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COMPOUND MATRICES FOR STABILITY ANALYSIS OF NONLINEAR SYSTEMS

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To my family.

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Abstract

Although Lyapunov exponents have been widely used to characterize the dynamics of nonlinear systems, few methods are available so far to obtain a-priori bounds on their magnitudes. In this thesis, sufficient conditions to rule out the existence of attractors with positive Lyapunov exponents and to ensure 2-contractivity of the system are derived via a Lyapunov approach based on the second additive compound matrices of the system Jacobian. Moreover, insights into this approach are provided by showing how several available techniques for computing Lyapunov functions can be fruitfully applied to Lorenz and Thomas systems to derive explicit conditions on their system parameters, which ensure that there are no attractors with positive Lyapunov exponents. Then, the approach is extended to the case of nonlinear systems with a first integral of motion and its application to the memristor Chua's circuit is discussed. Furthermore, sufficient small-gain like conditions for 2-contraction of feedback interconnected systems, on the basis of individual gains of suitable subsystems arising from a modular decomposition of the second additive compound equations, are introduced. The condition applies even to cases when individual subsystems might fail to be contractive (due to the extra margin of contraction afforded by the second additive compound matrix). Some examples are provided to illustrate the theory and show its degree of conservatism and scope of applicability. Finally, the second additive compound approach is also used to derive conditions, formulated in terms of 2-contractivity of the closed loop system, for designing a feedback control law to remove the dense set of Unstable Periodic Orbits (UPOs) and chaotic attractors, while preserving the system equilibrium points. Matrix inequalities for computing the control gain matrix are derived and applied to the Lorenz and Thomas systems.

Contents

Introduction

The long-term behavior of solutions of differential equations is one of the most investigated topic in the analysis of dynamical systems, also for the strict connection with the stability and instability properties [46]. Among several other approaches, Lyapunov exponents have been largely employed to characterize the attractors of finite dimensional continuous-time nonlinear systems [17]. Conditions for the existence of the Lyapunov exponents are known since long time [40, 44] and a-priori bounds on their location [26] are usually obtained through the spectral analysis of the variational equation, also exploiting the fact that they are invariant under fairly general coordinate transformations. Indeed, several algorithms are nowadays available to numerically compute the Lyapunov exponents for a given system [15, 16]. Conversely, it remains an open problem how to derive conditions on the system parameters which guarantee bounded Lyapunov exponents within prescribed regions of the complex plane. In this respect, the presence/absence of positive Lyapunov exponents is of particular importance as it allows to classify the nature of the system attractors (see [45] for a related discussion on this issue).

In this thesis, a new approach to rule out the existence of attractors with positive Lyapunov exponents in continuous-time nonlinear systems is proposed, extending previous conditions ensuring the non-existence of periodic and almost periodic solutions [1]. The approach is based on the properties pertaining to the second additive compound of the Jacobian matrix which have been characterized by James Muldowney in the seminal paper [36]. Muldowney ruled out the existence of non-constant periodic solutions by formulating conditions on the matrix norm of the second additive compound of the Jacobian. Indeed, compound matrices have been widely used in the study of nonlinear dynamics, mainly focused on the estimation of attractors dimension (see [23] and references therein). Up-to-date introductions to ad-

ditive and multiplicative compound matrices were recently reported in [6,53], where their application to contracting systems is thoroughly discussed. The approach here presented shows that the non-existence of attractors with positive exponents can be tackled via the Lyapunov's direct method, i.e., determining a Lyapunov function for the second additive compound of the Jacobian matrix. Notably, this means that to rule out the existence of positive Lyapunov exponents one can exploit the several available techniques for computing Lyapunov functions. In some cases, these conditions can be formulated as a feasibility Linear Matrix Inequality (LMI) problem involving matrices of dimension $\binom{n}{2} \times \binom{n}{2}$.
Recently, a lot of attention has been devoted the so called k-Contraction Theory [52], which provides conditions ensuring that arbitrary k-dimensional submanifolds of state space contract in volume along solutions of the system (see [6] for more details). This approach, for $k = 1$, recovers the classical Contraction Theory, while for $k = 2$ we obtain Muldowney's conditions.
At the same time, stability analysis of interconnected dynamical systems has also become a very active area of research. The general idea behind a Small-Gain Theorem is the formulation of a sufficient stability condition, for a feedback interconnected system of some sort, on the basis of the stability of its modular components, and the calculation of some notion of "loop gain," which, if sufficiently low (typically smaller than unity), is adequate for assessing the stability of the whole interconnection. Many versions of such result exist, ranging from Input-to-State-Stability (ISS) systems [21] to an LMI set-up [8], and passing for large-scale interconnected systems [13]. See [10] for a recent and up-to-date reference, where modular techniques for contraction analysis of large-scale networks are perfected and treated in depth. The special case of k-contraction for two cascaded systems is studied in [38], while in [39] the case of static nonlinear feedback (of the Lurie form) is considered.
In the present thesis, we formulate a small-gain theorem result for 2-contraction of feedback interconnected systems, based on the second additive compound matrix of individual subsystems and of an auxiliary coupling systems, which captures the dynamics of their interconnections. In particular, rather than resolving a unique LMI condition of size $\binom{n}{2} \times \binom{n}{2}$, we consider subsystems of dimension n_1 and n_2 (with $n_1 + n_2 = n$) and we solve 3 separate LMIs with unknowns of size $\binom{n_1}{2}$, $\binom{n_2}{2}$ and $n_1 \cdot n_2$ respectively.
Since the compound matrices allow expressing conditions directly on the

Lyapunov exponents of a system, one may think to exploit the machinery of compound matrices to formulate condition for removing chaotic behaviours from the system through a feedback control law. Chaos was experimentally observed for the first time by Edward N. Lorenz in his work on deterministic non-periodic flows [32], showing that a chaotic system exhibits high sensitivity with respect to initial conditions, i.e., the so called “butterfly effect”. Since then, chaotic systems were discovered in several fields of biological, ecological, chemical, physical sciences and engineering ([18]). Moreover, other peculiar aspects of chaotic systems started to be clear, such as the coexistence of unstable periodic orbits (UPOs) and the fact that, due to ergodicity, the neighborhood of each UPO is continuosly visited during the system time evolution ([29]).

Subsequently, the idea of taking advantage of chaos gained momentum. In particular, it was observed that many different oscillatory behaviours can be derived from a single chaotic system, without significant input injections in steady state, by stabilizing individual UPOs via the application of the right perturbation to the system. The first approach to stabilize a given UPO of a chaotic system was introduced by Ott-Grebogi-Yorke (OGY) in [41]. The method suggests to tackle the problem of stabilizing UPOs as a local stabilization of a fixed point of the system’s Poincarè map. Therefore, knowledge of the right perturbation to apply as well as of the considered Poincarè section is needed. Since then, the field of chaos control has gained ground and different alternative methods were introduced. For example, in [42] the author suggests the well known delayed feedback control (DFC), which has undergone several modifications over the years ([43]). The Pyragas’ approach needs a priori knowledge of the UPO period and it exploits as a control input a signal proportional to the difference of the actual system’s output and a delayed one, with some constant gain that has to be properly tuned numerically or experimentally. Unfortunately, the stability analysis is complex since the resulting controller and closed-loop system is not finite dimensional ([50]). To overcome this issue, in the case of harmonically forced nonlinear systems, finite-dimensional controllers have been introduced, which are obtained by solving linear matrix inequalities (LMIs) in [7] and [5]. It is important to note that both the OGY and DFC methods are non-invasive methods, since the stabilized periodic orbit coincides with the uncontrolled UPO.

