_2) \setminus \Gamma^{(a)} = \\ &= \left(\partial\Omega_1 \setminus \Gamma^{(a)} \right) \cup \left(\partial\Omega_2 \setminus \Gamma^{(a)} \right) = \\ &= \Gamma_1^{(i)} \cup \Gamma_2^{(i)}, \end{aligned}$$

which gives (8.3.23).

Step III. We conclude the proof of (8.3.14). Let

$$x \in \partial(\Omega_1 \setminus \bar{G}).$$

Let us distinguish two cases:

- (j) $x \in \mathbb{R}^n \setminus \bar{\Omega}_1$,
- (jj) $x \in \bar{\Omega}_1$.

Case (j) cannot occur, because if $x \in \mathbb{R}^n \setminus \overline{\Omega}_1$ then there exists $r > 0$ such that $B_r(x) \subset \mathbb{R}^n \setminus \overline{\Omega}_1$, hence

$$B_r(x) \cap (\Omega_1 \setminus \overline{G}) \subset B_r(x) \cap \Omega_1 = \emptyset,$$

but this cannot hold because $x \in \partial(\Omega_1 \setminus \overline{G})$.

Let us consider case (jj). If $x \in \Gamma_1^{(i)}$ then trivially $x \in \Gamma_1^{(i)} \cup (\Gamma_2^{(i)} \cap \partial G)$. Instead, if $x \notin \Gamma_1^{(i)}$ then, by (8.3.23), we have

$$x \in \Gamma_2^{(i)}. \quad (8.3.30)$$

Moreover, since, by (8.3.24), we have $x \notin \overset{\circ}{\Gamma}^{(a)}$ and since, by (8.2.4), we have $\partial\Omega_1 = \Gamma_1^{(i)} \cup \overset{\circ}{\Gamma}^{(a)}$, we get $x \notin \partial\Omega_1$ and, taking into account that $x \in \overline{\Omega}_1$, we have $x \in \Omega_1$. Hence there exists $r > 0$ such that

$$B_r(x) \subset \Omega_1. \quad (8.3.31)$$

Now, let $s \in (0, r]$ be arbitrary. Since $x \in \partial(\Omega_1 \setminus \overline{G})$ we have

$$\begin{aligned} \emptyset \subsetneq B_s(x) \cap [\mathbb{R}^n \setminus (\Omega_1 \setminus \overline{G})] &= \\ &= B_s(x) \cap (\Omega_1 \cap \overline{G}). \end{aligned}$$

Hence

$$B_s(x) \cap \overline{G} \neq \emptyset, \quad \forall s \in (0, r], \quad (8.3.32)$$

therefore

$$x \in \overline{G}. \quad (8.3.33)$$

On the other hand, since $x \in \partial(\Omega_1 \setminus \overline{G})$, we have for every $s > 0$

$$\emptyset \subsetneq B_s(x) \cap (\Omega_1 \setminus \overline{G}) \subset B_s(x) \cap (\mathbb{R}^n \setminus G).$$

Therefore

$$x \in \overline{\mathbb{R}^n \setminus G},$$

that combined with (8.3.33) gives $x \in \partial G$, which in turn combined with (8.3.30) concludes proof. ■

Chapter 9

The Hadamard example. Solvability of the Cauchy problem and continuous dependence by the data

9.1 The Hadamard example

We present the Hadamard example relating to the Cauchy problem.

Let us consider the following Cauchy problem

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, \\ u(x, 0) = \varphi(x), \quad \text{for } x \in (0, 1), \\ \partial_y u(x, 0) = \psi(x), \quad \text{for } x \in (0, 1). \end{cases} \quad (9.1.1)$$

If φ and ψ are analytic, by the Cauchy–Kovalevskaya Theorem, there exists a unique analytic solution to problem (9.1.1). Such a solution, by the Holmgren Theorem, is the unique solution of class C^2 in a neighborhood of $(0, 1) \times \{0\}$.

Let now

$$\varphi_\nu(x) = e^{-\sqrt{\nu}} \sin \nu x, \quad \text{and} \quad \psi_\nu(x) = 0, \quad \nu \in \mathbb{N}.$$

It is immediately checked that

$$u_\nu(x, y) = e^{-\sqrt{\nu}} \sin \nu x \sinh \nu y, \quad \nu \in \mathbb{N},$$

is the solution to the Cauchy problem

$$\begin{cases} \partial_x^2 u_\nu + \partial_y^2 u_\nu = 0, \\ u_\nu(x, 0) = \varphi_\nu(x), \quad \text{for } x \in (0, 1), \\ \partial_y u_\nu(x, 0) = \psi_\nu(x), \quad \text{for } x \in (0, 1). \end{cases} \quad (9.1.2)$$

Let us note that for every $k \in \mathbb{N}_0$

$$\sup_{(0,1)} \left| \frac{d^k \varphi_\nu}{dx^k} \right| \rightarrow 0, \quad \text{as } \nu \rightarrow \infty,$$

on the other hand, for every $a, b \in (0, 1)$, $a < b$ and for every $\delta > 0$ we have

$$\sup_{[a,b] \times [-\delta, \delta]} |u_\nu| \rightarrow +\infty, \quad \text{as } \nu \rightarrow \infty.$$

In other words, "**small errors**" on the data of Cauchy problem (9.1.1) **yield uncontrollable errors on the solution**. This phenomenon makes problem (9.1.1) essentially intractable for the applications. Actually, in any problem of an applied nature, the data, in the present case the initial data, are derived through measurements and these are necessarily approximated with some error, so in order to be able to practically use the mathematical solution it is necessary that it depends continuously by the data.

In a broad way, we may present the notion of **well-posed problem in the sense of Hadamard** as follows. Let X and Y be two metric spaces and be

$$A : X \rightarrow Y$$

a map from X to Y . Let us consider the problem of determining $x \in X$ such that

$$A(x) = f, \quad (9.1.3)$$

where $f \in Y$.

We say that problem (9.1.3) is well-posed in the sense of Hadamard provided that we have

1. **(Existence)** for any $f \in Y$ there exists at least one $x \in X$ such that $A(x) = f$.
2. **(Uniqueness)** for any $x_1, x_2 \in X$ which satisfy $A(x_1) = A(x_2)$ we have $x_1 = x_2$;

3. **(Continuous dependence by the data)** let us suppose that condition 2 is satisfied, then the map

$$A^{-1} : A(X) \rightarrow X$$

is continuous ($A(X)$ with the topology induced by Y);

In the next Section we will see that in the Cauchy problem there is an interesting relationship between the first two points above (existence and uniqueness) and the third point (continuous dependence by the data).

9.2 Solvability of the Cauchy problem and its relations to the continuous dependence on the data

In this Section we will use some theorems from Functional Analysis on topological vector spaces of which, however, we will not give the proof. As a reference book we will use W. Rudin's book [69] to which we refer for the above-mentioned proofs and for further consideration. We will quote detailed references from of [69] as we go along.

Let us recall the following Theorem of General Topology.

Theorem 9.2.1 (Baire). *Let \mathcal{X} be a complete metric space. Then for any countable family of closed subset, $\{F_n\}_{n \in \mathbb{N}}$, which satisfies*

$$\text{Int}(F_n) = \emptyset, \quad \forall n \in \mathbb{N}, \quad (9.2.1)$$

we have

$$\text{Int} \left(\bigcup_{n \in \mathbb{N}} F_n \right) = \emptyset. \quad (9.2.2)$$

Definition 9.2.2 (topological vector space). Let \mathcal{X} be a vector space on \mathbb{C} (or \mathbb{R}) equipped with a topology τ . We say that \mathcal{X} is a topological vector space

- (a) for every $x \in \mathcal{X}$, $\{x\}$ is closed w.r.t. τ
- (b) the maps

$$\mathcal{X} \times \mathcal{X} \ni (x, y) \rightarrow x + y \in \mathcal{X}$$

and

$$\mathcal{X} \times \mathbb{C} \ni (x, \lambda) \rightarrow \lambda x \in \mathcal{X}$$

(or, $\mathcal{X} \times \mathbb{R} \ni (x, \lambda) \rightarrow \lambda x \in \mathcal{X}$) are continuous.

Let \mathcal{X} be a vector space, we say that $p : \mathcal{X} \rightarrow [0, +\infty)$ is a **seminorm** on \mathcal{X} provided we have

$$p(x + y) \leq p(x) + p(y), \quad \forall x, y \in \mathcal{X}$$

and

$$p(\lambda x) \leq |\lambda|p(x), \quad \forall \lambda \in \mathbb{C}, \forall x \in \mathcal{X}.$$

Let \mathcal{X} be a topological vector space:

(i) We say that \mathcal{X} is **locally convex** if there exists a local base of neighborhoods of 0 whose members are convex. By the condition (b) of Definition 9.2.2 it is clear that if \mathcal{U} is a neighborhood of 0 then, for every $x \in \mathcal{X}$, $x + \mathcal{U}$ is a neighborhood of x and conversely. Hence, given a local base of neighborhoods of 0 it turns out defined trivially a local base of neighborhoods of each point of \mathcal{X} ;

(ii) We say that \mathcal{X} is a **F-space** if its topology is induced by a complete metric d which is invariant w.r.t. translation (i.e. $d(x + z, y + z) = d(x, y)$ for every $x, y, z \in \mathcal{X}$);

(iii) We say that \mathcal{X} is a **Fréchet space** if it is a locally convex F-space.

Let Ω be an open set of \mathbb{R}^n . If Ω is bounded, We set, as usual, for any $u \in C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$,

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{j=0}^k \sum_{|\alpha| \leq j} \max_{\overline{\Omega}} |\partial^\alpha u|.$$

As it is well known, $C^k(\overline{\Omega})$, equipped with $\|\cdot\|_{C^k(\overline{\Omega})}$, is a Banach space. Also, we recall that $C^{k,\sigma}(\overline{\Omega})$, $0 < \sigma \leq 1$, is a Banach space equipped with the norm

$$\|u\|_{C^{k,\sigma}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + [u]_{\Omega;k,\sigma}.$$

Now we equip $C^k(\Omega)$, where $k \in \mathbb{N}_0$, or $k = \infty$, with a topology that makes it a space of Fréchet space.

We start by $C^0(\Omega)$ ($C^0(\Omega, \mathbb{C})$ or $C^0(\Omega, \mathbb{R})$, [69, Ch. 1, Sect. 1.44]). Let $\{K_j\}_{j \in \mathbb{N}}$ be a family of compact sets contained in Ω such that $K_j \neq \emptyset$, $K_j \subset \overset{\circ}{K}_{j+1}$ for every $j \in \mathbb{N}$ and

$$\bigcup_{j=1}^{\infty} K_j = \Omega.$$

For any $f \in C^0(\Omega)$, let us denote by

$$p_{0,j}(f) = \max_{K_j} |f|.$$

$\{p_{0,j}\}_{j \in \mathbb{N}}$ is a family of **separating seminorms**, that is, for each $f \in C^0(\Omega)$, which does not vanish identically, there exists $j \in \mathbb{N}$ such that $p_j(f) \neq 0$. The collection of sets [69, Thm. 1.37]

$$\mathcal{V}_j = \left\{ f \in C^0(\Omega) : p_{0,j}(f) < \frac{1}{j} \right\},$$

make up a local base in $C^0(\Omega)$ of convex neighborhoods of 0 which in turn defines a topology induced by the distance

$$d_0(f, g) = \sum_{j=1}^{\infty} \frac{2^{-j} p_{0,j}(f - g)}{1 + p_{0,j}(f - g)}, \quad \forall f, g \in C^0(\Omega). \quad (9.2.3)$$

It is proved that $C^0(\Omega)$ with the metric (9.2.3) is a Fréchet space (it is quite simple and is left as an exercise).

In a similar way we proceed for $C^k(\Omega)$, k finite. More precisely, we start, rather than from the seminorms $p_{0,j}$, from the seminorms

$$p_{k,j}(f) = \max \left\{ \max_{K_j} |\partial^\alpha f| : |\alpha| \leq k \right\}$$

and we define the distance d_k on $C^k(\Omega)$ by substituting $p_{0,j}$ by $p_{k,j}$ in (9.2.3), that is

$$d_k(f, g) = \sum_{j=1}^{\infty} \frac{2^{-j} p_{k,j}(f - g)}{1 + p_{k,j}(f - g)}, \quad \forall f, g \in C^k(\Omega). \quad (9.2.4)$$

Similarly, it is proved that $C^k(\Omega)$ with the metric (9.2.4) is a Fréchet space.

Finally, concerning $C^\infty(\Omega)$, the following seminorms (with the corresponding metric), are defined

$$q_N(f) = \max \{ |\partial^\alpha f(x)| : x \in K_N, \quad |\alpha| \leq N \}.$$

$$d_\infty(f, g) = \sum_{N=1}^{\infty} \frac{2^{-N} q_N(f - g)}{1 + q_N(f - g)}, \quad \forall f, g \in C^\infty(\Omega). \quad (9.2.5)$$

in the sequel, when we are dealing with the convergence of sequences, it will be more convenient to use directly the seminorms instead of the distances

d_k , $0 \leq k \leq \infty$. For instance, the sequence $\{f_k\}$ in $C^\infty(\Omega)$ converges to f in the topology induced by the norm d_∞ if and only if

$$\lim_{k \rightarrow \infty} q_N(f_k - f) = 0, \quad \forall N \in \mathbb{N}.$$

Theorem 9.2.3 (closed graph). *Let \mathcal{X}, \mathcal{Y} be two F -spaces and*

$$\Lambda : \mathcal{X} \rightarrow \mathcal{Y} \tag{9.2.6}$$

be a linear map. Then Λ is continuous if and only if the graph of Λ

$$\mathcal{G} = \{(x, \Lambda x) : x \in \mathcal{X}\}$$

is closed in $\mathcal{X} \times \mathcal{Y}$.

We refer to [69, Prop. 2.14, Thm 2.15] for a proof. Keep in mind that the most significant implication of Theorem 9.2.3 consists of

$$\mathcal{G} \text{ closed} \implies \Lambda \text{ continuous.}$$

The reverse is true even if Λ is nonlinear, with \mathcal{X} and \mathcal{Y} topological spaces topological and \mathcal{Y} is a Hausdorff space

Let

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha, \tag{9.2.7}$$

a linear differential operator of order m and $a_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{C})$, for $|\alpha| \leq m$, we say that the Cauchy problem with initial surface $\{x_n = 0\}$ for $P(x, \partial)$ enjoys the **local uniqueness property (in 0)** provided there exists $\delta > 0$ such that we have: if $u \in C^m(\overline{B_\delta})$ satisfies

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_\delta, \\ \partial_n^j u(x', 0) = 0, & \text{for } j = 0, 1, \dots, m-1, \quad \forall x' \in B'_\delta, \end{cases} \tag{9.2.8}$$

then

$$u \equiv 0 \text{ in } B_\delta.$$

For instance, if the coefficients of $P(x, \partial)$ are analytic (in a neighborhood of 0) and $P(0, e_n) \neq 0$, the Holmgren Theorem implies that the **local uniqueness property** is satisfied.

We say that the following Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, \\ \partial_n^j u(x', 0) = g_j(x'), \quad \text{for } j = 0, 1, \dots, m-1, \end{cases} \quad (9.2.9)$$

is **locally solvable (in the origin)** in C^∞ if the following occurs: for every open neighborhood \mathcal{U} of 0 in \mathbb{R}^{n-1} and for every

$$g = (g_0, g_1, \dots, g_{m-1}) \in C^\infty(\mathcal{U}, \mathbb{C}^m)$$

there exists \mathcal{V} , open neighborhood of 0 in \mathbb{R}^n and there exists $u \in C^\infty(\mathcal{V})$ such that

$$\begin{cases} P(x, \partial)u = 0, & \text{in } \mathcal{V}, \\ \partial_n^j u = g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } \mathcal{V} \cap \{x_n = 0\}. \end{cases} \quad (9.2.10)$$

The definitions of the **local uniqueness property** and of the **solvability in the origin** of the Cauchy problem with initial surface the hyperplane

$$\{x \in \mathbb{R}^n : N \cdot x = 0\},$$

where N is a versor of \mathbb{R}^n , is formulated in obvious way or, simply, by reconstructing them to the case $N = e_n$ by means of an isometry of \mathbb{R}^n .

The following two theorems and their immediate consequences are known in the literature as the Lax–Mizohata Theorem. Here we present them in a slightly modified form. In particular, Theorem 9.2.4 is due to Lax, and Theorem 9.2.5 is due to Mizohata. The above theorems are treated, for instance, in [56].

Theorem 9.2.4. *Let $P(x, \partial)$ be operator (9.2.7) with $C^\infty(\mathbb{R}^n, \mathbb{C})$ coefficients. Let us suppose that $P(x, \partial)$ enjoys the **local uniqueness property** and that Cauchy problem (9.2.9) is **locally solvable** in C^∞ . Then for every \mathcal{U} , neighborhood of 0 in \mathbb{R}^{n-1} , there exists $r > 0$ such that for every*

$$g = (g_0, g_1, \dots, g_{m-1}) \in C^\infty(\mathcal{U}, \mathbb{C}^m),$$

there exists a unique $u \in C^m(\overline{B_r})$ such that

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_r, \\ \partial_n^j u = g_j & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases} \quad (9.2.11)$$

Remark 1. Let us observe that in Theorem 9.2.4, unlike in the definition of local solvability, r , hence the neighborhood of 0, B_r , **does not** depend on g . Of course r depends on \mathcal{U} . ♦

Proof. Let $\mathcal{U} \in \mathbb{R}^{n-1}$ be a neighborhood of 0 and let $\sigma \in (0, 1)$ be fixed. Since Cauchy problem (9.2.9) is locally solvable in 0, by local uniqueness property we have that, for every

$$g = (g_0, g_1, \dots, g_{m-1}) \in C^\infty(\mathcal{U}, \mathbb{C}^m),$$

there exists $\delta > 0$ such that $B'_\delta \subset \mathcal{U}$ and such that for every $\rho \in (0, \delta]$ there exists a unique solution $u \in C^{m, \sigma}(\overline{B_\rho})$ of problem $\mathcal{P}_{g, \rho}$:

$$(\mathcal{P}_{g, \rho}) \quad \begin{cases} P(x, \partial)u = 0, & \text{in } B_\rho, \\ \partial_n^j u = g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_\rho. \end{cases}$$

Let $\{\rho_k\}$ be a strictly decreasing sequence such that

$$\lim_{k \rightarrow \infty} \rho_k = 0.$$

For any $k, M \in \mathbb{N}$, let us consider the sets

$$\begin{aligned} \mathcal{A}_{k, M} &= \\ &= \left\{ g \in C^\infty(\mathcal{U}, \mathbb{C}^m) : \text{exists } u \text{ solution to } \mathcal{P}_{g, \rho_k} \text{ and } \|u\|_{C^{m, \sigma}(\overline{B_{\rho_k}})} \leq M \right\} \end{aligned}$$

It is evident that $\mathcal{A}_{k, M}$ is symmetric, for any $k, M \in \mathbb{N}$ ($g \in \mathcal{A}_{k, M} \Rightarrow -g \in \mathcal{A}_{k, M}$) and convex for every $k, M \in \mathbb{N}$.

Step 1. Let us check that

$$C^\infty(\mathcal{U}, \mathbb{C}^m) = \bigcup_{k, M \in \mathbb{N}} \mathcal{A}_{k, M}. \quad (9.2.12)$$

Of course, it suffices to check " \subset ". Let $g \in C^\infty(\mathcal{U}, \mathbb{C}^m)$. By the local solvability there exists $\mathcal{V} \subset \mathbb{R}^n$, neighborhood of 0, such that Cauchy problem (9.2.10) admits a solution $u \in C^\infty(\mathcal{V})$. It is enough then to choose $k \in \mathbb{N}$ such that $\overline{B_{\rho_k}} \subset \mathcal{V}$ and M such that $M \geq \|u\|_{C^{m, \sigma}(\overline{B_{\rho_k}})}$ and we have $g \in \mathcal{A}_{k, M}$.

Step 2. Now we prove that for every $k, M \in \mathbb{N}$, $\mathcal{A}_{k, M}$ is closed in $C^\infty(\mathcal{U}, \mathbb{C}^m)$ equipped with the topology induced by the metric d_∞ .

We fix $k, M \in \mathbb{N}$ and let $\{g_\nu\}_{\nu \in \mathbb{N}}$ be a sequence in $\mathcal{A}_{k,M}$ such that

$$\{g_\nu\} \rightarrow \tilde{g}, \quad \text{in } C^\infty(\mathcal{U}, \mathbb{C}^m). \quad (9.2.13)$$

let us check that $\tilde{g} \in \mathcal{A}_{k,M}$.

First of all, we have

$$\tilde{g} \in C^\infty(\mathcal{U}, \mathbb{C}^m). \quad (9.2.14)$$

Moreover, denoting by u_ν the solution of problem $\mathcal{P}_{g_\nu, \rho_k}$ we have, by the definition of $\mathcal{A}_{k,M}$,

$$\|u_\nu\|_{C^{m,\sigma}(\overline{B_{\rho_k}})} \leq M, \quad \forall \nu \in \mathbb{N}.$$

Hence, by the Arzelà–Ascoli Theorem there exists a subsequence $\{u_{\nu_q}\}$ of $\{u_\nu\}$ and $\tilde{u} \in C^{m,\sigma}(\overline{B_{\rho_k}})$ such that

$$u_{\nu_q} \rightarrow \tilde{u}, \quad \text{as } q \rightarrow \infty, \quad \text{in } C^m(\overline{B_{\rho_k}}). \quad (9.2.15)$$

Moreover we have

$$\|\tilde{u}\|_{C^{m,\sigma}(\overline{B_{\rho_k}})} \leq M, \quad (9.2.16)$$

Concerning the justification of the latter inequality, we notice that the sequence $\{u_{\nu_q}\}$ converges to \tilde{u} in $C^m(\overline{B_{\rho_k}})$, but not necessarily it converges in $C^{m,\sigma}(\overline{B_{\rho_k}})$, nevertheless (9.2.16) holds, since for $x \neq y \in \overline{B_{\rho_k}}$ we have, for any $q \in \mathbb{N}$,

$$\sum_{|\alpha|=m} \frac{|\partial^\alpha u_{\nu_q}(x) - \partial^\alpha u_{\nu_q}(y)|}{|x-y|^\sigma} \leq M - \|u_{\nu_q}\|_{C^m(\overline{B_{\rho_k}})};$$

hence, from the punctual convergence of $\{\partial^\alpha u_{\nu_q}\}$ for $|\alpha| = m$ and by

$$\|u_{\nu_q}\|_{C^m(\overline{B_{\rho_k}})} \rightarrow \|u\|_{C^m(\overline{B_{\rho_k}})}, \quad \text{as } q \rightarrow \infty,$$

we obtain (9.2.16).

By (9.2.15) we have easily

$$0 = P(x, \partial)u_{\nu_q} \rightarrow P(x, \partial)\tilde{u}, \quad \text{as } q \rightarrow \infty, \quad \text{in } C^0(\overline{B_{\rho_k}}).$$

Therefore

$$P(x, \partial)\tilde{u} = 0, \quad \text{in } \overline{B_{\rho_k}}. \quad (9.2.17)$$

By (9.2.13) and (9.2.15) we have, for $j = 0, 1, \dots, m-1$,

$$\partial_n^j u(x', 0) = \lim_{q \rightarrow \infty} \partial_n^j u_{\nu_q}(x', 0) = \lim_{q \rightarrow \infty} g_{j, \nu_q}(x') = \tilde{g}, \quad \forall x' \in \overline{B'_{\rho_k}}. \quad (9.2.18)$$

By (9.2.14), (9.2.16)–(9.2.18) we get $\tilde{g} \in \mathcal{A}_{k,M}$.

Step 3. Now, recalling that $C^\infty(\mathcal{U}, \mathbb{C}^m)$ is a Frechét space, by Theorem 9.2.1 we have that there exist $k_0, M_0 \in \mathbb{N}$ such that

$$\text{Int}(\mathcal{A}_{k_0, M_0}) \neq \emptyset. \quad (9.2.19)$$

To prove (9.2.19) we argue by contradiction. Let us assume (9.2.19) does not hold. Consequently we have

$$\text{Int}(\mathcal{A}_{k,M}) = \emptyset, \quad \forall k, M \in \mathbb{N},$$

and recalling that $\mathcal{A}_{k,M}$ is closed for every $k, M \in \mathbb{N}$, (9.2.12) and Theorem 9.2.1 imply

$$C^\infty(\mathcal{U}, \mathbb{C}^m) = \text{Int}(C^\infty(\mathcal{U}, \mathbb{C}^m)) = \text{Int}\left(\bigcup_{k, M \in \mathbb{N}} \mathcal{A}_{k, M}\right) = \emptyset.$$

This is clearly a contradiction, then (9.2.19) needs to hold.

On the other hand, (9.2.19) implies that there exists $\psi \in \text{Int}(\mathcal{A}_{k_0, M_0})$, and since \mathcal{A}_{k_0, M_0} is symmetric and convex, we have $-\psi \in \text{Int}(\mathcal{A}_{k_0, M_0})$ therefore

$$0 = \frac{1}{2}(\psi - \psi) \in \text{Int}(\mathcal{A}_{k_0, M_0}).$$

All in all, we have

$$0 \in \text{Int}(\mathcal{A}_{k_0, M_0}).$$

Consequently there exists $\mathcal{W}_0 \subset \text{Int}(\mathcal{A}_{k_0, M_0})$, where \mathcal{W}_0 is an element of local base of neighborhood of 0 in $C^\infty(\mathcal{U}, \mathbb{C}^m)$, hence it is of the type

$$\mathcal{W}_0 = \bigcap_{|\alpha| \leq h, 1 \leq j \leq h'} \left\{ f \in C^\infty(\mathcal{U}, \mathbb{C}^m) : \max_{K_j} |\partial^\alpha f| < \varepsilon_{\alpha, j} \right\},$$

where $h, h', \varepsilon_{\alpha, j}$ are suitable positive numbers and $\{K_j\}_{j \in \mathbb{N}}$ is a family of compact subset of \mathcal{U} which satisfies $K_j \neq \emptyset$, $K_j \subset \overset{\circ}{K}_{j+1}$ for every $j \in \mathbb{N}$ and

$$\bigcup_{j=1}^{\infty} K_j = \mathcal{U}.$$

We recall that the local uniqueness property implies that there exists $r > 0$, that we may assume less or equal to ρ_{k_0} , such that if $w \in C^m(\overline{B_r})$ satisfies

$$\begin{cases} P(x, \partial)w = 0, & \text{in } B_r, \\ \partial_n^j w = 0, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r, \end{cases}$$

then we have

$$w \equiv 0 \quad \text{in } B_r.$$

Now, let $g \in C^\infty(\mathcal{U})$, then there exists trivially $\lambda > 0$ such that

$$\lambda^{-1}g \in \mathcal{W}_0 \subset \text{Int}(\mathcal{A}_{k_0, M_0})$$

and by the definition of \mathcal{A}_{k_0, M_0} we have there exists $v \in C^m(\overline{B_{\rho_{k_0}}})$ (actually, $v \in C^{m, \sigma}(\overline{B_{\rho_{k_0}}})$) solution to $\mathcal{P}_{\lambda^{-1}g, \rho_{k_0}}$. Hence $v|_{B_r}$ is the unique solution in $C^m(\overline{B_r})$ to the Cauchy problem

$$\begin{cases} P(x, \partial)v = 0, & \text{in } B_r, \\ \partial_n^j v = \lambda^{-1}g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r, \end{cases}$$

Therefore, $u = \lambda v$ is the unique solution in $C^m(\overline{B_r})$ to the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_r, \\ \partial_n^j u = g_j & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases}$$

■

Remark 2. It is evident from the proof of Theorem 9.2.4 that if in the definition of local solvability we require the existence of u in $C^{m'}(\mathcal{V})$ with $m' > m$, then we reach the same conclusions. ◆

Theorem 9.2.5 (Lax–Mizohata). *Let us suppose that $P(x, \partial)$ satisfies the same assumption of Theorem 9.2.4. Let $\mathcal{U} \subset \mathbb{R}^{n-1}$ be a neighborhood of 0 and let $r > 0$ be defined in Theorem 9.2.4. Let Λ be the map*

$$\Lambda : C^\infty(\mathcal{U}, \mathbb{C}^m) \rightarrow C^m(\overline{B_r}),$$

$C^\infty(\mathcal{U}, \mathbb{C}^m) \ni g \rightarrow \Lambda(g) = u$ solution to the Cauchy problem :

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_r, \\ \partial_n^j u = g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases} \quad (9.2.20)$$

Then Λ is continuous.

Proof. Let us prove that the graph of Λ is closed and then we apply Theorem 9.2.3. Let $\{(g_\nu, u_\nu)\}_{\nu \in \mathbb{N}}$ be a sequence in $C^\infty(\mathcal{U}, \mathbb{C}^m) \times C^m(\overline{B_r})$ which satisfies

$$\begin{cases} P(x, \partial)u_\nu = 0, & \text{in } B_r, \\ \partial_n^j u_\nu = g_{\nu,j}, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r \end{cases} \quad (9.2.21)$$

and

$$\{(g_\nu, u_\nu)\} \rightarrow (g, v), \quad \text{in } C^\infty(\mathcal{U}, \mathbb{C}^m) \times C^m(\overline{B_r}). \quad (9.2.22)$$

Then

$$P(x, \partial)v = \lim_{\nu \rightarrow \infty} P(x, \partial)u_\nu = 0$$

and, for $0 \leq j \leq m-1$,

$$\partial_n^j v(x', 0) = \lim_{\nu \rightarrow \infty} \partial_n^j u_\nu(x', 0) = \lim_{\nu \rightarrow \infty} g_{\nu,j}(x') = g_j(x').$$

Hence v solves Cauchy problem (9.2.20) and by Theorem 9.2.4, we have

$$v = \Lambda g.$$

Therefore the graph of Λ is closed, hence Theorem 9.2.3 implies that Λ is continuous. ■

We observe that by Theorem 9.2.5 and the Hadamard example it follows (again) that the Cauchy problem for the Laplace equation, (9.1.1), is not locally solvable in C^∞ . Indeed, by the Holmgren Theorem, such a Cauchy problem for the Laplace equation enjoys the property of local uniqueness in C^2 , but, as shown in the Hadamard example, the map Λ defined for this Cauchy problem is not continuous.

Similarly, Theorem 9.2.5 may be applied to obtain some necessary condition for the local solvability of the Cauchy problem. Here we limit ourselves only to consider the case of the operators with constant coefficients which are equal to their principal part. We refer to [34, Ch.5, Sect.4], [36, Ch.12,

Sect.3] for the general operators with constant coefficients, and to [36, Ch.23, Sect.3] for the operators with C^∞ coefficients, warning the reader, however, that (especially in [36, Ch.23, Sect. 3]) quite advanced tools are used.

Let

$$P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad (9.2.23)$$

be a **homogeneous polynomial** of degree m with coefficients $a_\alpha \in \mathbb{C}$, for $|\alpha| = m$ and let N be a versor of \mathbb{R}^n . We say that P_m is hyperbolic with respect to the direction N provided we have

- (a) $P_m(N) \neq 0$
- (b) for every $\xi \in \mathbb{R}^n$ the algebraic equation in z

$$P_m(\xi + zN) = 0,$$

has real roots only.

Theorem 9.2.6. *Let N be a versor of \mathbb{R}^n and let $P_m(\partial)$ be the differential operator with constant coefficients*

$$P_m(\partial) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha. \quad (9.2.24)$$

Let us suppose that the Cauchy problem for operator (9.2.24) with initial surface

$$\{x \in \mathbb{R}^n : N \cdot x = 0\} \quad (9.2.25)$$

enjoys the local solvability and the local uniqueness property.

Then the polynomial $P_m(\xi)$ is hyperbolic w.r.t. the direction N .

Proof. We have already seen in Section 7.5.2 that by the local uniqueness for the Cauchy problem for operator (9.2.24) with initial surface (9.2.25), we have

$$P_m(N) \neq 0. \quad (9.2.26)$$

We argue by contradiction to prove that, for every $\xi \in \mathbb{R}^n$, there exist real roots only to the equation

$$P_m(\xi + zN) = 0. \quad (9.2.27)$$

Let us suppose that there exist $\xi_0 \in \mathbb{R}^n$ and $z_0 = \Re z_0 + i\Im z_0$, such that $\Im z_0 \neq 0$ and

$$P_m(\xi_0 + z_0 N) = 0. \quad (9.2.28)$$

Let us denote by

$$\eta = \xi_0 + \Re z_0 N, \quad \text{and} \quad \tau = \Im z_0$$

and by

$$u_\nu(x) = \exp \{ -|\nu|^{1/2} + \nu (i\eta \cdot x - \tau N \cdot x) \}, \quad \nu \in \mathbb{Z}.$$

We have, by (9.2.28) (recalling that $\xi_0 + z_0 N = \eta + i\tau N$ and $P_m(\xi)$ is a homogeneous polynomial)

$$\begin{aligned} P_m(\partial)u_\nu(x) &= u_\nu(x)P_m(\nu(i\eta - \tau N)) = \\ &= u_\nu(x)(i\nu)^m P_m(\eta + i\tau N) = 0. \end{aligned} \tag{9.2.29}$$

Set

$$g_{\nu,j} = \frac{\partial^j}{\partial N^j} u_\nu, \quad \text{for } \nu \in \mathbb{Z} \quad \text{and } j = 0, 1, \dots, m-1,$$

we have trivially that u_ν solves the Cauchy problem

$$\begin{cases} P(x, \partial)u_\nu = 0, & \text{in } \mathbb{R}^n, \\ \frac{\partial^j}{\partial N^j} u_\nu = g_{\nu,j}, & \text{for } j = 0, 1, \dots, m-1, \quad N \cdot x = 0. \end{cases}$$

On the other hand it is easy to check that

$$g_\nu \rightarrow 0, \quad \text{as } |\nu| \rightarrow +\infty, \quad \text{in } C^\infty.$$

Moreover, if $\tau N \cdot x > 0$ then we have

$$|u_\nu(x)| \rightarrow +\infty, \quad \text{as } \nu \rightarrow -\infty$$

and if $\tau N \cdot x < 0$ then we have

$$|u_\nu(x)| \rightarrow +\infty, \quad \text{as } \nu \rightarrow +\infty.$$

Hence, the map Λ defined in Theorem 9.2.5 is not continuous. Since $P_m(\partial)$ enjoys the local uniqueness property, $P_m(\partial)$ cannot enjoy at the same time the local solvability. Thus we have a contradiction. Therefore equation (9.2.27) has real roots only. ■

If the polynomial $P(\xi)$ is not homogeneous, one can prove (with more efforts) a theorem which is similar to Theorem 9.2.6 that we merely state here (see [34, Ch.5, Sect. 4], [36, Ch.12, Sect. 3])

Theorem 9.2.7. *Let $P(\xi)$ be a polynomial of degree m , N a versor of \mathbb{R}^n and $P(\partial)$ the following differential operator with constant coefficient*

$$P(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha, \quad (9.2.30)$$

where $a_\alpha \in \mathbb{C}$ for $|\alpha| \leq m$. Let us suppose that the Cauchy problem for operator (9.2.30) with initial surface

$$\{x \in \mathbb{R}^n : N \cdot x = 0\},$$

enjoys local solvability property and let us suppose

$$P_m(N) \neq 0, \quad (9.2.31)$$

where P_m is the principal part of P .

Then

$$\exists \tau_0 \in \mathbb{R} \text{ such that } \forall \xi \in \mathbb{R}^n \text{ and } \forall \tau < \tau_0 \quad P(i(\xi + i\tau N)) \neq 0. \quad (9.2.32)$$

A polynomial that enjoys the properties (9.2.31) and (9.2.32) is called **hyperbolic polynomial w.r.t. the direction N** . It is not difficult to prove (see the literature quoted above) that in the case where $P(\xi)$ is a homogeneous polynomial the two definitions of hyperbolicity coincide. Moreover, if $P(\xi)$ is a hyperbolic polynomial (not necessarily homogeneous), denoted by $P_m(\xi)$ its principal part, then we have

(a) $P_m(N) \neq 0$,

(b) for every $\xi \in \mathbb{R}^n$ the equation in z

$$P_m(\xi + zN) = 0,$$

has real roots only.

However, if the principal part of $P(\xi)$ satisfies (a) and (b) it does not imply that $P(x, \partial)$ is hyperbolic w.r.t. N . It can be proved that if the equation $P_m(\xi + zN) = 0$, has **simple real roots** only, then the hyperbolicity condition with respect to a direction N is also sufficient for the local solvability of the Cauchy problem with initial surface $\{N \cdot x = 0\}$.

9.3 Concluding Remarks

1. In this Chapter we have considered the properties of the local uniqueness and the local solvability for the Cauchy problem, but with simple and natural

changes one could consider the same properties for the one-sided Cauchy problem. Actually, it suffices to replace each neighborhood of $0 \in \mathbb{R}^n$, say \mathcal{V} , by $\mathcal{V}_+ = \mathcal{V} \cap \mathbb{R}^{n-1} \times [0, +\infty)$, and in defining the seminorms of $C^\infty(\mathcal{V}_+)$ we consider a family of compacts $K_j \subset \text{Int}_{\mathcal{V}_+}(K_{j+1})$ for every $j \in \mathbb{N}$ e $\bigcup_{j=1}^\infty K_j = \mathcal{V}_+$. In this way provided that the appropriate modifications are introduced, Theorems 9.2.5 and 9.2.4 preserve their validity, in particular, the ball B_r should be replaced with the half-ball $B_r \cap (\mathbb{R}^{n-1} \times [0, +\infty))$.

2. In Theorem 9.2.5 we have seen that if the properties of the local uniqueness and of the local solvability in C^{infy} hold true then the map Λ , defined in Theorem 9.2.5, is continuous. In what follows, we check that if the operator $P(x, \partial)$ has analytic coefficients, the converse also holds (in a sense).

Let us suppose that

$$P_m(0, e_n) \neq 0,$$

and let us suppose that there exists $r > 0$ such that only the null function solves (in $C^\infty(B_r)$) the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_r, \\ \partial_n^j u = 0, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases} \quad (9.3.1)$$

Then by the Cauchy-Kovalevskaya Theorem, there exists $\tilde{r} \leq r$ such that the following map $\tilde{\Lambda}$ is well-defined

$C^\omega(B'_r, \mathbb{C}^m) \ni g \rightarrow \tilde{\Lambda}(g) = u \in C^\omega(B_{\tilde{r}})$ solution to the Cauchy problem:

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_{\tilde{r}}, \\ \partial_n^j u = g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases} \quad (9.3.2)$$

We prove what follows: let us assume that $\tilde{\Lambda}$ is a continuous map provided that we equip the spaces $C^\omega(B'_r, \mathbb{C}^m)$ and $C^\omega(B_{\tilde{r}})$ with the metric d_∞ , we have that Cauchy problem (9.3.2) satisfies the local solvability property in C^∞ (i.e. if the initial data of the Cauchy belong to C^∞ there exist solutions to the Cauchy problem).

Let us suppose that $\tilde{\Lambda}$ is continuous and let $g \in C^\infty(B'_r, \mathbb{C}^m)$. Let $\{g_{,\nu}\}$ be the sequence in $C^\omega(B'_r, \mathbb{C}^m)$ defined as follows

$$g_{,\nu}(x') = \left(\frac{\nu}{2\pi}\right)^{(n-1)/2} \int_{B'_r} e^{-\frac{\nu|x'-y'|^2}{2}} g(x') dx'.$$

The sequence $\{g_{\nu}\}$ converges to g in $C^\infty(B'_r, \mathbb{C}^m)$ and since $g_{\nu} \in C^\omega(B'_r, \mathbb{C}^m)$, for every $\nu \in \mathbb{N}$, the Cauchy-Kovalevskaya Theorem yields the existence of a solution u_ν which is unique in $C^\omega(B_{\tilde{r}})$. By the continuity of $\tilde{\Lambda}$ and since $\{g_\nu\}_{\nu \in \mathbb{N}}$ is a Cauchy sequence in $C^\infty(B'_r, \mathbb{C}^m)$ we derive that $\{u_\nu\}$ is a Cauchy sequence in $C^\infty(B_{\tilde{r}}(0))$. The completeness of $C^\infty(B_{\tilde{r}})$ implies that $\{u_\nu\}_{\nu \in \mathbb{N}}$ converges to a function $u \in C^\infty(B_{\tilde{r}})$.

Moreover, by

$$P(x, \partial)u_\nu \rightarrow P(x, \partial)u, \quad \text{as } \nu \rightarrow \infty$$

and

$$\partial_n^j u_\nu(x', 0) = g_{j,\nu}(x') \rightarrow g_j(x'), \quad \partial_n^j u_\nu(x', 0) \rightarrow \partial_n^j u(x', 0), \quad \text{as } \nu \rightarrow \infty,$$

for $j = 0, 1, \dots, m-1$, we derive that $u \in C^\infty(B_{\tilde{r}})$ solves the Cauchy problem

$$\begin{cases} P(x, \partial)u = 0, & \text{in } B_{\tilde{r}}, \\ \partial_n^j u = g_j, & \text{for } j = 0, 1, \dots, m-1, \quad \text{in } B'_r. \end{cases}$$

3. Keep in mind that the definition of hyperbolicity that we provided above, involves not only the operator (or, more precisely, its symbol), but also the direction N . Let us consider, for instance, the wave operator in space dimension two

$$P(\partial_t, \partial_{x_1}, \partial_{x_2}) = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2).$$

The symbol of $P(\partial_t, \partial_{x_1}, \partial_{x_2})$ is the polynomial

$$P(\eta, \xi_1, \xi_2) = -\eta^2 + (\xi_1^2 + \xi_2^2),$$

which is hyperbolic w.r.t. the direction $(1, 0, 0)$, but **is not hyperbolic w.r.t. the directions** $(0, 1, 0)$ e $(0, 0, 1)$.

Elliptic operators (with constant coefficients), as is checked easily, are not hyperbolic with respect to any direction. Hence, the Cauchy problem for the elliptic operators does not enjoy the property of local solvability in C^∞ , nor, at the light of what was shown in Section 9.2, it may happen that there is a continuous dependence in C^∞ (or from C^∞ in C^m).

Chapter 10

Well-posed problems. Conditional stability

10.1 Introduction

In the previous Chapter, we introduced the notion of well-posed problem in the sense of Hadamard and we observed that some Cauchy problems are not well posed. In particular, we observed that in such problems may fail some kind of continuous dependence of the solutions with respect to the data. This phenomenon represents a serious obstacle in the study of the problems originating from the applications. Indeed, in these problems the measurements of the data are, apart for trivial cases, affected by the errors of approximation the effect of which must always be taken into account if the theoretical results obtained have any reasonable application.

Moreover, one should not believe that the phenomena of noncontinuous dependence of the solutions with respect to the data are present only in particularly complicated situations as is the case of Cauchy problems. Indeed, such phenomena are encountered even in the approximate calculation of a derivative or, to put ourselves in an "applicative" perspective, in the approximate calculation of the velocity from a given time law. Let us suppose that a certain object moves with rectilinear motion with a time law

$$x = s(t)$$

and let us let us suppose that we are interested in determining its velocity $v(t)$. As we know very well

$$v(t) = s'(t).$$

However, now, let us suppose that we only have an approximation of the time law. Let $\varepsilon > 0$ and let s_ε be an approximate measure of the time law

of our object. For instance let us suppose that

$$\sup_{t \in [0, T]} |s(t) - s_\varepsilon(t)| \leq \varepsilon, \quad (10.1.1)$$

where $T > 0$ is the time in which the motion is performed (initial time 0). It would be desired that $v_\varepsilon = s'_\varepsilon$ be itself an approximation of v . Nevertheless, this does not happen. As a matter of fact, let

$$s_\varepsilon(t) = s(t) + \varepsilon \sin(\varepsilon^{-2}t), \quad t \in [0, T].$$

Then (10.1.1) is satisfied, hence

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} |s(t) - s_\varepsilon(t)| = 0, \quad (10.1.2)$$

on the other side

$$v(t) - v_\varepsilon(t) = s'(t) - s'_\varepsilon(t) = \varepsilon^{-1} \cos(\varepsilon^{-2}t) \not\rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Thus, without additional information about the motion of the object, we cannot obtain an approximation of its velocity based only on an approximation of its time law.

The example we have just considered may be expressed more formally by saying that the operator

$$\frac{d}{dt} : X \rightarrow Y,$$

where

$$X = C^1([0, T]), \quad \text{equipped with norm } \|\cdot\|_{C^0([0, T])}$$

and

$$Y = C^0([0, T]), \quad \text{equipped with norm } \|\cdot\|_{C^0([0, T])}$$

is not continuous.

In what follows when we say that a problem is not well-posed, or ill-posed, in the sense of Hadarmard we will always mean (if we do not risk ambiguities) that although the existence and uniqueness of the solutions occur, the problem does not enjoy the continuous dependence with respect to the data. The continuous dependence should be considered with respect to the topologies suggested by the same nature of the applied problem under investigation.

The Functional Analysis is a rich repository of examples of not well-posed problems in the sense specified above. For instance, it is known that, if X is a Banach space and $A \in \mathcal{L}(X)$ (where $\mathcal{L}(X)$ denotes the space of linear and continuous operators from X into itself) is injective then

$$A^{-1} : \mathcal{R}(A) \rightarrow X,$$

where

$$\mathcal{R}(A) = \{Au : u \in X\},$$

is continuous if and only if $\mathcal{R}(A)$ is closed in X . In particular, if $\mathcal{R}(A)$ is not closed, then the problem

$$A(u) = f, \quad (10.1.3)$$

is not well-posed in the sense of Hadamard. This is the case of the integral operator

$$L^2(0, 1) \ni u \rightarrow (Au)(t) = \int_0^t u(s)ds \in L^2(0, 1). \quad (10.1.4)$$

Similarly, if X has not a finite dimension and A is a compact and injective operator from X into itself, then again A^{-1} is not continuous and therefore (10.1.3) is an ill-posed problem.

As it is well known, operator (10.1.4) is compact. More generally, if $k \in L^2([0, 1] \times [0, 1])$ then the operator

$$L^2(0, 1) \ni u \rightarrow (Ku)(t) = \int_0^1 k(t, s)u(s)ds \in L^2(0, 1), \quad (10.1.5)$$

is a compact operator. Hence the integral equation

$$\int_0^1 k(t, s)u(s)ds = f(t), \quad t \in [0, 1], \quad (10.1.6)$$

is certainly an ill-posed problem in the sense of Hadamard since, even if it admits solutions they do not depend continuously (in $L^2(0, 1)$) by f .

As we have already observed, a problem for which there is no continuous dependence with respect to the data, without further information, cannot be treated practically. In order to treat it, additional information is needed. This introduces the notion of **conditionally well-posed problem** whose formal definition formal is as follows.

Definition 10.1.1. Let (X, d_1) e (Y, d_2) be two metric spaces and let

$$A : X \rightarrow Y$$

be a map from X to Y . we say that the problem of determining $u \in X$ such that

$$A(u) = f, \quad (10.1.7)$$

where $f \in Y$, is a **conditional well-posed problem** or likewise **well-posed problem in the sense of Tikhonov** [73] with respect to $\mathcal{K} \subset X$ provided that we have

- (i) $A|_{\mathcal{K}} : \mathcal{K} \rightarrow Y$ is injective,
- (ii) $(A|_{\mathcal{K}})^{-1} : A(\mathcal{K}) \rightarrow \mathcal{K}$ is continuous.

Remark. Let us note that in the above Definition *there is no requirement for the existence of solutions* of the problem $A(u) = f$. In essence, the problem that is considered is the following one.

Given $f \in A(\mathcal{K})$, determine u such that

$$\begin{cases} A(u) = f, \\ u \in \mathcal{K}. \end{cases} \quad (10.1.8)$$

The introduction of the set \mathcal{K} into the definition of a well-posed problem in the sense of Tikhonov is equivalent to the introduction of a **additional information** or an **a priori information** (as it is often said in the literature) to the problem under investigation. We will synthesize the above requirements by saying

$$\begin{aligned} &\text{Determine } u \in X \text{ such that } A(u) = f \\ &\text{with the a priori bound } u \in \mathcal{K}. \end{aligned} \quad (10.1.9)$$

The a priori information pertains to the specific character of the problem under investigation and it is suggested by the applied nature of that problem. To find a **stability estimate for problem** (10.1.8) or (10.1.9) means to find an appropriate estimate of the modulus of continuity ω of $(A|_{\mathcal{K}})^{-1}$, which, we recall, is defined by

$$\omega(\delta) = \sup \{d_1(A^{-1}(f_1), A^{-1}(f_2)) : f_1, f_2 \in A(\mathcal{K}), \quad d_2(f_1, f_2) \leq \delta\}.$$

Obviously the best that one can do is to determine exactly ω , but very often to arrive at an accurate asymptotic estimate of $\omega(\delta)$ as δ goes to 0, can be considered a satisfactory result for many applications.



If the problem

$$A(u) = f,$$

is not well-posed in the sense of Hadamard and A is a continuous and injective map, there always exist sets \mathcal{K} for which problem (10.1.7) is conditionally well-posed. The following Theorem holds true.

Theorem 10.1.2 (Tikhonov). *Let \mathcal{K} and Y be two metric spaces. Let us assume \mathcal{K} is a compact. Moreover, let*

$$F : \mathcal{K} \rightarrow Y$$

which satisfies

(i) F is injective,

(ii) F is continuous.

Then

$$F^{-1} : F(\mathcal{K}) \rightarrow \mathcal{K},$$

is continuous.

Proof. Let C be a closed subset of \mathcal{K} , let us prove that

$$F(C) = (F^{-1})^{-1}(C)$$

is closed in Y . We have

$$\begin{aligned} C \subset \mathcal{K}, C \text{ closed } \mathcal{K} &\Rightarrow F(C) \text{ compact in } Y \Rightarrow \\ &\Rightarrow F(C) \text{ closed in } Y. \end{aligned}$$

Therefore F is continuous. ■

Example 1. To illustrate what we have said so far, let us return to the problem of calculating the derivative of a function and we consider a simplified version of it, i.e. the following one: calculate the derivative of $f \in C^1([0, 1])$ which satisfies

$$f'(0) = f'(1) = 0. \tag{10.1.10}$$

Let us note that also this problem is ill-posed in $L^2(0, 1)$. As a matter of fact, let

$$f_n(t) = \frac{1}{\sqrt{n}} \cos \pi n t, \quad t \in [0, 1], \quad n \in \mathbb{N}.$$

Then (10.1.10) is satisfied and

$$\|f_n\|_{L^2(0,1)} = \frac{1}{\sqrt{2n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

and

$$\|f'_n\|_{L^2(0,1)} = \frac{\pi\sqrt{n}}{\sqrt{2}} \rightarrow +\infty, \quad \text{as } n \rightarrow \infty.$$

We reformulate the problem as a functional equation. Let be

$$X = \{u \in C^0([0, 1]) : u(0) = u(1) = 0\},$$

$$Y = \{f \in C^1([0, 1]) : f'(0) = f'(1) = 0\}$$

and

$$T : X \rightarrow Y, \quad (T(u)) := \int_0^t u(s) ds.$$

Equip X and Y with the $L^2(0, 1)$ norm

The above problem becomes: given $f \in Y$ determine $u \in X$ such that

$$T(u) = f, \tag{10.1.11}$$

which has, trivially, a unique solution given by

$$u(t) = f'(t), \quad t \in [0, 1].$$

We have seen above that there is no continuous dependence of the solutions by the datum. On the other hand T is continuous and, setting

$$\mathcal{K} = \left\{ u \in X : u \in C^1([0, 1]), \quad \|u'\|_{L^2(0,1)} \leq E \right\},$$

where E is a positive number, since \mathcal{K} is a compact of X , the problem

$$\begin{cases} T(u) = f, \\ u \in \mathcal{K}, \end{cases} \tag{10.1.12}$$

is well-posed in the sense of Tikhonov. Hence, denoting by ω_E the modulus of continuity of $(T|_{\mathcal{K}})^{-1}$ we have

$$\begin{aligned} \|T^{-1}(f_1) - T^{-1}(f_2)\|_{L^2(0,1)} &= \|f'_1 - f'_2\|_{L^2(0,1)} \leq \\ &\leq \omega_E \left(\|f_1 - f_2\|_{L^2(0,1)} \right), \end{aligned} \tag{10.1.13}$$

for every $f_1, f_2 \in T(\mathcal{K})$.

An estimate of ω_E can be easily proved as follows. If $f \in C^2([0, 1])$ and $f'(0) = f'(1) = 0$ then integrating by parts and applying the Cauchy–Schwarz inequality we get

$$\begin{aligned} \int_0^1 f'^2(t) dt &= \int_0^1 f'(t) f'(t) dt = - \int_0^1 f''(t) f(t) dt \leq \\ &\leq \left(\int_0^1 f''^2(t) dt \right)^{1/2} \left(\int_0^1 f^2(t) dt \right)^{1/2}. \end{aligned}$$

Hence

$$\int_0^1 f'^2(t) dt \leq \left(\int_0^1 f''^2(t) dt \right)^{1/2} \left(\int_0^1 f^2(t) dt \right)^{1/2}. \quad (10.1.14)$$

Let now $f_1, f_2 \in T(\mathcal{K})$, satisfy

$$\|f_1 - f_2\|_{L^2(0,1)} \leq \varepsilon,$$

then by (10.1.14) we have

$$\|T^{-1}(f_1) - T^{-1}(f_2)\|_{L^2(0,1)} \leq (2E\varepsilon)^{1/2}. \quad (10.1.15)$$

Therefore

$$\omega_E(\varepsilon) \leq (2E\varepsilon)^{1/2}. \quad (10.1.16)$$

In particular, for fixed E , we have

$$\omega_E(\varepsilon) = \mathcal{O}((\varepsilon)^{1/2}) \quad \text{as } \varepsilon \rightarrow 0, \quad (10.1.17)$$

the reader is invited to check that the exponent $1/2$ in the estimate (10.1.17) cannot be improved, however on this kind of issue we will return to further in this Chapter.

10.2 Interpolation estimates for the derivatives of a function.

In this Section we will prove some estimates among functions and their derivatives. These estimates can be considered as conditional stability estimates of some not well-posed problem. We start by the following.

Proposition 10.2.1. *If $f \in C^2([a, b])$, where $a, b \in \mathbb{R}$, $a < b$, then we have*

$$\begin{aligned} \|f'\|_{L^\infty(a,b)} &\leq \\ &\leq c_0 \left(((b-a)^{-2} \|f\|_{L^\infty(a,b)} + \|f''\|_{L^\infty(a,b)})^{1/2} \|f\|_{L^\infty(a,b)}^{1/2} \right), \end{aligned} \quad (10.2.1)$$

where $c_0 \leq 8\sqrt{2}$ is a positive constant.

Proof. The proof of (10.2.1) can be reduced to the case where $[a, b] = [0, 1]$. To this aim it suffices to consider, instead of f , the function

$$[0, 1] \ni t \rightarrow f(a + (b-a)t) \in \mathbb{R}.$$

Let us continue to denote by f this function. Let us fix $x \in [0, \frac{1}{2}]$ and let $h \in (0, \frac{1}{2}]$. We have

$$f'(x) = \left(f'(x) - \frac{f(x+h) - f(x)}{h} \right) + \frac{f(x+h) - f(x)}{h}. \quad (10.2.2)$$

The Lagrange Theorem implies that there exist ξ, η such that

$$x < \eta < \xi < x + h$$

and

$$f'(x) - \frac{f(x+h) - f(x)}{h} = f'(x) - f'(\xi) = (x - \xi)f'(\eta).$$

Hence

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq h \|f''\|_{L^\infty(0,1)}. \quad (10.2.3)$$

By the just obtained inequality and by (10.2.2) we have, for every $x \in [0, \frac{1}{2}]$ and for every $h \in (0, \frac{1}{2}]$,

$$\begin{aligned} |f'(x)| &\leq h \|f''\|_{L^\infty(0,1)} + h^{-1} (|f(x+h)| + |f(x)|) \leq \\ &\leq h \|f''\|_{L^\infty(0,1)} + 2h^{-1} \|f\|_{L^\infty(0,1)}. \end{aligned} \quad (10.2.4)$$

Instead, if $x \in [\frac{1}{2}, 1]$, then it suffices to replace (10.2.2) by

$$f'(x) = \left(f'(x) - \frac{f(x-h) - f(x)}{-h} \right) + \frac{f(x-h) - f(x)}{-h},$$

for every $h \in (0, \frac{1}{2}]$ and we obtain

$$|f'(x)| \leq h \|f''\|_{L^\infty(0,1)} + 2h^{-1} \|f\|_{L^\infty(0,1)}, \quad (10.2.5)$$

for every $h \in (0, \frac{1}{2}]$. By (10.2.4) and (10.2.5) we get

$$\|f'\|_{L^\infty(0,1)} \leq h \|f''\|_{L^\infty(0,1)} + 2h^{-1} \|f\|_{L^\infty(0,1)}, \quad \forall h \in \left(0, \frac{1}{2}\right]. \quad (10.2.6)$$

Now, set

$$E = \|f''\|_{L^\infty(0,1)} \quad \text{e} \quad \varepsilon = \|f\|_{L^\infty(0,1)} \quad (10.2.7)$$

and let us determine the minimum of the function (in h variable) on the right-hand side of (10.2.6), i.e.

$$\left(0, \frac{1}{2}\right] \ni h \rightarrow \Phi(h) = hE + 2h^{-1}\varepsilon.$$

It turns out that if $(\frac{2\varepsilon}{E})^{1/2} \leq \frac{1}{2}$ then, for $h = h_0 := (\frac{2\varepsilon}{E})^{1/2}$,

$$\min_{[0,1/2]} \Phi = \Phi(h_0) = 2\sqrt{2E\varepsilon}, \quad (10.2.8)$$

while, if $(\frac{2\varepsilon}{E})^{1/2} \geq \frac{1}{2}$ then

$$\min_{[0,1/2]} \Phi = \Phi(1/2) = \frac{1}{2}E + 4\varepsilon,$$

but, since in this case $8\varepsilon \geq E$, we get

$$\min_{[0,1/2]} \Phi \leq 8\varepsilon. \quad (10.2.9)$$

By (10.2.8), (10.2.9) and (10.2.6) we have

$$\|f'\|_{L^\infty(0,1)} \leq 2\sqrt{2E\varepsilon} + 8\varepsilon \leq 8\sqrt{2}(E + \varepsilon)^{1/2}\varepsilon^{1/2}$$

and recalling (10.2.7) we obtain (10.2.1). ■

Remarks

1. By Proposition 10.2.1 it follows the equivalence of the norms

$$\|f\|_{C^2([a,b])} = \|f\|_{L^\infty(a,b)} + (b-a) \|f'\|_{L^\infty(a,b)} + (b-a)^2 \|f''\|_{L^\infty(a,b)},$$

and

$$\|f\| = \|f\|_{L^\infty(a,b)} + (b-a)^2 \|f''\|_{L^\infty(a,b)},$$

(reader check: use the inequality $2AB \leq A^2 + B^2$).

2. Inequality (10.2.1) is a **stability estimate for the calculation of the first derivative** provided we have the a priori information

$$(b-a)^2 \|f''\|_{L^\infty(a,b)} \leq E. \quad (10.2.10)$$

As a matter of fact, if

$$\|f\|_{L^\infty(a,b)} \leq \varepsilon, \quad (10.2.11)$$

then

$$\|f'\|_{L^\infty(a,b)} \leq \frac{c_0}{b-a} (E + \varepsilon)^{1/2} \varepsilon^{1/2}.$$

More precisely, setting

$$\mathcal{K}_E = \left\{ f \in C^2([a, b]) : (b-a)^2 \|f''\|_{L^\infty(a,b)} \leq E \right\}$$

and denoting by ω the modulus of continuity of

$$\mathcal{K}_E \ni f \rightarrow f' \in C^1([a, b]),$$

we have

$$\omega(\varepsilon) \leq \frac{c_0}{b-a} (E + \varepsilon)^{1/2} \varepsilon^{1/2}, \quad \forall \varepsilon > 0. \quad (10.2.12)$$

Also, we observe

$$\omega(\varepsilon) \geq \frac{1}{b-a} (E + \varepsilon)^{1/2} \varepsilon^{1/2}, \quad \forall \varepsilon > 0. \quad (10.2.13)$$

In order to check (10.2.13), for the sake of brevity, let us consider the case $[a, b] = [0, 1]$ and let us denote

$$f_\varepsilon(x) = \frac{E\varepsilon}{E + \varepsilon} \sin \frac{x}{\sqrt{\varepsilon/(\varepsilon + E)}}. \quad (10.2.14)$$

We have

$$\|f_\varepsilon\|_{L^\infty(0,1)} \leq \varepsilon, \quad \|f_\varepsilon''\|_{L^\infty(0,1)} \leq E.$$

Hence $f_\varepsilon \in \mathcal{K}_E$ and we have

$$\omega(\varepsilon) \geq \|f_\varepsilon'\|_{L^\infty(0,1)} = (E + \varepsilon)^{1/2} \varepsilon^{1/2},$$

from which we get (10.2.13).

Inequality (10.2.13) implies that the exponent $\frac{1}{2}$ in stability estimate (10.2.1) is **optimal**, that is it cannot be improved by a bigger exponent.

◆

In what follows we extend Proposition 10.2.1 to higher order derivatives. To this aim we need some notations. Let $a \in \mathbb{R}$. We recall that the translation operator, τ_a , is defined as

$$(\tau_a f)(x) = f(x - a) \quad (10.2.15)$$

where f is any one real variable function. Also we denote

$$\tau_a^* = \tau_{-a}. \quad (10.2.16)$$

Let us denote by I the identity operator. Moreover, if $h \in \mathbb{R}$ let us denote by $\Delta_h f$ the difference operator

$$(\Delta_h f)(x) = ((\tau_h^* - I) f)(x) = f(x + h) - f(x). \quad (10.2.17)$$

Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_j$, $j \in \mathbb{N}_0$, be the polynomials

$$\begin{aligned} \mathbf{e}_0(x) &= 1, \\ \mathbf{e}_1(x) &= x, \\ \mathbf{e}_2(x) &= x(x - h), \\ &\dots \end{aligned}$$

$$\mathbf{e}_j(x) = x(x - h) \cdots (x - (j - 1)h).$$

Let us note that \mathbf{e}_j , has degree j for every $j \in \mathbb{N}_0$. Let $h \neq 0$ and let us denote by P_{n-1} , $n \in \mathbb{N}$, the **Newton interpolation polynomial centered at x_0 and with degree $n - 1$** , that is

$$P_{n-1}(x) = \sum_{j=0}^{n-1} \frac{(\Delta_h^j f)(x_0)}{j! h^j} \mathbf{e}_j(x - x_0). \quad (10.2.18)$$

Let us notice that if $1 \leq j \leq n - 1$ and $s \in \{0, 1, \dots, n - 1\}$ then we have

$$\mathbf{e}_j(sh) = \begin{cases} 0, & \text{for } 0 \leq s \leq j - 1, \\ \binom{s}{j} j! h^j, & \text{for } j \leq s \leq n - 1. \end{cases} \quad (10.2.19)$$

We check (10.2.19). If $0 \leq s \leq j - 1$ then one of the factors of $\mathbf{e}_j(sh)$ vanishes. While, if $j \leq s \leq n - 1$ (where $1 \leq j \leq n - 1$) we have

$$\begin{aligned} \mathbf{e}_j(sh) &= (sh)(sh - h) \cdots (sh - (sh - (j - 1)h)) = \\ &= h^j s(s - 1) \cdots (h - (s - (j - 1))) = \\ &= h^j \binom{s}{j} j!. \end{aligned}$$

Also, let us note that if $0 \leq s \leq n-1$, we get

$$f(x_0 + sh) = ((\tau_h^*)^s f)(x_0) \quad (10.2.20)$$

and

$$(\tau_h^*)^s = (\Delta_h + I)^s = \sum_{j=0}^{n-1} \binom{s}{j} \Delta_h^j. \quad (10.2.21)$$

Hence (10.2.20) and (10.2.21) yield

$$\begin{aligned} f(x_0 + sh) &= \sum_{j=0}^{n-1} \binom{s}{j} (\Delta_h^j f)(x_0) = \\ &= \sum_{j=0}^{n-1} \frac{1}{j!h^j} (\Delta_h^j f)(x_0) \mathbf{e}_j(x_0 + sh - x_0) = \\ &= P_{n-1}(x_0 + sh). \end{aligned}$$

All in all, we have

$$f(x_0 + sh) = P_{n-1}(x_0 + sh), \quad \text{per } s = 0, 1, \dots, n-1. \quad (10.2.22)$$

We can now state and prove the following

Proposition 10.2.2. *Let $n \geq 2$ and $f \in C^n([a, b])$, where $a, b \in \mathbb{R}$, $a < b$. For $1 \leq k \leq n-1$ we have*

$$\begin{aligned} \|f^{(k)}\|_{L^\infty(a,b)} &\leq \\ &\leq c_{k,n} \left(((b-a)^{-n} \|f\|_{L^\infty(a,b)} + \|f^{(n)}\|_{L^\infty(a,b)} \right)^{\frac{k}{n}} \|f\|_{L^\infty(a,b)}^{1-\frac{k}{n}}, \end{aligned} \quad (10.2.23)$$

where $c_{k,n}$ is a positive constant which depends on k and n only.

Proof. As in the proof of Proposition 10.2.1 we may reduce to the case $[a, b] = [0, 1]$. We begin to prove (10.2.23) when $k = n-1$. Let $f \in C^n([0, 1])$. Fix $x_0 \in [0, \frac{1}{2}]$ and let $h \in (0, \frac{1}{2(n-1)}]$ and set

$$R(x) = f(x) - P_{n-1}(x), \quad x \in [0, 1].$$

By (10.2.22) we have

$$R(x_0) = R(x_0 + h) = \dots = R(x_0 + (n-1)h) = 0$$

from which, By repeatedly applying the Rolle Theorem, we have that there exists $\xi \in (x_0, x_0 + (n-1)h)$ such that

$$f^{(n-1)}(\xi) - P_{n-1}^{(n-1)}(\xi) = R^{(n-1)}(\xi) = 0.$$

Therefore there exists $\xi \in (x_0, x_0 + (n-1)h)$ such that

$$f^{(n-1)}(\xi) = P_{n-1}^{(n-1)}(\xi). \quad (10.2.24)$$

On the other hand we have

$$P_{n-1}^{(n-1)}(x) = \frac{\mathbf{e}_{n-1}^{(n-1)}(x-x_0) (\Delta_h^{n-1} f)(x_0)}{(n-1)! h^{n-1}},$$

and

$$\mathbf{e}_{n-1}^{(n-1)}(x) = (n-1)! .$$

By the latter and by (10.2.24) we have

$$f^{(n-1)}(\xi) = \frac{(\Delta_h^{n-1} f)(x_0)}{h^{n-1}}. \quad (10.2.25)$$

Therefore

$$f^{(n-1)}(x_0) = (f^{(n-1)}(x_0) - f^{(n-1)}(\xi)) + \frac{(\Delta_h^{n-1} f)(x_0)}{h^{n-1}} \quad (10.2.26)$$

and by the Lagrange Theorem, we get

$$\begin{aligned} |f^{(n-1)}(x_0) - f^{(n-1)}(\xi)| &\leq \|f^{(n)}\|_{L^\infty(0,1)} |x_0 - \xi| \leq \\ &\leq \|f^{(n)}\|_{L^\infty(0,1)} (n-1)h. \end{aligned} \quad (10.2.27)$$

Moreover

$$\begin{aligned} (\Delta_h^{n-1} f)(x_0) &= ((\tau_h^* - I)^{n-1} f)(x_0) = \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} f(x_0 + jh) \end{aligned} \quad (10.2.28)$$

therefore

$$|(\Delta_h^{n-1} f)(x_0)| \leq \sum_{j=0}^{n-1} \binom{n-1}{j} |f(x_0 + jh)| \leq 2^{n-1} \|f\|_{L^\infty(0,1)}. \quad (10.2.29)$$

By (10.2.26), (10.2.27) and (10.2.29) we have

$$|f^{(n-1)}(x_0)| \leq \|f^{(n)}\|_{L^\infty(0,1)} (n-1)h + \left(\frac{2}{h}\right)^{n-1} \|f\|_{L^\infty(0,1)}. \quad (10.2.30)$$

Applying (10.2.30) to $f(1-x)$ we also obtain the estimate for $x_0 \in [\frac{1}{2}, 1]$. All in all, we have for $0 < h \leq \frac{1}{2(n-1)}$,

$$\|f^{(n-1)}\|_{L^\infty(0,1)} \leq E(n-1)h + \left(\frac{2}{h}\right)^{n-1} \varepsilon, \quad (10.2.31)$$

here we set

$$E := \|f^{(n)}\|_{L^\infty(0,1)}, \quad \varepsilon := \|f\|_{L^\infty(0,1)}.$$

Now we find the minimum of the function

$$\left(0, \frac{1}{2(n-1)}\right] \ni \rightarrow \Phi(h) = E(n-1)h + \left(\frac{2}{h}\right)^{n-1} \varepsilon.$$

By elementary calculation we have

$$\min_{\left(0, \frac{1}{2(n-1)}\right]} \Phi \leq c_n (E + \varepsilon)^{1-\frac{1}{n}} \varepsilon^{\frac{1}{n}},$$

where c_n depends on n only. Hence

$$\|f^{(n-1)}\|_{L^\infty(0,1)} \leq c_n \left(\|f^{(n)}\|_{L^\infty(0,1)} + \|f\|_{L^\infty(0,1)} \right)^{1-\frac{1}{n}} \|f\|_{L^\infty(0,1)}^{\frac{1}{n}}. \quad (10.2.32)$$

Now, let $1 \leq k \leq n-1$, by iteration of (10.2.32) we obtain

$$\begin{aligned} \|f^{(k)}\|_{L^\infty(0,1)} &\leq c_{k+1} \left(\|f^{(k+1)}\|_{L^\infty(0,1)} + \|f\|_{L^\infty(0,1)} \right)^{\frac{k}{k+1}} \|f\|_{L^\infty(0,1)}^{\frac{1}{k+1}} \leq \\ &\leq c_{k+1} c_{k+2}^{\frac{1}{k+2}} \left(\|f^{(k+2)}\|_{L^\infty(0,1)} + \|f\|_{L^\infty(0,1)} \right)^{\frac{k}{k+2}} \|f\|_{L^\infty(0,1)}^{\frac{2}{k+2}} \leq \\ &\leq \dots \leq \\ &\leq c_{k,n} \left(\|f^{(n)}\|_{L^\infty(0,1)} + \|f\|_{L^\infty(0,1)} \right)^{\frac{k}{n}} \|f\|_{L^\infty(0,1)}^{1-\frac{k}{n}}. \end{aligned}$$

where $c_{k,n}$ depends on k and n only. By the above obtained inequality, coming back $[a, b]$ we get (10.2.23). ■

Estimates like (10.2.23) can be easily derived for L^p norm, $1 \leq p \leq \infty$.
Let

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \right)^{1/p}. \quad (10.2.33)$$

We have

Proposition 10.2.3. *Let $n \geq 2$ and $f \in C^n([a, b])$, where $a, b \in \mathbb{R}$, $a < b$. For $1 \leq k \leq n-1$ we have*

$$\begin{aligned} \|f^{(k)}\|_{L^p(a,b)} &\leq \\ &\leq c_{k,n} \left(((b-a)^{-n} \|f\|_{L^p(a,b)} + \|f^{(n)}\|_{L^p(a,b)})^{\frac{k}{n}} \|f\|_{L^p(a,b)}^{1-\frac{k}{n}} \right), \end{aligned} \quad (10.2.34)$$

where $c_{k,n}$ is a positive constant which depends on k and n only.

Proof. Similarly to the proof of the previous Proposition, we may reduce to the case $[a, b] = [0, 1]$. Let us prove (10.2.34) when $k = n-1$. Let $t \in [0, \frac{1}{2}]$ $h \in (0, \frac{1}{2(n-1)}]$, by (10.2.26) (with t replacing x_0) we have

$$f^{(n-1)}(t) = (f^{(n-1)}(t) - f^{(n-1)}(\xi)) + \frac{(\Delta_h^{n-1} f)(t)}{h^{n-1}}. \quad (10.2.35)$$

Set

$$\widetilde{f^{(n)}}(\tau) = \begin{cases} f^{(n)}(\tau), & \text{for } \tau \in [0, 1], \\ 0, & \text{for } \tau \notin [0, 1]. \end{cases}$$

We have

$$\begin{aligned} |f^{(n-1)}(t) - f^{(n-1)}(\xi)| &= \left| \int_{\xi}^t f^{(n)}(\tau) d\tau \right| \leq \\ &\leq \int_t^{t+(n-1)h} |f^{(n)}(\tau)| d\tau = \int_{\mathbb{R}} |\widetilde{f^{(n)}}(\tau)| \chi_{(0, (n-1)h)}(\tau - t) d\tau, \end{aligned}$$

where $\chi_{(0, (n-1)h)}$ is the characteristic function $(0, (n-1)h)$. Hence, by (10.2.35), for $t \in [0, \frac{1}{2}]$ and $h \in (0, \frac{1}{2(n-1)}]$, we have

$$|f^{(n-1)}(t)| \leq \int_{\mathbb{R}} |\widetilde{f^{(n)}}(\tau)| \chi_{(0, (n-1)h)}(\tau - t) d\tau + \frac{(\Delta_h^{n-1} f)(t)}{h^{n-1}}. \quad (10.2.36)$$

At this point we use the triangle inequality in L^p and the Young inequality for convolutions:

$$\|F \star G\|_{L^p(\mathbb{R})} \leq \|F\|_{L^p(\mathbb{R})} \|G\|_{L^1(\mathbb{R})}$$

where $F = \widetilde{f^{(n)}}$, $G = \chi_{(0,(n-1)h)}$ and we get

$$\left(\int_0^{1/2} |f^{(n-1)}(t)|^p dt \right)^{1/p} \leq (n-1)h \|f^{(n)}\|_{L^p(0,1)} + \left(\frac{2}{h}\right)^{n-1} \|f\|_{L^p(0,1)}.$$

Similarly, we have

$$\left(\int_{1/2}^1 |f^{(n-1)}(t)|^p dt \right)^{1/p} \leq (n-1)h \|f^{(n)}\|_{L^p(0,1)} + \left(\frac{2}{h}\right)^{n-1} \|f\|_{L^p(0,1)}.$$

Hence, by $h \in \left(0, \frac{1}{2^{(n-1)}}\right]$, we have

$$\|f^{(n-1)}\|_{L^p(0,1)} \leq (n-1)h \|f^{(n)}\|_{L^p(0,1)} + \left(\frac{2}{h}\right)^{n-1} \|f\|_{L^p(0,1)}.$$

From now on we proceed as in the proof of Proposition 10.2.2. ■

Remark 4. It can be proved (see exercise below) that the exponent $1 - \frac{k}{n}$ of the estimate (10.2.23) is optimal. Regarding the constant we report here, without a proof (we refer to [30]), the following sharp estimate ($k, m \in \mathbb{N}_0$, $m > 0$)

$$\|f^{(k)}\|_{L^\infty(a,b)} \leq 4e^{2k} m^k \|f\|_{L^\infty(a,b)}^{1-\frac{1}{m}} M_{km}^{\frac{1}{m}},$$

where

$$M_{km} = \max \left\{ \frac{(km)!}{(b-a)^{nm}} \|f\|_{L^\infty(a,b)}^{1-\frac{1}{m}}, \|f^{(km)}\|_{L^\infty(a,b)} \right\}.$$

◆

Exercise 1. Prove the optimality of the exponent $1 - \frac{k}{n}$ in inequalities (10.2.23) and (10.2.34). (hint: note that inequalities (10.2.23) and (10.2.34) hold for complex-valued functions. After that, instead of trigonometric functions like (10.2.14) use complex exponential). ♣

Exercise 2. (i) Let $0 < \alpha < \beta \leq 1$. Prove that for every $f \in C^{0,\beta}([a, b])$ we have

$$|f|_{\alpha,[a,b]} \leq C(b-a)^{-\alpha} \left[(b-a)^\beta |f|_{\beta,[a,b]} + \|f\|_{L^\infty(a,b)} \right]^{\frac{\alpha}{\beta}} \|f\|_{L^\infty(a,b)}^{1-\frac{\alpha}{\beta}},$$

where C depends on α and β only.

(ii) Let $0 < \alpha \leq 1$. Prove that for every $f \in C^{1,\alpha}([a, b])$ we have

$$\|f'\|_{L^\infty(a,b)} \leq C(b-a)^{-1} \left[(b-a)^{1+\alpha} |f'|_{\alpha,[a,b]} + \|f\|_{L^\infty(a,b)} \right]^{\frac{1}{1+\alpha}} \|f\|_{L^\infty(a,b)}^{\frac{\alpha}{1+\alpha}},$$

where C depends on α only.

hint to (ii): note that instead of (10.2.3) we have

$$\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| \leq |f|_{\alpha,[a,b]} h^\alpha.$$



We conclude this Section with two estimates for the derivatives of several variables functions.

Proposition 10.2.4. *Let $f \in C^2(\overline{B_1})$. We have*

$$\|\nabla f\|_{L^\infty(B_1)} \leq c \left(\|\partial^2 f\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} \right)^{\frac{1}{2}} \|f\|_{L^\infty(B_1)}^{\frac{1}{2}}, \quad (10.2.37)$$

where c is a positive constant which depends on n only.

Proof. Let $h \in (0, 1]$ be to choose later on and let $x \in B_1$. Set

$$\Omega_h(x) = B_h(x) \cap B_1.$$

For $j = 1, \dots, n$ we have

$$\begin{aligned} f_{x_j}(x) &= f_{x_j}(x) - \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} f_{x_j}(y) dy + \\ &+ \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} f_{x_j}(y) dy = \\ &= \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} (f_{x_j}(x) - f_{x_j}(y)) dy + \\ &+ \frac{1}{|\Omega_h(x)|} \int_{\partial\Omega_h(x)} f(y) \nu_j dS. \end{aligned} \quad (10.2.38)$$

We have, for a suitable \bar{x} on the segment of extremes x and y ,

$$f_{x_j}(x) - f_{x_j}(y) = \nabla f_{x_j}(\bar{x}) \cdot (x - y), \quad (10.2.39)$$

in addition we have

$$|\Omega_h(x)| \geq C_1 h^n, \quad |\partial\Omega_h(x)| \geq C_2 h^{n-1}, \quad (10.2.40)$$

where C_1 and C_2 depend on n only. From what was obtained in (10.2.38), (10.2.39) and (10.2.40) we get

$$|\nabla f(x)| \leq C \left(h \|\partial^2 f\|_{L^\infty(B_1)} + h^{-1} \|f\|_{L^\infty(B_1)} \right),$$

where C depends on n only. Now we minimize the function on the right-hand side of (10.2.40) and we obtain (10.2.37). ■

Proposition 10.2.5. *Let $f \in C^1(\overline{B_1})$. We have*

$$\|f\|_{L^\infty(B_1)} \leq c \left(\|\nabla f\|_{L^\infty(B_1)} + \|f\|_{L^2(B_1)} \right)^{\frac{n}{n+2}} \|f\|_{L^2(B_1)}^{\frac{n}{n+2}}, \quad (10.2.41)$$

where c depends on n only.

Proof. Let $h \in (0, 1]$ be to choose, $x \in B_1$ and $\Omega_h(x)$ like in the previous proof. We have

$$f(x) = \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} (f(x) - f(y)) dy + \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} f(y) dy. \quad (10.2.42)$$

By

$$|f(x) - f(y)| \leq \|\nabla f\|_{L^\infty(B_1)} |x - y|$$

we have

$$\frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} |f(x) - f(y)| dy \leq h \|\nabla f\|_{L^\infty(B_1)}. \quad (10.2.43)$$

On the other hand, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left| \frac{1}{|\Omega_h(x)|} \int_{\Omega_h(x)} |f(y)| dy \right| &\leq \frac{1}{|\Omega_h(x)|^{1/2}} \|f\|_{L^2(B_1)} \leq \\ &\leq \frac{1}{(c_1 h^n)^{1/2}} \|f\|_{L^2(B_1)}. \end{aligned} \quad (10.2.44)$$

Hence

$$|f(x)| \leq C \left(h^{-n/2} \|f\|_{L^2(B_1)} + h \|\nabla f\|_{L^\infty(B_1)} \right), \quad (10.2.45)$$

where C depends on n only. Now we minimize the function on the right-hand side of (10.2.45) and we get (10.2.41). ■

10.3 Stability estimates for the continuation of holomorphic functions

In what follows we will identify \mathbb{C} with \mathbb{R}^2 . Let us recall very quickly the definition and some properties of the holomorphic functions.

1. Let Ω be an open set of \mathbb{C} , $f : \Omega \rightarrow \mathbb{C}$ be a complex-valued function defined in Ω and $z_0 \in \Omega$. We say that f is **holomorphic in** z_0 if the following limit there exists (in \mathbb{C})

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}. \quad (10.3.1)$$

In such a case we denote by $f'(z_0)$ the value of limit (10.3.1) and we say that it is the **derivative of f in** z_0 . We say that f is **holomorphic in** Ω provided it is holomorphic in each point of Ω . For instance, z, z^n are holomorphic functions in \mathbb{C} while $\Re z, \Im z, \bar{z}$ are not.

2. From what we say in **1** it follows that if f is holomorphic in $z_0 = x_0 + iy_0$ then f , as a function of the real variables x and y , is differentiable in (x_0, y_0) and

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0, \quad \text{in } (x_0, y_0). \quad (10.3.2)$$

Denoting by $P = \Re f$, $Q = \Im f$, (10.3.2) we may write

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \quad \text{in } (x_0, y_0). \quad (10.3.3)$$

Equations (10.3.2) and (10.3.3) are known as the **Cauchy-Riemann equations (or condition)**. By introducing the notations

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad (10.3.4)$$

the Cauchy-Riemann conditions can be written as

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad \text{in } (x_0, y_0). \quad (10.3.5)$$

Also we have, setting $dz = dx + idy$, $d\bar{z} = dx - idy$ and by considering f as a function of z and \bar{z}

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \quad (10.3.6)$$

3. The Cauchy Theorem. If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic in Ω then the differential form

$$f(z)dz = f(x + iy)dx + if(x + iy)dy$$

is locally exact. That is, for every $(x_0, y_0) \in \Omega$ there exist $\delta > 0$ and

$$F : B_\delta(x_0, y_0) \rightarrow \mathbb{C},$$

F differentiable in $B_\delta(x_0, y_0)$ such that

$$\frac{\partial F}{\partial x} = f, \quad \frac{\partial F}{\partial y} = if.$$

4. It can be proved that if $f \in C^0(\Omega)$ is holomorphic in $\Omega \setminus L$, where L is a straight line then fdz is holomorphic in Ω . In particular, if $f \in C^0(\Omega)$ and f is holomorphic in $\Omega \setminus \{a\}$ where $a \in \Omega$, then fdz is locally exact. This in turn enables the proof of the

Cauchy integral formula. Let f be holomorphic in Ω . Let $a \in \Omega$ and $r > 0$ satisfy $\overline{B_r(a)} \subset \Omega$. Setting $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi)$, we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz. \quad (10.3.7)$$

5. The Cauchy formula implies that if f is holomorphic in B_ρ then f can be expanded in a power series in B_ρ , that is there exists $\{a_n\}_{n \geq 0}$, sequence of \mathbb{C} , such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in B_\rho. \quad (10.3.8)$$

Since a holomorphic function can be expanded in a power series in each point of an open set Ω , we have $f : \Omega \rightarrow \mathbb{C}$ is holomorphic in Ω if and only if f is a complex analytic function in Ω , i.e. if and only if for every $a \in \Omega$ there exists δ such that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n, \quad \forall z \in B_\delta(a). \quad (10.3.9)$$

The "if ... then" part of the equivalence follows immediately by the properties of differentiability of power series and by (10.3.1). Keep in mind, however, that the expression "analytic complex function" should not be confused with the expression "analytic complex-valued function". For instance

$f(z, \bar{z}) = z^2 - \bar{z}^2$ is complex-valued analytic function, but not analytic complex function, as it is not holomorphic.

6. From what we have said in **4**, we get the converse of the Cauchy Theorem. That is to say: if $f \in C^0(\Omega)$ and $f(z)dz$ is locally exact in Ω then f is holomorphic in Ω . As a matter of fact, if $f(z)dz$ is locally exact in Ω then it has locally a primitive hence, there exists locally, $F \in C^1$ such that $\frac{\partial F}{\partial x} = f$, $\frac{\partial F}{\partial y} = if$ from which we have that F satisfies the Cauchy-Riemann conditions, hence F is holomorphic and $f = F'$, on the other hand since the derivative of a complex analytic function is still a complex analytic function, hence holomorphic, $f = F'$ is holomorphic.

We also have that if $f \in C^0(\Omega)$ and $\overline{B_r(a)} \subset \Omega$ then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz, \quad \forall n \in \mathbb{N}_0, \quad (10.3.10)$$

where $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi)$. Moreover, again by (10.3.7), we obtain the **mean property**

$$f(a) = \frac{n!}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt, \quad (10.3.11)$$

as soon as $\overline{B_r(a)} \subset \Omega$. From the mean property it follows the

Maximum modulus principle. Let $\Omega \subset \mathbb{C}$ a bounded open set and $f \in C^0(\overline{\Omega})$ be a holomorphic function in Ω , then

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|.$$

Moreover, if Ω is connected and there exists $a \in \Omega$ such that

$$|f(a)| = \max_{\overline{\Omega}} |f|$$

then f is constant in Ω .

7. Let us now return our attention to the analyticity of holomorphic functions and let us recall what follows. If

$$f : \Omega \rightarrow \mathbb{C},$$

is holomorphic in Ω , connected open set of \mathbb{C} , then if $a \in \Omega$ we have

$$f^{(n)}(a) = 0, \quad \forall n \in \mathbb{N}_0 \implies f \equiv 0 \quad \text{in } \Omega. \quad (10.3.12)$$

From which we have that, if D is nonempty open set contained in Ω , then

$$f = 0, \quad \text{in } D \implies f \equiv 0 \quad \text{in } \Omega. \quad (10.3.13)$$

Moreover, if f does not vanish identically in Ω then **the set of zeros of f has no accumulation points in Ω** . As a matter of fact, if f does not vanish identically in Ω then (10.3.12) implies that for every $a \in \Omega$ there exists $k \in \mathbb{N}_0$ such that

$$f^{(k)}(a) \neq 0.$$

Denoting by $k_0 \in \mathbb{N}_0$ the minimum of such k , (10.3.9) gives

$$f(z) = (z - a)^{k_0} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^{n-k_0} := (z - a)^{k_0} \varphi(z), \quad \forall z \in B_\delta(a),$$

where $\varphi(a) \neq 0$, hence $f(z) \neq 0$ in $B_{\delta_1}(a) \setminus \{a\}$ for some $\delta_1 > 0$. The above point can also be expressed in the following way: let

$$\{z_n : n \in \mathbb{N}\}$$

a **infinite set which has at least an accumulation point in Ω** , then

$$f(z_n) = 0, \quad \forall n \in \mathbb{N} \implies f \equiv 0 \quad \text{in } \Omega \quad (10.3.14)$$

or also

$$f(z) = \mathcal{O}(|z - a|^N), \quad \forall N \in \mathbb{N} \implies f \equiv 0 \quad \text{in } \Omega. \quad (10.3.15)$$

Let us notice that (10.3.14) does not hold if $\{z_n : n \in \mathbb{N}\}$ has accumulation points on $\partial\Omega$ only. Let us consider, for instance,

$$\Omega = \{x + iy : x > 0, \quad |y| < x\}$$

and let

$$f(z) = e^{-\frac{1}{z}}.$$

We have $f(z) = \mathcal{O}(|z|^N)$, for every $N \in \mathbb{N}$, but $f \not\equiv 0$.

8. In this concluding part of this summary, we prove

Proposition 10.3.1. *Let Ω be a connected open set of \mathbb{C} . Let us suppose that*

$$\frac{o}{\Omega} = \Omega. \quad (10.3.16)$$

Let $z_0 \in \partial\Omega$ and $\Gamma = \partial\Omega \cap B_R(z_0) \neq \emptyset$, where $R > 0$. In addition, Let $f \in C^0(\Omega \cup \bar{\Gamma})$, f be holomorphic in Ω which satisfies

$$f = 0 \quad \text{on } \Gamma, \tag{10.3.17}$$

then $f \equiv 0$ in Ω .

Proof. In what follows we will need some simple topological relationships that we will prove (for the convenience of the reader) in the concluding part of the main proof. By (10.3.16) we have immediately

$$\partial\Omega = \bar{\Omega} \cap \overline{(\mathbb{C} \setminus \bar{\Omega})}. \tag{10.3.18}$$

Now, let us fix $\delta \in (0, \frac{R}{4})$. Since $z_0 \in \partial\Omega$, we have by (10.3.18)

$$B_\delta(z_0) \cap \Omega \neq \emptyset, \quad B_\delta(z_0) \cap (\mathbb{C} \setminus \bar{\Omega}) \neq \emptyset. \tag{10.3.19}$$

Now, let a and b be such that

$$b \in B_\delta(z_0) \cap \Omega, \quad a \in B_\delta(z_0) \cap (\mathbb{C} \setminus \bar{\Omega}).$$

Since

$$|b - a| \leq |b - z_0| + |z_0 - a| < 2\delta < R - 2\delta,$$

we get

$$b \in B_{R-2\delta}(a) \cap \Omega \subset B_R(z_0) \cap \Omega,$$

in particular $B_{R-2\delta}(a) \cap \Omega \neq \emptyset$, (the second inclusion relationship follows from the triangle inequality). Let us denote by

$$r = R - 2\delta.$$

We have also (see "concluding part")

$$\partial(B_r(a) \cap \Omega) \setminus \Gamma \subset (\partial B_r(a)) \cap \bar{\Omega}. \tag{10.3.20}$$

Set now

$$\varphi(z) = (z - a)^{-n} f(z), \quad n \in \mathbb{N}. \tag{10.3.21}$$

It turns out that φ is holomorphic in $B_r(a) \cap \Omega$ and continuous in $\overline{B_r(a) \cap \Omega}$. By the maximum modulus principle we obtain

$$|\varphi(z)| \leq \max_{\partial(B_r(a) \cap \Omega)} |\varphi|, \quad \forall z \in B_r(a) \cap \Omega \tag{10.3.22}$$

and by (10.3.20) we get, recalling that $f = 0$ on Γ ,

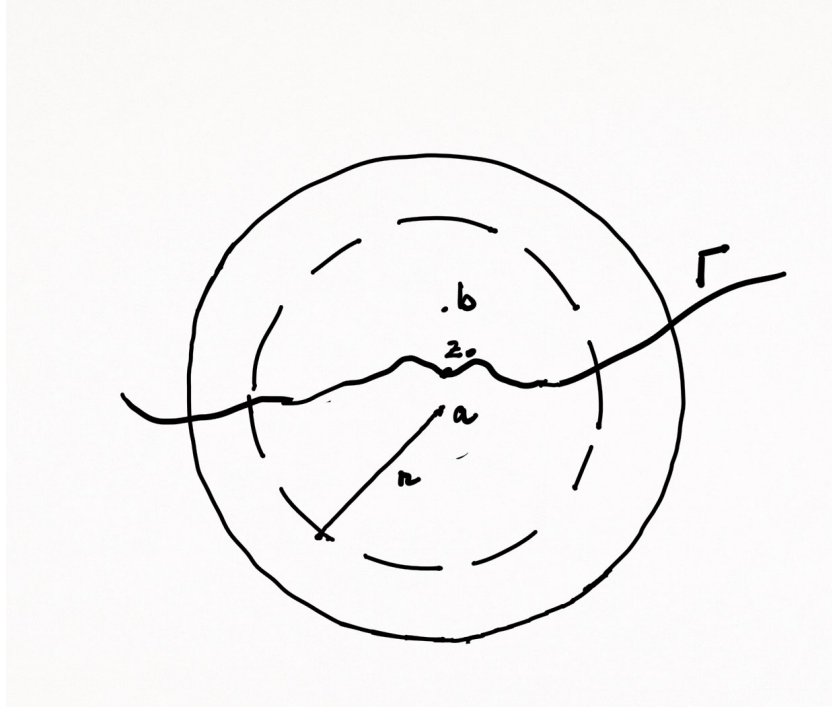


Figure 10.1:

$$\begin{aligned}
 \max_{\partial(B_r(a) \cap \Omega)} |\varphi| &\leq \max_{\partial(B_r(a) \cap \Omega) \setminus \Gamma} |\varphi| + \max_{\Gamma} |\varphi| \leq \\
 &\leq \max_{\partial(B_r(a)) \cap \bar{\Omega}} |\varphi| = \max_{\partial(B_r(a)) \cap \bar{\Omega}} (|z - a|^{-n} |f|) = \\
 &= r^{-n} \max_{\partial(B_r(a)) \cap \bar{\Omega}} |f|.
 \end{aligned} \tag{10.3.23}$$

Now let $z \in B_r(a) \cap \Omega$, by (10.3.21), (10.3.22) and (10.3.23) we derive

$$|f(z)| \leq \left(\frac{|z - a|}{r} \right)^n \max_{\partial(B_r(a)) \cap \bar{\Omega}} |f|, \quad \forall n \in \mathbb{N} \tag{10.3.24}$$

and passing to the limit as n goes to infinity we deduce

$$f(z) = 0, \quad \forall z \in B_r(a) \cap \Omega.$$

Since $B_r(a) \cap \Omega$ is a nonempty open set and Ω is a connected open set, by (10.3.13) we have $f \equiv 0$ in Ω .

Concluding part of the proof. We prove (10.3.20). Let us recall

$$\partial(A \cap B) \subset \partial A \cup \partial B. \quad (10.3.25)$$

Let now

$$x \in \partial(B_r(a) \cap \Omega) \setminus \Gamma,$$

we wish to prove that

$$x \in (\partial B_r(a)) \cap \overline{\Omega}. \quad (10.3.26)$$

First, we have trivially

$$\partial(B_r(a) \cap \Omega) \setminus \Gamma \subset \overline{\Omega} \quad (10.3.27)$$

By (10.3.25) we have

$$x \in (\partial B_r(a)) \cup \partial \Omega \quad \text{and} \quad x \notin \Gamma. \quad (10.3.28)$$

Now, by (10.3.28) we have that if $x \notin \partial B_r(a)$ then $x \in \partial \Omega$. Moreover we have $x \in B_r(a)$. Because, if $x \notin B_r(a)$, as $x \notin \partial B_r(a)$, we would have $x \notin \overline{B_r(a)}$, hence it would exist $\rho > 0$ such that $B_\rho(x) \cap B_r(a) = \emptyset$. Consequently, for such ρ we would have $B_\rho(x) \cap (B_r(a) \cap \Omega) = \emptyset$ which contradicts $x \in \partial(B_r(a) \cap \Omega)$.

All in all, if $x \notin \partial B_r(a)$ then

$$x \in \partial \Omega \cap B_r(a) \subset \partial \Omega \cap B_R(z_0) = \Gamma,$$

But this cannot occur because, by (10.3.28), $x \notin \Gamma$. Therefore, (10.3.28) implies that

$$x \in \partial B_r(a).$$

Finally, since (10.3.27) holds we get (10.3.26), hence (10.3.20) is proved. ■

Remark. Assumption $\frac{\partial}{\partial \Omega} = \Omega$ excludes, for instance, that $\Omega = B_1 \setminus \{z_1, \dots, z_n\}$ where $z_j \in B_R$ for $j = 1, \dots, n$, where $R < 1$. In this case, Proposition 10.3.1 does not hold for $R < 1$ and $\Gamma = \{z_1, \dots, z_n\}$. ♦

10.4 The Hadamard three circle inequality and other examples of stability estimates.

In the previous Section we focused on the unique continuation property for the holomorphic functions. As we have seen it takes on several facets corresponding to (10.3.12)–(10.3.15) and to Proposition 10.3.1. In particular, we have that if a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ is known in a set $D \subset \Omega$

which admits at least one accumulation point in Ω and if Ω is connected, then f is uniquely determined in Ω . The problem of determining effectively the values of f on Ω from $f|_D$ has an interest in applications. However this problem is not well posed in the sense of Hadamard as can be inferred from the following simple example.

Example 1.

Let $\Omega = B_1$ and $D = B_r$ where $r \in (0, 1)$. Then any holomorphic function on B_1 is uniquely determined by $f|_{B_r}$. Nevertheless, small errors in the evaluation of $f|_{B_r}$ may produce uncontrollable errors on f . Let indeed

$$f_n(z) = \frac{1}{n} \left(\frac{z}{r} \right)^n, \quad n \in \mathbb{N}.$$

We have

$$\max_{\overline{B_r}} |f_n| = \frac{1}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

on the other hand, if $|z| > r$, then we have

$$|f_n(z)| = \frac{1}{n} \left(\frac{|z|}{r} \right)^n \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$



The conditional stability question for the analytic extension problem may be formulated as follows

Let Ω be a connected open set of \mathbb{C} and $D \subset \Omega$ which has at least one accumulation point in Ω . Let f be any holomorphic function in Ω , continuous on $\overline{\Omega}$, which satisfies

$$\max_{\overline{\Omega}} |f| \leq E \tag{10.4.1}$$

and

$$\max_D |f| \leq \varepsilon. \tag{10.4.2}$$

We are interested in finding a stability estimate like the following one

$$|f(z)| \leq E\eta\left(\frac{\varepsilon}{E}; z\right), \quad \forall z \in \Omega, \tag{10.4.3}$$

where

$$\eta(s; z) \rightarrow 0 \quad \text{as } s \rightarrow 0, \quad \forall z \in \Omega.$$

There is a fairly general treatment of estimates of stability (10.4.3), but here we will examine only a few examples that are particularly significant.

Example 2: The Hadamard three circle inequality.

Let $0 < r < \rho < R$. Let f be a holomorphic function in B_R and continuous in $\overline{B_R}$. Let us denote by

$$M(s) := \max_{\overline{B_s}} |f|, \quad \text{for } 0 < s \leq R. \quad (10.4.4)$$

We have

$$M(\rho) \leq (M(r))^{\theta_0} (M(R))^{1-\theta_0}, \quad (10.4.5)$$

where

$$\theta_0 = \frac{\log \frac{R}{\rho}}{\log \frac{R}{r}}. \quad (10.4.6)$$

Proof of (10.4.5).

Let n and m be two integer numbers, $m > 0$. Let us consider the function

$$F(z) = z^{-n} (f(z))^m, \quad \text{for } z \in B_R \setminus \{0\}. \quad (10.4.7)$$

F is holomorphic in $B_R \setminus \{0\}$ and it is continuous in $\overline{B_R} \setminus B_r$. We can apply the maximum modulus principle. Set

$$\widetilde{M}(s) := \max_{\partial B_s} |f|, \quad \text{for } 0 < s \leq R.$$

We have, for any $\rho \in (r, R)$,

$$\begin{aligned} \rho^{-n} \left(\widetilde{M}(\rho) \right)^m &= \max_{\partial B_\rho} |F| \leq \\ &\leq \max \left\{ \max_{\partial B_r} |F|, \max_{\partial B_R} |F| \right\} = \\ &= \max \left\{ r^{-n} \left(\widetilde{M}(r) \right)^m, R^{-n} \left(\widetilde{M}(R) \right)^m \right\}, \end{aligned}$$

which gives

$$\widetilde{M}(\rho) \leq \max \left\{ \left(\frac{\rho}{r} \right)^{\frac{n}{m}} \widetilde{M}(r), \left(\frac{\rho}{R} \right)^{\frac{n}{m}} \widetilde{M}(R) \right\}. \quad (10.4.8)$$

Since \mathbb{Q} is dense in \mathbb{R} , by (10.4.8) we have

$$\widetilde{M}(\rho) \leq \max \left\{ \left(\frac{\rho}{r} \right)^\alpha \widetilde{M}(r), \left(\frac{\rho}{R} \right)^\alpha \widetilde{M}(R) \right\}, \quad \forall \alpha \in \mathbb{R}. \quad (10.4.9)$$

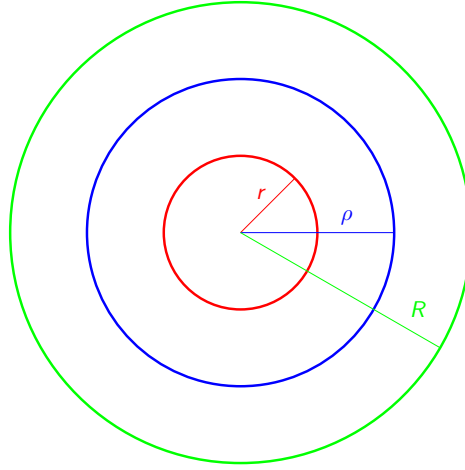


Figure 10.2:

Now, let us choose α in such a way that

$$\left(\frac{\rho}{r}\right)^\alpha \widetilde{M}(r) = \left(\frac{\rho}{R}\right)^\alpha \widetilde{M}(R),$$

that is, let

$$\alpha = \frac{\log\left(\frac{\widetilde{M}(R)}{\widetilde{M}(r)}\right)}{\log\frac{R}{\rho}}$$

and (10.4.9) implies

$$\widetilde{M}(\rho) \leq \left(\widetilde{M}(r)\right)^{\theta_0} \left(\widetilde{M}(R)\right)^{1-\theta_0}, \quad (10.4.10)$$

where θ_0 is given by (10.4.6). Finally, by the maximum modulus principle, we get (10.4.5). ■

Remarks

1. If f is holomorphic in $B_R \setminus \overline{B_r}$ and it is continuous in $\overline{B_R} \setminus B_r$, inequality (10.4.10) still applies.
2. It is evident that (10.4.5) is a stability estimate for the problem:
Determine $f \in C^0(\overline{B_R})$, f holomorphic in B_R which satisfies

$$\max_{\overline{B_r}} |f| \leq \varepsilon$$

and

$$\max_{\overline{B_R}} |f| \leq E.$$

3. Let us notice that inequality (10.4.5) is equivalent to the convexity of the function

$$t \rightarrow \log M(e^t).$$

4. The inequality (10.4.5) cannot be improved. More precisely, the following facts apply.

For every $C > 0$ independent of f we have

$$\theta_0 = \sup \left\{ \theta : M(\rho) \leq C (M(r))^\theta, \quad M(R) = 1 \right\}. \quad (10.4.11)$$

In other words, the exponent θ_0 in (10.4.5) is the best exponent. Moreover

$$\inf \left\{ C > 0 : M(\rho) \leq C (M(r))^{\theta_0}, \quad M(R) = 1 \right\} = 1, \quad (10.4.12)$$

that is the constant 1 in (10.4.5) is the best constant.

Proof of (10.4.11)

It suffices to prove that if $\theta \in \mathbb{R}$ satisfies

$$M(\rho) \leq C (M(r))^\theta, \quad (10.4.13)$$

for every $f \in C^0(\overline{B_R})$, f holomorphic in B_R and such that $M(R) = 1$ then

$$\theta \leq \theta_0. \quad (10.4.14)$$

Now, let

$$f_n(z) = \left(\frac{z}{R} \right)^n, \quad n \in \mathbb{N}.$$

We have

$$M(\rho) = \left(\frac{\rho}{R} \right)^n, \quad M(r) = \left(\frac{r}{R} \right)^n$$

and by (10.4.13) we have

$$n \log \frac{\rho}{R} \leq \log C + \theta n \log \frac{r}{R}, \quad \forall n \in \mathbb{N},$$

from which (recalling that $r < \rho < R$) we have

$$\frac{\log(\rho/R)}{\log(r/R)} \geq \theta + \frac{\log C}{n \log(r/R)}, \quad n \in \mathbb{N}$$

and passing to the limit as $n \rightarrow \infty$ we obtain (10.4.14).

Proof of (10.4.12)

It suffices to prove that if $C > 0$ satisfies

$$M(\rho) \leq C (M(r))^{\theta_0},$$

for every $f \in C^0(\overline{B_R})$, f holomorphic in B_R and $M(R) = 1$ then

$$C \geq 1. \quad (10.4.15)$$

It suffices to choose

$$f(z) = \frac{z}{R}$$

and we have trivially

$$(M(r))^{\theta_0} = \left(\frac{r}{R}\right)^{\theta_0} = \frac{\rho}{R} = M(\rho),$$

from which (10.4.15) follows.

5. It is interesting to note that the mere inequality (10.4.5) implies the following unique continuation property

$$f(z) = \mathcal{O}(|z|^N), \text{ as } z \rightarrow 0, \quad \forall N \in \mathbb{N} \implies f \equiv 0 \text{ in } B_R. \quad (10.4.16)$$

indeed, let us assume that

$$f(z) = \mathcal{O}(|z|^N), \text{ as } z \rightarrow 0, \quad \forall N \in \mathbb{N} \quad (10.4.17)$$

and, arguing by contradiction, let us suppose that

$$f \not\equiv 0 \text{ in } B_R. \quad (10.4.18)$$

Then there exists $\rho \in (0, R)$ such that

$$M(\rho) > 0. \quad (10.4.19)$$

On the other hand, (10.4.17) implies

$$M(r) \leq C_N \left(\frac{r}{R}\right)^N, \quad \forall N \in \mathbb{N}$$

for some constant C_N (independent of r). By this inequality and by (10.4.5) we have

$$\begin{aligned} \frac{M(\rho)}{M(R)} &\leq \left(\frac{M(r)}{M(R)} \right)^{\theta_0} \leq \\ &\leq \left(C_N \left(\frac{r}{R} \right)^N \right)^{\theta_0} = \\ &= \exp \left\{ \frac{\log R/\rho}{\log R/r} \left[-N \log \frac{R}{r} + \log C_N \right] \right\}, \quad \forall r \in (0, \rho), \forall N \in \mathbb{N} \end{aligned}$$

and passing to the limit as $r \rightarrow 0$, we have

$$\frac{M(\rho)}{M(R)} \leq \exp \left[-N \log \frac{R}{\rho} \right], \quad \forall N \in \mathbb{N} \quad (10.4.20)$$

now, again passing to the limit as $N \rightarrow \infty$ we have

$$M(\rho) = 0$$

which contradicts (10.4.19). Hence $f \equiv 0$ in B_R .

6. We can easily prove an inequality in L^2 similar to (10.4.5). More precisely the following inequalities ($0 < r < \rho < R$) hold true

$$\begin{aligned} \int_0^{2\pi} |f(\rho e^{i\phi})|^2 d\phi &\leq \\ &\leq \left(\int_0^{2\pi} |f(re^{i\phi})|^2 d\phi \right)^{\theta_0} \left(\int_0^{2\pi} |f(Re^{i\phi})|^2 d\phi \right)^{1-\theta_0}, \end{aligned} \quad (10.4.21)$$

$$\int_{B_\rho} |f|^2 dx dy \leq \left(\int_{B_r} |f|^2 dx dy \right)^{\theta_0} \left(\int_{B_R} |f|^2 dx dy \right)^{1-\theta_0}. \quad (10.4.22)$$

Proof of (10.4.21).

First, let us observe that

$$\rho = r^{\theta_0} R^{1-\theta_0}. \quad (10.4.23)$$

By the assumption on f we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| \leq R.$$

Hence, by (10.4.23) and by the Hölder inequality we have

$$\begin{aligned}
\int_0^{2\pi} |f(\rho e^{i\phi})|^2 d\phi &= \sum_{n=0}^{\infty} \rho^{2n} |a_n|^2 = \\
&= \sum_{n=0}^{\infty} (r^{2n} |a_n|^2)^{\theta_0} (R^{2n} |a_n|^2)^{1-\theta_0} \leq \\
&\leq \left(\sum_{n=0}^{\infty} r^{2n} |a_n|^2 \right)^{\theta_0} \left(\sum_{n=0}^{\infty} R^{2n} |a_n|^2 \right)^{1-\theta_0} = \\
&= \left(\int_0^{2\pi} |f(re^{i\phi})|^2 d\phi \right)^{\theta_0} \left(\int_0^{2\pi} |f(Re^{i\phi})|^2 d\phi \right)^{1-\theta_0}.
\end{aligned}$$

Now, **let us prove** (10.4.22). We have

$$\begin{aligned}
\int_{B_\rho} |f|^2 dx dy &= \int_0^\rho s \left(\int_0^{2\pi} |f(se^{i\phi})|^2 d\phi \right) ds = \\
&= \int_0^1 t\rho \left(\int_0^{2\pi} |f(t\rho e^{i\phi})|^2 d\phi \right) dt.
\end{aligned}$$

From which, by using (10.4.21), (10.4.23) and by Hölder inequality, we get

$$\begin{aligned}
\int_{B_\rho} |f|^2 dx dy &\leq \\
&\leq \rho \left\{ \int_0^1 \left[\int_0^{2\pi} t |f(tre^{i\phi})|^2 d\phi \right]^{\theta_0} \left[\int_0^{2\pi} t |f(tRe^{i\phi})|^2 d\phi \right]^{1-\theta_0} dt \right\} \leq \\
&\leq \rho \left[\int_0^1 \int_0^{2\pi} t |f(tre^{i\phi})|^2 d\phi dt \right]^{\theta_0} \left[\int_0^1 \int_0^{2\pi} t |f(tRe^{i\phi})|^2 d\phi dt \right]^{1-\theta_0} = \\
&= \left[\int_0^1 \int_0^{2\pi} tr |f(tre^{i\phi})|^2 d\phi dt \right]^{\theta_0} \left[\int_0^1 \int_0^{2\pi} tR |f(tRe^{i\phi})|^2 d\phi dt \right]^{1-\theta_0} = \\
&= \left(\int_{B_r} |f|^2 dx dy \right)^{\theta_0} \left(\int_{B_R} |f|^2 dx dy \right)^{1-\theta_0}.
\end{aligned}$$

One can also prove L^p versions of the inequalities (10.4.21) and (10.4.22), for these we refer the interested reader to [20, Ch. 1]. ♠

Exercise. Let u be a harmonic function in $B_R \subset \mathbb{R}^2$ such that $u \in C^0(\overline{B_R})$. Prove that if $0 < r < \rho < R$ then the following inequality holds true

$$\int_{\partial B_\rho} u^2 dS \leq \left(\int_{\partial B_r} u^2 dS \right)^{\theta_0} \left(\int_{\partial B_R} u^2 dS \right)^{1-\theta_0}, \quad (10.4.24)$$

$$\int_{B_\rho} u^2 dx dy \leq \left(\int_{B_r} u^2 dx dy \right)^{\theta_0} \left(\int_{B_R} u^2 dx dy \right)^{1-\theta_0}, \quad (10.4.25)$$

where θ_0 is given by (10.4.6). [Hint: recall the solution formula for Dirichlet problem in polar coordinates

$$u(\varrho, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \varrho^n (a_n \cos n\phi + b_n \sin n\phi)$$

and apply it to obtain (10.4.25)]. ♣

Example 3: Stability estimate on the bisector of an angle ([13]).

Let S be a bounded open set of \mathbb{C} whose boundary is made up of two segments, OA e OB such that $\widehat{AOB} = \pi\alpha$, $0 < \alpha < 2$ and by a Jordan curve Γ of extremes A and B . Let $z_0 \in S$ and let us assume that z_0 belongs to the bisector of the angle \widehat{AOB} . Let $f \in C^0(\overline{S})$ be holomorphic in S . Let us denote by

$$E = \max_{\overline{S}} |f|, \quad \varepsilon = \min_{\overline{\Gamma}} |f|.$$

Then

$$|f(z_0)| \leq E^{1 - \left(\frac{|z_0|}{R}\right)^{1/\alpha}} \varepsilon \left(\frac{|z_0|}{R}\right)^{1/\alpha}, \quad (10.4.26)$$

where R denotes the diameter of S .

Proof of (10.4.26).

Let $\sigma > 0$ be to choose and

$$F(z) = f(z) \exp \sigma \left(\frac{z}{z_0} \right)^{1/\alpha}.$$

Set $|z_0| = r$. We have, for $z = \rho e^{\pm \frac{i\alpha\pi}{2}}$,

$$\left| F\left(\rho e^{\pm \frac{i\alpha\pi}{2}}\right) \right| = \left| f\left(\rho e^{\pm \frac{i\alpha\pi}{2}}\right) \right|.$$

Hence

$$|F(z)| \leq E, \quad \text{on } OA \text{ and } OB.$$

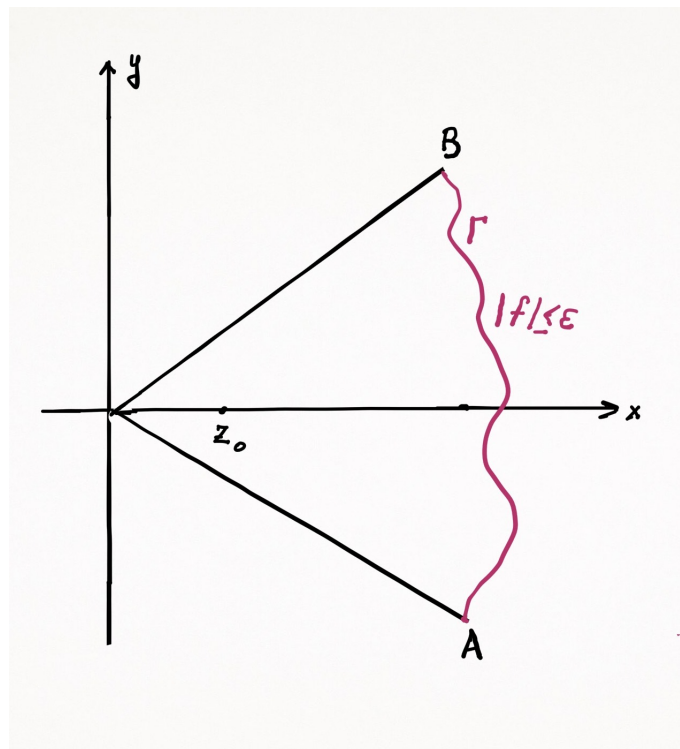


Figure 10.3:

Moreover

$$|F(z)| \leq \varepsilon \exp \sigma \left(\frac{R}{r} \right)^{1/\alpha}, \quad \text{on } \Gamma.$$

Hence, by the maximum modulus principle, we have

$$|f(z_0)|e^\sigma = |F(z_0)| \leq \max \left\{ E, \varepsilon \exp \sigma \left(\frac{R}{r} \right)^{1/\alpha} \right\},$$

from which we have

$$|f(z_0)| = e^{-\sigma} |F(z_0)| \leq \max \left\{ Ee^{-\sigma}, \varepsilon \exp \sigma \left[\left(\frac{R}{r} \right)^{1/\alpha} - 1 \right] \right\}.$$

Now, we choose σ such that

$$e^\sigma = \left(\frac{E}{\varepsilon} \right)^{\left(\frac{r}{R} \right)^{1/\alpha}}$$

and we obtain (10.4.26). ■

Stability estimates for the analytic continuation can be proved even for more general sets than those considered in examples 1 and 2. We report, without proof, the following result (see [48, cap. III], [38]):

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected open set whose boundary is of C^1 class. Let $\Gamma = \partial\Omega$ and let us assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 is a regular path such that $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let us assume that $f \in C^0(\overline{\Omega})$, f holomorphic in Ω satisfying

$$|f(z)| \leq \varepsilon, \quad \forall z \in \Gamma_1, \quad |f(z)| \leq E, \quad \forall z \in \Gamma_2$$

then

$$|f(z)| \leq E^{1-\omega(z)} \varepsilon^{\omega(z)}, \quad \forall z \in \Omega, \quad (10.4.27)$$

where $\omega(z)$ is the harmonic function in Ω such that

$$\omega(z) = 1, \quad \forall z \in \Gamma_1, \quad \omega(z) = 0, \quad \forall z \in \Gamma_2.$$

ω is called *harmonic measure associated to Γ_1 in Ω*

Example 4: Stability estimate for the continuation of real analytic functions.

In Theorem 6.2.2 we have seen that if $f : \Omega \rightarrow \mathbb{R}$ (or \mathbb{C}), where Ω is a connected open set of \mathbb{R}^n , is an analytic function and $D \subset \Omega$ is a (nonempty) open set then

$$f = 0 \quad \text{in } D \quad \implies \quad f = 0 \quad \text{in } \Omega. \quad (10.4.28)$$

That is, a real analytic function on a connected open Ω is determined by its values on any nonempty open set $D \subset \Omega$. Nevertheless small errors on $f|_D$ can have uncontrollable effects on $f(z)$ for $z \in \Omega \setminus \overline{D}$.

In the present Example 4, as application of Example 3, we will find an error estimate for the analytic continuation problem.

Let us consider the following particular situation: let Ω be a star shaped open set of \mathbb{R}^n w.r.t. $x_0 \in \Omega$ (i.e. for every $x \in \Omega$ we have $x_0 + t(x - x_0) \in \Omega$ for every $x \in \Omega$). Let us assume that $\overline{B_r(x_0)} \subset \Omega$ for some $r > 0$. Let $f : \Omega \rightarrow \mathbb{R}$ satisfy $(E, \varepsilon > 0)$

$$f \in \mathcal{C}_{E,\rho}(x), \quad \forall x \in \Omega, \quad (10.4.29)$$

that is (compare Definition 6.2.3)

$$|\partial^\alpha f(x)| \leq E\rho^{-|\alpha|}|\alpha|! \quad \forall \alpha \in \mathbb{N}_0^n, \quad \forall x \in \Omega, \quad (10.4.30)$$

and

$$|f(x)| \leq \varepsilon, \quad \forall x \in B_r. \quad (10.4.31)$$

We want to prove the following stability estimate

$$|f(x)| \leq (2E)^{1-\theta}\varepsilon^\theta \quad \forall x \in \Omega, \quad (10.4.32)$$

where $\theta \in (0, 1)$ and θ depends on n , $\frac{\rho}{r}$ and $\frac{\rho}{d}$ only, where d is the diameter of Ω .

Proof of (10.4.31).

It is not restrictive to assume $x_0 = 0$. The idea of the proof is as follows: let us fix $x \in \Omega \setminus \overline{B_r}$ and let us consider the function

$$\varphi(t) = f(tx), \quad t \in [0, 1]; \quad (10.4.33)$$

we extend such a function holomorphically to a function φ in a neighborhood (in \mathbb{C}) of $\{t + i0 : t \in [0, 1]\}$ and by (10.4.30), (10.4.31) and the result of Example 3, we reach (10.4.32).

By formula (1.2.6) we have, for $t_0 \in [0, 1]$ and $k \in \mathbb{N}_0$

$$\varphi^{(k)}(t_0) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha (\partial^\alpha f)(t_0x).$$

By (10.4.30) we have

$$\begin{aligned}
 |\varphi^{(k)}(t_0)| &\leq \sum_{|\alpha|=k} \frac{k!}{\alpha!} |x|^{|\alpha|} |(\partial^\alpha f)(t_0 x)| \leq \\
 &\leq Ek! \left(\frac{|x|}{\rho}\right)^k \sum_{|\alpha|=k} \frac{k!}{\alpha!} = \\
 &= Ek! \left(\frac{n|x|}{\rho}\right)^k.
 \end{aligned} \tag{10.4.34}$$

Hence the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \varphi^{(k)}(t_0) (z - t_0)^k, \quad z = t + i\tau \in \mathbb{C}, \tag{10.4.35}$$

converges in

$$B_h(t_0) = \{z \in \mathbb{C} : |z - t_0| < h\},$$

where $h = \frac{\rho}{2n|x|}$. Moreover the sum of power series (10.4.35) is holomorphic in $B_h(t_0)$. The above extension can be performed for every $t_0 \in [0, 1]$ therefore the function φ can be holomorphically extended in (Figure 10.4)

$$K = \{z \in \mathbb{C} : \text{dist}(z, I) < h\},$$

where $I := \{t + i0 : t \in [0, 1]\}$. The extension φ to K is formally written as $\varphi(t + i\tau)$ and by (10.4.34) we have

$$|\varphi(t + i\tau)| \leq 2E, \quad \text{for every } t + i\tau \in K. \tag{10.4.36}$$

On the other hand by (10.4.31) and by (10.4.33) we have

$$|\varphi(t + i0)| \leq \varepsilon, \quad \text{for } |t| \leq \frac{r}{|x|}. \tag{10.4.37}$$

At this point it suffices to prove an estimate from above of $|\varphi(t + i0)| = |f(x)|$. This estimate can be obtained by applying twice estimate (10.4.26). First we apply estimate (10.4.26) in the triangles

$$\begin{aligned}
 S_+ &= \left\{ t + i\tau : |t| \leq s, 0 \leq \tau \leq h \left(1 - \frac{|t|}{s}\right) \right\}, \\
 S_- &= \left\{ t + i\tau : |t| \leq s, -h \left(1 - \frac{|t|}{s}\right) \leq \tau \leq 0 \right\},
 \end{aligned}$$

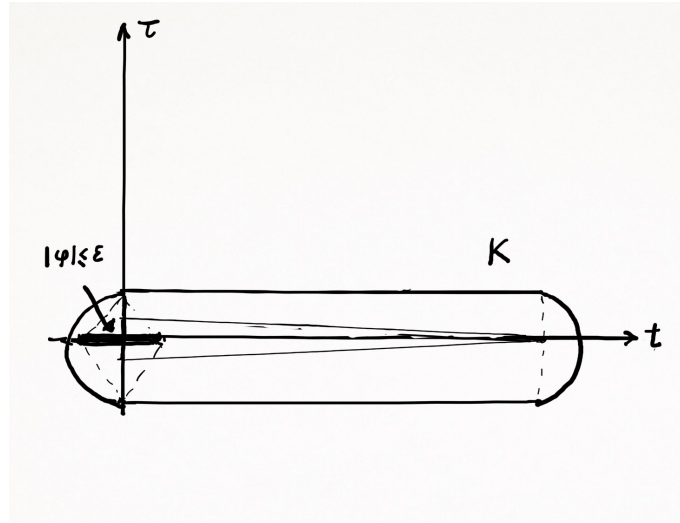


Figure 10.4:

where

$$s := \min\left\{\frac{r}{|x|}, h\right\}.$$

We get

$$|\varphi(i\tau)| \leq (2E)^{1-\vartheta} \varepsilon^\vartheta, \quad \text{for } |\tau| \leq \frac{h}{2}, \quad (10.4.38)$$

where

$$\vartheta = \left(\frac{h}{2R_0}\right)^{1/\alpha_0} = \frac{1}{\left(2\sqrt{\frac{s^2}{h^2} + 1}\right)^{1/\alpha_0}}, \quad (10.4.39)$$

$$\alpha_0 = \frac{2}{\pi} \arctan \frac{s}{h},$$

$$R_0 = \sqrt{s^2 + h^2},$$

(hence, ϑ depends on $\frac{r}{\rho}$ and n only).

Now, we use (10.4.38) and (10.4.36) to apply (10.4.26) in the triangle

$$T = \left\{ t + i\tau : |\tau| \leq \frac{h}{2}, 0 \leq t \leq (h+1) \left(1 - \frac{2|\tau|}{h}\right) \right\}.$$

To this aim, set

$$\alpha_1 = \frac{2}{\pi} \arctan \frac{h}{h+1},$$

$$R_1 = \sqrt{\left(\frac{h}{2}\right)^2 + (1+h)^2},$$

and we have

$$|\varphi(1+i0)| \leq (2E)^{1-\vartheta\tilde{\vartheta}} \varepsilon^{\vartheta\tilde{\vartheta}}, \quad (10.4.40)$$

where

$$\tilde{\vartheta} = \left(\frac{1}{R_1}\right)^{1/\alpha_1}.$$

Therefore we have proved (10.4.32) with $\theta = \vartheta\tilde{\vartheta}$. ■

Concluding Remarks.

Come back to the holomorphic functions. The Hadamard three circle inequality allows us to estimate $|f(z)|$ for $z \in B_R$ provided we know that f is holomorphic in B_R and, in addition, we know

$$\sup_{|z| \leq r} |f(z)| \leq \varepsilon \text{ (error) }, \quad (10.4.41)$$

and

$$\sup_{|z| \leq R} |f(z)| \leq E \text{ (a priori information) }. \quad (10.4.42)$$

As a matter of fact, we have

$$|f(z)| \leq \varepsilon^{\theta|z|} E^{1-\theta|z|}, \quad (10.4.43)$$

where

$$\theta_{|z|} = \frac{\log R/|z|}{\log R/r}. \quad (10.4.44)$$

It is immediately checked that the a priori information

$$\sup_{|z| \leq R} |f(z)| \leq E,$$

is not sufficient to control the error on $\{|z| = R\}$. It is enough to consider $f_n(z) = \left(\frac{z}{R}\right)^n$, obtaining

$$\sup_{|z| \leq r} |f_n(z)| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{and} \quad |f_n(z)| = 1, \text{ as } |z| = R.$$

We will now show that by "strengthening" the a priori information (10.4.42) we can find a stability estimate for $|f(z)|$ when $|z| = R$. Let us consider, for instance, the following a priori information: $f \in C^{0,\alpha}(\overline{B_R})$, $0 < \alpha \leq 1$

$$\sup_{|z| \leq R} |f(z)| + [f]_{0,\alpha} \leq E_\alpha, \quad (10.4.45)$$

where

$$[f]_{0,\alpha} = \sup_{z,w \in B_R, z \neq w} \frac{|f(z) - f(w)|}{|z - w|}. \quad (10.4.46)$$

Let us assume, for brevity, that $R = 1$. Let $z_0 \in \partial B_1$. Set $z_t = z_0(1 - t)$, with $t \in [0, 1)$ to be chosen, we have

$$\begin{aligned} |f(z_0)| &\leq |f(z_t) - f(z_0)| + |f(z_t)| \leq \\ &\leq E_\alpha |z_t - z_0|^\alpha + |f(z_t)| \leq \\ &\leq E_\alpha t^\alpha + |f(z_t)|. \end{aligned}$$

On the other hand by (10.4.43) we have

$$|f(z)| \leq \varepsilon^{\tilde{\theta}_t} E_\alpha^{1-\tilde{\theta}_t},$$

where

$$\tilde{\theta}_t = \frac{\log \frac{1}{1-t}}{\log 1/r}.$$

Hence

$$|f(z_0)| \leq E_\alpha \left(t^\alpha + \varepsilon_1^{\log \frac{1}{1-t}} \right), \quad \forall t \in [0, 1),$$

where

$$\varepsilon_1 = \left(\frac{\varepsilon}{E_\alpha} \right)^{\frac{1}{|\log r|}}. \quad (10.4.47)$$

Now we have

$$t^\alpha + \varepsilon_1^{\log \frac{1}{1-t}} = t^\alpha + \exp(\log(1-t) |\log \varepsilon_1|).$$

On the other hand, we have

$$\log(1-t) \leq -t.$$

Hence

$$|f(z_0)| \leq E_\alpha (t^\alpha + \exp(-t |\log \varepsilon_1|)), \quad \forall t \in [0, 1). \quad (10.4.48)$$

Now we note that, if $\varepsilon_1 < 1$ then

$$0 < |\log \varepsilon_1|^{-1} (\log |\log \varepsilon_1|) \leq e^{-1}$$

and we can choose

$$t = |\log \varepsilon_1|^{-1} (\log |\log \varepsilon_1|) \in [0, 1).$$

We obtain

$$\begin{aligned} |f(z_0)| &\leq E_\alpha (|\log \varepsilon_1|^{-\alpha} (\log |\log \varepsilon_1|) + |\log \varepsilon_1|^{-1}) \leq \\ &\leq CE_\alpha |\log \varepsilon_1|^{-\alpha} (\log |\log \varepsilon_1|), \end{aligned} \quad (10.4.49)$$

where C depends on α only. If $\varepsilon_1 \geq 1$ then we have trivially

$$|f(z_0)| \leq E_\alpha \leq E_\alpha \varepsilon_1. \quad (10.4.50)$$

Thus, by (10.4.49) and (10.4.50) we have the following **stability estimate**, for every $z_0 \in \partial B_1$

$$|f(z_0)| \leq \tilde{C} E_\alpha |\log \varepsilon_1|^{-\alpha} (\log |\log \varepsilon_1|). \quad (10.4.51)$$

where \tilde{C} depends on α and ε_1 is given by (10.4.47). \blacklozenge

Chapter 11

The John stability Theorem for the Cauchy problem for PDEs with analytic coefficients

11.1 Statement of the Theorem

The stability estimate that we present in this Chapter is due to F. John [40]. The basic elements of the proof are as follows.

1. The Green identity and the construction of an appropriate solution of the adjoint operator.
2. The stability estimates for the analytic continuation problem.

In what follows we will consider the following linear system

$$u_t(x, t) = \sum_{j=1}^n A_j(x, t)u_{x_j}(x, t) + A_0(x, t)u(x, t), \quad (11.1.1)$$

where $u := (u^1, \dots, u^N)^T$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $A_j(x, t)$, $j = 0, 1, \dots, n$ are $N \times N$ matrices. Moreover, let us introduce the following notations

$$\gamma(x) = (1 - |x|^2)^{n+1}, \quad (11.1.2)$$

For any $\lambda \in \mathbb{R}$ let us denote by S_λ the surface

$$S_\lambda = \{(x, \lambda\gamma(x)) | x \in B_1\} \quad (11.1.3)$$

and, for any $\lambda_1 < \lambda_2$, let

$$\mathcal{R}_{\lambda_1, \lambda_2} = \{(x, t) \in \mathbb{R}^{n+1} | x \in B_1 \quad \lambda_1 \gamma(x) < t < \lambda_2 \gamma(x)\}. \quad (11.1.4)$$

Theorem 11.1.1 (John stability estimate). *Let $c_0, L, M, \rho, E, \varepsilon$ positive numbers. Let $A_j, j = 1, \dots, n$ and B matrices $N \times N$ whose entries are analytic in $\overline{\mathcal{R}_{0,L}}$ and satisfy*

$$A_j \in \mathcal{C}_{M, \rho}(\overline{\mathcal{R}_{0,L}}), \quad j = 0, 1, \dots, n \quad \forall (\overline{x}, \overline{t}) \in \overline{\mathcal{R}_{0,L}}. \quad (11.1.5)$$

Set

$$A(x, \lambda) = I + \lambda \sum_{j=1}^n A_j(x, \lambda \gamma(x)) \gamma_{x_j}(x). \quad (11.1.6)$$

Let us assume that

$$|\det A(x, \lambda)| \geq c_0, \quad \forall x \in \overline{\mathcal{R}_{0,L}}, \quad \forall \lambda \in [0, L], \quad (11.1.7)$$

(that is S_λ is a noncharacteristic surface for every $\lambda \in [0, L]$).

Let $u \in C^{n+1}(\overline{\mathcal{R}_{0,L}})$ satisfy

$$u_t(x, t) = \sum_{j=1}^n A_j(x, t) u_{x_j}(x, t) + A_0(x, t) u(x, t), \quad \forall (x, t) \in \mathcal{R}_{0,L} \quad (11.1.8a)$$

$$\|u(\cdot, 0)\|_{L^\infty(B_1)} \leq \varepsilon, \quad (11.1.8b)$$

$$\|u\|_{C^{n+1}(\overline{\mathcal{R}_{0,L}})} \leq E. \quad (11.1.8c)$$

Then, for every $r \in (0, 1)$, we have

$$|u(x, t)| \leq \frac{C(E + 2\varepsilon)}{(1-r)^{n+1}} \left| \log \frac{\varepsilon}{E + 2\varepsilon} \right|^{-1}, \quad \forall x \in \overline{\mathcal{R}_{0,L}} \cap (\overline{B_{1-r}} \times \mathbb{R}), \quad (11.1.9)$$

where C depends on M, L, ρ, c_0 and n only.

11.2 Proof of the Theorem

Let us premise the following

Lemma 11.2.1. *Let A_j be as in Theorem 11.1.1. Let us assume that*

$$A_j, \in \mathcal{C}_{M_0, \rho_0}(\bar{x}, \bar{t}), \quad \forall (\bar{x}, \bar{t}) \in \bar{B}_1 \times [-\delta_0, \delta_0]. \quad (11.2.1)$$

Let $W \in \mathbb{R}^N$ be such that $|W| = 1$ and let $\xi \in \mathbb{R}^n$ arbitrary.

Let U be the solution to the Cauchy problem

$$\begin{cases} U_t = \sum_{j=1}^n A_j(x, t)U_{x_j} + A_0(x, t)U, \\ U(x, 0) = e^{-ix \cdot \xi}W, \quad \forall x \in B_1. \end{cases} \quad (11.2.2)$$

Then there exist M, ρ, δ positive numbers depending by M_0, ρ_0, δ_0 , but independent of ξ such that

$$U \in \mathcal{C}_{Me^{|\xi|}\rho}(\bar{x}, \bar{t}), \quad \forall (\bar{x}, \bar{t}) \in \bar{B}_1 \times [-\delta, \delta], \quad (11.2.3)$$

that is

$$|\partial^\alpha U| \leq Me^{|\xi|}\rho^{-|\alpha|}|\alpha|!, \quad \forall (\bar{x}, \bar{t}) \in \bar{B}_1 \times [-\delta, \delta], \quad \forall \alpha \in \mathbb{N}_0^{n+1}. \quad (11.2.4)$$

Proof of Lemma 11.2.1. Let

$$\psi(x) = e^{-|\xi|}e^{-i\xi \cdot x}W.$$

Let us consider the Cauchy problem

$$\begin{cases} V_t = \sum_{j=1}^n A_j(x, t)V_{x_j} + A_0(x, t)V, \\ V(x, 0) = \psi(x), \quad \forall x \in B_1. \end{cases} \quad (11.2.5)$$

We have, trivially,

$$U(x, t) = e^{|\xi|}V(x, t). \quad (11.2.6)$$

On the other hand

$$\begin{aligned} \left| \partial^{\alpha'} \psi(x) \right| &= \left| (i\xi)^{\alpha'} e^{-|\xi|} e^{-\xi \cdot x} W \right| = |\xi|^{|\alpha'|} e^{-|\xi|} \leq \\ &\leq |\alpha'|^{|\alpha'|} e^{-|\alpha'|} \leq |\alpha'|!, \quad \forall \alpha' \in \mathbb{N}_0^n, \quad \forall x \in \bar{B}_1, \end{aligned}$$

hence

$$\psi \in \mathcal{C}_{1,1}(\bar{x}), \quad \forall \bar{x} \in \bar{B}_1.$$

Therefore there exist M, ρ, δ which depend on M_0, ρ_0, δ_0 , but independent of ξ such that

$$V \in \mathcal{C}_{1,1}(\bar{x}, \bar{t}), \quad \forall (\bar{x}, \bar{t}) \in \overline{B_1} \times [-\delta, \delta]$$

and by (11.2.6) we have (11.2.3). ■

Proof of Theorem 11.1.1.

Step 1. The Green identity.

We have

$$\begin{aligned} v^T \left(u_t - \sum_{j=1}^n A_j u_{x_j} - A_0 u \right) &= \\ &= \partial_t (v^T u) - \partial_{x_j} \left(v^T \sum_{j=1}^n A_j u \right) - \\ &\quad - \left(v_t^T - \sum_{j=1}^n (v^T A_j)_{x_j} + v^T A_0 \right) u. \end{aligned} \quad (11.2.7)$$

Let now $v \in C^1(\overline{\mathcal{R}_{\lambda_1, \lambda_2}})$ be a solution to the adjoint system

$$v_t - \sum_{j=1}^n (A_j^T v)_{x_j} + A_0^T v = 0, \quad \text{in } \mathcal{R}_{\lambda_1, \lambda_2}. \quad (11.2.8)$$

Since u is a solution to system (11.1.8a), integrating both the sides of (11.2.7) over $\mathcal{R}_{\lambda_1, \lambda_2}$ we have

$$\begin{aligned} 0 &= \int_{\mathcal{R}_{\lambda_1, \lambda_2}} \left[\partial_t (v^T u) - \partial_{x_j} \left(v^T \sum_{j=1}^n A_j u \right) \right] dx dt = \\ &= \int_{\partial \mathcal{R}_{\lambda_1, \lambda_2}} \left[(v^T u)(\nu \cdot e_{n+1}) - \left(v^T \sum_{j=1}^n A_j u (\nu \cdot e_j) \right) \right] dS = \\ &= \int_{B_1} \left(v^T u + \lambda_2 v^T \sum_{j=1}^n \gamma_{x_j} A_j u \right) (x, \lambda_2 \gamma(x)) dx - \\ &\quad - \int_{B_1} \left(v^T u + \lambda_1 v^T \sum_{j=1}^n \gamma_{x_j} A_j u \right) (x, \lambda_1 \gamma(x)) dx. \end{aligned}$$

Hence, recalling (11.1.6) we get

$$\begin{aligned} \int_{B_1} v^T(x, \lambda_2 \gamma(x)) A(x, \lambda_2) u(x, \lambda_2 \gamma(x)) dx &= \\ = \int_{B_1} v^T(x, \lambda_1 \gamma(x)) A(x, \lambda_1) u(x, \lambda_1 \gamma(x)) dx. \end{aligned} \quad (11.2.9)$$

Step 2. Construction of an appropriate solution to (11.2.8). Let $W \in \mathbb{R}^N$ be such that $|W| = 1$ and $\xi \in \mathbb{R}^n$ arbitrary. Let us denote by w the function

$$w(x) = e^{-ix \cdot \xi} W. \quad (11.2.10)$$

Let $\lambda \in [0, L]$. Let us consider the following Cauchy problem

$$\begin{cases} v_t = \sum_{j=1}^n (A_j^T(x, t)v)_{x_j} - A_0^T(x, t)v, \\ v(x, \lambda\gamma(x)) = \gamma(x) (A^T(x, \lambda))^{-1} w(x), \quad x \in B_1. \end{cases} \quad (11.2.11)$$

Let us prove that there exists $\delta > 0$, depending on M, ρ, c_0 only, such that there exists the solution $v(x, t; \lambda)$ of (11.2.11), and it is analytic in $\overline{\mathcal{R}_{\lambda-\delta, \lambda+\delta}}$. To this purpose we perform some change of variables. First, we set

$$s = \frac{t}{\gamma(x)}, \quad v(x, t; \lambda) = \gamma(x) V \left(x, \frac{t}{\gamma(x)}; \lambda \right)$$

and we have

$$\begin{aligned} v_t(x, t; \lambda) &= V_s \left(x, \frac{t}{\gamma(x)}; \lambda \right). \\ v_{x_j} &= \gamma V_{x_j} + \gamma_{x_j} V - \frac{t\gamma_{x_j}}{\gamma} V_s = \\ &= \gamma V_{x_j} + \gamma_{x_j} V - s\gamma_{x_j} V_s. \end{aligned}$$

Inserting what obtained above in system (11.2.11), we get

$$\begin{aligned} V_s &= v_t = \sum_{j=1}^n A_j^T v_{x_j} + \left(\sum_{j=1}^n A_{j,x_j}^T - A_0^T \right) v = \\ &= \sum_{j=1}^n A_j^T (\gamma V_{x_j} - s\gamma_{x_j} V_s) + \left[\sum_{j=1}^n (A_j^T \gamma_{x_j} + A_{j,x_j}^T \gamma) - A_0^T \gamma \right] V, \end{aligned}$$

From which (recalling (11.1.6)), we get

$$\begin{aligned} A^T(x, s)V_s &= \gamma \sum_{j=1}^n A_j^T V_{x_j} + \left(\sum_{j=1}^n A_{j,x_j}^T - A_0^T \right) V = \\ &= \sum_{j=1}^n A_j^T (\gamma V_{x_j} - s\gamma_{x_j} V_s) + \left[\sum_{j=1}^n (A_j^T \gamma_{x_j} + A_{j,x_j}^T \gamma) - A_0^T \gamma \right] V. \end{aligned}$$

Now, set

$$\bar{A}_j(x, s) = \gamma(x) (A^T(x, s))^{-1} A_j^T(x, s\gamma(x)), \quad j = 1, \dots, n,$$

$$\begin{aligned} \bar{A}_0(x, s) &= \\ &= (A^T(x, s))^{-1} \left[\sum_{j=1}^n A_j^T(x, s\gamma(x)) \gamma_{x_j} + \gamma \left(\sum_{j=1}^n A_{j,x_j}^T(x, s\gamma(x)) - A_0^T(x, s\gamma(x)) \right) \right]. \end{aligned}$$

Therefore problem (11.2.11) can be written as

$$\begin{cases} V_s(x, s; \lambda) = \sum_{j=1}^n \bar{A}_j(x, s) V_{x_j}(x, s; \lambda) + \bar{A}_0(x, s) V(x, s; \lambda), \\ V(x, s; \lambda)|_{s=\lambda} = (A^T(x, \lambda))^{-1} w(x), \quad \forall x \in B_1. \end{cases} \quad (11.2.12)$$

Now we denote

$$Z(x, s; \lambda) = (A^T(x, \lambda))^{-1} V(x, s + \lambda; \lambda) \quad (11.2.13)$$

and by (11.2.10), (11.2.13) we have

$$\begin{cases} Z_s(x, s; \lambda) = \sum_{j=1}^n \tilde{A}_j(x, s; \lambda) Z_{x_j}(x, s; \lambda) + \tilde{A}_0(x, s; \lambda) Z(x, s; \lambda), \\ Z(x, 0; \lambda) = e^{-ix \cdot \xi} W, \quad x \in B_1, \end{cases} \quad (11.2.14)$$

where

$$\tilde{A}_j(x, s; \lambda) = A^T(x, \lambda) \bar{A}_j(x, s + \lambda) (A^T(x, \lambda))^{-1}, \quad j = 1, \dots, n,$$

and

$$\begin{aligned} \tilde{A}_0(x, s; \lambda) &= \\ &= A^T(x, \lambda) \left[\bar{B}(x, s + \lambda) (A^T(x, \lambda))^{-1} + \sum_{j=1}^n \bar{A}_j(x, s + \lambda) \partial_{x_j} (A^T(x, \lambda))^{-1} \right]. \end{aligned}$$

Now $\tilde{A}_j(x, s; \lambda)$, $j = 1, \dots, n$ and $\tilde{A}_0(x, s; \lambda)$ are analytic functions in (x, s, λ) . In addition, for every $(\bar{x}, \bar{s}, \bar{\lambda}) \in \bar{B}_1 \times [0, L] \times [0, L]$, we have

$$\tilde{A}_j \in \mathcal{C}_{M', \rho'}(\bar{x}, \bar{s}, \bar{\lambda}), \quad j = 0, 1, \dots, n,$$

where M' e ρ' depend on M, ρ, c_0 (and n , which we will omit in the sequel) only.

By Lemma 11.2.1, there exist M'', ρ'', δ depending on M, ρ, c_0 and L only such that there is Z which is the solution to (11.2.14), it is analitic in $\overline{B_1} \times [-\delta, \delta] \times [0, L]$ and satisfies

$$Z \in \mathcal{C}_{M''e^{|\xi|}, \rho''}(\overline{x}, \overline{s}, \overline{\lambda}), \quad \forall (\overline{x}, \overline{s}, \overline{\lambda}) \in \overline{B_1} \times [-\delta, \delta] \times [0, L]. \quad (11.2.15)$$

Coming back to problem (11.2.11), we have that there exists v , solution to (11.2.11) in $\mathcal{R}_{\lambda-\delta, \lambda+\delta}$, such that

$$|\partial_{x,t,\lambda}^\alpha v(x, t; \lambda)| \leq M_0 e^{|\xi|} \rho^{-|\alpha|} |\alpha|!, \quad \forall \alpha \in \mathbb{N}_0^{n+2}, \quad (11.2.16)$$

for all $(x, t; \lambda) \in \overline{\mathcal{R}_{\lambda-\delta, \lambda+\delta}} \times [0, L]$, where M_0, δ_0 and δ depend on M, ρ, c_0 and L only. Let us note that to obtain (11.2.16) for every $\lambda \in [0, L]$ it suffices to consider (11.2.15) and the similar relationships on the coefficients corresponding to $\lambda = 0$.

Step 3. Planning the concluding part of the proof.

We employ (11.2.9), where v is the solution to problem (11.2.11) for some $\lambda \in [0, L]$. Let $\lambda_0 \in [0, L]$ be fixed and let λ satisfy $|\lambda - \lambda_0| < \delta$. By (11.2.9) we have

$$\begin{aligned} g(\lambda) &:= \int_{B_1} \gamma(x) w^T(x) u(x, \lambda \gamma(x)) dx = \\ &= \int_{B_1} v^T(x, \lambda_0 \gamma(x); \lambda) A(x, \lambda_0) u(x, \lambda_0 \gamma(x)) dx. \end{aligned} \quad (11.2.17)$$

The function g is **analitic** because the integrand in the second integral of (11.2.17) depends analytically by λ .

We are first interested in proving an estimates from above for $g(\lambda)$ from which, subsequently, we will derive the estimates from above for u . Setting $\lambda_0 = 0$ in (11.2.17) we have

$$g(\lambda) = \int_{B_1} v^T(x, 0; \lambda) u(x, 0) dx, \quad \forall \lambda \in [0, \delta]. \quad (11.2.18)$$

By (11.1.8b) and by (11.2.16), we have, for $\alpha = 0$,

$$|g(\lambda)| \leq \int_{B_1} |v^T(x, 0; \lambda)| |u(x, 0)| dx \leq c M_0 e^{|\xi|} \varepsilon, \quad \forall \lambda \in [0, \delta], \quad (11.2.19)$$

where $c \geq 1$ depends on n only.

Let now λ_0 be an arbitrary point of $[0, L]$, by (11.2.16) and (11.2.17) we get

$$\begin{aligned} |g^{(k)}(\lambda)| &= \left| \int_{B_1} \partial_\lambda^k v^T(x, \lambda_0 \gamma(x); \lambda) A(x, \lambda_0) u(x, \lambda_0 \gamma(x)) dx \right| \leq \\ &\leq \int_{B_1} |\partial_\lambda^k v^T(x, \lambda_0 \gamma(x); \lambda)| |A(x, \lambda_0)| |u(x, \lambda_0 \gamma(x))| dx \leq \\ &\leq C_1 E M_0 e^{|\xi|} \rho_0^{-k} k!, \end{aligned}$$

where $C_1 \geq 1$ depends on M, ρ, c_0, L only.

Summarizing we have

$$|g(\lambda)| \leq c M_0 e^{|\xi|} \varepsilon, \quad \forall \lambda \in [0, \delta) \quad (11.2.20)$$

and

$$|g^{(k)}(\lambda)| \leq C_1 M_0 e^{|\xi|} \rho_0^{-k} k!, \quad \forall \lambda \in [0, L], k \in \mathbb{N}_0. \quad (11.2.21)$$

Inequality (11.2.21) implies that g can be extended analytically in a neighborhood of $[0, L] \times \{0\} \subset \mathbb{C}$. In addition, for any $\lambda_\star \in [0, L]$, we have that the power series

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(\lambda_\star)}{k!} (z - \lambda_\star)^k,$$

has the radius of convergence equal to ρ_0 and

$$\left| \sum_{k=0}^{\infty} \frac{g^{(k)}(\lambda_\star)}{k!} (z - \lambda_\star)^k \right| \leq C_1 M_0 e^{|\xi|} \frac{\rho_0}{\rho_0 - |z - \lambda_\star|}, \quad (11.2.22)$$

for $|z - \lambda_\star| < \rho_0$. Therefore the sum of the power series (10.4.35) is holomorphic in $B_{\frac{\rho_0}{2}}(\lambda_\star)$ and the function g can be extended holomorphically in (see Figure 8.1)

$$J = \left\{ z \in \mathbb{C} : \text{dist}(z, [0, L] \times \{0\}) < \frac{\rho_0}{2} \right\}.$$

The extension of g to J is formally written as $g(\lambda + i\tau)$ and by (11.2.22) we have

$$|g(\lambda + i\tau)| \leq 2C_1 E M_0 e^{|\xi|}, \quad \text{for } t + i\tau \in J,$$

$$|g(\lambda + i0)| \leq c M_0 e^{|\xi|} \varepsilon, \quad \forall \lambda \in [0, \delta).$$

Now, proceeding in a similar way to what we did to prove (10.4.40), we get

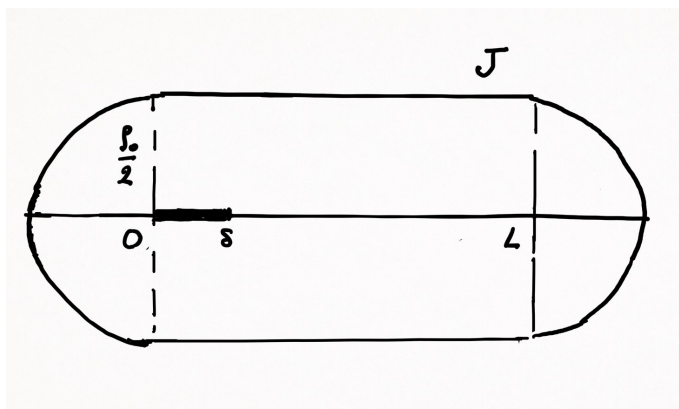


Figure 11.1:

$$\begin{aligned} |g(\lambda + i0)| &\leq (2C_1EM_0e^{|\xi|})^{1-\vartheta} (cEM_0e^{|\xi|\varepsilon})^{\vartheta} \leq \\ &\leq C_2e^{|\xi|}E^{1-\vartheta}\varepsilon^{\vartheta}, \quad \forall \lambda \in [0, L], \end{aligned}$$

where $\vartheta \in (0, 1)$ depends on $L\rho_0^{-1}$ only and $C_2 = 2cC_1M_0$. By the definition of g given in (11.2.17) and recalling that $w(x) = e^{-|\xi|x}W$, we have

$$\left| \int_{B_1} \gamma(x)W^T u(x, \lambda\gamma(x))e^{-i\xi \cdot x} dx \right| \leq C_2e^{|\xi|}E^{1-\vartheta}\varepsilon^{\vartheta}, \quad \forall \lambda \in [0, L]. \quad (11.2.23)$$

Step 4. Conclusion of the proof.

Let us fix $\lambda \in [0, L]$. Let $W = e_j$, for $j = 1, \dots, N$ and set

$$f_j(x) = \begin{cases} \gamma(x)u_j(x, \lambda\gamma(x)), & \text{for } |x| \leq 1, \\ 0, & \text{for } |x| \geq 1. \end{cases} \quad (11.2.24)$$

Let us fix $j = 1, \dots, N$ and, in the sequel, let us omit the index j by f_j . By (11.2.23) and (11.2.24) we get

$$\left| \widehat{f}(\xi) \right| = \left| \int_{\mathbb{R}^n} f(x)e^{-i\xi \cdot x} dx \right| \leq C_2e^{|\xi|}E^{1-\vartheta}\varepsilon^{\vartheta}, \quad \forall \xi \in \mathbb{R}^n, \quad (11.2.25)$$

(where \widehat{f} is the Fourier transform of f).

The proof will be completed as soon as we estimate $|f(x)|$ by means of (11.2.25). Let us recall that

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i\xi \cdot x} d\xi, \quad \forall x \in \mathbb{R}^n. \quad (11.2.26)$$

Let s be a positive number which we will choose later. By (11.2.25) and (11.2.26) we have, for every $x \in \mathbb{R}^n$

$$\begin{aligned} |f(x)| &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)| d\xi = \\ &= (2\pi)^{-n} \left(\int_{|\xi| \leq s} |\widehat{f}(\xi)| d\xi + \int_{|\xi| > s} |\widehat{f}(\xi)| d\xi \right) \leq \\ &\leq (2\pi)^{-n} \left(C_3 s^n e^s E^{1-\vartheta} \varepsilon^\vartheta + \int_{|\xi| > s} |\widehat{f}(\xi)| d\xi \right), \end{aligned} \quad (11.2.27)$$

where $C_3 = \frac{\omega_n}{n} C_2$ (ω_n is the measure of unit ball of \mathbb{R}^n).

To estimate from above the last integral in (11.2.27) we proceed as follows. First of all we note that by the definition of γ and f we have $\partial^\alpha f(x) = 0$ for every $x \in \partial B_1$ and for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq n$ from which we have, for $k = 1, \dots, n$, by using integration by parts

$$\begin{aligned} (-i\xi_k)^{n+1} \widehat{f}(\xi) &= \int_{\mathbb{R}^n} (-i\xi_k)^{n+1} e^{-i\xi \cdot x} f(x) dx = \\ &= \int_{B_1} \partial_k^{n+1} (e^{-i\xi \cdot x}) f(x) dx = \\ &= (-1)^{n+1} \int_{B_1} e^{-i\xi \cdot x} \partial_k^{n+1} f(x) dx. \end{aligned}$$

Hence (11.1.8c) implies

$$|\xi_k|^{n+1} |\widehat{f}(\xi)| \leq \int_{B_1} |\partial_k^{n+1} f(x)| dx \leq C_4 E, \quad (11.2.28)$$

where C_4 depends on L (and on n) only. So that, we have trivially

$$|\xi|^{n+1} |\widehat{f}(\xi)| \leq C_5 E, \quad (11.2.29)$$

where $C_5 = n^{\frac{n+1}{2}} C_4$. Now, by (11.2.28) we have

$$\begin{aligned} \int_{|\xi| > s} |\widehat{f}(\xi)| d\xi &= \int_{|\xi| > s} |\xi|^{-(n+1)} |\xi|^{n+1} |\widehat{f}(\xi)| d\xi \leq \\ &\leq C_5 E \int_{|\xi| > s} |\xi|^{-(n+1)} d\xi = \omega_n C_5 \frac{E}{s}. \end{aligned} \quad (11.2.30)$$

Now we use in (11.2.30) what we obtained in (11.2.30) and by the trivial inequality $E < E + 2\varepsilon$ we get

$$|f(x)| \leq C_6(E + 2\varepsilon) \left[\left(\frac{\varepsilon}{E + 2\varepsilon} \right)^\vartheta s^n e^s + \frac{1}{s} \right], \quad \forall x \in \mathbb{R}^n, \forall s > 0, \quad (11.2.31)$$

where C_6 depends on M, L, ρ, c_0 only.

In order to choose s we proceed as follows. Set

$$\sigma = \left(\frac{\varepsilon}{E + 2\varepsilon} \right)^\vartheta$$

and rewrite the term on the right-hand side of (11.2.31) as

$$\phi(s) := \exp(s + \log s - |\log \sigma|) + \frac{1}{s}.$$

Now we choose

$$s : s_0 = \frac{1}{2} |\log \sigma|, \quad (11.2.32)$$

taking into account that $0 < \sigma \leq 2^{-\vartheta} < 1$, we have

$$\phi(s_0) = \sqrt{\sigma} |\log \sqrt{\sigma}| + 2 |\log \sigma|^{-1} \leq c_\vartheta |\log \sigma|^{-1},$$

where c_ϑ depends on ϑ . Hence, by (11.2.32) and (11.2.31) we have

$$|f(x)| \leq C_7(E + 2\varepsilon) \left| \log \left(\frac{\varepsilon}{E + 2\varepsilon} \right) \right|^{-1}, \quad (11.2.33)$$

where $C_7 = C_6(c_\vartheta + 2)\vartheta^{-1}$. Finally, recalling that $f(x) = \gamma(x)u_j(x, \lambda\gamma(x))$ for $|x| \leq 1$ we have, for any $r \in (0, 1)$

$$|u(x, \lambda\gamma(x))| \leq \frac{C_7 N^{1/2}}{(1-r)^{n+1}} (E + 2\varepsilon) \left| \log \left(\frac{\varepsilon}{E + 2\varepsilon} \right) \right|^{-1}, \quad (11.2.34)$$

for $|x| \leq 1 - r$, from which the thesis follows. ■

Remarks.

We outline some changes that we should make in the case of nonhomogeneous system

1. Let us consider the case in which instead of (11.1.8) we have, for $(x, t) \in \mathcal{R}_{0,L}$,

$$u_t(x, t) = \sum_{j=1}^n A_j(x, t)u_{x_j}(x, t) + A_0(x, t)u(x, t) + F(x, t), \quad (11.2.35a)$$

$$\|u(\cdot, 0)\|_{L^\infty(B_1)} \leq \varepsilon, \quad (11.2.35b)$$

$$\|u\|_{C^{n+1}(\overline{\mathcal{R}_{0,L}})} \leq E, \quad (11.2.35c)$$

$$\|F\|_{L^\infty(\mathcal{R}_{0,L})} \leq \varepsilon_1, \quad (11.2.35d)$$

where F is not necessarily analytic. Let us continue to denote (even though $\lambda_1 > \lambda_2$) by $\mathcal{R}_{\lambda_1, \lambda_2}$ the subset of \mathbb{R}^{n+1} enclosed by S_{λ_1} and S_{λ_2} . In such a way (11.2.9) becomes

$$\begin{aligned} & \int_{B_1} v^T(x, \lambda_2 \gamma(x))A(x, \lambda_2)u(x, \lambda_2 \gamma(x))dx + \\ & + \operatorname{sgn}(\lambda_1 - \lambda_2) \int_{\mathcal{R}_{\lambda_1, \lambda_2}} v^T(x, t)F(x, t)dxdt = \\ & = \int_{B_1} v^T(x, \lambda_1 \gamma(x))A(x, \lambda_1)u(x, \lambda_1 \gamma(x))dx. \end{aligned} \quad (11.2.36)$$

We construct $v(x, t; \lambda)$ likewise the Step 2 of Theorem 11.2.1. Consequently, instead of (11.2.17), for a fixed λ_0 in $[0, L]$ and setting

$$\begin{aligned} \tilde{g}(\lambda) &= \int_{B_1} w^T(x)u(x, \lambda \gamma(x))dx + \\ &+ \operatorname{sgn}(\lambda_0 - \lambda) \int_{\mathcal{R}_{\lambda, \lambda_0}} v^T(x, t; \lambda)F(x, t)dxdt, \end{aligned}$$

we have

$$\tilde{g}(\lambda) = \int_{B_1} v^T(x, \lambda_0 \gamma(x); \lambda)A(x, \lambda_0)u(x, \lambda_0 \gamma(x))dx, \quad (11.2.37)$$

for $|\lambda - \lambda_0| < \delta$ and $\lambda \in [0, L]$. Exactly like the Step 3 of the proof of Theorem 11.2.1 we get the estimate

$$|\tilde{g}(\lambda + i0)| \leq C_2 M_0 e^{|\xi|} E^{1-\vartheta} \varepsilon^\vartheta, \quad \forall \lambda \in [0, L]. \quad (11.2.38)$$

Now by (11.2.16) and by (11.2.35d) we have

$$\begin{aligned} \left| \int_{\mathcal{R}_{\lambda, \lambda_0}} v^T(x, t; \lambda)F(x, t)dxdt \right| &\leq \varepsilon_1 \int_{\mathcal{R}_{\lambda, \lambda_0}} |v^T(x, t; \lambda)| dxdt \leq \\ &\leq cLM_0 e^{|\xi|} \varepsilon_1, \end{aligned} \quad (11.2.39)$$

where c depends on n only. By (11.2.38) and (11.2.39) we have

$$\left| \int_{B_1} \gamma(x) w^T(x) u(x, \lambda \gamma(x)) dx \right| \leq C_2 M_0 e^{|\xi|} E^{1-\vartheta} \varepsilon^\vartheta + c L M_0 e^{|\xi|} \varepsilon_1.$$

From now on we may argue like Step 4 of the proof of Theorem 11.1.1 and we find

$$|u(x, t)| \leq \frac{C(E + 2\varepsilon_2)}{(1 - r)^{n+1}} \left| \log \frac{\varepsilon_2}{E + 2\varepsilon_2} \right|^{-1} \tag{11.2.40}$$

for all x in $\overline{\mathcal{R}_{0,L}} \cap (\overline{B_{1-r}} \times \mathbb{R})$, where C depends on M, L, ρ, c_0 and n only and

$$\varepsilon_2 = E^{1-\vartheta} \varepsilon^\vartheta + \varepsilon_1.$$

2. The a priori information (11.1.8c) and γ (compare (11.1.2)) occur in the proof of stability Theorem especially to obtain (11.2.30), while in the other parts of the proof what is needed to know about u is only that

$$\|u\|_{L^\infty(\mathcal{R}_{0,L})} \leq E.$$

Now we prove that with some further arrangements we may define

$$\gamma(x) = 1 - |x|^2 \tag{11.2.41}$$

and, instead of the a priori information (11.1.8c) we require

$$\|u\|_{C^1(\overline{\mathcal{R}_{0,L}})} \leq E. \tag{11.2.42}$$

First of all, we notice that the functions f_j defined like (11.2.24) (with γ given by (11.2.41)) satisfy to (we omit the index j)

$$\|f\|_{C^1(B_1)} \leq CE, \tag{11.2.43}$$

where C depends on L only. Since $f = 0$ in $\mathbb{R}^n \setminus B_1$ we have

$$-i\xi_k \widehat{f}(\xi) = (-1) \int_{B_1} e^{-i\xi \cdot x} \partial_k f(x) dx, \quad k = 1, \dots, n.$$

By the latter, taking into account (11.2.43) we have

$$\int_{\mathbb{R}^n} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi = (2\pi)^{-n} \int_{B_1} |\nabla f(x)|^2 dx \leq \overline{C} E^2,$$

where \overline{C} depends on L (and n) only.

Therefore we have (recalling (11.2.25)), for every $s > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} |\widehat{f}(\xi)| d\xi &= \int_{|\xi| \leq s} |\widehat{f}(\xi)|^2 d\xi + \int_{|\xi| > s} |\widehat{f}(\xi)|^2 d\xi \leq \\ &\leq c (C_2 E^{1-\vartheta} \varepsilon^\vartheta)^2 s^n e^{2s} + \frac{\overline{C}^2 E^2}{s^2}, \end{aligned}$$

where c depends on n only.

All in all, by

$$\int_{B_1} |f(x)|^2 dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi,$$

we have

$$\int_{B_1} |f(x)|^2 dx \leq \overline{C}_3 \left((E^{1-\vartheta} \varepsilon^\vartheta)^2 s^n e^{2s} + E^2 s^{-2} \right),$$

where

$$\overline{C}_3 = \max \{ (2\pi)^n c C_2^2, (2\pi)^n \overline{C} \}.$$

Arguing similarly to the proof of (11.2.34), we get

$$\int_{B_1} |f(x)|^2 dx \leq \overline{C}_4 (E^2 + 2\varepsilon^2) \left| \log \frac{\varepsilon^2}{E^2 + 2\varepsilon^2} \right|^{-2}, \quad (11.2.44)$$

where $\overline{C}_4 = \overline{c}_\vartheta \overline{C}_3$ and \overline{c}_ϑ depends on ϑ .

By applying Proposition 10.2.5 and by (11.2.44), we have

$$|u(x, t)| \leq \frac{C(E + \sqrt{2}\varepsilon)}{1-r} \left| \log \frac{\varepsilon}{E + \sqrt{2}\varepsilon} \right|^{-\frac{1}{n+1}}, \quad (11.2.45)$$

for every $x \in \overline{\mathcal{R}_{0,L}} \cap (\overline{B_{1-r}} \times \mathbb{R})$, where C depends on M, L, ρ, c_0 and n only.

3. Let us examine the main modifications that we should make in the proof of Theorem 11.2.1 to deal with the case where the initial surface in the Cauchy problem is not a portion of the hyperplane $\{t = 0\}$. Let $\varphi \in C^2(\overline{B_1})$ satisfy $\varphi(0) = |\nabla\varphi(0)| = 0$. Let us consider the Cauchy problem

$$\begin{cases} u_t = \sum_{j=1}^n A_j(x, t) u_{x_j} + A_0(x, t) u + F(x, t), \\ u(x, \varphi(x)) = g(x), \quad x \in B_1, \end{cases} \quad (11.2.46)$$

Where $A_j, j = 0, 1, \dots, n$ are analytic functions. We require that the surface $\{t = \varphi(x) | x \in B_1\}$ is noncharacteristic. This is equivalent to require that the "algebraic" system

$$\begin{cases} u_t(x, \varphi(x)) - \sum_{j=1}^n A_j(x, \varphi(x))u_{x_j}(x, \varphi(x)) = \tilde{f}(x), \\ u_t(x, \varphi(x))\varphi_{x_i}(x) + u_{x_i}(x, \varphi(x)) = g_{x_i}(x), \quad i = 1, \dots, n, \end{cases} \quad (11.2.47)$$

has a unique solution $(u_t(x, \varphi(x)), u_{x_1}(x, \varphi(x)), \dots, u_{x_n}(x, \varphi(x)))$ for any $g(x)$, where

$$\tilde{f}(x) = A_0(x, \varphi(x))g(x) + F(x, \varphi(x)).$$

In turn this is equivalent to require the uniqueness of $u_t, u_{x_i}, i = 0, 1, \dots, n$ as solution to the system

$$\begin{cases} \left(I + \sum_{j=1}^n \varphi_{x_i}(xx)A_j(x, \varphi(x)) \right) u_t(x, \varphi(x)) = G(x), \\ u_{x_i}(x, \varphi(x)) = g_{x_i}(x) - u_t(x, \varphi(x))\varphi_{x_i}(x), \quad i = 1, \dots, n, \end{cases}$$

where

$$G(x) = \tilde{f}(x) + \sum_{j=1}^n A_j(x, \varphi(x))g_{x_j}(x).$$

From which we have that $\{t = \varphi(x) | x \in B_1\}$ is a characteristic surface if and only if

$$\det \left(I + \sum_{j=1}^n \varphi_{x_i}(x)A_j(x, \varphi(x)) \right) \neq 0, \quad \forall x \in B_1. \quad (11.2.48)$$

Let us first consider the case in which

$$g \equiv 0.$$

By mean the Holmgren transformation, (7.6.22), we may assume that φ is strictly convex. For any λ_1, λ_2 positive numbers, let us denote by $\mathcal{S}_{\lambda_1, \lambda_2}$ the subset of \mathbb{R}^{n+1} enclosed by hyperplanes $t = \lambda_1, t = \lambda_2$ and the graph of φ , let us suppose that λ_1 and λ_2 are small enough in such a way that $\mathcal{S}_{\lambda_1, \lambda_2}$ has a "lens shape" and let us apply the Green identity. We get

$$\begin{aligned} & \int_{B_1} v^T(x, \lambda_2)A(x, \lambda_2)u(x, \lambda_2\gamma(x))dx + \\ & + \operatorname{sgn}(\lambda_1 - \lambda_2) \int_{\mathcal{S}_{\lambda_1, \lambda_2}} v^T(x, t)F(x, t)dxdt = \\ & = \int_{B_1} v^T(x, \lambda_1)A(x, \lambda_1)u(x, \lambda_1)dx, \end{aligned} \quad (11.2.49)$$

Let $v(x, t; \lambda)$ the solution to

$$\begin{cases} v_t = \sum_{j=1}^n (A_j^T(x, t)v)_{x_j} - A_0^T(x, t)v, \\ v(x, \lambda) = W e^{-i\xi \cdot x}, \quad x \in B_1. \end{cases} \quad (11.2.50)$$

Now we set

$$g(\lambda) := \int_{B_1} e^{-i\xi \cdot x} W^T u(x, \lambda \gamma(x)) dx$$

and along the lines of the proof of Theorem 11.2.1 we obtain an estimate like (11.1.9) (reader take care of the details).

In the case in which g does not vanish we may reduce to the previous case by setting

$$\tilde{u}(x, t) = u(x, t) - g(x). \quad (11.2.51)$$

Let us examine the situation in some detail. First, let us assume, for the sake of brevity, that in (11.2.46) we have $F \equiv 0$. Furthermore, we assume that the solution u of Cauchy problem (11.2.46) there exists in an open set D , we assume that $u \in C^2(\overline{D})$ and that u satisfies the a priori information

$$\|u\|_{C^2(\overline{D})} \leq E. \quad (11.2.52)$$

In addition, let us assume that

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon. \quad (11.2.53)$$

We have that \tilde{u} satisfies

$$\begin{cases} \tilde{u}_t = \sum_{j=1}^n A_j(x, t)\tilde{u}_{x_j} + A_0(x, t)\tilde{u} + \tilde{F}(x, t), \\ \tilde{u}(x, \varphi(x)) = 0, \quad x \in B_1, \end{cases} \quad (11.2.54)$$

where

$$\tilde{F}(x, t) = \sum_{j=1}^n A_j(x, t)g_{x_j}(x) + A_0(x, t)g(x). \quad (11.2.55)$$

Now by Proposition 10.2.4 we have

$$\|\nabla g\|_{L^\infty(B_1)} \leq c \left(\|\partial^2 g\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)} \right)^{\frac{1}{2}} \|g\|_{L^\infty(B_1)}^{\frac{1}{2}}, \quad (11.2.56)$$

where c is a positive constant depending on n only. Inequality (11.2.56) allows us to estimate from above the first derivatives of g in terms of ε and the a priori bound (11.2.52). Concerning the latter it suffices to recall that $g(x) = u(x, \varphi(x))$ and to calculate the derivatives of g obtaining

$$\|\partial^2 g\|_{L^\infty(B_1)} \leq K_1 E, \quad (11.2.57)$$

where K_1 depends on $\|\varphi\|_{C^2(\overline{B_1})}$. By (11.2.53), (11.2.55), (11.2.56) and (11.2.57) we have

$$\|\tilde{F}\|_{C^2(\overline{D})} \leq K_2 (E + \varepsilon)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}. \quad (11.2.58)$$

Finally, taking into account that by (11.2.51) and (11.2.52) we have

$$\|\tilde{u}\|_{C^2(\overline{D})} \leq K_3 E. \quad (11.2.59)$$

By using what is obtained in the case $g \equiv 0$, we get, by (11.2.58) and (11.2.59), a stability estimate for \tilde{u} from which immediately follows a stability estimate for u . We invite the reader to write explicitly a stability estimate for the solution u to problem (11.2.46) provided the a priori information (11.2.52) is satisfied.

4. Stability estimate (11.1.9) is a *logarithmic estimate* and, while it is still a stability estimate, it is a rather modest estimate. John, in [40], called "well-behaved" the problems for which a Hölder conditional stability holds and "not well-behaved" the problems for which the conditional stability is at best of logarithmic type. This terminology is still in use today. Of course, in order to be able to say that a class of problems is "well-behaved" or "not well-behaved" with respect to certain a priori informations, it needs to be shown that the estimate in question is optimal in that class of problems with those certain a priori informations. Concerning Theorem 11.1.1, the class of problems is the class of the Cauchy problems for partial differential equations with analytic coefficients and the a priori bounds concern a finite numbers of derivatives of the solutions. Now, with respect to the class of problems and of the a priori informations that we have considered above, John himself, in [40], proved that the Cauchy problem is "not well-behaved." The example constructed by John concerns the following Cauchy problem for the wave equation

$$\begin{cases} u_{xx} + u_{yy} - u_{tt} = 0, & x^2 + y^2 < 1, \quad t \in \mathbb{R}, \\ u = g_0, & x \in \partial B_\rho \times \mathbb{R}, \\ \frac{\partial u}{\partial \nu} = g_1, & x \in \partial B_\rho \times \mathbb{R}. \end{cases} \quad (11.2.60)$$

where $\rho < 1$ and ν is the unit outward normal to $\partial B_\rho \times \mathbb{R}$.

More precisely, set

$$\varepsilon = \|g_0\|_{L^\infty(\partial B_\rho \times \mathbb{R})} + \|g_1\|_{L^\infty(\partial B_\rho \times \mathbb{R})}, \quad (11.2.61)$$

John has proved that for every $m \in \mathbb{N}$ there exists $u \in C^m(\overline{B_1} \times \mathbb{R})$ solution to (11.2.60), where g_0, g_1 satisfy (11.2.61), such that

$$\|u\|_{C^m(\overline{B_1} \times \mathbb{R})} = 1$$

and such that

$$\|u\|_{L^\infty(B_r \times \mathbb{R})} \geq C |\log \varepsilon|^{-\alpha},$$

where $r \in (\rho, 1)$, $C > 0$ and $\alpha > 0$ depend on r . ♦

Part III

CARLEMAN ESTIMATES AND UNIQUE CONTINUATION PROPERTIES

Chapter 12

PDEs with constant coefficients in the principal part

12.1 Introduction

We begin to study the unique continuation properties for **operators with non analytic coefficients**. First we give some definitions. Let Ω an open connected set of \mathbb{R}^n , we say that the linear differential equation

$$Lu = 0 \quad \text{in } \Omega, \quad (12.1.1)$$

enjoys the **weak unique continuation property** if for any open subset ω of Ω ,

$$P(x, \partial)u = 0 \quad \text{in } \Omega \text{ and } u = 0 \text{ in } \omega \quad \implies u \equiv 0.$$

We say that equation (12.1.1) enjoys the **strong unique continuation property** if for any point $x_0 \in \Omega$ and for any solution u which satisfies

$$\lim_{r \rightarrow 0} r^{-k} \int_{B_r(x_0)} u^2 = 0, \quad \forall k \in \mathbb{N},$$

it follows that

$$u \equiv 0, \quad \text{in } \Omega.$$

It is obvious that the strong unique continuation property implies the weak unique continuation property.

As we will see later, the weak unique continuation property is strictly related to the uniqueness of the Cauchy problem for equation (12.1.1).

In the present Chapter we consider the linear differential operators whose **principal part has constant coefficients**. In other words, we will consider the operators

$$Lu = P(D)u + M(x, D)u, \quad (12.1.2)$$

where $P(D)$ is a differential operator of order m whose coefficients are constant (real or complex) and $M(x, D)$ is a differential operator of order (less or equal to) $m - 1$ whose coefficients belong to L^∞ and

$$D_j = \frac{1}{i} \partial_j, \quad j = 1, \dots, n.$$

The latter notation is very convenient in this Chapter because we will be using extensively the Fourier transform.

One of the main purposes of this Chapter is to lay the ground for the Carleman estimates, which will be studied more systematically in the next chapters. These types of estimates were introduced by Carleman in [14] and [15] (in 1933 and 1939 respectively). With these estimates a very important qualitative step is accomplished in the investigation of the unique continuation properties for partial differential equations, particularly for the Cauchy problem. Indeed, by means of the Carleman estimates, one can prove the unique continuation properties for differential equations with nonanalytic coefficients. Actually, the estimates proved in [14] and [15] involve partial differential equations of two variables, but the idea introduced by Carleman has revealed to be very fruitful leading to the development of a technique that constitutes certainly the most general and powerful tool, though not unique, for dealing with unique continuation issues.

The Main Theorem which we will prove here is due to **Nirenberg**, (see Theorem 12.2.1), [60]. Subsequently, we will apply such a Theorem to obtain the weak unique continuation property for the equation

$$\Delta u - b(x) \cdot \nabla u - c(x)u = 0. \quad (12.1.3)$$

where $b = (b_1, \dots, b_n) \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$, $c \in L^\infty(\mathbb{R}^n, \mathbb{C})$. Moreover, we will illustrate other applications and relevant features of Theorem 12.2.1.

12.2 The Nirenberg Theorem

Let us introduce and recall some notations.

Let $P(D)$ be the operator

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad (12.2.1)$$

where $a_\alpha \in \mathbb{C}$, for every $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leq m$. Let

$$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \quad \forall \xi \in \mathbb{R}^n,$$

the symbol of $P(D)$. For each multi-index α we denote

$$P^{(\alpha)}(\xi) = \partial_\xi^\alpha P(\xi).$$

We set

$$Q_1 = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \dots, n\}.$$

We prove the following

Theorem 12.2.1 (Nirenberg). *Let $N \in \mathbb{R}^n$, $|N| = 1$. Then there exists a constant C , depending on n and m only, such that for every $\alpha \in \mathbb{N}_0^n$ we have*

$$\int_{Q_1} e^{2\tau N \cdot x} |P^{(\alpha)}(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.2.2)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

Remark 1. Estimate (12.2.2) is a prototype of the Carleman estimates. Let us notice that in such an estimate there is a "weight," $e^{2\tau N \cdot x}$ dependent on a parameter τ , and it is very important that such a parameter can be arbitrarily large.

Let us observe, in particular, that the at right-hand side of (12.2.2) it occurs the operator $P(D)$ applied to an arbitrary $u \in C_0^\infty(Q_1, \mathbb{C})$, *not* to a solution of some equation. \blacklozenge

In order to prove Theorem 12.2.1 we need some preliminary results.

First of all, let us recall the following one-dimensional Poincaré inequality

$$\int_{-1}^1 |u|^2 dt \leq \frac{4}{\pi^2} \int_{-1}^1 |u'|^2 dt, \quad \forall u \in C_0^\infty((-1, 1), \mathbb{C}). \quad (12.2.3)$$

Now let us prove

Lemma 12.2.2. *There exists $C_0 > 0$ such that for each $\gamma \in \mathbb{C}$ we have*

$$\int_{-1}^1 |u|^2 dt \leq C_0 \int_{-1}^1 |u' - \gamma u|^2 dt, \quad \forall u \in C_0^\infty((-1, 1), \mathbb{C}). \quad (12.2.4)$$

Proof. Let $\gamma = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. We have

$$\begin{aligned} |u' - \gamma u|^2 &= \alpha^2 |u|^2 + |u' - i\beta u|^2 - 2\alpha \Re((u' - i\beta u)\bar{u}) = \\ &= \alpha^2 |u|^2 + |(ue^{-i\beta t})'|^2 - \alpha (|u|^2)'. \end{aligned}$$

Hence, as $u \in C_0^\infty((-1, 1), \mathbb{C})$, taking into account (12.2.3), we get

$$\begin{aligned} \int_{-1}^1 |u' - \gamma u|^2 dt &= \int_{-1}^1 \left(\alpha^2 |u|^2 + |(ue^{-i\beta t})'|^2 \right) dt \geq \\ &\geq \int_{-1}^1 |(ue^{-i\beta t})'|^2 dt \geq \frac{\pi^2}{4} \int_{-1}^1 |ue^{-i\beta t}|^2 dt = \\ &= \frac{\pi^2}{4} \int_{-1}^1 |u|^2 dt. \end{aligned}$$

Therefore inequality (12.2.4) is proved with $C_0 = \frac{4}{\pi^2}$. ■

Let $a_1, \dots, a_k \in \mathbb{C}$, $a_k \neq 0$, and let

$$p(\eta) = \sum_{j=0}^k a_j \eta^j, \quad \eta \in \mathbb{C}. \quad (12.2.5)$$

Let us consider the differential operator

$$p(D_t) = \sum_{j=0}^k a_j D_t^j, \quad (12.2.6)$$

where $D_t = \frac{1}{i} \frac{d}{dt}$. Set

$$p'(D_t) = \sum_{j=1}^k j a_j D_t^{j-1}.$$

We have the following

Lemma 12.2.3. *Let $k \in \mathbb{N}$. Then there exists $C_1 > 0$ depending on k only, such that we have*

$$\int_{-1}^1 |p'(D_t)u|^2 dt \leq C_1 \int_{-1}^1 |p(D_t)u|^2 dt, \quad \forall u \in C_0^\infty((-1, 1), \mathbb{C}). \quad (12.2.7)$$

Proof. It is not restrictive to assume that $a_k = 1$. Let $\gamma_1, \dots, \gamma_k \in \mathbb{C}$ be the roots of the polynomial p , we have

$$p(\eta) = \prod_{1 \leq j \leq k} (\eta - \gamma_j).$$

Set

$$p_l(\eta) = \frac{1}{(\eta - \gamma_l)} \prod_{1 \leq j \leq k} (\eta - \gamma_j), \text{ for } l = 1, \dots, k.$$

We have

$$p'(\eta) = \sum_{l=1}^k p_l(\eta), \quad \forall \eta \in \mathbb{C}. \quad (12.2.8)$$

Let $l \in \{1, \dots, k\}$ be fixed, $u \in C_0^\infty((-1, 1), \mathbb{C})$ and let us denote

$$v_l = p_l(D_t)u.$$

Let us observe that

$$(D_t - \gamma_l)v_l = p(D_t)u.$$

Now we apply Lemma 12.2.2 to v_l (where $\gamma = i^{-1}\gamma_l$) and we get

$$\begin{aligned} \int_{-1}^1 |p_l(D_t)u|^2 dt &= \int_{-1}^1 |v_l|^2 dt \leq \\ &\leq C_0 \int_{-1}^1 |(D_t - \gamma_l)v_l|^2 dt = \\ &= C_0 \int_{-1}^1 |p(D_t)u|^2 dt. \end{aligned} \quad (12.2.9)$$

By (12.2.8) and (12.2.9) we have

$$\begin{aligned} \int_{-1}^1 |p'(D_t)u|^2 dt &\leq k \sum_{l=1}^k \int_{-1}^1 |p_l(D_t)u|^2 dt \leq \\ &\leq C_0 k^2 \int_{-1}^1 |p(D_t)u|^2 dt. \end{aligned} \quad (12.2.10)$$

Hence, inequality (12.2.7) is proved with $C_1 = k^2 C_0$. ■

Theorem 12.2.4 (Hörmander). *Let $P(D)$ be a differential operator of order m with constant coefficients. Then there exists a constant C_2 which depends on m and on n only, such that we have, for any $\alpha \in \mathbb{N}_0^n$,*

$$\int_{Q_1} |P^{(\alpha)}(D)u|^2 dx \leq C_2 \int_{Q_1} |P(D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}). \quad (12.2.11)$$

Proof. First of all we prove (12.2.11) for $\alpha = e_j$, $j = 1, \dots, n$. It is not restrictive to assume $j = n$. for any $f \in L^2(\mathbb{R}^n)$ we set

$$\widehat{f}(\xi', x_n) = \mathcal{F}_{\xi'}(f(\cdot, x_n)) = \int_{\mathbb{R}^{n-1}} f(x', x_n) e^{-ix'\xi'} dx', \quad \forall \xi' \in \mathbb{R}^{n-1}.$$

Let $u \in C_0^\infty(Q_1, \mathbb{C})$; we have

$$\mathcal{F}_{\xi'}(P(D)u) = P(\xi', D_n) \widehat{u}(\xi', x_n) \tag{12.2.12}$$

and by the Parseval identity, we have

$$\begin{aligned} \int_{Q_1} |P^{(e_n)}(D)u|^2 dx &= \\ &= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} d\xi' \int_{-1}^1 |P^{(e_n)}(\xi', D_n) \widehat{u}(\xi', x_n)|^2 dx_n. \end{aligned} \tag{12.2.13}$$

Now we apply Lemma 12.2.9 to the operator $p(D_n) = P(\xi', D_n)$, where $\xi' \in \mathbb{R}^{n-1}$ is fixed. We obtain

$$\begin{aligned} \int_{-1}^1 |P^{(e_n)}(\xi', D_n) \widehat{u}(\xi', x_n)|^2 dx_n &\leq \\ &\leq C_1 \int_{-1}^1 |P(\xi', D_n) \widehat{u}(\xi', x_n)|^2 dx_n. \end{aligned} \tag{12.2.14}$$

By (12.2.13) and (12.2.14) we have

$$\begin{aligned} \int_{Q_1} |P^{(e_n)}(D)u|^2 dx &\leq \frac{C_1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} d\xi' \int_{-1}^1 |P(\xi', D_n) \widehat{u}(\xi', x_n)|^2 dx_n = \\ &= C_1 \int_{Q_1} |P(D)u|^2 dx. \end{aligned}$$

Since the previous proof can be repeated for any indices. We have that for each multi-indices α such that $|\alpha| = 1$ the estimate following holds

$$\int_{Q_1} |P^{(\alpha)}(D)u|^2 dx \leq C_1 \int_{Q_1} |P(D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}). \tag{12.2.15}$$

By iteration of (12.2.15) we get, for any $\alpha \in \mathbb{N}_0^n$,

$$\int_{Q_1} |P^{(\alpha)}(D)u|^2 dx \leq C_1^{|\alpha|} \int_{Q_1} |P(D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}).$$

Hence inequality (12.2.11) is now proved with $C_2 = C_1^{|\alpha|}$. ■

Proof of Theorem 12.2.1.

Let $u \in C_0^\infty(Q_1, \mathbb{C})$. Setting $v = e^{\tau N \cdot x} u$, we obtain

$$D_j u = e^{-\tau N \cdot x} (D_j + i\tau N_j) v. \quad (12.2.16)$$

For each multi-index α , we have

$$D^\alpha u = e^{-\tau N \cdot x} (D + i\tau N)^\alpha v.$$

Hence

$$e^{\tau N \cdot x} P(D)u = P(D + i\tau N)v, \quad e^{\tau N \cdot x} P^{(\alpha)}(D)u = P^{(\alpha)}(D + i\tau N)v.$$

Therefore by (12.2.11) we get

$$\begin{aligned} \int_{Q_1} e^{2\tau N \cdot x} |P^{(\alpha)}(D)u|^2 dx &= \int_{Q_1} |P^{(\alpha)}(D + i\tau N)v|^2 dx \leq \\ &\leq C_2 \int_{Q_1} |P(D + i\tau N)v|^2 dx = \\ &= C_2 \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \end{aligned}$$

where C_2 is the same constant of (12.2.11). ■

Remark 1.

Since $P(D)$ is an operator of order m , we have that there exists $\alpha \in \mathbb{N}_0^n$ such that $P^{(\alpha)}(\xi) = \alpha! a_\alpha \neq 0$. Therefore (12.2.2) gives, in particular,

$$\int_{Q_1} |u|^2 dx \leq C_3 \int_{Q_1} |P(D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}), \quad (12.2.17)$$

where

$$C_3 = C_2 \left(\frac{1}{\alpha!} \min\{|a_\alpha| : a_\alpha \neq 0, |\alpha| = m\} \right)^2.$$

By the proof of Theorem 12.2.1 we observe that, if $M(\xi)$ is a polynomial for which there exists a constant $C_4 > 0$ such that

$$\frac{|M(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2} \leq C_4, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \tau \in \mathbb{R} \quad (12.2.18)$$

then there exists a constant C such that

$$\int_{Q_1} e^{2\tau N \cdot x} |M(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.2.19)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

Likewise, if $M_\tau(\xi)$ is a polynomial in the variable ξ depending by the parameter τ and if

$$\sup \left\{ \frac{|M_\tau(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2} : (\xi, \tau) \in \mathbb{R}^{n+1} \right\} < +\infty, \quad (12.2.20)$$

then, for a constant C (independent of τ and u), we have

$$\int_{Q_1} e^{2\tau N \cdot x} |M_\tau(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.2.21)$$

for any $u \in C_0^\infty(Q_1, \mathbb{C})$ and any $\tau \in \mathbb{R}$.

For instance, in the case

$$P(D) = -(D_1^2 + \cdots + D_n^2) = \Delta,$$

we have

$$P(\xi) = -(\xi_1^2 + \cdots + \xi_n^2), \quad \sum_{|\alpha| \leq 2} |P^{(\alpha)}(\xi)|^2 = |\xi|^4 + 4(n^2 + |\xi|^2),$$

from which we easily obtain that (12.2.18) is satisfied for all $N \in \mathbb{R}^n$, $|N| = 1$, provided $M(\xi) = \xi_j$ (as well as, of course, for $M(\xi) = 1$). Hence, we have

$$\int_{Q_1} (|u|^2 + |\nabla u|^2 + |D^2 u|^2) dx \leq C \int_{Q_1} |\Delta u|^2 dx, \quad (12.2.22)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$, where

$$|D^2 u|^2 = \sum_{j,k=1}^n |D_{jk}^2 u|^2 = \sum_{j,k=1}^n |\partial_{jk}^2 u|^2.$$

Moreover, we have

$$\sum_{|\alpha| \leq 2} |P^{(\alpha)}(\xi + i\tau N)|^2 = (|\xi|^2 - \tau^2)^2 + 4\tau^2(\xi \cdot N)^2 + 4(|\xi|^2 + \tau^2) + 4n^2$$

and, setting

$$M_{0,\tau}(\xi) = \tau, \quad M_{j,\tau}(\xi) = \xi_j, \quad j = 1, \dots, n,$$

we have (reader check), for $k = 1, \dots, n$,

$$\sup \left\{ \frac{|M_{k,\tau}(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq 2} |P^{(\alpha)}(\xi + i\tau N)|^2} : (\xi, \tau) \in \mathbb{R}^{n+1} \right\} < +\infty.$$

Hence,

$$\int_{Q_1} e^{2\tau N \cdot x} (\tau^2 |u|^2 + |\nabla u|^2) dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |\Delta u|^2 dx, \quad (12.2.23)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$. Trivially, also the following estimate holds

$$\int_{Q_1} e^{2\tau N \cdot x} (|u|^2 + |\nabla u|^2) dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |\Delta u|^2 dx, \quad (12.2.24)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

In what follows, we will exploit estimate (12.2.24) to prove some unique continuation property for the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U,$$

with $b = (b_1, \dots, b_n) \in L^\infty(\mathbb{R}^n)$, $c \in L^\infty(\mathbb{R}^n)$.

As it will be clear later on, the aforesaid unique continuation property results could be derived with a slightly less effort by using (12.2.23) instead of (12.2.24). However, part of the arguments that we will use employing (12.2.24) can be extended to differential operators which are more general and this, in a certain sense, will repay us for the greater effort we will put into using (12.2.24). ♦

12.3 Application of the Nirenberg Theorem to the Laplace operator

In this Section we will apply estimate (12.2.24) to obtain the weak unique continuation property and **the uniqueness** for the Cauchy problem to equation (12.1.3).

The steps to be done are fairly numerous and, in order to highlight the key points, we proceed gradually. First, we warn that we should not confuse the solution of equation (12.1.3) with u in inequality (12.2.24). Now, let us dwell on inequality (12.2.24) and we notice that, since $C_0^\infty(Q_1, \mathbb{C})$ is dense in $H_0^2(Q_1, \mathbb{C})$, estimate (12.2.24) holds true also for any $u \in H_0^2(Q_1, \mathbb{C})$.

Hence, we have

$$\int_{Q_1} e^{2\tau N \cdot x} (|u|^2 + |\nabla u|^2) dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |\Delta u|^2 dx, \quad (12.3.1)$$

for every $u \in H_0^2(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

Actually, it would not be difficult to prove that (12.2.24) holds for each $u \in H_0^1(Q_1, \mathbb{C})$ which satisfies $\Delta u \in L^2(Q_1, \mathbb{C})$. However, at least for the time being, let us omit further consideration on this point. Let us recall that (Theorem 4.6.1):

Proposition 12.3.1. *Let Ω be an open set of \mathbb{R}^n , $f \in L^2(\Omega, \mathbb{C})$ and let $U \in H^1(\Omega, \mathbb{C})$ satisfy*

$$\int_{\Omega} \nabla U \cdot \nabla \varphi dx = - \int_{\Omega} f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega, \mathbb{C}), \quad (12.3.2)$$

then $U \in H_{loc}^2(\Omega, \mathbb{C})$.

Let $b = (b_1, \dots, b_n) \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $c \in L^\infty(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$.

Let us start to consider the following Cauchy problem. Let

$$h(x') = 1 - \sqrt{1 - |x'|^2},$$

$$\Lambda = \{(x', x_n) \in B'_1 \times \mathbb{R} : h(x') < x_n < 1\}$$

and

$$\Gamma = \{(x', h(x')) : x' \in B'_1\}.$$

We say that $U \in H^2(\Lambda)$ is a solution of the Cauchy problem

$$\begin{cases} \Delta U = b(x) \cdot \nabla U + c(x)U + f(x), & \text{in } \Lambda, \\ U = 0, & \text{on } \Gamma, \\ \frac{\partial U}{\partial \nu} = 0, & \text{on } \Gamma, \end{cases} \quad (12.3.3)$$

provided $U \in H^2(\Lambda)$ and

$$\begin{cases} \Delta U = b(x) \cdot \nabla U + c(x)U + f(x), & \text{in } \Lambda \\ U\Psi \in H_0^2(\Lambda), \quad \forall \Psi \in C^\infty(\mathbb{R}^n), \quad \text{supp}\Psi \subset \mathbb{R}^{n-1} \times (-\infty, 1). \end{cases} \quad (12.3.4)$$

Let us observe that in formulation (12.3.4), we express the conditions $U = \frac{\partial U}{\partial \nu} = 0$ on Γ as " $U\Psi \in H_0^2(\Lambda)$, for every $\Psi \in C^\infty(\mathbb{R}^n)$, such that $\text{supp}\Psi \subset \mathbb{R}^{n-1} \times (-\infty, 1)$ ". Another way to express correctly these initial conditions is through the definition of the traces. In the case of initial surface Γ that we are considering, both the formulations are equivalent (due to the regularity of Γ), however formulation (12.3.4) is more elementary because it allows us to dispense with the notion of the trace.

Set

$$K = \|b\|_{L^\infty(\Lambda)} + \|c\|_{L^\infty(\Lambda)}, \quad \varepsilon = \|f\|_{L^2(\Lambda)}. \quad (12.3.5)$$

Our goal is to find an estimate that, roughly speaking, tells us that if ε "is small" then $\|U\|_{L^2(\Lambda)}$ "is small" and tell us that if $\varepsilon = 0$ (thus $f \equiv 0$) then $U \equiv 0$

Let us start by considering the simple case in which, in (12.3.3), K and ε are zero. In such a case we have

$$\begin{cases} \Delta U = 0, & \text{in } \Lambda, \\ U\Psi \in H_0^2(\Lambda), \quad \forall \Psi \in C^\infty(\mathbb{R}^n), \quad \text{supp}\Psi \subset \mathbb{R}^{n-1} \times (-\infty, 1). \end{cases} \quad (12.3.6)$$

Let $\delta \in (0, \frac{1}{3})$ and let $\zeta \in C^\infty(\mathbb{R})$ satisfy $0 \leq \zeta \leq 1$, (Figure 10.1)

$$\zeta(x_n) = 1, \text{ for } x_n \leq 1 - 2\delta, \quad \zeta(x_n) = 0 \text{ for } 1 - \delta \leq x_n < 1$$

and

$$|\zeta'(x_n)| \leq c\delta^{-1} \text{ e } |\zeta''(x_n)| \leq c\delta^{-2}, \text{ for } 1 - 2\delta \leq x_n \leq 1 - \delta,$$

where c is a constant.

Let us extend U to zero in $\{(x', x_n) \in B_1' \times \mathbb{R} : -1 < x_n < h(x')\}$, in such a way that this extension belongs to $H^2(Q_1)$, we continue to denote by U such an extension. We have $\zeta U \in H_0^2(Q_1)$. Let us apply (12.3.1) to

$$u = \zeta(x_n)U(x),$$

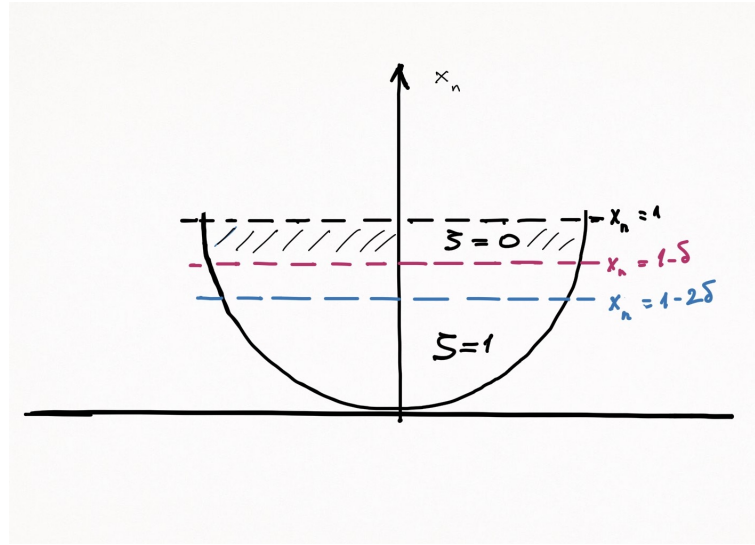


Figure 12.1:

where

$$N = -e_n, \text{ and } \tau > 0.$$

We have

$$\Delta(\zeta U) = \zeta(x_n)\Delta U + 2\zeta'(x_n)\partial_n U + \zeta''(x_n)U$$

as $\Delta U = 0$, we get

$$|\Delta(\zeta U)| \leq \chi_{(1-2\delta, 1-\delta)}(x_n) (2c\delta^{-1} |\partial_n U| + c\delta^{-2} |U|). \quad (12.3.7)$$

Hence by (12.3.1) and (12.3.7) we get

$$\begin{aligned} \int_{Q_1} e^{-2\tau x_n} (|\zeta U|^2 + |\nabla(\zeta U)|^2) dx &\leq C \int_{Q_1} e^{-2\tau x_n} |\Delta(\zeta U)|^2 dx \leq \\ &\leq C' \delta^{-4} \int_{Q_1 \cap \{1-2\delta < x_n < 1-\delta\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx \leq \\ &\leq C' e^{-2\tau(1-2\delta)} \delta^{-4} \int_{Q_1 \cap \{1-2\delta < x_n < 1-\delta\}} (|U|^2 + |\nabla U|^2) dx \leq \\ &\leq C' e^{-2\tau(1-2\delta)} \delta^{-4} \|U\|_{H^1(Q_1)}^2, \end{aligned} \quad (12.3.8)$$

for every $\tau > 0$.

Now we have, trivially, for every $\tau > 0$

$$\begin{aligned}
 & \int_{Q_1} e^{-2\tau x_n} (|\zeta U|^2 + |\nabla(\zeta U)|^2) dx \geq \\
 & \geq e^{-2\tau(1-3\delta)} \int_{\Lambda \cap \{x_n < 1-3\delta\}} (|U|^2 + |\nabla U|^2) dx.
 \end{aligned} \tag{12.3.9}$$

By (12.3.8) and (12.3.9) we obtain

$$\begin{aligned}
 & z \int_{\Lambda \cap \{x_n < 1-3\delta\}} (|U|^2 + |\nabla U|^2) dx \leq \\
 & \leq C' e^{-2\tau\delta} \delta^{-4} \|U\|_{H^1(Q_1)}^2 \rightarrow 0, \quad \text{as } \tau \rightarrow +\infty.
 \end{aligned}$$

From which we have $U = 0$ in $\Lambda \cap \{x_n < 1 - 3\delta\}$ and, as δ is arbitrary, we get $U = 0$ in Λ .

In the general case, particularly when b or c are not zero, one would like to argue in a similar manner, but one encounters an obstacle due to the fact that in expanding $|\Delta(\zeta U)|$, besides the terms present in (12.3.7), new terms, depending on U and ∇U , arise. To overcome this difficulty, we first perform a rescaling of estimate (12.3.1)

For any $r > 0$, set

$$Q_r = \{x \in \mathbb{R}^n : |x_j| < r, \quad j = 1, \dots, n\}.$$

Proposition 12.3.2. *Let $N \in \mathbb{R}^n$ ($|N| = 1$). There exists a constant $C > 0$ so that*

$$\int_{Q_r} e^{2\tau N \cdot x} (|u|^2 + r^2 |\nabla u|^2) dx \leq Cr^4 \int_{Q_r} e^{2\tau N \cdot x} |\Delta u|^2 dx, \tag{12.3.10}$$

for every $r > 0$, for every $u \in H_0^2(Q_r)$ and for every $\tau \in \mathbb{R}$.

Proof.

Let $u \in H_0^2(Q_r)$ and let us denote

$$\tilde{u}(y) = u(ry), \quad \forall y \in Q_1.$$

We apply (12.3.1) to \tilde{u} , replacing there τ by τr . We get

$$\int_{Q_1} e^{2\tau r N \cdot y} (|u(ry)|^2 + r^2 |(\nabla u)(ry)|^2) dy \leq Cr^4 \int_{Q_1} e^{2\tau r N \cdot y} |(\Delta u)(ry)|^2 dy.$$

Now, by performing the change of variables $y = r^{-1}x$, we have

$$\int_{Q_r} e^{2\tau N \cdot x} (|u(x)|^2 + r^2 |\nabla u(x)|^2) r^{-n} dx \leq Cr^4 \int_{Q_r} e^{2\tau N \cdot x} |\Delta u(x)|^2 r^{-n} dy.$$

By last inequality we immediately have (12.3.10). ■

Now let us go back to Cauchy problem (12.3.3). Let $r \in (0, 1)$ be a number to be chosen later, set

$$\rho = \frac{r^2}{2}, \quad R = \sqrt{2\rho - \rho^2}.$$

It is easy to check that $\rho, R < r$ hence

$$\Lambda_\rho := \{(x', x_n) \in B'_R \times \mathbb{R} : h(x') < x_n < \rho\} \subset Q_r.$$

Let $\delta \in (0, \frac{1}{3})$ and $\zeta \in C^\infty(\mathbb{R})$ satisfy $0 \leq \zeta \leq 1$,

$$\zeta(x_n) = 1, \text{ for } x_n \leq \rho(1 - 2\delta), \quad \zeta(x_n) = 0 \text{ for } \rho(1 - \delta) \leq x_n < \rho$$

and

$$|\zeta'(x_n)| \leq c(\delta\rho)^{-1} \text{ and } |\zeta''(x_n)| \leq c(\delta\rho)^{-2}, \text{ for } \rho(1 - 2\delta) \leq x_n \leq \rho(1 - \delta),$$

where c is a constant.

Now we extend U to zero in $\{(x', x_n) \in B'_R \times \mathbb{R} : -\rho < x_n < h(x')\}$, this extension belongs to $H^2(Q_r)$, we continue to denote it by U , it turns out that $\zeta U \in H_0^2(Q_r)$. Let us prepare to apply the estimate (12.3.10) to

$$u = \zeta(x_n)U(x),$$

where $N = -e_n$ and $\tau > 0$.

$$\begin{aligned} \Delta(\zeta U) &= \zeta(x_n)\Delta U + 2\zeta'(x_n)\partial_n U + \zeta''(x_n)U = \\ &= \zeta(x_n)(b \cdot \nabla U + cU + f) + \\ &\quad + 2\zeta'(x_n)\partial_n U + \zeta''(x_n)U. \end{aligned}$$

From which we have, taking into account (12.3.5),

$$\begin{aligned} |\Delta(\zeta U)| &\leq K\zeta(|U| + |\nabla U|) + \zeta|f| + \\ &\quad + c(\rho\delta)^{-2}\chi_I(x_n)(|\partial_n U| + |U|), \end{aligned} \tag{12.3.11}$$

where we set $I = (\rho(1 - 2\delta), \rho(1 - \delta))$.

Let us denote

$$J_1 = \int_{Q_r} e^{-2\tau x_n} \zeta^2 (|U|^2 + |\nabla U|^2) dx,$$

and

$$J_2 = \int_{Q_r \cap \{\rho(1-2\delta) < x_n < \rho(1-\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx.$$

Hence by (12.3.10) and (12.3.11) we have (recall that C denotes a constant that may change from line to line)

$$\begin{aligned} & \int_{Q_r} e^{-2\tau x_n} (|\zeta U|^2 + r^2 |\nabla(\zeta U)|^2) dx \leq \\ & \leq Cr^4 \int_{Q_r} e^{-2\tau x_n} |\Delta(\zeta U)|^2 dx \leq \\ & \leq CK^2 r^4 J_1 + Cr^4 (\rho\delta)^{-4} J_2 + \\ & + Cr^4 \int_{Q_r} e^{-2\tau x_n} \zeta^2 |f|^2 dx, \end{aligned} \tag{12.3.12}$$

for every $\tau > 0$.