In this thesis, a different perspective on chaos control is adopted by exploit-

ing the variational equation associated to the second additive compound of the Jacobian. In particular, the method allows designing nonlinear time-invariant state feedback controllers capable to remove the dense set of UPOs and chaotic attractors, while preserving equilibria, by making the closed loop system 2-contractive. Moreover, it will be shown how the feedback gain matrix can be obtained by solving some matrix inequalities which involve the second additive compound of the Jacobian of the closed loop system.
The aim of this thesis is to propose alternative techniques to address the problem of finding some convergence conditions for systems that exhibit multistability, i.e. the presence of stable and unstable fixed points. In particular, it is proposed to exploit the connection between compound matrices and dynamical systems for the analysis of the dynamics and the control of nonlinear systems in such a scenario, where most of the global stability techniques fail.
Initially, some known preliminary results and a background on compound matrices are introduced in Chapter 1. Then, in Chapter 2, it is presented a new technique to exclude attractors with positive Lyapunov exponents, based on the exponential contraction of the surface area spanned by arbitrary pairs of infinitesimal perturbations of initial conditions, propagated along the solutions of the system. Initially, it is shown the connection between the Lyapunov spectra of the standard variational equation and the compound variational equation of second order (Proposition 8), since it is a crucial point for the result in Lemma 1, Theorem 4 and Theorem 5. Then, some final remarks and insights on the results are discussed at the end of the chapter, together with the extension to systems with a first integral of motion in Theorem 6. Subsequently, in Chapter 3, suitable notions of gains for linear systems are formulated in terms of LMIs via the introduction of a small-gain theorem in Theorems 7 and 8. As a consequence, the results are extended to the case of nonlinear systems and state-dependent contraction metrics (Theorems 9 and 10), in order to provide sufficient small gain conditions for the assessment of 2-contraction for both linear and nonlinear systems. In Chapter 4 it is shown how the technique presented in Chapter 2 can be exploited to design a non-invasive feedback control law for controlling chaos ensuring the closed loop system to be 2-contractive. A notion of 2-contraction stabilizability is first defined, in order to highlight the key concept at the basis of the result in Theorem 11. Subsequently, Propositions 20 and 21 provide a new technique, arising from Theorem 4, to design a control

law with derivative feedback ensuring no positive Lyapunov exponents for the controlled system. Chapter 5 is devoted to some examples of application and case studies of the techniques presented in the previous chapters. Finally, some final conclusions as well as some future research directions end the thesis.

Chapter 1

Compound matrices and dynamical systems

This chapter provides the adopted notation and a brief introduction on the algebra of compound matrices. The first part introduces the notion of k-multiplicative and k-additive compound of a matrix, together with some important properties, while the second part is devoted to the connection between compound matrices and linear and nonlinear dynamical systems. The material of the chapter can be found in [6, 9, 33, 36].

1.1 Notation

- $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$: sets of nonnegative integers, real numbers, complex numbers;
- e_i: the i-th canonical basis vector;
- $J(x)$: Jacobian $\partial f(x)/\partial x$ of $f(x)$;
- $\text{Re}(\lambda)$, $\text{Im}(\lambda)$, $|\lambda|$; real part, imaginary part, magnitude of $\lambda \in \mathbb{C}$;
- I_n: $n \times n$ identity matrix;
- $A^\top$: transpose of matrix A;
- $A \geq 0$: positive semidefinite (resp. definite) matrix A;
- $\text{Ker}(A)$: kernel of matrix A;
- $\text{spec}(A)$: spectrum of matrix A;
- $\text{conv}(A_1, A_2, \ldots)$ (resp. $A > 0$): convex hull of matrices $A_1, A_2, \ldots$;
- $Q(k, n)$: set of increasing sequences of k nonnegative integers with values in $\{1, 2, \ldots, n\}$ listed in lexicographical order;

- $A(\alpha|\beta)$: minor of matrix A corresponding to the pair $(\alpha, \beta) \in Q(k, n) \times Q(k, m)$;
- $A^{(k)}$: k-multiplicative compound matrix of matrix A;
- $A^{[k]}$: k-additive compound matrix of matrix A;
- $[v_1, v_2, \ldots, v_m]$: matrix with column vectors $v_1, v_2, \ldots, v_m$;
- $v_1 \wedge v_2$: wedge product of vectors v_1 and v_2.

1.2 Multiplicative compound matrix

Definition 1 *[6] For a matrix $A \in \mathbb{R}^{n\times m}$ and any positive integer $k \leq \min\{n, m\}$, the k-th multiplicative compound matrix is the matrix $A^{(k)} \in \mathbb{R}^{\binom{n}{k}\times\binom{m}{k}}$ whose entries are all the k dimensional minors of A listed in lexicographical order.*

Example 1 *Let $A \in \mathbb{R}^{3\times 3}$ and $k = 2$. $Q(2, 3)$ denote the set of increasing sequences of $k = 2$ integers in lexicographical order, i.e.*

$$Q(2, 3) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

The 2-multiplicative compound of A reads:

$$A^{(2)} = \begin{bmatrix} A(\{1,2\}|\{1,2\}) & A(\{1,2\}|\{1,3\}) & A(\{1,2\}|\{2,3\}) \\ A(\{1,3\}|\{1,2\}) & A(\{1,3\}|\{1,3\}) & A(\{1,3\}|\{2,3\}) \\ A(\{2,3\}|\{1,2\}) & A(\{2,3\}|\{1,3\}) & A(\{2,3\}|\{2,3\}) \end{bmatrix} =$$

$$= \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{bmatrix}$$

Proposition 1 *[6] Let $A \in \mathbb{R}^{n\times m}$ and $k \leq \min\{n, m\}$ be a positive integer. The following properties hold true:*

1. *$A^{(1)} = A$.*
2. *If $n = m \implies A^{(n)} = det(A)$.*

3. $(AB)^{(k)} = A^{(k)}B^{(k)}$ *(Cauchy-Binet formula).*

4. *Let is* $I_n^{(k)} = I_r$, *where* $r := \binom{n}{k}$, *the identity matrix of* $\binom{n}{k}$*-order. If* A *is non-singular then* $(AA^{-1})^{(k)} = I_r$ *and* $(A^{-1})^{(k)} = (A^{(k)})^{-1}$*. Furthermore, if* A *is non-singular then so is* $A^{(k)}$*.*

5. $(A^{(k)})^T = (A^T)^{(k)}$*.*

Multiplicative compound matrices allow denoting in a synthetic way the *wedge* product between vectors, as it is shown in the next statement.

Proposition 2 *Let* v_i, $v_j \in \mathbb{C}^n$ *be any complex vectors. The wedge product (or exterior product) between vectors can be computed through the* 2*-multiplicative compound of the matrix that has the vectors* v_i *and* v_j *as columns, i.e.*

$$v_i \wedge v_j = [v_i, v_j]^{(2)}.$$

In the case of square matrices, multiplicative compound matrices have an important spectral property.

Proposition 3 *[6] Let* $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ *be the eigenvalues of* A*. Then, the spectrum of the* k*-multiplicative compound matrix of* A *is equal to:*

$$\{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},\ 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

Furthermore, if $v_1, v_2, \dots, v_k$ *are independent eigenvector of* A *corresponding to eigenvalues* $\lambda_1, \lambda_2, \dots, \lambda_k$*, then the wedge product* $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ *is an eigenvector of* $A^{(k)}$ *with corresponding eigenvalue* $\lambda_1\lambda_2 \dots \lambda_k$*.*

1.3 Additive compound matrix

Definition 2 *[6] Let* A *be a square matrix,* $A \in \mathbb{R}^n$*, and consider any positive integer* k *such that* $k \le n$*. The* k*-additive compound matrix is the matrix* $A^{[k]} \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$ *defined as:*

$$A^{[k]} = \frac{d}{d\varepsilon}(I + \varepsilon A)^{(k)}\bigg|_{\varepsilon=0}.$$

Definition 2 implies that $A^{[k]} = \frac{d}{d\varepsilon}(\exp[A\varepsilon])^{(k)}|_{\varepsilon=0}$ (see [6]), so that:

$$(I + \varepsilon A)^{(k)} = I + \varepsilon A^{[k]} + o(\varepsilon)\ .$$

Therefore, $A^{[k]}$ is the first-order approximation of the Taylor expansion of $(I+\varepsilon A)^{(k)}$. The next proposition gives an explicit formula to construct the k-additive compound of a matrix.

Proposition 4 *[6] Let $\alpha, \beta \in Q(k,n)$ with $\alpha = \{i_1, \ldots, i_k\}$ and $\beta = \{j_1, \ldots, j_k\}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Then, the entry of $A^{[k]}$ corresponding to (α, β) is obtained as:*

- $\sum_{l=1}^{k} a_{i_l i_l}$, *if* $i_l = j_l \quad \forall l \in \{1, \ldots, k\}$
- $(-1)^{l+m} a_{i_l i_m}$, *if all the elements of α and β agree except for one element of index $i_l \neq j_m$*
- 0*, otherwise*

Example 2 *Let $A \in \mathbb{C}^{3\times 3}$ and $k=2$, then the 2-additive compound $A^{[2]}$ is equal to:*

$$Q(2,3) = \{\{1,2\},\{1,3\},\{2,3\}\}$$

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix}$$

Proposition 5 *[6] Let $A \in \mathbb{R}^{n\times n}$ and $k \leq n$ be a positive integer. The following properties hold true:*

1. $(A+B)^{[k]} = A^{[k]} + B^{[k]} \implies$ *the map $A \to A^{[k]}$ is linear.*
2. $A^{[1]} = A$.
3. $A^{[n]} = tr(A)$.