At this point it is useful to note that compared to (12.3.8), here we have two new terms: J_1 and $r^4 \int_{Q_r} e^{-2\tau x_n} \zeta^2 |f|^2 dx$. Recalling the second equality in (12.3.5), we estimate from above the last term in (12.3.8) as follows

$$r^4 \int_{Q_r} e^{-2\tau x_n} \zeta^2 |f|^2 dx \leq r^4 \int_{Q_r} |f|^2 dx = r^4 \varepsilon^2, \tag{12.3.13}$$

for every $\tau > 0$.

Before considering the term J_1 , let us estimate from above J_2 basically in the same way as done in (12.3.8). We have

$$\begin{aligned} J_2 &= \int_{Q_r \cap \{\rho(1-2\delta) < x_n < \rho(1-\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx \leq \\ &\leq Ce^{-2\tau\rho(1-2\delta)} \int_{Q_r \cap \{\rho(1-2\delta) < x_n < \rho(1-\delta)\}} (|U|^2 + |\nabla U|^2) dx \leq \\ &\leq Ce^{-2\tau\rho(1-2\delta)} \|U\|_{H^1(Q_r)}^2, \end{aligned} \tag{12.3.14}$$

for every $\tau > 0$.

Hence, by (12.3.12)–(12.3.14), we get

$$\begin{aligned}
 \int_{Q_r} e^{-2\tau x_n} (|\zeta U|^2 + r^2 |\nabla(\zeta U)|^2) dx &\leq CK^2 r^4 J_1 + \\
 + C (r\rho^{-1}\delta^{-1})^4 e^{-2\tau\rho(1-2\delta)} \|U\|_{H^1(Q_r)}^2 &+ \\
 + Cr^4 \varepsilon^2, &
 \end{aligned} \tag{12.3.15}$$

for every $\tau > 0$.

By the manner in which ζ is defined, we have trivially (recall $r < 1$)

$$\begin{aligned}
 \int_{Q_r} e^{-2\tau x_n} (|\zeta U|^2 + r^2 |\nabla(\zeta U)|^2) dx &\geq \\
 \geq r^2 \int_{Q_r \cap \{x_n < \rho(1-2\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx, &
 \end{aligned} \tag{12.3.16}$$

for every $\tau > 0$.

Concerning J_1 , let us observe

$$\begin{aligned}
 J_1 &= \int_{Q_r \cap \{x_n < \rho(1-2\delta)\}} e^{-2\tau x_n} \zeta^2 (|U|^2 + |\nabla U|^2) dx + \\
 + \int_{Q_r \cap \{\rho(1-2\delta) < x_n < \rho(1-\delta)\}} e^{-2\tau x_n} \zeta^2 (|U|^2 + |\nabla U|^2) dx &\leq \\
 \leq \int_{Q_r \cap \{x_n < \rho(1-2\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx + & \\
 + e^{-2\tau\rho(1-2\delta)} \|U\|_{H^1(Q_r)}^2, &
 \end{aligned} \tag{12.3.17}$$

for every $\tau > 0$.

Now by (12.3.15)–(12.3.17) we have

$$\begin{aligned}
 r^2 (1 - CK^2 r^2) \int_{Q_r \cap \{x_n < \rho(1-2\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx &\leq Cr^4 \varepsilon^2 + \\
 + Cr^4 (\rho^{-1}\delta^{-1})^4 e^{-2\tau\rho(1-2\delta)} \|U\|_{H^1(Q_r)}^2, &
 \end{aligned}$$

for every $\tau > 0$.

Now, let us choose $r = r_0 < 1$ satisfying $1 - CK^2 r_0^2 \geq \frac{1}{2}$ (here, recall that C does not depend on r) and denoting by ρ_0 and R_0 the values of ρ and R correspondingly to this choice of r we get

$$\begin{aligned}
 \int_{Q_{r_0} \cap \{x_n < \rho_0(1-2\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx &\leq C\varepsilon^2 + \\
 + Ce^{-2\tau\rho_0(1-2\delta)} \|U\|_{H^1(\Lambda_{\rho_0})}^2, &
 \end{aligned} \tag{12.3.18}$$

for every $\tau > 0$, where C depends on K and δ only.

We have, trivially

$$\begin{aligned} & \int_{Q_{\tau_0} \cap \{x_n < \rho_0(1-2\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx \geq \\ & \geq \int_{Q_{\tau_0} \cap \{x_n < \rho_0(1-3\delta)\}} e^{-2\tau x_n} (|U|^2 + |\nabla U|^2) dx \geq \\ & \geq e^{-2\tau \rho_0(1-3\delta)} \|U\|_{H^1(\Lambda_{\rho_0(1-3\delta)})}^2, \end{aligned}$$

for every $\tau > 0$.

By the last obtained estimate and by (12.3.18) we get

$$\|U\|_{H^1(\Lambda_{\rho_0(1-3\delta)})}^2 \leq C \left[e^{2\tau \rho_0(1-3\delta)} \varepsilon^2 + e^{-2\tau \rho_0 \delta} \|U\|_{H^1(\Lambda_{\rho_0})}^2 \right], \quad (12.3.19)$$

for every $\tau > 0$.

Now, let us observe that if $\varepsilon = 0$, and if τ goes to $+\infty$, then by (12.3.19) we get $U = 0$ in $\Lambda_{\rho_0(1-3\delta)}$. Instead, if $\varepsilon > 0$ by (12.3.19) we can derive a **stability estimate** by choosing appropriately τ .

The choice of τ is driven by the idea of "balancing" the two right hand addends of (12.3.19) (or also, minimize with respect to τ the right-hand member of (12.3.19)). For this purpose it is convenient to rearrange the inequality and set

$$E = \|U\|_{H^1(\Lambda_{\rho_0})}.$$

By (12.3.19) we have, trivially

$$\|U\|_{H^1(\Lambda_{\rho_0(1-3\delta)})}^2 \leq C \left[e^{2\tau \rho_0(1-3\delta)} \varepsilon^2 + e^{-2\tau \rho_0 \delta} (E^2 + \varepsilon^2) \right], \quad (12.3.20)$$

for every $\tau > 0$.

Let us choose

$$\tau = \tau_0 = \frac{1}{2\rho_0(1-2\delta)} \log \left(\frac{E^2 + \varepsilon^2}{\varepsilon^2} \right).$$

We get

$$e^{2\tau_0 \rho_0(1-3\delta)} \varepsilon^2 = e^{-2\tau_0 \rho_0 \delta} (E^2 + \varepsilon^2) = (\varepsilon^2)^{\mu(\delta)} (E^2 + \varepsilon^2)^{1-\mu(\delta)},$$

where

$$\mu(\delta) = \frac{\delta}{1-2\delta}.$$

Hence, by (12.3.20) we have the following **stability estimate**

$$\|U\|_{H^1(\Lambda_{\rho_0(1-3\delta)})} \leq 2^{3/2} C \varepsilon^{\mu(\delta)} (E + \varepsilon)^{1-\mu(\delta)}. \quad (12.3.21)$$

It is obvious that **estimate (12.3.21) implies (putting there $\varepsilon = 0$) the uniqueness to Cauchy problem (12.3.3)**, however this estimate says something more about the Cauchy problem. Precisely tells us that if we have a bound of the norm H^1 of the solution U , then we can estimate the error on the solution U starting from the error on the datum f .

However, let us leave out the stability issue and we return to the uniqueness question for the Cauchy problem. So far we have obtained a local uniqueness result in the case where the initial surface is a semisphere (the graph of h). Recalling that the operator Δ is invariant under the rotations, the result of local uniqueness that we obtained can be easily extend to $\mathcal{U} \cap \Lambda$, where \mathcal{U} is a neighborhood of the graph of h . Actually, by further exploiting the particularity of the operator Δ we can easily obtain an intermediate result starting from which it will be easy to reach the global **uniqueness for the Cauchy problem for quite general initial surfaces**.

Such an intermediate result is proved in the following Proposition.

Proposition 12.3.3. *Let $K, R_2, R_1 > 0$, $R_1 < R_2$. Let us suppose that $U \in H^2(B_{R_2}(0) \setminus \overline{B_{R_1}})$ satisfy what follows*

$$|\Delta U| \leq K (|\nabla U| + |U|), \quad \text{in } B_{R_2} \setminus \overline{B_{R_1}} \quad (12.3.22)$$

$$U = 0, \quad \frac{\partial U}{\partial \nu} = 0, \quad \text{on } \partial B_{R_2}, \quad (12.3.23)$$

(i.e. $U \Psi \in H_0^2(B_{R_2} \setminus \overline{B_{R_1}})$ for every $\Psi \in H^2(\mathbb{R}^n)$ whose support is contained in $\mathbb{R}^n \setminus \overline{B_{R_1}}$) then

$$U = 0 \quad \text{in } B_{R_2} \setminus \overline{B_{R_1}}. \quad (12.3.24)$$

Proof. It suffices to reduce by dilation to the case in which $R_2 = 1$ and observe that to obtain (12.3.21) (with $\varepsilon = 0$) we may just use (12.3.22) instead of the equation $\Delta U = b \cdot \nabla U + cU$. Next, starting from (12.3.21) (with $\varepsilon = 0$) and using the invariance of Δ with respect to the rotations, we immediately obtain that there exists $r \in (0, R_2)$ such that $U = 0$ in $B_{R_2} \setminus \overline{B_{R_2-r}}$. From which iterating the obtained result, we get that $U \equiv 0$. ■

Remark 1. Let us observe that to require that

$$|\Delta U| \leq K (|\nabla U| + |U|), \quad \text{in } \Omega, \quad (12.3.25)$$

where Ω is an open set of \mathbb{R}^n is equivalent to require that there exist $b \in L^\infty(\Omega; \mathbb{C}^n)$ and $c \in L^\infty(\Omega; \mathbb{C})$ such that U is a solution of the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U, \quad \text{in } \Omega. \quad (12.3.26)$$

Indeed, it is obvious that if U satisfies (12.3.26) then U satisfies (12.3.25) with

$$K = \max \left\{ \|b\|_{L^\infty(\Omega; \mathbb{C}^n)}, \|c\|_{L^\infty(\Omega; \mathbb{C})} \right\}.$$

Conversely, if U satisfies (12.3.25), then we define

$$b(x) = \begin{cases} \frac{\overline{\nabla U(x)}(\Delta U(x))}{|\nabla U(x)|^2 + |U(x)|^2}, & \text{for } |\nabla U(x)|^2 + |U(x)|^2 > 0, \\ 0, & \text{for } |\nabla U(x)|^2 + |U(x)|^2 = 0 \end{cases}$$

and

$$c(x) = \begin{cases} \frac{\overline{U(x)}(\Delta U(x))}{|\nabla U(x)|^2 + |U(x)|^2}, & \text{for } |\nabla U(x)|^2 + |U(x)|^2 > 0, \\ 0, & \text{for } |\nabla U(x)|^2 + |U(x)|^2 = 0. \end{cases}$$

By (12.3.25) we have

$$\|b\|_{L^\infty(\Omega; \mathbb{C}^n)} \leq K, \quad \text{and} \quad \|c\|_{L^\infty(\Omega; \mathbb{C})} \leq K.$$

In addition, if $|\nabla U(x)|^2 + |U(x)|^2 = 0$ then (12.3.25) implies

$$\Delta U(x) = 0 = b(x) \cdot \nabla U(x) + c(x)U(x),$$

and, if $|\nabla U(x)|^2 + |U(x)|^2 > 0$ then

$$\begin{aligned} b(x) \cdot \nabla U(x) + c(x)U(x) &= \\ &= \frac{\overline{\nabla U(x)}(\Delta U(x))}{|\nabla U(x)|^2 + |U(x)|^2} \cdot \nabla U(x) + \frac{\overline{U(x)}(\Delta U(x))}{|\nabla U(x)|^2 + |U(x)|^2} U(x) = \\ &= \Delta U(x). \end{aligned}$$

◆

In the next Proposition we will use **the Kelvin transform** which is defined as follows. Let u be a sufficiently smooth function, let

$$v(y) = |y|^{2-n} u(y|y|^{-2}), \quad (12.3.27)$$

we have

$$\Delta v(y) = |y|^{-2-n} (\Delta_x u) (y|y|^{-2}). \quad (12.3.28)$$

Theorem 12.3.4 (weak unique continuation property). *Let $K, R, \rho > 0$, $\rho < R$. Let us assume that $U \in H^2(B_R)$ satisfies the inequality*

$$|\Delta U| \leq K (|\nabla U| + |U|), \quad \text{in } B_R. \quad (12.3.29)$$

We have that, if

$$U = 0, \quad \text{in } B_\rho, \quad (12.3.30)$$

then

$$U = 0 \quad \text{in } B_R. \quad (12.3.31)$$

Proof. It is not restrictive to assume $R = 1$ and, consequently, $\rho < 1$. Let us apply the Kelvin transform. Set

$$V(y) = |y|^{2-n} U (y|y|^{-2}), \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}, \quad (12.3.32)$$

by (12.3.30) we have

$$V = 0, \quad \text{in } \mathbb{R}^n \setminus \overline{B_{1/\rho}}. \quad (12.3.33)$$

Moreover $V \in H^2(B_{1/\rho} \setminus \overline{B_1})$ and

$$|(\nabla_x U) (y|y|^{-2})| \leq C (|y|^{n-1} |V(y)| + |y|^n |\nabla_y V(y)|)$$

hence, by this inequality, by (12.3.28) and by (12.3.29) we obtain

$$|\Delta V| \leq CK (|\nabla V| + |V|), \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}. \quad (12.3.34)$$

Now, by applying Proposition 12.3.3 we have

$$V = 0 \quad \text{in } B_{1/\rho} \setminus \overline{B_1}. \quad (12.3.35)$$

From which, taking into account (12.3.27), we immediately obtain the thesis. \blacksquare

We now prove the following

Theorem 12.3.5. *Let Ω be a (bounded) connected open set of \mathbb{R}^n , let $\Sigma \subset \partial\Omega$ be a local graph of a function of class C^2 . Let us assume that $b \in L^\infty(\Omega, \mathbb{C}^n)$, $c \in L^\infty(\Omega, \mathbb{C})$. Moreover, let $U \in H^2(\Omega, \mathbb{C})$ satisfy*

$$\begin{cases} \Delta U = b(x) \cdot \nabla U + c(x)U, & \text{in } \Omega, \\ U = 0, & \text{on } \Sigma, \\ \frac{\partial U}{\partial \nu} = 0, & \text{on } \Sigma. \end{cases} \quad (12.3.36)$$

Then we have

$$U = 0 \quad \text{in } \Omega. \quad (12.3.37)$$

Remark 2. As usual, the conditions $U = \frac{\partial U}{\partial \nu} = 0$ on Σ should be understood as: $U\Psi \in H_0^2(\Omega)$, for every $\Psi \in C^\infty(\mathbb{R}^n)$ such that $\Psi = \frac{\partial \Psi}{\partial \nu} = 0$ on $\partial\Omega \setminus \Sigma$. ♦

Proof. For any $M, r > 0$ let us denote

$$Q_{r,M} = B'_r \times (-Mr, Mr).$$

Up to rigid transformation of \mathbb{R}^n , we may assume there exist r_0 and $h \in C^2(\overline{B'_{r_0}})$ such that

$$h(0) = |\nabla_{x'} h(0)| = 0$$

and

$$\text{graph}(h) \subset \Sigma,$$

where $\text{graph}(h)$ is the graph of h . While eventually reducing r_0 we may assume that there exists $M_0 > 0$ such that

$$\|h\|_{C^2(\overline{B'_{r_0}})} \leq M_0 r_0$$

and

$$\Omega \cap Q_{r_0, 2M_0} = \{x \in B'_{r_0} \times \mathbb{R} : h(x') < x_n < 2M_0 r_0\}.$$

Now, let us denote

$$Q_{r_0, 2M_0}^- = \{x \in B'_{r_0} \times \mathbb{R} : -2M_0 r_0 < x_n \leq h(x')\}$$

and let

$$\tilde{\Omega} = \Omega \cup Q_{r_0, 2M_0}^-.$$

Moreover, let \tilde{U} be the extension of U to 0 in $Q_{r_0, 2M_0}^-$. We have $\tilde{U} \in H^2(\tilde{\Omega})$ and

$$\Delta \tilde{U} = \tilde{b}(x) \cdot \nabla \tilde{U} + \tilde{c}(x) \tilde{U},$$

where \tilde{b} e \tilde{c} are the extensions of b, c to 0 in $Q_{r_0, 2M_0}^-$. We have trivially $\tilde{b} \in L^\infty(\tilde{\Omega}, \mathbb{C}^n)$, $\tilde{c} \in L^\infty(\tilde{\Omega}, \mathbb{C})$.

From now on, we argue as we did in the proof of Theorem 7.6.5. For the convenience of the reader here we repeat the main steps of this proof.

First of all, we note that $\tilde{\Omega}$ is connected.

Then, set

$$A = \left\{ x \in \tilde{\Omega} : \exists \rho_x > 0 \text{ such that } \tilde{U} = 0, \text{ in } B_{\rho_x}(x) \right\}. \quad (12.3.38)$$

By the definition of \tilde{U} we have $\text{Int}(Q_{r_0, 2M_0}^-) \subset A$ hence $A \neq \emptyset$ and it turns out, trivially, that A is an open set in $\tilde{\Omega}$. To prove that A is also closed in $\tilde{\Omega}$ (from which, since $\tilde{\Omega}$ is connected, we get $A = \tilde{\Omega}$ and, consequently, $\tilde{U} \equiv 0$) we can follow exactly the same argument followed in the proof of Theorem 7.6.5, using, in this case, Proposition 12.3.4 instead of Theorem 7.6.4. ■

Remark 3. Let us observe that by the weak unique continuation property (Theorem 7.6.4) we have derived the uniqueness for Cauchy problem (12.3.36). Conversely, if we dispose of the uniqueness for Cauchy problem (12.3.36) we may derive the weak unique continuation property for the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U, \quad \text{in } \Omega,$$

where Ω is a connected open set of \mathbb{R}^n . Indeed, let ω a subset of Ω and let us assume that

$$U = 0, \quad \text{in } \omega.$$

Let $B_r(x_0) \Subset \omega$.

We have that U is a solution to the Cauchy problem

$$\begin{cases} \Delta U = b(x) \cdot \nabla U + c(x)U, & \text{in } \Omega, \\ U = 0, & \text{on } \partial B_r(x_0), \\ \frac{\partial U}{\partial \nu} = 0, & \text{on } \partial B_r(x_0). \end{cases}$$

Hence, by the uniqueness for the Cauchy problem, we have

$$U \equiv 0, \quad \text{in } \Omega.$$

◆

12.4 Necessary conditions

In the present Section we return to estimate (12.2.2) or, more generally, to estimate (12.2.19). We have already observed (Remark 1 of Section 12.2) that if (12.2.18) holds, then (12.2.19) holds. As we will see, the converse is also true. More precisely, we have the following

Theorem 12.4.1. *Let $M(D)$ and $P(D)$ be two linear differential operators of order r and m respectively. The following conditions are equivalent:*

(i) *There exists $C > 0$ such that*

$$\int_{Q_1} e^{2\tau N \cdot x} |M(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.4.1)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

(ii) *The following is true*

$$\sup_{(\xi, \tau) \in \mathbb{R}^{n+1}} \left\{ \frac{|M(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2} \right\} < +\infty. \quad (12.4.2)$$

In particular, if (12.4.1) holds true, then $r \leq m$.

In order to prove Theorem 12.4.1 we need a Lemma (Lemma 12.4.3) to which we premise the following

Proposition 12.4.2 (extension of the Leibniz formula). *Let $P(D)$ be a linear differential operator of order m , we have*

$$P(D)[fu] = \sum_{|\alpha| \leq m} \frac{D^\alpha f}{\alpha!} P^{(\alpha)}(D)u, \quad \forall f, u \in C^m(\mathbb{R}^n, \mathbb{C}). \quad (12.4.3)$$

Proof. By the Leibniz formula, we get

$$P(D)[fu] = \sum_{|\alpha| \leq m} (D^\alpha f) R_\alpha(D)u, \quad \forall f, u \in C^m(\mathbb{R}^n, \mathbb{C}), \quad (12.4.4)$$

where $R_\alpha(D)$ are linear differential operators of order (less or equal) to $m - |\alpha|$. Let $\xi, \eta \in \mathbb{R}^n$ be arbitrary and let $f(x) = e^{ix \cdot \xi}$, $u(x) = e^{ix \cdot \eta}$. By (12.4.4) we have

$$e^{ix \cdot (\xi + \eta)} P(\xi + \eta) = P(D)[e^{ix \cdot \xi} e^{ix \cdot \eta}] = e^{ix \cdot (\xi + \eta)} \sum_{|\alpha| \leq m} \xi^\alpha R_\alpha(\eta).$$

Hence

$$P(\xi + \eta) = \sum_{|\alpha| \leq m} \xi^\alpha R_\alpha(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n.$$

On the other hand, by the Taylor formula we have

$$P(\xi + \eta) = \sum_{|\alpha| \leq m} \frac{\xi^\alpha}{\alpha!} P^{(\alpha)}(\eta),$$

from which we obtain

$$R_\alpha(\eta) = \frac{1}{\alpha!} P^{(\alpha)}(\eta)$$

that gives (12.4.3). ■

Lemma 12.4.3. *For any $m \in \mathbb{N}_0$ and for any $\phi \in C_0^\infty(Q_1, \mathbb{C})$, which does not vanish identically, there exists a constant $C_\phi \geq 1$ such that for every linear differential operator $R(D)$ of order m we have*

$$C_\phi^{-1} \tilde{R}(\xi) \leq \left(\int_{Q_1} |R(D) [\phi(x) e^{ix \cdot \xi}]|^2 dx \right)^{1/2} \leq C_\phi \tilde{R}(\xi), \quad (12.4.5)$$

for every $\xi \in \mathbb{R}^n$, where

$$\tilde{R}(\xi) = \left(\sum_{|\alpha| \leq m} |R^{(\alpha)}(\xi)|^2 \right)^{1/2}. \quad (12.4.6)$$

Proof. Let $\phi \in C_0^\infty(Q_1, \mathbb{C})$ be a function not identically zero. Formula (12.4.3) gives

$$R(D) [\phi(x) e^{ix \cdot \xi}] = e^{ix \cdot \xi} \sum_{|\alpha| \leq m} \frac{1}{\alpha!} R^{(\alpha)}(\xi) D^\alpha \phi(x). \quad (12.4.7)$$

Set

$$I_{\alpha\beta}(\phi) = \int_{\mathbb{R}^n} \frac{1}{\alpha!} D^\alpha \phi(x) \frac{1}{\beta!} \overline{D^\beta \phi(x)} dx, \quad (12.4.8)$$

by (12.4.7), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |R(D) [\phi(x)e^{ix \cdot \xi}]|^2 dx &= \int_{\mathbb{R}^n} \left| \sum_{|\alpha| \leq m} \frac{1}{\alpha!} R^{(\alpha)}(\xi) D^\alpha \phi(x) \right|^2 dx = \\
&= \int_{\mathbb{R}^n} \sum_{|\alpha|, |\beta| \leq m} R^{(\alpha)}(\xi) \overline{R^{(\beta)}(\xi)} \frac{1}{\alpha!} D^\alpha \phi(x) \frac{1}{\beta!} \overline{D^\beta \phi(x)} dx = \\
&= \sum_{|\alpha|, |\beta| \leq m} R^{(\alpha)}(\xi) \overline{R^{(\beta)}(\xi)} I_{\alpha\beta}(\phi).
\end{aligned} \tag{12.4.9}$$

Let us consider the quadratic form

$$H(z) = \sum_{|\alpha|, |\beta| \leq m} I_{\alpha\beta}(\phi) z_\alpha \overline{z_\beta}, \tag{12.4.10}$$

where $z \in \mathbb{C}^{N(m,n)}$ ($N(m,n)$ the number of multi-indexes α such that $|\alpha| \leq m$). Let us notice that

$$H(z) = \int_{\mathbb{R}^n} \left| \sum_{|\alpha| \leq m} \frac{1}{\alpha!} z_\alpha D^\alpha \phi(x) \right|^2 dx. \tag{12.4.11}$$

From which, recalling (12.4.9) and choosing

$$z_\alpha^{(0)} = R^{(\alpha)}(\xi), \quad \text{for } |\alpha| \leq m, \tag{12.4.12}$$

we have

$$H(z^{(0)}) = \int_{\mathbb{R}^n} |R(D) [\phi(x)e^{ix \cdot \xi}]|^2 dx. \tag{12.4.13}$$

Of course, (12.4.11) gives $H(z) \geq 0$ for every $z \in \mathbb{C}^{N(m,n)}$. Actually, $H(z)$ is a **positive-definite form** as we are going to prove. Arguing by contradiction, let us suppose that there exists $\tilde{z} \neq 0$ such that $H(\tilde{z}) = 0$ then, applying inequality (12.2.17) to the operator

$$\tilde{M}(D) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \tilde{z}_\alpha D^\alpha,$$

we would have that there exists a constant \tilde{C} such that

$$0 = \tilde{C}H(\tilde{z}) = \tilde{C} \int_{Q_1} |\tilde{M}(D)\phi|^2 dx \geq \int_{Q_1} |\phi|^2 dx$$

and this would imply $\phi \equiv 0$, which contradicts the assumption that ϕ does not vanish identically. Since H is **positive-definite**, there exists a constant $C_\phi \geq 1$ (depending by ϕ) such that

$$C_\phi^{-1} \sum_{|\alpha| \leq m} |z_\alpha|^2 \leq H(z) \leq C_\phi \sum_{|\alpha| \leq m} |z_\alpha|^2, \quad \forall z \in \mathbb{C}^{N(m,n)}. \quad (12.4.14)$$

By the latter, recalling (12.4.12) and (12.4.13) we get (12.4.5). ■

Proof of Theorem 12.4.1.

We have already seen in Remark 1 of Section 12.2 that (12.4.2) implies (12.4.1). Now we prove the converse. Let C_\star be a positive constant, let us fix $\tau \in \mathbb{R}$ and let us assume that

$$\int_{Q_1} e^{2\tau N \cdot x} |M(D)u|^2 dx \leq C_\star \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.4.15)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$. Then, setting $v = e^{2\tau N \cdot x} u$ and arguing similarly to the proof of Theorem 12.2.1, we have

$$\begin{aligned} \int_{Q_1} |M(D + i\tau N)v|^2 dx &= \int_{Q_1} e^{2\tau N \cdot x} |M(D)u|^2 dx \leq \\ &\leq C_\star \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx = \\ &= C_\star \int_{Q_1} |P(D + i\tau N)v|^2 dx. \end{aligned}$$

Therefore

$$\int_{Q_1} |M(D + i\tau N)v|^2 dx \leq C_\star \int_{Q_1} |P(D + i\tau N)v|^2 dx, \quad (12.4.16)$$

for every $v \in C_0^\infty(Q_1, \mathbb{C})$.

Now let $\phi \in C_0^\infty(Q_1, \mathbb{C})$ be not identically zero. For instance, let

$$\phi(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right), & \text{for } |x| < 1, \\ 0, & \text{for } |x| \geq 1. \end{cases}$$

Let $v(x) = \phi(x)e^{ix \cdot \xi}$. By (12.4.5) and (12.4.16) we have, for every $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
& C^{-1} \sum_{|\alpha| \leq r} |M^{(\alpha)}(\xi + i\tau N)|^2 \leq \\
& \leq \int_{Q_1} |M(D + i\tau N) [\phi(x)e^{ix \cdot \xi}]|^2 dx \leq \\
& \leq C_* \int_{Q_1} |P(D + i\tau N) [\phi(x)e^{ix \cdot \xi}]|^2 dx \leq \\
& \leq CC_* \sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2.
\end{aligned} \tag{12.4.17}$$

where $C \geq 1$ does not depend on τ . By (12.4.17) we obtain

$$|M(\xi + i\tau N)|^2 \leq C^2 C_* \sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1}. \tag{12.4.18}$$

Therefore

$$\sup_{(\xi, \tau) \in \mathbb{R}^{n+1}} \left\{ \frac{|M(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2} \right\} \leq C^2 C_* < +\infty,$$

that concludes the proof. ■

Remark 1. By reviewing the proof of Theorem 12.4.1 it is easily seen that if τ_0 is a fixed real number the following conditions are equivalent:

(i') There exists $C > 0$ such that

$$\int_{Q_1} e^{2\tau_0 N \cdot x} |M(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau_0 N \cdot x} |P(D)u|^2 dx, \tag{12.4.19}$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$.

(ii') The following holds true

$$\sup_{\xi \in \mathbb{R}^n} \left\{ \frac{|M(\xi + i\tau_0 N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau_0 N)|^2} \right\} < +\infty. \tag{12.4.20}$$

In particular the following conditions are equivalent:

(i'') There exists $C > 0$ such that for every $u \in C_0^\infty(Q_1, \mathbb{C})$ we have

$$\int_{Q_1} |M(D)u|^2 dx \leq C \int_{Q_1} |P(D)u|^2 dx,$$

(ii'')

$$\sup_{\xi \in \mathbb{R}^n} \frac{|M(\xi)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi)|^2} < +\infty.$$

◆

Remark 2. In applying Theorem 12.2.1 for proving the uniqueness of Cauchy problem (12.3.3) it was sufficient to use estimate (12.2.2) for τ sufficiently large. More precisely, the estimate we actually used is

$$\int_{Q_1} e^{2\tau N \cdot x} |M(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.4.21)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$, where τ_0 is a positive number and, as well as C , does not depend on u . If $M(\xi)$ and $P(\xi)$ are **homogeneous polynomials**, it is simple to check that estimate (12.4.21) is equivalent to the estimate

$$\int_{Q_1} e^{-2\tau N \cdot x} |M(D)u|^2 dx \leq C \int_{Q_1} e^{-2\tau N \cdot x} |P(D)u|^2 dx, \quad (12.4.22)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$. It suffices to consider the simple change of variables $x \rightarrow -x$ (the reader takes care of the details). Taking into account what we said in **Remark 1** of this Section, estimate (12.4.21) (hence, estimate (12.4.22)) is equivalent to

$$\sup \left\{ \frac{|M(\xi + i\tau N)|^2}{\sum_{|\alpha| \leq m} |P^{(\alpha)}(\xi + i\tau N)|^2} : (\xi, \tau) \in \mathbb{R}^{n+1}, |\tau| \geq \tau_0 \right\} < +\infty. \quad (12.4.23)$$

◆

Remark 3. If $M_j(\xi)$ are polynomials of degree r_j , for $j = 1, \dots, J$ and $S_k(\xi)$ are polynomials of degree s_k , for $k = 1, \dots, K$, the necessary and sufficient conditions for the validity of estimate

$$\sum_{j=1}^J \int_{Q_1} e^{2\tau N \cdot x} |M_j(D)u|^2 dx \leq C \sum_{k=1}^K \int_{Q_1} e^{2\tau N \cdot x} |S_k(D)u|^2 dx, \quad (12.4.24)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$ (or for a fixed τ) can be obtained easily arguing as in the proof of Theorem 12.4.1. For instance, the estimate

$$\sum_{j=1}^J \int_{Q_1} |M_j(D)u|^2 dx \leq C \sum_{k=1}^K \int_{Q_1} |S_k(D)u|^2 dx, \quad (12.4.25)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$, is equivalent to

$$\sup_{\xi \in \mathbb{R}^n} \left\{ \frac{\sum_{j=1}^J |M_j(\xi)|^2}{\sum_{k=1}^K \sum_{|\alpha| \leq s_k} |S_k^{(\alpha)}(\xi)|^2} \right\} < +\infty. \quad (12.4.26)$$

From which, for $r \leq s$, where r, s are nonnegative integer numbers, we easily get the estimate

$$\int_{Q_1} |D^r u|^2 dx \leq C \int_{Q_1} |D^s u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}), \quad (12.4.27)$$

where, for a nonnegative integer p , we set

$$|D^p u|^2 = \sum_{|\alpha|=p} |D^\alpha u|^2.$$

Similarly it can be proved that

$$\int_{Q_1} e^{2\tau N \cdot x} |D^r u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |D^s u|^2 dx, \quad (12.4.28)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

More generally we have

$$\tau^{2(s-r)} \int_{Q_1} e^{2\tau N \cdot x} |D^r u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |D^s u|^2 dx, \quad (12.4.29)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

More attention is required to study the following two estimates. Let $P_m(\xi)$ be a homogeneous polynomial of degree $m \geq 1$ and let us consider the estimates

$$\int_{Q_1} |D^m u|^2 dx \leq C \int_{Q_1} |P_m(D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}) \quad (12.4.30)$$

and

$$\int_{Q_1} e^{2\tau N \cdot x} |D^m u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_m(D)u|^2 dx, \quad (12.4.31)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$, where τ_0 is a nonnegative integer number.

Let us begin by (12.4.30). We distinguish two cases:

(a) $P_m(D)$ is **elliptic**, that is

$$\xi \in \mathbb{R}^n, P_m(\xi) = 0 \Rightarrow \xi = 0; \quad (12.4.32)$$

(b) $P_m(D)$ is **not elliptic**, that is there exists $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ such that

$$P_m(\xi_0) = 0. \quad (12.4.33)$$

Let us check that **estimate (12.4.30) holds if and only if $P_m(D)$ is elliptic.**

Let us denote by

$$q(\xi) = \frac{|\xi|^{2m}}{\sum_{|\alpha| \leq m} |P_m^{(\alpha)}(\xi)|^2}. \quad (12.4.34)$$

Let us assume that (a) holds true. Since $P_m(\xi)$ is a homogeneous polynomial of degree m , there exists $\lambda \geq 1$ such that

$$\lambda^{-1} |\xi|^m \leq |P_m(\xi)| \leq \lambda |\xi|^m, \quad \forall \xi \in \mathbb{R}^n. \quad (12.4.35)$$

Hence

$$q(\xi) \leq \lambda^2, \quad \forall \xi \in \mathbb{R}^n$$

and, by what noted in **Remark 1** of this Section, we have that estimate (12.4.30) holds true.

In case (b), let $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ satisfy $P_m(\xi_0) = 0$. Let $\mu \in \mathbb{R}$. Then the numerator of $q(\mu\xi_0)$ is equal to $\mu^{2m} |\xi_0|^{2m}$ and the denominator has degree w.r.t. μ less or equal to $2m - 2$, as $P_m(\mu\xi_0) = 0$, for every $\mu \in \mathbb{R}$. Therefore

$$\lim_{\mu \rightarrow +\infty} q(\mu\xi_0) = +\infty$$

from which we have that in case (b), estimate (12.4.30) does not hold.

Let now consider estimate (12.4.31). We prove that **it does not hold in any case.**

For any $\xi \in \mathbb{R}^n$ and $\tau \geq \tau_0$, set

$$q(\xi, \tau) = \frac{(|\xi|^2 + \tau^2)^m}{\sum_{|\alpha| \leq m} \left| P_m^{(\alpha)}(\xi + i\tau N) \right|^2}. \quad (12.4.36)$$

Let us begin by case (a). Let $\xi \in \mathbb{R}^n$ be such that ξ and N are linearly independent. Let us consider the equation

$$P_m(\xi + zN) = 0, \quad z \in \mathbb{C}. \quad (12.4.37)$$

Let $a + ib$ be a solution of (12.4.37). Then $b \neq 0$, otherwise we would have $P_m(\xi + aN) = 0$, but since $P_m(D)$ is elliptic, we would have $\xi + aN = 0$ which contradicts the assumption of linear independence between ξ and N . Hence, either $b > 0$ or $b < 0$. Setting $\eta = \xi + aN$, we have $\eta \neq 0$ and

$$P_m(\eta + ibN) = 0. \quad (12.4.38)$$

Now, if $b > 0$, let $\mu \geq \frac{\tau_0}{b}$ and we get

$$P_m(\mu\eta + i\mu bN) = \mu^m P_m(\eta + ibN) = 0. \quad (12.4.39)$$

Hence the numerator of $q(\mu\eta, \mu\tau)$ is equal to $\mu^{2m} (|\eta|^2 + b^2)^m$, whereas, by (12.4.39), the denominator of $q(\mu\eta, \mu\tau)$ has degree w.r.t. μ less or equal to $2m - 2$. Therefore

$$\lim_{\mu \rightarrow +\infty} q(\mu\eta, \mu b) = +\infty. \quad (12.4.40)$$

If $b < 0$, it suffices to notice that by (12.4.38) it follows $P_m(-\eta + i(-b)N) = 0$ and similarly to (12.4.40) we have

$$\lim_{\mu \rightarrow +\infty} q(\mu(-\eta), \mu(-b)) = +\infty.$$

Hence

$$\sup \{q(\xi, \tau) : (\xi, \tau) \in \mathbb{R}^{n+1}, |\tau| \geq \tau_0\} = +\infty. \quad (12.4.41)$$

To conclude, in the elliptic case estimate (12.4.31) does not hold.

Let us consider case (b). Let $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ satisfy

$$P_m(\xi_0) = 0.$$

Let $\tau \geq \tau_0$ be fixed. We have

$$\lim_{\mu \rightarrow +\infty} q(\mu\xi_0, \mu\tau) = \lim_{\mu \rightarrow +\infty} \frac{\left(|\xi_0|^2 + (\tau\mu^{-1})^2 \right)^m}{h(\mu)} = +\infty,$$

where

$$h(\mu) = \sum_{|\alpha| \leq m-1} \mu^{-2(m-|\alpha|)} |P_m^{(\alpha)}(\xi_0 + i(\tau\mu^{-1})N)|^2 + |P_m(\xi_0 + i(\tau\mu^{-1})N)|^2.$$

From which we have that if $P_m(D)$ is not elliptic, estimate (12.4.31) does not hold. ♦

12.5 Examples and further considerations.

Remark 1 of Section 12.2 implies that, if $P_m(\xi)$, is a homogeneous polynomial of degree m , then we have

$$\int_{Q_1} e^{2\tau N \cdot x} |u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_m(D)u|^2 dx, \tag{12.5.1}$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

Let us assume $m \geq 1$. Let $N = -e_n$, let l be a positive number and

$$h : B'_1 \rightarrow \mathbb{R}$$

be a **strictly convex function** which satisfies

$$h(0) = 0, \quad \text{and} \quad l \leq \inf_{\partial B'_1} h. \tag{12.5.2}$$

Let

$$\Lambda = \{(x', x_n) \in B'_1 \times \mathbb{R} : h(x') < x_n < l\}$$

and

$$\Gamma = \{(x', h(x')) : x' \in B'_1(0)\}.$$

Moreover, let $a_0 \in L^\infty(\Lambda)$. By proceeding in similar manner as we did to prove the uniqueness for Cauchy problem (12.3.3), we prove the uniqueness for the problem

$$\begin{cases} P_m(D)U + a_0(x)U = 0, & \text{in } \Lambda, \\ U\Psi \in H_0^m(\Lambda), \quad \forall \Psi \in C^\infty(\mathbb{R}^n), \quad \text{supp } \Psi \subset \mathbb{R}^{n-1} \times (-\infty, 1). \end{cases} \tag{12.5.3}$$

We invite the reader to develop the details (remember to use the homothetic transformation $x \rightarrow rx$), however, on this issue we refer to [60, Theorem 1].

Of course, it is meaningful and interesting to ask what happens regarding the uniqueness for the Cauchy problem if one perturbs the operator with operators of order r with $1 \leq r \leq m - 1$. Keep in mind, however, that here we are basically considering the case of uniqueness for the Cauchy problem whose initial surface is a strictly convex function.

Let us consider the case $r = m - 1$. If the estimate holds true

$$\int_{Q_1} e^{2\tau N \cdot x} |D^{m-1}u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_m(D)u|^2 dx, \tag{12.5.4}$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$ ($\tau_0 \geq 0$), then we have the uniqueness of solutions to the Cauchy problem

$$\begin{cases} P_m(D)U + \sum_{|\alpha| \leq m-1} b_\alpha(x)U = 0, & \text{in } \Lambda, \\ U\Psi \in H_0^m(\Lambda), \quad \forall \Psi \in C^\infty(\mathbb{R}^n), \quad \text{supp}\Psi \subset \mathbb{R}^{n-1} \times (-\infty, 1), \end{cases} \tag{12.5.5}$$

where $b_\alpha \in L^\infty(\Lambda)$, for $|\alpha| \leq m - 1$. As a matter of fact if (12.5.4) holds, then by (12.4.28) we have

$$\sum_{j=0}^{m-1} \int_{Q_1} e^{2\tau N \cdot x} |D^j u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_m(D)u|^2 dx, \tag{12.5.6}$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$. In particular, let us observe that (12.5.6) and (12.5.4) are equivalent and, in addition, (12.5.6) allows us to treat (when $N = -e_n$) problem (12.5.5) in a manner similar to (12.3.3).

In **Remark 3** of Section 12.4, we have seen, that necessary and sufficient condition to be hold (12.5.4) is there exists $C > 0$ such that, for every $\xi \in \mathbb{R}^n$ and for every $\tau \geq \tau_0$, we have

$$q_{m-1}(\xi, \tau) = \frac{(|\xi|^2 + \tau^2)^{m-1}}{\sum_{|\alpha| \leq m} |P_m^{(\alpha)}(\xi + i\tau N)|^2} \leq C. \tag{12.5.7}$$

Now, in the next Proposition we give a simpler formulation of condition (12.5.7)

Proposition 12.5.1. *The following conditions are equivalent:*

- (a) Estimate (12.5.6) holds true
 (b) If $(\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}$, then

$$P_m(\xi + i\tau N) = 0 \Rightarrow \sum_{j=1}^n |P_m^{(j)}(\xi + i\tau N)|^2 > 0,$$

where $P_m^{(j)}(\xi + i\tau N) = P_m^{(e_j)}(\xi + i\tau N)$.

In order to prove Proposition 12.5.1 we will use two lemmas.

Lemma 12.5.2. *Let d be a positive integer number. Let K be a compact set of \mathbb{R}^d and let $f : K \rightarrow \mathbb{R}$ and $g : K \rightarrow \mathbb{R}$ be two continuous functions. The following conditions are equivalent:*

- (i) $X \in K, f(X) = 0 \Rightarrow g(X) > 0$;
 (ii) There exists $C > 0$ such that $C(f(X))^2 + g(X) > 0$ for every $X \in K$.

Proof. If $K = \emptyset$, the equivalence between (i) and (ii) is trivial. Let us suppose, accordingly, that $K \neq \emptyset$ and that (i) apply. Set

$$K_0 = \{X \in K : f(X) = 0\}.$$

By (i), by the continuity of g , and since K is compact, there exists an open set V_0 , of \mathbb{R}^d , which satisfies $K_0 \subset V_0$ and

$$g(X) > 0, \quad \forall X \in V_0 \cap K.$$

If $K \setminus V_0 = \emptyset$, then (ii) is trivially satisfied. If $K \setminus V_0 \neq \emptyset$, we set

$$M_1 = \min_{K \setminus V_0} f^2 > 0, \quad M_2 = \min_{K \setminus V_0} g$$

and let C be a positive number such that $CM_1 + M_2 > 0$. We have

$$C(f(X))^2 + g(X) \geq g(X) > 0, \quad \forall X \in V_0 \cap K$$

and

$$C(f(X))^2 + g(X) \geq CM_1 + M_2 > 0, \quad \forall X \in K \setminus V_0,$$

from which (ii) follows.

Let us suppose that (ii) holds, we have trivially that if $f(X) = 0$ then

$$g(X) = C(f(X))^2 + g(X) > 0.$$

■

Lemma 12.5.3. *Let $\tau_0 \in \mathbb{R}$ and let $M_j(\xi)$ be polynomials of degree r_j , for $j = 1, \dots, J$. The following conditions are equivalent:*

(a) *There exists $C > 0$ such that*

$$\sum_{j=1}^J \int_{Q_1} e^{2\tau_0 N \cdot x} |M_j(D)u|^2 dx \leq C \int_{Q_1} e^{2\tau_0 N \cdot x} |P_m(D)u|^2 dx, \quad (12.5.8)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$;

(b) *there exists $C > 0$ such that*

$$\sum_{j=1}^J \int_{Q_1} |M_j(D)u|^2 dx \leq C \int_{Q_1} |P_m(D)u|^2 dx, \quad (12.5.9)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$;

(c)

$$\sup_{\xi \in \mathbb{R}^n} \frac{\sum_{j=1}^J |M_j(\xi)|^2}{\sum_{|\alpha| \leq m} |P_m^{(\alpha)}(\xi)|^2} < +\infty. \quad (12.5.10)$$

Proof. Let us assume that (12.5.8) holds. For every $u \in C_0^\infty(Q_1, \mathbb{C})$ we have

$$\begin{aligned} \sum_{j=1}^J \int_{Q_1} |M_j(D)u|^2 dx &= \sum_{j=1}^J \int_{Q_1} e^{-2\tau_0 N \cdot x} e^{2\tau_0 N \cdot x} |M_j(D)u|^2 dx \leq \\ &\leq e^{2|\tau_0|\sqrt{n}} \sum_{j=1}^J \int_{Q_1} e^{2\tau_0 N \cdot x} |M_j(D)u|^2 dx \leq \\ &\leq C e^{2|\tau_0|\sqrt{n}} \int_{Q_1} e^{2\tau_0 N \cdot x} |P_m(D)u|^2 dx \leq \\ &\leq C e^{4|\tau_0|\sqrt{n}} \int_{Q_1} |P_m(D)u|^2 dx. \end{aligned} \quad (12.5.11)$$

Hence, (b) follows. Similarly, we can prove that (b) implies (a).

The equivalence between (b) and (c) was proved in **Remark 3** of Section 12.4, see (12.4.26). ■

Remark 1. Taking into account **Remark 2** of Section 12.4, we have that if (12.5.6) holds then

$$\sum_{j=0}^{m-1} \int_{Q_1} e^{2\tau N \cdot x} |D^j u|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_m(D)u|^2 dx, \quad (12.5.12)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$. Hence, Theorem 12.4.1 implies that estimate (12.5.6) holds if and only if there exists $C > 0$ such that for every $(\xi, \tau) \in \mathbb{R}^{n+1}$ we have

$$q_{m-1}(\xi, \tau) = \frac{(|\xi|^2 + \tau^2)^{m-1}}{\sum_{|\alpha| \leq m} |P_m^{(\alpha)}(\xi + i\tau N)|^2} \leq C. \quad (12.5.13)$$

◆

Proof of Proposition 12.5.1.

In order to prove that (a) implies (b) we argue by contradiction. We assume that (a) holds and that (b) does not hold. Hence, we assume that there exists $(\xi_*, \tau_*) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}$ satisfying:

$$P_m(\xi_* + i\tau_* N) = 0 \quad \text{and} \quad \sum_{j=1}^n |P_m^{(j)}(\xi_* + i\tau_* N)|^2 = 0.$$

From this, we easily obtain

$$\lim_{\mu \rightarrow +\infty} q(\mu\xi_*, \mu\tau_*) = +\infty$$

which contradicts (12.5.13). Hence, (a) implies (b).

Now, let us suppose that (b) holds true. Let

$$\mathbb{S}^n = \{(\xi, \tau) \in \mathbb{R}^{n+1} : |\xi|^2 + \tau^2 = 1\}.$$

By Lemma 12.5.2, there exists C , which we may assume larger than 1, which satisfies

$$C|P_m(\xi + i\tau N)|^2 + \sum_{j=1}^n |P_m^{(j)}(\xi + i\tau N)|^2 > 0, \quad \forall (\xi, \tau) \in \mathbb{S}^n. \quad (12.5.14)$$

Therefore, since the polynomials P_m and $P_m^{(j)}$ are homogeneous of degree m and $m - 1$ respectively, there exists $\lambda > 0$ such that, for each $(\xi, \tau) \in \mathbb{R}^{n+1}$ we have

$$\begin{aligned} \gamma(\xi, \tau) &:= \frac{|P_m(\xi + i\tau N)|^2}{(|\xi|^2 + \tau^2)} + \frac{1}{C} \sum_{j=1}^n |P_m^{(j)}(\xi + i\tau N)|^2 \geq \\ &\geq \lambda (|\xi|^2 + \tau^2)^{m-1}. \end{aligned} \quad (12.5.15)$$

On the other hand, we have trivially that, for some constant \tilde{C} , we get

$$q_{m-1}(\xi, \tau) \leq \tilde{C}, \quad |\xi|^2 + \tau^2 \leq 1$$

and by (12.5.15), we have

$$\begin{aligned} q_{m-1}(\xi, \tau) &\leq \frac{(|\xi|^2 + \tau^2)^{m-1}}{\sum_{|\alpha| \leq m-2} \left| P_m^{(\alpha)}(\xi + i\tau N) \right|^2 + \gamma(\xi, \tau)} \leq \\ &\leq \lambda^{-1}, \quad \text{for } |\xi|^2 + \tau^2 \geq 1. \end{aligned} \quad (12.5.16)$$

Hence

$$q_{m-1}(\xi, \tau) \leq \max \left\{ \tilde{C}, \lambda^{-1} \right\}, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1}.$$

Now, taking into account **Remark 1** of this Section, (a) follows. ■

Examples

1. Let $P_2(\xi) = \sum_{j=1}^n \xi_j^2$. We know that the corresponding operator is $P_2(D) = \sum_{j=1}^n D_j^2 = -\Delta$. We already know that the following estimate holds

$$\int_{Q_1} e^{2\tau N \cdot x} (|u|^2 + |\nabla u|^2) dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |\Delta u|^2 dx, \quad (12.5.17)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$. On the other hand, it is immediate to see that (b) of Proposition 12.5.1 is satisfied because

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi + i\tau N) \right|^2 = 4 (|\xi|^2 + \tau^2) > 0, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}.$$

♠

2. A similar argument applies to the wave operator $\square = D_0^2 - \Delta$ (here x_0 represents the time variable). Since the symbol of the operator \square is $P_2(\xi) = -\xi_0^2 + \sum_{j=1}^n \xi_j^2$, we have

$$\sum_{j=0}^n \left| P_2^{(j)}(\xi + i\tau N) \right|^2 = 4 (|\xi|^2 + \tau^2) > 0, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}.$$

Hence (b) of Proposition 12.5.1 is satisfied. ♠

3. Whereas, the fourth order operator Δ^2 , which is called *bilaplacian* or, in dimension 2, *plate operator*, does not satisfy (b) of Proposition 12.5.1. In this case we have

$$P_4(\xi) = \left(\sum_{j=1}^n \xi_j^2 \right)^2.$$

Hence

$$P_4(\xi + i\tau N) = (|\xi|^2 + 2i\tau\xi \cdot N - \tau^2)^2.$$

Which implies that $P_4(\xi + i\tau N) = 0$ holds if and only if

$$\begin{cases} |\xi|^2 - \tau^2 = 0, \\ \tau\xi \cdot N = 0. \end{cases} \tag{12.5.18}$$

The system above, clearly has non-zero (ξ, τ) solutions. On the other hand for these values we have

$$\sum_{j=1}^n \left| P_4^{(j)}(\xi + i\tau N) \right|^2 = 16 (|\xi|^2 + \tau^2) \left| |\xi|^2 + 2i\tau\xi \cdot N - \tau^2 \right|^2 = 0.$$

Therefore P_4 does not satisfy (b) of Proposition 12.5.1. ♠

4a. We will now consider more carefully the case in which $P_2(D)$ is a **second order elliptic operator with complex coefficients**.

$$P_2(D) = \sum_{j,k=1}^n a_{jk} D_{jk}^2, \tag{12.5.19}$$

where $a_{jk} \in \mathbb{C}$, $a_{jk} = a_{kj}$, for $j, k = 1, \dots, n$. Set

$$a(\xi, \eta) = \sum_{j,k=1}^n a_{jk} \xi_j \eta_k, \quad \xi, \eta \in \mathbb{R}^n. \tag{12.5.20}$$

Let $N \in \mathbb{R}^n$ a versor and let us suppose that

$$a(N, N) = 1. \tag{12.5.21}$$

Let us notice that we may always reduce to the situation (12.5.21) since by the ellipticity of $P_2(D)$, we have $a(N, N) = P_2(N) \neq 0$ and, consequently we may divide from the beginning all the coefficients of $P_2(D)$ by $a(N, N)$ leading us back to the assumption (12.5.21). Hence we have

$$P_2(\xi + i\tau N) = a(\xi, \xi) + 2i\tau a(\xi, N) - \tau^2. \quad (12.5.22)$$

Let us observe that, as P_2 is elliptic, we have that, if $(\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}$ satisfies

$$P_2(\xi + i\tau N) = 0, \quad (12.5.23)$$

then **both ξ and τ need to be different from zero**. As a matter of fact, if $\tau = 0$ then, as $P_2(D)$ is elliptic, the unique solution of equation

$$P_2(\xi + i0N) = 0, \quad (12.5.24)$$

is $\xi = 0$. On the other hand, if we had $\xi = 0$ then by the homogeneity of P_2 we would have

$$0 = P_2(i\tau N) = -\tau^2 P_2(N),$$

and by the ellipticity of P_2 we would have $\tau = 0$.

Moreover, by the homogeneity and the ellipticity of P_2 , we have that if $(\xi, \tau) \neq (0, 0)$ satisfies (12.5.23), then ξ and N must be linearly independent. Let us prove the last sentence arguing by contradiction. If ξ and N were not linearly dependent then there would exist $a, b \in \mathbb{R}$ not both zero, such that $a\xi + bN = 0$. Let us suppose, for instance, $a \neq 0$, then $\xi = -\frac{b}{a}N$. Hence

$$0 = P_2(\xi + i\tau N) = P_2\left(\left(-\frac{b}{a} + i\tau\right)N\right) = \left(-\frac{b}{a} + i\tau\right)^2 P_2(N),$$

from which $P_2(N) = 0$, on the other hand this cannot occur because P_2 is elliptic and $N \neq 0$. Similarly, we proceed assuming $b \neq 0$.

In the **case of real coefficients** it is easily seen that (b) of Proposition 12.5.1 is satisfied. Indeed, having to consider only the solutions $(\xi, \tau) \neq (0, 0)$ of equation (12.5.23), we would get $\xi \neq 0$ and $\tau \neq 0$ and then to establish (b) it suffices to check that equation (12.5.23), (considered in the unknown τ) has no nonzero real double roots. Let, therefore, $\xi_0 \neq 0$ and $\tau_0 \neq 0$ such that

$$P_2(\xi_0 + i\tau_0 N) = 0 \quad (12.5.25)$$

and let us assume that

$$\frac{d}{d\tau} P_2(\xi_0 + i\tau N)|_{\tau=\tau_0} = \sum_{j=1}^n P_2^{(j)}(\xi_0 + i\tau N) N_j = 0. \quad (12.5.26)$$

Since

$$\frac{d}{d\tau} P_2(\xi_0 + i\tau N)|_{\tau=\tau_0} = 2ia(\xi_0, N) - 2\tau_0,$$

by (12.5.26) we get $\tau_0 = ia(\xi_0, N)$ and, taking into account (12.5.22), we have

$$P_2(\xi_0 + i\tau_0 N) = a(\xi_0, \xi_0) - (a(\xi_0, N))^2.$$

On the other hand, since $\xi_0 \in N$ since ξ_0 and N are linearly independent, by the Cauchy–Schwarz inequality we have (recall $a(N, N) = 1$)

$$a(\xi_0, \xi_0) - (a(\xi_0, N))^2 > 0,$$

which contradicts (12.5.25). Therefore, if (12.5.25) holds true, then (12.5.26) cannot be true, consequently

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi_0 + i\tau_0 N) \right|^2 > 0.$$

Let us consider now the **case of complex coefficients**.

If $n = 2$ and we consider

$$P_2(D) = -D_1^2 + 2iD_{12}^2 + D_2^2 = (iD_1 + D_2)^2,$$

then $P_2(D)$ is elliptic, but it is easy to check that (b) of Proposition 12.5.1 does not hold. As a matter of fact, we have

$$|P_2(\xi + i\tau N)|^2 = ((\xi_1 + \tau N_2)^2 + (\xi_2 - \tau N_1)^2)^2$$

and

$$\sum_{j=1}^2 |P_2^{(j)}(\xi + i\tau N)|^2 = 8((\xi_1 + \tau N_2)^2 + (\xi_2 - \tau N_1)^2).$$

Hence, if $\xi_0 = (-N_2, N_1)$ and $\tau = 1$, then we have

$$|P_2(\xi_0 + iN)|^2 = 0$$

and

$$\sum_{j=1}^2 |P_2^{(j)}(\xi_0 + iN)|^2 = 0.$$

Now, let us prove that if $n \geq 3$ then (b) of Proposition 12.5.1 is satisfied. We prove, like in the case of the real coefficients, that if $(\xi_0, \tau_0) \neq (0, 0)$ satisfies

$$P_2(\xi_0 + i\tau_0 N) = 0, \tag{12.5.27}$$

then

$$\sum_{j=1}^n P_2^{(j)}(\xi_0 + i\tau_0 N) N_j = \frac{d}{d\tau} P_2(\xi_0 + i\tau N)|_{\tau=\tau_0} \neq 0. \quad (12.5.28)$$

We argue by contradiction. Let us assume $\xi_0 \in \mathbb{R}^n$ and $\tau_0 \in \mathbb{R}$, where $(\xi_0, \tau_0) \neq (0, 0)$, satisfy (12.5.27) and let us assume that (12.5.28) does not hold, i.e. let us assume τ_0 is a double root of equation in τ

$$\frac{d}{d\tau} P_2(\xi_0 + i\tau N)|_{\tau=\tau_0} = 0. \quad (12.5.29)$$

We already noticed that we must have $\xi_0 \neq 0$ and $\tau_0 \neq 0$. Since τ_0 is a double solution of the equation (12.5.22), the discriminant of that equation (when $\xi = \xi_0$) is null. Hence

$$a(\xi_0, \xi_0) = (a(\xi_0, N))^2 \quad (12.5.30)$$

and

$$\tau_0 = ia(\xi_0, N). \quad (12.5.31)$$

By (12.5.30) and (12.5.31) we have

$$a(\xi_0, \xi_0) = -\tau_0^2, \quad a(\xi_0, N) = -i\tau_0 \quad \text{and (recall)} \quad a(N, N) = 1. \quad (12.5.32)$$

Now, let η be a vector of \mathbb{R}^n such that ξ_0 , N and η be linearly independent (recall $n \geq 3$). Let \mathcal{B} a basis of \mathbb{R}^n which complete $\{\xi_0, N, \eta\}$ and let us write the matrix of bilinear form a w.r.t. \mathcal{B} . Set

$$\tilde{a}_{11} = a(\xi_0, \xi_0) = -\tau_0^2, \quad \tilde{a}_{12} = \tilde{a}_{21} = a(\xi_0, N), \quad \tilde{a}_{22} = a(N, N),$$

$$\tilde{a}_{33} = a(\eta, \eta) = \alpha_{33} + i\beta_{33},$$

and

$$\tilde{a}_{13} = \tilde{a}_{31} = a(\xi_0, \eta) := \alpha_{13} + i\beta_{13}, \quad \tilde{a}_{23} = \tilde{a}_{32} = a(N, \eta) := \alpha_{23} + i\beta_{23},$$

where $\alpha_{13}, \beta_{13}, \alpha_{23}, \beta_{23}, \alpha_{33}, \beta_{33}$, are real numbers.

Let us consider the vector v of $\mathbb{R}^n \setminus 0$ whose components with respect to the base \mathcal{B} have coordinates represented by the vector $(x, y, z, 0, \dots, 0)$. Thus, let us note that

$$v \neq 0 \Leftrightarrow (x, y, z) \neq (0, 0, 0).$$

Since $P_2(D)$ is elliptic and $v \neq 0$ we have

$$a(v, v) \neq 0, \tag{12.5.33}$$

in turn, by (12.5.32), this is equivalent to the fact that for $(x, y, z) \neq (0, 0, 0)$ we have

$$x^2 - 2i\tau_0xy - \tau_0^2y^2 + 2(\alpha_{13} + i\beta_{13})xz + 2(\alpha_{23} + i\beta_{23})yz + (\alpha_{33} + i\beta_{33})z^2 \neq 0,$$

for every $(x, y, z) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Now, the above condition is equivalent to the fact that $(0, 0, 0)$ is the unique solution to the algebraic system

$$\begin{cases} x^2 - \tau_0^2y^2 + 2\alpha_{13}xz + 2\alpha_{23}yz + \alpha_{33}z^2 = 0, \\ -2\tau_0xy + 2\beta_{13}xz + 2\beta_{23}yz + \beta_{33}z^2 = 0. \end{cases} \tag{12.5.34}$$

But this cannot occur. Let us see why.

First of all, let us recall that $\tau_0 \neq 0$. Moreover, let us suppose that $z \neq 0$. Then, if we set

$$X = \frac{x}{z}, \quad Y = \frac{y}{z},$$

system (12.5.35) become

$$\begin{cases} X^2 - \tau_0^2Y^2 + 2\alpha_{13}X + 2\alpha_{23}Y + \alpha_{33} = 0 \\ -2\tau_0XY + 2\beta_{13}X + 2\beta_{23}Y + \beta_{33} = 0 \end{cases} \tag{12.5.35}$$

and it is simple to check that system (12.5.35) admits *always* solutions. To convince yourself of this, it suffices to notice that the asymptotes of the first hyperbola (possibly degenerate) in (12.5.35) are parallel to the straight lines $X = \pm\tau_0Y$ which must necessarily meet the asymptotes of the second hyperbola (possibly degenerate) that are parallel to the coordinate axes. From above it follows that there exists $v \in \mathbb{R}^n \setminus 0$ such that $a(v, v) = 0$ and this contradicts (12.5.33). In summary if $n \geq 3$ then (12.5.28) must hold, and this implies that (b) of the Proposition 12.5.1 is satisfied. ♠

4b. We conclude by considering the case where $P_2(D)$ is a **non-elliptic** operator of second order, with **real coefficients**

In such a case there is at least one *characteristic direction*. We recall that $N \in \mathbb{R}^n \setminus 0$ is a characteristic direction with respect to the operator $P_2(D)$ provided we have

$$P_2(N) = 0.$$

In this case, the planes

$$N \cdot x = c,$$

where $c \in \mathbb{R}$ are characteristic surfaces. Let N be a characteristic direction, if we have

$$\sum_{j=1}^n \left| P_2^{(j)}(N) \right|^2 > 0,$$

we say that $N \cdot x = c$, $c \in \mathbb{R}$, is a **simple characteristic** If we have

$$P_2(N) = 0$$

and

$$\sum_{j=1}^n \left| P_2^{(j)}(N) \right|^2 = 0,$$

we say that $\{N \cdot x = c\}$, is a **double characteristic**

For instance, the wave operator has only simple characteristics (reader check) while **the heat operator** and **the Schrödinger operator** have double characteristics. The heat operator is given by (for $n > 1$)

$$P(D) = - \sum_{j=1}^{n-1} D_j^2 - iD_n = \Delta_{x'} - \partial_n.$$

So the principal part of the heat operator is

$$P_2(D) = - \sum_{j=1}^{n-1} D_j^2,$$

whose symbol is $-\sum_{j=1}^{n-1} \xi_j^2$. It is evident that the unique characteristic directions are those generated by the versor e_n which is a double characteristic direction. In the case of the Schrödinger operator we have.

$$P(D) = - \sum_{j=1}^{n-1} D_j^2 - D_n = \Delta_{x'} - \frac{1}{i} \partial_n$$

and it is once again clear that the only characteristic directions are those generated by the e_n versor and, as in the case of the heat operator, they are double.

Let us check that the operators with simple characteristics satisfy (b) of Proposition 12.5.1 for all the versors N .

Let N be a versor of \mathbb{R}^n such that

$$P_2(N) \neq 0. \quad (12.5.36)$$

Let $(\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{0\}$ satisfy

$$P_2(\xi + i\tau N) = 0. \quad (12.5.37)$$

Let us denote by A the symmetric matrix $\{a_{jk}\}_{j,k=1}^n$. Since the coefficients of $P_2(D)$ are real numbers, we have

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi + i\tau N) \right|^2 = 4(|A\xi|^2 + \tau^2|AN|^2). \quad (12.5.38)$$

Now, if $\tau = 0$, then (12.5.37) implies $P_2(\xi) = 0$, however $P_2(D)$ has only simple characteristics, hence

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi + i0N) \right|^2 > 0.$$

If $\tau \neq 0$, then $AN \neq 0$, otherwise, if it were $AN = 0$ we would have $P_2(N) = AN \cdot N = 0$ which would contradict the (12.5.36). Therefore from (12.5.38) we have

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi + i\tau N) \right|^2 \geq 4\tau^2|AN|^2 > 0.$$

Hence, if (12.5.36) holds true, then (b) of Proposition 12.5.1 holds true.

If N is a simple characteristic, then

$$P_2(N) = 0.$$

and, as N is a simple characteristic, we have

$$AN \neq 0. \quad (12.5.39)$$

Now, let $(\xi, \tau) \in \mathbb{R}^{n+1} \setminus \{0\}$ be a solution to (12.5.37). If $\tau = 0$, we have $P_2(\xi) = 0$, hence

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi) \right|^2 > 0.$$

On the other hand, if $\tau \neq 0$ from (12.5.39), as already seen above, we have

$$\sum_{j=1}^n \left| P_2^{(j)}(\xi + i\tau N) \right|^2 > 0.$$

Therefore, even when N is a simple characteristic direction, (b) of Proposition 12.5.1 is satisfied.

Finally, let us consider the case in which $P_2(D)$ has a double characteristic direction; be it η , then any way one chooses the versor N , we have

$$P_2(\eta + i0N) = 0 \quad (12.5.40)$$

and

$$\sum_{j=1}^n \left| P_2^{(j)}(\eta + i0N) \right|^2 = 0. \quad (12.5.41)$$

Therefore, in this case, (b) of Proposition 12.5.1 is not satisfied. This implies that the following estimate **does not hold**

$$\int_{Q_1} e^{2\tau N \cdot x} |u|^2 dx + \int_{Q_1} e^{2\tau N \cdot x} |Du|^2 dx \leq C \int_{Q_1} e^{2\tau N \cdot x} |P_2(D)u|^2 dx,$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \geq \tau_0$.

Let us notice that in the case of the heat operator and of Schrödinger operator, the principal part $P_2(D)$ excludes the term $iD_n u$. Actually, if we employ directly estimate (12.2.1) we have (reader check)

$$\begin{aligned} \int_{Q_1} e^{2\tau N \cdot x} |u|^2 dx + \int_{Q_1} e^{2\tau N \cdot x} |\nabla u|^2 dx &\leq \\ &\leq C \int_{Q_1} e^{2\tau N \cdot x} |\Delta_{x'} u - \partial_n|^2 dx, \end{aligned} \quad (12.5.42)$$

for every $u \in C_0^\infty(Q_1, \mathbb{C})$ and for every $\tau \in \mathbb{R}$.

As in previous situations, (12.5.42) implies the uniqueness for the Cauchy problem with **strictly convex initial surfaces** for the differential inequalities

$$|\Delta_{x'} U - \partial_n U| \leq M (|\nabla_{x'} U| + |U|), \quad (12.5.43)$$

where M is a positive number and U is enough regular.

When $n = 2$, one can exploit the particularity of the dimension two to prove the following unique continuation property, we refer to [60, Theorem 9] for the proof:

Let ω be an open of $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_t$ contained in a rectangle \mathcal{R} . For $t_0 \in \mathbb{R}$, we denote by s_{t_0} straight line of equation $t = t_0$ and set (Figure 12.2)

$$\mathcal{R}_\omega = \{(x, t_0) \in \mathcal{R} : s_{t_0} \cap \omega \neq \emptyset\}.$$

Let us assume

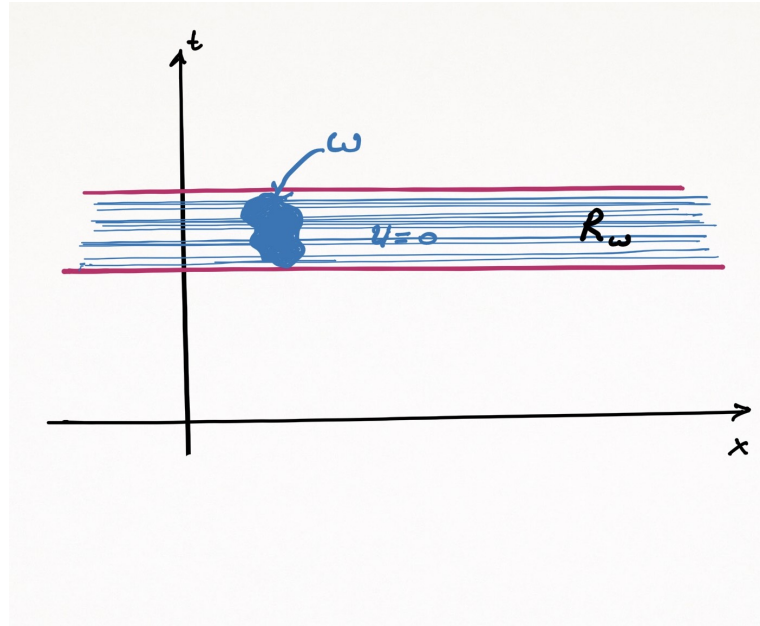


Figure 12.2:

$$|\partial_x^2 U - \partial_t U| \leq M (|\partial_x U| + |U|), \quad \text{in } \mathcal{R}, \quad (12.5.44)$$

then

$$U(x, t) = 0, \quad \forall (x, t) \in \mathcal{R}_\omega. \quad (12.5.45)$$

From the previous result it follows, in particular, that if U is regular enough (it is sufficient that $U, \partial_x U, \partial_t U, \partial_x^2 U$ are continuous in $(0, 1) \times (0, 1)$) and if U is a solution of the Cauchy problem

$$\begin{cases} \partial_x^2 U - \partial_t U = a(x, t)\partial_x U + b(x, t)U, & \text{in } (0, 1) \times (0, 1), \\ U(0, t) = 0, & \text{for } t \in (\alpha, \beta), \\ \partial_x U(0, t) = 0, & \text{for } t \in (\alpha, \beta), \end{cases} \quad (12.5.46)$$

where α, β are given numbers which satisfy $0 < \alpha < \beta < 1$ and

$$a, b \in L^\infty((0, 1) \times (0, 1)),$$

then

$$U = 0, \quad \text{in } (0, 1) \times (\alpha, \beta).$$

We do not enter into the details and refer the interested reader directly to [60, Theorem 9]. ♠

12.6 Chapter summary and conclusions

In this chapter we have proved estimate (12.2.2), in a relatively simple manner. The most relevant peculiarity of such an estimate is that in it there is a "weight" which depends on a parameter τ that may be arbitrarily large.

By applying estimate (12.2.2) to the Laplace operator and by exploiting some important invariance properties of this operator, we have proved, in Theorem 12.3.5, the global uniqueness for the Cauchy problem for the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U,$$

where $b, c \in L^\infty$.

We proved that the estimate (12.2.2) allows us to prove the global uniqueness for the Cauchy problem with strictly convex initial surface, for the operators $P_m(D) + a_0(x)$, where $P_m(D)$ is a homogeneous operator with constant coefficients and $a_0 \in L^\infty$.

We have shown with several examples and remarks related to Theorem 12.4.1, the strict connections that exists between an estimate of the type (12.4.1) and some properties of the symbols of operators $M(D)$ and PD . These connections makes it possible to transfer into the algebraic field the estimates under investigation in this Chapter.

The estimates considered in this Chapter have two remarkable **weaknesses** that we now briefly discuss.

1. The first weakness lies in the character of the weight exponent. This is because such an exponent is linear and, as we have seen, this greatly limits the geometry in which to apply our estimates. Let us consider, for instance the following Cauchy problem

$$P(D)U = 0, \quad \text{in } \mathbb{R}^n$$

and

$$U = 0, \quad \text{for } x_n \leq 0,$$

to prove the uniqueness we would be most helped by a weight whose level surfaces are "curved" with respect to the $x_n = c$ planes. More precisely, if instead of the weight $e^{-2\tau x_n}$ we dispose of estimates with weight $e^{2\tau(-x_n + \frac{\delta}{2}|x|^2)}$,

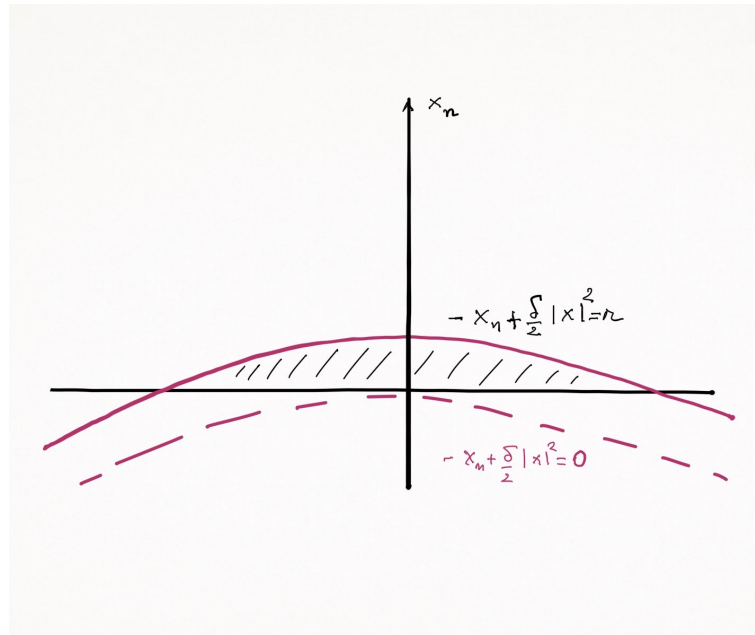


Figure 12.3:

with $\delta > 0$ (even small), it could be shown that U vanishes in regions of the type $\{-x_n + \frac{\delta}{2}|x|^2 < r, x_n > 0\}$ with $r > 0$ (Figure 12.3).

However, the proof of an estimate that corresponds to this nonlinear weight would not allow repeat, in a simple and immediate way, the proof of Theorem 12.2.1. To realize this, let us observe that, setting $v = e^{\tau(N \cdot x + \frac{\delta}{2}|x|^2)}u$, instead of the (12.2.16) we would have

$$D_j u = e^{-\tau N \cdot x} (D_j + i\tau (N_j + \delta x_j)) v$$

and we cannot use in *immediate manner* the Fourier transform. For the time being, we refer to [75] for further discussion.

2. The other weakness of estimate (12.2.2) consists in the fact that these estimates hold for operators with constant coefficients in the principal part.

Let $P_m(x, D)$ be the principal part of the operator. One might be tempted to consider $P_m(x, D)$, in a neighborhood \mathcal{U} of a point x_0 , as the perturbation of the operator with constant coefficients $P_m(x_0, D)$ i.e., we could write

$$P_m(x, D) = P_m(x_0, D) + (P_m(x_0, D) - P_m(x, D))$$

and, by exploiting the regularity of the coefficients of $P_m(x, D)$, we may consider that, for a suitable neighborhood \mathcal{U} , we have

$$|P_m(x_0, D)U - P_m(x, D)U| < \varepsilon |D^m U|, \quad \forall x \in \mathcal{U}. \quad (12.6.1)$$

To clarify the idea further let us show, in broad terms, that if $P_m(x, D)$ is elliptic with continuous coefficients then we have

$$\int_{Q_1} |D^m u|^2 dx \leq C \int_{Q_1} |P_m(x, D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_1, \mathbb{C}). \quad (12.6.2)$$

We have already seen that, in the case of elliptic operators with constant coefficients, (12.4.30) is valid. Now let $x_0 = 0$, $\varepsilon > 0$ and let r_ε be such that the (12.6.1) is satisfied for $\mathcal{U} = Q_{r_\varepsilon}$.

Then by estimate (12.4.30) (which we know is true in the elliptic case) we have

$$\int_{Q_1} |D^m u|^2 dx \leq 2C \int_{Q_1} |P_m(x, D)u|^2 dx + 2C\varepsilon^2 \int_{Q_1} |D^m u|^2 dx,$$

for every $u \in C_0^\infty(Q_{r_\varepsilon}, \mathbb{C})$. It is then evident that by choosing $\varepsilon = \frac{1}{2\sqrt{C}}$ we get

$$\int_{Q_{r_\varepsilon}} |D^m u|^2 dx \leq 4C \int_{Q_{r_\varepsilon}} |P_m(x, D)u|^2 dx, \quad \forall u \in C_0^\infty(Q_{r_\varepsilon}, \mathbb{C}).$$

A similar argument can be made in the neighborhood of the other points of Q_1 and, using a partition of the unity, one can obtain (12.6.2).

It is easily understood that a similar argument for the estimates of the type (12.5.6) – that is, considering $P_m(x, D)$ as a perturbation of order m in a neighborhood of x_0 of the operator with constant coefficients $P_m(x_0, D)$ – is difficult to realize even in the elliptic case. Actually, estimate (12.4.31) does not hold even for elliptic operators and thus the following "error term" that would follow from (12.6.1),

$$C\varepsilon^2 \int_{Q_1} e^{2\tau N \cdot x} |D^m u|^2 dx,$$

cannot be "absorbed" by the terms to the left of the sign of inequality. The development of the theory will show that the case of the variable coefficients (in the principal part) is more tricky than the one with constant coefficients, even in the case of the weight $e^{2\tau N \cdot x}$.

Chapter 13

Carleman estimates and the Cauchy problem I – Elliptic operators

13.1 Introduction

In the previous Chapter we gave a first insight into the Carleman estimates by showing how they are used in the investigation of the uniqueness of the Cauchy problem and of the unique continuation property for operators whose principal part has constant coefficients. In Section 12.6, we pointed out some important weaknesses of Theorem 12.2.1. Such weak points, briefly, consist of:

- (a) the linear character of the weight exponent;
- (b) estimate (12.2.2) of Theorem 12.2.1 cannot be easily extended for operators with variable coefficients.

For such reasons, here we begin a more systematic study of the Carleman estimates in order to extend somewhat the uniqueness results we have seen in Chapter 12. Although we will focus mainly on elliptic operators (Section 13.5), the introductory examples (Sect. 13.2) and the framework apply to other types of operators as well. In Chapter 14 we will consider the **second order operators whose principal part has real coefficients and that are not necessarily elliptic**.

Let us consider the operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad x \in \Omega, \quad (13.1.1)$$

where $D_j = \frac{1}{i}\partial_j$, $j = 1, \dots, n$, Ω is an open set of \mathbb{R}^n and $a_\alpha \in L^\infty(\Omega; \mathbb{C})$, for every $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$.

We will follow the classical approach developed in [34] by L. Hörmander. This approach is not only more elementary than the one based on the pseudodifferential operators ([36, vol. IV], [50]), but it allows more easily to reduce on the regularity assumptions of the coefficients of the principal parts of the operators.

Let us recall that the symbol of the operator $P(x, D)$ is

$$P(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \quad \forall \xi \in \mathbb{R}^n.$$

We further denote by $P_m(x, D)$ the principal part of $P(x, D)$, i.e.

$$P_m(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha \quad (13.1.2)$$

(of course we assume that $|\alpha| = m$, a_α is not identically zero for at least one α such that).

Let φ be a sufficiently regular real-valued function, say $\varphi \in C^\infty(\overline{\Omega})$, however in many cases a less regularity will suffice.

Let $\mu \geq 0$, we are interested in the Carleman estimates such as

$$\tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int_{\Omega} |P_m(x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.1.3)$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$, where C and τ_0 are constants independent of u and τ .

In this Chapter we will be interested in the case where

$$\mu > 0.$$

In this case, it is simple to check that the estimate (13.1.3) is equivalent to a similar estimate where $P(x, D)$ is replaced by $P_m(x, D)$. Indeed, let us suppose that estimate (13.1.3) holds and let us denote by

$$R(x, D) = P(x, D) - P_m(x, D),$$

we have

$$|R(x, D)u| = \left| \sum_{|\alpha| \leq m-1} a_\alpha(x) D^\alpha u \right| \leq M \sum_{|\alpha| \leq m-1} |D^\alpha u|, \quad (13.1.4)$$

where

$$M = \max_{|\alpha| \leq m-1} \left\{ \|a_\alpha\|_{L^\infty(\Omega)} \right\}.$$

Hence, by (13.1.3) we get

$$\begin{aligned} \tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx &\leq C \int_{\Omega} |P_m(x, D)u|^2 e^{2\tau\varphi} dx \leq \\ &\leq 2C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx + 2C \int_{\Omega} |R(x, D)u|^2 e^{2\tau\varphi} dx \leq \\ &\leq 2C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx + \tilde{C}M^2 \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx, \end{aligned}$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$. Moving the last sum to the left hand side, we have

$$\left(\tau^\mu - \tilde{C}M^2 \right) \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.1.5)$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$. Now let $\tau_1 \geq \tau_0$ be a number such that for every $\tau \geq \tau_1$ we have

$$\tau^\mu - \tilde{C}M^2 \geq \frac{\tau^\mu}{2},$$

by (13.1.5) we obtain

$$\tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq 4C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.1.6)$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_1$. Hence, if (13.1.3) holds then (13.1.6) holds. The converse (of course with different values of C and τ_0) can be similarly proved.

It should be observed at once that estimates of type (13.1.3) (or (13.1.6)) for $\mu > 0$ have a local character in the sense specified in the following

Lemma 13.1.1 (local character of the Carleman estimates). *Let $\mu > 0$. Let Ω be a bounded open set of \mathbb{R}^n and let $P(x, D)$ be a differential operator whose coefficients belong to $L^\infty(\Omega)$. Let us assume that for each $y \in \bar{\Omega}$ there exist $\delta_y > 0$, $C_y > 0$ and $\tau_y \in \mathbb{R}$ such that*

$$\begin{aligned} \tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega \cap B_{\delta_y}(y)} |D^\alpha u|^2 e^{2\tau\varphi} dx &\leq \\ &\leq C_y \int_{\Omega \cap B_{\delta_y}(y)} |P(x, D)u|^2 e^{2\tau\varphi} dx, \end{aligned} \quad (13.1.7)$$

for every $u \in C_0^\infty(\Omega \cap B_{\delta_y}(y))$ and for every $\tau \geq \tau_y$. Then there exist $C > 0$ and $\tau_0 \in \mathbb{R}$ such that

$$\tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.1.8)$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$.

Proof. Since $\bar{\Omega}$ is compact, there exists a finite set of points, y_1, \dots, y_N , such that

$$\bar{\Omega} \subset \bigcup_{j=1}^N B_{\delta_{y_j}}(y_j).$$

Let $\{\eta_j\}_{1 \leq j \leq N}$ be a partition of unity (Lemma 2.4.2) such that

$$\eta_j \in C_0^\infty(B_{\delta_{y_j}}(y_j)); \quad 0 \leq \eta_j \leq 1; \quad \sum_{j=1}^N \eta_j = 1, \quad \text{in } \bar{\Omega}.$$

Let $u \in C_0^\infty(\Omega)$ and let us denote by

$$u_j = u\eta_j, \quad \text{for } j = 1, \dots, N.$$

For every $\alpha \in \mathbb{N}_0^n$, by the Cauchy–Schwarz inequality we have

$$|D^\alpha u|^2 = \left| \sum_{j=1}^N D^\alpha u_j \right|^2 \leq N \sum_{j=1}^N |D^\alpha u_j|^2. \quad (13.1.9)$$

Now, we notice that

$$P(x, D)u_j = \eta_j P(x, D)u + R_{m-1}(x, D)u,$$

where $R_{m-1}(x, D)$ is an operator of order $m-1$ whose coefficients depend on the coefficients of $P(x, D)$ (but not on their derivatives) and on the functions η_j , $j = 1, \dots, N$ and their derivatives of order less or equal to m . We have

$$|P(x, D)u_j| \leq \eta_j |P(x, D)u| + M \sum_{|\alpha| \leq m-1} |D^\alpha u|, \quad (13.1.10)$$

where M depends on the L^∞ norms of the coefficients of $P(x, D)$ and on the L^∞ norms of the derivatives of η_j of order less or equal to m .

By (13.1.10), (13.1.7) and (13.1.9) we get

$$\begin{aligned} & \tau^\mu \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq \\ & \leq N^2 \tau^\mu \sum_{j=1}^N \sum_{|\alpha| \leq m-1} \int_{\Omega \cap B_{\delta y_j}(y_j)} |D^\alpha u_j|^2 e^{2\tau\varphi} dx \leq \\ & \leq N^2 \sum_{j=1}^N \int_{\Omega \cap B_{\delta y_j}(y_j)} |P(x, D)u_j|^2 e^{2\tau\varphi} dx \leq \\ & \leq C \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx + C \sum_{|\alpha| \leq m-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx, \end{aligned}$$

for every $\tau \geq \tau_0$, where

$$\tau_0 = \max_{1 \leq j \leq N} \tau_{y_j}$$

and C is a constant. Now, we move the last integral to the left hand side and we proceed exactly as did above to prove (13.1.6) and we obtain (13.1.8). ■

13.2 Introductory examples – the first order operators

This Section has essentially two purposes: the first one consists of showing, with simple examples concerning the first order operators (with constant and real coefficients), that certain conditions are needed (necessary or sufficient) on φ in order that it can be the exponent of a weight in a Carleman estimate of type (13.1.3). The other purpose is to show, again in the case of first order operators, how to apply the Carleman estimates to prove the uniqueness of the Cauchy problem.

Let $I = (-1, 1)$ and $\varphi \in C^2(\bar{I})$; let us begin by considering the following elementary Carleman estimate

$$\tau^\mu \int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.1)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$. We will show how to derive an estimate like (13.2.1) and what conditions on μ and φ are necessary for them to hold.

Let us start by the following

Proposition 13.2.1. *Let*

$$A = \{x \in \bar{I} : \varphi'(x) = 0\}. \quad (13.2.2)$$

If

$$x \in A \Rightarrow \varphi''(x) > 0, \quad (13.2.3)$$

then there exist τ_0 and C such that

$$\tau \int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.4)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$.

Remark. Of course if $A = \emptyset$ (13.2.2) is trivially satisfied. \blacklozenge

Proof. Set

$$v = e^{\tau\varphi}u, \quad (13.2.5)$$

we have

$$e^{\tau\varphi}u' = e^{\tau\varphi} (e^{-\tau\varphi}v)' = v' - \tau\varphi'v. \quad (13.2.6)$$

Hence estimate (13.2.1) is equivalent to (we omit for brevity the integration set)

$$\frac{\tau}{C} \int |v|^2 dx \leq \int |v' - \tau\varphi'v|^2 dx$$

for every $v \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$.

Now estimate from below the integral on the RHS. Spreading the square and integrating by parts we have

$$\begin{aligned} \int |v' - \tau\varphi'v|^2 dx &= \int \left(|v'|^2 - 2\tau\varphi'\Re(v'\bar{v}) + \tau^2\varphi'^2 |v|^2 \right) dx = \\ &= \int \left(|v'|^2 - \tau\varphi'(|v|^2)' + \tau^2\varphi'^2 |v|^2 \right) dx = \\ &= \int \left(|v'|^2 + \tau(\varphi'' + \tau\varphi'^2) |v|^2 \right) dx. \end{aligned} \quad (13.2.7)$$

Now, as (13.2.3) holds, by applying Lemma 12.5.2 (with $f = \varphi'$ and $g = \varphi''$) we have that there exists τ_0 such that

$$\tau_0 \varphi'^2(x) + \varphi''(x) > 0, \quad \forall x \in \bar{I}$$

and, setting

$$C^{-1} = \min_{\bar{I}} (\tau_0 \varphi'^2 + \varphi'') > 0,$$

we have, by (13.2.7)

$$\begin{aligned} \int |v' - \tau \varphi' v|^2 dx &\geq \int \tau (\varphi'' + \tau \varphi'^2) |v|^2 dx \geq \\ &\geq C^{-1} \tau \int |v|^2 dx. \end{aligned} \quad (13.2.8)$$

From which, taking into account (13.2.5) and (13.2.6), (13.2.4) follows. ■

Remarks.

1. If $A = \emptyset$ then a stronger version of (13.2.1) holds true. More precisely, we have

$$\tau^2 \int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.9)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$. In this case, indeed, we have

$$m := \min_{\bar{I}} |\varphi'| > 0$$

as $A = \emptyset$. On the other hand, setting

$$m_1 = \|\varphi''\|_{L^\infty(I)},$$

and taking into account (13.2.7), we get

$$\begin{aligned} \int |v' - \tau \varphi' v|^2 dx &= \int (|v'|^2 + \tau^2 (\tau^{-1} \varphi'' + \varphi'^2) |v|^2) dx \geq \\ &\geq \int (|v'|^2 + \tau^2 (-\tau^{-1} m_1 + m^2) |v|^2) dx \geq \\ &\geq \frac{\tau^2 m^2}{2} \int |v|^2 dx, \end{aligned} \quad (13.2.10)$$

for every $v \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$, where $\tau_0 = \frac{2m_1^2}{m^2}$ e $C = \frac{2}{m^2}$. From which (13.2.9) follows.

2. On the other hand, it is also evident that estimate (13.2.1) cannot be true for $\mu > 2$. As a matter of fact, we have

$$\int |v' - \tau\varphi'v|^2 dx \leq 2 \int \left(|v'|^2 + \tau^2 \|\varphi'\|_{L^\infty(I)}^2 |v|^2 \right) dx \quad (13.2.11)$$

and so, if (13.2.1), and $\mu > 2$, would imply

$$\begin{aligned} \int |v|^2 dx &\leq \\ &\leq 2\tau^{2-\mu} \int \left(\tau^{-2} |v'|^2 + \|\varphi'\|_{L^\infty(I)}^2 |v|^2 \right) dx \rightarrow 0, \quad \text{as } \tau \rightarrow +\infty \end{aligned}$$

which is evidently absurd since v is an arbitrary function of $C_0^\infty(I)$. \blacklozenge

We establish some necessary conditions for estimate (13.2.1).

Proposition 13.2.2. *Let*

$$A_0 = \{x \in I : \varphi'(x) = 0\}. \quad (13.2.12)$$

If there exist C and τ_0 such that

$$\int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.13)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$, then

$$x \in A_0 \Rightarrow \varphi''(x) \geq 0. \quad (13.2.14)$$

Proof. We first notice that, by density, (13.2.13) is satisfied for each $u \in H_0^1(I)$.

In order to prove the Proposition we argue by contradiction and we assume that (13.2.14) does not hold. Let $x_0 \in I$ satisfy $\varphi'(x_0) = 0$ and $\varphi''(x_0) < 0$. For the purpose of simplifying the notations, since x_0 is an interior point of I , we assume that $x_0 = 0$. Hence we have

$$\varphi'(0) = 0 \quad \text{and} \quad \varphi''(0) < 0. \quad (13.2.15)$$

Since (13.2.13) is trivially equivalent to

$$\int_I |u(x)|^2 e^{2\tau(\varphi(x)-\varphi(0))} dx \leq C \int_I |u'(x)|^2 e^{2\tau(\varphi(x)-\varphi(0))} dx, \quad (13.2.16)$$

for every $u \in H_0^1(I)$ and for every $\tau \geq \tau_0$, we may assume

$$\varphi(0) = 0.$$

Set

$$a = -\varphi''(0) > 0$$

and let $\psi \in H_0^1(I)$ be a function that we will choose later.

If $0 < \varepsilon \leq \tau_0^{-1/2}$, we have

$$x \rightarrow \psi(\varepsilon\sqrt{\tau}x) \in H_0^1(I), \quad \forall \tau \geq \varepsilon^{-2}.$$

Now, introducing the following notation in (13.2.16)

$$u(x) = \psi(\varepsilon\sqrt{\tau}x),$$

we have

$$\int_I |\psi(\varepsilon\sqrt{\tau}x)|^2 e^{2\tau\varphi(x)} dx \leq C\varepsilon^2\tau \int_I |\psi'(\varepsilon\sqrt{\tau}x)|^2 e^{2\tau\varphi(x)} dx.$$

By performing the change of variables $t = \sqrt{\tau}x$, we get

$$\int_{-\sqrt{\tau}}^{\sqrt{\tau}} |\psi(\varepsilon t)|^2 e^{2\tau\varphi(t/\sqrt{\tau})} dt \leq C\varepsilon^2\tau \int_{-\sqrt{\tau}}^{\sqrt{\tau}} |\psi'(\varepsilon t)|^2 e^{2\tau\varphi(t/\sqrt{\tau})} dt. \quad (13.2.17)$$

We notice that the Taylor formula gives

$$\varphi(x) = -\frac{a}{2}x^2 + \frac{x^2}{2}\omega(x), \quad \forall x \in I,$$

where

$$\lim_{x \rightarrow 0} \omega(x) = 0. \quad (13.2.18)$$

Now we choose

$$\tau = \varepsilon^{-2},$$

and by (13.2.17) we get

$$\int_{-1/\varepsilon}^{1/\varepsilon} |\psi(\varepsilon t)|^2 e^{-t^2(a-\omega(\varepsilon t))} dt \leq C \int_{-1/\varepsilon}^{1/\varepsilon} |\psi'(\varepsilon t)|^2 e^{-t^2(a-\omega(\varepsilon t))} dt. \quad (13.2.19)$$

Let us choose ψ such that

$$\psi(x) = \begin{cases} 2(x+1), & \text{for } x \in [-1, -\frac{1}{2}), \\ 1, & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}), \\ 2(-x+1), & \text{for } x \in [\frac{1}{2}, 1]. \end{cases} \quad (13.2.20)$$

By (13.2.19) and (13.2.20) we have

$$\begin{aligned} \int_{-1/2\varepsilon}^{1/2\varepsilon} e^{-t^2(a-\omega(\varepsilon t))} dt &\leq \int_{-1/\varepsilon}^{1/\varepsilon} |\psi(\varepsilon t)|^2 e^{-t^2(a-\omega(\varepsilon t))} dt \leq \\ &\leq C \int_{-1/\varepsilon}^{1/\varepsilon} |\psi'(\varepsilon t)|^2 e^{-t^2(a-\omega(\varepsilon t))} dt = \\ &= 8C \int_{1/2\varepsilon}^{1/\varepsilon} e^{-t^2(a-\omega(\varepsilon t))} dt. \end{aligned} \quad (13.2.21)$$

Passing to the limit as $\varepsilon \rightarrow 0$ and recalling (13.2.18), we get (by the Dominated Convergence Theorem)

$$0 < \int_{-\infty}^{+\infty} e^{-at^2} dt = \lim_{\varepsilon \rightarrow 0} \int_{-1/2\varepsilon}^{1/2\varepsilon} e^{-t^2(a-\omega(\varepsilon t))} dt \leq \lim_{\varepsilon \rightarrow 0} 8C \int_{1/2\varepsilon}^{1/\varepsilon} e^{-t^2(a-\omega(\varepsilon t))} dt = 0$$

Which is, evidently, absurd. ■

Proposition 13.2.3. *Let A_0 be as in Proposition 13.2.12. If there exist C_0 e τ_0 such that*

$$\tau \int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.22)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \tau_0$, then

$$x \in A_0 \Rightarrow \varphi''(x) \geq \frac{1}{2C} > 0. \quad (13.2.23)$$

Proof. As already noticed above (proof of Proposition 13.2.1), estimate (13.2.22) is equivalent to

$$\frac{\tau}{C} \int_I |v|^2 dx \leq \int_I |v' - \tau\varphi'v|^2 dx, \quad (13.2.24)$$

for every $v \in H_0^1(I)$ and for every $\tau \geq \tau_0$. Let us notice that

$$\begin{aligned} \int |v' - \tau\varphi'v|^2 dx &= \int |v' + \tau\varphi'v|^2 dx - 4\tau \int_I \varphi' \Re(\bar{v}v') dx = \\ &= \int |v' + \tau\varphi'v|^2 dx + 2\tau \int_I \varphi'' |v|^2 dx. \end{aligned} \quad (13.2.25)$$

We suppose, as in the proof of Proposition 13.2.2, that $\varphi'(0) = 0$ and we want to prove that $\varphi''(0) \geq \frac{1}{2C}$. We may also let us assume here that $\varphi(0) = 0$.

Let

$$v(x) = e^{-\tau\varphi(x)}\psi(\varepsilon\sqrt{\tau}x),$$

with $\psi \in H_0^1(I)$ to be chosen later and with $\tau \geq \varepsilon^{-2}$, $0 < \varepsilon < \tau_0^{-1/2}$. By (13.2.23) and (13.2.24) we have

$$\begin{aligned} \frac{1}{C} \int_I |\psi(\varepsilon\sqrt{\tau}x)|^2 e^{-2\tau\varphi} dx &\leq \\ &\leq 2 \int_I \varphi''(x) |\psi(\varepsilon\sqrt{\tau}x)|^2 e^{-2\tau\varphi} dx + \varepsilon^2 \int_I |\psi'(\varepsilon\sqrt{\tau}x)|^2 e^{-2\tau\varphi} dx. \end{aligned} \quad (13.2.26)$$

By Proposition 13.2.2, if we set

$$\alpha = \varphi''(0),$$

we have

$$\alpha \geq 0. \quad (13.2.27)$$

Now, by the Taylor formula and by performing the change of variables $t = \sqrt{\tau}x$ we obtain, (we argue as in the proof of Proposition 13.2.2),

$$\begin{aligned} \frac{1}{C} \int_{-\sqrt{\tau}}^{\sqrt{\tau}} |\psi(\varepsilon t)|^2 e^{-t^2(\alpha + \omega(t/\sqrt{\tau}))} dt &\leq \\ &\leq 2 \int_{-\sqrt{\tau}}^{\sqrt{\tau}} (\alpha + \omega_1(t/\sqrt{\tau})) |\psi(\varepsilon t)|^2 e^{-t^2(\alpha + \omega(t/\sqrt{\tau}))} dt + \\ &+ \varepsilon^2 \int_{-\sqrt{\tau}}^{\sqrt{\tau}} |\psi'(\varepsilon t)|^2 e^{-t^2(\alpha + \omega(t/\sqrt{\tau}))} dt, \end{aligned} \quad (13.2.28)$$

where $\omega(x)$ and $\omega_1(x)$ go to 0 as x goes to 0. Passing to the limit in (13.2.28) as $\tau \rightarrow +\infty$, we have

$$\begin{aligned} \frac{1}{C} \int_{-\infty}^{+\infty} |\psi(\varepsilon t)|^2 e^{-\alpha t^2} dt &\leq \\ &\leq 2 \int_{-\infty}^{+\infty} \alpha |\psi(\varepsilon t)|^2 e^{-\alpha t^2} dt + \varepsilon^2 \int_{-\infty}^{+\infty} |\psi'(\varepsilon t)|^2 e^{-\alpha t^2} dt. \end{aligned} \quad (13.2.29)$$

If it were $\alpha = 0$, then (13.2.29) would be written

$$\frac{1}{C} \int_{-\infty}^{+\infty} |\psi(\varepsilon t)|^2 dt \leq \varepsilon^2 \int_{-\infty}^{+\infty} |\psi'(\varepsilon t)|^2 dt.$$

The latter, by the change of variable $s = \varepsilon t$, implies

$$\frac{1}{C} \int_I |\psi(s)|^2 ds \leq \varepsilon^2 \int_I |\psi'(s)|^2 ds$$

which, passing to the limit as $\varepsilon \rightarrow 0$, leads to an absurd (just choose ψ not identically null). Therefore necessarily we have

$$\alpha > 0.$$

At this point, passing to the limit as $\varepsilon \rightarrow 0$ in (13.2.29), we obtain (by the Dominated Convergence Theorem)

$$\frac{1}{C} |\psi(0)|^2 \int_{-\infty}^{+\infty} e^{-\alpha t^2} dt \leq 2 |\psi(0)|^2 \alpha \int_{-\infty}^{+\infty} e^{-\alpha t^2} dt.$$

From which we have trivially

$$2\alpha \geq \frac{1}{C}.$$

■

Remark. If $1 < \mu \leq 2$, in (13.2.1), then $A_0 = \emptyset$ i.e. $\varphi'(x) \neq 0$ for every $x \in (-1, 1)$. As a matter of fact, if $\mu > 1$, we would have, for every $K > 0$

$$\tau K^{\mu-1} \int_I |u(x)|^2 e^{2\tau\varphi(x)} dx \leq C \int_I |u'(x)|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.30)$$

for every $u \in C_0^\infty(I)$ and for every $\tau \geq \max\{\tau_0, K\}$. Now, if $A_0 \neq \emptyset$, then there exists $x_0 \in I$ such that $\varphi'(x_0) = 0$, hence, by (13.2.23) we have

$$\varphi''(x_0) \geq \frac{K^{\mu-1}}{2C}, \quad \forall K > 0$$

from which we have

$$\varphi''(x_0) = +\infty.$$

which contradicts $\varphi \in C^2(\bar{I})$. \blacklozenge

We now consider the first-order operator

$$P_1(\partial) = \sum_{j=1}^n a_j \partial_j = a \cdot \nabla. \quad (13.2.31)$$

where $a = (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{0\}$ and $a_j, j = 1, \dots, n$ are constants. Let us propose to transfer to operator (13.2.31) what we established above for the derivative operator. We will reach a Carleman estimate of the type

$$\tau \int |u|^2 e^{2\tau\varphi(x)} dx \leq C \int |P_1(\partial)u|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.32)$$

for every $u \in C_0^\infty(\mathbb{R}^n)$ and for every τ large enough, where φ is a function which belongs to $C^2(\mathbb{R}^n)$ on which we will make further assumptions later.