Similar to multiplicative compound, also additive compound matrices have an important spectral property.

Proposition 6 *[6] Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of A, then:*

$$\{\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k},\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is the spectrum of $A^{[k]}$. Furthermore, if $v_1, v_2, \ldots, v_k$ are independent eigenvector of A corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the wedge product $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ is an eigenvector of $A^{[k]}$ with corresponding eigenvalue $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

Example 3 *Let* $A \in \mathbb{R}^{3\times 3}$ *and let* $\{\lambda_1, \lambda_2, \lambda_3\}$ *be the eigenvalues of* A*. Then, the eigenvalues of* $A^{[2]}$ *are:*

$$\text{spec}(A^{[2]}) = \{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}$$

Remark 1 *It is worth noting that, due to the spectral property of additive compound matrices in Proposition 6, the* 2*-additive compound matrix* $A^{[2]}$ *can be Hurwitz only if the matrix* A *has maximum one eigenvalue with positive real part.*

1.4 Connection between compound matrices and linear time-varying systems

Besides interesting algebraic properties, compound matrices are noteworthy for their links with differential equations. Indeed, let $X(t) \in \mathbb{R}^{n\times m}$ be a matrix solution of the differential equation

$$\dot{x}(t) = A(t)x(t), \tag{1.1}$$

where $A(t)$ is a square matrix of compatible dimension. For any positive integer $k \leq \min\{n, m\}$ the following holds:

$$\dot{x}^{(k)}(t) = A^{[k]}(t)x^{(k)}(t). \tag{1.2}$$

Equation (1.2) highlights how the k order minors of the matrix $X(t)$ evolve according to linear dynamics, which are determined by the k-additive compound of the matrix $A(t)$.

Remark 2 *[6] If* $A(t) \equiv A$*, then* $X(t) = \exp[At]$ *and* $X^{(k)}(t) = (\exp[At])^{(k)}$*, from which follows the property:*

$$(\exp[At])^{(k)} = \exp[A^{[k]}t].$$

Furthermore, for $k = n$ *we obtain*

$$\frac{d}{dt}\det(X(t)) = tr(A(t))\det(X(t))$$

that is the Abel-Jacobi-Liouville identity.

1.4.1 Lozinskiĭ logaritmic norm

Let A be a matrix of order $n \times n$. The Lozinskiĭ logaritmic norm is defined as the right-hand derivative [36]:

$$\mu(A) = \lim_{\varepsilon \to 0^+} |I + \varepsilon A|_{\varepsilon=0} \tag{1.3}$$

where $|\cdot|$ is any norm in $\mathbb{R}^n$. The Lozinskiĭ norm has the following property.

Proposition 7 *[36] Let x(t) be a solution of the LTV system (1.1). Then:*

$$|x(t)| \exp\left(-\int_{t_0}^{t} \mu(A)\, dt\right) \quad , \quad |x(t)| \exp\left(\int_{t_0}^{t} \mu(-A)\, dt\right)$$

are non-increasing and non-decreasing, respectively. Subsequently, we have the following properties:

- *If the LTV system (1.1) is stable* $\implies \int_{t_0}^{t} \mu(A)\, dt \leq K$ $, t_0 \leq t < \infty$ *(K independent of t)*
- *If the LTV system (1.1) is asymptotically stable* $\implies \lim_{t \to +\infty} \int_{t_0}^{t} \mu(A)\, dt = -\infty$
- *If the LTV system (1.1) is uniformly stable* $\implies \int_{s}^{t} \mu(A)\, dt \leq M$ $, t_0 \leq s \leq t < \infty$ *(M independent of t and s)*

Replacing $\mu(A)$ with $-\mu(-A)$, we obtain necessary conditions for the LTV system (1.1) to have the corresponding stability properties.

Remark 3 *[36] The Lozinskiĭ norm $\mu(A)$ depends on the used norm. For the main norms, we have the following explicit formulas:*

- $|x|_\infty = \sup_i |x_i| \quad \Rightarrow \quad \mu_\infty(A) = \sup_i (\mathrm{Re}(a_{ii}) + \sum_{j \neq i} |a_{ij}|)$
- $|x|_1 = \sum_i |x_i| \quad \Rightarrow \quad \mu_1(A) = \sup_j (\mathrm{Re}(a_{jj}) + \sum_{i \neq j} |a_{ij}|)$
- $|x|_2 = (\sum_i |x_i|^2)^{\frac{1}{2}} \quad \Rightarrow \quad \mu_2(A) = \lambda_1$, *where* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *are the eigenvalues of $(A^* + A)/2$, A^* conjugate transpose of A*

In general, with the additive compound of A of order $k = 1, 2, \ldots, n$, we have the formulas:

$$\begin{cases} \mu_\infty(A^{[k]}) = \sup\limits_{\alpha \in Q(k,n)} \Big(\mathrm{Re}(a_{i_1 i_1} + \cdots + a_{i_k i_k}) + \sum_{j \notin \alpha}(|a_{i_1 j}| + \cdots + |a_{i_k j}|)\Big) \\ \mu_1(A^{[k]}) = \sup\limits_{\alpha \in Q(k,n)} \big(\mathrm{Re}(a_{j_1 j_1} + \cdots + a_{j_k j_k}) + \sum_{i \notin \alpha}(|a_{i j_1}| + \cdots + |a_{i j_k}|)\big) \\ \mu_2(A^{[k]}) = \lambda_1 + \cdots + \lambda_k \end{cases} \tag{1.4}$$

An important result is given by J. S. Muldowney in [36], where the stability properties of system (1.1) are provided in terms of the Lozinskiĭ norm of the k-additive compound matrices of A.

Corollary 1 *[36] Suppose there exists a constant M such that*

$$\int_s^t \mu(A)\,dt \leq M \ , \quad 0 \leq s \leq t < \infty \ ,$$

where M is independent of s and t. Then (1.1) has an $(n-k+1)-$dimensional set of solutions x such that $\lim\limits_{t\to\infty} x(t) = 0$ if

$$\liminf_{t\to\infty} \int_0^t \mu(A^{[k]})\,dt = -\infty$$

and only if, for $1 \leq l \leq n$,

$$\lim_{t\to\infty} \int_0^t \mu(-A^{[l]})\,dt = \infty$$

1.4.2 Connections between compound and skew-symmetric matrices

Definition 3 *[30] If $A \in \mathbb{R}^{n\times n}$ satisfies $A = A^\top$, then A is called symmetric. If $A = -A^\top$, then A is called skew-symmetric.*

For a $n \times n$ skew-symmetric matrix X we denote by $\vec{X}$ the $\binom{n}{2}$ column vector:

$$\vec{X} = [x_{12}, x_{13}, \ldots, x_{1n}, x_{23}, x_{24}, \ldots, x_{2n}, \ldots, x_{(n-1)n}]^T. \tag{1.5}$$

Instead, for a $m \times n$ rectangular matrix X, we denote by $\mathrm{vec}(X)$ the $n \cdot m$ column vector:

$$\mathrm{vec}(X) = [x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}]^T. \tag{1.6}$$

In [37] it has been shown that for any skew symmetric matrix $X \in R^{n\times n}$ there exists a matrix $M_n \in \mathbb{R}^{n^2\times\binom{n}{2}}$ such that

$$\mathrm{vec}(X) = M_n\vec{X}. \tag{1.7}$$

In particular, M_n is given as:

$$M_n = \sum_{1\le i\ne j\le n} \mathrm{sign}(j-i)e_{[(i-1)n+j]}e^T_{k(i,j)}$$

where

$$k(i,j) = |i-j| + \binom{n}{2} - \binom{n+1-\min\{i,j\}}{2}.$$

For clarity, matrix $M_4 \in \mathbb{R}^{16\times 6}$ is shown next:

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Conversely, there exists a matrix $L_n \in \mathbb{R}^{\binom{n}{2}\times n^2}$ such that for any skew-symmetric matrix $X \in \mathbb{R}^{n\times n}$ (see [37]):

$$\vec{X} = L_n \mathrm{vec}(X), \tag{1.8}$$

where L_n is given as:

$$L_n = \sum_{1\le i<j\le n} e_{k(i,j)}e^T_{[(i-1)n+j]}.$$

As an example, $L_4 \in \mathbb{R}^{6\times 16}$ is of the form:

$$L_4 = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix} \tag{1.9}$$

Skew-symmetric matrices are useful due to the following result linking them to 2-additive compound matrices. Consider the linear system:

$$\dot{x} = A\,x, \tag{1.10}$$

and assume that a given skew-symmetric matrix function X fulfills the matrix differential equation:

$$\dot{X} = AX + XA^T. \tag{1.11}$$

It can be verified that the linear operator $L(X) = AX + XA^T$ preserves skew symmetry. In particular, $L(X)^T = -L(X)$ for all skew-symmetric X. Moreover, the vector $\vec{X}$ in (1.5) fulfills the differential equation:

$$\dot{\vec{X}} = A^{[2]}\vec{X}, \tag{1.12}$$

where $A^{[2]}$ is the 2-additive compound matrix of A. Some applications of this operators on the stability of LTV systems can be found in [3, 9, 33, 37] and references therein.