Let us suppose, for instance, that

$$a_n \neq 0. \quad (13.2.33)$$

Let, for $y \in \mathbb{R}^{n-1}$, $x = X(t, y)$, the equations of characteristic lines satisfying

$$\begin{cases} \partial_t X(t, y) = a \cdot X(t, y), \\ X(0, y) = (y, 0). \end{cases} \quad (13.2.34)$$

We have

$$\begin{cases} X_1(t, y) = a_1 t + y_1, \\ \dots, \\ X_{n-1}(t, y) = a_{n-1} t + y_{n-1}, \\ X_n(t, y) = a_n t. \end{cases} \quad (13.2.35)$$

X is a linear and bijective transformation from \mathbb{R}^n in itself since the absolute value of the determinant of the matrix associated to X is equal to $|a_n|$ and by (13.2.33) we have $a_n \neq 0$. Moreover, see Section 5.3, setting

$$z(t, y) = u((X(t, y))), \quad (13.2.36)$$

$$(a \cdot \nabla u)(X(t, y)) = \partial_t z(t, y) \quad (13.2.37)$$

and

$$\tilde{\varphi}(t, y) = \varphi(X(t, y)),$$

estimate (13.2.32) is equivalent to the estimate

$$\tau \int |z|^2 e^{2\tau\tilde{\varphi}(t,y)} dt dy \leq C \int |\partial_t z|^2 e^{2\tau\tilde{\varphi}(t,y)} dt dy, \quad (13.2.38)$$

for every $z \in C_0^\infty(\mathbb{R}^n)$ and for every τ large enough. Of course, if we are interested in estimate (13.2.32) for u supported in a bounded open Ω then estimate (13.2.38) will be established for z supported in a bounded open set. More precisely, set $\tilde{\Omega} = X^{-1}(\Omega)$, Proposition 13.2.1 yields what follows:

If

$$\partial_t^2 \tilde{\varphi}(t, y) > 0, \quad \text{for every } (t, y) \text{ such that } \partial_t \tilde{\varphi}(t, y) = 0, \quad (13.2.39)$$

then estimate (13.2.38) and (consequently) estimate (13.2.32) holds true.

Now we have,

$$\partial_t \tilde{\varphi}(t, y) = (\nabla \varphi)(X(t, y)) \cdot a(X(t, y)) = \sum_{j=1}^n (\partial_j \varphi a_j)(X(t, y)),$$

and

$$\begin{aligned} \partial_t^2 \tilde{\varphi}(t, y) &= \partial_t \left(\sum_{j=1}^n (\partial_j \varphi)(X(t, y)) a_j(X(t, y)) \right) = \\ &= \sum_{j,k=1}^n (\partial_{jk}^2 \varphi)(X(t, y)) (\partial_t X(t, y)) a_j(X(t, y)) + \\ &+ \sum_{j,k=1}^n (\partial_j \varphi)(X(t, y)) (\partial_{x_k} a_i)(X(t, y)) \partial_t X(t, y) = \\ &= \sum_{j,k=1}^n (\partial_{jk}^2 \varphi a_j a_k)(X(t, y)), \end{aligned} \quad (13.2.40)$$

(in the second to last step we used that $\partial_{x_k} a_i = 0$, as a is a constant vector). Therefore, with respect to the variables x_1, \dots, x_n , condition (13.2.39) can be written

$$a \cdot \nabla \varphi(x) = 0 \quad \Rightarrow \quad \sum_{j,k=1}^n \partial_{jk}^2 \varphi(x) a_j a_k > 0. \quad (13.2.41)$$

So if (13.2.41) holds, then for every bounded open set Ω , the following Carleman estimate holds

$$\tau \int |u|^2 e^{2\tau\varphi(x)} dx \leq C \int |P_1(\partial)u|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.42)$$

for every $u \in C_0^\infty(\Omega)$ and for every τ large enough.

Before applying this estimate to study of the uniqueness of the Cauchy problem we provide a geometric interpretation of condition (13.2.41). Let $x_0 \in \mathbb{R}^n$ and let us suppose that

$$\nabla \varphi(x_0) \neq 0. \quad (13.2.43)$$

Now, the assertion

$$a \cdot \nabla \varphi(x_0) = 0,$$

is equivalent to the assertion that the surface $\{\varphi(x) = \varphi(x_0)\}$ is a characteristic surface for the operator $P_1(\partial)$ in x_0 . Regarding the interpretation of the term

$$\sum_{j,k=1}^n \partial_{jk}^2 \varphi(x_0) a_j a_k,$$

we are helped by the calculations performed in (13.2.40). Actually, denoting by $x = \gamma(t)$ the parametric equation of the characteristic line passing through x_0 , for instance, set $\gamma(0) = x_0$, then we have

$$\frac{d^2 \varphi(\gamma(t))}{dt^2} \Big|_{t=0} = \sum_{j,k=1}^n \partial_{jk}^2 \varphi(x_0) a_j a_k.$$

Therefore, condition (13.2.41) states that if x_0 is a characteristic point of the level surface with respect to the operator P_1 , then there exists a neighborhood of 0, J , such that

$$\varphi(\gamma(t)) > \varphi(x_0), \quad \forall t \in J \setminus \{0\}$$

that is, the characteristic line $x = \gamma(t)$ remains locally confined to the region $\{\varphi(x) > \varphi(x_0)\}$ or, in other words, does not cross the level surface

$$\{\varphi(x) = \varphi(x_0)\}$$

in x_0 .

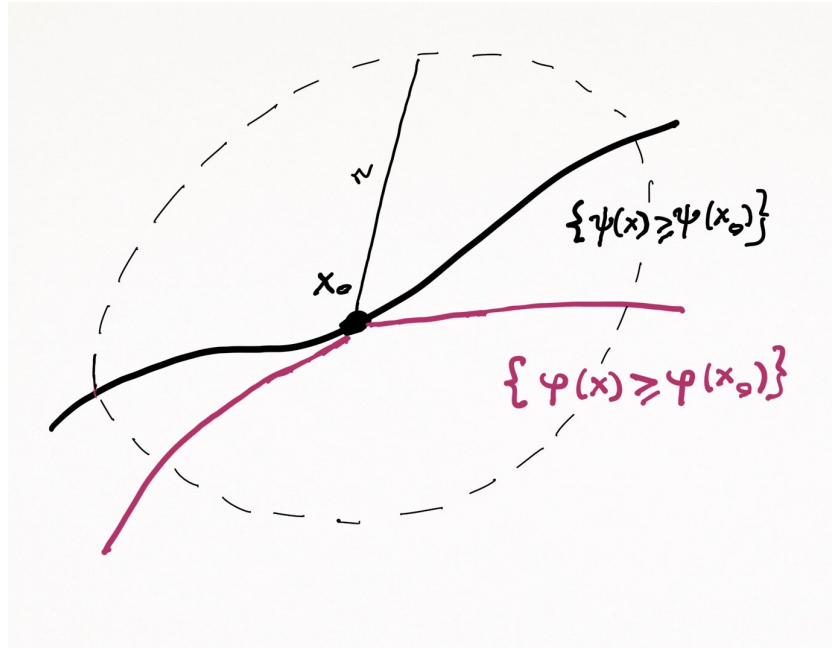


Figure 13.1:

Given Ω , an open set of \mathbb{R}^n , $x_0 \in \Omega$ and $\psi \in C^1(\overline{\Omega})$ a real-valued function such that

$$\nabla\psi(x) \neq 0, \quad \forall x \in \Gamma, \quad (13.2.44)$$

where

$$\Gamma = \{x \in \Omega : \psi(x) = \psi(x_0)\}. \quad (13.2.45)$$

We will say that a real-valued function $\varphi \in C^0(\overline{\Omega})$ with $\varphi(x_0) = \psi(x_0)$ enjoys the **property of convexification with respect to Γ in x_0** if $\varphi(x_0) = \psi(x_0)$ and there exists $r > 0$ such that (Figure 13.1)

$$\begin{aligned} & \{x \in B_r(x_0) : \varphi(x) \geq \varphi(x_0)\} \setminus \{x_0\} \subset \\ & \subset \{x \in B_r(x_0) : \psi(x) > \psi(x_0)\}. \end{aligned} \quad (13.2.46)$$

In the next Proposition, we set for a function $f \in C^2(\Omega)$

$$Q_f(x) = \sum_{j,k=1}^n \partial_{jk}^2 f(x) a_j a_k$$

The following holds true

Proposition 13.2.4. *Let Ω be an open set of \mathbb{R}^n and let $a \in \mathbb{R}^n \setminus \{0\}$, $c \in L^\infty(\Omega)$ (with values in \mathbb{C}), $x_0 \in \Omega$ and $\psi \in C^2(\overline{\Omega})$ a real-valued function satisfying the following conditions*

$$\nabla\psi(x_0) \neq 0 \quad (13.2.47)$$

and let us suppose that

$$a \cdot \nabla\psi(x_0) = 0 \quad \Rightarrow \quad Q_\psi(x_0) > 0. \quad (13.2.48)$$

Let $U \in H^1(\Omega)$ satisfy

$$\begin{cases} a \cdot \nabla U + c(x)U = 0, & \text{in } \Omega, \\ U(x) = 0 & \text{in } \{x \in \Omega : \psi(x) > \psi(x_0)\}. \end{cases} \quad (13.2.49)$$

Then there exists a neighborhood \mathcal{U}_{x_0} of x_0 such that

$$U = 0 \quad \text{in} \quad \mathcal{U}_{x_0}. \quad (13.2.50)$$

Proof. It is not restrictive to assume $x_0 = 0$,

$$\psi(0) = 0$$

and

$$|a| = 1.$$

Let

$$\varphi_\varepsilon(x) = \psi(x) - \frac{\varepsilon|x|^2}{2}, \quad (13.2.51)$$

where ε is a positive number to be chosen later.

Now we check that φ_ε satisfies (13.2.41) in a neighborhood of 0. We note, to this purpose, that (13.2.48) implies that there exists a constant $C_0 > 0$ such that

$$M_0 := C_0 (a \cdot \nabla\psi(0))^2 + Q_\psi(0) > 0.$$

We have easily that, for any $\varepsilon \leq \varepsilon_0 = \frac{M_0}{2}$,

$$C_0 (a \cdot \nabla\varphi_\varepsilon(0))^2 + Q_{\varphi_\varepsilon}(0) = M_0 - \varepsilon \geq \frac{M_0}{2}. \quad (13.2.52)$$

Let us choose $\varepsilon = \varepsilon_0$ and we omit from now on the subscript of φ . By (13.2.52), since $\varphi \in C^2(\Omega)$, there exists $R > 0$ such that

$$C_0 (a \cdot \nabla \varphi(x))^2 + Q_\varphi(x) \geq \frac{M_0}{4} > 0, \quad \forall x \in B_{2R}. \quad (13.2.53)$$

Therefore, (13.2.41) applies and, consequently, setting

$$Pu = a \cdot \nabla u + c(x)u,$$

we get

$$\tau \int |u|^2 e^{2\tau\varphi(x)} dx \leq C \int |Pu|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.54)$$

for every $u \in C_0^\infty(B_{2R})$ and for every $\tau \geq \tau_0$. By density (13.2.54) holds for every $u \in H_0^1(B_{2R})$.

Let $\eta \in C_0^\infty(B_{2R})$ be a function such that

$$0 \leq \eta(x) \leq 1, \quad \forall x \in B_R; \quad \eta(x) = 1, \quad \forall x \in B_{R/2}$$

and

$$\text{supp } \eta = \overline{B_R}.$$

Let us denote

$$C_1 = \|\nabla \eta\|_{L^\infty(B_R)}.$$

Now we apply (13.2.54) to

$$u = \eta U,$$

since

$$P(\eta U) = \eta(a \cdot \nabla U + c(x)U) + (a \cdot \nabla \eta)U,$$

we have

$$|P(\eta U)| \leq C_1 \chi_{B_R \setminus B_{R/2}} |U|,$$

where $\chi_{B_R \setminus B_{R/2}}$ is the characteristic function of $B_R \setminus B_{R/2}$. Hence by (13.2.54) we obtain

$$\tau \int_{B_R} |U\eta|^2 e^{2\tau\varphi(x)} dx \leq CC_1^2 \int_{B_R \setminus B_{R/2}} |U|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.55)$$

for every $\tau \geq \tau_0$.

Now, let

$$G = (\overline{B_R} \setminus B_{R/2}) \cap \text{supp } U,$$

$$M_1 = \max_G \varphi$$

and let us prove that

$$M_1 < 0. \quad (13.2.56)$$

For this purpose we first observe that φ is a convexification of

$$\Gamma := \{x \in B_R : \psi(x) = 0\}.$$

As a matter of fact, let us note that if

$$x \in \{y \in B_R : \varphi(y) \geq 0\} \setminus \{0\} \quad (13.2.57)$$

then

$$\psi(x) \geq \frac{\varepsilon_0 |x|^2}{2} > 0, \quad \text{for } x \neq 0.$$

Now, arguing by contradiction, let us suppose that (13.2.56) is false, i.e. let us suppose that

$$M_1 \geq 0.$$

Let $\bar{x} \in G$ satisfy $\varphi(\bar{x}) = M_1$. Since $G \subset \overline{B_R} \setminus B_{R/2}$ we have $\bar{x} \neq 0$. Hence, (13.2.57) implies $\psi(\bar{x}) > 0$ from which we get that there exists $\delta > 0$ such that

$$\psi(x) > 0, \quad \forall x \in B_\delta(\bar{x})$$

and, recalling that

$$\{x \in \Omega : \psi(x) > 0\} \subset \{x \in \Omega : U(x) = 0\},$$

we have $B_\delta(\bar{x}) \subset \{x \in \Omega : U(x) = 0\}$. Therefore

$$\bar{x} \notin \text{supp } U,$$

that contradicts $\bar{x} \in G \subset \text{supp } U$. Hence (13.2.56) holds true.

Now, by $\varphi(0) = 0$ and by (13.2.56) we have trivially that 0 is an interior point of

$$\{x \in \overline{B_R} : \varphi(x) > M_1\},$$

therefore there exists r , $0 < r \leq \frac{R}{2}$, such that

$$\overline{B_r} \subset \{x \in \overline{B_R} : \varphi(x) > M_1\}. \quad (13.2.58)$$

Let now

$$M_2 = \min_{\overline{B_r}} \varphi,$$

by (13.2.58) we have

$$M_2 > M_1.$$

Now, let us come back to (13.2.55). We have, trivially,

$$\int_{B_R \setminus B_{R/2}} |U|^2 e^{2\tau\varphi(x)} dx \leq e^{2\tau M_1} \int_G |U|^2 dx \quad (13.2.59)$$

and

$$\begin{aligned} \int_{B_R} |U\eta|^2 e^{2\tau\varphi(x)} dx &\geq \int_{B_R \cap \{\varphi > M_1\}} |U\eta|^2 e^{2\tau\varphi(x)} dx \geq \\ &\geq \int_{B_r} |U|^2 e^{2\tau\varphi(x)} dx \geq e^{2\tau M_2} \int_{B_r} |U|^2 dx. \end{aligned} \quad (13.2.60)$$

By (13.2.55), (13.2.59) and (13.2.60) we have

$$\int_{B_r} |U|^2 dx \leq CC_1^2 e^{-2\tau(M_2 - M_1)} \int_G |U|^2 dx$$

for every $\tau \geq \tau_0$, from which, passing to the limit as τ goes to $+\infty$, we obtain $U = 0$ in B_r . ■

Remarks.

1. The geometric part of the proof of Proposition (13.2.4) is to be considered standard and will occur again even in the case of more general operators than the ones considered so far. On the contrary, the path that we followed to arrive to estimate (13.2.42) is not extendable (or, at least, is not easily extendable) to more general situations. However, the reader can easily repeat, for the variable coefficient operator

$$P_1(x, \partial) = a(x) \cdot \nabla, \quad (13.2.61)$$

the calculations we did in the case where a is a constant vector. Of course, assuming, for instance, that

$$a_n(0) \neq 0,$$

instead of (13.2.34), the reader may consider

$$\begin{cases} \partial_t X(t, y) = a(X(t, y)), \\ X(0, y) = (y, 0). \end{cases}$$

obtaining, unlike the case in which a is constant, a local change of coordinates and, instead of (13.2.41), it will be found (compare with (13.2.40)) the following condition in \mathcal{U}_0 , where \mathcal{U}_0 is a neighborhood of 0.

$$\begin{aligned}
a(x) \cdot \nabla \varphi(x) = 0 &\Rightarrow \\
\Rightarrow \sum_{j,k=1}^n \partial_{jk}^2 \varphi(x) a_j(x) a_k(x) + \sum_{j,k=1}^n \partial_k a_j(x) a_k(x) \partial_j \varphi(x) &> 0. \quad (13.2.62)
\end{aligned}$$

By the procedure that we have outlined above, we have that if (13.2.62) holds then

$$\tau \int |u|^2 e^{2\tau\varphi(x)} dx \leq C \int |P_1(x, \partial)u|^2 e^{2\tau\varphi(x)} dx, \quad (13.2.63)$$

for every $u \in C_0^\infty(\mathcal{U}_0)$ and for every τ large enough.

2. We now derive estimate (13.2.63) by means of a procedure based on integrations by parts. This procedure will be extended in the next Sections to more general operators.

Let

$$P_{1,\tau}(x, \partial)v := e^{\tau\varphi} P_1(x, \partial)(e^{-\tau\varphi}v) = a \cdot \nabla v - \tau(a \cdot \nabla \varphi)v,$$

estimate (13.2.63) is equivalent to

$$\tau \int |v|^2 dx \leq C \int |P_{1,\tau}(x, \partial)v|^2 dx, \quad (13.2.64)$$

for every $v \in C_0^\infty(\mathcal{U}_0)$ and for every τ large enough. Since the coefficients of $P_1(x, \partial)$ are real-valued, to prove the (13.2.64) it suffices to consider v real-valued. We have

$$\begin{aligned}
\int |P_{1,\tau}(x, \partial)v|^2 dx &= \int |a \cdot \nabla v - \tau(a \cdot \nabla \varphi)v|^2 dx = \\
&= \int |a \cdot \nabla v|^2 dx + \tau^2 \int |(a \cdot \nabla \varphi)v|^2 dx - \\
&\quad - 2\tau \int (a \cdot \nabla \varphi)(a \cdot \nabla v) v dx. \quad (13.2.65)
\end{aligned}$$

Now, let us consider the third integral on the right hand side in (13.2.62); Integrating by parts we have

$$\begin{aligned}
-2\tau \int (a \cdot \nabla \varphi)(a \cdot \nabla v) v dx &= -\tau \int a \cdot \nabla (v^2)(a \cdot \nabla \varphi) dx = \\
&= \tau \int \operatorname{div}((a \cdot \nabla \varphi)a) v^2 dx.
\end{aligned}$$

By the the just obtained equality and by (13.2.62) we have.

$$\int |P_{1,\tau}(x, \partial)v|^2 dx \geq \int \left(\tau^2 (a \cdot \nabla \varphi) + \tau \tilde{Q}_\varphi \right) v^2 dx, \quad (13.2.66)$$

where

$$\begin{aligned} \tilde{Q}_\varphi &= \operatorname{div} ((a \cdot \nabla \varphi) a) = \\ &= \sum_{j,k=1}^n \partial_{jk}^2 \varphi a_j a_k + \sum_{j,k=1}^n \partial_k a_j a_k \partial_j \varphi + (a \cdot \nabla \varphi)(\operatorname{div} a). \end{aligned}$$

Now, proceeding as in the proof of Proposition 13.2.1 we have that if

$$a(x) \cdot \nabla \varphi(x) = 0 \quad \Rightarrow \quad \tilde{Q}_\varphi > 0, \quad \text{in } \overline{\mathcal{U}_0} \quad (13.2.67)$$

then, taking into account (13.2.66),

$$\int |P_{1,\tau}(x, \partial)v|^2 dx \geq \frac{\tau}{C} \int |v|^2 dx,$$

for every $v \in C_0^\infty(\mathcal{U}_0)$ and for $\tau \geq \tau_0$ (τ_0 independent of v). From the latter (13.2.63) follows. Let us note that (13.2.62) and (13.2.67) are equivalent.

13.3 Quadratic differential form and their integration by parts

As we have already seen in the simple examples of the previous Section, the first steps one makes to prove a Carleman estimate consists in setting, for an arbitrary function $u \in C_0^\infty(\Omega)$,

$$v = e^{\tau\varphi} u.$$

In this way, denoting, for the sake of brevity, by P the differential operator $P_m(x, D)$ (principal part of the operator $P(x, D)$) we introduce **the conjugate operator of P** which is defined by

$$P_\tau v = e^{\tau\varphi} P (e^{-\tau\varphi} v) = e^{\tau\varphi} P_m(x, D) (e^{-\tau\varphi} v); \quad (13.3.1)$$

after that, since

$$\int_\Omega |P_m(x, D)u|^2 e^{2\tau\varphi(x)} dx = \int_\Omega |P_\tau v|^2 dx,$$

we wish to prove a suitable estimate from below of $\int_{\Omega} |P_{\tau}v|^2 dx$, if τ is *large enough*. Of course, it is precisely this estimate from below the most tricky part of the proof of a Carleman estimate. In the examples we encountered in the previous Section we first spread the square $|P_{\tau}v|^2$ and then we integrate it by parts, but it is evident that unless appropriate arrangements are made, this procedure leads to great difficulty for the operators just a little slightly more general than those encountered in the previous Section. These difficulties also arise in the case of the Laplace operator. As a matter of fact, set

$$P = \Delta = - \sum_{j=1}^n D_j^2,$$

we have

$$P_{\tau}v = e^{\tau\varphi} \Delta (e^{-\tau\varphi}v) = \Delta v - \tau \nabla\varphi \cdot \nabla v - \tau \Delta\varphi + \tau^2 |\nabla\varphi|^2 v, \quad (13.3.2)$$

and, spreading the square we obtain an expression like

$$\sum_{|\alpha|, |\beta| \leq 2} \tau^{\gamma_{\alpha, \beta}} a_{\alpha\beta}(x) \partial^{\alpha} v \partial^{\beta} v.$$

To handle this kind of expressions, in the present Section we will study the quadratic forms

$$\sum_{\alpha, \beta} a_{\alpha\beta}(x) D^{\alpha} u \overline{D^{\beta} u}, \quad (13.3.3)$$

where the sum is finite, $a_{\alpha\beta}$ are complex-valued functions. We are particularly interested in the integration by parts of forms (13.3.3).

We recall that a **sesquilinear form** on a complex vector space V is a function

$$\Phi : V \times V \rightarrow \mathbb{C}$$

such that $f(\cdot, v)$ is linear for every $v \in V$ and $\Phi(u, \cdot)$ is antilinear for every $u \in V$. We say that a sesquilinear form on V is **hermitian**, if

$$\Phi(u, v) = \overline{\Phi(v, u)}, \quad \forall u, v \in V. \quad (13.3.4)$$

In the sequel of this Section we will denote by \mathcal{V} the space of sesquilinear forms on $C^{\infty}(\mathbb{R}^n, \mathbb{C})$.

Let us consider the sesquilinear forms

$$\Phi_{\alpha\beta} : C^{\infty}(\mathbb{R}^n, \mathbb{C}) \times C^{\infty}(\mathbb{R}^n, \mathbb{C}) \rightarrow C^{\infty}(\mathbb{R}^n, \mathbb{C}), \quad (13.3.5)$$

$$\Phi_{\alpha\beta}(u, v) = D^\alpha u \overline{D^\beta v}, \quad \forall u, v \in C^\infty(\mathbb{R}^n, \mathbb{C}). \quad (13.3.6)$$

In what follows, for any $\zeta \in \mathbb{C}^n$, $\zeta = (\zeta_1, \dots, \zeta_n)$ and $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, we will denote by

$$\zeta \cdot x = \sum_{j=1}^n \zeta_j x_j.$$

The following Proposition holds true

Proposition 13.3.1. *The family of sesquilinear forms $\{\Phi_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{N}_0^n}$ is linearly independent in \mathcal{V} .*

Proof. Let us consider a finite linear combination of the forms $\Phi_{\alpha\beta}$, $\alpha, \beta \in \Lambda$, where Λ is a finite subset of $\mathbb{N}_0^n \times \mathbb{N}_0^n$ and let us assume that it vanishes identically. We have, for some $c_{\alpha\beta} \in \mathbb{C}$,

$$\sum_{\alpha, \beta \in \Lambda} c_{\alpha\beta} \Phi_{\alpha\beta} = 0. \quad (13.3.7)$$

Now we prove that

$$c_{\alpha\beta} = 0, \quad \forall \alpha, \beta \in \Lambda. \quad (13.3.8)$$

Let us notice that (13.3.7) is equivalent to

$$\sum_{\alpha, \beta \in \Lambda} c_{\alpha\beta} D^\alpha u \overline{D^\beta v} = 0, \quad \forall u, v \in C^\infty(\mathbb{R}^n, \mathbb{C}). \quad (13.3.9)$$

Now, let $\zeta, \eta \in \mathbb{C}^n$ be arbitrary and put $u = e^{i\zeta \cdot x}$, $v = e^{i\eta \cdot x}$ in (13.3.9). We get

$$0 = \sum_{\alpha, \beta \in \Lambda} c_{\alpha\beta} D^\alpha (e^{i\zeta \cdot x}) \overline{D^\beta (e^{i\eta \cdot x})} = e^{i\zeta \cdot x} \overline{e^{i\eta \cdot x}} \sum_{\alpha, \beta \in \Lambda} c_{\alpha\beta} \zeta^\alpha \overline{\eta^\beta}.$$

Therefore

$$\sum_{\alpha, \beta \in \Lambda} c_{\alpha\beta} \zeta^\alpha \overline{\eta^\beta} = 0, \quad \forall \zeta, \eta \in \mathbb{C}^n.$$

From which we obtain (13.3.8). ■

Let us denote by \mathcal{W} the subspace of \mathcal{V} generated by $\{\Phi_{\alpha\beta}\}_{\alpha, \beta \in \mathbb{N}_0^n}$. We will call **sesquilinear differential form** any element of the space \mathcal{W} . Thus an arbitrary element of \mathcal{W} is

$$\Phi(u, v) = \sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta v}, \quad (13.3.10)$$

where the sum is finite, and $a_{\alpha\beta} \in \mathbb{C}$.

The following Proposition holds true.

Proposition 13.3.2. *Let $\Phi \in \mathcal{W}$ be given by (13.3.10) the following conditions are equivalent.*

$$\Phi \text{ is an hermitian form,} \quad (13.3.11a)$$

$$\Phi(u, u) \in \mathbb{R}, \quad \forall u \in C^\infty(\mathbb{R}^n, \mathbb{C}), \quad (13.3.11b)$$

$$a_{\alpha\beta} = \overline{a_{\beta\alpha}}, \quad \forall \alpha, \beta \in \mathbb{N}_0^n, \quad (13.3.11c)$$

$$\sum_{\alpha, \beta} a_{\alpha\beta} \zeta^\alpha \overline{\zeta^\beta} \in \mathbb{R}, \quad \forall \zeta \in \mathbb{C}^n. \quad (13.3.11d)$$

Proof. We follow the pattern

$$(13.3.11a) \iff (13.3.11b) \iff (13.3.11c) \iff (13.3.11d).$$

implication (13.3.11a) \implies (13.3.11b) is trivial.

Let us prove that (13.3.11b) \implies (13.3.11a).

Let $u, v \in C^\infty(\mathbb{R}^n, \mathbb{C})$ be arbitraries. By (13.3.11b) we have

$$\Phi(u + v, u + v) \in \mathbb{R}, \quad \Phi(u + iv, u + iv) \in \mathbb{R}. \quad (13.3.12)$$

Setting

$$z = \Phi(u, v), \quad w = \Phi(v, u),$$

we get, by (13.3.12),

$$z + w = \Phi(u + v, u + v) - \Phi(u, u) - \Phi(v, v) \in \mathbb{R}$$

and

$$-iz + iw = \Phi(u + iv, u + iv) - \Phi(u, u) - \Phi(v, v) \in \mathbb{R}.$$

From which we have $\Im(z + w) = 0$ and $\Re(z - w) = 0$; that is

$$\begin{cases} z + w - \overline{(z + w)} = 0, \\ z - w + \overline{(z - w)} = 0. \end{cases}$$

and adding member to member we have

$$2z - 2\overline{w} = 0.$$

Hence

$$\Phi(u, v) = \overline{\Phi(v, u)}.$$

Let us prove that (13.3.11b) \implies (13.3.11c).

Let us assume that (13.3.11b) holds true. By the equivalence proved previously we have

$$\Phi(u, v) = \overline{\Phi(v, u)}, \quad \forall u, v \in C^\infty(\mathbb{R}^n, \mathbb{C}).$$

Hence, for any $u, v \in C^\infty(\mathbb{R}^n, \mathbb{C})$ we get

$$\begin{aligned} \sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta v} &= \overline{\sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha v \overline{D^\beta u}} = \\ &= \sum_{\alpha, \beta} \bar{a}_{\alpha\beta} \overline{D^\alpha v} D^\beta u = \sum_{\alpha, \beta} \bar{a}_{\beta\alpha} \overline{D^\beta v} D^\alpha u \end{aligned}$$

(the last step is a mere change of indices). From what just obtained and setting

$$c_{\alpha, \beta} = a_{\alpha\beta} - \bar{a}_{\beta\alpha},$$

we have

$$\sum_{\alpha, \beta} c_{\alpha\beta} D^\alpha u \overline{D^\beta v} = 0 \tag{13.3.13}$$

and Proposition 13.3.1 gives

$$a_{\alpha\beta} - \bar{a}_{\beta\alpha} = c_{\alpha, \beta} = 0$$

for any $\alpha, \beta \in \mathbb{N}_0^n$. Therefore (13.3.11c) holds.

Let us prove that (13.3.11c) \implies (13.3.11b).

Let us suppose

$$a_{\alpha\beta} = \bar{a}_{\beta\alpha}.$$

Let $u \in C^\infty(\mathbb{R}^n, \mathbb{C})$ be arbitrary. We get

$$\begin{aligned} \sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta u} &= \sum_{\alpha, \beta} \bar{a}_{\beta\alpha} D^\alpha u \overline{D^\beta u} = \\ &= \overline{\sum_{\alpha, \beta} a_{\beta\alpha} \overline{D^\alpha u} D^\beta u} = \overline{\sum_{\alpha, \beta} a_{\alpha\beta} \overline{D^\beta u} D^\alpha u} = \overline{\Phi(u, u)}. \end{aligned}$$

the implication (13.3.11c) \implies (13.3.11d) can be proved in a similar way to the previous one (just replace ζ to D).

Let us prove that (13.3.11d) \implies (13.3.11c).

Let us assume that (13.3.11d) holds. Then

$$\begin{aligned} \sum_{\alpha,\beta} a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta &= \overline{\sum_{\alpha,\beta} a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta} = \\ &= \sum_{\alpha,\beta} \bar{a}_{\alpha\beta} \bar{\zeta}^\alpha \zeta^\beta = \sum_{\alpha,\beta} \bar{a}_{\beta\alpha} \bar{\zeta}^\beta \zeta^\alpha. \end{aligned}$$

From what obtained above and setting

$$c_{\alpha,\beta} = a_{\alpha\beta} - \bar{a}_{\beta\alpha},$$

we get

$$\sum_{\alpha,\beta} c_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta = 0, \quad \forall \zeta \in \mathbb{C}^n.$$

From which we have, for any $\gamma, \delta \in \mathbb{N}_0^n$,

$$\gamma! \delta! c_{\gamma\delta} = \partial_\zeta^\gamma \partial_{\bar{\zeta}}^\delta \left(\sum_{\alpha,\beta} c_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta \right) \Big|_{|\zeta|=0} = 0, \quad (13.3.14)$$

where, $\zeta = (\zeta_1, \dots, \zeta_n)$, $z_j = \xi_j + i\eta_j$,

$$\partial_{\zeta_j} = \frac{1}{2} (\partial_{\xi_j} - i\partial_{\eta_j}), \quad \partial_{\bar{\zeta}_j} = \frac{1}{2} (\partial_{\xi_j} + i\partial_{\eta_j}), \quad j = 1, \dots, n.$$

Finally, (13.3.14) gives (13.3.11c). ■

From here on, it is convenient to use the following notations: to denote a sesquilinear form with constant coefficients we will write

$$F(D, \bar{D})[u, \bar{v}] := \Phi(u, v) = \sum_{\alpha,\beta} a_{\alpha\beta} D^\alpha u \bar{D}^\beta v,$$

where $a_{\alpha\beta} \in \mathbb{C}$ are null except for a finite set of multi-indices.

We will call **differential quadratic form** with constant coefficients, the following form

$$F(D, \bar{D})[u, \bar{u}] := \Phi(u, u) = \sum_{\alpha,\beta} a_{\alpha\beta} D^\alpha u \bar{D}^\beta u. \quad (13.3.15)$$

In what follows we will do the convention of the "regrouped terms" according to to which the terms with the same indices α and β occur only one time. With this convention, the following polynomial in ζ and $\bar{\zeta}$

$$F(\zeta, \bar{\zeta}) = \sum_{\alpha, \beta} a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta, \quad (13.3.16)$$

is uniquely associated to the form F . As a matter of fact, it turns out

$$F(\zeta, \bar{\zeta}) = e^{-2(\Im \zeta) \cdot x} F(D, \bar{D}) \left[e^{i\zeta \cdot x}, \overline{e^{i\zeta \cdot x}} \right].$$

Hence, if

$$F(D, \bar{D})[u, \bar{u}] = 0, \quad \forall u \in C^\infty(\mathbb{R}^n), \quad (13.3.17)$$

then

$$F(\zeta, \bar{\zeta}) = 0, \quad \forall \zeta \in \mathbb{C}^n. \quad (13.3.18)$$

Conversely, if (13.3.18) holds, then we have

$$\gamma! \delta! a_{\gamma\delta} = \partial_\zeta^\gamma \partial_{\bar{\zeta}}^\delta \left(\sum_{\alpha, \beta} a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta \right) \Big|_{\zeta=0} = \partial_\zeta^\gamma \partial_{\bar{\zeta}}^\delta F(\zeta, \bar{\zeta}) \Big|_{\zeta=0} = 0,$$

from which we get (13.3.17). All in all (13.3.17) and (13.3.18) are equivalent.

We call the polynomial $F(\zeta, \bar{\zeta})$ the **symbol of the differential quadratic form** $F(D, \bar{D})$.

The following Proposition will be useful later on.

Proposition 13.3.3. *Let $F(D, \bar{D})[u, \bar{u}]$ be a differential quadratic form. We have*

$$\begin{aligned} \int_{\mathbb{R}^n} F(D, \bar{D})[u, \bar{u}] dx &= \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} F(\xi, \bar{\xi}) |\widehat{u}(\xi)|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \end{aligned} \quad (13.3.19)$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^n)$ and

$$F(D, \bar{D})[u, \bar{u}] = \sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta u}.$$

From the Parseval identity we have

$$\begin{aligned}
\int_{\mathbb{R}^n} F(D, \bar{D})[u, \bar{u}] dx &= \sum_{\alpha, \beta} a_{\alpha\beta} \int_{\mathbb{R}^n} D^\alpha u \overline{D^\beta u} dx = \\
&= \frac{1}{(2\pi)^n} \sum_{\alpha, \beta} a_{\alpha\beta} \int_{\mathbb{R}^n} \widehat{D^\alpha u} \overline{\widehat{D^\beta u}} d\xi = \\
&= \frac{1}{(2\pi)^n} \sum_{\alpha, \beta} a_{\alpha\beta} \int_{\mathbb{R}^n} \xi^{\alpha+\beta} |\widehat{u}(\xi)|^2 d\xi = \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} F(\xi, \xi) |\widehat{u}(\xi)|^2 d\xi.
\end{aligned}$$

■

We are interested to examine **under what conditions** $F(D, \bar{D})[u, \bar{u}]$ **can be "written as a divergence" of some vector field.**

More precisely, we are interested in examining under which conditions there exist some differential quadratic forms $G_k(D, \bar{D})$, $k = 1, \dots, n$ with constant coefficients, such that

$$F(D, \bar{D})[u, \bar{u}] = \sum_{k=1}^n \partial_k (G_k(D, \bar{D})[u, \bar{u}]). \quad (13.3.20)$$

For this purpose we consider the differential quadratic form with constant coefficients

$$G(D, \bar{D})[u, \bar{u}] = \sum_{\alpha, \beta} c_{\alpha\beta} D^\alpha u \overline{D^\beta u}$$

and we wish to express the symbol of $\partial_k (G(D, \bar{D})[u, \bar{u}])$ through the symbol of $G(D, \bar{D})[u, \bar{u}]$. We have

$$\begin{aligned}
\partial_k (G(D, \bar{D})[u, \bar{u}]) &= \sum_{\alpha, \beta} c_{\alpha\beta} \partial_k (D^\alpha u \overline{D^\beta u}) = \\
&= \sum_{\alpha, \beta} c_{\alpha\beta} \left(\partial_k D^\alpha u \overline{D^\beta u} + D^\alpha u \overline{\partial_k D^\beta u} \right) = \\
&= \sum_{\alpha, \beta} c_{\alpha\beta} \left(i D_k D^\alpha u \overline{D^\beta u} - i D^\alpha u \overline{D_k D^\beta u} \right).
\end{aligned}$$

Hence, the symbol associated to $\partial_k (G(D, \bar{D})[u, \bar{u}])$ is

$$\begin{aligned} \sum_{\alpha, \beta} c_{\alpha\beta} \left(i\zeta_k \zeta^\alpha \bar{\zeta}^\beta - i\bar{\zeta}_k \bar{\zeta}^\alpha \zeta^\beta \right) &= i(\zeta_k - \bar{\zeta}_k) \sum_{\alpha, \beta} c_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta = \\ &= i(\zeta_k - \bar{\zeta}_k) G(\zeta, \bar{\zeta}). \end{aligned}$$

Therefore, in order to have (13.3.20) it is necessary that

$$F(\zeta, \bar{\zeta}) = i \sum_{k=1}^n (\zeta_k - \bar{\zeta}_k) G_k(\zeta, \bar{\zeta}). \quad (13.3.21)$$

Now, setting $\zeta = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}^n$, we have

$$F(\xi + i\eta, \xi - i\eta) = -2 \sum_{k=1}^n \eta_k G_k(\xi + i\eta, \xi - i\eta). \quad (13.3.22)$$

In particular, we have

$$F(\xi, \xi) = 0, \quad \forall \xi \in \mathbb{R}^n \quad (13.3.23)$$

and

$$G_k(\xi, \xi) = -\frac{1}{2} \frac{\partial}{\partial \eta_k} F(\xi + i\eta, \xi - i\eta)|_{\eta=0}, \quad \forall \xi \in \mathbb{R}^n. \quad (13.3.24)$$

Therefore a **necessary condition to be true** (13.3.20) is that (13.3.23) be true. Below we will see that this condition is also sufficient, but first we give the definition of the double and total order of a differential quadratic form.

Definition 13.3.4. Let

$$F(D, \bar{D})[u, \bar{u}] = \sum_{\alpha, \beta} a_{\alpha\beta} D^\alpha u \bar{D}^\beta \bar{u}, \quad (13.3.25)$$

be a differential quadratic form with constant coefficients. We say that F has **double order** $(\mu; m)$ provided

$$a_{\alpha\beta} \neq 0 \implies |\alpha| + |\beta| \leq \mu; \quad |\alpha|, |\beta| \leq m. \quad (13.3.26)$$

μ is called the **total order** and m is called the **separated order** of the differential quadratic form F .

It is evident that

$$\mu \leq 2m.$$

Moreover, when adopting the convention of the grouped terms, the previous definition uniquely determines the order of the differential quadratic form. Here and in the sequel we will naturally extend the notions of the double order and the total order to the symbol of a differential quadratic form.

Lemma 13.3.5. *Let $F(D, \bar{D})[u, \bar{u}]$ be a differential quadratic form with constant coefficients. Let us suppose that*

$$F(\xi, \xi) = 0, \quad \forall \xi \in \mathbb{R}^n, \quad (13.3.27)$$

then there exist n differential quadratic forms $G_k(D, \bar{D})[u, \bar{u}]$, $k = 1, \dots, n$, such that

$$F(D, \bar{D})[u, \bar{u}] = \sum_{k=1}^n \partial_k (G_k(D, \bar{D})[u, \bar{u}]) \quad (13.3.28)$$

and we have

$$G_k(\xi, \xi) = -\frac{1}{2} \frac{\partial}{\partial \eta_k} F(\xi + i\eta, \xi - i\eta)|_{\eta=0}, \quad \forall \xi \in \mathbb{R}^n. \quad (13.3.29)$$

In addition, let us assume that $F(D, \bar{D})[u, \bar{u}]$ has a double order $(\mu; m)$, $m > 0$ then

(a) if $\mu < 2m$, the forms G_k , $k = 1, \dots, n$, can be chosen of double order $(\mu - 1; m - 1)$;

(b) if $\mu = 2m$, the forms G_k , $k = 1, \dots, n$, can be chosen of double order $(\mu - 1; m)$.

Proof. Let us assume that (13.3.27) holds. Let us consider the polynomial

$$\eta \rightarrow F(\xi + i\eta, \xi - i\eta)$$

and let us apply the Taylor formula at $\eta = 0$. We have, for suitable polynomials f_k , $k = 1, \dots, n$,

$$F(\xi + i\eta, \xi - i\eta) = \sum_{k=1}^n \eta_k f_k(\xi, \eta) = -\frac{i}{2} \sum_{k=1}^n (\zeta_k - \bar{\zeta}_k) f_k\left(\frac{\zeta + \bar{\zeta}}{2}, \frac{\zeta - \bar{\zeta}}{2i}\right).$$

Set

$$G_k(\zeta, \bar{\zeta}) = -\frac{1}{2} f_k\left(\frac{\zeta + \bar{\zeta}}{2}, \frac{\zeta - \bar{\zeta}}{2i}\right),$$

we get

$$F(\zeta, \bar{\zeta}) = i \sum_{k=1}^n (\zeta_k - \bar{\zeta}_k) G_k(\zeta, \bar{\zeta}) = -2 \sum_{k=1}^n \eta_k G_k(\xi + i\eta, \xi - i\eta), \quad (13.3.30)$$

from which we have

$$\frac{\partial}{\partial \eta_k} F(\xi + i\eta, \xi - i\eta) = -2G_k(\xi + i\eta, \xi - i\eta), \quad \forall \xi \in \mathbb{R}^n. \quad (13.3.31)$$

Hence

$$G_k(\xi, \xi) = -\frac{1}{2} \frac{\partial}{\partial \eta_k} F(\xi + i\eta, \xi - i\eta)|_{\eta=0}, \quad \forall \xi \in \mathbb{R}^n.$$

Moreover, from the first equality in (13.3.30) (retracing to backward the calculations that led to (13.3.21)) we have

$$F(D, \bar{D})[u, \bar{u}] = i \sum_{k=1}^n (D_k - \bar{D}_k) (G_k(D, \bar{D})[u, \bar{u}]) = \sum_{k=1}^n \partial_k (G_k(D, \bar{D})[u, \bar{u}]).$$

Proof of (a) and (b).

Case (a), $\mu < 2m$. Let us show that if $\alpha', \alpha'', \beta', \beta''$ are multi-indices such that

$$\begin{cases} \alpha' + \beta' = \alpha'' + \beta'' \\ |\alpha'| + |\beta'| = |\alpha''| + |\beta''| \leq \mu \leq 2m - 1 \\ |\alpha'|, |\beta'|, |\alpha''|, |\beta''| \leq m, \end{cases} \quad (13.3.32)$$

then

$$\zeta^{\alpha'} \bar{\zeta}^{\beta'} = \zeta^{\alpha''} \bar{\zeta}^{\beta''} + i \sum_{j=1}^n (\zeta_j - \bar{\zeta}_j) h_j(\zeta, \bar{\zeta}), \quad (13.3.33)$$

where $h_j(\zeta, \bar{\zeta})$, $j = 1, \dots, n$, have the total order less or equal than $\mu - 1$ and the separated order less or equal than $m - 1$.

Notice that by (13.3.32) we have either $|\alpha'| < m$ or $|\beta'| < m$ (likewise for $|\alpha''|$ and $|\beta''|$).

The proof consists of repeatedly applying both simple identities.

$$\bar{\zeta}_j = \zeta_j - (\zeta_j - \bar{\zeta}_j), \quad (13.3.34a)$$

$$\zeta_j = \bar{\zeta}_j + (\zeta_j - \bar{\zeta}_j). \quad (13.3.34b)$$

Let us consider $\zeta^{\alpha'} \bar{\zeta}^{\beta'}$ and let us suppose $|\alpha'| < m$. Identity (13.3.34a) allows us to move the factors from $\bar{\zeta}^{\beta'}$ to $\zeta^{\alpha'}$ as long as the exponent of ζ does not have modulus m , when this occurs identity (13.3.34b) is used. Let us see more in detail. If $|\alpha'| < m$ e $\beta' \neq 0$, for instance let $\beta'_j > 0$, then

$$\begin{aligned} \zeta^{\alpha'} \bar{\zeta}^{\beta'} &= \zeta^{\alpha'} \bar{\zeta}^{\beta' - e_j} \bar{\zeta}_j = \zeta^{\alpha'} \bar{\zeta}^{\beta' - e_j} (\zeta_j - (\zeta_j - \bar{\zeta}_j)) = \\ &= \zeta^{\alpha' + e_j} \bar{\zeta}^{\beta' - e_j} - (\zeta_j - \bar{\zeta}_j) \zeta^{\alpha'} \bar{\zeta}^{\beta' - e_j}. \end{aligned}$$

Let us notice that $\zeta^{\alpha'} \bar{\zeta}^{\beta' - e_j}$ has total order $|\alpha'| + |\beta' - e_j| \leq \mu - 1$ and separated order less or equal than $m - 1$ (recall $|\alpha'| < m$ and $|\beta'| \leq m$). If, on the other hand $|\beta'| < m$ (and this includes the case $\beta' = 0$ which was neglected previously) we use identity (13.3.34b) and proceeding as above we reach (assuming $\alpha'_j > 0$, for some j) to

$$\zeta^{\alpha'} \bar{\zeta}^{\beta'} = \zeta^{\alpha' - e_j} \bar{\zeta}^{\beta' - e_j} + (\zeta_j - \bar{\zeta}_j) \zeta^{\alpha' - e_j} \bar{\zeta}^{\beta'}. \quad (13.3.35)$$

Similarly to the case $|\alpha'| < m$, we have that $\zeta^{\alpha' - e_j} \bar{\zeta}^{\beta'}$ has total order less or equal than $\mu - 1$ and separated order less or equal than $m - 1$. Repeatedly applying the procedure used above we arrive to (13.3.33), which in turn implies that there exists \tilde{F} such that

$$F(\zeta, \bar{\zeta}) = \tilde{F}(\zeta, \bar{\zeta}) + i \sum_{j=1}^n (\zeta_j - \bar{\zeta}_j) G_j(\zeta, \bar{\zeta}), \quad (13.3.36)$$

where $G_j(\zeta, \bar{\zeta})$, $j = 1, \dots, n$, has double order $(\mu - 1; m - 1)$ and

$$\tilde{F}(\zeta, \bar{\zeta}) = \sum_{(\alpha, \gamma) \in \Lambda} c_\gamma \zeta^\alpha \bar{\zeta}^{\gamma - \alpha}, \quad (13.3.37)$$

where $c_\gamma \in \mathbb{C}$

$$\Lambda = \{(\alpha, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : |\gamma| \leq m, \alpha \leq \gamma\}$$

and also \tilde{F} has double order $(\mu; m)$. Notice that, thanks to (13.3.33) and (13.3.35), the summation in (13.3.37) is written in such a way that for a given sum of the multi-indices only one addend occurs.

By (13.3.27) and (13.3.36) we have

$$0 = \tilde{F}(\xi, \xi) = \sum_{|\gamma| \leq m} N_\gamma c_\gamma \xi^\gamma, \quad \forall \xi \in \mathbb{R}^n, \quad (13.3.38)$$

where N_γ is the cardinality of the set $\{\alpha \in \mathbb{N}_0^n : \alpha \leq \gamma\}$, hence, $c_\gamma = 0$ from which we have $\tilde{F} \equiv 0$. Therefore

$$F(\zeta, \bar{\zeta}) = i \sum_{j=1}^n (\zeta_j - \bar{\zeta}_j) G_j(\zeta, \bar{\zeta})$$

and (13.3.28) is proved in case (a).

Case (b), $\mu = 2m$.

Obviously, we can handle the terms of

$$F(\zeta, \bar{\zeta}) = \sum_{\alpha, \beta} a_{\alpha\beta} \zeta^\alpha \bar{\zeta}^\beta.$$

satisfying $|\alpha| + |\beta| < 2m$ (and $|\alpha|, |\beta| \leq m$) in the same way of case (a). Let us examine in which a way we can handle the terms such that $|\alpha| + |\beta| = 2m$. Since $|\alpha|, |\beta| \leq m$ we have $|\alpha| = |\beta| = m$.

By identities (13.3.34a) and (13.3.34b) we have

$$\zeta_j \bar{\zeta}_k = \zeta_k \bar{\zeta}_j + (\zeta_j - \bar{\zeta}_j) \bar{\zeta}_k - (\zeta_k - \bar{\zeta}_k) \bar{\zeta}_j,$$

for $j, k = 1, \dots, n$.

Now, let us suppose that $\alpha', \alpha'', \beta', \beta''$ satisfy

$$\begin{cases} \alpha' + \beta' = \alpha'' + \beta'', \\ |\alpha'| = |\beta'| = |\alpha''| = |\beta''| = m, \end{cases}$$

then we have (if $\alpha'_j > 0$ and $\beta'_k > 0$)

$$\begin{aligned} \zeta^{\alpha'} \bar{\zeta}^{\beta'} &= \zeta^{\alpha' - e_j} \bar{\zeta}^{\beta' - e_k} \zeta_j \bar{\zeta}_k = \zeta^{\alpha' - e_j + e_k} \bar{\zeta}^{\beta' - e_k + e_j} + \\ &+ (\zeta_j - \bar{\zeta}_j) h_1(\zeta, \bar{\zeta}) + (\zeta_k - \bar{\zeta}_k) h_2(\zeta, \bar{\zeta}), \end{aligned}$$

where

$$h_1(\zeta, \bar{\zeta}) = \zeta^{\alpha' - e_j} \bar{\zeta}^{\beta'}, \quad h_2(\zeta, \bar{\zeta}) = -\zeta^{\alpha' - e_j} \bar{\zeta}^{\beta' - e_k + e_j},$$

have double order $(\mu - 1, m)$. From now on, one may proceed as in case (a) and we reach the conclusion. ■

In the case of a differential quadratic forms with variable coefficients we have the following

Lemma 13.3.6. *Let*

$$F(x, D, \bar{D})[u, \bar{u}] = \sum_{\alpha, \beta} a_{\alpha\beta}(x) D^\alpha u \bar{D}^\beta \bar{u} \quad (13.3.39)$$

be a differential quadratic form with variable coefficients $a_{\alpha\beta} \in C^s(\Omega)$, where $s \in \mathbb{N}$. Let us suppose that F has double order $(\mu; m)$, $m > 0$, and that

$$F(x, \xi, \xi) = 0, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n. \quad (13.3.40)$$

Then there exists a differential quadratic form $G(x, D, \bar{D})[u, \bar{u}]$ whose coefficients belong to $C^{s-1}(\Omega)$ such that

$$\int_{\Omega} F(x, D, \bar{D})[u, \bar{u}] dx = \int_{\Omega} G(x, D, \bar{D})[u, \bar{u}] dx, \quad \forall u \in C_0^\infty(\Omega) \quad (13.3.41)$$

and such that:

(a) if $\mu < 2m$, then $G(x, D, \bar{D})$ can be chosen of double order $(\mu - 1; m - 1)$;

(b) if $\mu = 2m$, then $G(x, D, \bar{D})$ can be chosen of double order $(\mu - 1; m)$.
Moreover

$$G(x, \xi, \xi) = \frac{1}{2} \sum_{k=1}^n \partial_{x_k \eta_k}^2 F(x, \xi + i\eta, \xi - i\eta)|_{\eta=0}. \quad (13.3.42)$$

Proof. Let F_1, \dots, F_N be a basis of the vector space (of finite dimension) of all quadratic forms H of double order $(\mu; m)$ with constant coefficients and satisfying

$$H(\xi, \xi) = 0, \quad \forall \xi \in \mathbb{R}^n.$$

By Lemma 13.3.5 there exist differential quadratic forms with constant coefficients G_j^k , $j = 1, \dots, N$, $k = 1, \dots, n$, of double order $(\mu - 1; m')$, with $m' = m - 1$, provided $\mu < 2m$, and $m' = m$ provided $\mu = 2m$, such that

$$F_j(D, \bar{D})[u, \bar{u}] = \sum_{k=1}^n \partial_{x_k} (G_j^k(D, \bar{D})[u, \bar{u}]), \quad j = 1, \dots, N.$$

Now, (13.3.40) implies that there exist $c_j \in C^s(\Omega)$, $j = 1, \dots, N$, such that

$$F(x, D, \bar{D})[u, \bar{u}] = \sum_{j=1}^N c_j(x) F_j(D, \bar{D})[u, \bar{u}].$$

Hence, if $u \in C_0^\infty(\Omega)$, then integration by parts yields

$$\begin{aligned} \int_{\Omega} F(x, D, \bar{D})[u, \bar{u}] dx &= \sum_{j=1}^N \sum_{k=1}^n \int_{\Omega} c_j(x) \partial_{x_k} (G_j^k(D, \bar{D})[u, \bar{u}]) dx = \\ &= - \sum_{j=1}^N \sum_{k=1}^n \int_{\Omega} \partial_{x_k} c_j(x) G_j^k(D, \bar{D})[u, \bar{u}] dx. \end{aligned}$$

Thus, we can choose

$$G(x, D, \bar{D})[u, \bar{u}] = \sum_{j=1}^N \sum_{k=1}^n -\partial_{x_k} c_j(x) G_j^k(D, \bar{D})[u, \bar{u}].$$

From which we have

$$G(x, \xi, \xi) = \sum_{j=1}^N \sum_{k=1}^n -\partial_{x_k} c_j(x) G_j^k(\xi, \xi). \quad (13.3.43)$$

On the other hand by (13.3.29) we have

$$G_j^k(\xi, \xi) = -\frac{1}{2} \frac{\partial}{\partial \eta_k} F_j(\xi + i\eta, \xi - i\eta)|_{\eta=0}, \quad \forall \xi \in \mathbb{R}^n.$$

Now, by the last equality we get

$$\begin{aligned} G(x, \xi, \xi) &= \sum_{k=1}^n \sum_{j=1}^N -\partial_{x_k} c_j(x) G_j^k(\xi, \xi) = \\ &= \frac{1}{2} \sum_{k=1}^n \left(\partial_{x_k \eta_k}^2 \sum_{j=1}^N c_j(x) F_j(\xi + i\eta, \xi - i\eta) \right) \Big|_{\eta=0} = \\ &= \frac{1}{2} \sum_{k=1}^n \partial_{x_k \eta_k}^2 F(\xi + i\eta, \xi - i\eta)|_{\eta=0}. \end{aligned}$$

■

13.4 The conjugate of $P_m(x, D)$ – Set up of a Carleman estimate

In this Section we will consider the conjugate of the operator $P_m(x, D)$ which, we recall, is defined by

$$P_\tau v = e^{\tau\varphi} P_m(x, D) (e^{-\tau\varphi} v). \quad (13.4.1)$$

We first observe that

$$P_\tau v = P_m(x, D + i\tau\nabla\varphi(x))v. \quad (13.4.2)$$

As a matter of fact we have

$$\begin{aligned} e^{\tau\varphi} D_j (e^{-\tau\varphi} v) &= D_j v - \tau(D_j\varphi)v = D_j v + i\tau(\partial_j\varphi)v, \\ e^{\tau\varphi} D_k D_j (e^{-\tau\varphi} v) &= e^{\tau\varphi} D_k (e^{-\tau\varphi} (D_j v + i\tau(\partial_j\varphi)v)) = (D_k + i\tau(\partial_k\varphi))(D_j + i\tau(\partial_j\varphi))v, \\ &\vdots \\ e^{\tau\varphi} D^\alpha (e^{-\tau\varphi} v) &= (D + i\tau\nabla\varphi(x))^\alpha v, \end{aligned}$$

for every $\alpha \in \mathbb{N}_0^n$. Now, since

$$P_m(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha, \quad (13.4.3)$$

we have

$$\begin{aligned} e^{\tau\varphi} P_m(x, D) (e^{-\tau\varphi} v) &= \sum_{|\alpha|=m} a_\alpha(x) e^{\tau\varphi} D^\alpha (e^{-\tau\varphi} v) = \\ &= \sum_{|\alpha|=m} a_\alpha(x) (D + i\tau\nabla\varphi(x))^\alpha v = \\ &= P_m(x, D + i\tau\nabla\varphi(x))v. \end{aligned}$$

Now, let us consider the polynomial $P_m(x, \xi + i\tau\nabla\varphi(x))$ in the variable ξ and let us denote by

$$p_m(x, D, \tau) \text{ the operator whose symbol is } P_m(x, \xi + i\tau\nabla\varphi). \quad (13.4.4)$$

Let us note that, in general, operator (13.4.4) does not equal to the operator $P_m(x, D + i\tau\nabla\varphi)$. For instance, if

$$P_2(x, D) = D_1^2$$

we have

$$P_2(x, D + i\tau\nabla\varphi(x))v = D_1^2 v + 2i\tau\partial_1\varphi D_1 v - \tau^2(\partial_1\varphi)^2 v + \tau(\partial_1^2\varphi)v$$

hence

$$p_2(x, D, \tau) = P_2(x, D + i\tau\nabla\varphi(x)) - \tau\partial_1^2\varphi(x).$$

In general we have

$$p_m(x, D, \tau) = \sum_{|\alpha|+j=m} \tau^j b_{\alpha j}(x) D^\alpha, \quad (13.4.5)$$

where the coefficients $b_{\alpha j}(x)$ depend on $\nabla\varphi$, on coefficients of $P_m(x, D)$ (but not on their derivatives) and **do not depend** on τ . In addition we have

$$P_m(x, D + i\tau\nabla\varphi(x)) = p_m(x, D, \tau) + R_{m-1, \tau}(x, D, \tau), \quad (13.4.6)$$

where

$$R_{m-1, \tau}(x, D) = \sum_{|\alpha|+j \leq m-1} \tau^j \tilde{b}_{\alpha j}(x) D^\alpha,$$

the coefficients $\tilde{b}_{\alpha j}$ depend on the coefficients of $P_m(x, D)$ (but not on their derivatives), on $\nabla\varphi$ and on the higher-order derivatives of φ and **do not depend on** τ . The second term on the right-hand side in (13.4.6), as we will realize soon, may be regarded as a harmless perturbation of the operator $p_m(x, D, \tau)$.

Now let us deal with the square in the integral

$$\int |P_m(x, D + i\tau\nabla\varphi(x))v|^2 dx, \quad (13.4.7)$$

From here on, since v is supported in Ω , we omit the set of integration. First we notice that by (13.4.6) we have

$$\begin{aligned} \int |P_m(x, D + i\tau\nabla\varphi(x))v|^2 dx &\geq \frac{1}{2} \int |p_m(x, D, \tau)v|^2 dx - \\ &\quad - \int |R_{m-1, \tau}(x, D)v|^2 dx \geq \\ &\geq \frac{1}{2} \int |p_m(x, D, \tau)v|^2 dx - \\ &\quad - C \sum_{|\alpha| \leq m-1} \tau^{2(m-|\alpha|)-2} \int |D^\alpha v|^2 dx, \end{aligned} \quad (13.4.8)$$

where C depends by the L^∞ norms of the coefficients of $P_m(x, D)$.

We need some additional notation. Let $M(x, \xi)$ be a polynomial with respect to the variable ξ , let us suppose that the coefficients of $M(x, \xi)$ are differentiable. Let us set

$$M^{(j)}(x, \xi) = \partial_{\xi_j} M(x, \xi), \quad M_{,j}(x, \xi) = \partial_{x_j} M(x, \xi), \quad j = 1, \dots, n.$$

Let us denote by

$$\overline{M}(x, \xi)$$

the polynomial in ξ whose coefficients are the complex conjugate of the coefficients of $M(x, \xi)$. Keep in mind that if $\zeta \in \mathbb{C}^n$, then

$$\overline{M(x, \zeta)} = \overline{M}(x, \overline{\zeta}).$$

If $L(x, \xi)$ and $M(x, \xi)$ are two polynomials in the variable ξ with differentiable coefficients, we define their **Poisson brackets**

$$\begin{aligned} \{L(x, \xi), M(x, \xi)\} &= \\ &= \sum_{j=1}^n (L^{(j)}(x, \xi) M_{,j}(x, \xi) - L_{,j}(x, \xi) M^{(j)}(x, \xi)). \end{aligned} \quad (13.4.9)$$

Now we anticipate that in points 3 and 4 of the Remarks of the present Section, we will observe that $\overline{p}_m(x, D, \tau)$ is a suitable approximation of the adjoint of the operator $p_m(x, D, \tau)$.

Let

$$S(x, D, \tau) = \frac{1}{2} (p_m(x, D, \tau) + \overline{p}_m(x, D, \tau)), \quad (13.4.10a)$$

$$A(x, D, \tau) = \frac{1}{2} (p_m(x, D, \tau) - \overline{p}_m(x, D, \tau)), \quad (13.4.10b)$$

We have trivially

$$p_m(x, D, \tau) = S(x, D, \tau) + A(x, D, \tau). \quad (13.4.11)$$

Hence

$$\begin{aligned} \int |p_m(x, D, \tau)v|^2 dx &= \\ &= \int |S(x, D, \tau)v|^2 dx + \int |A(x, D, \tau)v|^2 dx + \\ &+ 2 \int \Re \left(S(x, D, \tau)v \overline{A(x, D, \tau)v} \right) dx. \end{aligned} \quad (13.4.12)$$

A crucial point in the proof of a Carleman estimate consists in finding an appropriate estimate from below of the third integral on the right hand side of (13.4.12).

Now, let us consider the differential quadratic form

$$F(x, D, \bar{D}, \tau) [v, \bar{v}] = 2\Re \left(S(x, D, \tau) \overline{vA(x, D, \tau)v} \right), \quad (13.4.13)$$

whose symbol is

$$F(x, \zeta, \bar{\zeta}, \tau) = 2\Re \left(S(x, \zeta, \tau) \overline{A(x, \zeta, \tau)} \right), \quad \forall \zeta \in \mathbb{C}^n.$$

By the definition of $p_m(x, D, \tau)$ we have that the symbol of $\bar{p}_m(x, D, \tau)$ is given by $\bar{P}_m(x, \xi - i\tau\nabla\varphi)$, from which we have, for $\zeta = \xi + i\eta \in \mathbb{C}^n$

$$\begin{aligned} F(x, \zeta, \bar{\zeta}, \tau) &= 2\Re \left(S(x, \zeta, \tau) \overline{A(x, \zeta, \tau)} \right) = \\ &= \frac{1}{2} \Re \left((P_m(x, \zeta + i\tau\nabla\varphi(x)) + \bar{P}_m(x, \zeta - i\tau\nabla\varphi(x))) \times \right. \\ &\quad \left. \times \overline{(P_m(x, \zeta + i\tau\nabla\varphi(x)) - \bar{P}_m(x, \zeta - i\tau\nabla\varphi(x)))} \right) = \\ &= \frac{1}{2} \left(|P_m(x, \zeta + i\tau\nabla\varphi(x))|^2 - |\bar{P}_m(x, \zeta - i\tau\nabla\varphi(x))|^2 \right). \end{aligned}$$

Hence

$$\begin{aligned} F(x, \zeta, \bar{\zeta}, \tau) &= \\ &= \frac{1}{2} \left(|P_m(x, \zeta + i\tau\nabla\varphi(x))|^2 - |\bar{P}_m(x, \zeta - i\tau\nabla\varphi(x))|^2 \right). \end{aligned} \quad (13.4.14)$$

By this equality we get

$$\begin{aligned} F(x, \xi, \xi, \tau) &= \frac{1}{2} \left(|P_m(x, \xi + i\tau\nabla\varphi(x))|^2 - |\bar{P}_m(x, \xi - i\tau\nabla\varphi(x))|^2 \right) = \\ &= \frac{1}{2} \left(|P_m(x, \xi + i\tau\nabla\varphi(x))|^2 - \left| \overline{P_m(x, \xi + i\tau\nabla\varphi(x))} \right|^2 \right) = 0. \end{aligned}$$

Hence, **assuming that the coefficients of $P_m(x, D)$ belong to $C^1(\bar{\Omega})$** we can apply Lemma 13.3.6. Using formula (13.3.42) and denoting by

$$G(x, \xi, \xi, \tau) := \frac{1}{2} \sum_{k=1}^n \partial_{x_k \eta_k}^2 F(x, \xi + i\eta, \xi - i\eta, \tau)|_{\eta=0}, \quad (13.4.15)$$

we have

$$\begin{aligned} 2 \int \Re \left(S(x, D, \tau) \overline{vA(x, D, \tau)v} \right) dx &= \int F(x, D, \bar{D}, \tau) [v, \bar{v}] dx = \\ &= \int G(x, D, \bar{D}, \tau) [v, \bar{v}] dx, \end{aligned} \quad (13.4.16)$$

for every $v \in C_0^\infty(\Omega)$.

Now we calculate the expression on the right hand side in (13.4.15). Although the calculation is elementary, let us perform it in detail. We have

$$\begin{aligned}
& \partial_{\eta_k} \frac{1}{2} \left(|P_m(x, \xi + i\eta + i\tau \nabla \varphi(x))|^2 - \right. \\
& \quad \left. - |\overline{P}_m(x, \xi + i\eta - i\tau \nabla \varphi(x))|^2 \right)_{|\eta=0} = \\
& = \partial_{\eta_k} \frac{1}{2} \left(|P_m(x, \xi + i\eta + i\tau \nabla \varphi(x))|^2 - \right. \\
& \quad \left. - |P_m(x, \xi - i\eta + i\tau \nabla \varphi(x))|^2 \right)_{|\eta=0} = \\
& = 2\Re \left(-i P_m(x, \xi + i\tau \nabla \varphi(x)) \overline{P_m^{(k)}(x, \xi + i\tau \nabla \varphi(x))} \right).
\end{aligned} \tag{13.4.17}$$

Set for short

$$\zeta = \xi + i\tau \nabla \varphi(x)$$

and let us differentiate what obtained in (13.4.17) w.r.t. x_k . By (13.4.15) we get

$$\begin{aligned}
G(x, \xi, \xi, \tau) & = \\
& = \tau \sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi(x) P_m^{(j)}(x, \zeta) \overline{P_m^{(k)}(x, \zeta)} + \\
& + \Im \left(\sum_{k=1}^n P_{m,k}(x, \zeta) \overline{P_m^{(k)}(x, \zeta)} \right) + \\
& + \Im \left[P_m(x, \zeta) \left(\sum_{k=1}^n \overline{P_{m,k}^{(k)}(x, \zeta)} - i\tau \sum_{j,k=1}^n \overline{P_m^{(k,j)}(x, \zeta)} \partial_{x_j x_k}^2 \varphi(x) \right) \right].
\end{aligned} \tag{13.4.18}$$

Let us observe that we have

$$\begin{aligned}
P_m(x, \xi + i\tau \nabla \varphi) & = 0 \Rightarrow \\
\Rightarrow G(x, \xi, \xi, \tau) & = \frac{i}{2} \left\{ P_m(x, \xi + i\tau \nabla \varphi), \overline{P_m(x, \xi + i\tau \nabla \varphi)} \right\},
\end{aligned} \tag{13.4.19}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket defined in (13.4.9). In order to check (13.4.19) it suffices to develop the Poisson bracket in (13.4.19), and to notice that the third term on the right hand side in (13.4.18) vanishes when $P_m(x, \xi + i\tau \nabla \varphi(x)) = 0$.

Let us notice that $G(x, \xi, \tau)$ is a homogeneous polynomial of degree $2m-1$ in the variables (ξ, τ) . However, we will be interested in more precise information about the differential quadratic form $G(x, D, \overline{D}, \tau)$ or, equivalently on its symbol $G(x, \zeta, \overline{\zeta}, \tau)$, to this end we prove

Proposition 13.4.1. *Let $P_m(x, D)$ the differential operator*

$$P_m(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha,$$

where $a_\alpha \in C^1(\overline{\Omega}, \mathbb{C})$, for $|\alpha| = m$. Let $F(x, \zeta, \overline{\zeta}, \tau)$ be defined by (13.4.14). Then there exists a differential quadratic form $G(x, D, \overline{D}, \tau)$ such that

$$\int F(x, D, \overline{D}, \tau) [v, \overline{v}] dx = \int G(x, D, \overline{D}, \tau) [v, \overline{v}] dx. \quad (13.4.20)$$

Moreover

$$G(x, D, \overline{D}, \tau) = \sum_{h=0}^{2m-1} \tau^h G^{(h)}(x, D, \overline{D}), \quad (13.4.21)$$

where $G^{(h)}(x, D, \overline{D})$ is a differential quadratic form which has double order $(2m-h-1; m)$, for $h = 0, 1, \dots, 2m-1$.

If the **coefficients a_α , for $|\alpha| = m$, are real valued functions** then (13.4.20) continues to hold true, but instead of (13.4.21) we have

$$G(x, D, \overline{D}, \tau) = \tau \sum_{h=0}^{2m-2} \tau^h G^{(h)}(x, D, \overline{D}), \quad (13.4.22)$$

where $G^{(h)}(x, D, \overline{D})$ is a differential quadratic form which has double order $(2m-h-2; m)$, $h = 0, \dots, 2m-2$.

In any case $G(x, \xi, \xi, \tau)$ is given by (13.4.18).

Proof. By the Taylor formula we get

$$P_m(x, \zeta + i\tau \nabla \varphi(x)) = \sum_{k=0}^m \tau^k q_{m-k}(x, \zeta), \quad \forall \zeta \in \mathbb{C}^n, \quad (13.4.23)$$

where, for $k = 0, 1, \dots, m$, $q_{m-k}(x, \zeta)$ are polynomials in the variable ζ of degree $m-k$. Moreover the coefficients of $q_{m-k}(x, \zeta)$ are of class $C^1(\overline{\Omega})$. We have

$$|P_m(x, \zeta + i\tau \nabla \varphi(x))|^2 = \sum_{k,j=0}^m \tau^{k+j} q_{m-k}(x, \zeta) \overline{q_{m-j}(x, \overline{\zeta})}.$$

Hence, by (13.4.14) we get

$$F(x, \zeta, \bar{\zeta}, \tau) = \sum_{k,j=0}^m \tau^{k+j} F_{kj}(x, \zeta, \bar{\zeta}),$$

where, for $j, k = 1, \dots, m$

$$F_{kj}(x, \zeta, \bar{\zeta}) = q_{m-k}(x, \zeta) \bar{q}_{m-j}(x, \bar{\zeta}) - q_{m-k}(x, \bar{\zeta}) \bar{q}_{m-j}(x, \zeta). \quad (13.4.24)$$

Each of the forms F_{kj} has double order $(2m - (k + j); m)$, furthermore by (13.4.24), since q_{00} has degree 0, we have

$$F_{mm}(x, \zeta, \bar{\zeta}) = 0, \quad \forall \zeta \in \mathbb{C}^n. \quad (13.4.25)$$

Moreover

$$F_{kj}(x, \xi, \xi) = 0, \quad \forall \xi \in \mathbb{R}^n$$

and by Lemma (13.3.6) – case (b) – we have that, for $j, k = 1, \dots, m$, where either j or k are different from m , there exist a differential quadratic form G_{kj} which have double order $(2m - (k + j) - 1; m)$ and satisfying

$$\int F_{kj}(x, D, \bar{D}) [v, \bar{v}] dx = \int G_{kj}(x, D, \bar{D}) [v, \bar{v}] dx, \quad \forall v \in C_0^\infty(\Omega),$$

of course, since (13.4.25) holds, we may choose

$$G_{mm} \equiv 0.$$

Therefore, by the last obtained equality and setting

$$G^{(h)}(x, D, \bar{D}) = \sum_{k+j=h} G_{kj}(x, D, \bar{D}), \quad h = 1, \dots, 2m - 1,$$

$$G(x, D, \bar{D}, \tau) [v, \bar{v}] = \sum_{h=0}^{2m-1} \tau^h G^{(h)}(x, D, \bar{D}) [v, \bar{v}],$$

we have that $G^{(h)}$ is a differential quadratic form which has double order

$(2m - h - 1; m)$ and

$$\begin{aligned}
\int F(x, D, \bar{D}, \tau) [v, \bar{v}] dx &= \sum_{k,j=0}^m \tau^{k+j} \int F_{kj}(x, D, \bar{D}) [v, \bar{v}] dx = \\
&= \sum_{k,j=0}^m \tau^{k+j} \int G_{jk}(x, D, \bar{D}) [v, \bar{v}] dx = \\
&= \sum_{h=0}^{2m-1} \tau^h \int G^{(h)}(x, D, \bar{D}) [v, \bar{v}] dx = \\
&= \int G(x, D, \bar{D}, \tau) [v, \bar{v}] dx.
\end{aligned} \tag{13.4.26}$$

If the coefficients of $P_m(x, D)$ are real-valued then also the coefficients of the polynomials $q_j(x, \zeta)$ in (13.4.23) are real-valued and (13.4.24) can be written as

$$F_{kj}(x, \zeta, \bar{\zeta}) = q_{m-k}(x, \zeta)q_{m-j}(x, \bar{\zeta}) - q_{m-k}(x, \bar{\zeta})q_{m-j}(x, \zeta). \tag{13.4.27}$$

Hence, besides (13.4.25), we have

$$F_{00}(x, \zeta, \bar{\zeta}) = 0, \quad \forall \zeta \in \mathbb{C}^n. \tag{13.4.28}$$

Therefore we have

$$F(x, \zeta, \bar{\zeta}, \tau) = \tau \sum_{h=0}^{2m-2} \tau^h F^{(h)}(x, \zeta, \bar{\zeta}),$$

where

$$F^{(h)}(x, \zeta, \bar{\zeta}) = \sum_{k+j=h+1} F_{kj}(x, \zeta, \bar{\zeta}).$$

Hence $F^{(h)}$, for $h = 0, \dots, 2m-2$, has double order $(2m-h-1; m)$. Therefore by applying Lemma (13.3.6) – case (a) – there exist $G^{(h)}(x, D, \bar{D})$, differential quadratic forms which have double order $(2m-h-2; m-1)$, such that

$$\int F^{(h)}(x, D, \bar{D}) [v, \bar{v}] dx = \int G^{(h)}(x, D, \bar{D}) [v, \bar{v}] dx, \quad \forall v \in C_0^\infty(\Omega),$$

and setting

$$G(x, D, \bar{D}, \tau) [v, \bar{v}] = \tau \sum_{h=0}^{2m-2} \tau^h G^{(h)}(x, D, \bar{D}) [v, \bar{v}]$$

we get

$$\int F(x, D, \bar{D}, \tau) [v, \bar{v}] dx = \int G(x, D, \bar{D}, \tau) [v, \bar{v}] dx.$$

■

Now, we **broadly outline** the main ideas that are involved in proving a Carleman estimate. We come back, then, to the third integral in (13.4.12). Let $x_0 \in \bar{\Omega}$ we have from (13.4.16) and (13.3.19),

$$\begin{aligned} 2 \int \Re \left(S(x, D, \tau) \overline{v A(x, D, \tau) v} \right) dx &= \int G(x, D, \bar{D}, \tau) [v, \bar{v}] dx = \\ &= \int G(x_0, D, \bar{D}, \tau) [v, \bar{v}] dx + \\ &+ \int (G(x, D, \bar{D}, \tau) - G(x_0, D, \bar{D}, \tau)) [v, \bar{v}] dx = \\ &= \frac{1}{(2\pi)^n} \int G(x_0, \xi, \xi, \tau) |\widehat{v}(\xi)|^2 d\xi + \\ &+ \underbrace{\int (G(x, D, \bar{D}, \tau) - G(x_0, D, \bar{D}, \tau)) [v, \bar{v}] dx}_{\mathcal{R}}. \end{aligned} \quad (13.4.29)$$

The main idea that we will follow consists essentially in what follows:

(a) Choice of φ . The choice of φ will be made so that we have

$$\begin{aligned} P_m(x, \xi + i\tau \nabla \varphi(x)) &= 0 \Rightarrow \\ \Rightarrow G(x, \xi, \xi, \tau) &> 0, \quad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R} \setminus (0, 0), \tau > 0 \end{aligned} \quad (13.4.30)$$

hence, by the homogeneity of G w.r.t. (ξ, τ) we get

$$\begin{aligned} P_m(x, \xi + i\tau \nabla \varphi(x)) &= 0 \Rightarrow \\ \Rightarrow G(x, \xi, \xi, \tau) &\geq C (|\xi|^2 + \tau^2)^{m-\frac{1}{2}}, \quad \forall x \in \bar{\Omega} \end{aligned} \quad (13.4.31)$$

for every $(\xi, \tau) \in \mathbb{R}^{n+1}$, $\tau > 0$.

(b) Next steps. Keeping in mind Lemma 13.1.1, we will exploit the local character of a Carleman estimate to focus on the case where **the support of v (and hence of u) is sufficiently small**. With this expedient, the term \mathcal{R} on the right-hand side in (13.4.29) can be treated as a kind of rest and can be

efficiently estimated from below by simultaneously exploiting the continuity of the coefficients of the quadratic form $G(x, D, \overline{D}, \tau)$ and Proposition 13.4.1. **In coarse words**, where $P_m(x, \xi + i\tau\nabla\varphi(x)) = 0$ (13.4.30) is used and where $P_m(x, \xi + i\tau\nabla\varphi(x)) \neq 0$ (so where one will not be able to exploit the property (13.4.30)) we will exploit the specific character of the operator P_m

Remarks.

1. Let us notice that if $P_m(x_0, \xi + i\tau\nabla\varphi(x_0))$ has some zero of multiplicity larger than 1 in $(\xi_0, \tau_0) \neq (0, 0)$, then (13.4.30) cannot be true. As a matter of fact in this case we have

$$P_m^{(j)}(x_0, \xi_0 + i\tau_0\nabla\varphi(x_0)) = 0, \quad j = 1, \dots, n,$$

hence by (13.4.18) we have $G(x_0, \xi_0, \xi_0, \tau_0) = 0$.

2. Let us notice that if the coefficients of $P_m(x, D)$ are constant, then we have

$$\begin{aligned} & \frac{i}{2\tau} \left\{ P_m(x, \xi + i\tau\nabla\varphi(x)), \overline{P_m(x, \xi + i\tau\nabla\varphi(x))} \right\} = \\ & = \frac{1}{\tau} G(x, \xi, \xi, \tau) = \\ & = \sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi(x) P_m^{(j)}(x, \xi + i\tau\nabla\varphi(x)) \overline{P_m^{(k)}(x, \xi + i\tau\nabla\varphi(x))}. \end{aligned} \quad (13.4.32)$$

3. Operators (13.4.10a), (13.4.10b) are the "first-order approximations" respectively, of the symmetric and the antisymmetric parts of the operator $p_m(x, D, \tau)$. Let us examine this issue in more detail. Let us suppose that the operator $P_m(x, D)$ has very regular coefficients, say C^∞ , and let us write $p_m(x, D, \tau)$ as follows

$$p_m(x, D, \tau) = \sum_{|\alpha|+j=m} c_{\alpha,j}(x) \tau^j D^\alpha = \tau^m \sum_{|\alpha|+j=m} c_{\alpha,j}(x) (\tau^{-1}D)^\alpha.$$

Let us consider the formal adjoint of $p_m(x, D, \tau)$ i.e. the operator $p_m^*(x, D, \tau)$ such that

$$\int (p_m(x, D, \tau)v\overline{w})dx = \int v \left(\overline{p_m^*(x, D, \tau)w} \right) dx, \quad \forall u, w \in C_0^\infty(\Omega).$$

We have, integrating by parts,

$$p_m^*(x, D, \tau)v = \tau^m \sum_{|\alpha|+j=m} (\tau^{-1}D)^\alpha (\bar{c}_{\alpha,j}v) = \sum_{|\alpha|+j=m} \tau^j D^\alpha (\bar{c}_{\alpha,j}v).$$

On the other hand

$$\bar{p}_m(x, D, \tau) = \sum_{|\alpha|+j=m} \bar{c}_{\alpha,j} \tau^j D^\alpha.$$

Hence we have

$$\begin{aligned} & (p_m^*(x, D, \tau) - \bar{p}_m(x, D, \tau))v = \\ &= \frac{1}{i} \sum_{|\alpha|+j=m-1} \tau^j \sum_{k=1}^n \binom{\alpha}{e_k} \partial_{x_k} c_{\alpha,j}(x) D^{\alpha-e_k} v + \\ &+ r_{m-2}(x, D, \tau)v, \end{aligned} \quad (13.4.33)$$

where

$$r_{m-2}(x, D, \tau)v = \sum_{|\alpha|+j \leq m-2} \tau^j \tilde{c}_{\alpha,j}(x) D^\alpha v$$

and $\tilde{c}_{\alpha,j}$ are suitable coefficients. By expressing (13.4.33) by means of the symbols of the operator, we have

$$p_m^*(x, \xi, \tau) = \overline{p_m(x, \xi, \tau)} + \frac{1}{i} \sum_{k=1}^n p_{m,j}^{(j)}(x, \xi, \tau) + r_{m-2}(x, \xi, \tau). \quad (13.4.34)$$

Relationship (13.4.33) and (13.4.34) expresses in a precise manner that $\bar{p}_m(x, D, \tau)$ approximates $p_m^*(x, D, \tau)$ to the first order.

4. It can be noticed that by spreading the square in (13.4.12) in a standard way one would leads to conclusions not unlike those seen above, in particular, with regard to (13.4.18). Here we give a brief mention referring the interested reader to [50, Ch. 4]. We warn, however, that this approach requires generally, assumptions of greater regularity on the coefficients of $P_m(x, D)$ than we will make in this Chapter.

Set

$$\begin{aligned} s(x, D, \tau) &= \frac{1}{2} (p_m(x, D, \tau) + p_m^*(x, D, \tau)), \\ a(x, D, \tau) &= \frac{1}{2} (p_m(x, D, \tau) - p_m^*(x, D, \tau)). \end{aligned}$$

We have trivially

$$p_m(x, D, \tau) = s(x, D, \tau) + a(x, D, \tau). \quad (13.4.35)$$

Denoting by p , s and a , respectively, $p_m(x, D, \tau)$, $s(x, D, \tau)$, $a(x, D, \tau)$ and denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega, \mathbb{C})$, we have:

$$\begin{aligned} \|p(v)\|_{L^2(\Omega)}^2 &= \langle p(v), p(v) \rangle = \\ &= \langle (s+a)(v), (s+a)(v) \rangle = \\ &= \|s(v)\|_{L^2(\Omega)}^2 + \|a(v)\|_{L^2(\Omega)}^2 + 2\Re\langle s(v), a(v) \rangle. \end{aligned} \quad (13.4.36)$$

Let us note that, denoting by

$$[s, a] = sa - as,$$

the **commutator of a and s** and taking into account that

$$s^* = s, \quad a^* = -a,$$

we have

$$\begin{aligned} 2\Re\langle s(v), a(v) \rangle &= \langle a(v), s(v) \rangle + \langle s(v), a(v) \rangle = \\ &= \langle s^*a(v), v \rangle + \langle a^*s(v), v \rangle = \\ &= \langle sa(v), v \rangle - \langle as(v), v \rangle = \\ &= \langle [s, a](v), v \rangle. \end{aligned} \quad (13.4.37)$$

Now, by (13.4.36) we have

$$\begin{aligned} \int |p_m(x, D, \tau)v|^2 dx &= \int |s(x, D, \tau)v|^2 dx + \int |a(x, D, \tau)v|^2 dx + \\ &+ 2 \int \Re \left(s(x, D, \tau) \overline{va(x, D, \tau)v} \right) dx. \end{aligned} \quad (13.4.38)$$

this, by (13.4.37), can be written as

$$\begin{aligned} \int |p_m(x, D, \tau)v|^2 dx &= \int |s(x, D, \tau)v|^2 dx + \int |a(x, D, \tau)v|^2 dx + \\ &+ 2 \int ([s(x, D, \tau), a(x, D, \tau)]v) \bar{v} dx. \end{aligned} \quad (13.4.39)$$

Now let us compare the integrals

$$2 \int \Re \left(S(x, D, \tau) v \overline{A(x, D, \tau) v} \right) dx, \quad 2 \int \Re \left(s(x, D, \tau) v \overline{a(x, D, \tau) v} \right) dx$$

which occur, respectively, as the third term on the right hand side in (13.4.12) and the third term on the right hand side in (13.4.38).

Set

$$R(x, D, \tau) = \frac{1}{2} \left(\frac{1}{i} \sum_{k=1}^n p_{m,j}^{(j)}(x, D, \tau) + r_{m-2}(x, D, \tau) \right),$$

we have

$$s(x, D, \tau) = S(x, D, \tau) + R(x, D, \tau) \quad \text{and} \quad a(x, D, \tau) = S(x, D, \tau) - R(x, D, \tau)$$

then

$$\begin{aligned} 2\Re \left(s(x, D, \tau) v \overline{a(x, D, \tau) v} \right) &= 2\Re \left(S(x, D, \tau) v \overline{A(x, D, \tau) v} \right) - \\ &\quad - q(x, D, \overline{D}, \tau) [v, \bar{v}] - |R(x, D, \tau) v|^2, \end{aligned}$$

where

$$q(x, D, \overline{D}, \tau) = 2\Re \left(S(x, D, \tau) v \overline{R(x, D, \tau) v} - R(x, D, \tau) v \overline{A(x, D, \tau) v} \right).$$

Recalling (13.4.10a) and (13.4.10b) we get

$$q(x, D, \overline{D}, \tau) [v, \bar{v}] = q_1(x, D, \overline{D}, \tau) [v, \bar{v}] + q_2(x, D, \overline{D}, \tau) [v, \bar{v}],$$

where

$$q_1(x, D, \overline{D}, \tau) [v, \bar{v}] = 2\Re \left(p_m(x, D, \tau) v \overline{R(x, D, \tau) v} \right)$$

and

$$\begin{aligned} q_2(x, D, \overline{D}, \tau) [v, \bar{v}] &= \\ &= \Re \left(\overline{p_m(x, D, \tau) v} \overline{R(x, D, \tau) v} - R(x, D, \tau) v \left(\overline{p_m(x, D, \tau) v} \right) \right) = 0. \end{aligned}$$

Therefore

$$\begin{aligned} 2\Re \left(s(x, D, \tau) v \overline{a(x, D, \tau) v} \right) &= \\ &= 2\Re \left(S(x, D, \tau) v \overline{A(x, D, \tau) v} \right) - \\ &\quad - 2\Re \left(p_m(x, D, \tau) v \overline{R(x, D, \tau) v} \right) - |R(x, D, \tau) v|^2 \end{aligned}$$

and by (13.4.16) we have

$$\begin{aligned}
& 2 \int \Re \left(s(x, D, \tau) v \overline{a(x, D, \tau) v} \right) dx = \\
& = \int G(x, D, \bar{D}, \tau) [v, \bar{v}] dx - \\
& - 2 \Re \int \left(p_m(x, D, \tau) v \overline{R(x, D, \tau) v} \right) dx - \\
& - \int |R(x, D, \tau) v|^2 dx.
\end{aligned} \tag{13.4.40}$$

This relationship allows (see **Exercise** subsequent to the proof of Theorem 13.5.1) to consider equivalent the approach we are following with the one outlined in this Remark (of course, when the coefficients of the operator are sufficiently regular). \blacklozenge

We conclude this Section with some lemma that will be useful later on.

Lemma 13.4.2. *Let $\varphi \in C^\infty(\bar{\Omega})$. Then for every $m \in \mathbb{N}_0$ there exists a constant $C > 1$ such that*

$$\begin{aligned}
C^{-1} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2 &\leq \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi} \leq \\
&\leq C \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2,
\end{aligned} \tag{13.4.41}$$

for every $u \in C^\infty(\bar{\Omega})$ and for every $\tau \geq 1$.

Proof. Both the inequalities are proved easily by means of Leibniz formula. Here we limit ourselves to prove

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi} \leq C \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2. \tag{13.4.42}$$

We use the induction principle. If $m = 0$, then (13.4.42) is trivial. Let us suppose that

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi} \leq C_m \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2,$$

where $C_m \geq 1$ and we have

$$\begin{aligned}
\sum_{|\alpha| \leq m+1} \tau^{2(m+1-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2 &= \sum_{|\alpha|=m+1} |D^\alpha (e^{\tau\varphi} u)|^2 + \\
&+ \tau^2 \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2 \geq \\
&\geq \sum_{|\alpha|=m+1} |D^\alpha (e^{\tau\varphi} u)|^2 + \\
&+ C_m^{-1} \sum_{|\alpha| \leq m} \tau^{2(m+1-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi}.
\end{aligned} \tag{13.4.43}$$

Let $\delta \in (0, 1)$ be to choose. Using the Leibniz formula we have, for $\tau \geq 1$,

$$\begin{aligned}
\sum_{|\alpha|=m+1} |D^\alpha (e^{\tau\varphi} u)|^2 &\geq \delta \sum_{|\alpha|=m+1} |D^\alpha (e^{\tau\varphi} u)|^2 \geq \\
&\geq \delta \sum_{|\alpha|=m+1} |D^\alpha u|^2 e^{2\tau\varphi} - \\
&- \delta \tilde{C}_m \sum_{|\alpha| \leq m} \tau^{2(m+1-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi},
\end{aligned} \tag{13.4.44}$$

where $\tilde{C}_m \geq 1$ is a suitable constant depending on m . By (13.4.43) and (13.4.44) we get

$$\begin{aligned}
\sum_{|\alpha| \leq m+1} \tau^{2(m+1-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2 &\geq \delta \sum_{|\alpha|=m+1} |D^\alpha u|^2 e^{2\tau\varphi} + \\
&+ \left(C_m^{-1} - \delta \tilde{C}_m \right) \sum_{|\alpha| \leq m} \tau^{2(m+1-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi}.
\end{aligned}$$

Now, we choose $\delta = \frac{1}{2C_m \tilde{C}_m}$ and we get

$$\sum_{|\alpha| \leq m+1} \tau^{2(m+1-|\alpha|)} |D^\alpha u|^2 e^{2\tau\varphi} \leq 2C_m \sum_{|\alpha| \leq m+1} \tau^{2(m+1-|\alpha|)} |D^\alpha (e^{\tau\varphi} u)|^2,$$

which concludes the proof. ■

Lemma 13.4.3. *Let $\varphi \in C^\infty(\overline{\Omega})$. Then for each $m \in \mathbb{N}_0$ there exists a constant $C > 1$ such that for every $v \in C_0^\infty(\Omega)$ and for every $\tau \geq 1$ we have*

$$\begin{aligned} C^{-1} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx &\leq \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi \leq \\ &\leq C \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx. \end{aligned} \quad (13.4.45)$$

Proof. We start with the first inequality in (13.4.45). By Lemma 13.4.2 and the Parseval identity we have

$$\begin{aligned} \sum_{|\alpha| \leq m} \int \tau^{2(m-|\alpha|)} |D^\alpha v|^2 dx &= \frac{1}{(2\pi)^n} \int \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |\xi^\alpha|^2 |\widehat{v}(\xi)|^2 d\xi \leq \\ &\leq C \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi. \end{aligned}$$

Concerning the second inequality in (13.4.41), we have similarly

$$\begin{aligned} \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi &= \sum_{k=0}^m \binom{m}{k} \tau^{2(m-k)} \int |\xi|^{2k} |\widehat{v}(\xi)|^2 d\xi \leq \\ &\leq 2^m \sum_{k=0}^m \tau^{2(m-k)} \int \sum_{|\alpha|=k} |\xi^\alpha|^2 |\widehat{v}(\xi)|^2 d\xi = \\ &= 2^m \int \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |\xi^\alpha|^2 |\widehat{v}(\xi)|^2 d\xi = \\ &= 2^m (2\pi)^n \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx. \end{aligned}$$

■

Lemma 13.4.4. *Let us assume that the coefficients of operator (13.4.3) belong to $C^0(\overline{\Omega})$. Let $\varphi \in C^\infty(\overline{\Omega})$ and let $p_m(x, D, \tau)$ be the operator defined by (13.4.4). Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\begin{aligned} \left| \frac{1}{(2\pi)^n} \int |p_m(x_0, \xi, \tau)|^2 |\widehat{v}(\xi)|^2 d\xi - \int |p_m(x, D, \tau)v|^2 dx \right| &\leq \\ &\leq \varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx, \end{aligned} \quad (13.4.46)$$

for every $v \in C_0^\infty(B_\delta(x_0) \cap \Omega)$, for every $\tau \in \mathbb{R}$ and for every $x_0 \in \overline{\Omega}$.

Proof. Let $x_0 \in \bar{\Omega}$. Let recall that by (13.4.5) we have

$$p_m(x, D, \tau) = \sum_{|\alpha|+j=m} \tau^j b_{\alpha j}(x) D^\alpha, \quad (13.4.47)$$

by the assumptions on φ and on the coefficients of $P_m(x, D)$, we have $b_{\alpha j} \in C^0(\bar{\Omega})$, for any α and j such that $|\alpha| + j = m$.

Let $\varepsilon > 0$ and $\delta > 0$ be such that for any α and j satisfying $|\alpha| + j = m$ we have

$$|b_{\alpha j}(x) - b_{\alpha j}(x_0)| < \varepsilon, \quad \forall x \in B_\delta(x_0) \cap \bar{\Omega}$$

(δ independent of x_0). We obtain

$$\begin{aligned} & |p_m(x, D, \tau)v - p_m(x_0, D, \tau)v| \leq \\ & \leq \sum_{|\alpha|+j=m} |\tau|^j |b_{\alpha j}(x) - b_{\alpha j}(x_0)| |D^\alpha v| \leq \\ & \leq C\varepsilon \sum_{|\alpha|+j=m} |\tau|^j |D^\alpha v|, \quad \forall x \in B_\delta(x_0) \cap \bar{\Omega}. \end{aligned} \quad (13.4.48)$$

On the other hand

$$|p_m(x, D, \tau)v| \leq C \sum_{|\alpha|+j=m} |\tau|^j |D^\alpha v|, \quad \forall x \in \bar{\Omega} \quad (13.4.49)$$

Now, taking into account the elementary inequality

$$||z|^2 - |w|^2| \leq (|z| + |w|)|z - w|, \quad \forall z, w \in \mathbb{C},$$

we have by (13.4.48) and (13.4.49), for every $x \in B_\delta(x_0) \cap \bar{\Omega}$,

$$||p_m(x, D, \tau)v|^2 - |p_m(x_0, D, \tau)v|^2| \leq C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha v|^2. \quad (13.4.50)$$

Therefore, for every $v \in C_0^\infty(B_\delta(x_0) \cap \Omega)$

$$\begin{aligned} & \frac{1}{(2\pi)^n} \int |p_m(x_0, \xi, \tau)|^2 |\widehat{v}(\xi)|^2 d\xi - \int |p_m(x, D, \tau)v|^2 dx = \\ & = \int (|p_m(x_0, D, \tau)v|^2 - |p_m(x, D, \tau)v|^2) dx \leq \\ & \leq C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx, \end{aligned} \quad (13.4.51)$$

and similarly, for every $v \in C_0^\infty(B_\delta(x_0) \cap \Omega)$,

$$\begin{aligned} \frac{1}{(2\pi)^n} \int |p_m(x_0, \xi, \tau)|^2 |\widehat{v}(\xi)|^2 d\xi - \int |p_m(x, D, \tau)v|^2 dx &\geq \\ &\geq -C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx. \end{aligned} \quad (13.4.52)$$

Finally (13.4.51) and (13.4.52) implies (13.4.46). ■

13.5 Carleman estimates for the elliptic operators

Let $m \in \mathbb{N}$, and let Ω be a bounded open set of \mathbb{R}^n . Let a_α complex-valued functions. We recall that the operator

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad (13.5.1)$$

is elliptic in a point x_0 if

$$P_m(x_0, \xi) = \sum_{|\alpha|=m} a_\alpha(x_0) \xi^\alpha \neq 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \quad (13.5.2)$$

We also say that $P(x, D)$ is elliptic in $\overline{\Omega}$ if (13.5.2) holds for every $x_0 \in \overline{\Omega}$. Let us note that if $a_\alpha \in C^0(\overline{\Omega}, \mathbb{C})$, for $|\alpha| = m$, the ellipticity condition for the operator $P(x, D)$ is equivalent to the existence of a constant $\lambda > 0$ such that

$$|P_m(x, \xi)| \geq \lambda |\xi|^m, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \overline{\Omega}. \quad (13.5.3)$$

For the sake of brevity, in the proof of Theorem below, for an open $\omega \subset \Omega$ we will identify $C_0^\infty(\omega)$ with the function space

$$\{u \in C_0^\infty(\Omega) : \text{supp } u \subset \omega\}.$$

Theorem 13.5.1 (Carleman–Hörmander). *Let $\varphi \in C^\infty(\overline{\Omega})$ be a real-valued function which satisfies*

$$\nabla \varphi(x) \neq 0, \quad \forall x \in \overline{\Omega}. \quad (13.5.4)$$

Let $P(x, D)$ be an operator of order m whose coefficients belong to $L^\infty(\Omega, \mathbb{C})$. Let us assume that the coefficients of the principal part $P_m(x, D)$ belong

to $C^1(\overline{\Omega}, \mathbb{C})$. Let us suppose that $P(x, D)$ satisfies the ellipticity condition (13.5.3) and that the following condition is satisfied:

(★) If

$$\begin{cases} P_m(x, \xi + i\sigma\nabla\varphi(x)) = 0, \\ x \in \overline{\Omega}, \\ (\xi, \sigma) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}, \end{cases} \quad (13.5.5)$$

then

$$\frac{i}{2\sigma} \left\{ P_m(x, \xi + i\sigma\nabla\varphi(x)), \overline{P_m(x, \xi + i\sigma\nabla\varphi(x))} \right\} > 0, \quad (13.5.6)$$

where $\{\cdot, \cdot\}$ is the Poisson bracket defined in (13.4.9).

Then there exist constants C and τ_0 such that

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int |P(x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.5.7)$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$.

Moreover C and τ_0 depend on λ , on the $L^\infty(\Omega, \mathbb{C})$ norms of a_α , $|\alpha| \leq m$, on the $L^\infty(\Omega, \mathbb{C})$ norms of ∇a_α , $|\alpha| = m$, and on the moduli of continuity of ∇a_α , for $|\alpha| = m$.

Remark 1. Let us notice that requiring that $(\xi, \sigma) \neq (0, 0)$ in (13.5.5) is equivalent to require that both ξ and σ are different from zero. As a matter of fact, if $\xi = 0$ then, by $P_m(x, \xi + i\sigma\nabla\varphi(x)) = 0$, we have $(i\sigma)^m P_m(x, \nabla\varphi(x)) = 0$ in addition, since $P_m(x, \nabla\varphi(x)) \neq 0$ and since $P_m(x, D)$ is elliptic and $\nabla\varphi(x) \neq 0$, we have $\sigma = 0$. Similarly, if $\sigma = 0$ by the ellipticity of $P_m(x, D)$ we have $\xi = 0$. ♦

Remark 2. Taking into account Remark 2 of Section 13.4, if the coefficients of $P_m(x, D)$ are constants (let us rename it $P_m(D)$), condition (★) become:

((★) – constant coefficients)

If

$$\begin{cases} P_m(\xi + i\sigma\nabla\varphi(x)) = 0, \\ (\xi, \sigma) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}, \end{cases} \quad (13.5.8)$$

then

$$\sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi(x) P_m^{(j)}(\xi + i\tau \nabla \varphi(x)) \overline{P_m^{(k)}(\xi + i\tau \nabla \varphi(x))} > 0.$$

In particular, if the Hessian matrix of φ is positive definite then **(★) – constant coefficients** is satisfied. \blacklozenge

Proof of Theorem 13.5.1.

Let $u \in C_0^\infty(\Omega)$, set

$$v = e^{-\tau\varphi} u.$$

As observed in the previous Section, we have

$$e^{\tau\varphi} P_m(x, D)u = e^{\tau\varphi} P_m(x, D)(e^{-\tau\varphi} v) = P_m(x, D + i\tau \nabla \varphi(x))v \quad (13.5.9)$$

and, denoting by $p_m(x, D, \tau)$ the operator whose symbol is $P_m(x, \xi + i\tau \nabla \varphi(x))$, by (13.4.8) we get

$$\begin{aligned} & \int |P_m(x, D + i\tau \nabla \varphi(x))v|^2 dx \geq \\ & \geq \frac{1}{2} \int |p_m(x, D, \tau)v|^2 dx - \\ & - C_1 \sum_{|\alpha| \leq m-1} \tau^{2(m-|\alpha|)-2} \int |D^\alpha v|^2 dx, \end{aligned} \quad (13.5.10)$$

where C_1 depends by the L^∞ norms of the coefficients of $P_m(x, D)$.