1.5 Connection between compound matrices and nonlinear systems

Consider the nonlinear autonomous system

$$\dot{x} = f(x) \tag{1.13}$$

where $f \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$. Let $x = x(t, x_0)$, $x_0 \in \mathbb{R}^n$ be a the solution of (1.13) which satisfies $x(0, x_0) = x_0$. We consider the linear time-varying system

$$\dot{y} = J^{[k]}(x(t, x_0))y\ , \quad k = 1, \ldots, n \tag{1.14}$$

where $y \in \mathbb{R}^{\binom{n}{k}}$ and $J^{[k]}(x(t,x_0))$ is the k-additive compound of the Jacobian calculated along the trajectory $x(t,x_0)$.
We observe that, for $k = 1$, (1.14) is the standard variational equation of (1.13), while the other cases of (1.14) define the compound variational equations. In particular, when $k = n$, we have the scalar linear equation

$$\dot{y} = \mathrm{div} f(x(t,x_0))y \tag{1.15}$$

Remark 4 *[36] The matrix $Y(t) = \partial x(t,x_0)/\partial x_0$ is a fundamental matrix for the standard variational equation and satisfies $Y(0) = I$. Thus, $Y^{(k)}(t)$, the matrix of Jacobian determinants $\partial(x_{i_1},\dots,x_{i_k})(t,x_0)/\partial(x_{0j_1},\dots,x_{0j_k})$, $1 \le i_1 < \cdots < i_k \le n$, $1 \le j_1 < \cdots < j_k \le n$ is the fundamental matrix for (1.14) satisfying $Y^{(k)}(0) = I$. From (1.15) we have the formula:*

$$\frac{\partial(x_1,\dots,x_n)}{\partial(x_{01},\dots,x_{0n})}(t,x_0) = \exp\left[\int_0^t div\, f(x(s,x_0))\, ds\right]$$

As it was observed in the linear case, equations (1.14) can be used to describe the local evolution in $\mathbb{R}^n$ of the measure of k-dimensional surfaces under the dynamics of the nonlinear system (1.13). The following two theorems extend the classical result of Bendixson and Poincarè to higher dimensions [36].

Theorem 1 *[36] Suppose that one of the inequalities*

$$\mu\left(\frac{\partial f^{[2]}}{\partial x}\right) < 0 \quad , \quad \mu\left(-\frac{\partial f^{[2]}}{\partial x}\right) < 0$$

holds for all $x \in \mathbb{R}^n$. Then the system (1.13) has no nonconstant periodic solutions.

Theorem 2 *[36] A sufficient condition for a periodic trajectory $\gamma = \{p(t) : 0 \le t \le \omega\}$ of (1.13) to be orbitally asymptotically stable is that the linear system*

$$\dot{y} = \frac{\partial f^{[2]}}{\partial x}(p(t))y$$

is asymptotically stable.

Corollary 2 *[36] Suppose that, for some Lozinskiĭ norm μ,*

$$\int_0^\omega \mu\left(\frac{\partial f^{[2]}}{\partial x}(p(t))\right) dt < 0\ .$$

Then, γ is orbitally asymptotically stable.

If some boundedness assumptions on the solutions of the nonlinear system (1.13) are enforced, then sufficient conditions ensuring convergence towards the equilibrium points can be given under the hypotheses of Theorem 1 [52].

Theorem 3 *[28] Consider the nonlinear time-invariant system (1.13), where* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $\mathcal{C}^1$*. Assume that its trajectories evolve on a convex and compact set* Ω*, and that*

$$\mu\left(J^{[2]}(x)\right) < 0 \text{ for all } x \in \Omega.$$

Then every solution emanating from Ω *converges to the set of equilibria. If in addition there exists a unique equilibrium* $e \in \Omega$ *then every solution emanating from* Ω *converges to* e*.*

Chapter 2

Connection between compound matrices and Lyapunov exponents

As discussed in the final part of the introduction, this chapter is devoted to highlight the relation between the algebra of compound matrices and the Lyapunov exponents of a nonlinear system, in order to tackle the problem of analysing some dynamical properties of the solutions in a multistability scenario, ruling out the presence of positive Lyapunov exponents, also including the case of convergence towards the equilibrium points. Firstly, the considered problem is formulated and the relation between the Lyapunov exponents of the standard variational equation and those of the second compound variational equation is introduced in Proposition 8. Then, the main results are presented in Theorem 4 and Theorem 5, which provide conditions ensuring the absence of positive Lyapunov exponents, while some remarks and insights are given in Subsection 2.2.1. Subsequently, the extension of the method to systems with a first integral of motion is provided in Theorem 6. The material of the Chapter has been collected and published in the scientific papers [34, 35].

2.1 Problem formulation

Consider the dynamical system described by the following set of first-order differential equations:

$$\dot{x}(t) = f(x(t)), \tag{2.1}$$

with state $x(t) \in X \subset \mathbb{R}^n$ for some open set X and initial condition $x(0) = x_0$ and $f : X \to \mathbb{R}^n$ of class $\mathcal{C}^1$. As customary, one may associate to 2.1 a *variational equation*:

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) \\ \dot{\delta}(t) & = & \frac{\partial f}{\partial x}(x(t))\delta(t), \end{array} \tag{2.2}$$

whose the component $\delta(t) \in \mathbb{R}^n$ propagates, along solutions of (2.1), the effect of infinitesimal perturbations of initial conditions in the direction of $\delta(0)$. A matrix version of (2.2) is also handy, and defined as follows:

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) \\ \dot{X}(t) & = & \frac{\partial f}{\partial x}(x(t))X(t), \\ & & \\ X(0) & = & I_n, \end{array} \tag{2.3}$$

where $X(t) \in \mathbb{R}^{n\times n}$ is the transition matrix of the 'time-varying' linear system describing the evolution of δ in (2.2) and I_n denotes the identity matrix of order n. In particular, $\delta(t)$, solution of (2.2) for a given $x(0)$ and $\delta(0)$, fulfills $\delta(t) = X(t)\delta(0)$, where $X(t)$ is a solution of (2.3) for the same initial condition $x(0)$. Indeed, we have

$$\frac{d}{dt}X(t)\delta(0) = \dot{X}(t)\delta(0) = \frac{\partial f}{\partial x}(x(t))X(t)\delta(0).$$

Definition 4 *Lyapunov Exponents: for a given initial condition $x(0)$ (and assuming $x(t)$ is uniformly bounded over $[0, +\infty)$), n non-zero (and orthogonal) vectors $v_1, v_2, \dots, v_n$, and n real numbers, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, are defined (the Lyapunov exponents), such that:*

$$\lambda_i = \lim_{t\to+\infty} \frac{\log(|X(t)v_i|)}{t}, \tag{2.4}$$

where $|\cdot|$ denotes the Euclidean norm of a vector.

Conditions for existence of the Lyapunov exponents are relatively mild and have been first established in [40] and later generalised in [44]. Lyapunov exponents are invariant under fairly general coordinate transformations and along solutions, thus providing an intrinsic characterization of how an attractor responds to perturbations of initial conditions in different directions. Finding a-priori bounds on their location yields insight into the dynamics of a system and is normally approached through the study of the variational equation (2.2), see for instance [26], where a direct estimation of the growth rate of the variational equation is proposed.