We now derive an appropriate estimate from below of the first term on the right-hand side in (13.5.10).

By (13.4.12) and (13.4.16) we get (by multiplying both equalities by τ)

$$\begin{aligned} \tau \int |p_m(x, D, \tau)v|^2 dx & \geq 2\tau \int \Re \left(S(x, D, \tau)v \overline{A(x, D, \tau)v} \right) dx = \\ & = \tau \int G(x, D, \overline{D}, \tau)[v, \overline{v}] dx, \end{aligned} \quad (13.5.11)$$

where $G(x, D, \overline{D}, \tau)$ has been defined in Proposition 13.4.1.

Let now $x_0 \in \overline{\Omega}$ be a fixed point. We may assume $x_0 = 0 \in \overline{\Omega}$. We get

$$\begin{aligned}
 \tau \int G(x, D, \bar{D}, \tau)[v, \bar{v}]dx &= \tau \int G(0, D, \bar{D}, \tau)[v, \bar{v}]dx + \\
 + \tau \int (G(x, D, \bar{D}, \tau) - G(0, D, \bar{D}, \tau)) [v, \bar{v}]dx &= \tag{13.5.12} \\
 = \frac{\tau}{(2\pi)^n} \int G(0, \xi, \xi, \tau) |\widehat{v}(\xi)|^2 d\xi + \tau \mathcal{R},
 \end{aligned}$$

where

$$\mathcal{R} = \int (G(x, D, \bar{D}, \tau) - G(0, D, \bar{D}, \tau)) [v, \bar{v}]dx.$$

By (13.4.21) we have

$$G(x, D, \bar{D}, \tau) = \sum_{h=0}^{2m-1} \tau^h G^{(h)}(x, D, \bar{D}), \tag{13.5.13}$$

where

$$G^{(h)}(x, D, \bar{D}) [v, \bar{v}] = \sum_{(\alpha, \beta) \in \Lambda_h} c_{\alpha\beta}^{(h)}(x) D^\alpha v \bar{D}^\beta \bar{v}$$

and

$$\Lambda_h = \{(\alpha, \beta) \in \mathbb{N}_0^n : |\alpha| \leq m, |\beta| \leq m, |\alpha| + |\beta| \leq 2m - h - 1\},$$

for $h = 0, 1, \dots, 2m - 1$ and, further, $c_{\alpha\beta}^{(h)} \in C^0(\bar{\Omega}, \mathbb{C})$ for $(\alpha, \beta) \in \Lambda_h$. Let ε be a positive number that we will choose later and let $\rho_1 > 0$ be such that

$$|c_{\alpha\beta}^{(h)}(x) - c_{\alpha\beta}^{(h)}(0)| < \varepsilon, \quad \forall x \in B_{\rho_1} \cap \bar{\Omega}, (\alpha, \beta) \in \Lambda_h, h = 0, 1, \dots, 2m - 1.$$

We have, for every $\tau \geq 1$ and for every $x \in B_{\rho_1} \cap \bar{\Omega}$,

$$\begin{aligned}
 &|\tau^{h+1} (G^{(h)}(x, D, \bar{D}) - G^{(h)}(0, D, \bar{D})) [v, \bar{v}]| \leq \\
 &\leq \sum_{(\alpha, \beta) \in \Lambda_h} \tau^{h+1} |c_{\alpha\beta}^{(h)}(x) - c_{\alpha\beta}^{(h)}(0)| |D^\alpha v| |\bar{D}^\beta \bar{v}| \leq \\
 &\leq \varepsilon \sum_{(\alpha, \beta) \in \Lambda_h} \tau^{2m - (|\alpha| + |\beta|)} |D^\alpha v| |D^\beta v| = \\
 &= \varepsilon \sum_{(\alpha, \beta) \in \Lambda_h} (\tau^{m - |\alpha|} |D^\alpha v|) (\tau^{m - |\beta|} |D^\beta v|) \leq \\
 &\leq C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m - |\alpha|)} |D^\alpha v|^2,
 \end{aligned}$$

where C depends on m only. Let us notice that in the second inequality we have exploited that, for $\tau \geq 1$,

$$(\alpha, \beta) \in \Lambda_h \Rightarrow h + 1 \leq 2m - (|\alpha| + |\beta|) \Rightarrow \tau^{h+1} \leq \tau^{2m - (|\alpha| + |\beta|)}.$$

Therefore, for every $x \in B_{\rho_1} \cap \bar{\Omega}$,

$$|\tau (G(x, D, \bar{D}, \tau) - G(0, D, \bar{D}, \tau)) [v, \bar{v}]| \leq C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} |D^\alpha v|^2.$$

Hence, by Lemma 13.4.3 we have for any $\tau \geq 1$,

$$\begin{aligned} |\tau \mathcal{R}| &\leq C\varepsilon \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx \leq \\ &\leq C\varepsilon \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi, \end{aligned} \quad (13.5.14)$$

for every $v \in C_0^\infty(B_{\rho_1} \cap \Omega)$,

Now, by (13.5.11), (13.5.12) and (13.5.14) we get

$$\begin{aligned} \tau \int |p_m(x, D, \tau)v|^2 dx &\geq \frac{\tau}{(2\pi)^n} \int G(0, \xi, \xi, \tau) |\widehat{v}(\xi)|^2 d\xi - \\ &- C\varepsilon \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi, \end{aligned} \quad (13.5.15)$$

for every $v \in C_0^\infty(B_{\rho_1} \cap \Omega)$ and for every $\tau \geq 1$.

Now we prove the following

Claim.

Set

$$N = \nabla \varphi(0),$$

there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} C_1 |\xi + i\sigma N|^{2m} &\leq \\ &\leq \sigma G(0, \xi, \xi, \sigma) + C_2 |P_m(0, \xi + i\sigma N)|^2, \quad \forall (\xi, \sigma) \in \mathbb{R}^{n+1}. \end{aligned} \quad (13.5.16)$$

Proof of the Claim. Let us denote

$$\mathbb{S}^n = \{(\xi, \sigma) \in \mathbb{R}^{n+1} : |\xi + i\sigma N| = 1\}$$

and

$$\eta = \frac{\xi}{|\xi + i\sigma N|}, \quad \mu = \frac{\sigma}{|\xi + i\sigma N|},$$

let us note that (13.4.18) implies, by homogeneity, that condition (\star) is equivalent to the following one

(\star') If

$$\begin{cases} P_m(0, \eta + i\mu N) = 0, \\ (\eta, \mu) \in \mathbb{S}^n, \end{cases} \quad (13.5.17)$$

then (recall Remark 1)

$$\mu G(0, \eta, \eta, \mu) > 0.$$

Furthermore, (13.5.16) is equivalent to

$$C_1 \leq \mu G(0, \eta, \eta, \mu) + C_2 |P_m(0, \eta + i\mu N)|^2, \quad \forall (\eta, \mu) \in \mathbb{S}^n. \quad (13.5.18)$$

Now, by (13.5.3), we have

$$(\mu G(0, \eta, \eta, \mu) + |P_m(0, \eta + i\mu N)|^2)_{|\mu=0} \geq \lambda, \quad \text{for } |\eta| = 1.$$

By the compactness of \mathbb{S}^n there exists $\mu_0 > 0$ such that

$$\mu G(0, \eta, \eta, \mu) + |P_m(0, \eta + i\mu N)|^2 \geq \frac{\lambda}{2}, \quad (13.5.19)$$

for every $(\eta, \mu) \in \mathbb{S}^n \cap \{|\mu| \leq \mu_0\}$.

Let us denote by K the compact set

$$K = \mathbb{S}^n \cap \{|\mu| \geq \mu_0\}$$

(of course, if $K = \emptyset$ the proof would be concluded). Since (\star') gives trivially

$$|P_m(0, \eta + i\mu N)| = 0, \quad (\eta, \mu) \in K \implies \mu G(0, \eta, \eta, \mu) > 0,$$

by Lemma 12.5.2 we have that there exists $C > 0$ such that

$$\mu G(0, \eta, \eta, \mu) + C |P_m(0, \eta + i\mu N)|^2 > 0, \quad \forall (\eta, \mu) \in K. \quad (13.5.20)$$

By (13.5.19) and (13.5.20) we have

$$\mu G(0, \eta, \eta, \mu) + (C + 1) |P_m(0, \eta + i\mu N)|^2 > 0, \quad \forall (\eta, \mu) \in \mathbb{S}^n \quad (13.5.21)$$

and (13.5.18) follows with

$$C_1 = \min_{(\eta, \mu) \in \mathbb{S}^n} (G(0, \eta, \eta, \mu) + (C + 1) |P_m(0, \eta + i\mu N)|^2)$$

and

$$C_2 = C + 1.$$

The proof of the Claim is concluded.

Now, we set

$$\gamma = \min \left\{ 1, \min_{\bar{\Omega}} |\nabla \varphi| \right\},$$

using (13.5.16) in (13.5.15) we get

$$\begin{aligned} \tau \int |p_m(x, D, \tau)v|^2 dx &\geq (2\pi)^{-n} C_1 \gamma^2 \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi - \\ &\quad - (2\pi)^{-n} C_2 \int |p_m(0, \xi, \tau)|^2 |\widehat{v}(\xi)|^2 d\xi - \\ &\quad - C\varepsilon \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi, \end{aligned} \quad (13.5.22)$$

for every $v \in C_0^\infty(B_{\rho_1} \cap \Omega)$ and for every $\tau \geq 1$. By Lemma 13.4.4 there exists $\rho_2 \leq \rho_1$ such that for every $v \in C_0^\infty(B_{\rho_2} \cap \Omega)$ and for every $\tau \geq 1$ we have

$$\begin{aligned} (2\pi)^{-n} \int |p_m(0, \xi, \tau)|^2 |\widehat{v}(\xi)|^2 d\xi &\leq \int |p_m(x, D, \tau)v|^2 dx + \\ &\quad + C\varepsilon \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi. \end{aligned} \quad (13.5.23)$$

By (13.5.22) and (13.5.23) we have

$$\begin{aligned} \tau \int |p_m(x, D, \tau)v|^2 dx &\geq \\ &\geq ((2\pi)^{-n} C_1 \gamma^2 - C\varepsilon) \int (|\xi|^2 + \tau^2)^m |\widehat{v}(\xi)|^2 d\xi - \\ &\quad - (2\pi)^{-n} C_2 \int |p_m(x, D, \tau)v|^2 dx, \end{aligned} \quad (13.5.24)$$

for every $v \in C_0^\infty(B_{\rho_2} \cap \Omega)$. Now, let us choose

$$\varepsilon = \varepsilon_0 := \frac{(2\pi)^{-n} C_1 \gamma^2}{2C}$$

and let us denote by $\bar{\rho}$ the value of ρ_2 when $\varepsilon = \varepsilon_0$. Moving the last integral of (13.5.24) to the left-hand side and recalling Lemma 13.4.3, we have

$$C_3\tau \int |p_m(x, D, \tau)v|^2 dx \geq \varepsilon_0 C^{-1} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx, \quad (13.5.25)$$

($C_3 = 1 + (2\pi)^{-n}C_2$) for every $v \in C_0^\infty(B_{\bar{\rho}} \cap \Omega)$, for every $\tau \geq 1$.

At this point we use (13.4.8) and we have

$$\begin{aligned} & C \sum_{|\alpha| \leq m-1} \tau^{2(m-|\alpha|)-1} \int |D^\alpha v|^2 dx + \\ & + 2C_3\tau \int |P_m(x, D + i\tau\nabla\varphi(x))v|^2 dx \geq \quad (13.5.26) \\ & \geq \varepsilon_0 C^{-1} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx, \end{aligned}$$

for every $v \in C_0^\infty(B_{\bar{\rho}} \cap \Omega)$ and for every $\tau \geq 1$. Now, in (13.5.26) we move on the right-hand side the first term which is on the left-hand side and we get

$$\begin{aligned} 2C_3\tau \int |P_m(x, D + i\tau\nabla\varphi(x))v|^2 dx & \geq \varepsilon_0 C^{-1} \sum_{|\alpha|=m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx + \\ & + \sum_{|\alpha| \leq m-1} \tau^{2(m-|\alpha|)} (C^{-1}\varepsilon_0 - C\tau^{-1}) \int |D^\alpha v|^2 dx, \end{aligned}$$

for every $v \in C_0^\infty(B_{\bar{\rho}} \cap \Omega)$ and for every $\tau \geq 1$. Hence, if $\tau \geq \tau_0$, where $\tau_0 = \max\{2C^2\varepsilon_0^{-1}, 1\}$, we have

$$2C_3\tau \int |P_m(x, D + i\tau\nabla\varphi(x))v|^2 dx \geq \frac{\varepsilon_0 C^{-1}}{2} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)} \int |D^\alpha v|^2 dx,$$

for every $v \in C_0^\infty(B_{\bar{\rho}} \cap \Omega)$ and for every $\tau \geq \tau_0$. By using Lemma 13.4.2 and by recalling (compare with (13.4.2))

$$P_m(x, D + i\tau\nabla\varphi(x))v = e^{\tau\varphi(x)} P_m(x, D)u,$$

we have

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int |P_m(x, D)u|^2 e^{2\tau\varphi} dx. \quad (13.5.27)$$

Estimate (13.5.7) follows by Lemma 13.1.1 and by the comments made at the beginning of the introduction to this Chapter. ■

Exercise. Prove Theorem 13.5.1 (assuming C^∞ coefficients in the principal part) by employing decomposition (13.4.35) instead of decomposition (13.4.11). [Hint: recall (13.4.40) and use

$$-2\Re(z\bar{w}) \geq -|z|^2 - |w|^2,$$

for $z, w \in \mathbb{C}$. ♣

13.5.1 Elliptic operators with Lipschitz continuous coefficients and the Cauchy problem

In Theorem 13.5.1 we have assumed that the coefficients of the principal part are of class $C^1(\bar{\Omega})$ and it turns out that the constants, C and τ_0 , in the estimate (13.5.7) depend on the **modulus of continuity** of the gradients of these coefficients. We will now see that with a relatively modest effort we can prove a Carleman estimate for elliptic operators with Lipschitz continuous coefficients in principal part. In this regard, it is useful to point out that this assumption cannot be substantially reduced as has been shown in the counterexamples of Mandache's Mandache [54] and of Plis [64].

For any $x \in \mathbb{R}^n$ and $R > 0$ let us denote by

$$Q_R(x) = \{y \in \mathbb{R}^n : |y_j - x_j| < R, \quad j = 1, \dots, n\}.$$

Let us introduce a special **partition of unity**.

Let $\vartheta_0 \in C_0^\infty(\mathbb{R})$ satisfy

$$\vartheta_0(t) = \begin{cases} 1, & \text{for } |t| \leq 1, \\ 0, & \text{for } |t| \geq 3/2. \end{cases}$$

Let, further, $0 \leq \vartheta \leq 1$ such that

$$\vartheta(x) = \vartheta_0(x_1) \cdots \vartheta_0(x_n),$$

we have

$$\vartheta(x) = \begin{cases} 1, & \text{for } x \in Q_1(0), \\ 0, & \text{for } x \in \mathbb{R}^n \setminus \overline{Q_{3/2}(0)}. \end{cases}$$

For any $\mu \geq 1$ and $g \in \mathbb{Z}^n$, let us denote

$$x_g = g/\mu$$

and

$$\vartheta_{g,\mu}(x) = \vartheta(\mu(x - x_g)).$$

Hence, we have

$$\text{supp } \vartheta_{g,\mu} \subset \overline{Q_{3/2\mu}(x_g)} \subset Q_{2/\mu}(x_g)$$

and

$$|D^k \vartheta_{g,\mu}| \leq C_1 \mu^k (\chi_{Q_{3/2\mu}(x_g)} - \chi_{Q_{1/\mu}(x_g)}), \quad k = 0, 1, \dots, m, \quad (13.5.28)$$

where $C_1 \geq 1$ depends on n only.

For any $g \in \mathbb{Z}^n$, set

$$A_g = \{g' \in \mathbb{Z}^n \mid \text{supp } \vartheta_{g',\mu} \cap \text{supp } \vartheta_{g,\mu} \neq \emptyset\},$$

then

$$\text{card}(A_g) \text{ depends only on } n. \quad (13.5.29)$$

Therefore we can define

$$\tilde{\vartheta}_\mu(x) := \sum_{g \in \mathbb{Z}^n} \vartheta_{g,\mu}(x) \geq 1, \quad \forall x \in \mathbb{R}^n. \quad (13.5.30)$$

By (13.5.28), we get

$$|D^k \tilde{\vartheta}_\mu| \leq C_2 \mu^k, \quad (13.5.31)$$

where $C_2 \geq 1$ depends on n only. Define

$$\eta_{g,\mu}(x) = \vartheta_{g,\mu}(x) / \tilde{\vartheta}_\mu(x), \quad \forall x \in \mathbb{R}^n,$$

we have thus

$$\left\{ \begin{array}{l} \eta_{g,\mu} \geq 0, \\ \sum_{g \in \mathbb{Z}^n} \eta_{g,\mu} = 1, \quad \text{in } \mathbb{R}^n, \\ \text{supp } \eta_{g,\mu} \subset \overline{Q_{3/2\mu}(x_g)} \subset Q_{2/\mu}(x_g), \\ |D^\alpha \eta_{g,\mu}| \leq C_3 \mu^{|\alpha|} \chi_{\overline{Q_{3/2\mu}(x_g)}}, \quad \forall \alpha \in \mathbb{N}^n, \quad 1 \leq |\alpha| \leq m, \end{array} \right. \quad (13.5.32)$$

where $C_3 \geq 1$ depends on n only.

Let $m \in \mathbb{N}$ and let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad (13.5.33)$$

be an elliptic operator. Let M_0, M_1, λ be positive constants and let us suppose that

$$\|a_\alpha\|_{L^\infty(Q_1)} \leq M_0, \quad \text{for } |\alpha| \leq m, \quad (13.5.34a)$$

$$|a_\alpha(x) - a_\alpha(y)| \leq M_1|x - y|, \quad \forall x, y \in Q_1, \quad \text{for } |\alpha| = m, \quad (13.5.34b)$$

$$|P_m(x, \xi)| \geq \lambda |\xi|^m, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \overline{Q_1}. \quad (13.5.34c)$$

Let $\varphi \in C^\infty(\overline{Q_1})$ and for $x, y \in \overline{Q_1}$, $(\xi, \sigma) \in \mathbb{R}^{n+1}$ set

$$\begin{aligned} \mathcal{G}(x, y; \xi, \sigma) &= \\ &= \sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi(x) P_m^{(j)}(y, \xi + i\sigma \nabla \varphi(x)) \overline{P_m^{(k)}(y, \xi + i\sigma \nabla \varphi(x))}. \end{aligned} \quad (13.5.35)$$

Theorem 13.5.2. *Let us assume that operator (13.5.33) satisfies ellipticity condition (13.5.34c) and that its coefficients satisfy conditions (13.5.34a) and (13.5.34b). Moreover, let us assume that*

$$\begin{cases} P_m(0, \xi + i\sigma \nabla \varphi(0)) = 0, \\ (\xi, \sigma) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\}, \end{cases} \implies \mathcal{G}(0, 0; \xi, \sigma) > 0. \quad (13.5.36)$$

Then there exist $\overline{R} \in (0, 1]$, $\delta_0 \in (0, 1]$, $C_0 \geq 1$ and τ_0 such that

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C_0 \int |P_m(\delta x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.5.37)$$

for every $\delta \in (0, \delta_0]$, for every $u \in C_0^\infty(Q_{\overline{R}})$ and for every $\tau \geq \tau_0$.

Proof. Since $|P_m(0, \xi + i\sigma \nabla \varphi(0))|^2$ and $(|\xi|^2 + \tau^2) \mathcal{G}(0, 0; \xi, \sigma)$ are homogeneous polynomials of degree $2m$, (13.5.36) is equivalent to the following property: there exist positive constants C_1 and C_2 such that (Lemma 12.5.2)

$$\begin{aligned} C_2 |P_m(0, \xi + i\sigma \nabla \varphi(0))|^2 + (|\xi|^2 + \sigma^2) \mathcal{G}(0, 0; \xi, \sigma) &\geq \\ &\geq C_1 (|\xi|^2 + \sigma^2)^m. \end{aligned} \quad (13.5.38)$$

for every $(\xi, \sigma) \in \mathbb{R}^{n+1}$.

Let us denote

$$H(x, y; \xi, \sigma) = C_2 |P_m(y, \xi + i\sigma \nabla \varphi(x))|^2 + (|\xi|^2 + \sigma^2) \mathcal{G}(x, y; \xi, \sigma) \quad (13.5.39)$$

and let us notice that H is a continuous function. Moreover, by (13.5.38) we have trivially

$$H(0, 0; \xi, \sigma) \geq C_1, \quad \text{for all } (\xi, \sigma) \text{ such that } |\xi|^2 + \sigma^2 = 1.$$

Hence, the continuity of H implies that there exists $\bar{R}_1 \in (0, 1]$ such that

$$H(x, y; \xi, \sigma) \geq \frac{C_1}{2}, \quad \text{for all } (\xi, \sigma) \text{ such that } |\xi|^2 + \sigma^2 = 1,$$

for every $x, y \in \bar{Q}_{\bar{R}_1}$.

Therefore

$$\begin{aligned} C_2 |P_m(y, \xi + i\sigma \nabla \varphi(x))|^2 + (|\xi|^2 + \sigma^2) \mathcal{G}(x, y; \xi, \sigma) &\geq \\ &\geq \frac{C_1}{2} (|\xi|^2 + \sigma^2)^m, \end{aligned} \quad (13.5.40)$$

for every $x, y \in \bar{Q}_{\bar{R}_1}$. By the previous inequality we have that for every $\tilde{y} \in Q_{\bar{R}_1}$ it occurs

$$\begin{cases} P_m(\tilde{y}, \xi + i\sigma \nabla \varphi(x)) = 0, \\ (\xi, \sigma) \in \mathbb{R}^{n+1} \setminus \{(0, 0)\} \end{cases} \implies \mathcal{G}(x, \tilde{y}; \xi, \sigma) > 0. \quad (13.5.41)$$

Now, for $y \in \bar{Q}_{\bar{R}_1}$ fixed and $\delta \in (0, 1]$ to be chosen, let us consider the operator with constant coefficients w.r.t. the variable x

$$P_m(\delta y, D_x) = \sum_{|\alpha|=m} a_\alpha(\delta y) D_x^\alpha. \quad (13.5.42)$$

Of course $\delta y \in \bar{Q}_{\bar{R}_1}$ (as $\delta \in (0, 1]$). Now, (13.5.41) (considered for $\tilde{y} = \delta y$) is nothing but (compare **Remark 2** after Theorem 13.5.1) condition (★) of Theorem 13.5.1. Hence there exist $C_3 > 0$ e τ_1 such that

$$\begin{aligned} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi(x)} dx &\leq \\ &\leq C_3 \int |P_m(\delta y, D)u|^2 e^{2\tau\varphi(x)} dx, \end{aligned} \quad (13.5.43)$$

for every $u \in C_0^\infty(Q_{\bar{R}_1})$ and for every $\tau \geq \tau_1$. Moreover $C_3 > 0$ and τ_1 **do not depend neither on $y \in \bar{Q}_{\bar{R}_1}$ nor on $\delta \in (0, 1]$.**

We now use the partition of unity introduced above with

$$\mu = \sqrt{\varepsilon\tau}, \quad (13.5.44)$$

for $\tau \geq \tau^{(\varepsilon)} := \max\{\varepsilon^{-1}, \tau_1\}$ where $\varepsilon \in (0, 1]$ is to be chosen.

Let $u \in C_0^\infty(Q_{\bar{R}_1})$. By the first relation of (13.5.32) we have

$$u = \sum_{g \in \mathbb{Z}^n} u\eta_{g,\mu}. \quad (13.5.45)$$

Now we apply (13.5.43) (for $y = x_g \in Q_{\bar{R}_1}$). We have, for every $\tau \geq \tau^{(\varepsilon)}$

$$\begin{aligned} & \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq \\ & \leq c \sum_{g \in \mathbb{Z}^n} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha (u\eta_{g,\mu})|^2 e^{2\tau\varphi} dx \leq \\ & \leq cC_3 \sum_{g \in \mathbb{Z}^n} \int |P_m(\delta x_g, D)(u\eta_{g,\mu})|^2 e^{2\tau\varphi} dx, \end{aligned} \quad (13.5.46)$$

where the constant c appearing in the second inequality, by (13.5.29), depends on n only.

Now, let us estimate from above the last term on the right-hand side of (13.5.46). We have

$$\begin{aligned} |P_m(\delta x_g, D)(u\eta_{g,\mu})|^2 & \leq 2|P_m(\delta x, D)(u\eta_{g,\mu})|^2 + \\ & + 2|(P_m(\delta x_g, D) - P_m(\delta x, D))(u\eta_{g,\mu})|^2. \end{aligned} \quad (13.5.47)$$

In order to estimate the **first term on the right-hand side in** (13.5.47) we notice that

$$\begin{aligned} P_m(\delta x, D)(u\eta_{g,\mu}) & = \eta_{g,\mu} \sum_{|\alpha| \leq m} a_\alpha(\delta x) D^\alpha u + \\ & + \sum_{|\alpha| \leq m} a_\alpha(\delta x) \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} \eta_{g,\mu} = \\ & = \eta_{g,\mu} P_m(\delta x, D)u + \tilde{P}(x, D, \mu)u \end{aligned} \quad (13.5.48)$$

where we set

$$\tilde{P}(x, D, \mu)u = \sum_{|\alpha| \leq m} a_\alpha(\delta x) \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} \eta_{g,\mu}.$$

Let us note that this operator has order $m - 1$. Moreover by (13.5.32) and (13.5.34a) we get

$$\left| \tilde{P}(x, D, \mu)u \right| \leq CM_0 \chi_{Q_{2/\mu}(x_g)} \sum_{|\beta| \leq m-1} |D^\beta u| \mu^{m-|\beta|}. \quad (13.5.49)$$

From (13.5.47), (13.5.49) and recalling (13.5.44) we have (for the first term on the right we use the trivial inequality $\eta_{g,\mu}^2 \leq \eta_{g,\mu}$)

$$\begin{aligned} |P_m(\delta x, D)(u\eta_{g,\mu})|^2 &\leq \eta_{g,\mu} |P_m(\delta x, D)u|^2 + \\ &+ CM_0^2 \chi_{Q_{2/\mu}(x_g)} \sum_{|\alpha| \leq m-1} (\varepsilon\tau)^{m-|\alpha|} |D^\alpha u|^2. \end{aligned} \quad (13.5.50)$$

We now estimate the second term on the right-hand side in (13.5.47). Proceeding in a similar way as above, we have

$$\begin{aligned} &|(P_m(\delta x_g, D) - P_m(\delta x, D))(u\eta_{g,\mu})| \leq \\ &\leq \sum_{|\alpha|=m} |(a_\alpha(\delta x) - a_\alpha(\delta x_g))| |D^\alpha(u\eta_{g,\mu})| = \\ &= \eta_{g,\mu} \sum_{|\alpha|=m} |(a_\alpha(\delta x) - a_\alpha(\delta x_g))| |D^\alpha u| + \\ &+ CM_0 \chi_{Q_{2/\mu}(x_g)} \sum_{|\alpha| \leq m-1} |D^\alpha u| \mu^{m-|\alpha|}. \end{aligned}$$

In order to estimate the second-to-last term, it must be taken into account that the estimate has to be done in the support of $\eta_{g,\mu}$. By (13.5.34b) we get, therefore,

$$\begin{aligned} &|(P_m(\delta x_g, D) - P_m(\delta x, D))(u\eta_{g,\mu})|^2 \leq \\ &\leq CM_1^2 \eta_{g,\mu} \frac{\delta^2}{\varepsilon\tau} \sum_{|\alpha|=m} |D^\alpha u|^2 + \\ &+ CM_0^2 \chi_{Q_{2/\mu}(x_g)} \sum_{|\alpha| \leq m-1} (\varepsilon\tau)^{m-|\alpha|} |D^\alpha u|^2. \end{aligned} \quad (13.5.51)$$

Now, we insert (13.5.50) and (13.5.51) into (13.5.47) and we get

$$\begin{aligned}
|P_m(\delta x_g, D)(u\eta_{g,\mu})|^2 &\leq 2\eta_{g,\mu} |P_m(\delta x, D)u|^2 + \\
&+ CM_1^2 \eta_{g,\mu} \frac{\delta^2}{\varepsilon\tau} \sum_{|\alpha|=m} |D^\alpha u|^2 + \\
&+ CM_0^2 \chi_{Q_{2/\mu}(x_g)} \sum_{|\alpha|\leq m-1} (\varepsilon\tau)^{m-|\alpha|} |D^\alpha u|^2.
\end{aligned}$$

Inserting the latter into (13.5.46) we obtain

$$\begin{aligned}
\sum_{|\alpha|\leq m-1} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx + \frac{1}{\tau} \sum_{|\alpha|=m} \int |D^\alpha u|^2 e^{2\tau\varphi} dx &\leq \\
&\leq cC_3 \int |P_m(\delta x, D)u|^2 e^{2\tau\varphi} dx + \\
&+ C_4 \frac{\delta^2}{\varepsilon\tau} \sum_{|\alpha|=m} \int |D^\alpha u|^2 e^{2\tau\varphi} dx + \\
&+ C_5 \sum_{|\alpha|\leq m-1} \int (\varepsilon\tau)^{m-|\alpha|} |D^\alpha u|^2 e^{2\tau\varphi} dx,
\end{aligned}$$

where C_4 depends on M_1 only and C_5 depends by M_0 only. From which we have

$$\begin{aligned}
\sum_{|\alpha|\leq m-1} \tau^{(m-|\alpha|)} (\tau^{m-|\alpha|-1} - C_5 \varepsilon^{m-|\alpha|}) \int |D^\alpha u|^2 e^{2\tau\varphi} dx + \\
+ \frac{1}{\tau} \left(1 - C_4 \frac{\delta^2}{\varepsilon}\right) \sum_{|\alpha|=m} \int |D^\alpha u|^2 e^{2\tau\varphi} dx &\leq \quad (13.5.52) \\
\leq cC_3 \int |P_m(\delta x, D)u|^2 e^{2\tau\varphi} dx.
\end{aligned}$$

Let us choose

$$\begin{aligned}
\varepsilon &= \varepsilon_0 := \frac{1}{2C_5}, \\
\delta &\leq \delta_0 := \sqrt{\frac{\varepsilon_0}{2C_4}}
\end{aligned}$$

and by (13.5.52) we get

$$\sum_{|\alpha|\leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq 2cC_3 \int |P_m(\delta x, D)u|^2 e^{2\tau\varphi} dx, \quad (13.5.53)$$

for every $u \in C_0^\infty(Q_{\bar{R}_1})$ and for every $\tau \geq \tau^{(\varepsilon_0)}$. Estimate (13.5.37) is proved. ■

Remark. Let us notice that (reader check) by the change of variables $X = \delta x$, (13.5.37) si become

$$\begin{aligned} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \delta^{m-|\alpha|} \int |D^\alpha u|^2 e^{2\tau\varphi(\delta^{-1}X)} dX &\leq \\ &\leq C_0 \int |P_m(X, D)u|^2 e^{2\tau\varphi(\delta^{-1}X)} dX, \end{aligned} \tag{13.5.54}$$

for every $u \in C_0^\infty(Q_{\delta\bar{R}})$ and for every $\tau \geq \tau_0$. ♦

In the following Theorem we will apply estimate (13.5.54) to prove a uniqueness result for the Cauchy problem.

Theorem 13.5.3. *Let $\psi \in C^1(\bar{Q}_1)$ be real-valued function such that*

$$\nabla\psi(0) \neq 0. \tag{13.5.55}$$

Let $P(x, D)$ be operator (13.5.33) and let us suppose that (13.5.34) holds true. Let $U \in H^m(Q_1)$ satisfy

$$\begin{cases} P(x, D)U = 0, & \text{in } Q_1, \\ U(x) = 0 & \text{in } \{x \in \bar{Q}_1 : \psi(x) > \psi(0)\}. \end{cases} \tag{13.5.56}$$

Let us suppose that for every $\xi \in \mathbb{R}^n \setminus \{0\}$ we have

$$\sigma \rightarrow P_m(0, \xi + i\sigma\nabla\psi(0)) \text{ has no real multiple roots.} \tag{13.5.57}$$

Then there exist a neighborhood \mathcal{U}_0 of 0 such that

$$U = 0 \text{ in } \mathcal{U}_0. \tag{13.5.58}$$

Remark. As it is easily checked, condition (13.5.57) can be expressed equivalently as follows

$$\begin{aligned} &\begin{cases} P_m(0, \xi + i\tau\nabla\psi(0)) = 0, \\ (\xi, \tau) \neq (0, 0), \end{cases} \implies \\ &\implies \sum_{j=1}^n P_m^{(j)}(0, \xi + i\tau\nabla\psi(0)) \partial_j \psi(0) \neq 0. \end{aligned} \tag{13.5.59}$$

We further observe that if $m = 2$ and the coefficients of $P_2(x, D)$ are real then the (13.5.57) is satisfied (see Example 4a, Section 12.5). \blacklozenge

Proof of Theorem 13.5.3.

It is not restrictive to assume $\psi(0) = 0$ and, since $\nabla\psi(0) \neq 0$, we may reduce to consider, up to isometries, the case where, for an appropriate $r_0 > 0$, we have

$$\{x \in Q_{r_0} : \psi(x) = 0\} = \{(x', f(x')) : x' \in Q'_{r_0}(0)\}, \quad (13.5.60)$$

($Q'_{r_0} = (-r_0, r_0)^{n-1}$) where $f \in C^1(Q'_{r_0})$, $f(0) = |\nabla f(0)| = 0$ and

$$\{x \in Q_1 : \psi(x) > 0\} \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} : x_n < f(x')\}.$$

Let us notice that in this way condition (13.5.57) becomes.

$$\sigma \rightarrow P_m(0, \xi - i\sigma e_n) \text{ has no real multiple roots.} \quad (13.5.61)$$

Now we use **Holmgren transformation** introduced in (7.6.22), that is we consider the transformation

$$\Lambda : \mathbb{R}_x^n \rightarrow \mathbb{R}_y^n, \quad x \rightarrow y = \Lambda(x', x_n) = \left(x', x_n + \frac{A}{2}|x'|^2\right), \quad (13.5.62)$$

(recall that Λ is a diffeomorphism) where $A > 0$ satisfies

$$A > \|\partial^2 f\|_{L^\infty(B'_{r_0})} \quad (13.5.63)$$

and where $\partial^2 f$ is the Hessian matrix of f . Let us fix A that satisfies (13.5.63) and we recall that, with this choice, the function

$$g(x') = f(x') + \frac{A}{2}|x'|^2, \quad (13.5.64)$$

is strictly convex and

$$g(0) = |\nabla g(0)| = 0. \quad (13.5.65)$$

Let us denote by $\tilde{P}(y, D_y)$ the transformed operator of $P(x, D_x)$ by mean of Λ . Since

$$P_m(x, \xi) = i^m \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$$

we have (compare with (7.3.4))

$$\tilde{P}_m(y, \eta) = i^m \sum_{|\alpha|=m} a_\alpha(\Lambda^{-1}(y)) ((\partial_x \Lambda(x))^t \eta)|_{x=\Lambda^{-1}(y)}^\alpha. \quad (13.5.66)$$

So the condition (13.5.61) is written (reader check)

$$\begin{aligned} & \text{for fixed } \xi \in \mathbb{R}^n \setminus \{0\}, \\ & \tilde{P}_m(0, \xi - i\sigma e_n) \text{ has no real multiple roots.} \end{aligned} \quad (13.5.67)$$

Moreover, (13.5.56) implies

$$\begin{cases} \tilde{P}(y, D_y) \tilde{U} = 0, & \text{in } Q_{r_0}, \\ \tilde{U}(y) = 0 & \text{in } \{(x', x_n) \in Q_{r_0} : x_n < g(x')\}. \end{cases} \quad (13.5.68)$$

where $\tilde{U}(y) = U(\Lambda^{-1}(y))$. It turns out $\tilde{U} \in H^m(Q_{r_0})$.

We agree from here on to omit " \sim " from P and U , and to rename " x " the variable " y ". Let

$$h(x_n) = -x_n + \frac{x_n^2}{2}$$

(let us notice that h is strictly decreasing in $0 \leq x_n \leq 1$)
and

$$\varphi(x) = h(\delta_0 x),$$

where δ_0 is defined in Theorem 13.5.2. We have

$$\nabla \varphi(0) = -\delta_0 e_n.$$

We have

$$\nabla \varphi(0) = -\delta_0 e_n$$

and also, (13.5.67) implies that if $\xi \in \mathbb{R}^n \setminus \{0\}$ and

$$P_m(0, \xi + i\sigma \nabla \varphi(0)) = P_m(0, \xi - i\sigma \delta_0 e_n) = 0,$$

then

$$\begin{aligned} \mathcal{G}(0, 0; \xi, \sigma) &= \\ &= \sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi(0) P_m^{(j)}(0, \xi + i\sigma \nabla \varphi(0)) \overline{P_m^{(k)}(0, \xi + i\sigma \nabla \varphi(0))} = \\ &= \delta_0^2 |P_m^{(n)}(0, \xi + i\sigma \nabla \varphi(0))|^2 = \delta_0^2 |P_m^{(n)}(0, \xi - i\sigma \delta_0 e_n)|^2 > 0. \end{aligned}$$

Therefore the assumptions of Theorem 13.5.2 are satisfied. Then Carleman estimate (13.5.54) holds. We may write such a Carleman estimate as (setting $\delta = \delta_0$)

$$\begin{aligned} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau h(x_n)} dx &\leq \\ &\leq C \int |P(x, D)u|^2 e^{2\tau h(x_n)} dx, \end{aligned} \quad (13.5.69)$$

for every $u \in C_0^\infty(Q_{\delta_0 \bar{R}})$ and for every $\tau \geq \bar{\tau}_0$ for a certain $\bar{\tau}_0 \geq \tau_0$. Set

$$r_1 = \min \{r_0, \delta_0 \bar{R}\}$$

and, for $\rho > 0$,

$$E_\rho = \{(x', x_n) \in Q_{r_1} : g(x') < x_n < \rho\}.$$

For the strict convexity of g and by (13.5.65), we have that there exists $\rho_1 > 0$ such that

$$\bar{E}_{\rho_1} \subset Q_{r_1}.$$

Let $\rho_2 \in (0, \rho_1)$ Let $\eta \in C^\infty(\mathbb{R})$ be a function such that

$$0 \leq \eta(x_n) \leq 1, \quad \forall x \in \mathbb{R}; \quad \eta(x_n) = 1, \quad \forall x_n \leq \rho_2; \quad \eta(x) = 0, \quad \forall x_n \geq \rho_1.$$

Let us assume that

$$\eta^{(k)}(x_n) \leq C (\rho_1 - \rho_2)^{-k}.$$

By density, (13.5.54) holds for every $u \in H_0^m(Q_{r_1})$, hence, in particular, (13.5.54) holds for $u(x) = U(x)\eta(x_n)$. From now on, the proof is quite standard, we present it for completeness. Since $P(x, D)U = 0$ in Q_{r_1} we have.

$$|P(x, D)(U\eta)| \leq CM_0 \chi_{\mathbb{R} \setminus (\rho_2, \rho_1)} \sum_{|\alpha| \leq m-1} (\rho_1 - \rho_2)^{-|\alpha|} |D^\alpha U|$$

and, for any $0 < \rho < \rho_2$, by (13.5.54), we get

$$\begin{aligned}
& e^{2\tau h(\rho)} \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int_{E_\rho} |D^\alpha U|^2 dx \leq \\
& \leq \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int_{E_{\rho_1}} |D^\alpha(U\eta)|^2 e^{2\tau h(x_n)} dx \leq \\
& \leq C \int_{E_{\rho_1}} |P(x, D)(U\eta)|^2 e^{2\tau h(x_n)} dx \leq \\
& \leq CM_0^2 e^{2\tau h(\rho_2)} \sum_{|\alpha| \leq m-1} \int_{E_{\rho_1} \setminus E_{\rho_2}} (\rho_1 - \rho_2)^{-|\alpha|} |D^\alpha U| dx,
\end{aligned}$$

for every $\tau \geq \bar{\tau}_0$. Hence

$$\begin{aligned}
& \sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int_{E_\rho} |D^\alpha U|^2 dx \leq \\
& \leq CM_0^2 e^{-2\tau(h(\rho)-h(\rho_2))} \sum_{|\alpha| \leq m-1} \int_{E_{\rho_1} \setminus E_{\rho_2}} (\rho_1 - \rho_2)^{-|\alpha|} |D^\alpha U| dx
\end{aligned}$$

for every $\tau \geq \bar{\tau}_0$. Passing to the limit as $\tau \rightarrow +\infty$ which goes to infinity, and taking into account that $h(\rho) - h(\rho_2) > 0$ we have $U = 0$ in $Q_{r_1} \cap \{x_n \leq \rho\}$. Theorem is proved. ■

Chapter 14

Carleman estimates and the Cauchy problems II – Second order operators

14.1 Introduction

In this Chapter we will consider the second-order operators whose principal part (not necessarily elliptic) is given by

$$P_2(x, \partial) = \sum_{j,k=1}^n g^{jk}(x) \partial_{x_j x_k}^2, \quad (14.1.1)$$

where the matrix of coefficients $\{g^{jk}(x)\}_{j,k=1}^n$ is a symmetric and invertible matrix, whose entries are the real-valued functions g^{jk} defined on a bounded open set $\Omega \subset \mathbb{R}^n$, on which we will make appropriate regularity assumptions. When we will refer to the symbol of the operator (14.1.1), here we will always refer to the polynomial in the variable ξ

$$P_2(x, \xi) = \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k. \quad (14.1.2)$$

We note that, with the notation for the derivatives used in the previous Sections, operator (14.1.1) can be written

$$P_2(x, \partial) = - \sum_{j,k=1}^n g^{jk}(x) D_{x_j x_k}^2 \quad (14.1.3)$$

and so polynomial (14.1.2) is simply the symbol of operator (14.1.3) with the sign changed. This abuse of notation will not create major problems,

in particular, it will not create problems when we compare procedures and results found in this Section with those in the previous sections.

The purpose of this Section is to derive the Carleman estimates in a more direct fashion than the last two sections. This will allow us, in particular, to write down explicitly the quadratic form $G(x, D, \bar{D}, \tau)$ given by Proposition (13.4.1) making it more easy to obtain estimates for the operators with $C^{0,1}$ coefficients.

14.2 The case of the Laplace operator

The case of the Laplace operator will serve us somewhat as a model for more general operators of type (14.1.1).

We begin by the following

Lemma 14.2.1 (The Rellich identity). *Let $\beta \in C^{0,1}(\Omega, \mathbb{R}^n)$, $\beta = (\beta^1, \dots, \beta^n)$ and $v \in C^2(\Omega)$, then*

$$\begin{aligned} 2(\beta \cdot \nabla v)\Delta v &= \operatorname{div} (2(\beta \cdot \nabla v)\nabla v - \beta|\nabla v|^2) + \\ &+ (\operatorname{div} \beta)|\nabla v|^2 - 2\partial_k \beta^j \partial_j v \partial_k v, \quad a.e. \ x \in \Omega, \end{aligned} \quad (14.2.1)$$

(in (14.2.1) we have used the Einstein notation of repeated indices).

Proof. We have

$$\begin{aligned} 2(\beta \cdot \nabla v)\Delta v &= 2(\beta^j \partial_j v)\Delta v = 2(\partial_k (\beta^j \partial_j v \partial_k v) - \partial_k (\beta^j \partial_j v) \partial_k v) = \\ &= 2(\partial_k (\beta^j \partial_j v \partial_k v - (\partial_k \beta^j) \partial_j v \partial_k v - \beta^j \partial_{jk}^2 v \partial_k v)) = \\ &= 2\operatorname{div} [(\beta \cdot \nabla v)\nabla v] - 2(\partial_k \beta^j) \partial_j v \partial_k v - \beta^j \partial_j (|\nabla v|^2) = \\ &= 2\operatorname{div} [(\beta \cdot \nabla v)\nabla v] - 2(\partial_k \beta^j) \partial_j v \partial_k v - \partial_j (\beta^j |\nabla v|^2) + (\operatorname{div} \beta)|\nabla v|^2 = \\ &= \operatorname{div} [2(\beta \cdot \nabla v)\nabla v - \beta|\nabla v|^2] - 2(\partial_k \beta^j) \partial_j v \partial_k v + (\operatorname{div} \beta)|\nabla v|^2. \end{aligned}$$

aalmost everywhere in Ω . ■

Remark. By (14.2.1) we have immediately

$$\int_{\Omega} 2(\beta \cdot \nabla v)\Delta v dx = \int_{\Omega} ((\operatorname{div} \beta)|\nabla v|^2 - 2\partial_k \beta^j \partial_j v \partial_k v) dx, \quad (14.2.2)$$

for every $v \in C_0^\infty(\Omega)$. On the other hand, as can be easily checked, if v is a real-valued function we have

$$2(\beta \cdot \nabla v)\Delta v = F(x, D, D)[v, \bar{v}],$$

where

$$F(x, D, D)[v, \bar{v}] = i \sum_{j,k=1}^n \beta^j \left(D_k^2 v \overline{D_j v} - D_j v \overline{D_k^2 v} \right)$$

which satisfies condition (13.3.40) of Lemma 13.3.6. In our case (13.3.41) takes the form (14.2.2). Let us notice that by (13.3.42) we obtain

$$G(x, \xi, \xi) = (\operatorname{div} \beta)|\xi|^2 - 2\partial_k \beta^j \xi_j \xi_k.$$



Let us review some key steps of the proof of Theorem 13.5.1 using Rellich identity (14.2.1) to perform the integrations by parts.

We begin by rewriting the statement of Theorem 13.5.1 in the case of the Laplace operator

Theorem 14.2.2. *Let Ω be a bounded open set of \mathbb{R}^n , and let $\varphi \in C^\infty(\bar{\Omega})$ be a real-valued function such that $\nabla \varphi \neq 0$ on $\bar{\Omega}$. Let us assume that the following implication holds true*

$$\begin{cases} |\xi|^2 = \tau^2 |\nabla \varphi(x)|^2, \\ \xi \cdot \nabla \varphi(x) = 0, \\ \tau \neq 0, \end{cases} \implies Q(x, \xi, \tau) = \sum_{j,k=1}^n \partial_{jk}^2 \varphi(x) \xi_j \xi_k + \tau^2 \sum_{j,k=1}^n \partial_{jk}^2 \varphi(x) \varphi \partial_j \varphi(x) \partial_k \varphi(x) > 0. \tag{14.2.3}$$

Then there exist constants C and τ_0 such that

$$\tau^3 \int_{\Omega} |u|^2 e^{2\tau\varphi} dx + \tau \int_{\Omega} |\nabla u|^2 e^{2\tau\varphi} dx + \tau^{-1} \int_{\Omega} |\partial^2 u|^2 e^{2\tau\varphi} dx \leq C \int_{\Omega} |\Delta u|^2 e^{2\tau\varphi} dx,$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$.

Proof. First, we observe that (14.2.3) is simply the rewriting of condition (★) of Theorem 13.5.1 in the case of the Laplace operator.

Let us denote by L the operator Δ . Let $u \in C_0^\infty(\Omega)$. Set $v = e^{\tau\varphi}u$. Let us calculate

$$\begin{aligned}\partial_j u &= e^{-\tau\varphi} (\partial_j v - \tau \partial_j \varphi v), \\ \partial_j^2 u &= e^{-\tau\varphi} (\partial_j^2 v - 2\tau \partial_j \varphi \partial_j v - \tau \partial_j^2 \varphi v + \tau^2 v (\partial_j \varphi)^2).\end{aligned}$$

We obtain

$$L_\tau v = e^{\tau\varphi} L(e^{-\tau\varphi} v) = \Delta v - \tau(\Delta\varphi)v - 2\tau \nabla\varphi \cdot \nabla v + \tau^2 |\nabla\varphi|^2 v. \quad (14.2.4)$$

Now in the setting provided in Section 13.4, in the first line, (see in particular (13.4.6)) we have neglected the term $-\tau(\Delta\varphi)v$ and, in the middle part of the proof of Theorem 13.5.1, we have focused on the operator $p_m(x, D, \tau)$, which in this special case is given by

$$\tilde{L}_\tau v = \Delta v - 2\tau \nabla\varphi \cdot \nabla v + \tau^2 |\nabla\varphi|^2 v. \quad (14.2.5)$$

Consequently, operators (13.4.10a) and (13.4.10b) in the case of the Laplace operator, are, respectively, given by

$$\begin{aligned}S_\tau v &= \Delta v + \tau^2 |\nabla\varphi|^2 v, \\ A_\tau v &= -2\tau \nabla\varphi \cdot \nabla v.\end{aligned}$$

Of course,

$$\tilde{L}_\tau v = S_\tau v + A_\tau v,$$

which implies

$$\int \left| \tilde{L}_\tau v \right|^2 dx = \int |S_\tau v|^2 dx + \int |A_\tau v|^2 dx + 2 \int S_\tau v A_\tau v dx, \quad (14.2.6)$$

(for brevity, we omit the domain of integration). As we saw, a crucial point in the proof of Theorem 13.5.1 consists to handle the third integral to the right-hand side of (14.2.6), which we will pursue here using the identity (14.2.2), where

$$\beta = \nabla\varphi.$$

We have

$$2 \int (\nabla\varphi \cdot \nabla v) \Delta v dx = \int [\Delta\varphi |\nabla v|^2 - 2\partial_{jk}^2 \varphi \partial_j v \partial_k v] dx.$$

Integrating by parts, we get

$$\begin{aligned}
2 \int S_\tau v A_\tau v dx &= -2\tau \int (2(\nabla\varphi \cdot \nabla v)\Delta v + 2\tau^2|\nabla\varphi|^2(\nabla\varphi \cdot \nabla v)v) dx = \\
&= -2\tau \int (\Delta\varphi|\nabla v|^2 - 2\partial_{jk}^2\varphi\partial_j v\partial_k v + \tau^2|\nabla\varphi|^2\nabla\varphi \cdot \nabla(v^2)) dx = \\
&= -2\tau \int (\Delta\varphi|\nabla v|^2 - 2\partial_{jk}^2\varphi\partial_j v\partial_k v - 2\tau^2\partial_{jk}^2\varphi\partial_j\varphi\partial_k\varphi v^2 - \tau^2|\nabla\varphi|^2\Delta\varphi v^2) dx = \\
&= -2\tau \int (\Delta\varphi(|\nabla v|^2 - \tau^2|\nabla\varphi|^2 v^2) - 2\partial_{jk}^2\varphi\partial_j v\partial_k v - 2\tau^2(\partial_{jk}^2\varphi\partial_j\varphi\partial_k\varphi)v^2) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
2 \int S_\tau v A_\tau v dx &= 4\tau \int (\partial_{jk}^2\varphi\partial_j v\partial_k v + \tau^2(\partial_{jk}^2\varphi\partial_j\varphi\partial_k\varphi)v^2) dx - \\
&\quad - 2\tau \int \Delta\varphi(|\nabla v|^2 - \tau^2|\nabla\varphi|^2 v^2) dx := I.
\end{aligned} \tag{14.2.7}$$

At this point one could continue without involving the Fourier transform, but for this approach we refer to [10].

Now, for any $x_0 \in \overline{\Omega}$, the integral on the right-hand side (14.2.7), which we denoted by I , can be written as

$$I = I_{x_0} + R_{x_0},$$

where

$$\begin{aligned}
I_{x_0} &= 4\tau \int (\partial_{jk}^2\varphi(x_0)\partial_j v\partial_k v + \tau^2\partial_{jk}^2\varphi(x_0)\partial_j\varphi(x_0)\partial_k\varphi(x_0)v^2) dx - \\
&\quad - 2\tau \int \Delta\varphi(x_0)(|\nabla v|^2 - \tau^2|\nabla\varphi(x_0)|^2 v^2) dx,
\end{aligned}$$

and, of course,

$$R_{x_0} = I - I_{x_0}.$$

Now, for $\varepsilon > 0$ to be chosen, there exists $\rho_1 > 0$ such that

$$|R_{x_0}| \leq \varepsilon \int (\tau|\nabla v|^2 + \tau^3|v|^2) dx, \quad \forall v \in C_0^\infty(B_{\rho_1}(x_0) \cap \Omega).$$

Now by the Parseval identity we have

$$(2\pi)^n I_{x_0} = \int q(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi,$$

where

$$q(\xi, \tau) = 4\tau (\partial_{jk}^2 \varphi(x_0) \xi_j \xi_k + \tau^2 \partial_{jk}^2 \varphi(x_0) \partial_j \varphi(x_0) \partial_k(x_0)) - 2\tau \Delta \varphi(x_0) (|\xi|^2 - \tau^2 |\nabla \varphi(x_0)|^2).$$

Now (14.2.2) implies there exist positive constants C_1 and C_2 , such that

$$C_1 |\xi + i\tau \nabla \varphi(x_0)|^4 \leq \tau q(\xi, \tau) + C_2 \left| \sum_{j=1}^n (\xi + i\tau \partial_j \varphi(x_0))^2 \right|^2, \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1}.$$

(as in (13.5.16)). At this point the most challenging part of the proof is done and it is not difficult to put the together the various "pieces" as in the proof of Theorem 13.5.1, we invite the reader to do so. ■

Examples. Let us examine some example of function satisfying (14.2.3).

Example 1. Let $\varphi \in C^\infty(\bar{\Omega})$ satisfy $\nabla \varphi \neq 0$ in $\bar{\Omega}$ and let Ω be a bounded open set of \mathbb{R}^2 . We wish to prove that condition (14.2.3) holds for each and only the functions φ which satisfy

$$\Delta \varphi > 0, \quad \forall x \in \bar{\Omega}. \quad (14.2.8)$$

Let us denote

$$N(x) = \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|}$$

and

$$Q(x, \xi, \tau) = \sum_{j,k=1}^2 \partial_{jk}^2 \varphi(\xi_j \xi_k + N_j N_k).$$

It is clear that (14.2.3) is equivalent to the following condition

(★'') if

$$\begin{cases} |\xi|^2 = 1, \\ N(x) \cdot \xi = 0, \end{cases} \quad (14.2.9)$$

then

$$Q(x, \xi, \tau) > 0.$$

Let us suppose, then, that (14.2.9) is true.

Since $\nabla\varphi \neq 0$, we may assume $N_1 \neq 0$. Recalling that $|N| = 1$, by (14.2.9) we obtain

$$\begin{cases} \xi_1^2 + \xi_2^2 = 1, \\ \xi_1 N_1 + \xi_2 N_2 = 0, \end{cases} \iff \begin{cases} \xi_2^2 \left(\frac{N_2^2}{N_1^2} + 1 \right) = 1, \\ \xi_1 = -\frac{\xi_2 N_2}{N_1}, \end{cases} \iff \begin{cases} |\xi_2| = |N_1|, \\ \xi_1 = -\frac{\xi_2 N_2}{N_1}. \end{cases}$$

Let us suppose that $\xi_2 = |N_1|$, which implies $\xi_1 = -\frac{N_2}{N_1}|N_1|$, we have

$$\begin{aligned} Q &= \partial_1^2\varphi\xi_1^2 + 2\partial_{12}^2\varphi\xi_1\xi_2 + \partial_2^2\varphi\xi_2^2 + \partial_1^2\varphi N_1^2 + 2\partial_{12}^2\varphi N_1 N_2 + \partial_2^2\varphi N_2^2 = \\ &= \partial_1^2\varphi N_2^2 + 2\partial_{12}^2\varphi \left(-\frac{N_2}{N_1}|N_1| \right) |N_1| + \partial_2^2\varphi N_1^2 + \partial_1^2\varphi N_1^2 + 2\partial_{12}^2\varphi N_1 N_2 + \partial_2^2\varphi N_2^2 = \\ &= \partial_1^2\varphi - 2\partial_{12}^2\varphi N_1 N_2 + \partial_2^2\varphi + 2\partial_{12}^2\varphi N_1 N_2 = \Delta\varphi. \end{aligned}$$

If $\xi_2 = -|N_1|$, we get a similar result. Therefore (\star'') is equivalent to (14.2.8). ♠

Example 2. Let $\varphi = e^{\lambda\psi}$, where $\lambda \in \mathbb{R}$ and

$$|\nabla\psi(x)| \neq 0, \quad \forall x \in \bar{\Omega}.$$

Let us look at whether there are any λ for which (14.2.3) applies.

Let us calculate

$$\begin{aligned} \partial_j\varphi &= \lambda e^{\lambda\psi} \partial_j\psi, \\ \partial_{jk}^2\varphi &= \lambda e^{\lambda\psi} \partial_{jk}^2\psi + \lambda^2 e^{\lambda\psi} \partial_j\psi \partial_k\psi. \end{aligned}$$

Hence (14.2.3) becomes

$$\begin{cases} |\xi|^2 = \tau^2 \lambda^2 e^{2\lambda\psi} |\nabla\psi|^2, \\ \nabla\psi \cdot \xi = 0, \\ \tau \neq 0, \end{cases} \implies Q_\lambda(x, \xi, \tau) > 0, \quad (14.2.10)$$

where

$$\begin{aligned} Q_\lambda(x, \xi, \tau) &= \sum_{j,k=1}^n \lambda e^{\lambda\psi} \partial_{jk}^2\psi \xi_j \xi_k + \tau^2 \sum_{j,k=1}^n \lambda^3 e^{3\lambda\psi} \partial_{jk}^2\psi \partial_j\psi \partial_k\psi + \\ &+ \lambda^2 e^{\lambda\psi} (\nabla\psi \cdot \xi)^2 + \tau^2 \lambda^4 e^{3\lambda\psi} |\nabla\psi|^2. \end{aligned} \quad (14.2.11)$$

In order to examine (14.2.10) let us suppose

$$\begin{cases} |\xi|^2 = \tau^2 \lambda^2 e^{2\lambda\psi} |\nabla\psi|^2, \\ \nabla\psi \cdot \xi = 0, \\ \tau \neq 0, \end{cases} \quad (14.2.12)$$

from which we have $\nabla\psi \cdot \xi = 0$. Hence, if (14.2.12) holds true, we have

$$Q_\lambda = \sum_{j,k=1}^n \lambda e^{\lambda\psi} (\partial_{jk}^2 \psi \xi_j \xi_k + \lambda^2 \tau^2 e^{2\lambda\psi} \partial_{jk}^2 \psi \partial_j \psi \partial_k \psi) + \tau^2 \lambda^4 |\nabla\psi|^2 e^{3\lambda\psi}.$$

Set

$$\eta = \frac{\xi}{\tau \lambda e^{\lambda\psi} |\nabla\psi|}.$$

In this way (14.2.12) it is rewritten as

$$\begin{cases} |\eta|^2 = 1 \\ \nabla\psi \cdot \eta = 0 \end{cases} \implies \tilde{Q}_\lambda = \sum_{j,k=1}^n \lambda \left(\partial_{jk}^2 \psi \eta_j \eta_k + \partial_{jk}^2 \psi \frac{\partial_j \psi}{|\nabla\psi|} \frac{\partial_k \psi}{|\nabla\psi|} \right) + \lambda^2 > 0.$$

It is clear now that if $|\lambda|$ is sufficiently large, then (14.2.10) is satisfied. More precisely, set

$$M = \max_{|\eta|=1} \left| \sum_{j,k=1}^n \partial_{jk}^2 \psi \left(\eta_j \eta_k + \frac{\partial_j \psi}{|\nabla\psi|} \frac{\partial_k \psi}{|\nabla\psi|} \right) \right|,$$

we have that, if

$$|\lambda| > M$$

then condition (14.2.3) of Theorem 14.2.2 is satisfied. ♠

Example 3. Let us consider a radial function

$$\varphi(x) = f(|x|), \quad (14.2.13)$$

in $\Omega = B_1 \setminus \overline{B_r}$, $r \in (0, 1)$.

We would like to find some functions f such that φ satisfies condition (14.2.3). We will see that in some cases this is not possible. Let us proceed in a similar manner to the previous example. Let us calculate

$$\partial_j \varphi(x) = \frac{x_j}{|x|} f'(|x|),$$

$$\begin{aligned}\partial_{jk}^2\varphi(x) &= \left(\frac{\delta_{jk}}{|x|} - \frac{x_jx_k}{|x|^3}\right) f'(|x|) + \frac{x_jx_k}{|x|^2} f''(|x|), \\ \partial_{jk}^2\varphi(x)\xi_j\xi_k &= \left(\frac{|\xi|^2}{|x|} - \frac{(\xi \cdot x)^2}{|x|^3}\right) f'(|x|) + \frac{(\xi \cdot x)^2}{|x|^2} f''(|x|), \\ \partial_{jk}^2\varphi(x)\partial_j\varphi(x)\partial_k\varphi(x) &= \left(\frac{|\nabla\varphi(x)|^2}{|x|} - \frac{(\nabla\varphi(x) \cdot x)^2}{|x|^3}\right) f'(|x|) + \\ &\quad + \frac{(\nabla\varphi(x) \cdot x)^2}{|x|^2} f''(|x|).\end{aligned}$$

Since

$$|\nabla\varphi(x)|^2 = f'^2(|x|)$$

and

$$\nabla\varphi(x) \cdot x = |x|f'(|x|)$$

we have

$$\partial_{jk}^2\varphi(x)\partial_j\varphi(x)\partial_k\varphi(x) = (f'(|x|))^2 f''(|x|).$$

Then by (14.2.3) we can write Q as follows

$$Q = \left(\frac{|\xi|^2}{|x|} - \frac{(\xi \cdot x)^2}{|x|^3}\right) f'(|x|) + \frac{(\xi \cdot x)^2}{|x|^2} f''(|x|) + \tau^2 (f'(|x|))^2 f''(|x|).$$

Let us suppose that the antecedent of condition (14.2.3) holds, i.e., let us suppose that

$$\begin{cases} |\xi|^2 = \tau^2 |\nabla\varphi(x)|^2 = \tau^2 (f'(|x|))^2, \\ \tau \nabla\varphi(x) \cdot \xi = \tau \frac{(\xi \cdot x)^2}{|x|^2} f'(|x|) = 0, \\ \tau \neq 0, \end{cases}$$

namely

$$\begin{cases} |\xi|^2 = \tau^2 (f'(|x|))^2, \\ \xi \cdot x = 0, \\ \tau \neq 0, \end{cases}$$

Hence Q can be written as follows

$$Q = \frac{\tau^2 (f'(|x|))^2}{|x|} f'(|x|) + \tau^2 (f'(|x|))^2 f''(|x|).$$

We get

$$\tau^{-2}Q = \frac{(f'(|x|))^2}{|x|} f'(|x|) + (f'(|x|))^2 f''(|x|).$$

We characterize the functions

$$f : (0, 1) \rightarrow (0, +\infty)$$

which satisfy (14.2.3) and for which we have

$$\lim_{t \rightarrow 0} f(t) = +\infty$$

and

$$f' < 0, \quad \text{in } (0, 1).$$

The condition $Q > 0$, for $t \in (0, 1)$, becomes

$$Q = \frac{f'}{t} + f'' > 0.$$

To solve this differential inequality we set

$$f(t) = \psi(\log t),$$

from which we have

$$f'(t) = \psi'(\log t) \frac{1}{t},$$

$$f''(t) = \psi''(\log t) \frac{1}{t^2} - \psi'(\log t) \frac{1}{t^2}.$$

Hence the differential inequality can be written

$$\frac{f'}{t} + f'' = \frac{1}{t^2} \psi''(\log t) > 0 \iff \psi''(\log t) > 0.$$

Let $s = \log t$. Then, as $t \in (0, 1)$, $s \in (-\infty, 0)$, condition $Q > 0$ becomes:

$$\psi''(s) > 0 \iff \frac{d^2}{ds^2} (f(e^s)) > 0 \quad \forall s \in (-\infty, 0).$$

Let us observe that there are functions that do not satisfy this condition. For $\alpha > 0$, we consider functions $f(t)$ of the type

$$f(t) = \left(\log \frac{1}{t} \right)^\alpha.$$

Let us calculate

$$f'(t) = \alpha \left(-\frac{1}{t} \right) \left(\log \frac{1}{t} \right)^{\alpha-1},$$

$$f'(t) = \frac{\alpha}{t^2} \left(\log \frac{1}{t} \right)^{\alpha-1} + \frac{\alpha(\alpha-1)}{t^2} \left(\log \frac{1}{t} \right)^{\alpha-2},$$

from which it follows that the condition on Q can be written as

$$Q = \frac{f'}{t} + f'' = \frac{\alpha(\alpha-1)}{t^2} \left(\log \frac{1}{t} \right)^{\alpha-2} > 0.$$

Consequently if $\alpha < 1$, we have

$$\frac{f'}{t} + f'' < 0.$$

If $\alpha = 1$, we have

$$\varphi = \log \frac{1}{|x|},$$

Therefore

$$Q = 0.$$



Exercise. Prove that if $U \in H^2(B_1)$ is a solution to the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U = 0, \quad \text{in } B_1,$$

where $b \in L^\infty(B_1; \mathbb{R}^n)$ and $c \in L^\infty(B_1)$ then the following three sphere inequality holds true

$$\|U\|_{L^2(B_\varrho(0))} \leq C \|U\|_{L^2(B_r)}^\vartheta \|U\|_{L^2(B_1)}^{1-\vartheta}, \quad (14.2.14)$$

for $0 < r \leq \varrho \leq C^{-1}$, where $C \leq 1$ e $\vartheta \in (0, 1)$ are constants depending on $\|b\|_{L^\infty(B_1; \mathbb{R}^n)}$, $\|c\|_{L^\infty(B_1)}$, on ϱ and on r .

[Hint: apply Theorem 14.2.2 where φ is a suitable radial function.] ♣

14.3 Second order operators I – constant coefficients in the principal part

We now consider a more general case related to the operator (14.1.1). Let M_0 and let us M_1 be positive numbers given and let us assume that

$$\|g^{jk}\|_{L^\infty(\Omega)} \leq M_0, \quad \text{for } j, k = 1, \dots, n, \quad (14.3.1a)$$

$$|g^{jk}(x) - g^{jk}(y)| \leq M_1|x - y|, \quad \text{for } j, k = 1, \dots, n, \quad \forall x, y \in \Omega. \quad (14.3.1b)$$

Let us note that with conditions (14.3.1a) and (14.3.1b), operator (14.1.1) is not necessarily elliptic. As we noted in Section 13.1, to establish a Carleman estimate for operator (14.1.1), under assumption (14.3.1b), is equivalent to establish a Carleman estimate for the operator

$$L_g u = \partial_j (g^{jk}(x) \partial_k u). \quad (14.3.2)$$

As a matter of fact we have

$$(L_g - P_2)u = \partial_j (g^{jk}(x)) \partial_k u,$$

which is a first order operator with bounded coefficients.

We begin by establishing an identity analogous to (14.2.1). To this purpose we introduce some notations. We set

$$\xi^{(g)} = \{g^{jk}(x)\xi_k\}_{j=1}^n$$

and for any function v , sufficiently regular, we set

$$\nabla^{(g)} v = \{g^{jk}(x)\partial_k v\}_{j=1}^n.$$

We further set,

$$\mathbf{g}(\xi, \eta) = g^{jk}(x)\xi_j\eta_k.$$

Using these notations we have

$$L_g u = \operatorname{div} (\nabla^{(g)} u). \quad (14.3.3)$$

Let us notice that if $g^{jk}(x) = \delta^{jk}$ then $\mathbf{g}(\xi, \eta) = \xi \cdot \eta$ and $L_g = \Delta$ and, if

$$\{g^{jk}(x)\}_{j,k=1}^n = \operatorname{diag} (1, \dots, 1, -1),$$

then

$$\mathbf{g}(\xi, \eta) = \xi' \cdot \eta' - \xi_n \eta_n$$

and

$$L_g u = \Delta_{x'} u - \partial_n^2 u = \square u.$$

Lemma 14.3.1 (generalized Rellich identity). *Let $\beta \in C^{0,1}(\Omega, \mathbb{R}^n)$, $\beta = (\beta^1, \dots, \beta^n)$ and $v \in C^2(\Omega)$, then we have*

$$\begin{aligned} 2\mathbf{g}(\beta, \nabla v) L_g v &= \operatorname{div} (2\mathbf{g}(\beta, \nabla v) \nabla v - \mathbf{g}(\nabla v, \nabla v) \beta) + \\ &+ (\operatorname{div} \beta) \mathbf{g}(\nabla v, \nabla v) - 2\partial_l \beta^k g^{lj} \partial_j v \partial_k v + \\ &+ \beta^k (\partial_k g^{lj}) \partial_l v \partial_j v, \quad \text{a.e. } x \in \Omega \end{aligned} \quad (14.3.4)$$

and

$$\begin{aligned}
2 \int_{\Omega} \mathbf{g}(\beta, \nabla v) L_g v dx &= \int_{\Omega} (\operatorname{div} \beta) \mathbf{g}(\nabla v, \nabla v) - 2 \partial_l \beta^k g^{lj} \partial_j v \partial_k v + \\
&+ \int_{\Omega} \beta^k (\partial_k g^{lj}) \partial_l v \partial_j v dx,
\end{aligned} \tag{14.3.5}$$

Proof. First, we observe that identity (14.3.5) is an immediate consequence of (14.3.4) after its integration over Ω . Hence, it suffices to prove (14.3.4).

We have

$$\begin{aligned}
2 \mathbf{g}(\beta, \nabla v) L_g v &= 2(\beta^k \partial_k v) \partial_l (g^{lj} \partial_j v) = \\
&= 2 \partial_l (\beta^k \partial_k v g^{lj} \partial_j v) - 2 \partial_l (\beta^k \partial_k v) g^{lj} \partial_j v = \\
&= \operatorname{div} (2 \mathbf{g}(\beta, \nabla v) \nabla^{(g)} v) - 2 \partial_l (\beta^k \partial_k v) g^{lj} \partial_j v = \\
&= \operatorname{div} (2 \mathbf{g}(\beta, \nabla v) \nabla^{(g)} v) - 2 (\partial_l \beta^k v) g^{lj} \partial_k v \partial_j v - \\
&- 2 \beta^k g^{lj} \partial_{lk}^2 v \partial_j v.
\end{aligned} \tag{14.3.6}$$

Now, we notice that

$$\begin{aligned}
\partial_k (g^{lj} \partial_l v \partial_j v) &= g^{lj} \partial_{lk}^2 v \partial_j v + g^{lj} \partial_l v \partial_{jk}^2 v + \partial_k (g^{lj}) \partial_l v \partial_j v = \\
&= 2 g^{lj} \partial_{lk}^2 v \partial_j v + \partial_k (g^{lj}) \partial_l v \partial_j v,
\end{aligned}$$

from which we have

$$2 g^{lj} \partial_{lk}^2 v \partial_j v = \partial_k (g^{lj} \partial_l v \partial_j v) - \partial_k (g^{lj}) \partial_l v \partial_j v.$$

Using this identity, we transform the last term on the right-hand side of (14.3.6). We have

$$\begin{aligned}
-2 \beta^k g^{lj} \partial_{lk}^2 v \partial_j v &= -\beta^k \partial_k (g^{lj} \partial_l v \partial_j v) + \beta^k \partial_k g^{lj} \partial_l v \partial_j v = \\
&= -\partial_k (\beta^k g^{lj} \partial_l v \partial_j v) + (\operatorname{div} \beta) g^{lj} \partial_l v \partial_j v + \beta^k \partial_k g^{lj} \partial_l v \partial_j v.
\end{aligned}$$

Hence

$$\begin{aligned}
-2 \beta^k g^{lj} \partial_{lk}^2 v \partial_j v &= -\operatorname{div} (\mathbf{g}(\nabla v, \nabla v)) + \\
&+ (\operatorname{div} \beta) \mathbf{g}(\nabla v, \nabla v) + \beta^k \partial_k g^{lj} \partial_l v \partial_j v
\end{aligned}$$

and using the just obtain equality in (14.3.6) we obtain (14.3.4). \blacksquare

In the present Subsection we consider the **case of constant coefficients**. In this case the operator is given by

$$P_2(\partial) = g^{jk} \partial_{jk}^2, \quad (14.3.7)$$

where $g^{jk} = g^{kj}$ are **real constants**. We begin by giving some definitions whose geometric meaning will be explained later (see Section 14.6).

Definition 14.3.2 (pseudo-convex functions). Let Ω be a bounded open set of \mathbb{R}^n and let $\phi \in C^2(\overline{\Omega})$ satisfy

$$\nabla\phi(x) \neq 0, \quad \forall x \in \overline{\Omega}. \quad (14.3.8)$$

We say that ϕ is **pseudo-convex w.r.t. the operator** (14.3.7) in the point $x \in \overline{\Omega}$, if we have

$$\left\{ \begin{array}{l} P_2(\xi) = 0, \\ P_2^{(j)}(\xi) \partial_j \phi(x) = 0, \\ \xi \neq 0, \end{array} \right. \implies \partial_{jk}^2 \phi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) > 0. \quad (14.3.9)$$

We say that ϕ is **pseudo-convex w.r.t. operator** $P_2(\partial)$ if (14.3.9) holds true for every $x \in \overline{\Omega}$.

Remark. Using the notations introduced above, (14.3.9) can be written

$$\left\{ \begin{array}{l} \mathbf{g}(\xi, \xi) = 0, \\ \mathbf{g}(\xi, \nabla\phi(x)) = 0, \\ \xi \neq 0, \end{array} \right. \implies \partial^2 \phi(x) \xi^{(g)} \cdot \xi^{(g)} > 0. \quad (14.3.10)$$

Let us notice that **if the matrix** $\{g^{jk}\}_{j,k=1}^n$ **is singular do not exist any pseudo-convex functions** because there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\xi^g = g \cdot \xi = 0$. It should also be noticed that if the operator $P_2(\partial)$ is elliptic, then condition (14.3.9) is trivially satisfied since the antecedent is false. \blacklozenge

Definition 14.3.3 (strong pseudo-convex functions). Let Ω , ϕ and $P_2(\partial)$ be as in Definition 14.3.2. We say that ϕ is **strongly pseudo-convex w.r.t. the operator P_2** in the point $x \in \overline{\Omega}$ if ϕ is pseudo-convex w.r.t. P_2 and, further, we have

$$\left\{ \begin{array}{l} P_2(\xi + i\tau\nabla\phi(x)) = 0, \\ P_2^{(j)}(\xi + i\tau\nabla\phi(x))\partial_j\phi(x) = 0, \\ \tau \neq 0, \end{array} \right. \implies \quad (14.3.11)$$

$$\implies \partial_{jk}^2\phi(x)P_2^{(j)}(\xi + i\tau\nabla\phi(x))\overline{P_2^{(k)}(\xi + i\tau\nabla\phi(x))} > 0.$$

We say that ϕ is **strongly pseudo-convex w.r.t. the operator $P_2(\partial)$** , if it is strongly pseudo-convex w.r.t. the operator $P_2(\partial)$ in each point $x \in \overline{\Omega}$.

Remarks.

1. With the notations introduced above, (14.3.11) can be written

$$\left\{ \begin{array}{l} \mathbf{g}(\xi, \xi) = \tau^2\mathbf{g}(\nabla\phi(x), \nabla\phi(x)), \\ \mathbf{g}(\xi, \nabla\phi(x)) = 0, \\ \mathbf{g}(\nabla\phi(x), \nabla\phi(x)) = 0, \\ \tau \neq 0 \end{array} \right. \implies \quad (14.3.12)$$

$$\implies \partial^2\phi(x)\xi^{(g)} \cdot \xi^{(g)} + \tau^2\partial^2\phi(x)\nabla^{(g)}\phi(x) \cdot \nabla^{(g)}\phi(x) > 0.$$

2. As we will easily check, **in the case of the real coefficients we are considering, the definitions of pseudo-convexity and strong pseudo-convexity are equivalent.** As a matter of fact, if ϕ is strongly pseudo-convex with respect to $P_2(\partial)$, it is trivially pseudo-convex. We prove that if ϕ is pseudo-convex then it is strongly pseudo-convex. Let us suppose, hence, that (14.3.9) is satisfied in $x_0 \in \overline{\Omega}$ and let us prove that (14.3.12) is satisfied in x_0 . If $\mathbf{g}(\nabla\phi(x_0), \nabla\phi(x_0)) \neq 0$, then (14.3.12) is trivially satisfied as the antecedent of the implication (14.3.12) is false. If, on the other hand, we have

$$\mathbf{g}(\nabla\phi(x_0), \nabla\phi(x_0)) = 0,$$

then the antecedent of condition (14.3.12) becomes (in x_0)

$$\left\{ \begin{array}{l} \mathbf{g}(\xi, \xi) = 0, \\ \mathbf{g}(\xi, \nabla\phi(x_0)) = 0, \\ \mathbf{g}(\nabla\phi(x_0), \nabla\phi(x_0)) = 0, \\ \tau \neq 0. \end{array} \right. \quad (14.3.13)$$

Now, by the first two conditions in (14.3.13) and the pseudo-convexity of ϕ we have

$$\partial^2\phi(x_0)\xi^{(g)} \cdot \xi^{(g)} > 0. \quad (14.3.14)$$

Moreover, since $\mathbf{g}(\nabla\phi(x_0), \nabla\phi(x_0)) = 0$ and $\nabla\phi(x_0) \neq 0$, setting $\xi_0 = \nabla\phi(x_0)$ we get by (14.3.9) trivially

$$\left\{ \begin{array}{l} \mathbf{g}(\xi_0, \xi_0) = 0, \\ \mathbf{g}(\xi_0, \nabla\phi(x_0)) = 0, \\ \xi_0 \neq 0, \end{array} \right.$$

hence by (14.3.10) we have

$$\partial^2\phi(x_0)\nabla^{(g)}\phi(x_0) \cdot \nabla^{(g)}\phi(x_0) > 0$$

and taking into account that (by (14.3.13)) $\tau \neq 0$, we have

$$\tau^2\partial^2\phi(x_0)\nabla^{(g)}\phi(x_0) \cdot \nabla^{(g)}\phi(x_0) > 0. \quad (14.3.15)$$

Now, by (14.3.14) and (14.3.15) we have

$$\partial^2\phi(x_0)\xi^{(g)} \cdot \xi^{(g)} + \tau^2\partial^2\phi(x_0)\nabla^{(g)}\phi(x_0) \cdot \nabla^{(g)}\phi(x_0) > 0.$$

Hence, we have proved that (14.3.12) is satisfied and, therefore, we have the equivalence of the definitions 14.3.2 and 14.3.3

Let us note that **in the elliptic case**, each $\phi \in C^2(\overline{\Omega})$ such that

$$\nabla\phi(x) \neq 0, \quad \forall x \in \overline{\Omega}$$

is trivially pseudo-convex (hence, it is strongly pseudo-convex). \blacklozenge

Warning about definitions 14.3.2 and 14.3.3. It is important to point out that generally the definitions of pseudo-convexity and strong pseudo-convexity are referred to the **level surfaces** $\{\phi(x) = \phi(x_0)\}$, where $x_0 \in \Omega$. And using the term "surface" we also want to emphasize the invariant character of the definition (see [34, §8.6]). Our modification of the terminology is only due to the purpose of to lighten the exposition a little. \blacktriangle

To obtain the Carleman estimate of Theorem 14.3.7 (see below) we need a condition more stringent than the strong pseudo-convexity. This condition, has the same form of condition (\star) of Theorem 13.5.1.

Definition 14.3.4. Let Ω , ϕ and P_2 as in Definition 14.3.3 We say that ϕ satisfies condition (**S**) w.r.t. the operator $P_2(\partial)$, if ϕ is pseudo-convex w.r.t. $P_2(\partial)$ and we have

$$\begin{aligned} \begin{cases} P_2(\xi + i\tau\nabla\phi(x)) = 0, \\ \tau \neq 0, \end{cases} & \implies \\ & \implies \partial_{jk}^2\phi(x)P_2^{(j)}(\xi + i\tau\nabla\phi(x))\overline{P_2^{(k)}(\xi + i\tau\nabla\phi(x))} > 0. \end{aligned} \quad (14.3.16)$$

Remark. With the notations introduced above (14.3.11), we write condition (14.3.16) as follows

$$\begin{aligned} \begin{cases} \mathbf{g}(\xi, \xi) = \tau^2\mathbf{g}(\nabla\phi(x), \nabla\phi(x)), \\ \mathbf{g}(\xi, \nabla\phi(x)) = 0, \\ \tau \neq 0, \end{cases} & \implies \\ & \implies Q_\phi := \partial^2\phi(x)\xi^{(g)} \cdot \xi^{(g)} + \tau^2\partial^2\phi(x)\nabla^{(g)}\phi(x) \cdot \nabla^{(g)}\phi(x) > 0. \end{aligned} \quad (14.3.17)$$

\blacklozenge

It is evident that if ϕ satisfies condition (**S**) then it is strongly pseudo-convex. However, the converse does not hold. Let us consider, for instance, the function $\phi(x) = \log \frac{1}{|x|}$; this function is strongly pseudo-convex, but, as we saw in **Example 3** of this Section, it does not satisfy condition (**S**).

However the following Proposition holds

Proposition 14.3.5. *Let Ω be a bounded open set of \mathbb{R}^n and let $\phi \in C^2(\overline{\Omega})$ strongly pseudo-convex w.r.t. operator (14.3.7), then $\varphi = e^{\lambda\phi}$ satisfies condition (S) if λ is large enough.*

Proof. If $P_2(\partial)$ is elliptic we can readily reduce to what we have done in Example 2. Hence let us suppose that $P_2(\partial)$ is **not elliptic**. We first prove that $\varphi = e^{\lambda\phi}$ is pseudo-convex. Let us calculate

$$\nabla\varphi = \lambda e^{\lambda\phi} \nabla\phi,$$

$$\partial_{jk}^2\varphi = \lambda e^{\lambda\phi} \partial_{jk}^2\phi + \lambda^2 e^{\lambda\phi} \partial_j\phi \partial_k\phi, \quad j, k = 1, \dots, n$$

and

$$\begin{aligned} \partial_{jk}^2\varphi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) &= \\ &= \lambda e^{\lambda\phi} \left[\partial_{jk}^2\phi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) + \lambda \left| P_2^{(j)}(\xi) \partial_j\phi \right|^2 \right]. \end{aligned} \quad (14.3.18)$$

Set

$$X = \overline{\Omega} \times \{ \xi \in \mathbb{R}^n : P_2(\xi) = 0, \quad |\xi| = 1 \}.$$

Since $P_2(\partial)$ is not elliptic we have that X is a nonempty compact subset of \mathbb{R}^{2n} . Now, by the definition of pseudo-convexity in $\overline{\Omega}$ we have

$$(x, \xi) \in X, \quad P_2^{(j)}(\xi) \partial_j\phi(x) = 0 \implies \partial_{jk}^2\phi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) > 0$$

and this, by Lemma 12.5.2, implies that there exists $\lambda_0 > 0$ such that

$$\lambda_0 \left| P_2^{(j)}(\xi) \partial_j\phi(x) \right|^2 + \partial_{jk}^2\phi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) > 0, \quad (14.3.19)$$

for every $(x, \xi) \in X$.

Inequality (14.3.19), in turn, implies (taking into account that the polynomial (in ξ) on the left-hand side is homogeneous of degree 2)

$$\lambda \left| P_2^{(j)}(\xi) \partial_j\phi(x) \right|^2 + \partial_{jk}^2\phi(x) P_2^{(j)}(\xi) P_2^{(k)}(\xi) > 0 \quad (14.3.20)$$

for every $x \in \overline{\Omega}$, for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and for every $\lambda \geq \lambda_0$. By (14.3.18) and (14.3.20) we have that (14.3.9) is satisfied by the function φ .

Now we prove (14.3.16). Let us introduce the following notation. For every $\xi \in \mathbb{R}^n$, for every $t \in \mathbb{R}$ and for every function $f \in C^2(\overline{\Omega})$ such that $\nabla f \neq 0$, in $\overline{\Omega}$, set

$$\zeta_{t,f} = \xi + it\nabla f.$$

Similarly to (14.3.18) we have

$$\begin{aligned} \partial_{jk}^2 \varphi P_2^{(j)}(\zeta_{\tau,\varphi}) \overline{P_2^{(k)}(\zeta_{\tau,\varphi})} &= \\ &= \lambda e^{\lambda\phi} \left[\partial_{jk}^2 \phi(x) P_2^{(j)}(\zeta_{\tau\lambda,\phi}) \overline{P_2^{(k)}(\zeta_{\tau\lambda,\phi})} + \lambda \left| P_2^{(j)}(\zeta_{\tau\lambda,\phi}) \partial_j \phi \right|^2 \right]. \end{aligned} \quad (14.3.21)$$

Set

$$X_1 = \{(x, \xi, t) \in \overline{\Omega} \times \mathbb{R}^n : P_2(\zeta_{t,\phi(x)}) = 0, \quad |\zeta_{t,\phi(x)}| = 1\}. \quad (14.3.22)$$

It turns out that $X_1 \neq \emptyset$ and that X_1 is a compact of \mathbb{R}^{2n+1} (since $\nabla\phi(x) \neq 0$ in $\overline{\Omega}$). We now check that we have

$$\begin{aligned} (x, \xi, t) \in X_1, \quad P_2^{(j)}(\zeta_{t,\phi(x)}) \partial_j \phi(x) = 0 &\implies \\ \implies \partial_{jk}^2 \phi(x) P_2^{(j)}(\zeta_{t,\phi(x)}) \overline{P_2^{(k)}(\zeta_{t,\phi(x)})} &> 0. \end{aligned} \quad (14.3.23)$$

As a matter of fact, for $t = 0$, (14.3.23) is nothing but (14.3.9) and so it is satisfied. Now, if $t \neq 0$ then (14.3.23) is satisfied because ϕ is strongly pseudo-convex. Hence by Lemma 12.5.2, there exists $\lambda_1 \geq \lambda_0$ such that

$$\lambda_1 \left| P_2^{(j)}(\zeta_{t,\phi(x)}) \partial_j \phi(x) \right|^2 + \partial_{jk}^2 \phi(x) P_2^{(j)}(\zeta_{t,\phi(x)}) \overline{P_2^{(k)}(\zeta_{t,\phi(x)})} > 0,$$

for every $(x, \xi, t) \in X_1$, which in turn implies

$$\lambda \left| P_2^{(j)}(\zeta_{t,\phi(x)}) \partial_j \phi(x) \right|^2 + \partial_{jk}^2 \phi(x) P_2^{(j)}(\zeta_{t,\phi(x)}) \overline{P_2^{(k)}(\zeta_{t,\phi(x)})} > 0, \quad (14.3.24)$$

for every $\lambda \geq \lambda_1$, for every $x \in \overline{\Omega}$ and for every $(\xi, t) \in \mathbb{R}^n \setminus \{(0, 0)\}$. Trivially (14.3.24) holds true for $t = \tau\lambda e^{\lambda\phi(x)}$, $\tau \neq 0$. Hence, recalling (14.3.21), we have that φ satisfies the condition

$$\begin{aligned} \left\{ \begin{array}{l} P_2(\xi + i\tau\nabla\varphi(x)) = 0, \\ \tau \neq 0, \end{array} \right. &\implies \\ \implies \partial_{jk}^2 \varphi(x) P_2^{(j)}(\xi + i\tau\nabla\varphi(x)) \overline{P_2^{(k)}(\xi + i\tau\nabla\varphi(x))} &> 0. \end{aligned} \quad (14.3.25)$$

The proof is concluded. ■

In the sequel we will need some notations and a Lemma.

Let $k \in \mathbb{R}$ and let $N \in \mathbb{R}^n \setminus \{0\}$. For any $f \in C_0^\infty(\Omega)$, let us denote by

$$\|f\|_{k,\tau}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi + i\tau N|^{2k} |\widehat{f}(\xi)|^2 d\xi. \quad (14.3.26)$$

Remark. By (14.3.26) and by the Parseval identity we have

$$\|f\|_{-1,\tau} \leq |\tau N|^{-1} \|f\|_{L^2(\Omega)}, \quad (14.3.27)$$

for every $f \in C_0^\infty(\Omega)$. ◆

The following Lemma holds

Lemma 14.3.6. *Let $\rho > 0$ and let h be a Lipschitz continuous function defined in $B_\rho(x_0)$. Let*

$$A = [h]_{0,1,\overline{B_\rho(x_0)}},$$

the Lipschitz constant of h . Let us suppose that

$$h(x_0) = 0.$$

We have, for every $w \in C_0^\infty(B_\rho(x_0))$

$$\|h(\partial_j w - \tau N_j w)\|_{-1,\tau} \leq A(\rho + |\tau N|^{-1}) \|w\|_{L^2(B_\rho(x_0))}. \quad (14.3.28)$$

Proof. We have

$$h(x)(\partial_j w - \tau N_j w) = (\partial_j - \tau N_j)(hw) - w\partial_j h.$$

Hence

$$\|h(\partial_j - \tau N_j w)\|_{-1,\tau} \leq \|(\partial_j - \tau N_j)(hw)\|_{-1,\tau} + \|w\partial_j h\|_{-1,\tau}. \quad (14.3.29)$$

Let us notice that

$$|h(x)| = |h(x) - h(x_0)| \leq A\rho, \quad \forall x \in B_\rho(x_0). \quad (14.3.30)$$

Now, the triangle inequality and (14.3.30) yield

$$\begin{aligned}
& \|(\partial_j w - \tau N_j)(hw)\|_{-1,\tau}^2 = \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{|\xi_j + i\tau N_j|^2 |\widehat{hw}(\xi)|^2}{|\xi + i\tau N|^2} d\xi \leq \\
&\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\widehat{hw}(\xi)|^2 d\xi = \\
&= \int_{\mathbb{R}^n} |h(x)w(x)|^2 dx \leq A^2 \rho^2 \|w\|_{L^2(B_\rho(x_0))}^2.
\end{aligned} \tag{14.3.31}$$

On the other hand

$$\begin{aligned}
\|w\partial_j h\|_{-1,\tau}^2 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{|\widehat{w\partial_j h}(\xi)|^2}{|\xi + i\tau N|^2} d\xi \leq \\
&\leq \frac{|\tau N|^{-2}}{(2\pi)^n} \|\widehat{w\partial_j h}\|_{L^2(\mathbb{R}^n)}^2 = \\
&= |\tau N|^{-2} \|w\partial_j h\|_{L^2(B_\rho(x_0))}^2 \leq \\
&\leq A^2 |\tau N|^{-2} \|w\|_{L^2(B_\rho(x_0))}^2.
\end{aligned} \tag{14.3.32}$$

Therefore, by (14.3.29), (14.3.31) and (14.3.32) we get (14.3.28). ■

We now state and prove the following

Theorem 14.3.7. *Let Ω be a bounded open set of \mathbb{R}^n and let*

$$P_2(\partial) = g^{jk} \partial_{jk}^2,$$

where the matrix $\{g^{jk}\}_{j,k=1}^n$ is real, constant and symmetric. Let us suppose that $\varphi \in C^2(\bar{\Omega})$ satisfies condition **(S)** w.r.t. $P_2(\partial)$.

Then there exist constants C and τ_0 such that

$$\tau^3 \int_{\Omega} |u|^2 e^{2\tau\varphi} dx + \tau \int_{\Omega} |\nabla u|^2 e^{2\tau\varphi} dx \leq C \int_{\Omega} |P_2(\partial)u|^2 e^{2\tau\varphi} dx, \tag{14.3.33}$$

for every $u \in C_0^\infty(\Omega)$ and for every $\tau \geq \tau_0$.

Proof. Let us denote

$$L = P_2(\partial).$$

Let $u \in C_0^\infty(\Omega)$ and set $v = e^{\tau\varphi}u$. We have

$$\begin{aligned}
P_{2,\tau}v &:= e^{\tau\varphi}L(e^{-\tau\varphi}v) = \\
&= L(v) + \tau^2 \mathbf{g}(\nabla\varphi, \nabla\varphi)v - 2\tau \mathbf{g}(\nabla\varphi, \nabla v) - \tau L(\varphi)v.
\end{aligned}$$

let us define

$$S_\tau v = L(v) + \tau^2 \mathbf{g}(\nabla \varphi, \nabla \varphi) v$$

and

$$A_\tau v = -2\tau \mathbf{g}(\nabla \varphi, \nabla v)$$

and set

$$p_2(x, \partial, \tau)v = S_\tau v + A_\tau v, \quad (14.3.34)$$

Let us note that

$$P_{2,\tau} v - p_2(x, \partial, \tau)v = -\tau L(\varphi)v. \quad (14.3.35)$$

In what follows, for the sake of brevity, we will omit the domain and the element of integration.

Let μ be a positive number which we will choose later, we have trivially

$$\begin{aligned} & \int |p_2(x, \partial, \tau)v|^2 e^{2\mu\varphi} = \\ & = \int |S_\tau v|^2 e^{2\mu\varphi} + \int |A_\tau v|^2 e^{2\mu\varphi} + \\ & + 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} \geq \\ & \geq 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} = \\ & = -2\tau \int 2(L(v) + \tau^2 \mathbf{g}(\nabla \varphi, \nabla \varphi)v) \mathbf{g}(\nabla \varphi, \nabla v) e^{2\mu\varphi} := \\ & := -2\tau J_1 - 2\tau^3 J_2, \end{aligned} \quad (14.3.36)$$

where

$$J_1 = \int 2L(v) \mathbf{g}(\nabla \varphi, \nabla v) e^{2\mu\varphi} \quad (14.3.37)$$

and

$$J_2 = \int 2\mathbf{g}(\nabla \varphi, \nabla \varphi) \mathbf{g}(\nabla \varphi, \nabla v) v e^{2\mu\varphi}. \quad (14.3.38)$$

Let us examine J_1 .

Let us apply (14.3.4) with

$$\beta = e^{2\mu\varphi} \nabla^{(g)} \varphi.$$

We get

$$\begin{aligned}
J_1 &= \int (\operatorname{div} (e^{2\mu\varphi} \nabla^{(g)} \varphi)) \mathbf{g}(\nabla v, \nabla v) - \\
&\quad - 2\partial_l (e^{2\mu\varphi} g^{kj} \partial_j \varphi) g^{ls} \partial_s v \partial_k v = \\
&= \int (e^{2\mu\varphi} (L(\varphi)) \mathbf{g}(\nabla v, \nabla v) + 2\mu e^{2\mu\varphi} \mathbf{g}(\nabla \varphi, \nabla \varphi) \mathbf{g}(\nabla v, \nabla v) - \\
&\quad - 2e^{2\mu\varphi} g^{kj} \partial_{ij}^2 \varphi g^{ls} \partial_s v \partial_k v - 4\mu e^{2\mu\varphi} \partial_l \varphi g^{kj} \partial_j \varphi g^{ls} \partial_s v \partial_k v) = \\
&= \int e^{2\mu\varphi} ((L(\varphi)) \mathbf{g}(\nabla v, \nabla v) - 2\partial^2 \varphi \nabla^{(g)} v \cdot \nabla^{(g)} v + \\
&\quad + 2\mu \mathbf{g}(\nabla \varphi, \nabla \varphi) \mathbf{g}(\nabla v, \nabla v) - 4\mu (\mathbf{g}(\nabla \varphi, \nabla v))^2).
\end{aligned} \tag{14.3.39}$$

Let us examine J_2 .

$$\begin{aligned}
J_2 &= \int 2e^{2\mu\varphi} \mathbf{g}(\nabla \varphi, \nabla \varphi) \mathbf{g}(\nabla \varphi, \nabla v) v = \\
&= \int e^{2\mu\varphi} \mathbf{g}(\nabla \varphi, \nabla \varphi) g^{ls} \partial_l \varphi \partial_s (v^2) = \\
&= - \int \partial_s (e^{2\mu\varphi} \mathbf{g}(\nabla \varphi, \nabla \varphi) g^{ls} \partial_l \varphi) v^2 = \\
&= - \int e^{2\mu\varphi} (2\partial^2 \varphi \nabla^{(g)} \varphi \cdot \nabla^{(g)} \varphi + \mathbf{g}(\nabla \varphi, \nabla \varphi) L(\varphi) + 2\mu (\mathbf{g}(\nabla \varphi, \nabla \varphi))^2) v^2.
\end{aligned}$$

By the above obtained equality and by (14.3.39) we have

$$\begin{aligned}
2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} &= -2\tau J_1 - 2\tau^3 J_2 = \\
&= 2\tau \int e^{2\mu\varphi} q_\mu(x, v, \nabla v, \tau),
\end{aligned} \tag{14.3.40}$$

where

$$q_\mu(x, v, \nabla v, \tau) = q_0(x, v, \nabla v, \tau) + \mu q_1(x, v, \nabla v, \tau), \tag{14.3.41}$$

$$\begin{aligned}
q_0(x, v, \nabla v, \tau) &= \\
&= 2 [\partial^2 \varphi(x) \nabla^{(g)} v \cdot \nabla^{(g)} v + \tau^2 \partial^2 \varphi(x) \nabla^{(g)} \varphi(x) \cdot \nabla^{(g)} \varphi(x) v^2] + \\
&\quad + L(\varphi) [\tau^2 \mathbf{g}(\nabla \varphi(x), \nabla \varphi(x)) v^2 - \mathbf{g}(\nabla v, \nabla v)],
\end{aligned} \tag{14.3.42}$$

and

$$\begin{aligned} q_1(x, v, \nabla v, \tau) &= \\ &= -2\mathbf{g}(\nabla\varphi(x), \nabla\varphi(x))\mathbf{g}(\nabla v, \nabla v) + 4(\mathbf{g}(\nabla\varphi(x), \nabla v))^2 + \\ &+ 2\tau^2(\mathbf{g}(\nabla\varphi(x), \nabla\varphi(x)))^2 v^2. \end{aligned} \quad (14.3.43)$$

Trivially, for any $x_0 \in \overline{\Omega}$, the right-hand side of (14.3.40) can be written

$$2\tau \int e^{2\mu\varphi} q_\mu(x, v, \nabla v, \tau) = I_{x_0} + R_{x_0},$$

where

$$I_{x_0} = 2\tau e^{2\mu\varphi(x_0)} \int q_\mu(x_0, v, \nabla v, \tau), \quad (14.3.44)$$

and

$$R_{x_0} = 2\tau \int (e^{2\mu\varphi(x)} q_\mu(x, v, \nabla v, \tau) - e^{2\mu\varphi(x_0)} q_\mu(x_0, v, \nabla v, \tau)). \quad (14.3.45)$$

Now, let $\rho \in (0, 1]$, to be chosen. We get easily

$$|R_{x_0}| \leq C\rho(\mu + 1)^2 e^{2\mu\Phi_0} \int (\tau|\nabla v|^2 + \tau^3|v|^2) dx, \quad (14.3.46)$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$, where

$$\Phi_0 = \max_{x \in \overline{\Omega}} |\varphi|,$$

C does not depend by μ , but depends on the $C^2(\overline{\Omega})$ norm of φ and by

$$M_0 = \max_{1 \leq j, k \leq n} |g^{jk}|.$$

Now, the Parseval identity gives

$$I_{x_0} = \frac{2\tau e^{2\mu\varphi(x_0)}}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi, \quad (14.3.47)$$

where

$$Q_\mu(\xi, \tau) = Q_0(\xi, \tau) + \mu Q_1(\xi, \tau), \quad (14.3.48)$$

$$\begin{aligned} Q_0(\xi, \tau) &= 2[\partial^2\varphi(x_0)\xi^{(g)} \cdot \xi^{(g)} + \tau^2\partial^2\varphi(x_0)\nabla^{(g)}\varphi(x_0) \cdot \nabla^{(g)}\varphi(x_0)] + \\ &+ L(\varphi)(x_0) [\tau^2\mathbf{g}(\nabla\varphi(x_0), \nabla\varphi(x_0)) - \mathbf{g}(\xi, \xi)] \end{aligned}$$

and

$$Q_1(\xi, \tau) = -2\mathbf{g}(\nabla\varphi(x_0), \nabla\varphi(x_0)) (\mathbf{g}(\xi, \xi) - \tau^2\mathbf{g}(\nabla\varphi(x_0), \nabla\varphi(x_0))) + 4(\mathbf{g}(\nabla\varphi(x_0), \xi))^2.$$

Claim.

There exist C_1, C_2 and μ , positive number, such that

$$|\xi + i\tau\nabla\varphi(x_0)|^2 \leq C_1 Q_\mu(\xi, \tau) + C_2 \frac{|P_2(\xi + i\tau\nabla\varphi(x_0))|^2}{|\xi + i\tau\nabla\varphi(x_0)|^2}, \quad (14.3.49)$$

for every $(\xi, \tau) \in \mathbb{R}^{n+1}$.

Proof of the Claim.

If P_2 is elliptic, we can proceed in a manner similar to what we did in the proof of Theorem 13.5.1. For completeness we provide the proof. First, we notice that if $\xi = 0$ then (14.3.16)–(14.3.17) are trivially satisfied in x_0 , because P_2 is elliptic and if we had $\xi = 0$ we would have in the first condition of (14.3.16)–(14.3.17), $P_2(i\tau\nabla\varphi(x_0)) = 0$ from which we would have $\tau = 0$ arriving to a contradiction.

Then for $\xi = 0$ the antecedent of (14.3.16)–(14.3.17) is false. Now, we choose $\mu = 0$ and set

$$\tilde{\Sigma} = \{(\xi, \tau) \in \mathbb{R}^{n+1} : |\xi + i\tau\nabla\varphi(x_0)| = 1\}.$$

By (14.3.16) we have

$$(\xi, \tau) \in \tilde{\Sigma}, \quad P_2(\xi + i\tau\nabla\varphi(x_0)) = 0 \implies Q_0(\xi, \tau) > 0.$$

Hence, by Lemma 12.5.2 we have that there exists $B > 0$ such that

$$\frac{B |P_2(\xi + i\tau\nabla\varphi(x_0))|^2}{|\xi + i\tau\nabla\varphi(x_0)|^2} + Q_0(\xi, \tau) > 0, \quad \forall (\xi, \tau) \in \tilde{\Sigma} \quad (14.3.50)$$

and, by homogeneity, (14.3.49) follows, with $\mu = 0$.

If P_2 is not elliptic, the set

$$K_1 = \{\xi \in \mathbb{R}^n : \mathbf{g}(\xi, \xi) = 0, \quad |\xi| = 1\},$$

is nonempty. Let us denote

$$K_2 = \{\xi \in K_1 : \mathbf{g}(\xi, \nabla\varphi(x_0)) = 0\}.$$

If $K_2 = \emptyset$, then we have trivially

$$|\mathbf{g}(\xi, \nabla\varphi(x_0))| > 0, \quad \forall \xi \in K_1$$

hence, by the compactness of K_1 , there exists $m_1 > 0$ such that

$$|\mathbf{g}(\xi, \nabla\varphi(x_0))| \geq m_1, \quad \forall \xi \in K_1.$$

Now we set

$$C_0 = 1 + \max_{|\xi|=1} |\partial^2\varphi(x_0)\xi^{(g)} \cdot \xi^{(g)}|$$

and we have, for any $\xi \in K_1$ and

$$\mu = \mu_0 := \frac{C_0}{m_1^2},$$

$$\begin{aligned} Q_\mu(\xi, 0) &= 2\partial^2\varphi(x_0)\xi^{(g)} \cdot \xi^{(g)} + 4\mu (\mathbf{g}(\nabla\varphi(x_0), \xi))^2 \geq \\ &\geq -2C_0 + 4\mu m_1^2 \geq 2C_0. \end{aligned} \quad (14.3.51)$$

Let us fix $\mu = \mu_0$ and let $(\xi, \tau) \in \tilde{\Sigma}$ satisfy

$$P_2(\xi + i\tau\nabla\varphi(x_0)) = 0;$$

we have what follows:

- (a) if $\tau = 0$ then we have, by (14.3.51), $Q_\mu(\xi, 0) > 0$;
- (b) if $\tau \neq 0$, then we have, by (14.3.16), $Q_\mu(\xi, \tau) > 0$.

Hence

$$(\xi, \tau) \in \tilde{\Sigma}, \quad P_2(\xi + i\tau\nabla\varphi(x_0)) = 0 \implies Q_\mu(\xi, \tau) > 0$$

and by Lemma 12.5.2 we derive that there exists $B > 0$ such that

$$\frac{B|P_2(\xi + i\tau\nabla\varphi(x_0))|^2}{|\xi + i\tau\nabla\varphi(x_0)|^2} + Q_\mu(\xi, \tau) > 0, \quad \forall (\xi, \tau) \in \tilde{\Sigma}. \quad (14.3.52)$$

Hence, if $K_2 = \emptyset$, then (14.3.49) is satisfied for $\mu = \mu_0$.

Now, let us suppose that $K_2 \neq \emptyset$. By (14.3.10) we have

$$\xi \in K_2 \implies Q_0(\xi, 0) > 0. \quad (14.3.53)$$

Since Q_0 is continuous and K_2 is compact, there exists $\delta_1 > 0$ such that

$$\xi \in K_2 \implies Q_0(\xi, 0) \geq \delta_1. \quad (14.3.54)$$

By compactness of K_2 , by continuity of Q_0 and by (14.3.54) it follows that there exists $d_0 > 0$ such that

$$\xi \in K_1, \quad |\mathbf{g}(\xi, \nabla\varphi(x_0))| \leq d_0 \implies Q_0(\xi, 0) \geq \frac{\delta_1}{2}. \quad (14.3.55)$$

Let us denote, like before,

$$C_0 = 1 + \max_{|\xi|=1} |\partial^2\varphi(x_0)\xi^{(g)} \cdot \xi^{(g)}|.$$

We notice that if

$$\xi \in K_1 \quad \text{e} \quad |\mathbf{g}(\xi, \nabla\varphi(x_0))| \geq d_0,$$

then, for any $\mu \geq \frac{C_0}{d_0^2}$ we have

$$\begin{aligned} Q_\mu(\xi, 0) &= 2\partial^2\varphi(x_0)\xi^{(g)} \cdot \xi^{(g)} + 4\mu (\mathbf{g}(\nabla\varphi(x_0), \xi))^2 \geq \\ &\geq -2C_0 + 4\mu d_0^2 \geq 2C_0 > 0. \end{aligned} \quad (14.3.56)$$

Now, let us fix

$$\mu = \mu_1 := \frac{C_0}{d_0^2}$$

and by (14.3.55) and (14.3.56) we obtain

$$\xi \in K_1 \implies Q_\mu(\xi, 0) \geq \delta_2 > 0, \quad (14.3.57)$$

where

$$\delta_2 = \min \left\{ 2C_0, \frac{\delta_1}{2} \right\}.$$

From now on, we proceed as we already did to prove the (14.3.52). **The proof of the Claim is concluded.**

From now on, we **fix a value** μ for which (14.3.49) is satisfied (recalling that, however, μ depends on x_0). Set $N = \nabla\varphi(x_0)$. Using the notations introduced in (14.3.26), we have by (14.3.49)

$$\|v\|_{1,\tau}^2 \leq \frac{C_1}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi + C_2 \|p_2(x_0, \partial, \tau)v\|_{-1,\tau}^2, \quad (14.3.58)$$

where $p_2(x, \partial, \tau)$ is defined in (14.3.34). We have easily

$$|p_2(x, \partial, \tau)v - p_2(x_0, \partial, \tau)v| \leq C\rho (\tau|\nabla v| + \tau^2|v|), \quad (14.3.59)$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$. Now, by (14.3.27), (14.3.59) and by the triangle inequality, we get

$$\begin{aligned} \|p_2(x_0, \partial, \tau)v\|_{-1, \tau} &\leq \frac{1}{\tau|\nabla\varphi(x_0)|} \|p_2(x_0, \partial, \tau)v\|_{L^2(\Omega)} \leq \\ &\leq \frac{1}{\tau|\nabla\varphi(x_0)|} \left(\|p_2(\cdot, \partial, \tau)v - p_2(x_0, \partial, \tau)v\|_{L^2(\Omega)} + \|p_2(\cdot, \partial, \tau)v\|_{L^2(\Omega)} \right) \leq \\ &\leq \frac{1}{\tau|\nabla\varphi(x_0)|} \left[C\rho\tau \|\tau|\nabla v| + \tau|v|\|_{L^2(\Omega)} + \|p_2(\cdot, \partial, \tau)v\|_{L^2(\Omega)} \right] \leq \\ &\leq C\rho \|v\|_{1, \tau} + \frac{1}{\tau|\nabla\varphi(x_0)|} \|p_2(\cdot, \partial, \tau)v\|_{L^2(\Omega)}. \end{aligned}$$

In sum, we have that

$$\|p_2(x_0, \partial, \tau)v\|_{-1, \tau} \leq C \left(\rho \|v\|_{1, \tau} + \frac{1}{\tau} \|p_2(\cdot, \partial, \tau)v\|_{L^2(\Omega)} \right), \quad (14.3.60)$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$, where C depends on $C^1(\overline{\Omega})$ norm of φ , on M_0 and on

$$m_2 = \min_{\overline{\Omega}} |\nabla\varphi|.$$

By (14.3.60) and by (14.3.58) we obtain

$$\begin{aligned} 2\tau \|v\|_{1, \tau}^2 &\leq \frac{2\tau C_1}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi + \\ &\quad + 2\tau C\rho^2 \|v\|_{1, \tau}^2 + C\tau^{-1} \|p_2(\cdot, \partial, \tau)v\|_{L^2(\Omega)}^2, \end{aligned} \quad (14.3.61)$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$.

Now we need to estimate the first term on the right-hand side in (14.3.61). From (14.3.44), (14.3.45) and (14.3.48) we have

$$\begin{aligned}
& \frac{2\tau}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi = 2\tau \int q_\mu(x_0, v, \nabla v, \tau) dx = \\
& = 2\tau e^{-2\mu\varphi(x_0)} \int e^{2\mu\varphi(x_0)} q_\mu(x_0, v, \nabla v, \tau) dx = \\
& = 2\tau e^{-2\mu\varphi(x_0)} \int (e^{2\mu\varphi(x_0)} q_\mu(x_0, v, \nabla v, \tau) - \\
& - e^{2\mu\varphi(x)} q_\mu(x, v, \nabla v, \tau)) dx + \\
& + 2\tau e^{-2\mu\varphi(x_0)} \int e^{2\mu\varphi(x)} q_\mu(x, v, \nabla v, \tau) dx = \\
& = R_{x_0} + 2\tau e^{-2\mu\varphi(x_0)} \int e^{2\mu\varphi(x)} q_\mu(x, v, \nabla v, \tau) dx.
\end{aligned} \tag{14.3.62}$$

From what we have just obtained, from (14.3.40) and from (14.3.46) we have (recall that we have fixed μ)

$$\begin{aligned}
& \frac{2\tau}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi \leq \\
& \leq C\rho(\mu + 1)^2 e^{2\mu\Phi_0} \int (\tau |\nabla v|^2 + \tau^3 |v|^2) dx + \\
& + C \int (S_\tau v A_\tau v) e^{2\mu\varphi} dx \leq \\
& \leq C\rho\tau \|v\|_{1,\tau}^2 + C \int |p_2(x, \partial, \tau)v|^2 e^{2\mu\varphi} dx,
\end{aligned} \tag{14.3.63}$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$. In the last estimate from above we used that (see (14.3.36))

$$2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} dx \leq \int |p_2(x, \partial, \tau)v|^2 e^{2\mu\varphi} dx.$$

By (14.3.61) and (14.3.63) (recalling that $\tau \geq 1$) we get

$$2\tau \|v\|_{1,\tau}^2 \leq C\rho\tau \|v\|_{1,\tau}^2 + C(1 + \tau^{-1}) \int |p_2(x, \partial, \tau)v|^2 dx,$$

for every $v \in C_0^\infty(B_\rho(x_0) \cap \Omega)$. Now, let us choose

$$\rho = \rho_0 := \min \left\{ \frac{1}{C}, 1 \right\}$$

and we get

$$\tau \|v\|_{1,\tau}^2 \leq C \int |p_2(x, \partial, \tau)v|^2 dx,$$

for every $v \in C_0^\infty(B_{\rho_0}(x_0) \cap \Omega)$ and for every $\tau \geq 1$.

Now, recalling (14.3.35) and applying Lemmas 13.4.2 and 13.4.3 we obtain

$$\tau^3 \int_\Omega |u|^2 e^{2\tau\varphi} dx + \tau \int_\Omega |\nabla u|^2 e^{2\tau\varphi} dx \leq C \int_\Omega |P_2(\partial)u|^2 e^{2\tau\varphi} dx, \quad (14.3.64)$$

for every $u \in C_0^\infty(B_{\rho_0}(x_0) \cap \Omega)$ and for every $\tau \geq \tau^*$, where C and τ^* are suitable positive numbers depending by x_0 . At this point we have only to apply Lemma 13.1.1 to conclude the proof. ■

14.4 Second order operators II – Lipschitz coefficients in the principal part

In the following Theorem we will consider the operator

$$P_2(x, \partial) = g^{jk}(x) \partial_{jk}^2, \quad \text{in } B_1, \quad (14.4.1)$$

where $\{g^{jk}(x)\}_{j,k=1}^n$ is a real symmetric matrix-valued function. Recall that $\phi \in C^2(\overline{\Omega})$ satisfies

$$\nabla\phi(x) \neq 0, \quad \forall x \in \overline{B_1}.$$

Moreover, ϕ satisfies condition (S) in 0 w.r.t. the operator $P_2(x, \partial)$ if we have

$$\begin{cases} P_2(0, \xi) = 0, \\ P_2^{(j)}(\xi) \partial_j \phi(0) = 0, \\ \xi \neq 0, \end{cases} \implies \implies \partial_{jk}^2 \phi(0) P_2^{(j)}(0, \xi) P_2^{(k)}(0, \xi) > 0 \quad (14.4.2)$$

and

$$\begin{cases} P_2(0, \xi + i\tau \nabla\phi(0)) = 0, \\ \tau \neq 0, \end{cases} \implies \implies \partial_{jk}^2 \phi(0) P_2^{(j)}(0, \xi + i\tau \nabla\phi(0)) \overline{P_2^{(k)}(0, \xi + i\tau \nabla\phi(0))} > 0. \quad (14.4.3)$$

Concerning (14.4.2) and (14.4.3), keep in mind, respectively, the Remarks that follow definitions 14.3.2 and 14.3.4.

Theorem 14.4.1. *Let $P_2(x, \partial)$ be operator (14.4.1). Let us assume that (14.3.1a) and (14.3.1b) are satisfied with $\Omega = B_1$. Let us suppose that $\varphi \in C^2(\overline{B_1})$ satisfies condition (S) w.r.t. $P_2(x, \partial)$ in $x = 0$.*

Then there exist $\rho_0 \in (0, 1]$, $\delta_0 \in (0, 1]$, $C \geq 1$ and $\tau_0 \geq 1$ such that

$$\begin{aligned} \tau^3 \int_{B_1} |u|^2 e^{2\tau\varphi} dx + \tau \int_{B_1} |\nabla u|^2 e^{2\tau\varphi} dx &\leq \\ &\leq C \int_{B_1} |P_2(\delta x, \partial)u|^2 e^{2\tau\varphi} dx, \end{aligned} \quad (14.4.4)$$

for every $\delta \in (0, \delta_0]$, for every $u \in C_0^\infty(B_{\rho_0})$ and for every $\tau \geq \tau_0$.

Proof. For the most part of the proof we repeat the steps we did in the proof of Theorem 14.3.7 by paying special attention to the additional terms that come up because now the coefficients of the operator are variables.

Let us denote

$$g_\delta^{jk}(x) = g^{jk}(\delta x), \quad j, k = 1, \dots, n.$$

Let $u \in C_0^\infty(B_1)$ and set $v = e^{\tau\varphi}u$. We have

$$\begin{aligned} P_2(\delta x, \partial, \tau)v &:= e^{\tau\varphi}P_2(\delta x, \partial)(e^{-\tau\varphi}v) = \\ &= P_2(\delta x, \partial)v - \tau(P_2(\delta x, \partial)\varphi)v + \tau^2 \mathbf{g}_\delta(\nabla\varphi, \nabla\varphi)v - 2\tau \mathbf{g}_\delta(\nabla\varphi, \nabla v) = \\ &= \partial_j \left(g_\delta^{jk}(x) \partial_k \right) - 2\tau \mathbf{g}_\delta(\nabla\varphi, \nabla v) - \\ &\quad - \tau(P_2(\delta x, \partial)\varphi)v - \partial_j \left(g_\delta^{jk}(x) \right) \partial_k v. \end{aligned}$$

Let us denote

$$L_\delta(v) = \partial_j \left(g_\delta^{jk}(x) \partial_k v \right), \quad (14.4.5)$$

$$p_2(x, \partial, \tau)v = S_\tau v + A_\tau v, \quad (14.4.6)$$

where

$$S_\tau v = L(v) + \tau^2 \mathbf{g}_\delta(\nabla\varphi, \nabla\varphi)v$$

and

$$A_\tau v = -2\tau \mathbf{g}_\delta(\nabla\varphi, \nabla v).$$

Let us notice that

$$P_2(\delta x, \partial, \tau)v - p_2(x, \partial, \tau)v = -\tau(P_2(\delta x, \partial)\varphi)v - \partial_j \left(g_\delta^{jk}(x) \right) \partial_k v. \quad (14.4.7)$$

For the sake of brevity, we omit the domain and the element of integration. Let μ be a positive number that we will choose later. We have trivially

$$\begin{aligned} \int |p_2(x, \partial, \tau)v|^2 e^{2\mu\varphi} &= \int |S_\tau v|^2 e^{2\mu\varphi} + \int |A_\tau v|^2 e^{2\mu\varphi} + \\ &+ 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} \geq \\ &\geq 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi}. \end{aligned} \quad (14.4.8)$$

Set

$$J_1 = \int 2(L_\delta v) \mathbf{g}_\delta(\nabla\varphi, \nabla v) e^{2\mu\varphi} \quad (14.4.9)$$

and

$$J_2 = \int 2\mathbf{g}_\delta(\nabla\varphi, \nabla\varphi) \mathbf{g}_\delta(\nabla\varphi, \nabla v) v e^{2\mu\varphi}, \quad (14.4.10)$$

we have

$$2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} = -2\tau J_1 - 2\tau^3 J_2. \quad (14.4.11)$$

Now, to handle J_1 , we apply (14.3.4) with

$$\beta_\delta = e^{2\mu\varphi} \nabla^{(g_\delta)} \varphi.$$

It should be kept in mind that now, with respect to proof of the Theorem 14.3.7, the coefficients of the operator $P_2(\delta x, \partial)$ depend on x , more precisely they are of the type $f(\delta x)$ where f is a Lipschitz continuous function. Hence we have

$$\begin{aligned} J_1 &\leq \int e^{2\mu\varphi} ((L_\delta(\varphi)) \mathbf{g}_\delta(\nabla v, \nabla v) - 2\partial^2 \varphi \nabla^{(g_\delta)} v \cdot \nabla^{(g_\delta)} v + \\ &+ 2\mu \mathbf{g}_\delta(\nabla\varphi, \nabla\varphi) \mathbf{g}_\delta(\nabla v, \nabla v) - 4\mu (\mathbf{g}_\delta(\nabla\varphi, \nabla v))^2) + C\delta \int e^{2\mu\varphi} |\nabla v|^2 \end{aligned}$$

and, similarly,

$$\begin{aligned} J_2 &\leq - \int e^{2\mu\varphi} (2\partial^2 \varphi \nabla^{(g_\delta)} \varphi \cdot \nabla^{(g_\delta)} \varphi + \mathbf{g}_\delta(\nabla\varphi, \nabla\varphi) L_\delta(\varphi) + \\ &+ 2\mu (\mathbf{g}_\delta(\nabla\varphi, \nabla\varphi))^2) v^2 + C\delta \int e^{2\mu\varphi} v^2, \end{aligned}$$

in the last two inequalities, C depends on M_0 , M_1 and $\|\varphi\|_{C^2(\overline{B_1})}$, but does not depend on μ . By these inequalities and by (14.4.11) we get

$$\begin{aligned} 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi} &= -2\tau J_1 - 2\tau^3 J_2 \geq \\ &\geq 2\tau \int e^{2\mu\varphi} q_\mu^{(\delta)}(x, v, \nabla v, \tau) - \\ &\quad - C\delta\tau \int e^{2\mu\varphi} (\tau^2 v^2 + |\nabla v|^2), \end{aligned} \quad (14.4.12)$$

where

$$q_\mu^{(\delta)}(x, v, \nabla v, \tau) = q_0^{(\delta)}(x, v, \nabla v, \tau) + \mu q_1^{(\delta)}(x, v, \nabla v, \tau), \quad (14.4.13)$$

$$\begin{aligned} q_0^{(\delta)}(x, v, \nabla v, \tau) &= \\ &= 2 [\partial^2 \varphi(x) \nabla^{(g_\delta)} v \cdot \nabla^{(g_\delta)} v + \\ &\quad \tau^2 (\partial^2 \varphi(x) \nabla^{(g_\delta)} \varphi(x) \cdot \nabla^{(g_\delta)} \varphi(x)) v^2] + \\ &\quad + L_\delta(\varphi) [\tau^2 \mathbf{g}_\delta(\nabla \varphi(x), \nabla \varphi(x)) v^2 - \mathbf{g}_\delta(\nabla v, \nabla v)] \end{aligned} \quad (14.4.14)$$

and

$$\begin{aligned} q_1^{(\delta)}(x, v, \nabla v, \tau) &= \\ &= -2\mathbf{g}_\delta(\nabla \varphi(x), \nabla \varphi(x)) \mathbf{g}_\delta(\nabla v, \nabla v) + 4(\mathbf{g}_\delta(\nabla \varphi(x), \nabla v))^2 + \\ &\quad + 2\tau^2 (\mathbf{g}_\delta(\nabla \varphi(x), \nabla \varphi(x)))^2 v^2. \end{aligned} \quad (14.4.15)$$

Now we **examine the first addend on the right-hand side of** (14.4.12), namely

$$2\tau \int e^{2\mu\varphi} q_\mu^{(\delta)}(x, v, \nabla v, \tau).$$

Likewise to the proof of Theorem 14.3.7 we write

$$2\tau \int e^{2\mu\varphi} q_\mu^{(\delta)}(x, v, \nabla v, \tau) = I_0 + R_0,$$

where

$$I_0 = 2\tau e^{2\mu\varphi(0)} \int q_\mu^{(\delta)}(0, v, \nabla v, \tau) \quad (14.4.16)$$

and

$$R_0 = 2\tau \int (e^{2\mu\varphi(x)} q_\mu^{(\delta)}(x, v, \nabla v, \tau) - e^{2\mu\varphi(0)} q_\mu^{(\delta)}(0, v, \nabla v, \tau)). \quad (14.4.17)$$

Let $\rho \in (0, 1]$, to be chosen; we get easily (recall $\delta \leq 1$)

$$|R_0| \leq C\rho(\mu + 1)^2 e^{2\mu\Phi_0} \int (\tau|\nabla v|^2 + \tau^3|v|^2) dx, \quad (14.4.18)$$

for every $v \in C_0^\infty(B_\rho)$, where

$$\Phi_0 = \max_{x \in \overline{B_1}} |\varphi|,$$

C does not depend on μ , but depends on the $C^2(\overline{B_1})$ norm of φ and on M_0 .

Let us denote

$$g_0^{jk} = g^{jk}(0).$$

By the Parseval identity we have

$$I_0 = \frac{2\tau e^{2\mu\varphi(0)}}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi, \quad (14.4.19)$$

where

$$Q_\mu(\xi, \tau) = Q_0(\xi, \tau) + \mu Q_1(\xi, \tau), \quad (14.4.20)$$

$$\begin{aligned} Q_0(\xi, \tau) = & 2 [\partial^2 \varphi(0) \xi^{(g_0)} \cdot \xi^{(g_0)} + \tau^2 \partial^2 \varphi(0) \nabla^{(g_0)} \varphi(0) \cdot \nabla^{(g_0)} \varphi(0)] + \\ & + L_0(\varphi)(0) [\tau^2 \mathbf{g}_0(\nabla \varphi(0), \nabla \varphi(0)) - \mathbf{g}_0(\xi, \xi)] \end{aligned}$$

and

$$\begin{aligned} Q_1(\xi, \tau) = & -2\mathbf{g}_0(\nabla \varphi(0), \nabla \varphi(0)) (\mathbf{g}_0(\xi, \xi) - \tau^2 \mathbf{g}_0(\nabla \varphi(0), \nabla \varphi(0))) + \\ & + 4(\mathbf{g}_0(\nabla \varphi(0), \xi))^2. \end{aligned}$$

Similarly to what was done in the proof of Theorem 14.3.7 can be proved the existence of constants C_1 , C_2 and μ such that

$$|\xi + i\tau \nabla \varphi(0)|^2 \leq C_1 Q_\mu(\xi, \tau) + C_2 \frac{|P_2(0, \xi + i\tau \nabla \varphi(0))|^2}{|\xi + i\tau \nabla \varphi(0)|^2}, \quad (14.4.21)$$

for every $(\xi, \tau) \in \mathbb{R}^{n+1}$.

From now on we fix a value μ for which (14.4.21) is satisfied. Adopting the notations introduced in (14.3.26) with $N = \nabla \varphi(0)$, by (14.4.21) we have

$$\begin{aligned}
 2\tau \|v\|_{1,\tau}^2 &\leq \\
 &\leq \frac{2\tau C_1}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi + 2\tau C_2 \|p_2(0, \partial, \tau)v\|_{-1,\tau}^2.
 \end{aligned}
 \tag{14.4.22}$$

Now, **we estimate from above the last term on the right-hand side in** (14.4.22). For this purpose, we will repeatedly use the triangle inequality and Lemma 14.3.6.

First of all, set

$$\widetilde{p}_2(x, \partial, \tau)v = g_\delta^{jk}(x)\partial_{jk}^2 v - 2\tau g_\delta^{jk}(x)\partial_j\varphi(0)\partial_k v + \tau^2 g_\delta^{jk}(x)\partial_j\varphi(0)\partial_k\varphi(0)v$$

and let us note that

$$\widetilde{p}_2(0, \partial, \tau) = p_2(0, \partial, \tau).$$

By this equality and by the triangle inequality we have

$$\begin{aligned}
 \|p_2(0, \partial, \tau)v\|_{-1,\tau}^2 &= \|\widetilde{p}_2(0, \partial, \tau)v\|_{-1,\tau}^2 \leq 2\|\widetilde{p}_2(\cdot, \partial, \tau)v\|_{-1,\tau}^2 + \\
 &\quad + 2\|\widetilde{p}_2(\cdot, \partial, \tau)v - \widetilde{p}_2(0, \partial, \tau)v\|_{-1,\tau}^2.
 \end{aligned}
 \tag{14.4.23}$$

Let us estimate from above the first term on the right-hand side in (14.4.23). By the triangle inequality and by the definition of $\|\cdot\|_{-1,\tau}$ we obtain

$$\begin{aligned}
 \|\widetilde{p}_2(\cdot, \partial, \tau)v\|_{-1,\tau}^2 &\leq C\tau^{-2} \|\widetilde{p}_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2 \leq \\
 &\leq C\tau^{-2} \|\widetilde{p}_2(\cdot, \partial, \tau)v - p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2 + \\
 &\quad + C\tau^{-2} \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2,
 \end{aligned}
 \tag{14.4.24}$$

on the other hand,

$$\begin{aligned}
 \widetilde{p}_2(x, \partial, \tau)v - p_2(x, \partial, \tau)v &= \\
 &= -\partial_j \left(g_\delta^{jk}(x) \right) \partial_k v + 2\tau g_\delta^{jk}(x) (\partial_j\varphi(x) - \partial_j\varphi(0)) \partial_k v + \\
 &\quad + \tau^2 g_\delta^{jk}(x) (\partial_j\varphi(x)\partial_k\varphi(x) - \partial_j\varphi(0)\partial_k\varphi(0)) v.
 \end{aligned}$$

From the latter and the triangle inequality, we deduce easily that

$$\begin{aligned}
 \tau^{-2} \|\widetilde{p}_2(\cdot, \partial, \tau)v - p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2 &\leq \\
 &\leq C\delta^2\tau^{-2} \|\nabla v\|_{L^2(B_1)}^2 + C\rho^2 \|v\|_{1,\tau}^2
 \end{aligned}
 \tag{14.4.25}$$

for every $v \in C_0^\infty(B_\rho)$.

By (14.4.24) and (14.4.25) we get

$$\begin{aligned} \|\tilde{p}_2(\cdot, \partial, \tau)v\|_{-1, \tau}^2 &\leq C\tau^{-2} \|\nabla v\|_{L^2(B_1)}^2 + C\rho^2 \|v\|_{1, \tau}^2 + \\ &+ C\tau^{-2} \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2. \end{aligned} \quad (14.4.26)$$

Now, let us estimate from above the second term on the right-hand side in (14.4.23).

We notice

$$\begin{aligned} \tilde{p}_2(x, \partial, \tau)v - \tilde{p}_2(0, \partial, \tau)v &= \\ &= \left(g_\delta^{jk}(x) - g_\delta^{jk}(0) \right) (\partial_j - \tau\partial_j\varphi(0)) (\partial_k - \tau\partial_k\varphi(0)) v. \end{aligned}$$

and, for a fixed $k = 1, \dots, n$, we set

$$w_k = (\partial_k - \tau\partial_k\varphi(0)) v.$$

By applying Lemma 14.3.6 we have, for every $v \in C_0^\infty(B_\rho)$,

$$\begin{aligned} \left\| \left(g_\delta^{jk}(x) - g_\delta^{jk}(0) \right) (\partial_j - \tau\partial_j\varphi(0)) w_k \right\|_{-1, \tau}^2 &\leq \\ &\leq C \left(M_1^2 \delta^2 \rho^2 + |\tau\nabla\varphi(0)|^{-2} \right) \|w_k\|_{L^2(B_1)}^2. \end{aligned} \quad (14.4.27)$$

On the other hand, for $k = 1, \dots, n$,

$$\|w_k\|_{L^2(B_1)}^2 = \|(\partial_k - \tau\partial_k\varphi(0)) v\|_{L^2(B_1)}^2 \leq C \|v\|_{1, \tau}^2.$$

By the latter and by (14.4.27) we get

$$\|\tilde{p}_2(\cdot, \partial, \tau)v - \tilde{p}_2(0, \partial, \tau)v\|_{-1, \tau}^2 \leq C \left(M_1^2 \rho^2 + \tau^{-2} \right) \|v\|_{1, \tau}^2. \quad (14.4.28)$$

By (14.4.23), (14.4.26) and (14.4.28) we have

$$\|p_2(0, \partial, \tau)v\|_{-1, \tau}^2 \leq C \left(\rho^2 + \tau^{-2} \right) \|v\|_{1, \tau}^2 + C\tau^{-2} \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2,$$

for every $v \in C_0^\infty(B_\rho)$, where C depends on M_0 and M_1 .

By the latter and by (14.4.22) we have

$$\begin{aligned} 2\tau \|v\|_{1, \tau}^2 &\leq \frac{2\tau C_1}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi + \\ &+ C \left(\rho^2 \tau + \tau^{-1} \right) \|v\|_{1, \tau}^2 + C\tau^{-1} \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2, \end{aligned} \quad (14.4.29)$$

for every $v \in C_0^\infty(B_\rho)$.

In order to estimate from above the first term on the right-hand side in (14.4.29) we proceed as in the proof of Theorem 14.3.7 (see (14.3.62)) and by (14.4.18), (14.4.19) we get

$$\begin{aligned} \frac{2\tau}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi &= R_0 + 2\tau e^{-2\mu\varphi(0)} \int e^{2\mu\varphi(x)} q_\mu^{(\delta)}(x, v, \nabla v, \tau) dx \leq \\ &\leq C\rho\tau \|v\|_{1,\tau}^2 + 2\tau e^{-2\mu\varphi(0)} \int e^{2\mu\varphi(x)} q_\mu^{(\delta)}(x, v, \nabla v, \tau) dx, \end{aligned}$$

on the other hand, by (14.4.12) we have

$$2\tau \int e^{2\mu\varphi} q_\mu^{(\delta)}(x, v, \nabla v, \tau) dx \leq C\delta\tau \|v\|_{1,\tau}^2 + 2 \int (S_\tau v A_\tau v) e^{2\mu\varphi},$$

hence, recalling (14.4.8),

$$\frac{2\tau}{(2\pi)^n} \int Q_\mu(\xi, \tau) |\widehat{v}(\xi)|^2 d\xi \leq C(\delta + \rho)\tau \|v\|_{1,\tau}^2 + C \int |p_2(x, \partial, \tau)v|^2 e^{2\mu\varphi},$$

for every $v \in C_0^\infty(B_\rho)$. By inserting the latter in (14.4.29) we have

$$2\tau \|v\|_{1,\tau}^2 \leq C((\rho + \delta)\tau + \tau^{-1}) \|v\|_{1,\tau}^2 + C(1 + \tau^{-1}) \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2,$$

for every $v \in C_0^\infty(B_\rho)$. Let

$$\rho_0 = \delta_0 = \frac{1}{4C},$$

then

$$\tau \|v\|_{1,\tau}^2 \leq 2C \|p_2(\cdot, \partial, \tau)v\|_{L^2(B_1)}^2, \quad (14.4.30)$$

for every $v \in C_0^\infty(B_{\rho_0})$, for every $\tau \geq 2$ and for every $\delta \leq \delta_0$. Taking into account (14.4.7), by (14.4.30) we easily deduce that there exists τ^* such that

$$\tau \|v\|_{1,\tau}^2 \leq 2C \|P_2(x, \partial, \tau)v\|_{L^2(B_1)}^2,$$

for every $v \in C_0^\infty(B_{\rho_0})$ and for every $\tau \geq \tau^*$. Finally, applying Lemmas 13.4.2 and 13.4.3 we obtain (14.4.4). ■

14.4.1 Application to the Cauchy problem

By Carleman estimate (14.4.4), we can obtain an uniqueness result for the Cauchy problem for the operator

$$P(x, \partial)u = g^{jk}(x)\partial_{jk}^2 u + b_k(x)\partial_k u + c(x)u, \quad (14.4.31)$$

where matrix $\{g^{jk}\}$ has real entries and satisfy conditions (14.3.1a) and (14.3.1b), $b_k \in L^\infty(B_1, \mathbb{C})$, $k = 1, \dots, n$, $c \in L^\infty(B_1, \mathbb{C})$. Below we state the theorem; the proof is only briefly mentioned as it is carried out in an analogous way to that of Theorem 13.5.3. Theorem 14.4.2 has been proved by **Calderón** in 1957 for more general (but with coefficients C^∞) than those considered here (for further discussion, we refer to [50, Ch. 3]).

Theorem 14.4.2 (Calderón). *Let $\psi \in C^1(\overline{B_1})$ be a real-valued function such that*

$$\nabla\psi(0) \neq 0.$$

Let $P(x, \partial)$ be operator (14.4.31). Let $U \in H^2(B_1)$ such that

$$\begin{cases} P(x, \partial)U = 0, & \text{in } B_1, \\ U(x) = 0, & \text{in } \{x \in B_1 : \psi(x) > \psi(0)\}. \end{cases}$$

Let us suppose that

$$\begin{cases} P_2(0, \xi + i\tau\nabla\psi(0)) = 0, \\ (\xi, \tau) \neq (0, 0), \end{cases} \implies \quad (14.4.32)$$

$$\implies P_2^{(j)}(0, \xi + i\tau\nabla\psi(0))\partial_j\psi(0) \neq 0,$$

$(P_2(x, \xi) = g^{jk}(x)\xi_j\xi_k)$.

Then there exists a neighborhood \mathcal{U}_0 of 0 such that

$$U = 0 \quad \text{in } \mathcal{U}_0.$$

Proof. The proof is carried out in a manner similar to that of Theorem 13.5.3. Therefore, first of all we write Carleman estimate (14.4.4) in the form

$$\begin{aligned} \tau^3\delta^4 \int |u|^2 e^{2\tau\varphi(\delta^{-1}X)} dX + \tau\delta^2 \int |\nabla u|^2 e^{2\tau\varphi(\delta^{-1}X)} dX &\leq \\ &\leq C_0 \int |P_2(X, \partial)u|^2 e^{2\tau\varphi(\delta^{-1}X)} dX, \end{aligned}$$

for every $\delta \in (0, \delta_0]$, for every $u \in C_0^\infty(B_{\tilde{\rho}_0}(0))$ and for every $\tau \geq \tilde{\tau}_0$.

Next, by means of a diffeomorphism we reduce to the case in which $\{x \in B_{r_0} : \psi(x) \leq 0\}$ is the epigraph of a function f strictly convex and such that $f(0) = |\nabla f(0)| = 0$. Like the proof of Theorem 13.5.3 we introduce the function

$$\varphi(x) = h(\delta_0 x),$$

where

$$h(x_n) = -x_n + \frac{x_n^2}{2}$$

and we check that φ satisfies condition **(S)**. To check this, first we check that φ is pseudo-convex. For this purpose, it suffices to notice that by (14.4.32) we have that if $\tau = 0$ then $\xi \neq 0$. Therefore

$$\begin{cases} P_2(0, \xi) = 0, \\ \xi \neq 0, \end{cases} \implies P_2^{(n)}(0, \xi) \neq 0,$$

the latter, in particular, implies that the antecedent of implication (14.4.2) is not satisfied by φ in 0 which, in turn, implies that the condition (14.3.9) holds. Regarding (14.4.3), we have that if (recall $\nabla\varphi(0) = -\delta_0 e_n$)

$$\begin{cases} P_2(0, \xi - i\tau\delta_0 e_n) = 0, \\ \tau \neq 0, \end{cases}$$

then, (14.4.32) implies

$$P_2^{(n)}(0, \xi - i\tau\delta_0 e_n) \neq 0,$$

hence

$$\partial_{jk}^2 \varphi(0) P_2^{(j)}(0, \xi - i\tau\delta_0 e_n) \overline{P_2^{(k)}(\xi - i\tau\delta_0 e_n)} = \delta_0^2 \left| P_2^{(n)}(0, \xi - i\tau\delta_0 e_n) \right|^2 > 0.$$

The remaining part of the proof is identical to that of Theorem 13.5.3. ■

Let us examine condition (14.4.32).

Let us first notice that if $N := \nabla\psi(0)$ satisfies (14.4.32), then N cannot be a characteristic direction. As a matter of fact, let us suppose the opposite, that is let us suppose that

$$P_2(0, N) = 0. \quad (14.4.33)$$

Let $\xi = 0$, then for every $\tau \neq 0$ we have

$$P_2(0, 0 + i\tau N) = 0,$$

hence the antecedent of (14.4.32) holds true, but (by Euler Theorem on homogeneous function)

$$P_2^{(j)}(0, 0 + i\tau N)N_j = 2i\tau P_2(0, N) = 0.$$

Therefore, **a necessary condition in order that (14.4.32) holds true is that the surface $\{\psi(x) = \psi(0)\}$ is noncharacteristic in 0.** Nevertheless, as we are going to see, the converse is not true, i.e. the condition $P_2(0, N) \neq 0$ is not sufficient for the validity of (14.4.32).

Let us start with the following

Proposition 14.4.3. *If $N := \nabla\psi(0)$ satisfies*

$$P_2(0, N) \neq 0. \quad (14.4.34)$$

then condition (14.4.32) is equivalent to (we set $g^{jk} = g^{jk}(0)$, $j, k = 1, \dots, n$)

$$\begin{cases} \mathbf{g}(\xi, \xi) = 0, \\ \xi \not\parallel N, \end{cases} \implies \mathbf{g}(\xi, N) \neq 0, \quad (14.4.35)$$

$\xi \not\parallel N$ mean " ξ and N linearly independent" .

Proof. We begin by noticing that the (14.4.32) is equivalent to

$$\begin{cases} P_2(0, \xi + i\tau N) = 0, \\ \xi \not\parallel N, \\ (\xi, \tau) \neq (0, 0), \end{cases} \implies P_2^{(j)}(0, \xi + i\tau N)N_j \neq 0. \quad (14.4.36)$$

Indeed, if (14.4.32) holds then trivially (14.4.36) holds. Let us suppose now that (14.4.36) is true, we have again that (14.4.32) is trivially satisfied, for any $\xi \not\parallel N$. Instead, if $\xi = \lambda N$, $\lambda \in \mathbb{R}$, we have that

$$P_2(0, \xi + i\tau N) = 0,$$

implies

$$0 = P_2(0, \lambda N + i\tau N) = (\lambda + i\tau)^2 P_2(0, N)$$

from which, taking into account that $P_2(0, N) \neq 0$, we have $\tau = \lambda = 0$, that is $\xi = 0$ and $\tau = 0$, consequently the antecedent of (14.4.32) is false and thus the condition (14.4.32) is satisfied.

Now we prove that (14.4.35) and (14.4.36) are equivalent. Let us begin assuming that (14.4.36) holds true. To prove (14.4.35) let us suppose that $\xi \not\parallel N$ and that

$$\mathbf{g}(\xi, \xi) = 0.$$

Now, if it were

$$\mathbf{g}(\xi, N) = 0, \tag{14.4.37}$$

we would have at the same time

$$\xi \not\parallel N,$$

$$P_2(0, \xi + i0N) = \mathbf{g}(\xi, \xi) = 0$$

and

$$P_2^j(0, \xi + i0N)N_j = 2\mathbf{g}(\xi, N) = 0.$$

So there would be a contradiction with (14.4.36), therefore (14.4.37) does not hold. Thus, if (14.4.36) holds true then (14.4.35) holds true.

Now let us suppose that (14.4.35) holds and let us suppose that

$$\begin{cases} P_2(0, \xi + i\tau N) = 0, \\ \xi \not\parallel N, \\ (\xi, \tau) \neq (0, 0). \end{cases} \tag{14.4.38}$$

If $\tau = 0$, then, (14.4.38) implies $\xi \not\parallel N$ and $\mathbf{g}(\xi, \xi) = P_2(0, \xi + i0N) = 0$, moreover (14.4.35) implies $\mathbf{g}(\xi, N) \neq 0$. Hence

$$P_2^{(j)}(0, \xi + i0N) = 2\mathbf{g}(\xi, N) \neq 0.$$

If $\tau \neq 0$, recalling that N is a noncharacteristic direction – so that $\mathbf{g}(N, N) = P_2(0, N) \neq 0$ – we have

$$P_2^{(j)}(0, \xi + i\tau N)N_j = 2\mathbf{g}(\xi, N) + i\tau\mathbf{g}(N, N) \neq 0.$$

All in all, if (14.4.35) holds then (14.4.36) holds. The proof is complete. ■

Remarks and Examples.

1. If $P_2(x, \partial)$ is elliptic (with real coefficients), then (14.4.32) is satisfied, as already proved in Example 4a of Section 12.5.
2. Let us consider the wave operator

$$P_2(\partial) = \Delta_{x'} - \partial_{x_n}^2. \quad (14.4.39)$$

Let us check for which $N \neq 0$ condition (14.4.32) is satisfied. It is not restrictive to assume

$$|N| = 1. \quad (14.4.40)$$

We first need to assume that N is a noncharacteristic direction for $P_2(\partial)$, i.e.

$$P_2(N) = |N'|^2 - N_n^2 \neq 0. \quad (14.4.41)$$

Now, let us see when (14.4.35) is satisfied (which, we recall, is equivalent to (14.4.32)). This condition can be written

$$\begin{cases} |\xi'|^2 - \xi_n^2 = 0, \\ \xi \nparallel N, \end{cases} \implies \xi' \cdot N' - \xi_n N_n \neq 0. \quad (14.4.42)$$

Let us first examine the **case** $n = 2$. In this case condition (14.4.41) can be written

$$N_1^2 - N_2^2 \neq 0. \quad (14.4.43)$$

the first condition of the antecedent of (14.4.42) can be written $\xi_1^2 - \xi_2^2 = 0$ and it is equivalent to

$$\xi_2 = \pm \xi_1.$$

From which, by (14.4.41) and $\xi \neq 0$, we have

$$\xi_1 N_1 - \xi_2 N_2 = \xi_1 (N_1 \mp N_2) \neq 0.$$

Therefore, if $n = 2$, (14.4.32) is satisfied for all $N \in \mathbb{R}^2$ that satisfy (14.4.43), i.e. that are not a characteristic direction.

Let us consider now the **case** $n \geq 3$. Since the scalar product in \mathbb{R}^{n-1} is invariant w.r.t. the rotations, we may assume that

$$N = N_1 e_1 + N_n e_n.$$

In this way, conditions (14.4.40) and (14.4.41), can be written, respectively;

$$N_1^2 + N_n^2 = 1$$

and

$$N_1^2 - N_n^2 \neq 0.$$

Let us distinguish two cases

(a) $N_1^2 - N_n^2 < 0$, i.e. $|N'| < |N_n|$;

(b) $N_1^2 - N_n^2 > 0$, i.e. $|N'| > |N_n|$;

Case (a). Let $\xi \not\parallel N$ such that

$$|\xi'|^2 - \xi_n^2 = 0.$$

In particular, we have $\xi_n \neq 0$ and $\xi' \neq 0$ (as a matter of fact, if one of them is zero the other is also zero) moreover

$$|\xi'| = |\xi_n|.$$

Hence

$$|\xi' \cdot N'| \leq |\xi'| |N'| = |\xi'| |N_1| < |\xi'| |N_n| = |\xi_n| |N_n|,$$

from which we have

$$\xi' \cdot N' - \xi_n N_n \neq 0.$$

Therefore, in case (a), (14.4.35) is satisfied.

In case (b) it is simple to check that (14.4.35) is not satisfied. To check this, Let

$$\xi_0 = e_1 N_n + e_2 \sqrt{N_1^2 - N_n^2} + e_n N_1.$$

We have

$$\begin{cases} |\xi_0'|^2 - \xi_{0,n}^2 = 0, \\ \xi_0 \not\parallel N, \end{cases}$$

but

$$\xi' \cdot N' - \xi_n N_n = N_n N_1 - N_1 N_n = 0.$$

Now it is interesting to point out (we refer to [50, Ch. 6]) that it has been proved that there exist $u, q \in C^\infty(\mathbb{R}^3, \mathbb{C})$ such that

$$\begin{cases} \partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u + q(x, t)u = 0, \\ \text{supp } u = \{x_2 \geq 0\}, \end{cases} \quad (14.4.44)$$

or, in other words, **does not hold uniqueness** for the Cauchy problem with initial surface $\Gamma := \{x_2 = 0\}$, for the equation

$$\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u + q(x, t)u = 0$$

where

$$q \in C^\infty(\mathbb{R}^3, \mathbb{C}).$$

Keep in mind that, when q is **analytic**, the Holmgren Theorem provides uniqueness for the Cauchy problem for the equation

$$\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u + q(x, t)u = 0,$$

with initial surface Γ (as it is a noncharacteristic surface).

3. We check that if $\theta \in (0, 1)$, then

$$\phi(x) = \frac{1}{2} (|x'|^2 - \theta^2 x_n^2), \quad (14.4.45)$$

is a pseudo-convex function w.r.t. wave operator (14.4.39) in the open set

$$\Omega = (B_R \setminus \overline{B_r}) \times (-T, T),$$

for every $0 < r < R$ and $T > 0$.

We only need to check (14.3.10). Calculate

$$\nabla \phi(x) = (x', -\theta^2 x_n) \neq 0, \quad \text{in } \overline{\Omega},$$

$$\partial^2 \phi(x) = \text{diag } (1, \dots, 1, -\theta^2 x_n).$$

Now let us suppose

$$\begin{cases} |\xi'|^2 - \xi_n^2 = 0, \\ \xi' \cdot x' + \theta^2 \xi_n x_n = 0, \\ \xi \neq 0 \end{cases} \quad (14.4.46)$$

and let us check that

$$\partial^2 \phi(x) \tilde{\xi} \cdot \tilde{\xi} > 0, \quad (14.4.47)$$

where

$$\tilde{\xi} = (\xi', -\theta^2 \xi_n).$$

We have

$$\partial^2 \phi(x) \tilde{\xi} \cdot \tilde{\xi} = |\xi'|^2 - \theta^2 \xi_n^2.$$

On the other hand, by the first condition of (14.4.46) we have $|\xi'|^2 = \xi_n^2$ hence, taking into account that if ξ satisfies at the same time $\xi \neq 0$ and $|\xi'|^2 - \xi_n^2 = 0$, then $\xi' \neq 0$, we have

$$\partial^2 \phi(x) \tilde{\xi} \cdot \tilde{\xi} = |\xi'|^2 (1 - \theta^2) > 0.$$

Therefore, we have proved that (14.4.46) implies (14.4.47). Hence φ is a pseudo-convex function in Ω . \blacklozenge

In the following Theorem we prove the uniqueness for a Cauchy problem under the assumption of pseudo-convexity of the initial surface.

Theorem 14.4.4. *Let us suppose that the coefficients of the principal part $P_2(\partial)$, of operator (14.4.31) are constants. Let $U \in H^2(B_1(x_0))$ satisfy*

$$\begin{cases} P(\partial)U = 0, & \text{in } B_1(x_0) \\ U(x) = 0 & \text{in } \{x \in B_1(x_0) : \psi(x) > \psi(x_0)\}. \end{cases}$$

Let $\psi \in C^2(\overline{B_1(x_0)})$ be a real-valued function and pseudo-convex w.r.t. $P_2(\partial)$ in x_0 . Then there exists a neighborhood \mathcal{U}_{x_0} of 0 such that

$$U = 0 \quad \text{in } \mathcal{U}_{x_0}.$$

Proof. The proof is very similar to that of Proposition 13.2.47. Therefore, here we merely point out the most important differences inviting the reader to care the details.

It is not restrictive to assume that $x_0 = 0$ and

$$\psi(0) = 0.$$

Since ψ is pseudo-convex in 0, by Proposition 14.3.5 we have that the function

$$\varphi(x) = e^{\lambda\psi(x)} - 1,$$

satisfies condition (S) in 0 for any λ large enough. Let

$$\varphi_\varepsilon(x) = \varphi(x) - \frac{\varepsilon|x|^2}{2}, \quad (14.4.48)$$

where ε is a positive number that can be chosen in such a way that φ_ε satisfies condition **(S)** (the reader cure the details). We fix this ε and from Theorem 14.3.7 we have that there exists $R \in (0, 1/2)$ and there exist two constants C and τ_0 such that

$$\tau^3 \int_{B_1} |u|^2 e^{2\tau\varphi_\varepsilon} dx + \tau \int_{B_1} |\nabla u|^2 e^{2\tau\varphi_\varepsilon} dx \leq C \int_{B_1} |P_2(\partial)u|^2 e^{2\tau\varphi_\varepsilon} dx, \quad (14.4.49)$$

for every $u \in C_0^\infty(B_{2R}(0))$ and for every $\tau \geq \tau_0$. Starting from this point one repeats, with obvious modifications, what we did in the proof of Proposition 13.2.4. ■

14.5 Stability estimate for the wave equation in a cylinder

In this Section we adopt the traditional notations: the "spatial coordinates" are denoted by x_1, \dots, x_n , the time coordinate is denoted by t and, therefore, the wave operator is

$$\square = \partial_t^2 - \Delta_x = \partial_t^2 - (\partial_{x_1}^2 + \dots + \partial_{x_n}^2). \quad (14.5.1)$$

Let us denote by

$$\nabla_x = (\partial_{x_1}, \dots, \partial_{x_n}), \quad \nabla_{x,t} = (\nabla_x, \partial_t).$$

Let $T > 1$, set

$$S_T = \mathbb{R}^n \times (-T, T).$$

The following Theorem holds true (see also [66])

Theorem 14.5.1 (stability estimate for the wave equation). *Let $a \in L^\infty(S_T, \mathbb{R}^n)$, $b \in L^\infty(S_T)$ and $c \in L^\infty(S_T)$. Let $M \geq 1$. Let us assume that*

$$\|a\|_{L^\infty(S_T, \mathbb{R}^n)} + \|b\|_{L^\infty(S_T)} + \|c\|_{L^\infty(S_T)} \leq M. \quad (14.5.2)$$

Let $F \in L^2(S_T)$ and $U \in C^\infty(S_T)$ satisfy

$$\begin{cases} \square U + a \cdot \nabla_x U + b \partial_t U + cU = F, & \text{in } S_T, \\ U(x, t) = 0, & \text{for } |x| > 1, t \in (-T, T). \end{cases} \quad (14.5.3)$$

Then

$$\int_{-T}^T \int_{B_1} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq C \|F\|_{L^2(B_1 \times (-T, T))}^2, \quad (14.5.4)$$

where C depends on M and T .

The proof of Theorem 14.5.1 is based on what follows:

- (a) an **energy estimate** for the equation in (14.5.3)
- (b) Carleman estimate (14.3.33).

Lemma 14.5.2 (energy estimate). Let $U \in C^2(\overline{B_1} \times (-T, T))$ satisfy

$$\begin{cases} \square U + a \cdot \nabla_x U + b \partial_t U + cU = F, & \text{in } B_1 \times (-T, T), \\ U(x, t) = 0, & \text{for } (x, t) \in \partial B_1 \times (-T, T), \end{cases} \quad (14.5.5)$$

where a, b, c and F satisfy the same assumption of Theorem 14.5.1, then the following inequality holds true

$$\begin{aligned} & \int_{-T}^T \int_{B_1} (U^2(x, t) + U_t^2(x, t) + |\nabla_x U(x, t)|^2) dxdt \leq \\ & \leq CT\delta^{-1} \int_{-\delta}^{\delta} \int_{B_1} (U^2(x, t) + U_t^2(x, t) + |\nabla_x U(x, t)|^2) dxdt + \\ & + C \|F\|_{L^2(B_1 \times (-T, T))}^2, \end{aligned} \quad (14.5.6)$$

where C depends on M and T only.

Proof of Lemma 14.5.2. By applying Lemma 14.3.1 with

$$\beta = (0, \dots, 0, 1)$$

and

$$g = \text{diag}(-1, \dots, -1, 1),$$

we get

$$(\square U)U_t = \frac{1}{2} \partial_t (U_t^2 + |\nabla_x U|^2) - \text{div}_x (U_t \nabla_x U). \quad (14.5.7)$$

Moreover, by (14.5.5) and taking into account (14.5.2) we have easily

$$|(\square U)U_t| \leq C(M+1)(U^2 + U_t^2 + |\nabla_x U|^2 + F^2). \quad (14.5.8)$$

Let $s, \sigma \in (-T, T)$, $s \leq \sigma$. By integrating both the sides of (14.5.7) over $B_1 \times [s, \sigma]$, we have from the divergence Theorem and from (14.5.8)

$$\begin{aligned} & \int_{B_1} (U_t^2(x, \sigma) + |\nabla_x U(x, \sigma)|^2) dx - \\ & - \int_{B_1} (U_t^2(x, s) + |\nabla_x U(x, s)|^2) dx = \\ & = \int_s^\sigma \int_{B_1} \partial_t (U_t^2 + |\nabla_x U|^2) dx dt = \\ & = 2 \int_s^\sigma \int_{B_1} (\square U)U_t dx dt \leq \\ & \leq C(M+1) \int_s^\sigma \int_{B_1} (U^2 + U_t^2 + |\nabla_x U|^2 + F^2) dx dt. \end{aligned} \quad (14.5.9)$$

Let us note that by the first Poincaré inequality (Theorem 3.4.2) we have

$$\begin{aligned} C_*^{-1} \int_{B_1} |\nabla_x U(x, t)|^2 dx & \leq \int_{B_1} (U^2(x, t) + |\nabla_x U(x, t)|^2) dx \leq \\ & \leq C_* \int_{B_1} |\nabla_x U(x, t)|^2 dx, \end{aligned} \quad (14.5.10)$$

for every $t \in (-T, T)$, where $C \geq 1$ is a constant. Therefore, setting

$$E(t) = \int_{B_1} (U_t^2(x, t) + |\nabla_x U(x, t)|^2) dx,$$

by (14.5.9) e (14.5.10) we have

$$E(\sigma) \leq E(s) + C_1 \|F\|_{L^2(B_1 \times (-T, T))}^2 + C_1 \int_s^\sigma E(t) dt$$

($C_1 = C_* C(M+1)$) and by the Gronwall inequality we obtain

$$E(\sigma) \leq \left(E(s) + C_1 \|F\|_{L^2(B_1 \times (-T, T))}^2 \right) e^{2C_1 T}. \quad (14.5.11)$$

We notice that if $\sigma \leq s$ we obtain similarly the previous estimate if that we integrate both the sides of (14.5.7) over $B_1 \times [\sigma, s]$ and we interchange σ and s . Therefore we have, for each $s, \sigma \in (-T, T)$,

$$E(\sigma) \leq \left(E(s) + C_1 \|F\|_{L^2(B_1 \times (-T, T))}^2 \right) e^{2C_1 T}. \tag{14.5.12}$$

Now, by integrating with respect to s both members of (14.5.12) over $(-\delta, \delta)$, where $\delta \in (0, T)$, we have, for each $\sigma \in (-T, T)$,

$$2\delta E(\sigma) \leq 2C_2 \int_{-\delta}^{\delta} E(s) ds + 2\delta C_2 \|F\|_{L^2(B_1 \times (-T, T))}^2 \tag{14.5.13}$$

($C_2 = e^{2C_1 T}$). Finally, by integrating both the sides of (14.5.13) with respect to σ over $(-T, T)$ and taking into account (14.5.10), we obtain (14.5.6). ■

Remark 4. As can be seen immediately from the proof, it is not necessary for Lemma 14.5.6 that T be greater than 1. ♦

Proof of Theorem 14.5.1. In Remark 3 of the previous Section we have proved that

$$\phi(x, t) = -\theta^2 t^2 + |x|^2,$$

is a pseudo-convex function if $\theta \in (0, 1)$ in $(B_R \setminus \overline{B_r}) \times (-T, T)$ for every $0 < r < R$. Let us fix θ in such a way that

$$\frac{1}{T} < \theta < 1. \tag{14.5.14}$$

(recall that $T > 1$) and let $\rho \in (0, 1/10)$ satisfy

$$\rho < \frac{\theta T - 1}{8}. \tag{14.5.15}$$

Proposition 14.3.5 implies that if λ is sufficiently large, then the functions

$$\varphi_0(x, t) = e^{\lambda\phi(x, t)}, \quad \varphi_1(x, t) = e^{\lambda\phi(x - 8\rho e_1, t)}, \tag{14.5.16}$$

satisfy condition **(S)** in $(B_R \setminus \overline{B_r}) \times (-T, T)$ for every $0 < r < 1 < R$.

Let us fix $\lambda > 0$ in such a way that φ_0 satisfy condition **(S)** in $(B_2 \setminus \overline{B_\rho}) \times (-T, T)$ (and, consequently φ_1 satisfies condition **(S)** in $(B_2(8\rho e_1) \setminus \overline{B_\rho(8\rho e_1)}) \times (-T, T)$).

For any $s > 0$ set

$$Z_{0,s} = \{(x, t) \in \overline{B_1} \times (-T, T) : \varphi_0(x, t) \geq e^{\lambda s}\}$$

and

$$Z_{1,s} = \{(x, t) \in \overline{B_1} \times (-T, T) : \varphi_1(x, t) \geq e^{\lambda s}\}.$$

Let us check that

$$\varphi_0(x, \pm T) \leq 1, \quad \varphi_1(x, \pm T) \leq 1, \quad \forall x \in B_1, \quad (14.5.17a)$$

$$\overline{B_1} \times [-2\rho, 2\rho] \subset Z_{0,s} \cup Z_{1,s}, \quad \forall s \in (0, 12\rho^2]. \quad (14.5.17b)$$

The first of (14.5.17a) is an immediate consequence of (14.5.14). Concerning the second of (14.5.17a), we observe that from (14.5.15) we have, for each $x \in B_1$

$$|x - 8\rho e_1| \leq 1 + 8\rho < \theta T$$

which implies

$$\varphi_1(x, \pm T) = e^{\lambda(-\theta^2 T^2 + |x - 8\rho e_1|^2)} \leq 1,$$

for every $x \in B_1$.

Now let us check (14.5.17b). Set

$$\psi(x) = \max \{|x|^2, |x - 8\rho e_1|^2\},$$

we obtain easily

$$Z_{0,s} \cup Z_{1,s} = \left\{ (x, t) \in \overline{B_1} \times (-T, T) : e^{\lambda(-\theta^2 t^2 + \psi(x))} \geq e^{\lambda s} \right\}. \quad (14.5.18)$$

Let us note now that ψ can be written as

$$\psi(x) = \begin{cases} |x|^2, & \text{for } x_1 \geq 4\rho, \\ (x_1 - 8\rho)^2 + x_2^2 + \cdots + x_n^2, & \text{for } x_1 < 4\rho, \end{cases}$$

from which we have

$$\psi(x) \geq 16\rho^2, \quad \forall x \in \mathbb{R}^n.$$

Now, if $(x, t) \in \overline{B_1} \times [-2\rho, 2\rho]$, then

$$-\theta^2 t^2 + \psi(x) \geq -\theta^2 t^2 + 16\rho^2 > -4\rho^2 + 16\rho^2 = 12\rho^2 \geq s, \quad \forall s \in (0, 12\rho^2],$$

by the latter and by (14.5.18), we get $(x, t) \in Z_{0,s} \cup Z_{1,s}$ for $0 \leq s \leq 12\rho^2$. Hence (14.5.17b) is proved.

Let us apply Carleman estimate (14.3.33) to the operator \square where $\varphi = \varphi_0$. Set

$$Q_T = \overline{B_1} \times (-T, T),$$

we get

$$\begin{aligned} \tau^3 \int_{Q_T} |u|^2 e^{2\tau\varphi_0} dxdt + \tau \int_{Q_T} |\nabla_{x,t} u|^2 e^{2\tau\varphi_0} dxdt &\leq \\ &\leq C \int_{Q_T} |\square u|^2 e^{2\tau\varphi_0} dxdt, \end{aligned} \quad (14.5.19)$$

for every $u \in C_0^\infty(\mathbb{R}^{n+1})$, such that $\text{supp } u \subset Q_T = \overline{B_1} \times (-T, T)$ and for every $\tau \geq \tau_0$. Let $\tilde{\eta} \in C^\infty(\mathbb{R})$ satisfy

$$\tilde{\eta}(r) = 0, \quad r \leq 9\rho^2; \quad 0 \leq \tilde{\eta}(r) \leq 1, \quad 9\rho^2 < r < 10\rho^2; \quad \tilde{\eta}(r) = 1, \quad r \geq 10\rho^2;$$

$$\left| \frac{d\tilde{\eta}}{dr} \right| \leq C\rho^{-2}, \quad \left| \frac{d^2\tilde{\eta}}{dr^2} \right| \leq C\rho^{-4}, \quad (14.5.20)$$

where C is a constant (independent by ρ). Set

$$\eta(x, t) = \tilde{\eta}(-\theta^2 t^2 + |x|^2)$$

and let us apply estimate (14.5.19) to $U\eta$. By (14.5.2) we have

$$\begin{aligned} |\square(\eta U)| &= |\eta \square U + 2(\partial_t \eta \partial_t U - \nabla_x \eta \cdot \nabla_x U) + U(\square \eta)| \leq \\ &\leq \eta M(|\nabla_{x,t} U| + |U|) + \eta |F| + C\rho^{-2} \chi_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} |\nabla_{x,t} U| + \\ &+ C\rho^{-4} \chi_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} |U|, \end{aligned}$$

where C depends on T . Let us observe that (by inserting what was obtained in (14.5.19)), we have

$$\begin{aligned} \int_{Q_T} (\tau^3 |U\eta|^2 + \tau |\nabla_{x,t}(\eta U)|^2) e^{2\tau\varphi_0} dxdt &\leq \\ &\leq CM^2 \int_{Q_T} \eta^2 (|\nabla_{x,t} U|^2 + |U|^2) e^{2\tau\varphi_0} dxdt + \\ &+ C \int_{Q_T} |F|^2 e^{2\tau\varphi_0} dxdt + \\ &+ C\rho^{-8} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} |U|^2 e^{2\tau\varphi_0} dxdt + \\ &+ C\rho^{-4} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} |\nabla_{x,t} U|^2 e^{2\tau\varphi_0} dxdt, \end{aligned} \quad (14.5.21)$$

for every $\tau \geq \tau_0$.

Now, let us estimate from below the left-hand side of (14.5.21)

$$\begin{aligned} & \int_{Q_T} (\tau^3 |U\eta|^2 + \tau |\nabla_{x,t}(\eta U)|^2) e^{2\tau\varphi_0} dxdt \geq \\ & \geq \tau \int_{Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) e^{2\tau\varphi_0} dxdt \end{aligned} \quad (14.5.22)$$

and let us estimate from above first integral on the right-hand side as follows

$$\begin{aligned} & \int_{Q_T} \eta^2 (|\nabla_{x,t}U|^2 + |U|^2) e^{2\tau\varphi_0} dxdt = \\ & = \int_{Z_{0,10\rho^2}} (|\nabla_{x,t}U|^2 + |U|^2) e^{2\tau\varphi_0} dxdt + \\ & + \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} \eta^2 (|\nabla_{x,t}U|^2 + |U|^2) e^{2\tau\varphi_0} dxdt \leq \\ & \leq \int_{Z_{0,10\rho^2}} (|\nabla_{x,t}U|^2 + |U|^2) e^{2\tau\varphi_0} dxdt + \\ & + e^{2\tau e^{10\lambda\rho^2}} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} (|\nabla_{x,t}U|^2 + |U|^2) dxdt. \end{aligned} \quad (14.5.23)$$

Using (14.5.22) and (14.5.23) in (14.5.21), we obtain, by simple calculations (recall $\rho < 1$)

$$\begin{aligned} & (\tau - CM^2) \int_{Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) e^{2\tau\varphi_0} dxdt \leq \\ & \leq C \int_{Q_T} |F|^2 e^{2\tau\varphi_0} dxdt + \\ & + C\rho^{-8} e^{2\tau e^{10\lambda\rho^2}} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} ((|U|^2 + |\nabla_{x,t}U|^2) dxdt, \end{aligned} \quad (14.5.24)$$

for every $\tau \geq \tau_0$. We set $\tau_1 = \max\{\tau_0, (2CM^2)^{-1}\}$ and by (14.5.24) we obtain

$$\begin{aligned} & \int_{Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) e^{2\tau\varphi_0} dxdt \leq C \int_{Q_T} |F|^2 e^{2\tau\varphi_0} dxdt + \\ & + C\rho^{-8} e^{2\tau e^{10\lambda\rho^2}} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} ((|U|^2 + |\nabla_{x,t}U|^2) dxdt, \end{aligned} \quad (14.5.25)$$

for every $\tau \geq \tau_1$. Now, in (14.5.25), we estimate trivially from below the integral on the left-hand side and we estimate trivially the integrals on the right-hand side. We get

$$\begin{aligned} & e^{2\tau e^{12\lambda\rho^2}} \int_{Z_{0,12\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq \\ & \leq \int_{Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) e^{2\tau\varphi_0} dxdt \leq \\ & \leq C e^{2\tau e^\lambda} \int_{Q_T} |F|^2 e^{2\tau\varphi_0} dxdt + \\ & + C \rho^{-8} e^{2\tau e^{10\lambda\rho^2}} \int_{Z_{0,9\rho^2} \setminus Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt, \end{aligned}$$

which implies

$$\begin{aligned} & \int_{Z_{0,12\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq \\ & \leq \int_{Z_{0,10\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) e^{2\tau\varphi_0} dxdt \leq \\ & \leq C e^{2\tau(e^\lambda - e^{12\lambda\rho^2})} \int_{Q_T} |F|^2 dxdt + \\ & + C \rho^{-8} e^{2\tau(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \int_{Z_{0,10\rho^2} \setminus Z_{0,9\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq \\ & \leq C e^{2\tau(e^\lambda - e^{12\lambda\rho^2})} \int_{Q_T} |F|^2 dxdt + \\ & + C \rho^{-8} e^{2\tau(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \int_{Q_T} (|U|^2 + |\nabla_{x,t}U|^2) dxdt, \end{aligned} \tag{14.5.26}$$

for every $\tau \geq \tau_1$. By Lemma 14.5.2 and by (14.5.26) we have

$$\begin{aligned} & \int_{Z_{0,12\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq C e^{2\tau(e^\lambda - e^{12\lambda\rho^2})} \|F\|_{L^2(B_1 \times (-T, T))}^2 + \\ & + C \rho^{-9} e^{2\tau(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \int_{-2\rho}^{2\rho} \int_{B_1} (|U|^2 + |\nabla_{x,t}U|^2) dxdt, \end{aligned} \tag{14.5.27}$$

for every $\tau \geq \tau_1$, where C depends on M and T . At this point we note that, by using (14.3.33) for operator \square with $\varphi = \varphi_1$, we obtain an estimate similar to (14.5.27). More precisely we have

$$\begin{aligned}
& \int_{Z_{1,12\rho^2}} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq \\
& \leq C e^{2\tau(e^\lambda - e^{12\lambda\rho^2})} \|F\|_{L^2(B_1 \times (-T, T))}^2 + \\
& + C \rho^{-9} e^{2\tau(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \int_{-2\rho}^{2\rho} \int_{B_1} (|U|^2 + |\nabla_{x,t}U|^2) dxdt,
\end{aligned} \tag{14.5.28}$$

for every $\tau \geq \tau_1$. By (14.5.17b), (14.5.27) e (14.5.28) we have

$$\begin{aligned}
& \int_{-2\rho}^{2\rho} \int_{B_1} (|U|^2 + |\nabla_{x,t}U|^2) dxdt \leq \\
& \leq C e^{2\tau(e^\lambda - e^{12\lambda\rho^2})} \|F\|_{L^2(B_1 \times (-T, T))}^2 + \\
& + C \rho^{-9} e^{2\tau(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \int_{-2\rho}^{2\rho} \int_{B_1} (|U|^2 + |\nabla_{x,t}U|^2) dxdt,
\end{aligned}$$

For every $\tau \geq \tau_1$.

Let us choose $\tau = \tau_2 \geq \tau_1$, where τ_2 satisfies

$$C \rho^{-9} e^{2\tau_2(e^{10\lambda\rho^2} - e^{12\lambda\rho^2})} \leq \frac{1}{2}.$$

By this choice of τ the second term on the right-hand side of (14.5.27) is absorbed on the left-hand side, and we get (14.5.4). ■

Remark. Theorem 14.5.1 can be proved under less restrictive assumptions on U , but here we do not go into this question. Instead, we want to illustrate a simple and direct application to the proof **of the uniqueness of the following inverse problem** – which we discuss here only at the formal level – let F a function depending only on the variable x , let U be the solution to the following direct problem

$$\begin{cases} \partial_t^2 U - \Delta U = F(x), & \text{in } (x, t) \in B_1 \times (-T, T), \\ U(x, t) = h_0, & \text{for } (x, t) \in \partial B_1 \times (-T, T), \\ U(x, 0) = U_0(x), \quad U_t(x, 0) = U_1(x), & \text{for } x \in B_1, \end{cases} \tag{14.5.29}$$

where h_0, U_0, U_1 and F given functions. The proof of the uniqueness and the existence of solution to the direct problem (14.5.29), for a given $F \in L^2(B_1)$, can be found, for instance, in [23, Ch. 7]. Let us assume now to know also

$$\frac{\partial U}{\partial \nu} |_{\partial B_1 \times (-T, T)} = h_1,$$

we wish to determine F . Here we give a sketch of the proof of the uniqueness for the problem of determining F (inverse problem) by mean of h_0, h_1, U_0, U_1 . Since the problem is linear, it is enough to prove that if

$$U_0 = U_1 = 0, \quad h_0 = h_1 = 0,$$

then $F \equiv 0$.

Let U satisfy

$$\begin{cases} \partial_t^2 U - \Delta U = F(x), & \text{in } (x, t) \in B_1 \times (-T, T), \\ U(x, t) = 0, \quad \frac{\partial U}{\partial \nu} = 0 & \text{for } (x, t) \in \partial B_1 \times (-T, T), \\ U(x, 0) = 0, \quad U_t(x, 0) = 0, & \text{for } x \in B_1 \end{cases} \quad (14.5.30)$$

Set

$$w = \partial_t U$$

and differentiate both the sides of the equation with respect to t . Since F does not depend on t we have

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & \text{in } (x, t) \in B_1 \times (-T, T), \\ w(x, t) = 0, \quad \frac{\partial w}{\partial \nu} = 0 & \text{for } (x, t) \in \partial B_1 \times (-T, T). \end{cases} \quad (14.5.31)$$

Applying to w estimate (14.5.4) we get

$$w \equiv 0, \quad \text{in } B_1 \times (-T, T).$$

Hence

$$U(x, t) = U(x, 0) + \int_0^t w(x, s) ds = 0, \quad \text{in } B_1 \times (-T, T).$$

Therefore by (14.5.30) we have

$$F(x) = \partial_t^2 U - \Delta U = 0.$$

This proves the uniqueness for the inverse problem. The idea that we have outlined is only a miniature of a more general method of proving uniqueness and stability results for inverse problems related to evolution equations in which it is required the determination of the time independent coefficients of some equation. For further study we refer to [37], [39]. \blacklozenge

14.6 Geometric meaning of the pseudo – convexity. Some remarks on the necessary conditions.

In this Section we briefly discuss some necessary conditions on the weight exponent for a Carleman estimate to be valid. We state without proof a Hörmander Theorem [34, Theorem 8.1.1]. In some sense, Proposition 13.2.2 is a "miniature" of such a Theorem

Theorem 14.6.1 (necessary condition for the Carleman estimate).

Let Ω be a bounded open set of \mathbb{R}^n . Let $\varphi \in C^\infty(\bar{\Omega})$ satisfy $\nabla\varphi \neq 0$ in $\bar{\Omega}$. Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad (14.6.1)$$

be a linear differential operator of order m , such that $a_\alpha \in L^\infty(\Omega)$, $|\alpha| \leq m$ and $a_\alpha \in C^1(\bar{\Omega})$, $|\alpha| = m$.

Let us suppose that there exist $K_1 > 0$ and τ_0 such that for every $u \in C_0^\infty(\Omega, \mathbb{C})$ and for every $\tau \geq \tau_0$ we have

$$\tau \sum_{|\alpha| \leq m-1} \binom{m-1}{\alpha} \int_{\Omega} |D^\alpha u|^2 e^{2\tau\varphi} dx \leq K_1 \int_{\Omega} |P(x, D)u|^2 e^{2\tau\varphi} dx. \quad (14.6.2)$$

Then, setting $\zeta = \xi + i\sigma\nabla\varphi(x)$ where $x \in \Omega$, $\xi \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$, we have

$$\begin{aligned} & \begin{cases} P_m(x, \zeta) = 0 \\ \sigma \neq 0 \end{cases} \implies \\ & \implies |\zeta|^2 \leq 2K_1 \left[\sum_{j,k=1}^n \partial_{jk}^2 \varphi(x) P_m^{(j)}(x, \zeta) \overline{P_m^{(k)}(x, \zeta)} + \right. \\ & \left. + \sigma^{-1} \Im \sum_{k=1}^n P_{m,k}(x, \zeta) \overline{P_m^{(k)}(x, \zeta)} \right]. \end{aligned} \quad (14.6.3)$$

As already noted in (13.4.18) the expression on the right-hand side of the implication (14.6.3) can be written by Poisson brackets and it is equal to

$$\frac{i}{2\sigma} \left\{ P_m(x, \xi + i\sigma\nabla\varphi), \overline{P_m(x, \xi + i\sigma\nabla\varphi)} \right\}.$$

Remark. Let us observe that if $P(x, D)$ is an elliptic operator and estimate (14.6.2) holds true, then Theorem 14.6.1 implies that necessary condition (14.6.3) holds true in any open set $\tilde{\Omega}$ compactly contained in Ω this, in turn, implies that condition (★) of Theorem 13.5.1 is satisfied in $\tilde{\Omega}$. Therefore such a Theorem 13.5.1 gives the estimate

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \int |D^\alpha u|^2 e^{2\tau\varphi} dx \leq C \int |P(x, D)u|^2 e^{2\tau\varphi} dx,$$

for every $u \in C_0^\infty(\tilde{\Omega})$ and for every τ large enough. ♦

In Theorem 14.6.1 we provided necessary conditions for the validity of Carleman estimates. To establish necessary conditions for the unique continuation property we would need deeper the exploration of the notion of pseudo-convexity which we mentioned in the previous Section for second-order operators. For this kind of further investigation we refer to Chapters 4, 5, 6 of the book [50].

Geometric interpretation of pseudo-convexity condition.

Let

$$P_2(x, \xi) = \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k. \tag{14.6.4}$$

where $\{g^{jk}\}_{j,k=1}^n$ is a nonsingular symmetric matrix, whose entries are **real constants**. Let $\phi \in C^2(\bar{\Omega})$ where Ω is a bounded open set of \mathbb{R}^n , and let us suppose that $\nabla\phi \neq 0$ in $\bar{\Omega}$. Let $x_0 \in \Omega$ and set

$$\xi_0 = \nabla\phi(x_0). \tag{14.6.5}$$

Let us consider the Hamiltonian system associated to $P_2(x, \xi)$ which, as operator P_2 has constant coefficients, is

$$\begin{cases} \dot{x}^j = P_2^{(j)}(\xi(t)), & j = 1, \dots, n, \\ \dot{\xi}_j = -P_{2,j}(\xi(t)) (= 0), & j = 1, \dots, n. \end{cases} \tag{14.6.6}$$

A solution to (14.6.6) on an interval J is the line $(x(t), \xi(t))$ of $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ which is called **bicharacteristic line** of the operator $P_2(D)$. Let us note that by (14.6.6) we have

$$\frac{d}{dt} P_2(\xi(t)) = \dot{\xi}_j P_2^{(j)}(x(t), \xi(t)) = 0, \quad \forall t \in J. \tag{14.6.7}$$

If

$$P_2(\xi(t)) = 0, \quad \forall t \in J,$$

we say that $(x(t), \xi(t))$ is a **null bicharacteristic line**. In particular, we have that if $(x(t), \xi(t))$ is a null bicharacteristic line in a point t_0 then it is null bicharacteristic line in the whole interval J . As a matter of fact, if

$$P_2(\xi(t_0)) = 0,$$

then (14.6.7) implies $P_2(\xi(t)) = 0$, for every $t \in J$. Let us notice that in each point $(x(t), \xi(t))$ of a null bicharacteristic line, $\xi(t)$ is a characteristic direction for operator P_2 in the point $x(t)$. The projection $x(\cdot)$, on \mathbb{R}_x^n of a null bicharacteristic line is called **ray** of P_2 (see Section 5.5). Now, if $x(t)$ is a ray of P_2 , then

$$\frac{d}{dt}\phi(x(t)) = \partial_j\phi(x(t))\frac{dx^j}{dt} = P_2^{(j)}(\xi(t))\partial_j\phi(x(t)) \quad (14.6.8)$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\phi(x(t)) &= \frac{d}{dt}\left(P_2^{(j)}(\xi(t))\partial_j\phi(x(t))\right) = \\ &= \partial_{jk}^2\phi(x(t))\frac{dx^k}{dt}P_2^{(j)}(\xi(t)) + P_2^{(jk)}(\xi(t))\frac{d\xi_k}{dt}\partial_j\phi(x(t)) = \\ &= \partial_{jk}^2\phi(x(t))P_2^{(j)}(\xi(t))P_2^{(k)}(\xi(t)). \end{aligned} \quad (14.6.9)$$

Let us suppose that in $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ we have

$$P_2(\xi_0) = 0, \quad (14.6.10)$$

let $x_0 \in \Omega$ and let us consider the solution $(x(t), \xi(t))$ of system (14.6.6) satisfying the initial condition in $t_0 \in J$

$$x(t_0) = x_0, \quad \xi(t_0) = \xi_0. \quad (14.6.11)$$

We quickly notice that (14.6.8) allows us to write condition (14.4.32) of Theorem 14.4.2 (see also Proposition 14.4.3) assuming there (14.6.10) and (14.6.11) as follows

$$\frac{d}{dt}\phi(x(t))|_{t=t_0} \neq 0. \quad (14.6.12)$$

This is equivalent to the fact that the ray $x(t)$ passing through x_0 is transverse to the level surface

$$\Gamma_\phi = \{x \in \Omega : \phi(x) = \phi(x_0)\}.$$

Furthermore, (14.6.8) and (14.6.9) allow us to write the (14.3.9) (in x_0) in the form

$$\frac{d}{dt}\phi(x(t))|_{t=t_0} = 0 \implies \frac{d^2}{dt^2}\phi(x(t))|_{t=t_0} > 0, \quad (14.6.13)$$

this implies that if t_0 is a critical point of $\phi(x(t))$ then it is a proper minimum point of the function $\phi(x(t))$. Condition (14.6.13) can also be formulated in the following manner: let us consider the above defined level surface Γ_ϕ and the level set

$$\Omega_\phi^+ = \{x \in \Omega : \phi(x) > \phi(x_0)\},$$

then (14.6.13) says that if the ray $x(t)$ is tangent in x_0 to the level surface Γ_ϕ (this is expressed by the antecedent of implication (14.6.13)), then for every t in a neighborhood of t_0 , we have, for $t \neq t_0$, $x(t) \in \Omega_\phi^+$. In other words, the ray $x(t)$ cannot "cross" the level surface Γ_ϕ at points where $x(t)$ is tangent to Γ_ϕ .

Chapter 15

Optimal three sphere and doubling inequality for second order elliptic equations

15.1 Introduction

In this Chapter we will prove the **strong unique continuation property** for the second order elliptic equations with real coefficients (in the principal part). Now we recall briefly this property and provide an introduction to the Chapter.

Let $\{a^{ij}(x)\}_{i,j=1}^n$ be a **symmetric matrix of real-valued functions**. We assume that the following uniform ellipticity condition is satisfied

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in B_1,$$

where $\lambda \geq 1$. Let us assume that the function a^{ij} , $i, j = 1, \dots, n$ are Lipschitz continuous

$$|a^{ij}(x) - a^{ij}(y)| \leq \Lambda |x - y|, \quad \text{for } i, j \in \{1, \dots, n\}, \quad \forall x, y \in B_1. \quad (15.1.1)$$

Let $b^i \in L^\infty(B_1)$, $i = 1, \dots, n$ and $c \in L^\infty(B_1)$ (these coefficients can also be complex-valued) satisfy

$$\|b^i\|_{L^\infty(B_1)} \leq M, \quad \text{for } i = 1, \dots, n$$

and

$$\|c\|_{L^\infty(B_1)} \leq M. \quad (15.1.2)$$

We recall that the equation

$$Lu = \sum_{i,j=1}^n a^{ij}(x) \partial_{x^i x^j}^2 u + \sum_{i=1}^n b^i(x) \partial_{x^i} u + c(x)u = 0, \quad \text{in } B_1, \quad (15.1.3)$$

enjoys the **strong unique continuation property** provided that any solution u to (15.1.3) satisfying the conditions

$$\int_{B_r} |u|^2 dx = \mathcal{O}(r^m), \quad \text{as } r \rightarrow 0, \quad \forall m \in \mathbb{N}, \quad (15.1.4)$$

identically vanishes. We will prove the strong unique continuation property as a consequence of an **optimal three sphere inequality**. A prototype of such an inequality is the Hadamard three circle inequality for the holomorphic functions that we first encountered in Section 10.4.

Generally speaking, a three sphere inequality for solutions to the equation (15.1.3) is an inequality of the type

$$\int_{B_\rho} |u|^2 dx \leq C \left(\int_{B_R} |u|^2 dx \right)^{1-\theta} \left(\int_{B_r} |u|^2 dx \right)^\theta, \quad (15.1.5)$$

where $0 < r < \rho < R \leq 1$, C and $\theta \in (0, 1)$ depend by λ , Λ , M and R , ρ (C and θ do not depend on u).

We say that (15.1.5) is an **optimal three spheres inequality** provided C **does not depend** on r and, for fixed R, ρ , we have

$$\theta \sim |\log r|^{-1}, \quad \text{as } r \rightarrow 0. \quad (15.1.6)$$

We recall that by $f(r) \sim g(r)$, as $r \rightarrow 0$ we means

$$0 < \lim_{r \rightarrow 0} \frac{f(r)}{g(r)} < +\infty.$$

First author that proved (15.1.5) was Landis in [45].

Arguing like in Remark 5 of Section 10.4, it is easy to check that if a function u satisfies an optimal three sphere inequality, then whenever u satisfies (15.1.4) it vanishes identically. Therefore, if an optimal three sphere inequality holds true for equation (15.1.4) then such an equation satisfies the strong unique continuation property.

Another type of inequality that implies the strong unique continuation property is the so-called **doubling inequality**. Such an inequality occurs in the form

$$\int_{B_{2r}} |u|^2 dx \leq K \int_{B_r} |u|^2 dx, \quad \forall r \in \left(0, \frac{1}{2}\right), \quad (15.1.7)$$

where K depends on u but **does not depend on** r . We will prove later on in which a way (15.1.7) implies the strong unique continuation property. The main idea may be expressed as follows. Iterating inequality (15.1.7) we have, for every $j \in \mathbb{N}$

$$\int_{B_{1/2}} |u|^2 dx \leq K \int_{B_{1/4}} |u|^2 dx \leq \dots \leq K^{j-1} \int_{B_{1/2^j}} |u|^2 dx$$

which, together with (15.1.4), provides, for each $j, m \in \mathbb{N}$ (C_m depends on m only),

$$\int_{B_{1/2}} |u|^2 dx \leq K^{j-1} C_m \left(\frac{1}{2^j}\right)^m = C_m K^{-1} \left(\frac{K}{2^m}\right)^j. \quad (15.1.8)$$

Let now m satisfy

$$2^m > K$$

and passing to the limit as $j \rightarrow \infty$, we get by (15.1.8)

$$\int_{B_{1/2}(0)} |u|^2 dx = 0.$$

Both the optimal three sphere inequality and the doubling inequality will be obtained from an appropriate Carleman estimate for the elliptic operator L (or, equivalently, for the principal part of that operator). We will approach this question in two phases: first we will study the case of the Laplace operator Δ and then we will study the case of equations with variable coefficients. In both the cases, the proofs of the Carleman estimates will start from rewriting the elliptic operators in polar coordinates: in the case of Laplace operator, in Euclidean polar coordinates; in the case of variable coefficients, in geodesic polar coordinates w.r.t. the Riemannian structure induced by a metric conforming to

$$a_{ij} dx^i \otimes dx^j,$$

($\{a_{ij}(x)\}_{i,j=1}^n$ is the inverse of the matrix $\{a^{ij}(x)\}_{i,j=1}^n$). We warn that throughout this Chapter we will use the Einstein convention of repeated indices. We will, in addition, adhere more scrupulously to the notation on indices of the components of a tensor. Actually, these notations are mostly needed in the Sections 15.6 and 15.7, but we will adopt it in the preceding sections as well. The proof in the case of the Laplace operator presents most of the main

difficulties that we will encounter in the case of variable coefficient, which, of course, presents additional technical difficulties. In addition to the proofs that we give here, There exist other proofs in the literature, e.g. [22], [35].

15.2 Formulas for the change of variables of second order operators

We begin by deriving a formula to the change of variables of the operator

$$\operatorname{div}(A(x)\nabla u(x)), \quad (15.2.1)$$

where $A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ is a symmetric matrix, a^{ij} are sufficiently smooth functions for $i, j = 1, \dots, n$ (will suffice $a^{ij} \in C^{0,1}(\mathbb{R}^n)$).

Let us consider the case where $A(x)$ is the identity matrix. Let Λ be an open set of \mathbb{R}^n and let $\Phi \in C^1(\bar{\Lambda}, \mathbb{R}^n)$ be a injective map such that

$$\det(J\Phi(x)) \neq 0, \quad \forall x \in \bar{\Lambda}, \quad (15.2.2)$$

where $J\Phi(x)$ is the jacobian matrix of Φ in x

$$J\Phi(x) = \begin{pmatrix} \partial_{x^1}\Phi^1(x) & \cdots & \partial_{x^n}\Phi^1(x) \\ \vdots & \ddots & \vdots \\ \partial_{x^1}\Phi^n(x) & \cdots & \partial_{x^n}\Phi^n(x) \end{pmatrix}.$$

Set

$$\Omega = \Phi(\Lambda)$$

and

$$\Psi = \Phi^{-1},$$

consequently $\Psi \in C^1(\bar{\Omega}, \mathbb{R}^n)$. Let us prove that, if $u \in C^2(\bar{\Lambda})$, then

$$(\Delta u)(\Psi(y)) = \frac{1}{|\det(J\Psi(y))|} \operatorname{div}(B(y)v(y)), \quad (15.2.3)$$

where

$$v(y) = u(\Psi(y)) \quad (15.2.4)$$

and

$$B(y) = |\det(J\Psi(y))| (J\Psi(y))^{-1} ((J\Psi(y))^{-1})^T. \quad (15.2.5)$$

More generally, for operator (15.2.1) we have

$$\operatorname{div}(A(x)\nabla u(x))|_{x=\Psi(y)} = \frac{1}{|\det(J\Psi(y))|} \operatorname{div}(\tilde{A}(y)v(y)), \quad (15.2.6)$$

where v is given by (15.2.4) and

$$\tilde{A}(y) = |\det(J\Psi(y))| (J\Psi(y))^{-1} A(\Psi(y)) ((J\Psi(y))^{-1})^T. \quad (15.2.7)$$

Since (15.2.6) can be proved similarly, we limit ourselves to prove (15.2.3).

Proof of (15.2.3). Let $w \in C_0^\infty(\Omega)$ and let

$$\tilde{w}(x) = w(\Phi(x)), \quad \forall x \in \Lambda.$$

Integrating by parts, we have

$$-\int_{\Lambda} \Delta u(x) \tilde{w}(x) dx = \int_{\Lambda} \nabla u(x) \cdot \nabla \tilde{w}(x) dx \quad (15.2.8)$$

and by the formula of change of variables for multiple integrals, we have

$$\begin{aligned} \int_{\Lambda} \nabla_x u(x) \cdot \nabla_x \tilde{w}(x) dx &= \\ &= \int_{\Omega} (\nabla_x u)(\Psi(y)) \cdot (\nabla_x \tilde{w})(\Psi(y)) |\det(J\Psi(y))| dy. \end{aligned} \quad (15.2.9)$$

Now, (15.2.4) gives

$$\nabla_y v(y) = (J\Psi(y))^T (\nabla_x u)(\Psi(y)),$$

($\nabla_x u$ and $\nabla_y v$ are column vectors). Hence

$$(\nabla_x u)(\Psi(y)) = ((J\Psi(y))^{-1})^T \nabla_y v(y), \quad (15.2.10)$$

similarly

$$(\nabla_x \tilde{w})(\Psi(y)) = ((J\Psi(y))^{-1})^T \nabla_y w(y). \quad (15.2.11)$$

Substituting (15.2.10) and (15.2.11) into the integral on the right-hand side of (15.2.9) and recalling (15.2.8), we have

$$\begin{aligned} -\int_{\Lambda} \Delta u(x) \tilde{w}(x) dx &= \int_{\Omega} B(y) \nabla_y v(y) \cdot \nabla_y w(y) dy = \\ &= -\int_{\Omega} \operatorname{div}_y (B(y) \nabla_y v(y)) w(y) dy. \end{aligned} \quad (15.2.12)$$

Now, we change the variables in the integral on the left-hand side of (15.2.12) and we find

$$- \int_{\Omega} (\Delta u)(\Psi(y))w(y) |\det(J\Psi(y))| dy = - \int_{\Omega} \operatorname{div}_y (B(y)\nabla_y v(y)) w(y) dy.$$

Since w is arbitrary in $C_0^\infty(\Omega)$, we get (15.2.3).

Exercise 1. Apply formula (15.2.3) to write the Laplace operator, in \mathbb{R}^2 , in polar coordinates showing that, setting

$$(x^1, x^2) = \Psi(\varrho, \vartheta) = (\varrho \cos \vartheta, \varrho \sin \vartheta), \quad (\varrho, \vartheta) \in (0, +\infty) \times [0, 2\pi)$$

and

$$v(\varrho, \vartheta) = u(\Psi(\varrho, \vartheta)) = u(\varrho \cos \vartheta, \varrho \sin \vartheta),$$

we have

$$(\Delta u)(\varrho \cos \vartheta, \varrho \sin \vartheta) = \partial_\varrho^2 v + \frac{1}{\varrho} \partial_\varrho v + \frac{1}{\varrho^2} \partial_\vartheta^2 v. \quad (15.2.13)$$



Exercise 2. Apply formula (15.2.3) to write the Laplace operator, in \mathbb{R}^3 , in polar coordinates showing that, setting

$$(x^1, x^2, x^3) = \Psi(\varrho, \vartheta, \phi) = (\varrho \sin \vartheta \cos \phi, \varrho \sin \vartheta \sin \phi, \varrho \cos \vartheta), \quad (15.2.14)$$

where for $(\varrho, \vartheta, \phi) \in (0, +\infty) \times (0, \pi) \times (0, 2\pi)$ and

$$v(\varrho, \vartheta, \phi) = u(\Psi(\varrho, \vartheta, \phi)) = u(\varrho \sin \vartheta \cos \phi, \varrho \sin \vartheta \sin \phi, \varrho \cos \vartheta),$$

we have

$$\begin{aligned} (\Delta u)(\varrho \sin \vartheta \cos \phi, \varrho \sin \vartheta \sin \phi, \varrho \cos \vartheta) &= \\ &= \partial_\varrho^2 v + \frac{2}{\varrho} \partial_\varrho v + \frac{1}{\varrho^2 \sin \vartheta} \partial_\vartheta (\sin \vartheta \partial_\vartheta v) + \frac{1}{\varrho^2 \sin^2 \vartheta} \partial_\phi^2 v. \end{aligned} \quad (15.2.15)$$



15.3 Polar coordinates in \mathbb{R}^n . The Laplace–Beltrami operator on the sphere

In the n -dimensional case, it is possible to express the Laplace operator in polar coordinates by means some formulas similar to (15.2.13) and to (15.2.15). Nevertheless, in this way we come to write quite cumbersome formulas. For

this reason we present here a procedure, which consists, first of all, in defining the Laplace operator on the sphere

$$\Sigma = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

Concerning the polar coordinates in \mathbb{R}^n if one wishes to follow the path suggested in Exercise 1 and 2 of the previous Section, it would be convenient to write a transformation similar to (15.2.14). For this purpose it would suffice to use the following recursive formula

$$\left\{ \begin{array}{l} r_n = |x| \\ x^n = r_n \cos \vartheta^n, \quad \vartheta^n \in [0, 2\pi), \\ r_{n-1} = r_n \sin \vartheta^n, \quad \vartheta^n \in [0, 2\pi), \\ x^{n-1} = r_{n-1} \cos \vartheta^{n-1}, \quad \vartheta^{n-1} \in [0, \pi) \\ \dots \\ x^1 = r_1 \cos \vartheta^2, \quad \vartheta^2 \in [0, \pi). \end{array} \right. \quad (15.3.1)$$

In this way, the "angular coordinates" are $(\vartheta^2, \vartheta^3, \dots, \vartheta^n)$. By using (15.3.1) one reaches to the following formula of change of variables in polar coordinates (f integrable function)

$$\begin{aligned} \int_{B_R} f(x) dx &= \int_0^R \left(\int_{\Sigma} f(\rho\omega) \rho^{n-1} dS \right) d\rho = \\ &= \int_0^R \left(\int_{\partial B_\rho} f(y) dS_y \right) d\rho, \end{aligned} \quad (15.3.2)$$

where $R > 0$.

In order to write the Laplace operator in polar coordinates we proceed in two steps:

Step I. We will write the Laplace–Beltrami operator on the sphere Σ ;

Step II. We will complete the transformation begun in Step I and we will prove Theorem 15.3.6.

Step I. Let $u(\omega)$ be a real-valued function on Σ . We associate to u the homogeneous function of degree 0 defined by

$$\tilde{u} : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}.$$

$$\tilde{u}(x) = u\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\}. \quad (15.3.3)$$

If $\tilde{u} \in C^k(\mathbb{R}^n \setminus \{0\})$, we say that u belongs to $C^k(\Sigma)$. We define the scalar product of two functions $u, v \in C^0(\Sigma)$ as

$$(u, v)_\Sigma = \int_\Sigma u(\omega)v(\omega)d\omega \quad (15.3.4)$$

($d\omega := dS$) and we set

$$\|u\|_{L^2(\Sigma)} = \int_\Sigma |u(\omega)|^2 d\omega; \quad (15.3.5)$$

Hence $C^0(\Sigma)$, equipped with the scalar scalar product (15.3.4), is a prehilbertian space and $L^2(\Sigma)$ is the completion of that space.

For any $1 \leq i \leq n$ we define the operator d_i as follows. Let $u \in C^1(\Sigma)$, let us denote

$$(d_i u)(\omega) = (\partial_{x^i} \tilde{u}(x))|_{x=\omega}, \quad \omega \in \Sigma. \quad (15.3.6)$$

Other symbols which are used in literature to denote $d_i u$ are: $\Omega_i u, \partial_{\omega^i} u$.

Remarks.

1. By (15.3.6) we have for $j, k = 1, \dots, n$

$$\begin{aligned} (d_i u)(\omega) &= \left(\left(\partial_{x^i} \frac{x^k}{|x|} \right) (\partial_{x^k} \tilde{u}) \left(\frac{x}{|x|} \right) \right) |_{x=\omega} = \\ &= \frac{1}{|x|} \left((\partial_{x^i} \tilde{u}) \left(\frac{x}{|x|} \right) - \frac{x^i}{|x|} \frac{x^k}{|x|} (\partial_{x^k} \tilde{u}) \left(\frac{x}{|x|} \right) \right) |_{x=\omega}. \end{aligned} \quad (15.3.7)$$

Let us note that the function written in the brackets in the second line is the i -th component of

$$(\nabla \tilde{u}) \left(\frac{x}{|x|} \right) - \frac{x}{|x|} \left(\frac{x}{|x|} \cdot (\nabla \tilde{u}) \left(\frac{x}{|x|} \right) \right), \quad (15.3.8)$$

which, in turn, is the i -th tangential component on $\{|x| = 1\}$ of the gradient of \tilde{u} .

2. The operators $d_i, 1 \leq i \leq n$, **are not independent**. As a matter of fact, by (15.3.6) and by the fact that \tilde{u} is a homogeneous function of degree 0, we have

$$\sum_{i=1}^n \omega^i (d_i u)(\omega) = \sum_{i=1}^n \left(\frac{x^i}{|x|} (\partial_{x^i} \tilde{u}) \left(\frac{x}{|x|} \right) \right) |_{x=\omega} = 0. \quad (15.3.9)$$

Moreover, by (15.3.6) we have, for every $u, v \in C^1(\Sigma)$,

$$d_i(uv) = (d_i u)v + u(d_i v), \quad 1 \leq i \leq n. \quad (15.3.10)$$

Also let us note

$$\partial_{x^i} \tilde{u}(x) = \frac{1}{|x|} (d_i u) \left(\frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\}, \quad 1 \leq i \leq n. \quad (15.3.11)$$

As a matter of fact, by homogeneity we have

$$\tilde{u}(x) = \tilde{u}(\lambda x) \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

Hence

$$\partial_{x^i} \tilde{u}(x) = \lambda (\partial_{x^i} \tilde{u})(\lambda x), \quad 1 \leq i \leq n$$

from which, choosing $\lambda = \frac{1}{|x|}$, we get

$$\partial_{x^i} \tilde{u}(x) = \frac{1}{|x|} (\partial_{x^i} \tilde{u}) \left(\frac{x}{|x|} \right) = \frac{1}{|x|} (d_i u) \left(\frac{x}{|x|} \right), \quad 1 \leq i \leq n,$$

from which (15.3.11) follows. \blacklozenge

Proposition 15.3.1. *Let $n > 1$. We have*

$$(d_i u, v)_\Sigma = (u, d_i^* v)_\Sigma, \quad \forall u, v \in C^1(\Sigma), \quad 1 \leq i \leq n, \quad (15.3.12)$$

where

$$d_i^* = (n-1)\omega^i - d_i, \quad 1 \leq i \leq n. \quad (15.3.13)$$

d_i^* is the formal adjoint of d_i with respect to the scalar product (15.3.4).

Proof. First, let us notice that, for any $1 \leq i \leq n$, $\partial_{x^i} \tilde{u}(x)$ is integrable over B_1 because

$$|\partial_{x^i} \tilde{u}(x)| \leq \frac{1}{|x|} \max_\Sigma |\nabla \tilde{u}|, \quad 1 \leq i \leq n$$

and $n > 1$.

Now, let us check that, for any $1 \leq i \leq n$, we have

$$\int_\Sigma d_i u(\omega) v(\omega) d\omega = (n-1) \int_{B_1} \partial_{x^i} \tilde{u}(x) \tilde{v}(x) dx. \quad (15.3.14)$$

We check (15.3.14). If $\varepsilon \in (0, 1)$ and $1 \leq i \leq n$, we have by (15.3.2) and by (15.3.11),

$$\begin{aligned} \int_{B_1 \setminus B_\varepsilon} \partial_{x^i} \tilde{u}(x) \tilde{v}(x) dx &= \int_{B_1 \setminus B_\varepsilon} \frac{1}{|x|} (d_i u) \left(\frac{x}{|x|} \right) v \left(\frac{x}{|x|} \right) dx = \\ &= \int_\varepsilon^1 \rho^{n-2} \int_\Sigma d_i u(\omega) v(\omega) d\omega = \\ &= \frac{1 - \varepsilon^{n-1}}{n-1} \int_\Sigma d_i u(\omega) v(\omega) d\omega. \end{aligned}$$

Passing to the limit as $\varepsilon \rightarrow 0$ we obtain (15.3.14).