We propose to study the Lyapunov exponents by using an alternative, possibly higher-order, linearized dynamics, involving the 2-additive compound matrix $\frac{\partial f}{\partial x}^{[2]}$ of the Jacobian. In particular,

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) \\ \dot{\eta}(t) & = & \frac{\partial f}{\partial x}^{[2]}(x(t))\eta(t), \end{array} \tag{2.5}$$

where $\eta(t) \in \mathbb{R}^{\binom{n}{2}}$, can be interpreted, when ascribable as $\delta_1(t) \wedge \delta_2(t) := [\delta_1(t), \delta_2(t)]^{(2)}$ for some solutions δ_1, δ_2 of (2.2), as the propagation in time of an infinitesimal area perturbation in the direction of $\delta_1(0)$, $\delta_2(0)$. In particular, if $\eta(0) = \delta_1(0) \wedge \delta_2(0)$, then $\eta(t) = \delta_1(t) \wedge \delta_2(t)$. A matrix version of (2.5) can also be introduced as follows:

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) \\ \dot{Y}(t) & = & \frac{\partial f}{\partial x}^{[2]}(x(t))Y(t), \\ Y(0) & = & I_{\binom{n}{2}} \end{array} \tag{2.6}$$

where $Y(t) \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}}$. Notice that $Y(t) = X^{(2)}(t)$, the 2-multiplicative compound matrix of $X(t)$. Our objective is to use (2.6) and its Lyapunov exponents in order to gain insight into the Lyapunov exponents of (2.3).

Definition 5 *[34] Lyapunov Exponents of (2.6): for a given initial condition* $x(0)$ *(and assuming* $x(t)$ *is uniformly bounded over* $[0, +\infty)$*), there exists* $\binom{n}{2}$ *non-zero (and orthogonal) vectors* $w_{ij} \in \mathbb{R}^{\binom{n}{2}}$, $1 \leq i < j \leq n$, *and* $\binom{n}{2}$ *real numbers,* μ_{ij}, *(the Lyapunov exponents of (2.6)), such that:*

$$\mu_{ij} = \lim_{t \to +\infty} \frac{\log(|Y(t)w_{ij}|)}{t}, \tag{2.7}$$

where $|\cdot|$ *denotes the Euclidean norm of a vector.*

More specifically, we aim to find a bound on the maximal Lyapunov exponent of (2.2) by expressing suitable conditions to bound the Lyapunov exponents of (2.5). While the proposed approach is tight, dimension of Y grows quadratically in n, potentially making its application challenging in high-dimensional examples.

2.2 Ruling out positive Lyapunov exponents by means of 2-additive compound

In order to accomplish our aims, deriving a link between the Lyapunov spectra of (2.2) and (2.5) is essential.

Proposition 8 *[34] For any given initial condition $x(0)$, such that $x(t)$ is uniformly bounded, the following relationship holds, (for a suitable ordering of the μ_{ij} and rescaling of vectors) :*

$$\begin{aligned} \mu_{ij} &= \lambda_i + \lambda_j, \qquad 1 \leq i < j \leq n \\ w_{ij} &= v_i \wedge v_j \end{aligned} \tag{2.8}$$

In order to make full use of the powerful algebraic machinery of compound matrices, we adopt the following equivalent characterization of Lyapunov exponents. This will be suitable for addressing the proof of Proposition 8.

Proposition 9 *For any given initial condition $x(0)$ the Lyapunov exponents of a matrix variational equation of solution $X(t)$ and the corresponding directions are given by the spectrum of the symmetric real matrix:*

$$\bar{H} := \lim_{t \to +\infty} \frac{\log\left(X(t)^T X(t)\right)}{2t} \tag{2.9}$$

and by its associated orthonormal basis of eigenvectors.

Accordingly, the Lyapunov exponents of the 2-additive compound variational equation (2.6) can be computed as follows.

Proposition 10 *[34] For any given initial condition $x(0) = x_0$ the Lyapunov exponents of (2.6) and the corresponding orthogonal directions are given by the spectrum of the symmetric real matrix*

$$\bar{M} := \lim_{t \to +\infty} \frac{\log\left(Y(t)^T Y(t)\right)}{2t} \tag{2.10}$$

and by its associated orthonormal basis of eigenvectors.

The equivalence between the definition of the Lyapunov exponents in Definition 5 and the one in Proposition 9 can be found in [15] and [44], where the Lyapunov exponents are defined as the logarithm of the eigenvalues of the following limit matrix $\Lambda(x_0)$

$$\Lambda(x_0) = \lim_{t\to+\infty} (X(t,x_0)^T X(t,x_0))^{\frac{1}{2t}} \tag{2.11}$$

where the notation $X(t, x_0)$ is adopted to remark that the solution of the matrix varational equation depends on the initial condition $x(0) = x_0$.
Proof of Proposition 8 is here reported since it is useful for a better understanding of the analysis.
Proof.(Proposition 8) By observing that $Y(t) = X^{(2)}(t)$ and exploiting the definition of the related Lyapunov exponents in Proposition 10 we have

$$\begin{aligned}
\bar{M} &= \lim_{t\to+\infty} \frac{\log\left(Y^T(t)Y(t)\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\log\left(\left(X^{(2)}(t)\right)^T X^{(2)}(t)\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\log\left(\left(X^T(t)\right)^{(2)} X^{(2)}(t)\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\log\left(\left(X^T(t)X(t)\right)^{(2)}\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\log\left(\left(\exp^{\log(X^T(t)X(t))}\right)^{(2)}\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\log\left(\exp^{\left[\log(X^T(t)X(t))\right]^{[2]}}\right)}{2t} \\
&= \lim_{t\to+\infty} \frac{\left[\log\left(X^T(t)X(t)\right)\right]^{[2]}}{2t} = \bar{H}^{[2]}
\end{aligned}$$

where the third and fifth properties in Proposition 1, the exponential property in Remark 2 and the definition of the Lyapunov exponents in Proposition 9 have been exploited. The first part of the claim is completed by recalling the spectral property of 2-additive compound. Let v_i and v_j be eigenvectors of $\bar{H}$ associated to eigenvalues λ_i and λ_j respectively. Then:

$$\bar{H}[v_i, v_j] = [v_i, v_j]\mathrm{diag}(\lambda_i, \lambda_j).$$

We exploit the latter equation in the following derivation:

$$
\begin{aligned}
\bar{M}(v_i \wedge v_j) = \bar{H}^{[2]}[v_i, v_j]^{(2)} &= \frac{d}{d\varepsilon}(I + \varepsilon \bar{H})^{(2)}\Big|_{\varepsilon=0} [v_i, v_j]^{(2)} \\
&= \frac{d}{d\varepsilon}\big((I + \varepsilon \bar{H})[v_i, v_j]\big)^{(2)}\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\big([v_i, v_j](I + \varepsilon\, \mathrm{diag}(\lambda_i, \lambda_j))\big)^{(2)}\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}[v_i, v_j]^{(2)}(I + \varepsilon\, \mathrm{diag}(\lambda_i, \lambda_j))^{(2)}\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}[v_i, v_j]^{(2)}(1 + \varepsilon\lambda_i)(1 + \varepsilon\lambda_j)\Big|_{\varepsilon=0} = (\lambda_i + \lambda_j)(v_i \wedge v_j).
\end{aligned}
$$

This concludes the proof of the Proposition. ■

Proposition 8 characterizes the Lyapunov exponents of the variational equation (2.5), i.e. of its matrix version (2.6). Our goal, however, is to find a bound to the maximal Lyapunov exponent of (2.2), i.e. of its matrix version (2.3). To this end we exploit that whenever the ω-limit set is non trivial (see e.g. [46] for a definition) there always exists a 0 Lyapunov exponent. This well known fact is recalled in the Lemma that follows for the sake of completeness.

Lemma 1 *Consider the non-linear dynamical system with its associate variational equation (2.3) with $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathcal{C}^1$. Let $\omega(x_0)$ be the ω-limit set and let $f(x) \neq 0$ for all $x \in \omega(x_0)$ (i.e. the ω-limit set does not contain equilibria). Then, there exists $i \in \{1, \ldots, n\}$ such that $\lambda_i = 0$, where $\lambda_1, \ldots, \lambda_n$ are the Lyapunov exponents of the system (2.1).*

Proof. First, we observe that the function $f(x(t))$ is a solution of the variational equation, since

$$
\dot{f}(x(t)) = \frac{\partial f}{\partial x}(x(t))\dot{x}(t) = \frac{\partial f}{\partial x}(x(t))f(x(t)) \ .
$$

Hence, it follows that

$$
f(x(t)) = X(t)f(x(0)).
$$

For systems with bounded solutions, the ω-limit set is compact. Therefore $f(x)$ is uniformly bounded for $x \in \omega(x_0)$ and, since by hypothesis $f(x) \neq 0$ in $\omega(x_0)$, we have for sufficiently small ε and sufficiently large M:

$$
0 < \varepsilon \leq |f(x(t))| \leq M, \qquad \forall\, t \geq 0.
$$

Considering the direction $v = f(x(0))$, we have that

$$\lambda = \lim_{t\to+\infty} \frac{\log(|X(t)v|)}{t} = \lim_{t\to+\infty} \frac{\log(|f(x(t))|)}{t}$$
$$\leq \lim_{t\to+\infty} \frac{\log(M)}{t} = 0$$

and

$$\lim_{t\to+\infty} \frac{\log(|f(x(t))|)}{t} \geq \lim_{t\to+\infty} \frac{\log(\varepsilon)}{t} = 0 \ .$$

Hence, it follows that

$$0 = \lim_{t\to+\infty} \frac{\log(\varepsilon)}{t} \leq \lambda \leq \lim_{t\to+\infty} \frac{\log(M)}{t} = 0$$

which implies $\lambda = 0$. Therefore, there exists at least one Lyapunov exponent equal to zero. ■

We are now ready to state our main result.

Theorem 4 *[34] Consider the system*

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{\delta}^{(2)}(t) &= \frac{\partial f}{\partial x}^{[2]}(x(t))\delta^{(2)}(t), \end{aligned} \tag{2.12}$$

with state $x(t) \in \mathbb{R}^n$ and initial condition $x(0) = x_0$. Assume further that $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$ and $f(x) \neq 0$ for all $x \in \omega(x_0)$. Let $V : \mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}$ of class $\mathcal{C}^1$ and let $\alpha_1,\ \alpha_2 : [0, +\infty) \to [0, +\infty)$ functions of class $\mathcal{K}_\infty$ such that

- $\alpha_1\left(\left|\delta^{(2)}\right|\right) \leq V\left(x, \delta^{(2)}\right) \leq \alpha_2\left(\left|\delta^{(2)}\right|\right)$
- $\dot{V}\left(x, \delta^{(2)}\right) \leq 0$

for all $x \in \mathbb{R}^n$ and $\delta^{(2)} \in \mathbb{R}^{\binom{n}{2}}$, where

$$\dot{V}(x, \delta^{(2)}) := \frac{\partial V}{\partial x}(x, \delta^{(2)})f(x) + \frac{\partial V}{\partial \delta^{(2)}}(x, \delta^{(2)})\frac{\partial f}{\partial x}^{[2]}(x)\delta^{(2)} \ .$$

Then, all Lyapunov exponents relative to the considered solution are less or equal than 0.

Proof. By Lemma 1 it follows that 0 is a Lyapunov exponent of (2.2) and an eigenvalue of $\left(\lim_{t\to+\infty} \frac{\log\left(X(t)^T X(t)\right)}{2t}\right)$.
From the assumptions on the candidate Lyapunov function V, it follows that along any solution of (2.12).

$$\begin{aligned}\alpha_1(|\delta^{(2)}(t)|) &\leq V(x(t), \delta^{(2)}(t)) \\ &\leq V(x(0), \delta^{(2)}(0)) \leq \alpha_2(|\delta^{(2)}(0)|).\end{aligned}$$

Hence $|\delta^{(2)}(t)| \leq \alpha_1^{-1}(\alpha_2(|\delta^{(2)}(0)|))$, proving uniform boundedness of $|\delta^{(2)}(t)|$. By Proposition 8 follows that

$$\mu_{ij} = \lim_{t\to+\infty} \frac{\log\left(|Y(t)w_{ij}|\right)}{t} = \lambda_i + \lambda_j$$

where $w_{ij} = v_i \wedge v_j$ and λ_i, λ_j with $1 \leq i < j \leq n$ are the Lyapunov exponents of the system (2.1). Hence, denoting by $M := \alpha_1^{-1}(\alpha_2(|Y(0)w_{ij}|))$ the uniform bound on $|Y(t)w_{ij}|$, we see that:

$$\lim_{t\to+\infty} \frac{\log\left(|Y(t)w_{ij}|\right)}{t} \leq \lim_{t\to+\infty} \frac{\log\left(M\right)}{t} = 0$$

for all $1 \leq i < j \leq n$. From the last inequality it follows that

$$\lambda_i + \lambda_j \leq 0 \quad \forall\, i \neq j \ .$$

In particular, by Lemma 1, there exists at least one $\lambda_i = 0$, and therefore we have

$$0 + \lambda_j \leq 0 \quad \forall\, j \ .$$

As a consequence, all Lyapunov exponents are less than or equal to zero, which concludes the proof of the claim. ∎

When the Lyapunov function V in Theorem 4 is quadratic with respect to $\delta^{(2)}$, one can formulate the condition on $\dot{V}$ through suitable Linear Matrix Inequalities involving the 2-additive compound matrix of the Jacobian. We formulate this in the next Theorem.

Theorem 5 *[34] Consider system (2.1) with state $x \in \mathbb{R}^n$ and initial condition $x(0) = x_0$. Assume that $f(x) \neq 0$ for all $x \in \omega(x_0)$. Consider a state-dependent symmetric matrix $P(x) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ such that:*

$$P(x) \geq \varepsilon I \tag{2.13}$$

for all x in a neighborhood of $\omega(x_0)$, and assume:

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P(x) + P(x)\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) + \dot{P}(x) \leq 0 \qquad (2.14)$$

then all Lyapunov exponents relative to the considered solution are less or equal than 0*.*

Proof. The result simply follows by defining function $V(x, \delta^{(2)}) := (\delta^{(2)})^T P(x)\delta^{(2)}$ and applying Theorem 4. Indeed taking derivatives of V along solutions of (2.12) we see that:

$$\dot{V} = \delta^{(2)T}\left[\frac{\partial f}{\partial x}^{[2]T} P(x) + P(x)\frac{\partial f}{\partial x}^{[2]} + \dot{P}(x)\right]\delta^{(2)} \leq 0,$$

where the last inequality follows by (2.14). ■

2.2.1 Some insights on the result

Remark 5 *Theorem 5 provides a sufficient condition to the problem of assessing if, inside some given invariant set $\mathcal{D}$ of (2.1), all the attractors whose ω-limit set does not contain equilibria, have no positive Lyapunov exponents. Furthermore, if condition (2.14) holds with the strict inequality, then exploiting the results in [1], it can be shown that (2.1) is a* non-oscillatory system *inside $\mathcal{D}$. Hence, in this case all the solutions $x(t)$ with $x(0) = x_0 \in \mathcal{D}$ converge towards the equilibria in $\mathcal{D}$, except for some particular dynamical behaviors such as certain kinds of homoclinic and heteroclinic orbits. Clearly, if all the solutions $x(t)$ of (2.1) with $x_0 \notin \mathcal{D}$ converge towards the invariant set $\mathcal{D}$, then such a dynamical scenario is enjoyed by system (2.1).*

We provide now further insights on condition (2.14) which are useful for the successive developments. It is worth noting that for each $x \in \mathcal{D}$ condition (2.14) amounts to solve a (Lyapunov-like) Linear Matrix Inequality (LMI). In particular, the following result holds for the equilibrium points of system (2.1).

Proposition 11 *[35] Let $x_e \in \mathcal{D}$ be an equilibrium point for system (2.1).*

1. *If the Jacobian $\frac{\partial f}{\partial x}(x_e)$ is Hurwitz, i.e., x_e is an asymptotically stable equilibrium point, then also $\frac{\partial f}{\partial x}^{[2]}(x_e)$ is Hurwitz.*

2. *Condition (2.14) holds for some $P(x)$ only if the 2-additive compound* $\dfrac{\partial f}{\partial x}^{[2]}(x_e)$ *is marginally stable.*

Proof. Condition 1) directly follows from the spectral relations between the Jacobian $\dfrac{\partial f}{\partial x}(x_e)$ and its 2-additive compound $\dfrac{\partial f}{\partial x}^{[2]}(x_e)$. In fact, we have that if $\lambda_1, \ldots, \lambda_n$, are the eigenvalues of $\dfrac{\partial f}{\partial x}(x_e)$ then the eigenvalues of $\dfrac{\partial f}{\partial x}^{[2]}(x_e)$ are $\mu_{ij} = \lambda_i + \lambda_j$, $1 \le i < j \le n$.
For condition 2) it can be observed that at the equilibrium point, $\dot{P}(x_e) = \sum_{i=1}^{n} \dfrac{\partial P}{\partial x_i}(x_e) \cdot f_i(x_e) = 0$ and hence condition (2.14) boils down to

$$\left(\frac{\partial f}{\partial x}^{[2]}(x_e)\right)^T P(x_e) + P(x_e)\left(\frac{\partial f}{\partial x}^{[2]}(x_e)\right) \le 0 \; . \tag{2.15}$$