Now, by (15.3.14) and integrating by parts we have

$$\begin{aligned} \frac{1}{n-1} (d_i u, v)_\Sigma &= \int_{B_1} \partial_{x^i} \tilde{u}(x) \tilde{v}(x) dx = \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} \partial_{x^i} \tilde{u}(x) \tilde{v}(x) dx = \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} (\partial_{x^i} (\tilde{u}\tilde{v}) - \tilde{u} \partial_{x^i} \tilde{v}) dx = \tag{15.3.15} \\ &= \int_\Sigma u(\omega) v(\omega) \omega^i d\omega - \\ &\quad - \lim_{\varepsilon \rightarrow 0} \left(\int_{\partial B_\varepsilon} \tilde{u}(x) \tilde{v}(x) \frac{x^i}{|x|} dS + \int_{B_1 \setminus B_\varepsilon} \tilde{u}(x) \partial_{x^i} \tilde{v}(x) dx \right). \end{aligned}$$

On the other hand

$$\int_{\partial B_\varepsilon} \tilde{u}(x) \tilde{v}(x) \frac{x^i}{|x|} dS = \mathcal{O}(\varepsilon^{n-1}), \quad \text{as } \varepsilon \rightarrow 0,$$

hence we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \tilde{u}(x) \tilde{v}(x) \frac{x^i}{|x|} dS = 0.$$

Therefore (15.3.15) gives

$$\frac{1}{n-1} (d_i u, v)_\Sigma = \int_\Sigma u(\omega) v(\omega) \omega^i d\omega - \int_{B_1} \tilde{u}(x) \partial_{x^i} \tilde{v}(x) dx$$

and employing (15.3.14) (interchange u with v in the latter) we get

$$\begin{aligned} \frac{1}{n-1} (d_i u, v)_\Sigma &= \int_\Sigma u(\omega) \left(v(\omega) \omega^i - \frac{1}{n-1} d_i v(\omega) \right) d\omega = \\ &= \frac{1}{n-1} (u, (n-1)\omega^i v - d_i v)_\Sigma. \end{aligned}$$

By the just obtained equality, (15.3.12) immediately follows. ■

Definition 15.3.2 (Laplace – Beltrami operator on the sphere). The Laplace–Beltrami operator on the sphere is defined as

$$\Delta_\Sigma = \sum_{i=1}^n d_i^2. \quad (15.3.16)$$

Proposition 15.3.3. *We have*

$$(\Delta_\Sigma u, v)_\Sigma = - \sum_{i=1}^n (d_i u, d_i v)_\Sigma, \quad \forall u, v \in C^2(\Sigma). \quad (15.3.17)$$

Proof. By (15.3.9) and (15.3.12) we get

$$\begin{aligned} (\Delta_\Sigma u, v)_\Sigma &= \sum_{i=1}^n (d_i^2 u, v)_\Sigma = \sum_{i=1}^n (d_i u, d_i^* v)_\Sigma = \\ &= \sum_{i=1}^n (d_i u, (n-1)\omega^i v - d_i v)_\Sigma = \\ &= (n-1) \left(\sum_{i=1}^n \omega^i d_i u, v \right)_\Sigma - \sum_{i=1}^n (d_i u, d_i v)_\Sigma = \\ &= - \sum_{i=1}^n (d_i u, d_i v)_\Sigma. \end{aligned}$$

■

Remark. Proposition (15.3.3) implies

$$(\Delta_\Sigma u, u)_\Sigma = - \sum_{i=1}^n \|d_i u\|_{L^2(\Sigma)}^2 \leq 0, \quad \forall u \in C^2(\Sigma) \quad (15.3.18)$$

and

$$\int_{\Sigma} \Delta_{\Sigma} u(\omega) d\omega = (\Delta_{\Sigma} u, 1)_{\Sigma} = 0. \quad (15.3.19)$$

◆

Proposition 15.3.4. *Si ha, per $i, j = 1, \dots, n$,*

$$d_j d_i u - d_i d_j u = \omega_j d_i u - \omega_i d_j u, \quad \forall u \in C^2(\Sigma). \quad (15.3.20)$$

Proof. By (15.3.11) we have

$$(d_i u) \left(\frac{x}{|x|} \right) = |x| \partial_{x^i} \tilde{u}(x), \quad 1 \leq i \leq n, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Hence

$$\begin{aligned} d_j d_i u(\omega) &= \partial_{x^j} (|x| \partial_{x^i} \tilde{u}(x))|_{x=\omega} = \\ &= ((\partial_{x^j} |x|) \partial_{x^i} \tilde{u}(x) + |x| \partial_{x^i x^j}^2 \tilde{u}(x))|_{x=\omega} = \\ &= \left(\frac{x^j}{|x|} \partial_{x^i} \tilde{u}(x) \right)|_{x=\omega} + (\partial_{x^i x^j}^2 \tilde{u})(\omega) = \\ &= \omega^j d_i u(\omega) + (\partial_{x^i x^j}^2 \tilde{u})(\omega) \end{aligned} \quad (15.3.21)$$

and, similarly,

$$d_i d_j u(\omega) = \omega^i d_j u(\omega) + (\partial_{x^j x^i}^2 \tilde{u})(\omega). \quad (15.3.22)$$

Subtracting (15.3.22) by (15.3.21) we obtain (15.3.20). ■

Step II. We associate to $u \in C^2(B_R \setminus \{0\})$, where $R > 0$, the function $U \in C^2((0, R) \times (\mathbb{R}^n \setminus \{0\}))$

$$U(\rho, y) = u \left(\rho \frac{y}{|y|} \right), \quad \rho \in (0, R), \quad y \in \mathbb{R}^n \setminus \{0\}. \quad (15.3.23)$$

It is obvious that if $\rho = |x|$ and $\omega = \frac{x}{|x|}$, then we have

$$U(\rho, \omega) = u \left(|x| \frac{x}{|x|} \right) = u(x). \quad (15.3.24)$$

Proposition 15.3.5. *Let U be defined by (15.3.23). Then we have, for any $i = 1, \dots, n$,*

$$\partial_{x^i} u(x) = \omega^i \partial_{\rho} U(\rho, \omega) + \frac{1}{\rho} d_i U(\rho, \omega). \quad (15.3.25)$$

where $\rho = |x|$ and $\omega = \frac{x}{|x|}$

Proof. By (15.3.24) we have

$$\begin{aligned}
 \partial_{x^i} u(x) &= (\partial_{x^i} |x|) \partial_\rho U \left(|x|, \frac{x}{|x|} \right) + \\
 &+ \sum_{k=1}^n \partial_{x^i} \left(\frac{x_k}{|x|} \right) \partial_{y^k} U \left(|x|, \frac{x}{|x|} \right) = \\
 &= \frac{x^i}{|x|} \partial_\rho U \left(|x|, \frac{x}{|x|} \right) + \\
 &+ \sum_{k=1}^n \left[\frac{\delta^{ik}}{|x|} - \frac{x^i x^k}{|x|^3} \right] \partial_{y^k} U \left(|x|, \frac{x}{|x|} \right).
 \end{aligned} \tag{15.3.26}$$

Now, (15.3.7) implies

$$\sum_{k=1}^n \left[\frac{\delta^{ik}}{|x|} - \frac{x^i x^k}{|x|^3} \right] \partial_{y^k} U \left(|x|, \frac{x}{|x|} \right) = \frac{1}{|x|} d_i U(\rho, \omega).$$

By the latter and by (15.3.26) we get (15.3.25). ■

Theorem 15.3.6. *Let $u \in C^2(B_R \setminus \{0\})$ and let U be defined by (15.3.23). Set*

$$U(\rho, \omega) = u(\rho\omega), \quad \forall (\rho, \omega) \in (0, R) \times \Sigma. \tag{15.3.27}$$

We have $v \in C^2((0, R) \times \Sigma)$ and

$$(\Delta u)(\rho\omega) = \partial_\rho^2 v(\rho, \omega) + \frac{n-1}{\rho} \partial_\rho v(\rho, \omega) + \frac{1}{\rho^2} \Delta_\Sigma v(\rho, \omega) \tag{15.3.28}$$

Proof. We denote, for the sake of brevity,

$$\rho(x) = |x|, \quad \omega(x) = \frac{x}{|x|}.$$

Let $i \leq i \leq n$; (15.3.25) implies

$$\partial_{x^i}^2 u(x) = \underbrace{\partial_{x^i} \left(\omega^i(x) \partial_\rho U(\rho(x), \omega(x)) \right)}_{J_1} + \underbrace{\partial_{x^i} \left(\frac{1}{\rho(x)} d_i U(\rho(x), \omega(x)) \right)}_{J_2}. \tag{15.3.29}$$

Computation of J_1 .

We have

$$\begin{aligned}
J_1 &= \partial_{x^i} (\omega^i(x)) \partial_\rho U(\rho(x), \omega(x)) + \omega^i(x) \partial_{x^i} (\partial_\rho U(\rho(x), \omega(x))) = \\
&= \left(\frac{1}{|x|} - \frac{(x^i)^2}{|x|^3} \right) \partial_\rho U(\rho(x), \omega(x)) + \\
&+ \omega^i(x) \left[\omega^i(x) \partial_\rho^2 U(\rho(x), \omega(x)) + \frac{1}{\rho(x)} (d_i \partial_\rho U)(\rho(x), \omega(x)) \right] = \quad (15.3.30) \\
&= \left(\frac{1}{\rho(x)} - \frac{\omega_i^2(x)}{\rho(x)} \right) \partial_\rho U(\rho(x), \omega(x)) + \\
&+ (\omega^i(x))^2 \partial_\rho^2 U(\rho(x), \omega(x)) + \frac{\omega^i(x)}{\rho(x)} (d_i \partial_\rho U)(\rho(x), \omega(x)).
\end{aligned}$$

Computation of J_2 .

We have

$$\begin{aligned}
J_2 &= \partial_{x^i} \left(\frac{1}{\rho(x)} \right) d_i U(\rho(x), \omega(x)) + \frac{1}{\rho(x)} \partial_{x^i} ((d_i U)(\rho(x), \omega(x))) = \\
&= -\frac{\omega^i(x)}{\rho^2(x)} d_i U(\rho(x), \omega(x)) + \quad (15.3.31) \\
&+ \frac{\omega^i(x)}{\rho(x)} \partial_\rho (d_i U)(\rho(x), \omega(x)) + \frac{1}{\rho^2(x)} (d_i^2 U)(\rho(x), \omega(x)),
\end{aligned}$$

in the last equality we have applied (15.3.25) to $(d_i U)(\rho(x), \omega(x))$.

Computation of Δ .

Now, adding up (15.3.30) and (15.3.31) we obtain (we omit, the variables)

$$\begin{aligned}
\Delta u &= \sum_{i=1}^n \partial_{x^i}^2 u = \\
&= \frac{n-1}{\rho} \partial_\rho U + \partial_\rho^2 U + \frac{1}{\rho} \sum_{i=1}^n \omega^i d_i (\partial_\rho U) - \\
&- \frac{1}{\rho^2} \sum_{i=1}^n \omega^i d_i U + \frac{1}{\rho} \sum_{i=1}^n \omega^i \partial_\rho (d_i U) + \frac{1}{\rho^2} \Delta_\Sigma U. \quad (15.3.32)
\end{aligned}$$

Now, (15.3.9) implies

$$\sum_{i=1}^n \omega_i d_i U = 0, \quad \sum_{i=1}^n \omega^i d_i (\partial_\rho U) = 0;$$

by the first equality we get

$$\sum_{i=1}^n \omega_i \partial_\rho (d_i U) = \partial_\rho \left(\sum_{i=1}^n \omega^i d_i U \right) = 0.$$

Therefore, (15.3.32) gives

$$\Delta u(x) = \partial_\rho^2 U(\rho(x), \omega(x)) + \frac{n-1}{\rho(x)} \partial_\rho U(\rho(x), \omega(x)) + \frac{1}{\rho^2(x)} (\Delta_\Sigma U)(\rho(x), \omega(x))$$

and by (15.3.27) we get (15.3.28). ■

Remarks.

1. Generally speaking, even if $u \in C^2(B_R)$, $U(\rho, \omega)$ is not differentiable w.r.t. ρ in 0. However (15.3.9) and (15.3.25) give

$$\partial_\rho U(\rho(x), \omega(x)) = \nabla u(x) \cdot \omega(x), \quad (15.3.33)$$

from which we have

$$|\partial_\rho U(\rho, \omega)| \leq \max_{\overline{B_r}} |\nabla u|, \quad \text{for } \rho \leq r \leq R. \quad (15.3.34)$$

2. Let us observe that

$$\begin{aligned} \partial_\rho^2 U(\rho(x), \omega(x)) &= \sum_{ij=1}^n \partial_{x^i x^j}^2 u(x) \frac{x^i x^j}{|x|^2} = \sum_{|\alpha|=2} \frac{2!}{\alpha!} \partial^\alpha u(x) \left(\frac{x}{|x|} \right)^\alpha = \\ &= \underbrace{\partial^2 u(x)}_{\text{Hessian matrix}} \cdot \frac{x}{|x|} \cdot \frac{x}{|x|}. \end{aligned} \quad (15.3.35)$$

Indeed we have

$$\begin{aligned} \partial_\rho^2 U(\rho(x), \omega(x)) &= \sum_{k=1}^n \omega^k(x) \partial_{x^k} \left(\sum_{j=1}^n \omega^j(x) \partial_{x^j} u \right) = \\ &= \sum_{k=1}^n \frac{x^k}{|x|} \sum_{j=1}^n \left[\left(\frac{\delta^{jk}}{|x|} - \frac{x^j x^k}{|x|^3} \right) \partial_{x^j} u + \frac{x^j}{|x|} \partial_{x^j x^k}^2 u \right] = \\ &= \sum_{k=1}^n \frac{x^k}{|x|} \left[\frac{1}{|x|} \partial_{x^k} u - \frac{x^k}{|x|^2} \left(\frac{x}{|x|} \cdot \nabla u \right) + \sum_{j=1}^n \frac{x^j}{|x|} \partial_{x^j x^k}^2 u \right] = \\ &= \sum_{ij=1}^n \partial_{x^i x^j}^2 u(x) \frac{x^i x^j}{|x|^2}. \end{aligned}$$

Similarly can be checked that

$$\partial_\rho^m U(\rho(x), \omega(x)) = \frac{1}{|x|^m} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \partial^\alpha u(x) x^\alpha. \quad (15.3.36)$$

3. Of course, by (15.3.33) we have (for $r \leq R$)

$$\frac{\partial u}{\partial \nu}(x) = \partial_\rho U(r, \omega), \quad \text{for } x \in \partial B_r, \quad (15.3.37)$$

where ν is the unit outward normal to ∂B_r . Similarly we have by (15.3.36),

$$\frac{\partial^m u}{\partial \nu^m}(x) := \frac{d^m}{dt^m} u(x + t\nu)|_{t=0} = \partial_\rho^m U(r, \omega).$$



Exercise 1. Let $u \in C^2(B_R)$ and $U(\rho, \omega) = u(\rho\omega)$. Apply formula (15.3.28) to prove that for any $r < R$ we have

$$\int_\Sigma \partial_\rho U(\rho, \omega) d\omega = \frac{1}{r^{n-1}} \int_{B_r} \Delta u(x) dx.$$



Exercise 2. Let u be a harmonic function in B_R . Prove that for any $r < R$ we have

$$\int_{\partial B_r} \frac{\partial^m u}{\partial \nu^m}(x) dS = 0.$$



15.4 The case of Laplace leading operator

In this Section and in the sequel we will use the following **Hardy inequality**, [33], [72].

Lemma 15.4.1 (the Hardy inequality). *If $f \in C_0^\infty(0, +\infty)$, then*

$$\int_0^{+\infty} \frac{f^2(s)}{s^2} ds \leq 4 \int_0^{+\infty} (f'(s))^2 ds. \quad (15.4.1)$$

Note. *the number 4 on the right-hand side of (15.4.1) is the best constant.*

Proof. We set

$$g(s) = s^{-\frac{1}{2}}f(s). \quad (15.4.2)$$

and we have

$$f(s) = s^{\frac{1}{2}}g(s), \quad f'(s) = \frac{1}{2}s^{-\frac{1}{2}}g(s) + s^{\frac{1}{2}}g'(s).$$

Hence (15.4.1) is equivalent to

$$\int_0^{+\infty} s^{-1}g^2(s)ds \leq 4 \int_0^{+\infty} \left(\frac{1}{2}s^{-\frac{1}{2}}g(s) + s^{\frac{1}{2}}g'(s) \right)^2 ds. \quad (15.4.3)$$

Now, we have

$$\begin{aligned} & 4 \int_0^{+\infty} \left(\frac{1}{2}s^{-\frac{1}{2}}g(s) + s^{\frac{1}{2}}g'(s) \right)^2 ds = \\ & = 4 \int_0^{+\infty} \left(\frac{1}{4}s^{-1}g^2(s) + g(s)g'(s) + s(g'(s))^2 \right) ds = \\ & = 4 \int_0^{+\infty} \left(\frac{1}{4}s^{-1}g^2(s) + \frac{1}{2}(g^2)'(s) + s(g'(s))^2 \right) ds = \\ & = \int_0^{+\infty} \left(s^{-1}g^2(s) + 4s(g'(s))^2 \right) ds \geq \\ & \geq \int_0^{+\infty} s^{-1}g^2(s)ds. \end{aligned}$$

The proof is complete. ■

Theorem 15.4.2 (Carleman estimate for Δ). *Let $\epsilon \in (0, 1]$. Let us define*

$$\rho(x) = \phi_\epsilon(|x|), \quad \text{for } x \in B_1, \quad (15.4.4)$$

where

$$\phi_\epsilon(s) = \frac{s}{(1+s^\epsilon)^{1/\epsilon}}. \quad (15.4.5)$$

Then there exist $\tau_1 > 1$ and $C > 1$, depending on ϵ only, such that

$$\begin{aligned} \tau^3 \int \rho^{\epsilon-2\tau}|u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau}|\nabla u|^2 dx &\leq \\ &\leq C \int \rho^{4-2\tau}|\Delta u|^2 dx, \end{aligned} \quad (15.4.6)$$

for every $u \in C_0^\infty(B_1 \setminus \{0\})$ and for every $\tau \geq \tau_1$.

Moreover, there exist $\tau_2 > 1$, $C > 1$, depending on ϵ only, such that

$$\begin{aligned} \tau^3 \int \rho^{\epsilon-2\tau} |u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau} |\nabla u|^2 dx + \\ + \tau^2 r \int \rho^{-1-2\tau} u^2 dx \leq C \int \rho^{4-2\tau} |\Delta u|^2 dx, \end{aligned} \quad (15.4.7)$$

for every $r \in (0, 1)$, for every $u \in C_0^\infty(B_1 \setminus \overline{B}_{r/4})$ and for every $\tau \geq \tau_2$.

Remark. We have

$$\frac{|x|}{2^{1/\epsilon}} \leq \rho(x) \leq |x|, \quad \forall x \in B_1. \quad (15.4.8)$$

◆

Proof. It is not restrictive to assume that u is a real-valued function. First we prove (15.4.6), afterwards, with a few modifications we will prove (15.4.7).

Let u be an arbitrary function of $C_0^\infty(B_1 \setminus \{0\})$ and let us consider the n -dimensional Laplace operator in the polar coordinates (ϱ, ω) , that is (recalling that $\Sigma = \partial B_1$)

$$\Delta u = u_{\varrho\varrho} + \frac{n-1}{\varrho} u_\varrho + \frac{1}{\varrho^2} \Delta_\Sigma u, \quad \forall (\varrho, \omega) \in (0, \infty) \times \Sigma. \quad (15.4.9)$$

Let us perform the following change of variables

$$\varrho = e^t, \quad \tilde{u}(t, \omega) = u(e^t, \omega), \quad \forall (t, \omega) \in (-\infty, 0) \times \Sigma.$$

We have, for every $(t, \omega) \in (-\infty, 0) \times \Sigma$,

$$e^{2t}(\Delta u)(e^t, \omega) = \mathcal{L}\tilde{u} := \tilde{u}_{tt} + (n-2)\tilde{u}_t + \Delta_\Sigma \tilde{u}. \quad (15.4.10)$$

For the sake of brevity, for any function $h \in C_0^\infty(B_1 \setminus \{0\})$ we will write h' , h'' , ... instead of h_t , h_{tt} ,

By (15.4.4) we have (we omit the subscript ϵ from now on)

$$\varphi(t) := \log(\phi(e^t)) = t - \epsilon^{-1} \log(1 + e^{t\epsilon}), \quad \forall t \in (-\infty, 0). \quad (15.4.11)$$

We get

$$\varphi'(t) = \frac{1}{1 + e^{\epsilon t}}, \quad \varphi''(t) = -\frac{\epsilon e^{\epsilon t}}{(1 + e^{\epsilon t})^2}, \quad \forall t \in (-\infty, 0). \quad (15.4.12)$$

Let

$$f(t, \omega) = e^{-\tau\varphi}\tilde{u}(t, \omega), \quad \forall(t, \omega) \in (-\infty, 0) \times \Sigma.$$

We have

$$\mathcal{L}_\tau f := e^{-\tau\varphi}\mathcal{L}(e^{\tau\varphi}f) = \underbrace{b_0f + b_1f'}_{\mathcal{A}_\tau f} + \underbrace{a_0f + f'' + \Delta_\Sigma f}_{\mathcal{S}_\tau f}, \quad (15.4.13)$$

where

$$a_0 = \tau^2\varphi'^2 + \tau(n-2), \quad b_0 = \tau\varphi'', \quad b_1 = 2\tau\varphi' + (n-2). \quad (15.4.14)$$

Let us denote by $\int(\cdot)$ the integral $\int_{-\infty}^0 \int_\Sigma(\cdot)d\omega dt$ and set

$$\gamma := \frac{1}{\varphi'} = 1 + e^{e^t}. \quad (15.4.15)$$

We obtain

$$\int \gamma |\mathcal{L}_\tau f|^2 \geq 2 \int \gamma \mathcal{A}_\tau f \mathcal{S}_\tau f + \int \gamma |\mathcal{A}_\tau f|^2 \quad (15.4.16)$$

and

$$\begin{aligned} 2 \int \gamma \mathcal{A}_\tau f \mathcal{S}_\tau f &= 2 \underbrace{\int \gamma (b_0f + b_1f') \Delta_\Sigma f}_{I_1} + \\ &+ 2 \underbrace{\int \gamma (b_0f + b_1f') (a_0f + f'')}_{I_2}. \end{aligned} \quad (15.4.17)$$

We examine I_1 .

For any function f, g on $\in C_0^\infty(B_1 \setminus \{0\})$, let us denote by

$$\langle \nabla_\Sigma f, \nabla_\Sigma g \rangle = \sum_{i=1}^n d_i f d_i g, \quad |\nabla_\Sigma f|^2 = \langle \nabla_\Sigma f, \nabla_\Sigma f \rangle.$$

Integraing by parts, using Proposition (15.3.3) and taking into account (15.4.15), we have

$$\begin{aligned} I_1 &= 2 \int (\gamma b_0 f \Delta_\Sigma f + \gamma b_1 f' \Delta_\Sigma f) = \\ &= 2 \int (-\gamma b_0 |\nabla_\Sigma f|^2 - \gamma b_1 \langle \nabla_\Sigma f, \nabla_\Sigma f' \rangle) = \\ &= 2 \int \left(-\gamma b_0 |\nabla_\Sigma f|^2 - \frac{1}{2} \gamma b_1 (|\nabla_\Sigma f|^2)' \right) = \\ &= 2 \int \left(-\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) |\nabla_\Sigma f|^2. \end{aligned}$$

By (15.4.12), (15.4.14) and (15.4.15) we obtain

$$\begin{aligned}
-\gamma b_0 + \frac{1}{2}(\gamma b_1)' &= -\gamma\tau\varphi'' + \frac{1}{2}\gamma b_1' + \frac{1}{2}\gamma' b_1 = \\
&= -\gamma\tau\varphi'' + \gamma\tau\varphi'' + \frac{1}{2}(2\tau\varphi' + n - 2)\gamma' = \\
&= \frac{\epsilon e^{t\tau}}{2} \left(\frac{2\tau}{1 + e^{t\tau}} + n - 2 \right) \geq \frac{\tau}{2}\epsilon e^{t\tau}, \quad \forall \tau > 0.
\end{aligned} \tag{15.4.18}$$

Hence, we have

$$I_1 \geq \int \tau \epsilon e^{t\tau} |\nabla_{\Sigma} f|^2, \quad \forall \tau > 0. \tag{15.4.19}$$

Now we examine I_2 .

Integration by parts gives

$$\begin{aligned}
I_2 &= 2 \int \gamma (b_0 f + b_1 f') (a_0 f + f'') = \\
&= 2 \int \gamma (a_0 b_0 f^2 + b_0 f f'' + b_1 a_0 f' f + b_1 f' f'') = \\
&= 2 \int \gamma a_0 b_0 f^2 - (\gamma b_0 f)' f' + \frac{1}{2} \gamma b_1 a_0 (f^2)' + \frac{1}{2} \gamma b_1 (f'^2)' = \\
&= 2 \int \left[\gamma a_0 b_0 - \frac{1}{2} (\gamma b_1 a_0)' \right] f^2 - (\gamma b_0)' f f' - \\
&\quad - \int \gamma b_0 f'^2 + \frac{1}{2} (\gamma b_1)' f'^2 = \\
&= 2 \int \underbrace{\left[\gamma a_0 b_0 - \frac{1}{2} (\gamma b_1 a_0)' + \frac{1}{2} (\gamma b_0)'' \right]}_{H_1} f^2 - \\
&\quad - \int \underbrace{\left[\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right]}_{H_2} f'^2.
\end{aligned} \tag{15.4.20}$$

Let now examine H_1 .

Since by (15.4.14), H_1 is a polynomial of third degree w.r.t. τ , we begin by evaluating the coefficient of τ^3 .

Let us notice that the terms of H_1 have the following behavior, as $\tau \rightarrow +\infty$

$$\begin{aligned}
\gamma a_0 b_0 &= \mathcal{O}(\tau^3), \\
-\frac{1}{2}(\gamma b_1 a_0)' &= \mathcal{O}(\tau^3), \\
\frac{1}{2}(\gamma b_0)'' &= \mathcal{O}(\tau).
\end{aligned}$$

Hence, let us first examine the term

$$\tilde{H}_1 := \gamma a_0 b_0 - \frac{1}{2} (\gamma b_1 a_0)'.$$

By (15.4.12), (15.4.14) and (15.4.15) we have

$$\begin{aligned} \tilde{H}_1 &= \gamma a_0 b_0 - \frac{1}{2} (\gamma b_1 a_0)' = \\ &= \gamma \left(\tau^2 \varphi'^2 + \tau(n-2)\varphi' \right) \tau \varphi'' - \frac{1}{2} b_1' (\gamma a_0) - \frac{1}{2} b_1 (\gamma a_0)' = \\ &= \gamma \left(\tau^2 \varphi'^2 + \tau(n-2)\varphi' \right) \tau \varphi'' - \\ &\quad - \frac{1}{2} (2\tau\varphi' + (n-2))' \gamma \left(\tau^2 \varphi'^2 + \tau(n-2)\varphi' \right) - \frac{1}{2} b_1 (\gamma a_0)' = \\ &= -\frac{1}{2} b_1 (\gamma a_0)' = \tag{15.4.21} \\ &= -\frac{1}{2} \left(\frac{2\tau}{1+e^{\epsilon t}} + n-2 \right) \left[\gamma \left(\tau^2 \varphi'^2 + \tau(n-2)\varphi' \right) \right]' = \\ &= -\left(\frac{\tau}{1+e^{\epsilon t}} + \frac{n-2}{2} \right) \left(\tau^2 \frac{1}{1+e^{\epsilon t}} + \tau(n-2) \right)' = \\ &= \left(\tau^3 \frac{1}{1+e^{\epsilon t}} + \tau^2 \frac{n-2}{2} \right) \frac{\epsilon e^{\epsilon t}}{(1+e^{\epsilon t})^2}. \end{aligned}$$

Then, using the trivial inequality

$$\frac{1}{1+e^{\epsilon t}} \geq \frac{1}{2}, \quad \forall t \in (-\infty, 0),$$

we have (for $t \in (-\infty, 0)$)

$$\tilde{H}_1 \geq \frac{\tau^3}{8} \epsilon e^{\epsilon t}, \quad \forall \tau > 0. \tag{15.4.22}$$

and

$$\frac{1}{2} (\gamma b_0)'' = \frac{\tau}{2} \left(\frac{-\epsilon e^{\epsilon t}}{1+e^{\epsilon t}} \right)'' = -\frac{\tau \epsilon^3 e^{\epsilon t} (1-e^{\epsilon t})}{2(1+e^{\epsilon t})^3} \geq -\frac{\tau}{2} \epsilon^3 e^{\epsilon t}, \quad \forall \tau > 0. \tag{15.4.23}$$

Inequalities (15.4.22) and (15.4.23) give

$$H_1 = \tilde{H}_1 + \frac{1}{2} (\gamma b_0)'' \geq \frac{\tau^3}{8} \epsilon e^{\epsilon t} - \frac{\tau}{2} \epsilon^3 e^{\epsilon t} \geq \frac{\tau^3}{16} \epsilon e^{\epsilon t}, \quad \forall \tau > \sqrt{8\epsilon}. \tag{15.4.24}$$

Now, let us consider H_2 .

$$\begin{aligned}
H_2 &= \gamma b_0 + \frac{1}{2} (\gamma b_1)' = \\
&= \tau(1 + e^{\epsilon t})\varphi'' + \frac{1}{2} [2\tau(1 + e^{\epsilon t})\varphi' + (n-2)(1 + e^{\epsilon t})]' = \\
&= -\tau \frac{\epsilon e^{\epsilon t}}{1 + e^{\epsilon t}} + \frac{1}{2} [2\tau + (n-2)(1 + e^{\epsilon t})]' = \\
&= -\tau \frac{\epsilon e^{\epsilon t}}{1 + e^{\epsilon t}} + \frac{(n-2)\epsilon e^{\epsilon t}}{2}.
\end{aligned} \tag{15.4.25}$$

Hence

$$-2H_2 \geq \frac{\tau}{2} \epsilon e^{\epsilon t}, \quad \forall \tau \geq 2(n-2). \tag{15.4.26}$$

By (15.4.20), (15.4.24) and (15.4.26) we have

$$I_2 \geq \int \frac{\tau^3}{8} \epsilon e^{\epsilon t} f^2 + \frac{\tau}{2} \epsilon e^{\epsilon t} f'^2, \quad \forall \tau \geq \tau_1, \tag{15.4.27}$$

where $\tau_1 = \max\{\sqrt{8\epsilon}, 2(n-2)\}$.

Now, (15.4.17), (15.4.19) and (15.4.27) give

$$2 \int \gamma \mathcal{A}_\tau f \mathcal{S}_\tau f \geq \frac{\epsilon}{8} \int \left(\tau^3 f^2 + \tau (f'^2 + |\nabla_\Sigma f|^2) \right) e^{\epsilon t}, \quad \forall \tau \geq \tau_1. \tag{15.4.28}$$

By (15.4.16) and (15.4.28) we have

$$\int \gamma |\mathcal{L}_\tau f|^2 \geq \frac{\epsilon \tau^3}{8} \int f^2 + \frac{\epsilon \tau}{8} \int \left(f'^2 + |\nabla_\Sigma f|^2 \right) e^{\epsilon t}, \tag{15.4.29}$$

for every $\tau \geq \tau_1$ and for every $f \in C_0^\infty((-\infty, 0) \times \Sigma)$.

Now, in order to obtain (15.4.6) we come back to u and to the original variables. Let us recall that $f(t, \omega) = e^{-\tau\varphi} u(e^t, \omega)$. By using (15.4.4), (15.4.10) and (15.4.13), we get

$$\begin{aligned}
\int_{-\infty}^0 \int_\Sigma |\mathcal{L}_\tau f|^2 d\omega dt &= \int_{-\infty}^0 \int_\Sigma e^{-2\tau\varphi(t)} e^{4t} |(\Delta u)(e^t, \omega)|^2 d\omega dt = \\
&= \int_0^1 \int_\Sigma e^{-2\tau\varphi(\log \varrho)} \varrho^3 |(\Delta u)(\varrho, \omega)|^2 d\omega d\varrho = \\
&= \int_{B_1} \rho^{-2\tau} |x|^{4-n} |\Delta u|^2 dx.
\end{aligned} \tag{15.4.30}$$

Similarly, we get

$$\int_{-\infty}^0 \int_{\Sigma} f^2 e^{\epsilon t} d\omega dt = \int_{B_1} \rho^{-2\tau} |x|^{\epsilon-n} u^2 dx. \tag{15.4.31}$$

Concerning the second integral on the right-hand side of (15.4.29), let $\delta \in (0, 1)$ be a number that we will choose later, we have

$$\begin{aligned} & \int_{-\infty}^0 \int_{\Sigma} e^{\epsilon t} \left(f'^2 + |\nabla_{\Sigma} f|^2 \right) d\omega dt \geq \\ & \geq \delta \int_{-\infty}^0 \int_{\Sigma} e^{\epsilon t} \left(f'^2 + |\nabla_{\Sigma} f|^2 \right) d\omega dt \geq \\ & \geq \frac{\delta}{2} \int_{-\infty}^0 \int_{\Sigma} e^{\epsilon t} e^{-2\tau\varphi(t)} \left(|u_{\rho}(e^t, \omega)|^2 e^{2t} + \right. \\ & \quad \left. + |\nabla_{\Sigma} u(e^t, \omega)|^2 - 2\tau^2 |u(e^t, \omega)|^2 \right) d\omega dt = \\ & = \frac{\delta}{2} \int_{B_1} \rho^{-2\tau} |x|^{\epsilon-n} \left(|x|^2 |\nabla u|^2 - 2\tau^2 |u|^2 \right) dx. \end{aligned} \tag{15.4.32}$$

Now, let us choose $\delta = \frac{1}{2}$ so that, by (15.4.29) and (15.4.30)–(15.4.32), we have

$$\begin{aligned} \int_{B_1} \rho^{-2\tau} |x|^{4-n} |\Delta u|^2 dx & \geq \frac{\epsilon\tau}{16} \int_{B_1} \rho^{-2\tau} |x|^{\epsilon+2-n} |\nabla u|^2 dx + \\ & \quad + \frac{\epsilon\tau^3}{16} \int_{B_1} \rho^{-2\tau} |x|^{\epsilon-n} u^2 dx, \end{aligned} \tag{15.4.33}$$

for every $u \in C_0^{\infty}(B_1 \setminus \{0\})$ and for every $\tau \geq \tau_1$. Finally, by (15.4.8), we substitute in (15.4.33) τ by $(\tau - \frac{n}{2})$ and we find inequality (15.4.6).

Let us now look at (15.4.7). Let $u \in C_0^{\infty}(B_1 \setminus \overline{B}_{r/4})$.

By (15.4.16) and (15.4.29) we have

$$\int |\mathcal{L}_{\tau} f|^2 \geq \frac{\epsilon}{8} \int \left(\tau^3 f^2 + \tau \left(f'^2 + |\nabla_{\Sigma} f|^2 \right) \right) e^{\epsilon t} + \int \gamma |\mathcal{A}_{\tau} f|^2. \tag{15.4.34}$$

To obtain the first term on the left-hand side of (15.4.7) we estimate from below the last term on the right-hand side of (15.4.34).

Let us note that by the trivial inequality $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$ and by (15.4.12), (15.4.15) we have

$$\begin{aligned} \int \gamma |\mathcal{A}_\tau f|^2 &\geq \frac{1}{2} \int \gamma (2\tau\varphi' + n - 2)^2 f'^2 - \int \gamma \tau^2 \varphi'^2 f^2 \geq \\ &\geq \tau^2 \int f'^2 - \epsilon^2 \tau^2 \int e^{2\epsilon t} f^2, \quad \forall \tau > 0 \end{aligned} \quad (15.4.35)$$

Using inequality (15.4.35) in (15.4.34), we have

$$\begin{aligned} \int |\mathcal{L}_\tau f|^2 &\geq \tau^2 \int f'^2 + \epsilon \tau^3 \int \left(\frac{1}{8} - \epsilon \tau^{-1} e^{\epsilon t} \right) e^{\epsilon t} f^2 + \\ &\quad + \frac{\epsilon}{8} \tau \int \left(f'^2 + |\nabla_\Sigma f|^2 \right) e^{\epsilon t}, \quad \forall \tau \geq \tau_1. \end{aligned} \quad (15.4.36)$$

Now, since $(\frac{1}{8} - \epsilon \tau^{-1} e^{\epsilon t}) \geq \frac{1}{16}$ for every $\tau \geq 4\epsilon$, by (15.4.36), we get

$$\int |\mathcal{L}_\tau f|^2 \geq \tau^2 \int f'^2 + \frac{\epsilon \tau^3}{16} \int e^{\epsilon t} f^2 + \frac{\epsilon}{8} \tau \int \left(f'^2 + |\nabla_\Sigma f|^2 \right) e^{\epsilon t}, \quad (15.4.37)$$

for every $\tau \geq \tau_2$, where $\tau_2 = \max\{4\epsilon, \tau_1\}$.

Proposition 15.4.1 implies

$$\begin{aligned} \int_{-\infty}^0 f^2(t, \omega) e^{-t} dt &= \int_0^1 s^{-2} f^2(\log s, \omega) ds \leq \\ &\leq 4 \int_0^1 \left| \frac{\partial}{\partial s} f(\log s, \omega) \right|^2 ds = \\ &= 4 \int_{-\infty}^0 f'^2(t, \omega) e^{-t} dt, \quad \forall \omega \in \Sigma. \end{aligned} \quad (15.4.38)$$

On the other hand, since $u \in C_0^\infty(B_1 \setminus \overline{B}_{r/4})$, we have $f(t, \omega) = 0$ for every $t \leq \log(r/4)$ and for every $\omega \in \Sigma$, (15.4.38) gives

$$\int_{-\infty}^0 f^2(t, \omega) e^{-t} dt \leq 4 \int_{-\infty}^{\log \frac{r}{4}} f'^2(t, \omega) e^{-t} dt \leq \frac{16}{r} \int_{-\infty}^0 f'^2(t, \omega) dt, \quad \forall \omega \in \Sigma.$$

Let us integrate over Σ both the sides of the just obtained inequality and let us use (15.4.37) to obtain

$$\int f^2 e^{-t} \leq \frac{16}{r} \int f'^2 \leq \frac{16}{\tau^2 r} \int |\mathcal{L}_\tau f|^2, \quad \forall \tau \geq \tau_2. \quad (15.4.39)$$

By (15.4.39) and (15.4.37) we have

$$\begin{aligned} C \int |\mathcal{L}_\tau f|^2 &\geq \epsilon \tau^3 \int e^{\epsilon t} f^2 + \epsilon \tau \int \left(f'^2 + |\nabla_\Sigma f|^2 \right) e^{\epsilon t} + \\ &\quad + \tau^2 r \int f^2 e^{-t}, \quad \forall \tau \geq \tau_2, \end{aligned} \quad (15.4.40)$$

where C is a constant.

Finally, by (15.4.33), (15.4.8) and by

$$\int_{-\infty}^0 \int_{\Sigma} f^2 e^{-t} d\omega dt = \int_{B_1} \rho^{-2\tau} |x|^{-1-n} u^2 dx, \tag{15.4.41}$$

we obtain (15.4.7). ■

15.5 Proof of the optimal three sphere and the doubling inequality

In the next Theorem we will consider the equation

$$\Delta U = b(x) \cdot \nabla U + c(x)U, \quad \text{in } B_1, \tag{15.5.1}$$

where $b \in L^\infty(B_1; \mathbb{R}^n)$ and $c \in L^\infty(B_1)$. Moreover, set

$$M = \max \left\{ \|b\|_{L^\infty(B_1; \mathbb{R}^n)}, \|c\|_{L^\infty(B_1)} \right\}. \tag{15.5.2}$$

Theorem 15.5.1 (optimal three sphere and doubling inequality). *Let us assume that $U \in H^2(B_1)$ is a solution to equation (15.5.1). Let $x_0 \in B_1$ and $0 < R_0 \leq 1 - |x_0|$. Then there exists $C > 1$ depending on M only, such that, if $0 < 2r < R < \frac{R_0}{2}$ then*

$$\int_{B_R(x_0)} U^2 \leq C \left(\frac{R_0}{R} \right)^C \left(\int_{B_r(x_0)} U^2 \right)^\theta \left(\int_{B_{R_0}(x_0)} U^2 \right)^{1-\theta}, \tag{15.5.3}$$

where

$$\theta = \frac{\log \frac{R_0}{2R}}{\log \frac{2R_0}{r}}. \tag{15.5.4}$$

Moreover, if U does not vanish identically in $B_{R_0/4}(x_0)$ then the following doubling inequality holds

$$\int_{B_{2r}(x_0)} U^2 \leq CN_{x_0, R_0}^3 \int_{B_r(x_0)} U^2, \tag{15.5.5}$$

where

$$N_{x_0, R_0} = \frac{\int_{B_{R_0}(x_0)} U^2}{\int_{B_{R_0/4}(x_0)} U^2}. \tag{15.5.6}$$

In order to prove Theorem 15.5.1 we need the following

Lemma 15.5.2. *Under the same assumption of Theorem 15.5.1 we have, for every $x_0 \in B_1$, R and for every R, r such that $0 < 2r < R < \frac{R_0}{2}$, where $R_0 \leq 1 - |x_0|$,*

$$\begin{aligned} & R(2r)^{-2\tau} \int_{B_{2r}(x_0)} U^2 + R^{1-2\tau} \int_{B_R(x_0)} U^2 \leq \\ & \leq C\bar{M}^2 \left[\left(\frac{r}{4}\right)^{-2\tau} \int_{B_r(x_0)} U^2 + \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}(x_0)} U^2 \right], \end{aligned} \quad (15.5.7)$$

for every $\tau \geq \tilde{\tau}_2$, where $\tilde{\tau}_2$ and $C \geq 1$ depend on M only.

Proof of Lemma. By a translation we may assume that $x_0 = 0$. Let r, R satisfy

$$0 < 2r < R < \frac{R_0}{2}. \quad (15.5.8)$$

Let $\eta \in C_0^\infty((0, R_0))$ satisfy

$$0 \leq \eta \leq 1, \quad (15.5.9)$$

$$\eta = 0, \quad \text{in } \left(0, \frac{r}{4}\right) \cup \left(\frac{2R_0}{3}, R_0\right); \quad \eta = 1, \quad \text{in } \left[\frac{r}{2}, \frac{R_0}{2}\right], \quad (15.5.10)$$

$$\left| \frac{d^k \eta}{dt^k}(t) \right| \leq Cr^{-k}, \quad \text{in } \left(\frac{r}{4}, \frac{r}{2}\right), \quad \text{for } 0 \leq k \leq 2, \quad (15.5.11)$$

$$\left| \frac{d^k \eta}{dt^k}(t) \right| \leq CR_0^{-k}, \quad \text{in } \left(\frac{R_0}{2}, \frac{2R_0}{3}\right), \quad \text{for } 0 \leq k \leq 2. \quad (15.5.12)$$

We define

$$\xi(x) = \eta(|x|). \quad (15.5.13)$$

Exploiting Carleman estimate (15.4.7) and fixing there $\epsilon = 1$, we get

$$\begin{aligned} & \tau^3 \int_{B_{R_0}} \rho^{1-2\tau} u^2 dx + \tau \int_{B_{R_0}} \rho^{3-2\tau} |\nabla u|^2 + \\ & + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} u^2 \leq C \int_{B_{R_0}} \rho^{4-2\tau} |\Delta u|^2, \end{aligned} \quad (15.5.14)$$

for every $u \in C_0^\infty(B_{R_0} \setminus \bar{B}_{r/4})$ and for every $\tau \geq \tau_2$ (we recall that τ_2 and C depend neither on r nor on R_0 and that the value C may change from line to line).

Since $\xi U \in H_0^2(B_{R_0})$, by density we can apply Carleman estimate (15.5.14) to $u = \xi U$. Hence we find

$$\begin{aligned} & \tau^3 \int_{B_{R_0}} \rho^{1-2\tau} \xi^2 U^2 + \tau \int_{B_{R_0}} \rho^{3-2\tau} |\nabla(\xi U)|^2 + \\ & + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 \leq C \int_{B_{R_0}} \rho^{4-2\tau} |\Delta(\xi U)|^2, \end{aligned} \quad (15.5.15)$$

for every $\tau \geq \tau_2$.

Since we have

$$|\Delta(\xi U)|^2 \leq 2\xi^2 |\Delta U|^2 + C \left(|\partial^2 \xi|^2 U^2 + |\nabla \xi|^2 |\nabla U|^2 \right), \quad (15.5.16)$$

setting

$$J_0 = \int_{B_{r/2} \setminus B_{r/4}} \rho^{4-2\tau} (r^{-4} U^2 + r^{-2} |\nabla U|^2), \quad (15.5.17)$$

$$J_1 = \int_{B_{2R_0/3} \setminus B_{R_0/2}} \rho^{4-2\tau} (U^2 + |\nabla U|^2), \quad (15.5.18)$$

we get

$$\begin{aligned} & \tau^3 \int_{B_{R_0}} \rho^{1-2\tau} \xi^2 U^2 dx + \tau \int_{B_{R_0}} \rho^{3-2\tau} |\nabla(\xi U)|^2 dx + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 dx \leq \\ & \leq C \int_{B_{R_0}} \rho^{4-2\tau} |\Delta U|^2 + C (J_0 + J_1), \end{aligned} \quad (15.5.19)$$

for every $\tau \geq \tau_2$.

Now we perform what follows: we use (15.5.8)–(15.5.13) and (15.5.16), we estimate trivially from below the left-hand side of (15.5.19) and we estimate trivially from above the right-hand side of (15.5.19), obtaining

$$\begin{aligned} & \tau^3 \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{1-2\tau} U^2 + \tau \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{3-2\tau} |\nabla U|^2 + \\ & + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 dx \leq \\ & \leq CM^2 \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{4-2\tau} (U^2 + |\nabla U|^2) + \\ & + C\bar{M}^2 (J_0 + J_1), \end{aligned} \quad (15.5.20)$$

for every $\tau \geq \tau_2$, where $\bar{M} = \sqrt{M^2 + 1}$.

By (15.5.20), we get

$$\begin{aligned} & \int_{B_{R_0/2} \setminus B_{r/2}} (\tau^3 - CM^2 \rho^3) \rho^{1-2\tau} U^2 + \\ & + \int_{B_{R_0/2} \setminus B_{r/2}} (\tau - CM^2 \rho) \rho^{3-2\tau} |\nabla U|^2 + \\ & + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 \leq C \bar{M}^2 (J_0 + J_1). \end{aligned} \quad (15.5.21)$$

By the latter, taking into account that $\rho \leq 1$ in B_{R_0} , we have

$$\begin{aligned} & \frac{\tau^3}{2} \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{1-2\tau} U^2 + \frac{\tau}{2} \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{3-2\tau} |\nabla U|^2 + \\ & + \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 dx \leq C \bar{M}^2 (J_0 + J_1), \end{aligned} \quad (15.5.22)$$

for every $\tau \geq \tilde{\tau}_2$, where (recall $\bar{\tau} \geq 1$)

$$\tilde{\tau}_2 = \min \{2CM^2, \tau_2\}.$$

Now, we estimate from above J_0 and J_1 . By the Caccioppoli inequality (Theorem 4.5.1) and recalling (15.4.8), we have

$$\begin{aligned} J_0 &= \int_{B_{r/2} \setminus B_{r/4}} \rho^{4-2\tau} (r^{-4} U^2 + r^{-2} |\nabla U|^2) \leq \\ &\leq C \left(\frac{r}{4}\right)^{-2\tau} \int_{B_{r/2}} (U^2 + r^2 |\nabla U|^2) \leq C \left(\frac{r}{4}\right)^{-2\tau} \int_{B_r} U^2, \end{aligned} \quad (15.5.23)$$

where C depends on M only.

Similarly we get

$$J_1 \leq C \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}} U^2. \quad (15.5.24)$$

By (15.5.22) – (15.5.24) we have

$$\begin{aligned} & \tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 + \tau^3 \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{1-2\tau} U^2 dx \leq \\ & \leq C \bar{M}^2 \left(\left(\frac{r}{4}\right)^{-2\tau} \int_{B_r} U^2 + \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}} U^2 \right), \end{aligned} \quad (15.5.25)$$

for every $\tau \geq \tilde{\tau}_2$.

Now, recalling that $2r < R < \frac{R_0}{2}$, by (15.5.10) we have

$$\tau^2 r \int_{B_{R_0}} \rho^{-1-2\tau} \xi^2 U^2 \geq (2r)^{-2\tau} \int_{B_{2r} \setminus B_{r/2}} U^2, \quad (15.5.26)$$

and

$$\tau^3 \int_{B_{R_0/2} \setminus B_{r/2}} \rho^{1-2\tau} U^2 \geq R^{1-2\tau} \int_{B_R \setminus B_{r/2}} U^2. \quad (15.5.27)$$

By (15.5.25), (15.5.26) and (15.5.25) we have

$$\begin{aligned} & (2r)^{-2\tau} \int_{B_{2r} \setminus B_{r/2}} U^2 + R^{1-2\tau} \int_{B_R \setminus B_{r/2}} U^2 \leq \\ & \leq C\bar{M}^2 \left[\left(\frac{r}{4}\right)^{-2\tau} \int_{B_r} U^2 + \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}} U^2 \right], \end{aligned} \quad (15.5.28)$$

for every $\tau \geq \tilde{\tau}$. Now, we add to both the sides of (15.5.28) the quantity

$$R(2r)^{-2\tau} \int_{B_{r/2}} U^2 + (R)^{1-2\tau} \int_{B_{r/2}} U^2,$$

and we find (15.5.7) for $r < R/2$ and $R < R_0/2$. ■

Proof of Theorem 15.5.1.

Let us suppose $x_0 = 0$, (15.5.7) gives, for $0 < 2r < R < \frac{R_0}{2}$,

$$R^{1-2\tau} \int_{B_R} U^2 \leq C\bar{M}^2 \left[\left(\frac{r}{4}\right)^{-2\tau} \int_{B_r} U^2 + \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}} U^2 \right], \quad (15.5.29)$$

for every $\tau \geq \tilde{\tau}_2$.

Set

$$A(s) = \int_{B_s} U^2.$$

By (15.5.29) we have

$$A(R) \leq CR^{-1}\bar{M}^2 \left[\left(\frac{4R}{r}\right)^{2\tau} A(r) + \left(\frac{2R}{R_0}\right)^{2\tau} A(R_0) \right], \quad (15.5.30)$$

for every $\tau \geq \tilde{\tau}_2$. Let

$$\widehat{\tau} = \frac{\log \frac{A(R_0)}{A(r)}}{2 \log \frac{2R_0}{r}}. \quad (15.5.31)$$

If

$$\widehat{\tau} \geq \widetilde{\tau}_2, \quad (15.5.32)$$

we choose $\tau = \widehat{\tau}$ in (15.5.30) and since

$$\left(\frac{4R}{r}\right)^{2\widehat{\tau}} A(r) = \left(\frac{2R}{R_0}\right)^{2\widehat{\tau}} A(R_0),$$

we have

$$\begin{aligned} A(R) &\leq CR^{-1}\overline{M}^2 \left[\left(\frac{4R}{r}\right)^{2\widehat{\tau}} A(r) + \left(\frac{2R}{R_0}\right)^{2\widehat{\tau}} A(R_0) \right] = \\ &= 2CR^{-1}\overline{M}^2 \left(\frac{4R}{r}\right)^{2\widehat{\tau}} A(r) = R^{-1}\overline{M}^2 (A(r))^\theta (A(R_0))^{1-\theta}, \end{aligned} \quad (15.5.33)$$

where θ is given by (15.5.4). Whereas, if (15.5.32) does not hold, then

$$\frac{\log \frac{A(R_0)}{A(r)}}{\log \frac{2R_0}{r}} \leq 2\widetilde{\tau}_2$$

and multiplying both the sides of the last inequality by $\log \frac{R_0}{2R}$, we have

$$(A(R_0))^\theta \leq \left(\frac{R_0}{2R}\right)^{2\widetilde{\tau}_2} (A(r))^\theta, \quad (15.5.34)$$

from which we have trivially

$$\begin{aligned} A(R) &\leq A(R_0) = (A(R_0))^\theta (A(R_0))^{1-\theta} \leq \\ &\leq \left(\frac{R_0}{2R}\right)^{2\widetilde{\tau}_2} (A(r))^\theta (A(R_0))^{1-\theta}, \end{aligned} \quad (15.5.35)$$

which, together with (15.5.33), gives (15.5.3).

Now we prove (15.5.5).

Let us fix $R = \frac{R_0}{4}$ in (15.5.7). We have

$$\begin{aligned} &\frac{(2r)^{-2\tau}}{4} \int_{B_{2r}} U^2 + \left(\frac{R_0}{4}\right)^{1-2\tau} \int_{B_{R_0/4}} U^2 \leq \\ &\leq C\overline{M}^2 \left[\left(\frac{r}{4}\right)^{-2\tau} \int_{B_r} U^2 + \left(\frac{R_0}{2}\right)^{-2\tau} \int_{B_{R_0}} U^2 \right], \end{aligned} \quad (15.5.36)$$

for every $\tau \geq \tilde{\tau}_2$.

Now, by choosing $\tau = \tau_0$, where

$$\tau_0 = \tilde{\tau} + \log_4 \left(4C\bar{M}^2 N \right)$$

and

$$N = \frac{\int_{B_{R_0}} U^2}{\int_{B_{R_0/4}} U^2}, \tag{15.5.37}$$

we have

$$\left(\frac{R_0}{4} \right)^{1-2\tau_0} \int_{B_{R_0/4}} U^2 \geq C\bar{M}^2 \left(\frac{R_0}{2} \right)^{-2\tau_0} \int_{B_{R_0}} U^2.$$

Hence, by (15.5.36), we obtain

$$\frac{(2r)^{-2\tau_0}}{4} \int_{B_{2r}} U^2 \leq C\bar{M}^2 \left(\frac{r}{4} \right)^{-2\tau_0} \int_{B_r} U^2. \tag{15.5.38}$$

By using (15.5.37) and (15.5.38), we have

$$\int_{B_{2r}} U^2 \leq CN^3 \int_{B_r} U^2, \tag{15.5.39}$$

where C depends on M only.

The proof is complete. ■

Corollary 15.5.3 (strong unique continuation for Laplace operator).

Let $U \in H^2(B_1)$ be a solution to equation (15.5.1). Let $x_0 \in B_1$ and $0 < R_0 \leq 1 - |x_0|$.

If U does not vanish identically in $B_{R_0/4}(x_0)$, then we have, for every $r < s \leq \frac{R_0}{16}$,

$$\int_{B_s(x_0)} U^2 \leq CN_{x_0, R_0}^3 \left(\frac{s}{r} \right)^{\log_2(CN_{x_0, R_0}^3)} \int_{B_r(x_0)} U^2, \tag{15.5.40}$$

where N_{x_0, R_0} is defined by (15.5.6).

Moreover, if

$$\int_{B_r(x_0)} U^2 = \mathcal{O}(r^m), \quad \text{as } r \rightarrow 0, \quad \forall m \in \mathbb{N}, \tag{15.5.41}$$

then

$$U \equiv 0, \quad \text{in } B_1. \tag{15.5.42}$$

Proof. We prove (15.5.40). Let us suppose that $x_0 = 0$ and let $r < s \leq \frac{R_0}{16}$. Set $j = [\log_2(sr^{-1})]$ (we recall that $[a]$ is the integer part of a). We have

$$2^j r \leq s < 2^{j+1} r$$

and applying repeatedly (15.5.5) we obtain

$$\int_{B_s} U^2 \leq \int_{B_{2^{j+1}r}} U^2 \leq (CN^3)^{j+1} \int_{B_r} U^2 \leq CN^3 \left(\frac{s}{r}\right)^{\log_2(CN^3)} \int_{B_r} U^2.$$

From which we get (15.5.40).

Now, let us suppose that (15.5.41) holds true. Hence let us suppose that there exists a sequence C_m such that

$$\int_{B_r} U^2 \leq C_m r^m, \quad \text{for } r < 1, \quad \forall m \in \mathbb{N}. \quad (15.5.43)$$

Set

$$r_0 = \sup \left\{ r \in [0, 1] : \int_{B_r} U^2 = 0 \right\} \quad (15.5.44)$$

(let us note that in (15.5.44) the "sup" is, actually, the maximum).

We distinguish two cases

- (i) $r_0 = 0$,
- (ii) $r_0 \in (0, 1]$.

In case (i) we have

$$\int_{B_r} U^2 > 0, \quad \forall r \in (0, 1]. \quad (15.5.45)$$

Hence, setting

$$K = \log_2(CN_{0,1}^3)$$

by (15.5.40) and (15.5.43), we get, for r, s such that $r < s \leq \frac{1}{4}$

$$\int_{B_s} U^2 \leq CN_{0,1}^3 \left(\frac{s}{r}\right)^K \int_{B_r} U^2 \leq CC_m N_{0,1}^3 s^K r^{m-K}. \quad (15.5.46)$$

Let $m > K$. Passing to the limit in (15.5.46) as r goes to 0. We obtain

$$\int_{B_s} U^2 = 0, \quad \text{for } s \leq \frac{1}{4}$$

which contradicts (15.5.45).

Let us consider case (ii).

If $r_0 = 1$ there is nothing to prove. Let, therefore, $r_0 \in (0, 1)$. By the definition of r_0 we have

$$\int_{B_{r_0}} U^2 = 0. \tag{15.5.47}$$

Let

$$\delta < \min \left\{ r_0, \frac{1 - r_0}{15} \right\}$$

and let \bar{x} be a point such that $|\bar{x}| = r_0 - \delta$. Setting $\bar{R} = 1 - |\bar{x}|$ we have

$$r_1 := r_0 - \delta + \frac{\bar{R}}{16} = r_0 + \frac{1 - r_0 - 15\delta}{16} > r_0.$$

Now, since (15.5.47) trivially implies

$$\int_{B_r(\bar{x})} U^2 = \mathcal{O}(r^m), \quad \text{as } r \rightarrow 0, \quad \forall m \in \mathbb{N},$$

repeating the argument of case (i) in the ball $B_{\bar{R}}(\bar{x})$, we reach

$$\int_{B_{\bar{R}/16}(\bar{x})} U^2 = 0.$$

Finally, since this equality holds for each \bar{x} such that $|\bar{x}| = r_0 - \delta$, taking into account (15.5.47), we have

$$\int_{B_{r_1}} U^2 = 0,$$

which, as $r_1 > r_0$, contradicts the definition of r_0 given in (15.5.44). ■

Remarks and comments.

1. The optimal three sphere inequality can be obtained by the doubling inequality, (15.5.40), in a simple and direct way. As a matter of fact, by (15.5.40), using the elementary properties of the logarithmic function, we have, for $2r \leq s \leq \frac{R_0}{16}$,

$$\int_{B_s(x_0)} U^2 \leq (CN_{x_0, R_0}^3)^{2 \log_2 \frac{s}{r}} \int_{B_r(x_0)} U^2. \tag{15.5.48}$$

Now, by (15.5.6) we have trivially

$$N_{x_0, R_0} = \frac{\int_{B_{R_0}(x_0)} U^2}{\int_{B_{R_0/4}(x_0)} U^2} \leq \frac{\int_{B_{R_0}(x_0)} U^2}{\int_{B_s(x_0)} U^2}.$$

By the latter and by (15.5.48) we get

$$\left(\int_{B_s(x_0)} U^2 \right)^{1+6\log_2 \frac{s}{r}} \leq \left(C \int_{B_{R_0}(x_0)} U^2 \right)^{6\log_2 \frac{s}{r}} \int_{B_r(x_0)} U^2,$$

which gives

$$\int_{B_s(x_0)} U^2 \leq \left(C \int_{B_{R_0}(x_0)} U^2 \right)^{1-\tilde{\theta}} \left(\int_{B_r(x_0)} U^2 \right)^{\tilde{\theta}}$$

where

$$\tilde{\theta} = \frac{1}{1 + 6\log_2 \frac{s}{r}}.$$

Let us notice that also $\tilde{\theta}$ is an optimal exponent in the sense that, for fixed s , (15.1.6) holds.

2. Let us observe that three sphere inequality (15.5.3) has been proved using Carleman estimate (15.4.6), while to prove the doubling inequality we have used Carleman estimate (15.4.7), which differs from the estimate (15.4.6) for the occurrence of the term

$$\tau^2 r \int \rho^{-1-2\tau} u^2 dx.$$

The idea of including this term is indebted to Bakri [8], [9] and this idea simplifies the proof of the doubling inequality with respect to the previous proofs based on the Carleman estimates. It should also be pointed out that in the literature there are other methods to prove the doubling inequality, see for instance [27], [42]. We will briefly discuss the main ideas underlying such methods in 16.3.1. ♦

15.6 The geodesic polar coordinates

In this Section we give the definitions and the main properties of geodesic polar coordinates introduced by Aronszajn, Krzywicki and Szarski in [7].

Let n be an integer number, $n \geq 2$. For any $r > 0$ we denote by \tilde{B}_r the set $B_r \setminus \{0\}$.

For any $A = \{a_{ij}\}_{i,j=1}^n$ real matrix, we denote by $|A|$ the norm of A , i.e.

$$|A|^2 = \sum_{i,j=1}^n a_{ij}^2.$$

In what follows we will use the Einstein convention of the repeated indices.

By I_n we denote the $n \times n$ identity matrix. Given two vectors $x, y \in \mathbb{R}^n$, $x = (x^1, \dots, x^n)$, $y = (y^1, \dots, y^n)$ we denote by

$$x \cdot y = \delta^{ij} x^i y^j = x^i y^i,$$

their Euclidean scalar product and by $|x| = \sqrt{x \cdot x}$ the Euclidean norm.

Let $\lambda, \lambda \geq 1, \Lambda$ be positive numbers. Let $G(x) = \{g_{ij}(x)\}_{i,j=1}^n$ be a nonsingular symmetric real matrix whose entries are functions that belong to $C^\infty(\bar{B}_2)$. Let us denote by $G^{-1}(x) = \{g^{ij}(x)\}_{i,j=1}^n$ the inverse of $G(x)$. Let us suppose that

$$\lambda^{-1} |\xi|^2 \leq G(x) \xi \cdot \xi \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in B_2, \quad (15.6.1)$$

$$|\partial_{x^k} G| \leq \Lambda, \quad \text{for } k \in \{1, \dots, n\}, \quad \text{in } B_2. \quad (15.6.2)$$

Set

$$r(x) = |x|$$

and let us denote by

$$\mu_0(x) = G^{-1}(x) \nabla r(x) \cdot \nabla r(x), \quad \text{for } x \in \tilde{B}_2, \quad (15.6.3)$$

and

$$\tilde{G}(x) = \mu_0(x) G(x), \quad \text{for } x \in \tilde{B}_2. \quad (15.6.4)$$

Let us denote by $\tilde{g}_{ij}(x)$, $i, j \in \{1, \dots, n\}$ the entries of the matrix $\tilde{G}(x)$.

We wish to introduce the geodesic polar coordinates with respect to the metric tensor

$$\tilde{g}_{ij}(x) dx^i \otimes dx^j.$$

Then we will express in such geodesic polar coordinates the Laplace–Beltrami operator

$$\Delta_g(\cdot) = \frac{1}{\sqrt{g(x)}} \partial_{x^i} \left(\sqrt{g(x)} g^{ij}(x) \partial_{x^j} \cdot \right), \quad \text{in } B_2, \quad (15.6.5)$$

where

$$g(x) = \det G(x).$$

To perform the above mentioned transformation we assume also

$$G(0) = I_n, \quad (15.6.6)$$

For any $\bar{x} \in \overline{B_1} \setminus \{0\}$ let us denote by $\Gamma(\sigma; \bar{x})$ the global solution of the following Cauchy problem

$$\begin{cases} \dot{\Gamma}(\sigma; \bar{x}) = \tilde{G}^{-1}(\Gamma(\sigma; \bar{x})) \nabla r(\Gamma(\sigma; \bar{x})), \\ \Gamma(r(\bar{x}), \bar{x}) = \bar{x}, \end{cases} \quad (15.6.7)$$

where $\dot{\Gamma}$ is the derivative of $\Gamma(\sigma; \bar{x})$ w.r.t. σ .

Remark. Let us observe that for any $\bar{x} \in \overline{B_1} \setminus \{0\}$, Γ is a geodesic line w.r.t. the Riemannian metric $\tilde{g}_{ij}(x) dx^i \otimes dx^j$. Indeed, by (15.6.3) and (15.6.4) we have trivially that function $r(\cdot)$ is a solution to eikonal equation

$$\tilde{g}^{ij}(x) \partial_{x^i} r \partial_{x^j} r = 1.$$

Hence, denoting by $p(\sigma) = \nabla r(\Gamma(\sigma; \bar{x}))$ we have that, see Section 5.5, (Γ, p) is a solution to Hamilton-Jacobi equations

$$\begin{cases} \dot{\Gamma} = \nabla_p H(\Gamma, p), \\ \dot{p} = -\nabla_x H(\Gamma, p), \end{cases}$$

where

$$H(x, p) = \frac{1}{2} \tilde{g}^{ij}(x) p_i p_j$$

is the Hamiltonian. Therefore Γ solves the Euler equation, see Section 5.6,

$$\frac{d}{d\sigma} \nabla_q L(\Gamma, \dot{\Gamma}) = L_x(\Gamma, \dot{\Gamma}),$$

where

$$L(x, q) = \frac{1}{2} \tilde{g}_{ij}(x) q^i q^j.$$

Hence $\Gamma(\cdot, \bar{x})$ is a geodesic line w.r.t. the metric

$$\tilde{g}_{ij}(x) dx^i \otimes dx^j.$$

◆

The following Proposition holds true.

Proposition 15.6.1. *Let $\Gamma(\cdot; \bar{x})$ be the global solution to Cauchy problem (15.6.7). Then*

$$\Gamma(\cdot; \bar{x}) \text{ is defined in the interval } (0, 2), \quad (15.6.8)$$

and

$$r(\Gamma(\sigma; \bar{x})) = \sigma, \text{ for every } \sigma \in (0, 2). \quad (15.6.9)$$

Proof. The proof is the same of that given in the Remark after Theorem 5.6.5, we repeat with different notation for the reader's convenience. Let us denote by J the interval on which $\Gamma(\cdot; \bar{x})$ is defined.

Claim.

We have

$$r(\Gamma(\sigma; \bar{x})) = \sigma, \quad \forall \sigma \in J. \quad (15.6.10)$$

Proof of the Claim

To prove (15.6.10) we note that equation (15.6.7) gives (we omit \bar{x} in Γ)

$$\frac{d}{d\sigma} r(\Gamma(\sigma)) = \frac{d\Gamma^i(\sigma)}{d\sigma} \partial_{x^i} r(\Gamma(\sigma)) = \tilde{g}^{ij}(\Gamma(\sigma)) \partial_{x^i} r(\Gamma(\sigma)) r \partial_{x^j} r(\Gamma(\sigma)) = 1,$$

for every $\sigma \in J$. Therefore, there exists a constant c , such that

$$r(\Gamma(\sigma)) = \sigma + c, \quad \forall \sigma \in J.$$

By initial condition we have

$$\Gamma(r(\bar{x}), \bar{x}) = \bar{x},$$

hence

$$r(\bar{x}) = r(\Gamma(r(\bar{x}), \bar{x})) = r(\bar{x}) + c,$$

consequently $c = 0$ which implies (15.6.10). Claim is proved.

By (15.6.10) and by standard results of general theory of ordinary differential equations we have that Γ can be defined in the whole interval $(0, 2)$ hence (15.6.8) is proved. ■

In order to introduce the geodesic polar coordinates we need some additional notations. Set $\Sigma = \partial B_1$. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathcal{J}}$ be a finite family of local maps which define an oriented C^∞ differentiable structure on Σ . For any $\alpha \in \mathcal{J}$, set $V_\alpha = \varphi_\alpha(U_\alpha)$. Let us denote by Φ the map

$$\Phi : B_1 \setminus \{0\} \rightarrow (0, 1) \times \Sigma, \quad (15.6.11)$$

such that

$$\Phi(x) = (|x|, \Gamma(1; x)), \quad \forall x \in B_1 \setminus \{0\}. \quad (15.6.12)$$

By (15.6.7) we have easily that Φ is bijective, moreover

$$\Phi^{-1}(\varrho, p) = \Gamma(\varrho; p), \quad \forall (\varrho, p) \in (0, 1) \times \Sigma. \quad (15.6.13)$$

For any $\alpha \in \mathcal{J}$, let us consider the following geodesic sector

$$\mathcal{I}(U_\alpha) = \{\Gamma(\varrho; p) : \varrho \in (0, 1), p \in U_\alpha\}, \quad (15.6.14)$$

let us denote by Φ_α the map Φ expressed in the local coordinates, i.e.

$$\Phi_\alpha : \mathcal{I}(U_\alpha) \rightarrow (0, 1) \times V_\alpha, \quad (15.6.15)$$

$$\Phi_\alpha(x) = (|x|, \varphi_\alpha(\Gamma(1; x))), \quad \forall x \in \mathcal{I}(U_\alpha). \quad (15.6.16)$$

For any $\alpha \in \mathcal{J}$ let us denote by

$$\Gamma_\alpha(\varrho, \theta) = \Gamma(\varrho, \varphi_\alpha^{-1}(\theta)), \quad \forall (\varrho, \theta) \in (0, 1) \times V_\alpha. \quad (15.6.17)$$

We have

$$\Phi_\alpha^{-1}(\varrho, \theta) = \Gamma_\alpha(\varrho, \theta), \quad \forall (\varrho, \theta) \in (0, 1) \times V_\alpha. \quad (15.6.18)$$

Let us note that, for any $\alpha, \alpha' \in \mathcal{J}$, we have

$$(\Phi_\alpha \circ \Phi_{\alpha'}^{-1})(\varrho, \theta) = (\varrho, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)), \quad \forall (\varrho, \theta) \in \Phi_\alpha(\mathcal{I}(U_\alpha \cap U_{\alpha'})).$$

Let us note that $\mathcal{I}(U_\alpha) \cap \mathcal{I}(U_{\alpha'}) = \mathcal{I}(U_\alpha \cap U_{\alpha'})$, hence $\{\mathcal{I}(U_\alpha), \Phi_\alpha\}_{\alpha \in \mathcal{J}}$ defines an oriented C^∞ differentiable structure on $B_1 \setminus \{0\}$.

Let us observe that by (15.6.7) and (15.6.17) we have

$$\begin{aligned} \partial_\varrho \Gamma_\alpha(\varrho, \theta) &= \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \nabla r(\Gamma_\alpha(\varrho, \theta)) = \\ &= \frac{1}{\varrho} \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \Gamma_\alpha(\varrho, \theta) \end{aligned} \quad (15.6.19)$$

and

$$\Gamma_\alpha(1, \theta) = \varphi_\alpha^{-1}(\theta). \quad (15.6.20)$$

Moreover, by (15.6.9) we have

$$|\Gamma_\alpha(\varrho, \theta)| = \varrho. \quad (15.6.21)$$

To save the sum index convention, in the next Proposition and in the sequel, we denote by θ^{j+1} the j -th component of θ , so $\theta = (\theta^2, \dots, \theta^n)$.

Let $\alpha \in \mathcal{J}$ be fixed. Let $\eta = \{\eta^h\}_{h=2}^n$ a vector, set

$$y_\eta(\varrho, \theta) = \frac{1}{\varrho} \partial_\theta \Gamma_\alpha(\varrho, \theta) \eta, \quad (15.6.22)$$

where $\partial_\theta(\cdot)$ denotes the jacobian matrix w.r.t. the variables $\theta_2, \dots, \theta_n$. Let us check that by (15.6.19) and (15.6.20) we obtain, respectively,

$$\partial_\varrho y_\eta = \frac{1}{\varrho} \left(\tilde{G}^{-1}(\Gamma_\alpha) - I_n \right) y_\eta + \frac{1}{\varrho} (\partial_{x^k} \tilde{G}^{-1}(\Gamma_\alpha)) y_\eta^k \Gamma_\alpha, \quad (15.6.23)$$

and

$$y_\eta(1, \theta) = \partial_\theta \varphi_\alpha^{-1}(\theta) \eta. \quad (15.6.24)$$

Equality (15.6.24) is an immediate consequence of (15.6.20). Concerning (15.6.23), first we set

$$\tilde{y}_h = \frac{1}{\varrho} \partial_{\theta^h} \Gamma_\alpha(\varrho, \theta), \quad h = 2, \dots, n,$$

and we have, for $h = 2, \dots, n$,

$$\begin{aligned} \partial_\varrho \tilde{y}_h &= \partial_\varrho \left(\frac{1}{\varrho} \partial_{\theta^h} \Gamma_\alpha(\varrho, \theta) \right) = \\ &= -\frac{1}{\varrho^2} \partial_{\theta^h} \Gamma_\alpha(\varrho, \theta) + \frac{1}{\varrho} \partial_{\varrho \theta^h}^2 \Gamma_\alpha(\varrho, \theta) = \\ &= -\frac{1}{\varrho} \tilde{y}_h + \frac{1}{\varrho} \partial_{\theta^h} (\partial_\varrho \Gamma_\alpha(\varrho, \theta)) = \\ &= -\frac{1}{\varrho} \tilde{y}_h + \frac{1}{\varrho^2} \partial_{\theta^h} \left(\tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \Gamma_\alpha(\varrho, \theta) \right) = \\ &= -\frac{1}{\varrho} \tilde{y}_h + \frac{1}{\varrho^2} \partial_{\theta^h} \left(\tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \right) \Gamma_\alpha(\varrho, \theta) + \\ &\quad + \frac{1}{\varrho^2} \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \partial_{\theta^h} \Gamma_\alpha(\varrho, \theta) = \\ &= -\frac{1}{\varrho} \tilde{y}_h + \frac{1}{\varrho^2} \left(\partial_{x^k} \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \partial_{\theta^h} \Gamma_\alpha^k(\varrho, \theta) \right) \Gamma_\alpha(\varrho, \theta) + \\ &\quad + \frac{1}{\varrho} \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \tilde{y}_h. \end{aligned}$$

Hence, for $h = 2, \dots, n$, we have

$$\partial_\varrho \tilde{y}_h = \frac{1}{\varrho} \left(\tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) - I_n \right) \tilde{y}_h + \frac{1}{\varrho^2} \left(\partial_{x^k} \tilde{G}^{-1}(\Gamma_\alpha(\varrho, \theta)) \partial_{\theta^h} \Gamma_\alpha^k(\varrho, \theta) \right) \Gamma_\alpha(\varrho, \theta).$$

By multiplying both the sides of the last equality by η^h and adding up the index h , we obtain (15.6.23). ■

Lemma 15.6.2. *Let us assume that (15.6.1), (15.6.2) and (15.6.6) hold true. Let Γ_α , $\alpha \in \mathcal{J}$ be defined by (15.6.17). We have*

$$C^{-1} |\partial_\theta \varphi_\alpha^{-1}(\theta)\eta| \leq |y_\eta(\varrho, \theta)| \leq C |\partial_\theta \varphi_\alpha^{-1}(\theta)\eta|, \quad (15.6.25)$$

for every $(\varrho, \theta) \in (0, 1) \times V_\alpha$ and for every $\eta \in \mathbb{R}^{n-1}$, where C and $C \geq 1$, depend on λ and Λ only (here and in the sequel we omit the dependence on n).

Proof. Let us omit the index α . By (15.6.1), (15.6.4), (15.6.6), and (15.6.21) we have

$$\begin{aligned} \left| \partial_{x^k} \tilde{G}^{-1}(\Gamma) y_\eta^k \right| &\leq C |y_\eta|, \\ \left| \tilde{G}^{-1}(\Gamma) - I_n \right| &\leq C \varrho, \end{aligned}$$

where C depends on λ and Λ only. Therefore, by (15.6.23) we have

$$|\partial_\varrho y_\eta| \leq C |y_\eta|, \quad (15.6.26)$$

where C depends on λ and Λ only.

By (15.6.24) and (15.6.26) we have

$$|y_\eta(\varrho, \theta)| \leq |\partial_\theta \varphi^{-1}(\eta)| + C \int_\varrho^1 |y_\eta(s, \theta)| ds, \quad \forall \varrho \in (0, 1), \quad (15.6.27)$$

By (15.6.27) and by the Gronwall inequality we get the second inequality of (15.6.25).

Now we prove the first inequality of (15.6.25). Let $\bar{\varrho}$ be fixed in $(0, 1)$. We have

$$|y_\eta(\varrho, \theta)| \leq |y_\eta(\bar{\varrho}, \theta)| + \int_{\bar{\varrho}}^\varrho |\partial_s y_\eta(s, \theta)| ds, \quad \forall \varrho \in [\bar{\varrho}, 1],$$

hence by (15.6.26) we get

$$|y_\eta(\varrho, \theta)| \leq |y_\eta(\bar{\varrho}, \theta)| + C \int_{\bar{\varrho}}^\varrho |y_\eta(s, \theta)| ds, \quad \forall \varrho \in [\bar{\varrho}, 1].$$

By the Gronwall inequality we have

$$|y_\eta(\varrho, \theta)| \leq |y_\eta(\bar{\varrho}, \theta)| e^C, \quad \forall \varrho \in [\bar{\varrho}, 1], \quad (15.6.28)$$

where C depends on λ and Λ only. By (15.6.24) and (15.6.28) we have

$$|\partial_\theta \varphi^{-1}(\theta)\eta| e^{-C} \leq |y_\eta(\bar{\varrho}, \theta)|,$$

that gives the first inequality of (15.6.25). ■

The following Proposition holds true.

Proposition 15.6.3. *For any $\alpha \in \mathcal{J}$ let us denote by $\tilde{b}_{\alpha,hk}$, $h, k \in \{1, \dots, n\}$, the components of the metric tensor $\tilde{g}_{ij}(x) dx^i \otimes dx^j$ with respect to the local coordinates $(\mathcal{I}(U_\alpha), \Phi_\alpha)$. We have*

$$\tilde{b}_{\alpha,hk}(\varrho, \theta) = \tilde{g}_{ij}(\Gamma_\alpha(\varrho, \theta)) \partial_{\theta^h} \Gamma_\alpha^i \partial_{\theta^k} \Gamma_\alpha^j, \quad \text{for } h, k \in \{2, \dots, n\}, \quad (15.6.29)$$

$$\tilde{b}_{\alpha,h1}(\varrho, \theta) = \tilde{b}_{\alpha,1h}(\varrho, \theta) = 0, \quad \text{for } h \in \{2, \dots, n\} \quad (15.6.30)$$

and

$$\tilde{b}_{\alpha,11}(\varrho, \theta) = 1. \quad (15.6.31)$$

Proof. To simplify the notation, in what follows we omit the index α . Equality (15.6.29) are nothing but the rule of transformation of the components of the metric tensor.

Let us prove (15.6.30).

By (15.6.19) and (15.6.21) we get, for every $h \in \{2, \dots, n\}$,

$$\begin{aligned} \tilde{b}_{h1} &= \tilde{g}_{ij}(\Gamma) \partial_{\theta^h} \Gamma^i \partial_\varrho \Gamma^j = \tilde{g}_{ij}(\Gamma) \partial_{\theta^h} \Gamma^i \tilde{g}^{jk}(\Gamma) \partial_{x^k} r(\Gamma) = \\ &= \delta_i^k \partial_{\theta^h} \Gamma^i \partial_{x^k} r(\Gamma) = \partial_{\theta^k} (r(\Gamma)) = \partial_{\theta^k} \varrho = 0, \end{aligned}$$

since \tilde{b}_{ij} is symmetric, we obtain (15.6.30).

Now, let us prove (15.6.31). We have

$$\begin{aligned} \tilde{b}_{11} &= \tilde{g}_{ij}(\Gamma) \partial_\varrho \Gamma^i \partial_\varrho \Gamma^j = \tilde{g}_{ij}(\Gamma) \tilde{g}^{ik}(\Gamma) \partial_{x^k} r(\Gamma) \tilde{g}^{jh}(\Gamma) \partial_{x^h} r(\Gamma) = \\ &= \delta_j^k \tilde{g}^{jh}(\Gamma) \partial_{x^k} r(\Gamma) \partial_{x^h} r(\Gamma) = \tilde{g}^{kh}(\Gamma) \partial_{x^k} r(\Gamma) \partial_{x^h} r(\Gamma) = 1. \end{aligned}$$

■

In formulas (15.6.32)–(15.6.37) below, we introduce some notations.

Set, for any $\alpha \in \mathcal{J}$,

$$\mu_\alpha = \mu_0 \circ \Phi_\alpha^{-1}, \quad (15.6.32)$$

$$\tilde{b}_\alpha = \det \left\{ \tilde{b}_{\alpha,ij} \right\}_{i,j=1}^n, \quad (15.6.33)$$

$$\beta_{\alpha,hk} = \frac{1}{\varrho^2} \tilde{b}_{\alpha,hk}, \quad \text{for } h, k \in \{2, \dots, n\}, \quad (15.6.34)$$

$$\{\beta_\alpha^{hk}\}_{h,k=2}^n = \left(\{\beta_{\alpha,hk}\}_{h,k=2}^n \right)^{-1}, \quad (15.6.35)$$

$$\beta_\alpha = \det \{\beta_{\alpha,hk}\}_{h,k=2}^n. \quad (15.6.36)$$

In addition, let

$$\beta_\alpha = \varrho^{-2(n-1)} \tilde{b}_\alpha. \quad (15.6.37)$$

The following Proposition holds true.

Proposition 15.6.4. *For every $\bar{\varrho} \in (0, 1)$, $\beta_{\alpha,hk}(\bar{\varrho}, \theta)$, $h, k \in \{2, \dots, n\}$, are the components of a metric tensor on Σ with respect to the local maps $(U_\alpha, \varphi_\alpha)$.*

Proof. Let $\bar{p} \in \Sigma$ and let $(U_\alpha, \varphi_\alpha)$, $(U_{\alpha'}, \varphi_{\alpha'})$ be two coordinate neighborhoods such that $\bar{p} \in U_\alpha \cap U_{\alpha'}$. Let p be an arbitrary point of $U_\alpha \cap U_{\alpha'}$. Set

$$\theta^{(p)} = \varphi_\alpha(p), \quad \widehat{\theta}^{(p)} = \varphi_{\alpha'}(p).$$

We have trivially,

$$p = \varphi_\alpha^{-1}(\theta^{(p)}) = \varphi_{\alpha'}^{-1}(\widehat{\theta}^{(p)}).$$

Moreover, since $\Gamma_\alpha(\cdot, \theta^{(p)})$ and $\Gamma_{\alpha'}(\cdot, \widehat{\theta}^{(p)})$ are solutions to equation (15.6.19) and

$$\Gamma_\alpha(1, \theta^{(p)}) = \Gamma_{\alpha'}(1, \widehat{\theta}^{(p)}) = p,$$

we have

$$\Gamma_\alpha(\cdot, \theta^{(p)}) = \Gamma_{\alpha'}(\cdot, \widehat{\theta}^{(p)}).$$

Therefore

$$\Gamma_{\alpha'}(\bar{\varrho}, \widehat{\theta}^{(p)}) = \Gamma_\alpha(\bar{\varrho}, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\widehat{\theta}^{(p)})).$$

Hence, if

$$\theta \in \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'}),$$

then

$$\Gamma_{\alpha'}(\bar{\varrho}, \theta) = \Gamma_\alpha(\bar{\varrho}, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)).$$

Differentiating w.r.t. θ^l both the sides of the last equality, we obtain

$$\partial_{\theta^l} \Gamma_{\alpha'}(\bar{\varrho}, \theta) = (\partial_{\theta^k} \Gamma_\alpha)(\bar{\varrho}, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)) \partial_{\theta^l} (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})^k(\theta), \quad (15.6.38)$$

for every $l \in \{2, \dots, n\}$, $\theta \in \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'})$.

Now

$$\beta_{\alpha',lm}(\bar{\varrho}, \theta) = \bar{\varrho}^{-2} \tilde{g}_{ij}(\Gamma_{\alpha'}(\bar{\varrho}, \theta)) \partial_{\theta^l} \Gamma_{\alpha'}^i \partial_{\theta^m} \Gamma_{\alpha'}^j, \quad (15.6.39)$$

for $l, m \in \{2, \dots, n\}$, $\theta \in \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'})$.

Therefore by (15.6.38) and (15.6.39) we obtain

$$\begin{aligned} \beta_{\alpha',lm}(\bar{\varrho}, \theta) &= \\ &= \beta_{\alpha',ks}(\bar{\varrho}, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)) \partial_{\theta^l} (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})^k(\theta) \partial_{\theta^m} (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})^s(\theta), \end{aligned} \quad (15.6.40)$$

for every $l, m \in \{2, \dots, n\}$ and for every $\theta \in \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'})$.

Equality (15.6.40) proves that $\beta_{\alpha,hk}(\bar{\varrho}, \theta)$, $h, k \in \{2, \dots, n\}$, are the components of a tensor which is a metric tensor because the matrix $\{\beta_{\alpha,hk}(\bar{\varrho}, \theta)\}_{h,k=2}^n$ is symmetric and positive. The proof is complete. ■

Now we begin to derive the expression of operator (15.6.5) in the polar coordinates introduced above.

Let $u \in C^\infty(B_2)$ and let us denote by w the function

$$(0, 1) \times \Sigma \ni (\varrho, p) \rightarrow w(\varrho, p) = u(\Gamma(\varrho, p)). \quad (15.6.41)$$

Set

$$w_\alpha = u \circ \Phi_\alpha^{-1}, \quad (15.6.42)$$

by (15.6.18) we have

$$w(\varrho, \varphi_\alpha^{-1}(\theta)) = w_\alpha(\varrho, \theta), \quad \forall (\varrho, \theta) \in (0, 1) \times V_\alpha, \quad (15.6.43)$$

hence, for any fixed $\varrho \in (0, 1)$, $w_\alpha(\varrho, \cdot)$ is the expression of $w(\varrho, \cdot)$ in the local coordinates $(U_\alpha, \varphi_\alpha)$.

Now, for any fixed $\bar{\varrho} \in (0, 1)$, Proposition 15.6.4 allows us to define on Σ the Riemannian structure, [11], induced by metric tensor whose components with respect coordinate neighborhood $(U_\alpha, \varphi_\alpha)$ are equals to $\beta_{\alpha,hk}(\bar{\varrho}, \theta)$ for $h, k \in \{2, \dots, n\}$.

Let us denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|_\Sigma$, respectively, the inner product and the associated norm on the above defined Riemannian structure. Let us denote by ∇_Σ and div_Σ the gradient and the divergence operators on Σ respectively.

We have

$$(\nabla_\Sigma w(\bar{\varrho}, \cdot)) \circ \varphi_\alpha^{-1}(\theta) = \{\beta_\alpha^{hk}(\bar{\varrho}, \theta) \partial_{\theta^k} w_\alpha(\bar{\varrho}, \theta)\}_{h=2}^n, \quad \forall \theta \in V_\alpha. \quad (15.6.44)$$

Set

$$\mu(\varrho, p) = \mu_0(\Gamma(\varrho, p)), \quad \forall (\varrho, \theta) \in (0, 1) \times \Sigma,$$

we have

$$\begin{aligned} \text{div}_\Sigma (\mu^{1-\frac{n}{2}}(\bar{\varrho}, \cdot) \nabla_\Sigma w(\bar{\varrho}, \cdot)) \circ \varphi_\alpha^{-1}(\theta) &= \\ &= \frac{1}{\sqrt{\beta_\alpha(\bar{\varrho}, \theta)}} \partial_{\theta^h} \left(\mu_\alpha^{1-\frac{n}{2}}(\bar{\varrho}, \theta) \sqrt{\beta_\alpha(\bar{\varrho}, \theta)} \beta_\alpha^{hk}(\bar{\varrho}, \theta) \partial_{\theta^k} w_\alpha(\bar{\varrho}, \theta) \right), \end{aligned} \quad (15.6.45)$$

for every $\theta \in V_\alpha$.

Let us note that Proposition (15.6.3) implies that the derivatives

$$\partial_\varrho \log \sqrt{\beta_\alpha(\varrho, \theta)}, \quad \text{for } \alpha \in \mathcal{J},$$

are the expression in the coordinate neighborhoods of a C^∞ function on Σ .

Let us observe that by the equality

$$\Gamma_\alpha(\varrho, \theta) = \Gamma_{\alpha'}(\varrho, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)) \quad \forall \theta \in \varphi_{\alpha'}(U_\alpha \cap U_{\alpha'}),$$

we get

$$\partial_{\theta^h} \Gamma_\alpha(\varrho, \theta) = \partial_{\theta^k} \Gamma_{\alpha'}(\varrho, (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})(\theta)) \partial_{\theta^h} (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})^k(\theta).$$

Let us denote, respectively, by \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L} the operators

$$\mathcal{L}_1 w = \mu \left(\frac{\partial^2 w}{\partial \varrho^2} + \frac{n-1}{\varrho} \frac{\partial w}{\partial \varrho} + \frac{1}{\varrho^2 \mu^{1-n/2}} \operatorname{div}_\Sigma (\mu^{1-n/2} \nabla_\Sigma w) \right), \quad (15.6.46)$$

$$\mathcal{L}_2 w = \mu \frac{\partial}{\partial \varrho} \left(\log \left(\mu^{1-n/2} \sqrt{\beta} \right) \right) \partial_\varrho w \quad (15.6.47)$$

and

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2. \quad (15.6.48)$$

We have the following

Proposition 15.6.5. (Transformation of the operator Δ_g) *The following equality holds true*

$$((\Delta_g u) \circ \Phi_\alpha^{-1})(\varrho, \theta) = (\mathcal{L} w)(\varrho, \varphi_\alpha^{-1}(\theta)), \quad \forall (\varrho, \theta) \in (0, 1) \times V_\alpha. \quad (15.6.49)$$

Proof. Let us denote by $\{b_\alpha^{ij}\}_{i,j=1}^n$ the inverse matrix of $\{\mu_\alpha^{-1} \tilde{b}_{\alpha,ij}\}_{i,j=1}^n$ (recall that $\{\tilde{b}_{\alpha,ij}\}_{i,j=1}^n$ is defined in Proposition (15.6.3)). Let us recall that $w_\alpha = u \circ \Phi_\alpha^{-1}$. By (15.6.4) and (15.6.30) we have

$$\begin{aligned} (\Delta_g u)(\Phi_\alpha^{-1}(\varrho, \theta)) &= \\ &= \frac{1}{\sqrt{b_\alpha}} \left(\partial_\varrho \left(\sqrt{b_\alpha} \mu_\alpha \partial_\varrho w_\alpha \right) + \partial_{\theta^h} \left(\sqrt{b_\alpha} b_\alpha^{hk} \partial_{\theta^k} w_\alpha \right) \right), \end{aligned} \quad (15.6.50)$$

where $b_\alpha = \det \left\{ \mu_\alpha^{-1} \tilde{b}_{\alpha,ij} \right\}_{i,j=1}^n$.

We have

$$b_\alpha = \mu_\alpha^{-n} \varrho^{2(n-1)} \beta_\alpha, \tag{15.6.51}$$

$$b_\alpha^{hk} = \varrho^{-2} \mu_\alpha \beta_\alpha^{hk}, \quad \text{for } h, k \in \{2, \dots, n\}. \tag{15.6.52}$$

By (15.6.45), (15.6.50), (15.6.51) and (15.6.52) we get

$$\begin{aligned} & (\Delta_g u) \circ \Phi_\alpha^{-1} = \\ & = \left(\mu \left(\partial_\varrho^2 w + \frac{n-1}{\varrho} \partial_\varrho w + \frac{1}{\varrho^2 \mu^{1-n/2}} \operatorname{div}_\Sigma \mu^{1-n/2} \nabla_\Sigma w \right) \right) \circ \varphi_\alpha^{-1} \\ & + \left(\mu \partial_\varrho \left(\log \mu^{1-n/2} \sqrt{\beta} \right) \partial_\varrho w \right) \circ \varphi_\alpha^{-1}. \end{aligned}$$

■

In the next propositions we will estimate the tensors which occur in the transformed operator \mathcal{L} .

Proposition 15.6.6. *Let $\mu_\alpha, \{\beta_{\alpha,hk}\}_{h,k=2}^n, \beta_\alpha$ be defined by (15.6.32), (15.6.34), (15.6.37). For any $\alpha \in \Gamma$ and $(\varrho, \theta) \in (0, 1) \times V_\alpha$, we have*

$$\lambda^{-1} \leq \mu_\alpha(\varrho, \theta) \leq \lambda, \tag{15.6.53}$$

$$|\partial_\varrho \beta_{\alpha,hk} \eta^h \eta^k| \leq C \beta_{\alpha,hk}(\varrho, \theta) \eta^h \eta^k, \quad \text{for } \eta \in \mathbb{R}^{n-1}, \tag{15.6.54}$$

$$|\partial_\varrho \log \sqrt{\beta_\alpha}| \leq C, \tag{15.6.55}$$

$$|\partial_\varrho \mu_\alpha(\varrho, \theta)| \leq C, \tag{15.6.56}$$

where C depends on λ and Λ only.

Proof. Let us omit the index α . By (15.6.2) and (15.6.3) we have

$$\lambda^{-1} \leq \mu_0(x) \leq \lambda,$$

by these inequalities and by (15.6.32) we obtain (15.6.53).

Now, we prove (15.6.54). For any vector $\{\eta^h\}_{h=2}^n$, let y_η be defined by (15.6.22).

By (15.6.22), (15.6.29) and (15.6.34) we have

$$\beta_{hk}(\varrho, \theta) \eta^h \eta^k = \tilde{G}(\Gamma(\varrho, \theta)) y_\eta \cdot y_\eta. \tag{15.6.57}$$

Therefore

$$\partial_\varrho \beta_{hk}(\varrho, \theta) \eta^h \eta^k = 2\tilde{G}(\Gamma) \partial_\varrho y_\eta \cdot y_\eta + \left(\partial_{x^k} \tilde{G}(\Gamma) \partial_\varrho \Gamma^k y_\eta \right) \cdot y_\eta. \tag{15.6.58}$$

Let us recall that

$$|\partial_\varrho y_\eta| \leq C |y_\eta|.$$

By (15.6.1), (15.6.2) (15.6.4), (15.6.21), (15.6.58) and by the last inequality we get

$$\begin{aligned} |\partial_\varrho \beta_{hk}(\varrho, \theta) \eta^h \eta^k| &\leq C |y_\eta|^2 \leq \\ &\leq C \lambda^2 \tilde{G}(\Gamma) y_\eta \cdot y_\eta = \\ &= C \lambda^2 \beta_{hk}(\varrho, \theta) \eta^h \eta^k, \end{aligned} \tag{15.6.59}$$

where C depends on λ and Λ only. Therefore (15.6.54) is proved.

In order to prove (15.6.55), recall that if $A(s)$ is a matrix-valued function of class C^1 such that $\det A(s) \neq 0$ then we have the following equality (we denote by $\text{tr}(\cdot)$ the trace of the matrix in the brackets)

$$\frac{d}{ds} \log |\det A(s)| = \text{tr} \left(\frac{dA(s)}{ds} A^{-1}(s) \right).$$

This equality and (15.6.54) give

$$\begin{aligned} \left| \partial_\varrho \log \sqrt{\beta} \right| &= \frac{1}{2} |(\partial_\varrho \beta_{ij}) \beta^{ji}| = \frac{1}{2} |(\partial_\varrho \beta_{hk}) \delta_i^h \delta_j^k \beta^{lm} \delta_l^i \delta_m^j| \leq \\ &\leq C \beta_{hk} \delta_i^h \delta_j^k \beta^{lm} \delta_l^i \delta_m^j = C \beta_{ij} \beta^{ji} = C(n-1), \end{aligned}$$

where C depends on λ and Λ only. Therefore (15.6.55) follows. ■

In order to prove Propositions 15.6.7 and 15.6.8 stated below we need a partition of unity $\{\zeta_\alpha\}_{\alpha \in \mathcal{J}}$ subordinate to the (finite) covering $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{J}}$. By this, we mean that for each $\alpha \in \mathcal{J}$, $\zeta_\alpha \in C^\infty(\Sigma)$, $\zeta_\alpha \geq 0$, $\text{supp } \zeta_\alpha \subset U_\alpha$ and

$$\sum_{\alpha \in \mathcal{J}} \zeta_\alpha(p) = 1, \quad \forall p \in \Sigma.$$

Let us denote by $\widehat{\zeta}_\alpha$ the function $\zeta_\alpha \circ \varphi_\alpha^{-1}$ and set $\widetilde{\zeta}_\alpha(x) = \zeta_\alpha(\Gamma(1; x, \cdot))$. We have

$$\sum_{\alpha \in \mathcal{J}} \widetilde{\zeta}_\alpha(x) = 1, \quad \forall x \in \widetilde{B}_1.$$

Proposition 15.6.7. *For every $\varrho \in (0, 1)$, let us denote by $d\Omega_\varrho$ the element of volume on Σ .*

Let f be a function belonging to $C^0(\overline{B}_1)$. We have

$$\int_{B_1} f(x) \sqrt{\widetilde{g}(x)} dx = \int_0^{R_0} d\varrho \int_\Sigma f(\Gamma(\varrho, p)) \varrho^{n-1} d\Omega_\varrho. \tag{15.6.60}$$

Proof. For any $\sigma \in (0, 1)$ we have

$$\int_{B_1 \setminus B_\sigma} f(x) \sqrt{\tilde{g}(x)} dx = \sum_{\alpha \in \mathcal{J}} \int_{\mathcal{I}(U_\alpha) \setminus B_\sigma} f(x) \sqrt{\tilde{g}(x)} \tilde{\zeta}_\alpha(x) dx. \quad (15.6.61)$$

Now, in the integral on the right-hand side of (15.6.61), we perform the following change of variables:

$$x = \Phi_\alpha^{-1}(\varrho, \theta).$$

By such a change of variables and by (15.6.18), (15.6.33), (15.6.37) we get

$$\begin{aligned} & \int_{\mathcal{I}(U_\alpha) \setminus B_\sigma} f(x) \sqrt{\tilde{g}(x)} \tilde{\zeta}_\alpha(x) dx = \\ &= \int_\sigma^1 \int_{V_\alpha} f(\Gamma_\alpha(\varrho; \varphi_\alpha^{-1}(\theta))) \varrho^{n-1} \sqrt{\beta_\alpha(\varrho, \theta)} \hat{\zeta}_\alpha(\theta) d\theta d\varrho. \end{aligned} \quad (15.6.62)$$

By (15.6.61) and (15.6.62) we get

$$\begin{aligned} & \int_{B_1 \setminus B_\sigma} f(x) \sqrt{\tilde{g}(x)} dx = \\ &= \int_\sigma^1 d\varrho \sum_{\alpha \in \mathcal{J}} \int_{V_\alpha} f(\Gamma(\varrho; \varphi_\alpha^{-1}(\theta))) \varrho^{n-1} \sqrt{\beta_\alpha(\varrho, \theta)} \hat{\zeta}_\alpha(\theta) d\theta = \\ &= \int_\sigma^1 d\varrho \int_{\Sigma} f(\Gamma(\varrho, p)) \varrho^{n-1} d\Omega_\varrho. \end{aligned}$$

Finally, by the latter we have

$$\begin{aligned} \int_{B_1} f(x) \sqrt{\tilde{g}(x)} dx &= \lim_{\sigma \rightarrow 0} \int_{B_1 \setminus B_\sigma} f(x) \sqrt{\tilde{g}(x)} dx = \\ &= \int_0^1 d\varrho \int_{\Sigma} f(\Gamma(\varrho, p)) \varrho^{n-1} d\Omega_\varrho. \end{aligned}$$

■

For any $\varrho \in (0, 1)$ denotes by $\Xi^{(\varrho)}$ the covariant tensor of order 2 whose components with respect to coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ are equal to

$$\partial_\varrho \beta_\alpha^{lm}(\varrho, \cdot) \beta_{\alpha, lh}(\varrho, \cdot) \beta_{\alpha, mk}(\varrho, \cdot), \quad \text{for } h, k = 2, \dots, n.$$

Let us denote by $\ell(\varrho, \cdot)$ the $C^\infty(\Sigma)$ function whose expressions with respect to the coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ are equal to

$$\partial_\varrho \log \sqrt{\beta_\alpha(\varrho, \cdot)}.$$

In the following Proposition we will denote by $\int(\cdot)$ the integral $\int_0^{\varrho_0} d\varrho \int_\Sigma(\cdot) d\Omega_\varrho$.

Proposition 15.6.8. *Let $v_1, v_2 \in C^\infty((0, \varrho_0) \times \Sigma)$. Let us suppose either v_1 or v_2 of compact support. Let $h \in C^\infty((0, \varrho_0))$.*

Then we have

$$\int v_1 \partial_\varrho v_2 = - \int (v_1 \ell + \partial_\varrho v_1) v_2. \quad (15.6.63)$$

If v_1 has compact support then

$$\begin{aligned} & \int h(\varrho) \partial_\varrho v_1 \operatorname{div}_\Sigma (\mu^{1-\frac{n}{2}} \nabla_\Sigma v_1) = \\ &= \frac{1}{2} \int h(\varrho) \mu^{1-\frac{n}{2}} \Xi^{(\varrho)} (\nabla_\Sigma v_1, \nabla_\Sigma v_1) + \\ &+ \frac{1}{2} \int (\partial_\varrho (h(\varrho) \mu^{1-\frac{n}{2}}) + h(\varrho) \mu^{1-\frac{n}{2}} \ell) |\nabla_\Sigma v_1|_\Sigma^2. \end{aligned} \quad (15.6.64)$$

Proof. For every $\alpha \in \mathcal{J}$ we denote, respectively, by $v_{1,\alpha}(\varrho, \theta)$ and $v_{2,\alpha}(\varrho, \theta)$, the functions $v_1(\varrho, \varphi_\alpha^{-1}(\theta))$ and $v_2(\varrho, \varphi_\alpha^{-1}(\theta))$.

Let us prove (15.6.63).