It is known that Lyapunov equation (2.15) is solved for some positive definite $P(x_e)$ only if $\dfrac{\partial f}{\partial x}^{[2]}(x_e)$ is marginally stable. ■

Remark 6 *It is worth noting that, from Proposition 11, the 2-additive compound* $\dfrac{\partial f}{\partial x}^{[2]}(x_e)$ *has to be marginally stable for all the equilibrium points inside the invariant set $\mathcal{D}$. Therefore, as a consequence of Remark 1, the Jacobian of the system* $\dfrac{\partial f}{\partial x}(x_e)$*, in every equilibrium point inside $\mathcal{D}$, can have only one eigenvalue with positive real part.*

The matrix $P(x)$ in condition (2.14) can be any state-dependent symmetric matrix, but taking suitable forms can simplify the computational effort. In this thesis, two forms of the matrix $P(x)$ are considered, one constant $P(x) = P$ and one state-dependent $P(x) = P\exp[V(x)]$ where $V(x) : \mathbb{R}^n \to \mathbb{R}$.

Proposition 12 *[35]*

1. *If $P(x) = P$, $P \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ symmetric and positive definite, then condition (2.14) boils down to*

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P + P\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) \le 0 \quad \forall x \in \mathcal{D} \; . \tag{2.16}$$

2. If $P(x) = P \cdot \exp[V(x)]$, $P \in \mathbb{R}^{\binom{n}{2} \times \binom{n}{2}}$ symmetric and positive definite, and $V(x) : \mathbb{R}^n \to \mathbb{R}$ is of class C^1. Then, condition (2.14) reduces to

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P + P\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) + \dot{V}(x) P \leq 0 \quad \forall x \in \mathcal{D} \ . \tag{2.17}$$

Proof. It readily follows by computing $\dot{P}(x)$ in both cases. ∎

Remark 7 *It is worth noting that condition (2.16) is solved for some constant symmetric and positive definite matrix P only if the 2-additive compound of the Jacobian $\frac{\partial f}{\partial x}^{[2]}(x)$ is marginally stable for all $x \in \mathcal{D}$. Indeed, the sought matrix P exists if all the 2-additive compounds $\frac{\partial f}{\partial x}^{[2]}(x)$, $x \in \mathcal{D}$, share a common quadratic Lyapunov function. Also, since $\dot{V}(x)$ vanishes at the equilibrium points $x_e \in \mathcal{D}$, it turns out that condition (2.17) holds only if the 2-additive compounds $\frac{\partial f}{\partial x}^{[2]}(x_e)$, $x_e \in \mathcal{D}$, share a common quadratic Lyapunov function.*

Remark 8 *It is worth noting that if $\frac{\partial f}{\partial x}^{[2]}(x)$ depends affine linearly on x and the invariant set $\mathcal{D}$ is the convex hull of given vertices $x^{(i)} \in \mathbb{R}^n$, $i = 1, \ldots, l$, condition (2.16) greatly simplifies from a computational point of view. Indeed, condition (2.16) boils down to*

$$\left(\frac{\partial f}{\partial x}^{[2]}(x^{(i)})\right)^T P + P\left(\frac{\partial f}{\partial x}^{[2]}(x^{(i)})\right) \leq 0 \quad \forall x^{(i)},\, i = 1, \ldots, l \ , \tag{2.18}$$

which amounts to solve a finite number of LMIs, a problem for which efficient software is available. Finally, observe that a similar conclusion can be reached for condition (2.17) once also $\dot{V}(x)$ depends affine linearly on x.

2.3 Extension of the method to systems with a first integral of motion

Dynamic systems with a first integral of motion are characterized by some non-trivial function of state which is constant along solutions. This implies that, for each initial condition, the system's state is confined to evolve on a

single leaf of manifolds foliation. Such a peculiar structure of the system's state space makes it possible to investigate its dynamics in a space of reduced dimension. In particular, this is very useful when the 2-additive compound of the Jacobian is employed, since its dimension is equal to $r = \binom{n}{2}$ with n the dimension of the system. Hence, it grows quadratically as n increases. For example, a system with dimension $n = 4$ and a first integral of motion evolves in a submanifold of dimension $n = 3$, so that the 2-additive compound of the Jacobian is of dimension 3×3 instead of 6×6.
Consider the nonlinear autonomous system (2.1) with its associated variational equation (2.2), i.e.

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{\delta}(t) &= \frac{\partial f}{\partial x}(x(t))\delta(t) \end{aligned} \tag{2.19}$$

and assume that for some $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ of class C^1 the following equality

$$\frac{\partial g}{\partial x}(x) \cdot f(x) = 0 \ , \quad \forall x \in \mathbb{R}^n \ , \tag{2.20}$$

holds with non-zero gradient (globally or on some forward invariant open set). Then, it follows that the variation in time of $g(x)$ is equal to zero. That is

$$\frac{d}{dt}g(x(t)) = \frac{\partial g}{\partial x}(x(t)) \cdot \dot{x}(t) = \frac{\partial g}{\partial x}(x(t)) \cdot f(x(t)) = 0 \ , \quad \forall t \geq 0 \ . \tag{2.21}$$

Hence, the solution $x(t)$ of (2.19) with initial condition $x(0) = x_0$ is constrained to evolve $\forall t \geq 0$ onto the invariant manifold described by

$$\{x \in \mathbb{R}^n : g(x) = g(x_0)\} \ . \tag{2.22}$$

The next result shows that also the variational equation in (2.19) satisfies a similar constraint.

Lemma 2 *Consider system (2.19) and let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ (of class C^2) satisfy (2.20). Then,*

$$\frac{\partial g}{\partial x}(x(t)) \cdot \delta(t) = \frac{\partial g}{\partial x}(x(0)) \cdot \delta(0) \ , \quad \forall t \geq 0 \ . \tag{2.23}$$

Proof. Observe that from (2.20) the following equality

$$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}(x) \cdot f(x)\right) = \frac{\partial g}{\partial x}(x)\frac{\partial f}{\partial x}(x) + f^\top(x)\frac{\partial^2 g}{\partial x^2}(x) = 0 \tag{2.24}$$

can be derived. Since the time-derivative of the left-hand side of (2.23) reads

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial g}{\partial x}(x(t))\cdot\delta(t)\right)=f^{\top}(x(t))\frac{\partial^2 g}{\partial x^2}(x(t))\delta(t)+\frac{\partial g}{\partial x}(x(t))\frac{\partial f}{\partial x}(x(t))\delta(t)\\&=\left(\frac{\partial g}{\partial x}(x(t))\frac{\partial f}{\partial x}(x(t))+f^{\top}(x(t))\frac{\partial^2 g}{\partial x^2}(x(t))\right)\delta(t)\end{aligned},$$

from (2.24) it follows that

$$\frac{d}{dt}\left(\frac{\partial g}{\partial x}(x(t))\cdot\delta(t)\right)=0\ ,$$

thus proving (2.23). ■

Remark 9 *It is worth noting that if $\delta(0)$ is tangential to the invariant manifold at $x = x_0$, viz. $\frac{\partial g}{\partial x}(x(0))\delta(0) = 0$, condition (2.23) boils down to $\dfrac{\partial g}{\partial x}(x(t)) \cdot \delta(t) = 0$, $\forall t \geq 0$.*