We have

$$\begin{aligned} & \int_0^{\varrho_0} d\varrho \int_\Sigma v_{1,\alpha} \partial_\varrho v_{2,\alpha} d\Omega_\varrho = \sum_{\alpha \in \mathcal{J}} \int_0^{\varrho_0} d\varrho \int_{V_\alpha} v_{1,\alpha} (\partial_\varrho v_{2,\alpha}) \sqrt{\beta_\alpha} \widehat{\zeta}_\alpha d\theta = \\ &= - \sum_{\alpha \in \mathcal{J}} \int_0^{\varrho_0} d\varrho \int_{V_\alpha} (v_{1,\alpha} \partial_\varrho \log \sqrt{\beta_\alpha} + \partial_\varrho v_{1,\alpha}) v_{2,\alpha} \sqrt{\beta_\alpha} \widehat{\zeta}_\alpha d\theta = \\ &= - \int_0^{\varrho_0} d\varrho \int_\Sigma (v_1 \ell + \partial_\varrho v_1) v_2 d\Omega_\varrho. \end{aligned}$$

Let us prove (15.6.64). Let us suppose v_1 with compact support. By the divergence Theorem on the Riemannian manifold Σ we have

$$\begin{aligned} & \int_0^{\varrho_0} d\varrho \int_\Sigma f(\varrho) \partial_\varrho v_1 \operatorname{div}_\Sigma (\mu^{1-\frac{n}{2}} \nabla_\Sigma v_1) d\Omega_\varrho = \\ &= - \int_0^{\varrho_0} d\varrho \int_\Sigma f(\varrho) \mu^{1-\frac{n}{2}} \langle \nabla_\Sigma v_1, \nabla_\Sigma \partial_\varrho v_1 \rangle d\Omega_\varrho. \end{aligned} \quad (15.6.65)$$

Now

$$\langle \nabla_{\Sigma} v_1, \nabla_{\Sigma} \partial_{\varrho} v_1 \rangle = \frac{1}{2} \partial_{\varrho} \langle \nabla_{\Sigma} v_1, \nabla_{\Sigma} v_1 \rangle - \frac{1}{2} \Xi^{(\varrho)} (\nabla_{\Sigma} v_1, \nabla_{\Sigma} v_1),$$

that, with (15.6.63) gives (15.6.64). ■

15.7 The case of variable coefficients

In this Section we will prove the Carleman estimate of Aronszajn–Krzywicki–Szarski, [7]. Basically we will proceed in a similar way to Section 15.4, however, instead of the Euclidean polar coordinates we will use the geodesic polar coordinates introduced in Section 15.6. Compared with the original proof of [7], the one we will prove here has some simplification.

Precisely we prove

Theorem 15.7.1 (Carleman estimate for Δ_g). *Let us suppose that the matrix $G = \{g_{ij}(x)\}_{i,j=1}^n$ satisfies to (15.6.1), (15.6.2), (15.6.6) and $g_{ij} \in C^\infty(B_2)$, for $i, j = 1, \dots, n$.*

Let $\epsilon \in (0, 1]$. We define

$$\rho(x) = \phi_\epsilon(|x|), \quad \forall x \in B_1 \setminus \{0\}, \tag{15.7.1}$$

where

$$\phi_\epsilon(s) = \frac{s}{(1 + s^\epsilon)^{1/\epsilon}}. \tag{15.7.2}$$

Then there exist $r_0 \in (0, 1)$, $\bar{\tau} > 1$ and $C > 1$, which depend on ϵ , λ and Λ only, such that

$$\begin{aligned} \tau^3 \int \rho^{\epsilon-2\tau} |u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau} |\nabla u|^2 dx + \\ + \tau^2 r \int \rho^{-1-2\tau} u^2 dx \leq C \int \rho^{4-2\tau} |\Delta_g u|^2 dx, \end{aligned} \tag{15.7.3}$$

for every $r \in (0, r_0)$, for every $\tau \geq \bar{\tau}$ and for every $u \in C_0^\infty(B_{r_0} \setminus \bar{B}_{r/4})$.

We start by the following simple estimation of the first order operator \mathcal{L}_2 defined in (15.6.47).

Proposition 15.7.2. *The following estimate holds true*

$$|\mathcal{L}_2 w| \leq C |\partial_{\varrho} w| \quad \forall w \in C^\infty((0, 1) \times \Sigma), \tag{15.7.4}$$

where C depends on λ and Λ only.

Proof. By (15.6.55) and (15.6.56) we have

$$\begin{aligned} & |(\mathcal{L}_2 w)(\varrho, \varphi_\alpha^{-1}(\theta))| = \\ & = \left| \mu_\alpha \left(\left(1 - \frac{n}{2}\right) \partial_\varrho \log \mu_\alpha + \partial_\varrho \log \sqrt{\beta_\alpha} \right) \partial_\varrho w_\alpha \right| \leq \\ & \leq \left(\left| \mu_\alpha \left(1 - \frac{n}{2}\right) \partial_\varrho \log \mu_\alpha \right| + \left| \partial_\varrho \log \sqrt{\beta_\alpha} \right| \right) |\partial_\varrho w_\alpha| \leq \\ & \leq C |\partial_\varrho w_\alpha|, \end{aligned}$$

where C depends on λ and Λ only. ■

Let u be an arbitrary function that belongs to $C_0^\infty(B_1 \setminus \{0\})$ and let us denote by w the function (recall (15.6.41))

$$(0, 1) \times \Sigma \ni (\varrho, p) \rightarrow w(\varrho, p) = u(\Gamma(\varrho, p)). \quad (15.7.5)$$

where Γ is defined in Proposition 15.6.1.

Now, we carry out the following change of variables

$$\varrho = e^t, \quad \tilde{w}(t, p) = w(e^t, p), \quad \forall (t, p) \in (-\infty, 0) \times \Sigma$$

and we adopt the following conventions: for any function $h(\varrho, \cdot)$ (or for every tensor) in which the variable ϱ occurs, we denote by $\tilde{h}(t, \cdot)$ the function (or tensor) $h(e^t, \cdot)$. We will continue to denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|_\Sigma$, respectively, the inner product and the norm associated with it in the Riemannian structure induced by the metric tensor

$$\left\{ \tilde{\beta}_{\alpha, hk} \right\}_{h,k=2}^n.$$

We will still denote by ∇_Σ , $\operatorname{div}_\Sigma$ and $d\Omega_t$, respectively, the gradient, the divergence operators and the element of volume on Σ in the above mentioned structure. In particular, the local expression of $d\Omega_t$ is equal to $\sqrt{\tilde{\beta}_\alpha} d\theta$. Moreover we set

$$\mathcal{M}(\cdot) = \operatorname{div}_\Sigma \left(\tilde{\mu}^{1-\frac{n}{2}} \nabla_\Sigma \cdot \right). \quad (15.7.6)$$

We have

$$e^{2t} \tilde{\mu}^{-1} (\mathcal{L}_1 w)(e^t, p) = \mathcal{P} \tilde{w}(t, p), \quad \forall (t, p) \in (-\infty, 0) \times \Sigma, \quad (15.7.7)$$

where

$$\mathcal{P} \tilde{w} = \tilde{w}_{tt} + (n-2) \tilde{w}_t + \frac{1}{\tilde{\mu}^{1-n/2}} \mathcal{M} \tilde{w}. \quad (15.7.8)$$

For the reader's convenience, we reformulate Propositions 15.6.6 and 15.6.8 in the Riemannian structure induced by the metric tensor $\left\{ \tilde{\beta}_{\alpha, hk} \right\}_{h,k=2}^n$.

Recall that $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathcal{J}}$ is a (finite) family of coordinate neighborhoods defining on Σ a structure of C^∞ oriented differentiable manifold, where, for any $\alpha \in \mathcal{J}$, we set $V_\alpha = \varphi_\alpha(U_\alpha)$.

Proposition 15.7.3. *For any $\alpha \in \mathcal{J}$ and for any $(t, \theta) \in (-\infty, 0) \times V_\alpha$, we have*

$$\left| \partial_t \tilde{\beta}_{\alpha, hk} \eta^h \eta^k \right| \leq C e^t \tilde{\beta}_{\alpha, hk}(\varrho, \theta) \eta^h \eta^k, \quad \forall \eta \in \mathbb{R}^{n-1}, \quad (15.7.9)$$

$$\left| \partial_t \log \sqrt{\tilde{\beta}_\alpha} \right| \leq C e^t, \quad (15.7.10)$$

$$|\partial_t \tilde{\mu}_\alpha(t, \theta)| \leq C e^t, \quad (15.7.11)$$

where C depends on λ and Λ only.

By the convention introduced above, for any $t \in (-\infty, 0)$, let us denote by $\tilde{\Xi}^{(t)}$ the covariant tensor satisfying

$$\tilde{\Xi}^{(t)} = \Xi^{(e^t)}$$

and let us denote by

$$\tilde{\ell}(t, \cdot) = \ell(e^t, \cdot).$$

In the next Proposition let us denote by $\int(\cdot)$ the integral $\int_{-\infty}^{t_0} dt \int_\Sigma(\cdot) d\Omega_t$.

Proposition 15.7.4. *Let $v_1, v_2 \in C^\infty((-\infty, t_0) \times \Sigma)$. Let us suppose either v_1 or v_2 of compact support. Let $h \in C^\infty((-\infty, t_0))$. Then we have*

$$\int v_1 \partial_t v_2 = - \int \left(e^t v_1 \tilde{\ell} + \partial_t v_1 \right) v_2. \quad (15.7.12)$$

If v_1 has compact support then

$$\begin{aligned} & \int h(t) \partial_t v_1 \operatorname{div}_\Sigma \left(\tilde{\mu}^{1-\frac{n}{2}} \nabla_\Sigma v_1 \right) = \\ &= \frac{1}{2} \int e^t h(t) \tilde{\mu}^{1-\frac{n}{2}} \tilde{\Xi}^{(t)} \left(\nabla_\Sigma v_1, \nabla_\Sigma v_1 \right) + \\ &+ \frac{1}{2} \int \left[\partial_t \left(h(t) \tilde{\mu}^{1-\frac{n}{2}} \right) + e^t h(t) \tilde{\mu}^{1-\frac{n}{2}} \tilde{\ell} \right] |\nabla_\Sigma v_1|_\Sigma^2. \end{aligned} \quad (15.7.13)$$

Proposition 15.7.5. *We have*

$$\left| \tilde{\ell}(t, p) \right| \leq C, \quad \forall (t, p) \in (-\infty, 0) \times \Sigma, \quad (15.7.14)$$

$$\left| \tilde{\Xi}^{(t)}(\nabla_\Sigma v, \nabla_\Sigma v) \right| \leq C |\nabla_\Sigma v|_\Sigma^2, \quad \forall v \in C^\infty((-\infty, 0) \times \Sigma), \quad (15.7.15)$$

where C depends on λ and Λ only.

Proof. Inequalities (15.7.14) and (15.7.15) are an immediate consequences of Proposition 15.6.6. ■

Proof of Theorem 15.7.1.

For any smooth function v we write v' , v'' , ... instead of $\partial_t v$, $\partial_{tt} v$,

By (15.4.11) we have, similarly to the proof of Theorem 15.7.1 (here and in the sequel we omit the subscript ϵ)

$$\varphi(t) := \log(\phi(e^t)) = t - \epsilon^{-1} \log(1 + e^{\epsilon t}), \quad \forall t \in (-\infty, 0) \quad (15.7.16)$$

and

$$\varphi'(t) = \frac{1}{1 + e^{\epsilon t}}, \quad \varphi''(t) = -\frac{\epsilon e^{\epsilon t}}{(1 + e^{\epsilon t})^2}, \quad \forall t \in (-\infty, 0). \quad (15.7.17)$$

Let

$$f(t, p) = e^{-\tau\varphi} w(t, p), \quad \forall (t, p) \in (-\infty, 0) \times \Sigma, \quad (15.7.18)$$

where w is defined in (15.7.5).

We have

$$\mathcal{P}_\tau f := e^{-\tau\varphi} \mathcal{P}(e^{\tau\varphi} f) = \underbrace{b_0 f + b_1 f'}_{\mathcal{A}_\tau f} + \underbrace{a_0 f + f'' + \frac{1}{m} \mathcal{M} f}_{\mathcal{S}_\tau f}, \quad (15.7.19)$$

where

$$m = \tilde{\mu}^{1 - \frac{n}{2}}$$

and

$$a_0 = \tau^2 \varphi'^2 + \tau(n - 2), \quad b_0 = \tau \varphi'', \quad b_1 = 2\tau \varphi' + (n - 2). \quad (15.7.20)$$

Let us note that (15.6.53) gives

$$\lambda^{\frac{n}{2} - 1} \leq m \leq \lambda^{1 - \frac{n}{2}}. \quad (15.7.21)$$

Set

$$\gamma := \frac{1}{\varphi'} = 1 + e^{\epsilon t}. \quad (15.7.22)$$

We have

$$\int m \gamma |\mathcal{P}_\tau f|^2 \geq 2 \int m \gamma \mathcal{A}_\tau f \mathcal{S}_\tau f + \int m \gamma |\mathcal{A}_\tau f|^2, \quad (15.7.23)$$

$$\begin{aligned}
 2 \int m\gamma \mathcal{A}_\tau f \mathcal{S}_\tau f &= 2 \underbrace{\int \gamma (b_0 f + b_1 f') \mathcal{M}f}_{I_1} + \\
 &+ 2 \underbrace{\int m\gamma (b_0 f + b_1 f') (a_0 f + f'')}_{I_2}.
 \end{aligned}
 \tag{15.7.24}$$

We examine I_1 .

We have

$$I_1 = 2 \int (\gamma b_0 f \mathcal{M}f + \gamma b_1 f' \mathcal{M}f) = 2 \underbrace{\int \gamma b_0 f \mathcal{M}f}_{I_{11}} + 2 \underbrace{\int \gamma b_1 f' \mathcal{M}f}_{I_{12}}.
 \tag{15.7.25}$$

By the divergence Theorem we obtain

$$I_{11} = 2 \int \gamma b_0 f \operatorname{div}_\Sigma (m \nabla_\Sigma f) = -2 \int m \gamma b_0 |\nabla_\Sigma f|_\Sigma^2.
 \tag{15.7.26}$$

By (15.7.13) we get

$$\begin{aligned}
 I_{12} &= 2 \int \gamma b_1 f' \mathcal{M}f = \int \left[(m\gamma b_1)' + e^t m\gamma b_1 \tilde{\ell} \right] |\nabla_\Sigma f|_\Sigma^2 + \\
 &+ \int e^t m\gamma b_1 \tilde{\Xi}^{(t)} (\nabla_\Sigma f, \nabla_\Sigma f).
 \end{aligned}
 \tag{15.7.27}$$

By (15.7.25), (15.7.26) and (15.7.27) we have

$$\begin{aligned}
 I_1 = I_{11} + I_{12} &= \int \left\{ -2m\gamma b_0 + \left[(m\gamma b_1)' + e^t m\gamma b_1 \tilde{\ell} \right] \right\} |\nabla_\Sigma f|_\Sigma^2 + \\
 &+ \int e^t m\gamma b_1 \tilde{\Xi}^{(t)} (\nabla_\Sigma f, \nabla_\Sigma f).
 \end{aligned}
 \tag{15.7.28}$$

We get (compare with (15.4.18))

$$\begin{aligned}
 &- 2m\gamma b_0 + \left[(m\gamma b_1)' + e^t m\gamma b_1 \tilde{\ell} \right] = \\
 &= 2m \left(-\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) + m' \gamma b_1 + e^t m\gamma b_1 \tilde{\ell} \geq \\
 &\geq \tau \epsilon e^{t\tau} + m' \gamma b_1 + e^t m\gamma b_1 \tilde{\ell}.
 \end{aligned}
 \tag{15.7.29}$$

Now, by (15.7.11) we have

$$|m'| \leq C e^t,
 \tag{15.7.30}$$

where C depends on λ and Λ only. By this inequality, by (15.7.15) and by (15.7.28), taking into account (15.7.14), we have

$$I_1 \geq \tau \int e^{\epsilon t} (\epsilon - C_* e^{(1-\epsilon)t}) |\nabla_{\Sigma} f|_{\Sigma}^2, \quad \forall \tau \geq 1, \quad (15.7.31)$$

where C_* depends on λ and Λ only.

Now, for every $t \geq T_1 := \frac{1}{1-\epsilon} \log \frac{\epsilon}{2C_*}$ we have

$$\epsilon - C_* e^{(1-\epsilon)t} \geq \frac{\epsilon}{2}$$

By this inequality and by (15.7.31) we have, for every $f \in C_0^{\infty}((-\infty, T_1) \times \Sigma)$,

$$I_1 \geq \frac{\epsilon \tau}{2} \int e^{\epsilon t} |\nabla_{\Sigma} f|_{\Sigma}^2, \quad \forall \tau \geq 1. \quad (15.7.32)$$

We examine I_2 .

By (15.7.12) we have

$$\begin{aligned} I_2 &= 2 \int m\gamma (b_0 f + b_1 f') (a_0 f + f'') = \\ &= 2 \int m\gamma (a_0 b_0 f^2 + b_0 f f'' + b_1 a_0 f' f + b_1 f' f'') = \\ &= 2 \int m\gamma a_0 b_0 f^2 - (m\gamma b_0 f)' f' + \frac{1}{2} m\gamma b_1 a_0 (f^2)' + \frac{1}{2} m\gamma b_1 (f'^2)' - \\ &\quad - 2 \int e^t m\gamma b_0 \tilde{\ell} f f' = \\ &= 2 \int m\gamma a_0 b_0 f^2 - (m\gamma b_0 f)' f' - \frac{1}{2} (m\gamma b_1 a_0)' f^2 - \frac{1}{2} (m\gamma b_1)' f'^2 - \\ &\quad - 2 \int e^t m\gamma \tilde{\ell} \left(b_0 f f' + \frac{1}{2} b_1 a_0 f^2 + \frac{1}{2} b_1 f'^2 \right) = 2 \int m \left[\left(\gamma a_0 b_0 + \frac{1}{2} (\gamma b_1 a_0)' \right) f^2 + \right. \\ &\quad \left. + \left(\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) f'^2 - (\gamma b_0)' f f' \right] - \\ &\quad - 2 \int e^t m\gamma \tilde{\ell} \left(b_0 f f' - \frac{1}{2} b_1 a_0 f^2 - \frac{1}{2} b_1 f'^2 \right) - \\ &\quad - 2 \int m' \gamma \left(b_0 f f' + \frac{1}{2} b_1 a_0 f^2 + \frac{1}{2} b_1 f'^2 \right) = \end{aligned}$$

$$\begin{aligned}
&= 2 \int m \left[\left(\gamma a_0 b_0 + \frac{1}{2} (\gamma b_1 a_0)' \right) f^2 + \right. \\
&\quad \left. + \left(\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) f'^2 - (\gamma b_0)' f f' \right] - \\
&\quad - 2 \int \gamma \left(m e^t \tilde{\ell} + m' \right) \left(b_0 f f' + \frac{1}{2} b_1 a_0 f^2 + \frac{1}{2} b_1 f'^2 \right) = I_{21} + I_{22}.
\end{aligned}$$

Where we set

$$\begin{aligned}
I_{21} &= 2 \int m \left[\left(\gamma a_0 b_0 + \frac{1}{2} (\gamma b_1 a_0)' \right) f^2 + \right. \\
&\quad \left. + \left(\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) f'^2 - (\gamma b_0)' f f' \right]
\end{aligned}$$

and

$$I_{22} = -2 \int \gamma \left(m e^t \tilde{\ell} + m' \right) \left(b_0 f f' + \frac{1}{2} b_1 a_0 f^2 + \frac{1}{2} b_1 f'^2 \right)$$

Now, let us estimate I_{21} e I_{22} .

We have (compare with (15.4.21) and (15.4.22)), for every $\tau > 0$

$$\gamma a_0 b_0 - \frac{1}{2} (\gamma b_1 a_0)' = \left(\tau^3 \frac{1}{1 + e^t} + \tau^2 \frac{n-2}{2} \right) \frac{\epsilon e^t}{(1 + e^t)^2} \geq \frac{\tau^3}{8} \epsilon e^t, \quad (15.7.33)$$

and (compare with (15.4.25) and (15.4.26)) for every $\tau > 2(n-2)$

$$-\left(\gamma b_0 + \frac{1}{2} (\gamma b_1)' \right) = \tau \frac{\epsilon e^t}{1 + e^t} - \frac{(n-2)\epsilon e^t}{2} \geq \frac{\epsilon \tau}{4} e^t. \quad (15.7.34)$$

Moreover

$$|(\gamma b_0)' f f'| = \tau \frac{\epsilon e^t}{(1 + e^t)^2} |f f'| \leq \epsilon e^t \left(\tau^2 f^2 + f'^2 \right).$$

By this inequality and by (15.7.33), (15.7.34) we have

$$I_{21} \geq \epsilon \int m e^t \left(\frac{\tau^3}{8} f^2 + \frac{\tau}{4} f'^2 \right), \quad \forall \tau \geq \bar{\tau}_1, \quad (15.7.35)$$

where $\bar{\tau}_1 = \max\{16, 2(n-2)\}$.

Now we estimate I_{22} e I_{23} .

By (15.7.14), (15.7.20), (15.7.21), and (15.7.30) we have

$$|I_{22}| \leq C_* \int e^t \left(\tau^3 f^2 + \tau f'^2 \right), \quad \forall \tau \geq \bar{\tau}_1, \quad (15.7.36)$$

where C_* depends on λ and Λ only.

By (15.7.35) and (15.7.36) we have, for every $\tau \geq \bar{\tau}_1$

$$I_2 \geq I_{21} - |I_{22}| \geq \int e^{\epsilon t} \left(\frac{\epsilon \lambda^{\frac{n}{2}-1}}{8} - C_* e^{(1-\epsilon)t} \right) (\tau^3 f^2 + \tau f'^2). \quad (15.7.37)$$

Notice that for every

$$t \geq T_2 := \frac{1}{1-\epsilon} \log \left(\frac{\epsilon \lambda^{\frac{n}{2}-1}}{16C_*} \right)$$

we have

$$\frac{\epsilon \lambda^{\frac{n}{2}-1}}{8} - C_* e^{(1-\epsilon)t} \geq \frac{\epsilon \lambda^{\frac{n}{2}-1}}{16},$$

this inequality and (15.7.37) imply that

$$I_2 \geq \frac{\epsilon \lambda^{\frac{n}{2}-1}}{16} \int e^{\epsilon t} (\tau^3 f^2 + \tau f'^2), \quad (15.7.38)$$

for every $f \in C_0^\infty((-\infty, T_2) \times \Sigma)$ and for every $\tau \geq \bar{\tau}_1$. By (15.7.24), (15.7.32) and (15.7.38) we have,

$$2 \int m\gamma \mathcal{A}_\tau f \mathcal{S}_\tau f \geq \frac{\epsilon\tau}{2} \int e^{\epsilon t} |\nabla_\Sigma f|_\Sigma^2 + \frac{\epsilon \lambda^{\frac{n}{2}-1}}{16} \int e^{\epsilon t} (\tau^3 f^2 + \tau f'^2), \quad (15.7.39)$$

for every $\tau \geq \bar{\tau}_1$ and for every $f \in C_0^\infty((-\infty, T_3) \times \Sigma)$, where $T_3 = \min\{T_1, T_2\}$.

Set

$$\epsilon_0 = \epsilon \min\left\{ \frac{1}{2}, \frac{\lambda^{\frac{n}{2}-1}}{16} \right\},$$

by (15.7.23) and (15.7.39) we get

$$\begin{aligned} \int m\gamma |\mathcal{P}_\tau f|^2 &\geq \\ &\geq \epsilon_0 \int \left(\tau^3 f^2 + \tau (f'^2 + |\nabla_\Sigma f|_\Sigma^2) \right) e^{\epsilon t} + \int m\gamma |\mathcal{A}_\tau f|^2, \end{aligned} \quad (15.7.40)$$

for every $\tau \geq \tau_1$ and for every $f \in C_0^\infty((-\infty, T_3) \times \Sigma)$.

In order to obtain the third term on the left-hand side of (15.7.3) we argue as in the proof of Theorem 15.4.2. For the sake of clarity, we repeat the most important steps.

By the trivial inequality $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$ we have

$$\begin{aligned} \int m\gamma |\mathcal{A}_\tau f|^2 &\geq \frac{\lambda^{\frac{n}{2}-1}}{2} \int \gamma (2\tau\varphi' + n - 2)^2 f'^2 - \lambda^{\frac{n}{2}-1} \int \gamma \tau^2 \varphi'^2 f^2 \geq \\ &\geq \lambda^{\frac{n}{2}-1} \tau^2 \int f'^2 - \epsilon^2 \lambda^{\frac{n}{2}-1} \tau^2 \int e^{2\epsilon t} f^2, \quad \forall \tau > 0. \end{aligned}$$

Plugging this inequality into (15.7.40) we have

$$\begin{aligned} \int m\gamma |\mathcal{P}_\tau f|^2 &\geq \lambda^{\frac{n}{2}-1} \tau^2 \int f'^2 + \tau^3 \int (\epsilon_0 - \epsilon \lambda^{\frac{n}{2}-1} \tau^{-1} e^{\epsilon t}) e^{\epsilon t} f^2 + \\ &+ \epsilon_0 \tau \int (f'^2 + |\nabla_\Sigma f|^2) e^{\epsilon t}, \end{aligned} \tag{15.7.41}$$

for every $\tau \geq \tau_1$ and for every $f \in C_0^\infty((-\infty, T_3) \times \Sigma)$.

By (15.7.41) we have

$$\begin{aligned} \int m\gamma |\mathcal{P}_\tau f|^2 &\geq \\ &\geq \lambda^{\frac{n}{2}-1} \tau^2 \int f'^2 + \frac{\epsilon_0 \tau^3}{2} \int e^{\epsilon t} f^2 + \epsilon_0 \tau \int (f'^2 + |\nabla_\Sigma f|_\Sigma^2) e^{\epsilon t}, \end{aligned} \tag{15.7.42}$$

for every $\tau \geq \bar{\tau}_2$, where $\bar{\tau}_2 = \max\{2\epsilon\epsilon_0^{-1}\lambda^{\frac{n}{2}-1}, \bar{\tau}_1\}$ and for every $f \in C_0^\infty((-\infty, T_3) \times \Sigma)$.

Let $r \leq 4e^{T_3}$. Since $u = 0$ in $B_{r/4}$ we have $f(t, p) = 0$ for every $t \leq \log(r/4)$ and for every $p \in \Sigma$. Proceeding as in the proof of (15.4.40) we obtain

$$\begin{aligned} C \int |\mathcal{P}_\tau f|^2 &\geq \tau^3 \int e^{\epsilon t} f^2 + \tau \int (f'^2 + |\nabla_\Sigma f|_\Sigma^2) e^{\epsilon t} + \\ &+ \tau^2 r \int f^2 e^{-t}, \quad \forall \tau \geq \bar{\tau}_2 \end{aligned} \tag{15.7.43}$$

where C depends on ϵ , λ and Λ only.

Now we come back to the original coordinates. By (15.7.5), (15.7.7), (15.7.18) and (15.7.19) we have

$$\begin{aligned}
& \int_{-\infty}^0 dt \int_{\Sigma} |\mathcal{P}_{\tau} f|^2 d\Omega_t = \\
& = \int_{-\infty}^0 dt \int_{\Sigma} e^{-2\tau\varphi(t)} e^{4t} |\tilde{\mu}^{-1}(\mathcal{L}_1 w)(e^t, p)|^2 d\Omega_t = \\
& = \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^3 |\mu^{-1} \mathcal{L}_1 w(\varrho, p)|^2 d\Omega_{\varrho}.
\end{aligned} \tag{15.7.44}$$

By propositions 15.6.5 and 15.7.2 and by (15.7.44) we have

$$\begin{aligned}
\int_{-\infty}^0 dt \int_{\Sigma} |\mathcal{P}_{\tau} f|^2 d\Omega_t & \leq C \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^3 |\mathcal{L}w|^2 d\Omega_{\varrho} + \\
& + C \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^3 |\partial_{\varrho} w|^2 d\Omega_{\varrho},
\end{aligned} \tag{15.7.45}$$

where C depends on λ and Λ only.

Moreover we have

$$\int_{-\infty}^0 dt \int_{\Sigma} f^2 e^{-t} d\Omega_t = \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^{-2} |w|^2 d\Omega_{\varrho} \tag{15.7.46}$$

and

$$\int_{-\infty}^0 dt \int_{\Sigma} f^2 e^{\epsilon t} d\Omega_t = \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^{\epsilon-1} |w|^2 d\Omega_{\varrho}. \tag{15.7.47}$$

Concerning the second integral on the right-hand side of (15.7.44), let $\delta \in (0, 1)$ be to choose later, we get

$$\begin{aligned}
& \int_{-\infty}^0 dt \int_{\Sigma} e^{\epsilon t} \left(f'^2 + |\nabla_{\Sigma} f|_{\Sigma}^2 \right) d\Omega_t \geq \\
& \geq \delta \int_{-\infty}^0 dt \int_{\Sigma} e^{\epsilon t} \left(f'^2 + |\nabla_{\Sigma} f|_{\Sigma}^2 \right) d\Omega_t \geq \\
& \geq \frac{\delta}{2} \int_{-\infty}^0 dt \int_{\Sigma} e^{\epsilon t} e^{-2\tau\varphi(t)} \left(|w_{\varrho}(e^t, p)|^2 e^{2t} + \right. \\
& \left. + |\nabla_{\Sigma} w(e^t, p)|_{\Sigma}^2 - 2\tau^2 |w(e^t, p)|^2 \right) d\Omega_t = \\
& = \frac{\delta}{2} \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \left(|w_{\varrho}(\varrho, p)|^2 + \right. \\
& \left. + \varrho^{-2} |\nabla_{\Sigma} w(\varrho, p)|_{\Sigma}^2 \right) \varrho^{\epsilon+1} d\Omega_{\varrho} - \\
& - \frac{\delta\tau^2}{2} \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} |w(\varrho, p)|^2 \varrho^{\epsilon-1} d\Omega_{\varrho}.
\end{aligned} \tag{15.7.48}$$

Now, plugging (15.7.45), (15.7.46), (15.7.47) and (15.7.48) into (15.7.43), we have

$$\begin{aligned}
 & \tau^3 \left(1 - \frac{\delta}{2}\right) \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} |w|^2 \varrho^{\epsilon-1} d\Omega_{\varrho} + \\
 & + \tau^2 r \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} |w|^2 \varrho^{-2} d\Omega_{\varrho} + \\
 & + \frac{\delta}{2} \tau \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} (|w_{\varrho}|^2 + \varrho^{-2} |\nabla_{\Sigma} w|_{\Sigma}^2) \varrho^{\epsilon+1} d\Omega_{\varrho} \leq \quad (15.7.49) \\
 & \leq C \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^3 |\mathcal{L}w|^2 d\Omega_{\varrho} + \\
 & + C \int_0^1 d\varrho \int_{\Sigma} e^{-2\tau\varphi(\log \varrho)} \varrho^3 |\partial_{\varrho} w|^2 d\Omega_{\varrho},
 \end{aligned}$$

for every $\tau \geq \tau_2$ and for every $f \in C_0^{\infty}((0, r_0) \times \Sigma)$, where $r_0 = e^{T_3}$. Now, let us choose $\delta = \frac{1}{2}$. It turns out that the last integral on the right-hand side of (15.7.49) can be absorbed by the third integral on the left-hand side. Finally, applying Proposition 15.6.7 and, taking into account that $2^{-1/\epsilon}|x| \leq \rho(x) \leq |x|$ and replacing τ by $(\tau - \frac{n}{2})$, we obtain inequality (15.7.3). ■

Corollary 15.7.6. *Let us assume that the entries of the matrix $G = \{g_{ij}(x)\}_{i,j=1}^n$ are of class $C^{0,1}(B_2)$ and that G satisfies (15.6.1), (15.6.6) and (instead of (15.6.2)) satisfies*

$$|G(x) - G(y)| \leq \Lambda |x - y|, \quad \forall x, y \in B_2, \quad (15.7.50)$$

then Carleman estimate (15.7.3) continue to hold.

More precisely, there exist C and $\widehat{\tau}_*$, depending on λ and Λ only, such that

$$\begin{aligned}
 & \tau^3 \int \rho^{\epsilon-2\tau} |u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau} |\nabla u|^2 dx + \\
 & + \tau^2 r \int \rho^{-1-2\tau} u^2 dx \leq C \int \rho^{4-2\tau} |\Delta_g u|^2 dx, \quad (15.7.51)
 \end{aligned}$$

for every $r \in (0, r_0)$, for every $\tau \geq \widehat{\tau}_*$ and for every $u \in C_0^{\infty}(B_{r_0} \setminus \overline{B}_{r/4})$.

Proof. Let $\psi \in C_0^{\infty}(\mathbb{R}^n)$ satisfy $\text{supp } \psi \subset B_1$, $\psi \geq 0$ and $\int_{\mathbb{R}^n} \psi dx = 1$. Set

$$\psi_{\nu}(x) = \nu^n \psi(\nu x), \quad \nu \in \mathbb{N}$$

and

$$G_\nu(x) = (G \star \psi_\nu)(x) = \int_{\mathbb{R}^n} G(x-y)\psi_\nu(y)dy, \quad \nu \in \mathbb{N}.$$

We have that G_ν satisfies to (15.6.1), (15.6.2), (15.6.6). Moreover

$$G_\nu \in C^\infty(B_2)$$

and

$$\|G_\nu - G\|_{L^\infty(B_1)} \rightarrow 0, \quad \text{as } \nu \rightarrow \infty. \quad (15.7.52)$$

Let $r \in (0, r_0)$ and let u be an arbitrary function belonging to $C_0^\infty(B_{r_0} \setminus \bar{B}_{r/4})$. By (15.7.3) we have, for every $\nu \in \mathbb{N}$ and for every $\tau \geq \bar{\tau}$,

$$\begin{aligned} & \tau^3 \int \rho^{\epsilon-2\tau}|u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau}|\nabla u|^2 dx + \\ & + \tau^2 r \int \rho^{-1-2\tau}u^2 dx \leq C \int \rho^{4-2\tau}|\Delta_{g_\nu}u|^2 dx. \end{aligned} \quad (15.7.53)$$

On the other hand, by (15.6.2) we have (by using the convention of repeated index)

$$\int \rho^{4-2\tau}|\Delta_{g_\nu}u|^2 dx \leq 2 \int \rho^{4-2\tau}|g_\nu^{ij}\partial_{x^i x^j}^2 u|^2 dx + C \int \rho^{4-2\tau}|\nabla u|^2 dx,$$

where C **depends only on** λ and Λ . By the just obtained inequality and by (15.7.53) we have, for every $\nu \in \mathbb{N}$,

$$\begin{aligned} & \tau^3 \int \rho^{\epsilon-2\tau}|u|^2 dx + \int \rho^{2+\epsilon-2\tau}(\tau - C\rho^{2-\epsilon})|\nabla u|^2 dx + \\ & + \tau^2 r \int \rho^{-1-2\tau}u^2 dx \leq C \int \rho^{4-2\tau}|g_\nu^{ij}\partial_{x^i x^j}^2 u|^2 dx, \end{aligned} \quad (15.7.54)$$

where C depends on λ and Λ only. Now let $\bar{\tau}_* \geq \bar{\tau}$ satisfy (recall that $\rho \leq 1$ in B_1) for every $\tau \geq \bar{\tau}_*$,

$$\tau - C\rho^{2-\epsilon} \geq \frac{\tau}{2}.$$

By (15.7.54) we have, for every $\nu \in \mathbb{N}$,

$$\begin{aligned} & \tau^3 \int \rho^{\epsilon-2\tau}|u|^2 dx + \frac{\tau}{2} \int \rho^{2+\epsilon-2\tau}|\nabla u|^2 dx + \\ & + \tau^2 r \int \rho^{-1-2\tau}u^2 dx \leq C \int \rho^{4-2\tau}|g_\nu^{ij}\partial_{x^i x^j}^2 u|^2 dx. \end{aligned} \quad (15.7.55)$$

Passing to the limit as $\nu \rightarrow \infty$ in (15.7.55) we obtain

$$\begin{aligned} & \tau^3 \int \rho^{\epsilon-2\tau} |u|^2 dx + \frac{\tau}{2} \int \rho^{2+\epsilon-2\tau} |\nabla u|^2 dx + \\ & + \tau^2 r \int \rho^{-1-2\tau} u^2 dx \leq C \int \rho^{4-2\tau} |g^{ij} \partial_{x^i x^j}^2 u|^2 dx. \end{aligned}$$

By the just obtained inequality, employing

$$\int \rho^{4-2\tau} |g^{ij} \partial_{x^i x^j}^2 u|^2 dx \leq 2 \int \rho^{4-2\tau} |\Delta_g u|^2 dx + C \int \rho^{4-2\tau} |\nabla u|^2 dx$$

and repeating the arguments already used above, we have that there exists $\widehat{\tau}_* \geq \bar{\tau}_*$, where $\widehat{\tau}_*$ depends on λ and Λ only, such that

$$\begin{aligned} & \tau^3 \int \rho^{\epsilon-2\tau} |u|^2 dx + \tau \int \rho^{2+\epsilon-2\tau} |\nabla u|^2 dx + \\ & + \tau^2 r \int \rho^{-1-2\tau} u^2 dx \leq C \int \rho^{4-2\tau} |\Delta_g u|^2 dx, \end{aligned}$$

for every $r \in (0, r_0)$, for every $\tau \geq \widehat{\tau}_*$ and for every $u \in C_0^\infty(B_{r_0} \setminus \bar{B}_{r/4})$. ■

Now we can state the analog of Theorem (15.5.1) and of Corollary 15.5.3 for the solutions $U \in H^2(B_1)$ to the equation

$$LU = \sum_{i,j=1}^n a^{ij}(x) \partial_{x^i x^j}^2 U + \sum_{i=1}^n b^i(x) \partial_{x^i} U + c(x)U = 0, \quad \text{in } B_1, \quad (15.7.56)$$

where $A(x) = \{a^{ij}(x)\}_{i,j=1}^n$ is a symmetric matrix whose entries are real-valued functions, such that

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in B_1, \quad (15.7.57)$$

where $\lambda \geq 1$. Also we assume

$$|a^{ij}(x) - a^{ij}(y)| \leq \Lambda |x - y|, \quad \text{for } i, j \in \{1, \dots, n\}, \quad \forall x, y \in B_1. \quad (15.7.58)$$

Moreover, $b^i \in L^\infty(B_1)$, $i = 1, \dots, n$ and $c \in L^\infty(B_1)$ (these may be complex-valued coefficients) and set

$$M = \max \left\{ \|b\|_{L^\infty(B_1, \mathbb{R}^n)}, \|c\|_{L^\infty(B_1)} \right\}, \quad (15.7.59)$$

where $b = (b^1, \dots, b^n)$.

Theorem 15.7.7. *Let us assume that $U \in H^2(B_1)$ is a solution to the equation (15.7.57). Let $x_0 \in B_1$ and $0 < R_0 \leq 1 - |x_0|$. Then there exist $C_1 \geq C \geq 1$ depending on λ, Λ and M only, such that, if $0 < r < \frac{R}{C} < \frac{R_0}{C_1}$ then*

$$\int_{B_R(x_0)} U^2 \leq C \left(\frac{R_0}{R}\right)^C \left(\int_{B_r(x_0)} U^2\right)^\theta \left(\int_{B_{R_0}(x_0)} U^2\right)^{1-\theta}, \quad (15.7.60)$$

where

$$\theta = \frac{\log \frac{R_0}{C_1 R}}{\log \frac{C_1 R_0}{r}}. \quad (15.7.61)$$

Moreover, if U does not vanish identically in $B_{R_0/C}(x_0)$ the following doubling inequality holds true

$$\int_{B_{2r}(x_0)} U^2 \leq CN_{x_0, R_0}^k \int_{B_r(x_0)} U^2, \quad (15.7.62)$$

where

$$N_{x_0, R_0} = \frac{\int_{B_{R_0}(x_0)} U^2}{\int_{B_{R_0/C}(x_0)} U^2} \quad (15.7.63)$$

and k is a positive number ($k \geq 3$).

Proof. For fixed $x_0 \in B_1$, since $A(x_0)$ is a symmetric matrix, there exists a linear map S such that

$$SA(x_0)S^T = I_n.$$

Hence, we may perform the change of variables $y = S^{-1}(x - x_0)$ in the equation (15.7.56) that allows to apply, after some simple modifications, the Carleman estimate (15.7.51) in a manner quite similar to what was done in the proof of Theorem (15.5.1). To obtain the analogon of Lemma 15.5.2, we may fix, for instance $\epsilon = \frac{1}{2}$. We leave the details to the reader. ■

Also, the following Corollary can be proved similarly to Corollary 15.5.3

Corollary 15.7.8 (strong unique continuation for elliptic equations). *Let $U \in H^2(B_1)$ be a solution to equation (15.7.57). Let $x_0 \in B_1$ and $0 < R_0 \leq 1 - |x_0|$. There exists C depending on λ, Λ and M only such that we have what follows. If U does not vanish identically in $B_{R_0/C}(x_0)$ then we have, for every $r < s \leq \frac{R_0}{C}$,*

$$\int_{B_s(x_0)} U^2 \leq CN_{x_0, R_0}^k \left(\frac{s}{r}\right)^{\log_2(CN_{x_0, R_0}^k)} \int_{B_r(x_0)} U^2, \quad (15.7.64)$$

where N_{x_0, R_0} is defined by (15.7.63) and k is the same number that occurs in (15.7.62).

Moreover, if

$$\int_{B_r(x_0)} U^2 = \mathcal{O}(r^m), \quad \text{as } r \rightarrow 0, \quad \forall m \in \mathbb{N}, \quad (15.7.65)$$

then

$$U \equiv 0, \quad \text{in } B_1. \quad (15.7.66)$$

Chapter 16

Miscellanea

16.1 Introduction

In this final Chapter we will first give (Section 16.3) a brief outline of two methods, alternative to the Carleman estimates, for dealing within the unique continuation issue. These methods are generally called the log – convexity and the frequency function method. We will see that they are intimately related. Next, in Section 16.4 , we will give a little mention of A_p weights, pointing out some applications of them to inverse problems. In Section 16.5 we will consider the Runge property for the Laplace operator.

16.2 The backward problem for the heat equation

Let us consider a rod of heat conducting material. Let π be the length of the rod, let us assume that its temperature is zero at its extremes and that the heat flows only in the direction of the axis of the rod. Let $u(x, t)$ be the temperature of the rod at the point x and the time t . If the initial temperature is $f(x)$, then u is a solution of the following + initial–boundary value problem for the heat equation

$$\begin{cases} u_t - u_{xx} = 0, & \text{for } (x, t) \in (0, \pi) \times (0, +\infty), \\ u(0, t) = u(\pi, t) = 0, & \text{for } t \in [0, +\infty), \\ u(x, 0) = f(x), & \text{for } x \in [0, \pi]. \end{cases} \quad (16.2.1)$$

Also, we assume

$$f \in C^1([0, \pi]), \quad \text{and} \quad f(0) = f(\pi) = 0. \quad (16.2.2)$$

Let us recall that, [77], there exists a unique solution to problem (16.2.1) in the class $C^0([0, \pi] \times [0, +\infty)) \cap C^2((0, \pi) \times (0, +\infty))$ and it is given by

$$u(x, t) = \sum_{k=1}^{\infty} f_k \sin kx e^{-k^2 t}, \quad (16.2.3)$$

where

$$f_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx dx. \quad (16.2.4)$$

Moreover we have

$$\int_0^{\pi} u^2(x, t) dx \leq \frac{\pi}{2} \int_0^{\pi} f^2(x) dx, \quad \forall t \geq 0$$

and, more generally

$$\int_0^{\pi} |\partial_x^m u(x, t)| dx \leq C_{m,t} \int_0^{\pi} f^2(x) dx, \quad \forall t > 0.$$

These inequalities imply a continuous dependence of the solution of problem (16.2.1) by the initial datum f .

In the **backward problem** we are interested in determining the temperature u , if we know, instead of initial temperature, the temperature at an instant $t > 0$, say $t = 1$. Set

$$g(x) = u(x, 1), \quad \text{in } [0, \pi]. \quad (16.2.5)$$

It is evident that, for $t > 1$ by the translation $t' = t - 1$ we reduce to problem (16.2.1). when $t < 1$ we will examine what happens for what concerns the uniqueness and continuous dependence of u by g .

Uniqueness. By the linearity of the problem, it suffices to check that if $g \equiv 0$ then $u \equiv 0$. Now, since u is given by (16.2.3), we may consider the equation (of the unknown f)

$$\sum_{k=1}^{\infty} f_k \sin kx e^{-k^2} = 0, \quad \forall x \in [0, \pi], \quad (16.2.6)$$

from which, multiplying both the sides by $\sin mx$, for $m \in \mathbb{N}$ and integrating over $[0, \pi]$, we obtain

$$\frac{\pi}{2} f_m e^{-m^2} = 0, \quad \forall m \in \mathbb{N}.$$

Therefore $f_m = 0$ for every $m \in \mathbb{N}$. Hence

$$u \equiv 0.$$

Continuous dependence and conditional stability. Let us consider the sequence of functions

$$g_\nu(x) = e^\nu \sin \pi \nu x, \quad \nu \in \mathbb{N}.$$

It is easily checked that

$$u_\nu(x, t) = e^{\nu^2(1-t)} e^\nu \sin \pi \nu x, \quad \nu \in \mathbb{N}, \quad \nu \in \mathbb{N}.$$

Hence

$$\|g_\nu^{(n)}\|_{L^2(0, \pi)} \rightarrow 0, \quad \text{as } \nu \rightarrow \infty, \quad \forall n \in \mathbb{N},$$

($g_\nu^{(n)}$ is the n -th derivative of g), but

$$\|u_\nu(\cdot, t)\|_{L^2(0, \pi)} \rightarrow +\infty, \quad \text{as } \nu \rightarrow \infty, \quad \forall t \in (0, 1).$$

In plain words, even if we had the estimates of the error of all derivatives of the datum g we could not control the error on $u(\cdot, t)$ when $t < 1$.

Let us denote by

$$\varepsilon := \|g\|_{L^2(0, \pi)} \tag{16.2.7}$$

and let us suppose the temperature at the initial time is bounded (in the $L^2(0, \pi)$ norm) by a known constant. More precisely, we suppose that

$$\|f\|_{L^2(0, \pi)} = \|u(\cdot, 0)\|_{L^2(0, \pi)} \leq E, \tag{16.2.8}$$

where $E > 0$ is known. By (16.2.3) and (16.2.5) we have

$$\sum_{k=1}^{\infty} f_k \sin kx e^{-k^2} = g(x), \quad \forall x \in [0, \pi],$$

from which we have

$$f_m = \frac{2}{\pi} g_m e^{-m^2}, \quad \forall m \in \mathbb{N},$$

where

$$g_m = \frac{2}{\pi} \int_0^\pi f(x) \sin mx dx, \quad \forall m \in \mathbb{N}.$$

Therefore (16.2.7) and condition (16.2.8) are expressed, respectively, by

$$\frac{\pi}{2} \sum_{k=1}^{\infty} f_k^2 e^{-2k^2} = \varepsilon^2$$

and

$$\frac{\pi}{2} \sum_{k=1}^{\infty} f_k^2 \leq E^2.$$

On the other hand, we are interested in estimating

$$\|u(\cdot, t)\|_{L^2(0, \pi)}^2 = \frac{\pi}{2} \sum_{k=1}^{\infty} f_k^2 e^{-2k^2 t}$$

for $t \in (0, 1)$.

Applying the Hölder inequality we get

$$\begin{aligned} \frac{\pi}{2} \sum_{k=1}^{\infty} f_k^2 e^{-2k^2 t} &= \frac{\pi}{2} \sum_{k=1}^{\infty} |f_k|^{2(1-t)} \left(|f_k|^2 e^{-2k^2} \right)^t \leq \\ &\leq \frac{\pi}{2} \left(\sum_{k=1}^{\infty} |f_k|^2 \right)^{1-t} \left(\sum_{k=1}^{\infty} |f_k|^2 e^{-2k^2} \right)^t \leq \\ &\leq E^{2(1-t)} \varepsilon^{2t}. \end{aligned}$$

Hence we have proved the following conditional stability estimate

$$\|u(\cdot, t)\|_{L^2(0, \pi)} \leq E^{1-t} \varepsilon^t, \quad \forall t \in (0, 1). \quad (16.2.9)$$

Remark. It is easily checked that estimate (16.2.9) cannot be improved. Furthermore, (16.2.9) implies the log-convexity of the function

$$[0, \pi] \ni t \rightarrow \|u(\cdot, t)\|_{L^2(0, \pi)}$$

By this we mean that the function

$$F(t) = \log \|u(\cdot, t)\|_{L^2(0, \pi)},$$

is convex. \blacklozenge

16.3 The log-convexity method and the frequency function method

16.3.1 The log-convexity method

At the base of the **log-convexity method** there are the following simple considerations.

Let us suppose that $F \in C^2([0, 1])$ is a nonnegative function, $F'' \geq 0$ in $[0, 1]$ and let us suppose

$$F''(t)F(t) - F'^2(t) \geq 0, \quad \forall t \in [0, 1]. \quad (16.3.1)$$

It is immediately checked that this inequality is equivalent to

$$F''(t)(F(t) + \gamma) - F'^2(t) \geq 0, \quad \forall t \in [0, 1], \quad \forall \gamma > 0$$

which, in turn, is equivalent to the log-convexity of $F + \gamma$ in $[0, 1]$. As a matter of fact we have,

$$\frac{d^2}{dt^2} \log(F(t) + \gamma) = \frac{F''(t)(F(t) + \gamma) - F'^2(t)}{F^2(t)} \geq 0.$$

Now, the log-convexity of $F + \gamma$ is equivalent to inequality

$$F(t) + \gamma \leq (F(0) + \gamma)^{1-t}(F(1) + \gamma)^t, \quad \forall t \in [0, 1], \quad \forall \gamma > 0$$

hence

$$F(t) \leq (F(0))^{1-t}(F(1))^t, \quad \forall t \in [0, 1]. \quad (16.3.2)$$

By the latter inequality we derive that if one of values $F(0)$, $F(1)$ is zero then F vanishes identically.

Let us at once see an application of the aforementioned idea for proving the uniqueness and a conditional stability estimate for the following backward problem

$$\begin{cases} u_t - (a(x)u_x)_x = 0, & \text{for } (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = u(1, t) = 0, & \text{for } t \in [0, T], \\ u(x, T) = g(x), & \text{for } x \in [0, 1] \end{cases} \quad (16.3.3)$$

where $T > 0$. Let us suppose that $a \in C^1([0, 1])$ and that there exists $u \in C^2([0, 1] \times [0, T])$ solution to (16.3.3).

We define

$$F(t) = \int_0^1 u^2(x, t) dx, \quad t \in [0, T] \quad (16.3.4)$$

and we have

$$F'(t) = 2 \int_0^1 u(x, t) u_t(x, t) dx. \quad (16.3.5)$$

Now, by the equation $u_t - (a(x)u_x)_x = 0$, taking into account that $u(0, t) = u(1, t) = 0$ and integrating by parts, we get

$$2 \int_0^1 u(x, t) u_t(x, t) dx = 2 \int_0^1 u(x, t) (a(x)u_x)_x dx = -2 \int_0^1 a(x) u_x^2(x, t) dx.$$

Hence

$$F'(t) = -2 \int_0^1 a(x) u_x^2(x, t) dx.$$

By using the just obtained equality, we calculate the second derivative of F

$$F''(t) = -4 \int_0^1 a(x) u_x(x, t) u_{xt}(x, t) dx.$$

Now we integrate by parts, and we recall that $u_t(0, t) = u_t(1, t) = 0$ (which is obtained by differentiating $u(0, t) = u(1, t) = 0$ with respect to t), by using again the equation, we get

$$\begin{aligned} -4 \int_0^1 a(x) u_x(x, t) u_{xt}(x, t) dx &= 4 \int_0^1 (a(x) u_x(x, t))_x u_t(x, t) dx = \\ &= 4 \int_0^1 u_t^2(x, t) dx. \end{aligned}$$

Hence

$$F''(t) = 4 \int_0^1 u_t^2(x, t) dx \geq 0. \quad (16.3.6)$$

Now, by (16.3.4), (16.3.5) and (16.3.6) we have

$$\begin{aligned} F''(t)F(t) - F'^2(t) &= 4 \int_0^1 u_t^2(x, t) dx \int_0^1 u^2(x, t) dx - \\ &\quad - 4 \left(\int_0^1 u(x, t) u_t(x, t) dx \right)^2 \geq 0, \end{aligned}$$

where the last inequality follows by the Cauchy–Schwarz inequality. Hence, by (16.3.2) we have

$$\int_0^1 u^2(x, t) dx \leq \left(\int_0^1 u^2(x, 0) dx \right)^{1-\frac{t}{T}} \left(\int_0^1 u^2(x, T) dx \right)^{\frac{t}{T}}, \quad \forall t \in [0, T]$$

from which, recalling $u(x, T) = g(x)$ in $[0, 1]$,

$$\int_0^1 u^2(x, t) dx \leq \left(\int_0^1 u^2(x, 0) dx \right)^{1-\frac{t}{T}} \left(\int_0^1 g^2(x) dx \right)^{\frac{t}{T}}, \quad (16.3.7)$$

for every $t \in [0, T]$. By the last estimate we obtain the uniqueness for the backward problem (16.3.3). As a matter of fact, if $g \equiv 0$ then (16.3.7) implies $u \equiv 0$. Moreover, if we have the information

$$\int_0^1 g^2(x) dx \leq \varepsilon^2, \quad (\text{error})$$

and

$$\int_0^1 u^2(x, 0) dx \leq E^2, \quad (\text{a priori information})$$

then we get the following conditional stability estimate

$$\left(\int_0^1 u^2(x, t) dx \right)^{1/2} \leq E^{1-\frac{t}{T}} \varepsilon^{\frac{t}{T}}, \quad \forall t \in [0, T]. \quad (16.3.8)$$

Remark. Unlike the method based on the Carleman estimates, in the log-convexity method, the equation and the initial and boundary data are used directly. We refer to [1] and [61] for more details on this topic. The elegance of the method and the simplicity of the proof that we have just given should not lead us to believe that the procedure does not have its asperities. To make a rough comparison with the method based on Carleman estimates, one could say that as, in the latter, the choice of weight is crucial (and non trivial), in the log-convexity method, the choice of the function F is crucial (and non trivial). ♦

We reconsider the backward problem with a depending on x and t . That is, we consider

$$\begin{cases} u_t - (a(x, t)u_x)_x = 0, & \text{for } (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = u(1, t) = 0, & \text{for } t \in [0, T], \\ u(x, T) = g(x), & \text{for } x \in [0, 1] \end{cases} \quad (16.3.9)$$

where $a \in C^1([0, 1] \times [0, +\infty))$. We denote

$$\lambda = \min_{(x,t) \in [0,1] \times [0,T]} a(x,t) > 0 \quad (16.3.10)$$

and

$$M = \max_{(x,t) \in [0,1] \times [0,T]} |a_t(x,t)|. \quad (16.3.11)$$

We are searching for a function μ , such that

$$\mu : [0, 1] \rightarrow [0, T], \quad (16.3.12)$$

bijjective, increasing, which satisfies $\mu \in C^2([0, 1])$ and such that

$$[0, 1] \ni s \rightarrow \Phi(s) := F(\mu(s)),$$

is log-convex, where F is given by

$$F(t) = \int_0^1 u^2(x,t) dx, \quad t \in [0, T].$$

We get

$$\dot{\Phi}(s) = F'(\mu(s))\dot{\mu}(s), \quad \forall s \in [0, 1],$$

where $\dot{\Phi}$ denotes the derivative of Φ w.r.t. s . In addition

$$\ddot{\Phi}(s) = F''(\mu(s))\dot{\mu}^2(s) + F'(\mu(s))\ddot{\mu}(s), \quad \forall s \in [0, 1]. \quad (16.3.13)$$

Hence

$$\begin{aligned} \ddot{\Phi}(s)\Phi(s) - \dot{\Phi}^2(s) &= (F''(\mu(s))F(\mu(s)) - F'^2(\mu(s)))\dot{\mu}^2(s) + \\ &+ F'(\mu(s))F(\mu(s))\ddot{\mu}(s), \quad \forall s \in [0, 1]. \end{aligned} \quad (16.3.14)$$

Now, by the equation $u_t = (a(x,t)u_x)_x$, we get

$$\begin{aligned} F'(t) &= 2 \int_0^1 u(x,t)u_t(x,t) dx = \\ &= 2 \int_0^1 u(x,t) (a(x,t)u_x(x,t))_x dx = \\ &= -2 \int_0^1 a(x,t)u_x^2(x,t) dx \end{aligned}$$

and

$$\begin{aligned}
 F''(t) &= -4 \int_0^1 a(x, t) u_x(x, t) u_{xt}(x, t) dx - 2 \int_0^1 a_t(x, t) u_x^2(x, t) dx = \\
 &= 4 \int_0^1 u_t(x, t) (a(x, t) u_x(x, t))_x dx - 2 \int_0^1 a_t(x, t) u_x^2(x, t) dx = \\
 &= 4 \int_0^1 u_t^2(x, t) dx - 2 \int_0^1 a_t(x, t) u_x^2(x, t) dx.
 \end{aligned}$$

Recalling (16.3.11), we have

$$F''(t) \geq 4 \int_0^1 u_t^2(x, t) dx - 2M \int_0^1 u_x^2(x, t) dx, \quad \forall t \in [0, T]. \quad (16.3.15)$$

By the last equality and by (16.3.14) we get (for the sake of brevity, we omit the variables)

$$\begin{aligned}
 \ddot{\Phi}(s)\Phi(s) - \dot{\Phi}^2(s) &\geq \left[\left(4 \int_0^1 u_t^2 dx - 2M \int_0^1 u_x^2 dx \right) \left(\int_0^1 u^2 dx \right) - \right. \\
 &\quad \left. - 4 \left(\int_0^1 u_t u dx \right)^2 \right] \dot{\mu}^2(s) + \\
 &\quad + 2 \left(\int_0^1 u_t u dx \right) \left(\int_0^1 u^2 dx \right) \ddot{\mu}(s) = \\
 &= 4 \left[\left(\int_0^1 u_t^2 dx \right) \left(\int_0^1 u^2 dx \right) - \left(\int_0^1 u_t u dx \right)^2 \right] \dot{\mu}^2(s) + \\
 &\quad + \left[-4M \dot{\mu}^2(s) \int_0^1 u_x^2 dx + 2 \left(\int_0^1 u_t u dx \right) \ddot{\mu}(s) \right] \left(\int_0^1 u^2 dx \right).
 \end{aligned}$$

By applying the the Cauchy-Schwarz inequality to the expression in the first square bracket, we obtain

$$\begin{aligned}
 \ddot{\Phi}(s)\Phi(s) - \dot{\Phi}^2(s) &\geq \left[-4M \dot{\mu}^2(s) \int_0^1 u_x^2 dx + \right. \\
 &\quad \left. + 2 \left(\int_0^1 u_t u dx \right) \ddot{\mu}(s) \right] \left(\int_0^1 u^2 dx \right).
 \end{aligned} \quad (16.3.16)$$

On the other hand

$$\int_0^1 u_t u dx = \int_0^1 (a(x, t) u_x)_x u dx = - \int_0^1 a(x, t) u_x^2 dx \geq -\lambda \int_0^1 u_x^2 dx.$$

Now, proposing to find μ **concave**, by the last inequality and by (16.3.16) we have

$$\ddot{\Phi}(s)\Phi(s) - \dot{\Phi}^2(s) \geq -2 \left[2M\dot{\mu}^2(s) + \lambda\ddot{\mu}(s) \right] \left(\int_0^1 u^2 dx \right) \left(\int_0^1 u_x^2 dx \right).$$

Hence, in order to

$$\ddot{\Phi}(s)\Phi(s) - \dot{\Phi}^2(s) \geq 0 \quad (16.3.17)$$

it suffices that μ satisfies the following conditions

$$\begin{cases} 2M\dot{\mu}^2(s) + \lambda\ddot{\mu}(s) \leq 0, & \forall s \in [0, T], \\ \dot{\mu}(s) \geq 0, & \forall s \in [0, T], \\ \ddot{\mu}(s) \leq 0, & \forall s \in [0, T], \\ \mu(0) = 0, \quad \mu(1) = T. \end{cases} \quad (16.3.18)$$

Let us notice that the condition $\ddot{\mu} \leq 0$ implies (see (16.3.13)) $\ddot{\Phi} \geq 0$.

Setting

$$\alpha = \frac{2M}{\lambda},$$

it is simple to check that

$$\mu(s) = T + \frac{1}{\alpha} \log [e^{-\alpha T} + (1 - e^{-\alpha T})], \quad s \in [0, 1], \quad (16.3.19)$$

satisfies all conditions (16.3.18). In particular we get

$$2M\dot{\mu}^2(s) + \lambda\ddot{\mu}(s) = 0, \quad \forall s \in [0, T].$$

All in all, if we have

$$\int_0^1 g^2(x) dx \leq \varepsilon^2$$

and

$$\int_0^1 u^2(x, 0) dx \leq E^2,$$

we obtain the conditional stability estimate

$$\left(\int_0^1 u^2(x, t) dx\right)^{1/2} \leq E^{1-\mu^{-1}(t)} e^{\mu^{-1}(t)}, \quad \forall t \in [0, T]. \quad (16.3.20)$$

where

$$\mu^{-1}(t) = \frac{e^{-\alpha(T-t)} - e^{-\alpha t}}{1 - e^{-\alpha T}}.$$

Let us note that as α goes to 0 (corresponding to the case in which a does not depend on t) $\mu^{-1}(t)$ goes to $\frac{t}{T}$, i.e. the exponent of the estimate (16.3.8).

The log-convexity method can also be applied to prove the uniqueness and conditional stability for the Cauchy problem. Perhaps, the first author to use it was M. M. Lavrent'ev in 1956, [47]. He applied the method to the Cauchy problem for the Laplace equation in a convex region. With some minor simplification, the situation considered is as follows (we consider only the uniqueness)

$$\begin{cases} u_{yy}(x, y) + \Delta_x u(x, y) = 0, & \forall (x, y) \in B_1 \times (0, 1), \\ u(x, y) = 0 & \forall (x, y) \in \partial B_1 \times [0, 1], \\ u(x, 0) = u_y(x, 0) = 0, & \forall x \in \overline{B_1}, \end{cases} \quad (16.3.21)$$

where

$$\Delta_x u(x, y) = \sum_{j=1}^n u_{x_j x_j}(x, y)$$

and we suppose that $u \in C^2(\overline{B_1} \times [0, 1])$.

Set

$$F(y) = \int_{B_1} u^2(x, y) dx. \quad (16.3.22)$$

We have

$$F'(y) = 2 \int_{B_1} u(x, y) u_y(x, y) dx \quad (16.3.23)$$

and

$$F''(y) = 2 \int_{B_1} (u_y^2(x, y) + u(x, y) u_{yy}(x, y)) dx. \quad (16.3.24)$$

Now, let us prove

$$\int_{B_1} u(x, y) u_{yy}(x, y) dx = \int_{B_1} u_y^2(x, y) dx. \quad (16.3.25)$$

First, we note that, by the equation $u_{yy} + \Delta_x u = 0$ and by the condition $u(x, y) = 0$ on $\partial B_1 \times [0, 1]$ we have

$$\begin{aligned} \int_{B_1} u(x, y) u_{yy}(x, y) dx &= - \int_{B_1} \Delta_x u(x, y) u(x, y) dx = \\ &= \int_{B_1} |\nabla_x u(x, y)|^2 dx. \end{aligned} \quad (16.3.26)$$

On the other hand

$$\begin{aligned} \frac{d}{dy} \int_{B_1} u_y^2 dx &= 2 \int_{B_1} u_{yy} u_y dx = \\ &= -2 \int_{B_1} (\Delta_x u) u_y dx = \\ &= -2 \int_{B_1} [\operatorname{div}_x (\nabla_x u u_y) - \nabla_x u \cdot \nabla_x u_y] dx = \\ &= 2 \int_{B_1} \nabla_x u \cdot \nabla_x u_y dx = \frac{d}{dy} \int_{B_1} |\nabla_x u(x, y)|^2 dx. \end{aligned}$$

By the just obtained equality and by (16.3.26) we get

$$\frac{d}{dy} \left(\int_{B_1} u_y^2(x, y) dx - \int_{B_1} |\nabla_x u(x, y)|^2 dx \right) = 0. \quad (16.3.27)$$

Now, since we have

$$u(x, 0) = u_y(x, 0) = 0$$

, we get

$$\int_{B_1} u_y^2(x, 0) dx - \int_{B_1} |\nabla_x u(x, 0)|^2 dx = 0$$

and by (16.3.27) we have

$$\int_{B_1} u_y^2(x, y) dx = \int_{B_1} |\nabla_x u(x, y)|^2 dx, \quad \forall y \in (0, 1).$$

By the latter and by (16.3.26) we have (16.3.25) which, in turn (recall (16.3.24)), gives

$$F''(y) = 4 \int_{B_1} u_y^2(x, y) dx.$$

By the just obtained equality, by (16.3.22), (16.3.23) and by the Cauchy-Schwarz inequality we have

$$F''(y)F(y) - F'^2(y) = 4 \left(\int_{B_1} u_y^2 dx \right) \left(\int_{B_1} u^2 dx \right) - 4 \left(\int_{B_1} uu_y dx \right)^2 \geq 0.$$

Hence $F(y)$ is a log-convex function, consequently we have

$$\int_{B_1} u^2(x, y) dx \leq \left(\int_{B_1} u^2(x, 0) dx \right)^y \left(\int_{B_1} u^2(x, 1) dx \right)^{1-y} = 0$$

for every $y \in [0, 1]$.

Therefore

$$u \equiv 0.$$

16.3.2 The frequency function method

In Section 10.4 we saw various versions of the Hadamard three circle inequality for the holomorphic functions and for the harmonic functions in two variables (for the latter, see (10.4.24) and (10.4.25)). In Chapter 15, using the Carleman estimates, we have extended this inequality to the solutions of the second-order elliptic equations and we proved the doubling inequality. The frequency function method was used for the first time in [27] for the second-order elliptic equations with variable coefficients. Here, in order to present the main ideas of this method, we consider the case of the Laplace equation.

Let us propose to prove inequality (10.4.24) for the harmonic functions of n variables. Then, let u be a solution of the Laplace equation.

$$\Delta u = 0, \quad \text{in } B_{R_0} \subset \mathbb{R}^n. \tag{16.3.28}$$

The inequality we are interested in can be written

$$H(r_2) \leq (H(r_1))^{\theta_0} (H(r_3))^{1-\theta_0}, \tag{16.3.29}$$

for $0 < r_1 < r_2 < r_3 < R_0$, where

$$H(r) = \int_{\partial B_r} u^2 dS, \tag{16.3.30}$$

and

$$\vartheta = \frac{\log\left(\frac{r_3}{r_2}\right)}{\log\left(\frac{r_3}{r_1}\right)}. \tag{16.3.31}$$

Now it is important to observe that (16.3.29) is equivalent to the log-convexity of

$$F(t) = H(e^t), \quad t \in (-\infty, \log R_0). \quad (16.3.32)$$

Therefore the derivative of $\log F(t)$ needs to be increasing. Let us calculate such a derivative

$$\frac{d}{dt} \log F(t) = \frac{F'(t)}{F(t)} = \frac{e^t H'(e^t)}{H(e^t)}, \quad t \in (-\infty, \log R_0).$$

This equality implies that (16.3.29) is equivalent to the fact that the function

$$\frac{rH'(r)}{H(r)}, \quad r \in (0, R_0), \quad (16.3.33)$$

is increasing. Now we calculate $H'(r)$. First we notice that

$$H(r) = \frac{1}{r} \int_{B_r} \operatorname{div}(xu^2) dx, \quad (16.3.34)$$

as a matter of fact, by the divergence Theorem we obtain

$$H(r) = \frac{1}{r} \int_{\partial B_r} \left(x \cdot \frac{x}{|x|} \right) u^2 dS = \frac{1}{r} \int_{\partial B_r} (x \cdot \nu) u^2 dS = \frac{1}{r} \int_{B_r} \operatorname{div}(xu^2) dx.$$

Now, set

$$I(r) = \int_{B_r} |\nabla u|^2 dx, \quad r \in (0, R_0) \quad (16.3.35)$$

and let us notice that

$$I(r) = \int_{B_r} \nabla u \cdot \nabla u dx = \int_{B_r} \operatorname{div}(u \nabla u) dx = \int_{\partial B_r} u \frac{\partial u}{\partial \nu} dS. \quad (16.3.36)$$

We have

$$\begin{aligned} H'(r) &= \frac{1}{r} \int_{\partial B_r} \operatorname{div}(xu^2) dS - \frac{1}{r^2} \int_{B_r} \operatorname{div}(xu^2) dx \\ &= \frac{1}{r} \left\{ n \int_{\partial B_r} u^2 dS + 2 \int_{\partial B_r} (x \cdot \nabla u) u dS \right\} - \frac{1}{r} H(r) \\ &= \frac{n-1}{r} \int_{\partial B_r} u^2 dS + 2 \int_{\partial B_r} u \frac{\partial u}{\partial \nu} dS = \frac{n-1}{r} H(r) + 2I(r). \end{aligned}$$

Therefore

$$H'(r) = \frac{n-1}{r}H(r) + 2I(r), \quad r \in (0, R_0). \tag{16.3.37}$$

Hence, function (16.3.33) can be written

$$\frac{rH'(r)}{H(r)} = n - 1 + 2\frac{rI(r)}{H(r)}, \quad r \in (0, R_0)$$

In sum, the log-convexity of F is equivalent to the fact that the function

$$N(r) = \frac{rI(r)}{H(r)}, \quad r \in (0, R_0) \tag{16.3.38}$$

is increasing. $N(\cdot)$ is called the **frequency function** of u . The frequency function was introduced by Almgren (1977), [5] and has taken this name because for a homogeneous harmonic polynomial of degree m , we have $N(r) = m$ for all r . For instance, in dimension 2, the homogeneous harmonic polynomials of degree m are (in polar coordinates) of the type

$$p_m(\varrho, \theta) = A\varrho^m \cos m\phi + B\varrho^m \sin m\phi$$

and it is easy to verify what has been asserted.

In the sequel, we will assume that $H(r) > 0$ for every $r \in (0, R_0)$, otherwise we can employ the device shown at the beginning of Section 16.3.1. We can also notice that if there exists $\bar{r} \in (0, R_0)$ such that $H(\bar{r}) = 0$ then $u = 0$ on $\partial B_{\bar{r}}$ from which, $u = 0$ in $B_{\bar{r}}$ and by the unique continuation property we have $u \equiv 0$, which make the (16.3.29) trivial. The second device (i.e., assuming a unique continuation property) is certainly legitimate, but it is somewhat reductive because by the method we are illustrating we can obtain independent proof of unique continuation property.

At this point, we state and prove the following.

Proposition 16.3.1. *If u is a non identically zero solution to (16.3.28) then $N(r)$ is an increasing function.*

Proof. First we prove that

$$I'(r) = 2 \int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 dS + \frac{n-2}{r}I(r). \tag{16.3.39}$$

For this purpose we use the Rellich identity (Lemma 14.2.1) for $v \in C^2(B_{R_0})$ and $\beta(x) = x$,

$$2(x \cdot \nabla v)\Delta v = \operatorname{div}[2(x \cdot \nabla v)\nabla v - x|\nabla v|^2] + (n-2)|\nabla v|^2. \tag{16.3.40}$$

We get by (16.3.35)

$$I'(r) = \int_{\partial B_r} |\nabla u|^2 dS. \quad (16.3.41)$$

Now we apply identity (16.3.40) to the function u , we integrate both the sides of (16.3.40) over B_r and we recall that $\Delta u = 0$, obtaining

$$\begin{aligned} 0 &= \int_{\partial B_r} \left\{ 2(x \cdot \nabla u) \nabla u \cdot \frac{x}{|x|} - x \cdot \frac{x}{|x|} |\nabla u|^2 \right\} dS + (n-2) \int_{B_r} |\nabla u|^2 dx \\ &= \int_{\partial B_r} \left\{ 2r \left(\frac{\partial u}{\partial \nu} \right)^2 - r |\nabla u|^2 \right\} dS + (n-2) \int_{B_r} |\nabla u|^2 dx, \end{aligned}$$

from which, taking into account (16.3.41), we have

$$\begin{aligned} I'(r) &= \int_{\partial B_r} |\nabla u|^2 dS = 2 \int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 dS + \frac{n-2}{r} \int_{B_r} |\nabla u|^2 dx \\ &= 2 \int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 dS + \frac{n-2}{r} I(r). \end{aligned} \quad (16.3.42)$$

hence (16.3.39) is proved.

Differentiating both the sides of (16.3.38), we get

$$N'(r) = \frac{I(r)}{H(r)} + r \frac{I'(r)}{H(r)} - r \frac{I(r)H'(r)}{H^2(r)}. \quad (16.3.43)$$

Unless u is constant (and, therefore (16.3.29) becomes trivial) we have

$$N(r) > 0$$

and we divide both the sides of (16.3.43) by $N(r)$, obtaining

$$\frac{N'(r)}{N(r)} = \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)},$$

from which, taking into account (16.3.37) and (16.3.39), we have

$$\begin{aligned} \frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{\frac{n-1}{r}H(r) + 2I(r)}{H(r)} = \\ &= \frac{2-n}{r} + \frac{I'(r)}{I(r)} - 2 \frac{I(r)}{H(r)} = \\ &= 2 \left\{ \frac{\int_{\partial B_r} \left(\frac{\partial u}{\partial \nu} \right)^2 dS}{I(r)} - \frac{I(r)}{H(r)} \right\}. \end{aligned} \quad (16.3.44)$$

Now we express $I(r)$ by means of (16.3.36) and, by (16.3.44), we get

$$\begin{aligned} \frac{N'(r)}{N(r)} &= 2 \left\{ \frac{\int_{\partial B_r} \left(\frac{\partial u}{\partial \nu}\right)^2 dS}{\int_{\partial B_r} \frac{\partial u}{\partial \nu} u dS} - \frac{\int_{\partial B_r} \frac{\partial u}{\partial \nu} u dS}{\int_{\partial B_r} u^2 dS} \right\} = \\ &= \frac{2}{I(r)H(r)} \left\{ \left(\int_{\partial B_r} \left(\frac{\partial u}{\partial \nu}\right)^2 dS \right) \left(\int_{\partial B_r} u^2 dS \right) - \left(\int_{\partial B_r} \frac{\partial u}{\partial \nu} u dS \right)^2 \right\}. \end{aligned}$$

Now, by the Cauchy–Schwarz inequality we have

$$\left(\int_{\partial B_r} \left(\frac{\partial u}{\partial \nu}\right)^2 dS \right) \left(\int_{\partial B_r} u^2 dS \right) - \left(\int_{\partial B_r} \frac{\partial u}{\partial \nu} u dS \right)^2 \geq 0$$

from which the thesis follows. ■

Let us summarize what has been obtained so far. By Proposition 16.3.1 we get the log-convexity of function (16.3.32), from which inequality (16.3.29) follows. By the latter, proceeding as done to prove (10.4.22) and using by (16.3.29), we obtain,

$$\int_{B_{r_2}} u^2 dx \leq \left(\int_{B_{r_1}} u^2 dx \right)^{\theta_0} \left(\int_{B_{r_3}} u^2 dx \right)^{1-\theta_0}, \quad (16.3.45)$$

for $0 < r_1 < r_2 < r_3 < R_0$, where θ is given by (16.3.31).

By the properties of the frequency function we derive a doubling inequality. Indeed the following holds true.

Proposition 16.3.2. *Let u be a non identically zero solution to (16.3.28) then*

$$\int_{B_{2r}} u^2 dx \leq 2^{2^n N(R_0)+1} \int_{B_r} u^2 dx, \quad \forall r \in \left(0, \frac{R_0}{2}\right] \quad (16.3.46)$$

and

$$\int_{B_{2r}} u^2 dx \leq \frac{2 \int_{B_{R_0}} u^2 dx}{\int_{B_{R_0/4}} u^2 dx} \int_{B_r} u^2 dx, \quad \forall r \in \left[0, \frac{R_0}{4}\right], \quad (16.3.47)$$

In the proof of Proposition 16.3.2 we use

Lemma 16.3.3. *Let u be a solution to (16.3.28) then we have*

$$\frac{1}{r} \int_{B_r} u^2 dx \leq \int_{\partial B_r} u^2 dS \leq \frac{C}{r} \int_{B_{2r}} u^2 dx, \quad r \in \left(0, \frac{R_0}{2}\right], \quad (16.3.48)$$

where C depends on n only.

Proof of Lemma. Let us prove the first inequality of (16.3.48). By (16.3.37) we have $H'(r) \geq 0$. Hence

$$\begin{aligned} \int_{B_r} u^2 dx &= \int_0^r \left(\int_{\partial B_t} u^2 dS \right) dt \leq \\ &\leq \int_0^r \left(\int_{\partial B_r} u^2 dS \right) dt = \\ &= r \int_{\partial B_r} u^2 dS. \end{aligned} \quad (16.3.49)$$

Concerning the second inequality of (16.3.48), by (16.3.34) and by the Caccioppoli inequality, (4.5.1), we have

$$\begin{aligned} H(r) &= \frac{1}{r} \int_{B_r} \operatorname{div}(xu^2) dx = \\ &= \frac{n}{r} \int_{B_r} u^2 dx + \frac{1}{r} \int_{B_r} u(x \cdot \nabla u) dx \leq \\ &\leq \frac{n}{r} \int_{B_r} u^2 dx + \left(\int_{B_r} u^2 dx \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 dx \right)^{1/2} \leq \\ &\leq \frac{n}{r} \int_{B_r} u^2 dx + \frac{\tilde{C}}{r} \left(\int_{B_r} u^2 dx \right)^{1/2} \left(\int_{B_{2r}} u^2 dx \right)^{1/2} \leq \\ &\leq \frac{\tilde{C} + n}{r} \int_{B_{2r}} u^2 dx, \end{aligned}$$

where \tilde{C} depends on n only. The second inequality in (16.3.48) is proved with $C = \tilde{C} + n$. ■

Proof of Proposition 16.3.2. By (16.3.37) and (16.3.38) we get

$$\frac{d}{dr} \left(\log \frac{H(r)}{r^{n-1}} \right) = \frac{2N(r)}{r}. \quad (16.3.50)$$

Let $\rho \in (0, \frac{R_0}{2}]$. Integrating both the sides of (16.3.50) over $[\rho, 2\rho]$ and recalling that N is increasing, we have

$$\log \left(\frac{H(2\rho)}{2^{n-1}H(\rho)} \right) = 2 \int_{\rho}^{2\rho} \frac{2N(r)}{r} \leq (2 \log 2)N(R_0).$$

Hence

$$H(2\rho) \leq 2^{2^n N(R_0)} H(\rho), \quad \forall \rho \in \left(0, \frac{R_0}{2} \right],$$

Integrating over $[0, r]$, $r \in (0, \frac{R_0}{2}]$, we get (16.3.46).

Let now prove (16.3.47). Let $\rho \in (0, \frac{R_0}{4}]$ and $R \in [\frac{R_0}{4}, \frac{R_0}{2}]$. Integrating both the sides of (16.3.50) over $[\rho, 2\rho]$ and recalling that N is an increasing function, we have

$$\log \left(\frac{H(2\rho)}{2^{n-1}H(\rho)} \right) \leq (2 \log 2)N(R).$$

By the just obtained inequality, taking into account (16.3.50), we have

$$\frac{1}{R} \log \left(\frac{H(2\rho)}{2^{n-1}H(\rho)} \right) \leq (2 \log 2) \frac{N(R)}{R} = (\log 2) \frac{d}{dR} \left(\log \frac{H(R)}{R^{n-1}} \right).$$

Again we integrate both the sides of the last inequality w.r.t. R over $[\frac{R_0}{4}, \frac{R_0}{2}]$ so that we have

$$\log \left(\frac{H(2\rho)}{2^{n-1}H(\rho)} \right) \leq \log \left(\frac{H(\frac{R_0}{2})}{2^{n-1}H(\frac{R_0}{4})} \right).$$

Now, Lemma 16.3.3 gives

$$H(2\rho) \leq \frac{\int_{B_{R_0}} u^2 dx}{\int_{B_{R_0/4}} u^2 dx} H(\rho), \quad \forall \rho \in \left(0, \frac{R_0}{4} \right].$$

From which integration of both the sides over $[0, r]$, $r \in [0, \frac{R_0}{4}]$, of last inequality gives

$$\int_{B_{2r}} u^2 dx \leq \frac{2 \int_{B_{R_0}} u^2 dx}{\int_{B_{R_0/4}} u^2 dx} \int_{B_r} u^2 dx, \quad \forall r \in \left[0, \frac{R_0}{4} \right].$$

Therefore, we get (16.3.47). ■

Final Remark. The proof of Proposition 16.3.1, based on the Rellich identity, differs from the proof given in [27] which is based on the transformation of the elliptic operator in polar coordinates. The proof based on the Rellich identity was given by [42] for the second-order elliptic operators. ♦

16.4 A brief review about the A_p weights.

In this Section we will provide a brief summary of the A_p weights by referring to [17] and [25, Chapter 4] for further reading. In Section 2.5.10 we have introduced the maximal function related to a function $f \in L^1(\mathbb{R}^n)$. Basically equivalent to it is the following definition. In what follows we will denote by Q a closed cube whose sides are parallel to the axes. Let $f \in L^1(\mathbb{R}^n)$. We define the maximal Hardy-Littlewood function as

$$\mathcal{M}(f)(x) = \sup \left\{ \int_Q |f(y)| dy : Q \ni x \right\}, \quad (16.4.1)$$

where, we recall,

$$\int_Q |f(y)| dx = \frac{1}{|Q|} \int_Q |f(y)| dy.$$

It is simple to check that

$$\frac{\omega_n}{2^{nn}} M(f)(x) \leq \mathcal{M}(f)(x) \leq \omega_n n^{\frac{1}{2}n-1} M(f)(x), \quad \forall x \in \mathbb{R}^n, \quad (16.4.2)$$

where $M(f)$ is defined in (2.5.10). From inequalities (16.4.2) one can prove for $\mathcal{M}(f)$ properties similar to those of $M(f)$ some of which have been proved or presented in Section 2.5. In particular, the following apply (compare with Lemma 2.5.4 and (2.5.15), respectively)

$$|E_t| \leq \frac{C_n}{t} \int_{\mathbb{R}^n} |f(x)| dx, \quad \forall f \in L^1(\mathbb{R}^n), \forall t > 0$$

(C_n depends on n only), where

$$E_t = \{x \in \mathbb{R}^n : M(f)(x) > t\}$$

and

$$\|M(f)\|_{L^p(\mathbb{R}^n)} \leq C_{n,p} \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p(\mathbb{R}^n).$$

($C_{n,p}$ depends on n and p only).

The A_p weight were introduced by Muckenhoupt, [58], and by Coifman and Fefferman, [17] to answer to the following question:

Let $p \in (1, +\infty)$, determine all measurable and nonnegative functions w such that

$$\int_{\mathbb{R}^n} |\mathcal{M}(f)(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx, \quad \forall f \in L^p(\mathbb{R}^n), \quad (16.4.3)$$

where C depends on w only. The functions w which enjoys property (16.4.3) are called A_p **weight**. If $p = 1$, then we say that w is an A_1 weight if

$$\int_{\tilde{E}_t} w(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|w(x)dx, \quad \forall f \in L^1(\mathbb{R}^n),$$

where

$$\tilde{E}_t = \{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}.$$

The following Theorem can be proved ([25, Chap. 4, Sect 2]).

Theorem 16.4.1. *Let $w \in L^1_{loc}(\mathbb{R}^n)$, $w \geq 0$. The following conditions are equivalent*

- (a) *there exists $p \in [1, +\infty)$ such that $w \in A_p$;*
 (b) *if $p \in (1, +\infty)$, then there exists $C > 0$ such that*

$$\left(\int_Q w(x)dx \right) \left(\int_Q w^{-\frac{1}{p-1}}(x)dx \right)^{p-1} \leq C, \quad \text{for every cube } Q \quad (16.4.4)$$

if $p = 1$, then we have

$$\left(\int_Q w(x)dx \right) \text{esssup}(w^{-1}) \leq C, \quad \text{for every cube } Q; \quad (16.4.5)$$

- (c) *there exist $\delta > 0$ and $C > 0$ such that*

$$\left(\int_Q w^{1+\delta}(x)dx \right)^{\frac{1}{1+\delta}} \leq C \int_Q w(x)dx \quad \text{for every cube } Q; \quad (16.4.6)$$

(d) *there exist $s > 0$ and $C > 0$ such that, for every cube Q and for every $E \subset Q$, E Lebesgue measurable set, we have*

$$\frac{|E|}{|Q|} \leq C \left(\frac{\int_E w(x)dx}{\int_Q w(x)dx} \right)^s. \quad (16.4.7)$$

Comments and Remarks. An inequality like (16.4.6) is called "**reverse Hölder inequality**". It can be proved that Theorem 16.4.1 can be reformulated for weight functions w defined on an open set Ω , as long as we replace, in (b), (c) and (d) "for every cube Q " by "for every cube $Q \subset \Omega$ ". The following remark turns out to be very useful in proving quantitative estimates in inverse problems. Below we illustrate this idea in a simplified way.

In Theorem 15.7.7 we proved that a nonzero if u is a solution of the elliptic equation

$$\sum_{i,j=1}^n a^{ij}(x) \partial_{x^i x^j}^2 u + \sum_{i=1}^n b^i(x) \partial_{x^i} u + c(x)u = 0, \quad (16.4.8)$$

whose coefficients satisfy the hypotheses (15.7.57), (15.7.58) and (15.7.59) then u satisfies doubling inequality (15.7.62). Let us suppose that u satisfies (16.4.8) in $B_{\bar{R}}$, $\bar{R} > 1$ and let us write inequality (15.7.62) in the form

$$\int_{B_{2r}(x_0)} u^2 \leq CN_{x_0}^k \int_{B_r(x_0)} u^2, \quad x_0 \in B_1. \quad (16.4.9)$$

$$N_{x_0} = \frac{\int_{B_1(x_0)} u^2 dx}{\int_{B_{r_0}(x_0)} u^2 dx}. \quad (16.4.10)$$

where r_0 is a suitable point of $(0, 1)$ (here, $R_0 = 1$ and $r_0 = 1/C$, in (15.7.62)). Let us assume for simplicity

$$\bar{R} = \max\{1 + 2\sqrt{n}, 16\}.$$

Let us denote by

$$F = \frac{\int_{B_{\bar{R}}} u^2 dx}{\int_{B_{r_0/2}} u^2 dx}. \quad (16.4.11)$$

By the Caccioppoli inequality and the Sobolev Embedding Theorem, proceeding similarly to what was done in Lemma 4.8.5, we have

$$\left(\int_{Q_r(x_0)} |u|^q dx \right)^{1/q} \leq \left(\int_{Q_{2r}(x_0)} u^2 dx \right)^{1/2}, \quad \forall x_0 \in B_{r_0/2}, \quad (16.4.12)$$

where q is an arbitrary number of $(1, +\infty)$ when $n = 2$, and it is equal to $\frac{2n}{n-2}$ when $n \geq 3$. On the other hand (see Corollary 15.7.8),

$$\begin{aligned} \left(\int_{Q_{2r}(x_0)} u^2 dx \right)^{1/2} &\leq \left(\int_{B_{2\sqrt{n}r}(x_0)} u^2 dx \right)^{1/2} \leq \\ &\leq (CN_{x_0})^{\tilde{k}} \left(\int_{B_r(x_0)} u^2 dx \right)^{1/2} \leq \\ &\leq (CN_{x_0})^{\tilde{k}} \left(\int_{Q_r(x_0)} u^2 dx \right)^{1/2}, \quad \forall x_0 \in B_{r_0/2}, \end{aligned}$$

where $\tilde{k} = \frac{k}{2}(1 + \log_2(2\sqrt{2n}))$. By the just obtained inequality and by (16.4.12), we have

$$\left(\int_{Q_r(x_0)} |u|^q dx\right)^{1/q} \leq (CN_{x_0})^{\tilde{k}} \left(\int_{Q_r(x_0)} u^2 dx\right)^{1/2}, \quad \forall x_0 \in B_{r_0/2}.$$

Moreover, by (16.4.11) we have trivially

$$N_{x_0} = \frac{\int_{B_1(x_0)} u^2 dx}{\int_{B_{r_0}(x_0)} u^2 dx} \leq \frac{\int_{B_{\bar{R}}} u^2 dx}{\int_{B_{r_0/2}} u^2 dx}.$$

Therefore we have, in particular, for any cube $Q \subset B_{r_0/2}$

$$\left(\int_Q |u|^q dx\right)^{1/q} \leq F^{\tilde{k}} \left(\int_Q u^2 dx\right)^{1/2}. \quad (16.4.13)$$

Recalling that $q > 2$, we get by (16.4.13) that u^2 satisfies a reverse Hölder inequality, consequently u^2 is an A_p weight. In particular, Theorem 16.4.1 yields that for every $Q \subset B_{r_0/2}$ and for every $E \subset Q$, E Lebesgue measurable set, we have

$$\frac{|E|}{|Q|} \leq C \left(\frac{\int_E u^2 dx}{\int_Q u^2 dx}\right)^s, \quad (16.4.14)$$

where C depends by F . Let us suppose, now that the set E has positive measure, then, if we have some bounds on F (generally obtainable from values at the boundary of u), estimate (16.4.14) can be trivially rewritten

$$\int_Q u^2 dx \leq \left(\frac{C|Q|}{|E|}\right)^{1/s} \int_E u^2 dx \quad (16.4.15)$$

This estimate implies, in particular, that if u vanishes on a set E of positive measure then u vanishes identically in $B_{\bar{R}}$. Actually, (16.4.15) also allows us to control, in terms of the measure E only, the propagation of the error

$$\int_E u^2 dx \leq \varepsilon^2$$

On a "small" cube Q and from there on the whole $B_{\bar{R}}$.

Another trivial translation of (16.4.14) is

$$|E| \leq C|Q| \left(\frac{\int_E u^2 dx}{\int_Q u^2 dx}\right)^s,$$

which allows us to estimate the Lebesgue measure of E by the integral of u^2 on E itself. The latter observation is useful for finding **size estimates** of unknown inclusions in problems of the type considered in Section 4.8 (for details, see [3]). \blacklozenge

16.5 The Runge property

As an introduction to the main topic of this Section, we show by an example that it is not always possible to extend a harmonic function u from B_1 to an open set, Ω , such that $B_1 \Subset \Omega$. The example we present here is due to Hadamard, [31]. Let us consider the function whose expression in polar coordinates is given by

$$u(\rho, \phi) = \sum_{n=1}^{\infty} 2^{-n} \rho^{4^n} \sin(4^n \phi). \quad (16.5.1)$$

It is simple to check that $u \in C^0(\overline{B_1}) \cap C^2(B_1)$ and that u is harmonic in B_1 . Now we check that

$$u \notin C^1(\overline{B_1}).$$

For this purpose we show that

$$\lim_{r \rightarrow 1^-} \int_{B_r} |\nabla u|^2 dx dy = \lim_{r \rightarrow 1^-} \int_0^r d\rho \int_0^{2\pi} (u_\rho^2 + \rho^{-2} u_\phi^2) \rho d\phi = +\infty. \quad (16.5.2)$$

We get

$$\int_0^{2\pi} (u_\rho^2 + \rho^{-2} u_\phi^2) \rho d\phi = 2\pi \sum_{n=1}^{\infty} 4^n \rho^{2 \cdot 4^n - 1}$$

Hence

$$\int_0^r d\rho \int_0^{2\pi} (u_\rho^2 + \rho^{-2} u_\phi^2) \rho d\phi = \pi \sum_{n=1}^{\infty} r^{2 \cdot 4^n},$$

from which (16.5.2) follows. It is therefore evident that u cannot be extended to a harmonic function in an open set containing $\overline{B_1}$.

However, it is of interest to know whether u can be approximated by functions that are harmonic in an open set containing $\overline{B_1}$. A property of this kind is called **Runge property** for the operator Δ . This issue has been studied for operators which are more general than the Laplace operator, but here we limit ourselves to the Laplace operator only, referring to the final comments for hints on further consideration.

We have

Theorem 16.5.1. *Let $\Omega_1 \Subset \Omega_2$ be two open sets of \mathbb{R}^n , where $\partial\Omega_2$ is of class $C^{1,1}$. Let us assume that $\Omega_2 \setminus \overline{\Omega_1}$ connected. Then for every u such that*

$$\Delta u = 0, \quad \text{in } \Omega_1$$

and for every $\varepsilon > 0$ there exists $v \in H^1(\Omega_2)$ such that

$$\Delta v = 0, \quad \text{in } \Omega_2$$

and

$$\|u - v\|_{L^2(\Omega_1)} < \varepsilon.$$

Proof. Set

$$\mathcal{S}_1 = \{u \in H^1(\Omega_1) : \Delta u = 0 \text{ in } \Omega_1\}$$

and

$$\mathcal{S}_2 = \left\{v|_{\Omega_1} : \Delta v = 0 \text{ in } \Omega_2\right\}. \quad (16.5.3)$$

The property that we wish to prove is equivalent to the fact that \mathcal{S}_2 is dense in \mathcal{S}_1 , with respect to the topology induced by $L^2(\Omega_1)$. We now prove this density property. To this aim, it suffices to prove that if $u \in \mathcal{S}_1$ and

$$\int_{\Omega_1} u v dx = 0, \quad \forall v \in \mathcal{S}_2, \quad (16.5.4)$$

then

$$u \equiv 0, \quad \text{in } \Omega_1.$$

Let \tilde{u}

$$\tilde{u}(x) = \begin{cases} u, & \text{in } \Omega_1, \\ 0, & \text{in } \Omega_2 \setminus \Omega_1. \end{cases}$$

and let w satisfy

$$\begin{cases} \Delta w = \tilde{u}, & \text{in } \Omega_2, \\ w \in H_0^1(\Omega_2). \end{cases} \quad (16.5.5)$$

Since $\tilde{u} \in L^2(\Omega_2)$ and $\partial\Omega_2$ is of class $C^{1,1}$, by Theorem 4.6.5 we have

$$w \in H^2(\Omega_2).$$

Now for an arbitrary $v \in \mathcal{S}_2$ we have

$$\begin{aligned}
 0 &= \int_{\Omega_1} uvdx = \int_{\Omega_2} \tilde{u}vdx = \int_{\Omega_2} \Delta wvdx = \\
 &= \int_{\partial\Omega_2} \frac{\partial w}{\partial \nu} v dS + \int_{\Omega_2} w \Delta v dx = \\
 &= \int_{\partial\Omega_2} \frac{\partial w}{\partial \nu} v dS.
 \end{aligned} \tag{16.5.6}$$

Hence

$$\int_{\partial\Omega_2} \frac{\partial w}{\partial \nu} v dS = 0, \quad \forall v \in \mathcal{S}_2.$$

By the latter and by Theorem 4.3.1 we have

$$\int_{\partial\Omega_2} \frac{\partial w}{\partial \nu} \varphi dS = 0, \quad \forall \varphi \in H^{1/2}(\partial\Omega_2), \tag{16.5.7}$$

hence for $\varphi = \frac{\partial w}{\partial \nu}$ we have

$$\frac{\partial w}{\partial \nu} = 0, \quad \text{on } \partial\Omega_2.$$

Therefore we have

$$\begin{cases} \Delta w = \tilde{u}, & \text{in } \Omega_2 \setminus \bar{\Omega}_1, \\ w = 0, & \text{on } \partial\Omega_2, \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega_2. \end{cases} \tag{16.5.8}$$

Now, since $\Omega_2 \setminus \bar{\Omega}_1$ is connected, (16.5.8) implies, by the unique continuation property,

$$w \equiv 0, \quad \text{in } \Omega_2 \setminus \bar{\Omega}_1.$$

By this relationship, recalling that $w \in H^2(\Omega_2)$ we have that $w \in H_0^2(\Omega_1)$ from which we have

$$\int_{\Omega_1} u^2 dx = \int_{\Omega_1} u \Delta w dx = \int_{\Omega_1} \Delta u w dx = 0$$

Hence

$$u \equiv 0, \quad \text{in } \Omega_1.$$

As we desired to prove. ■

Remarks.

1. Theorem 16.5.1 continues to be true also if, instead of $\partial\Omega_2 \in C^{1,1}$, we assume only $\partial\Omega_2 \in C^{0,1}$. To prove this, one must first keep in mind that in (16.5.6) occurs

$$\left\langle \frac{\partial w}{\partial \nu}, v \right\rangle_{H^{-1/2}(\partial\Omega_2), H^{1/2}(\partial\Omega_2)}$$

instead of

$$\int_{\partial\Omega_2} \frac{\partial w}{\partial \nu} v dS$$

and, consequently, instead of (16.5.7), we have

$$\left\langle \frac{\partial w}{\partial \nu}, \varphi \right\rangle_{H^{-1/2}(\partial\Omega_2), H^{1/2}(\partial\Omega_2)} = 0, \quad \forall \varphi \in H^{1/2}(\partial\Omega_2).$$

Therefore, we likewise have that $\frac{\partial w}{\partial \nu} = 0$. However, since $\frac{\partial w}{\partial \nu} \in H^{-1/2}(\partial\Omega_2)$, it will be necessary to first reformulate Cauchy problem (16.5.8) in a weak form and then to prove the uniqueness for such a Cauchy problem, for both of which we refer the reader to the paper [2].

2. The assumption that $\Omega_2 \setminus \bar{\Omega}_1$ is connected cannot be dropped as the following simple **counterexample** shows. Let

$$\Omega_1 = (B_7 \setminus \bar{B}_5) \cup (B_3 \setminus \bar{B}_1), \quad \Omega_2 = B_8,$$

let us notice that $\Omega_2 \setminus \bar{\Omega}_1$ is not connected.

Let u be the following function

$$u(x) = \begin{cases} 1, & \text{for } x \in B_3 \setminus \bar{B}_1, \\ 0, & \text{for } x \in B_7 \setminus \bar{B}_5, \end{cases} \quad (16.5.9)$$

of course u is harmonic in Ω_1 . Let ε be a given positive number which we will choose later and let $v \in H^1(\Omega_2)$ be a harmonic function satisfying

$$\|u - v\|_{L^2(\Omega_1)} < \varepsilon. \quad (16.5.10)$$

Let $x_0 \in \partial B_2$, taking into account that $B_1(x_0) \subset \Omega_1$, by (16.5.10) we have

$$\begin{aligned}
|v(x_0) - 1| &= \left| \frac{1}{|B_1(x_0)|} \int_{B_1(x_0)} (v - 1) dx \right| \leq \\
&\leq \frac{1}{|B_1(x_0)|^{1/2}} \left(\int_{B_1(x_0)} (v - 1)^2 dx \right)^{1/2} \leq \\
&\leq \frac{\varepsilon}{c_n},
\end{aligned}$$

where $c_n = |B_1|^{1/2}$. Hence we have

$$1 - \frac{\varepsilon}{c_n} < v(x) < 1 + \frac{\varepsilon}{c_n}, \quad \forall x \in \partial B_2. \quad (16.5.11)$$

Similarly we get

$$-\frac{\varepsilon}{c_n} < v(x) < \frac{\varepsilon}{c_n}, \quad \forall x \in \partial B_6. \quad (16.5.12)$$

Now, by (16.5.11), (16.5.12) and by using maximum principle we have

$$1 - \frac{\varepsilon}{c_n} < v(x) \leq \max_{x \in \partial B_6} v < \frac{\varepsilon}{c_n}, \quad \forall x \in \partial B_6,$$

that leads to a contradiction provided $\varepsilon < \frac{c_n}{2}$.

3. Theorem 16.5.1 can be extended to a large class of operators; in particular, it can be extended to the second-order elliptic operators with real coefficients whose formal adjoint enjoys the unique continuation property. Let us specify this a little. Let us consider, for instance, the operator

$$Lu = \operatorname{div}(A(x)\nabla u), \quad (16.5.13)$$

where $A \in L^\infty(\mathbb{R}^n; \mathbb{M}(n))$ is a not necessarily symmetric matrix and such that ($\lambda \geq 1$)

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n.$$

Then the formal adjoint of L is the operator

$$L^*u = \operatorname{div}(A^T(x)\nabla u). \quad (16.5.14)$$

We say that L enjoys the **Runge property** provided it occurs what follows.

Let $\Omega_1 \Subset \Omega_2$ be two open sets of \mathbb{R}^n , like Theorem 16.5.1, then for every u such that

$$Lu = 0, \quad \text{in } \Omega_1$$

and for every $\varepsilon > 0$ there exists $v \in H^1(\Omega_2)$ such that

$$Lv = 0, \quad \text{in } \Omega_2$$

and

$$\|u - v\|_{L^2(\Omega_1)} < \varepsilon.$$

Lax in [49] has proved that the following conditions are equivalent

- (a) L enjoys Runge property
- (b) L^* enjoys the unique continuation property.

The proof of (b) \implies (a) is analogous to the proof of Theorem 16.5.1 and it is left to the reader as an exercise. Concerning the implication (a) \implies (b), we refer to [49].

4. The quantitative versions of the Runge property are also of interest (especially in the stability issue of inverse problems). That is, it is of interest to estimate appropriately from above in terms of ε , the quantity

$$\|v\|_{H^{1/2}(\partial\Omega_2)},$$

(by Hadamard example, illustrated at the beginning of this Section, we should expect that, in general, as ε goes to 0 we should have $\|v\|_{H^{1/2}(\partial\Omega_2)}$ goes to infinity). A result in this regard is proven in [70]. \blacklozenge

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