To characterize the reduced dynamics onto the invariant manifolds we consider a local change of coordinates for the infinitesimal perturbation in the direction of δ:

$$\delta(t) := T(x(t))\tilde{\delta}(t)\ , \tag{2.25}$$

where $\tilde{\delta} \in \mathbb{R}^n$ are the new coordinates. Specifically, the transformation matrix $T(x)$ is defined as

$$T(x) := \left(\left(\frac{\partial g}{\partial x}\right)^{\perp}, \left(\frac{\partial g}{\partial x}\right)^{\top}\right)\ , \tag{2.26}$$

where the columns of the matrix $\left(\dfrac{\partial g}{\partial x}\right)^{\perp} \in \mathbb{R}^{n\times n-1}$ represent one of the several possible local bases for the tangent plane at the invariant manifold, i.e. $\left(\dfrac{\partial g}{\partial x}\right)\cdot\left(\dfrac{\partial g}{\partial x}\right)^{\perp} = 0$. From Remark 9 it follows that all the perturbations $\delta(t)$ belonging to the invariant manifold are obtained from (2.26) once $\tilde{\delta}(t)$ has the following form

$$\tilde{\delta}(t) = \left((\tilde{\delta}_*(t))^{\top}, 0\right)^{\top}\ , \quad \tilde{\delta}_* \in \mathbb{R}^{n-1}\ , \tag{2.27}$$

i.e. its nth-coordinate is equal to zero. By computing the time-derivative of (2.25) we derive the following equalities

$$\begin{aligned}&\dot{\delta}(t) = T(x(t))\dot{\tilde{\delta}}(t) + \dot{T}(x(t))\tilde{\delta}(t) \Rightarrow \frac{\partial f}{\partial x}(x(t))\delta(t) = T(x(t))\dot{\tilde{\delta}}(t) + \dot{T}(x(t))\tilde{\delta}(t) \\ &\Rightarrow \frac{\partial f}{\partial x}(x(t))T(x)\tilde{\delta}(t) = T(x(t))\dot{\tilde{\delta}}(t) + \dot{T}(x(t))\tilde{\delta}(t) \ . \end{aligned} \tag{2.28}$$

From the last one we get that the variational equation in the new coordinates $\tilde{\delta}$ enjoys the following form

$$\dot{\tilde{\delta}}(t) = H(x(t))\tilde{\delta}(t) \ , \tag{2.29}$$

where $H(x) \in \mathbb{R}^{n\times n}$ is given by

$$H(x) := T^{-1}(x(t))\left(\frac{\partial f}{\partial x}(x(t))T(x(t)) - \dot{T}(x(t))\right) \ . \tag{2.30}$$

Clearly, (2.29) holds also when $\tilde{\delta}$ has the form (2.27), i.e., $\delta(t)$ is constrained to lie on the invariant manifold. This implies that $H(x)$ enjoys the following structure:

$$H(x) = \begin{pmatrix} * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} ,$$

which in turn yields that the variational equation onto the invariant manifold is completely characterized by the following reduced form

$$\dot{\tilde{\delta}}_*(t) = \tilde{J}(x(t))\tilde{\delta}_*(t) \tag{2.31}$$

where $\tilde{J}(x) \in \mathbb{R}^{n-1\times n-1}$ is obtained eliminating the nth-row and nth-column of the matrix $H(x)$ in (2.30) and it represents the Jacobian of reduced dimension. Hence, associating the variational equation (2.31) to system (2.1) allows one to state the next result, which has validity onto the invariant manifolds.

Theorem 6 *[35] Consider system (2.1) with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a positively invariant set of (2.1) and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be such that (2.20) holds. Suppose that there exists a state-dependent* *symmetric matrix* $P(x) \in \mathbb{R}^{\binom{n-1}{2} \times \binom{n-1}{2}}$ *which is positive definite for all* $x \in \mathcal{D}$ *and such that:*

$$\left(\tilde{J}^{[2]}(x)\right)^{\top} P(x) + P(x)\tilde{J}^{[2]}(x) + \dot{P}(x) \leq 0 \quad \forall x \in \mathcal{D} \ , \tag{2.32}$$

where $\tilde{J}^{[2]}(x)$ *is the 2-additive compound of* $\tilde{J}(x)$ *in (2.31). Then, all the Lyapunov exponents relative to each solution* $x(t)$ *of (2.1), with initial condition* $x_0 \in \mathcal{D}$ *such that* $f(x) \neq 0$ *for all* $x \in \omega(x_0)$*, are less or equal than* 0.

Proof. It follows by observing that the variational equation (2.31) plays the role of (2.2) in Theorem 4. For the last Lyapunov exponent, recall that:

$$H(x) = T^{-1}(x) \left(\frac{\partial f}{\partial x}(x)T(x) - \dot{T}(x) \right) \ . \tag{2.33}$$

We aim to compute:

$$H_{nn}(x) = e_n^T H(x) e_n \tag{2.34}$$

Notice that $T(x)e_n = \frac{\partial g}{\partial x}(x)^T$. Also:

$$\frac{\partial^2 g}{\partial x^2}(x) f(x) + \frac{\partial f}{\partial x}(x) \frac{\partial g}{\partial x}(x)^T = 0 \tag{2.35}$$

Since $T(x) = \left[\frac{\partial g}{\partial x}(x)^{\perp}, \frac{\partial g}{\partial x}(x)^T\right]$ we have:

$$e_n^T T^{-1}(x) = \frac{\frac{\partial g}{\partial x}(x)}{\left|\frac{\partial g}{\partial x}(x)\right|^2}. \tag{2.36}$$

Hence, exploiting the previous expressions yields:

$$\begin{aligned} H_{nn}(x) &= e_n^T T^{-1}(x) \left(\frac{\partial f}{\partial x}(x) T(x) - \dot{T}(x) \right) e_n \\ &= \frac{\frac{\partial g}{\partial x}(x)}{\left|\frac{\partial g}{\partial x}(x)\right|^2} \left(\frac{\partial f}{\partial x}(x) \frac{\partial g}{\partial x}(x)^T - \dot{\overline{\frac{\partial g}{\partial x}}}^T \right) \\ &= -2 \frac{\frac{\partial g}{\partial x}(x) \dot{\overline{\frac{\partial g}{\partial x}}}(x)^T}{\left|\frac{\partial g}{\partial x}(x)\right|^2} = -\frac{\partial}{\partial x} \log\left(\left|\frac{\partial g}{\partial x}(x)\right|^2 \right) \cdot f(x) \end{aligned} \tag{2.37}$$

In particular then:

$$\int_0^T H_{nn}(x(t))dt = \log\left(\left|\frac{\partial g}{\partial x}(x(0))\right|^2\right) - \log\left(\left|\frac{\partial g}{\partial x}(x(T))\right|^2\right) \ , \tag{2.38}$$

so that:

$$\lim_{T\to+\infty} \frac{\int_0^T H_{nn}(x(t))dt}{T} = 0, \tag{2.39}$$

whenever $x(t)$ is bounded. We exploit this result to show that the remaining Lyapunov exponent is 0. Let $X(t)$ be the solution of:

$$\dot{X}(t) = H(x(t))X(t), \qquad X(0) = I_n. \tag{2.40}$$

Since H is upper block-triangular, $X(t)$ has the same structure. In particular:

$$e_n^T \dot{X}(t) = H_{nn}(x(t))e_n^T X(t). \tag{2.41}$$

Integration of this scalar equation yields:

$$e_n^T X(t) = e^{\int_0^t H_{nn}(x(\tau))\,d\tau} e_n^T \ , \tag{2.42}$$

where we exploited $e_n^T X(0) = e_n^T$. Hence, e_n is a left eigenvector of $X(t)$, relative to the eigenvalue $e^{\int_0^t H_{nn}(x(\tau))\,d\tau}$. Let $v(t)$ be the corresponding right eigenvector of unitary norm (and assume $v(t)$ is a continuous function without loss of generality). We see that:

$$X(t)v(t) = e^{\int_0^t H_{nn}(x(\tau))\,d\tau} v(t). \tag{2.43}$$

Let $t_k \to +\infty$ be a sequence of times such that $v(t_k) \to \bar{v}$ as $k \to +\infty$. Such a sequence exists, by compactness of the unit sphere. Then:

$$\lim_{k\to+\infty} \frac{\log(|X(t_k)\bar{v}|)}{t_k} = \lim_{k\to+\infty} \frac{\log(|X(t_k)v(t_k)|)}{t_k} = \lim_{k\to+\infty} \frac{\int_0^{t_k} H_{nn}(s)\,ds}{t_k} = 0. \tag{2.44}$$

■

Remark 10 *It is worth noting that the results introduced in the paragraph 2.2.1 can be rewritten also in the case of system with a first integral of motion. Specifically, the analogue of Proposition 11, Proposition 12, Remark 7 and Remark 8 can be readily state once the 2-additive compound* $\frac{\partial f}{\partial x}^{[2]}(x)$ *is replaced with the 2-additive compound of* $\tilde{J}(x)$.

Chapter 3

A small-gain theorem for 2-contraction

As soon as the dimension of the system n grows, the dimension of the 2-additive compound grows as $\binom{n}{2}$, i.e. quadratically in n. Therefore, the size of the problem that has to be solved grows rapidly. In this chapter it is shown how the 2-additive approach can be exploited to find small-gain like conditions considering the system as the interconnection of two different subsystems and, as a consequence, it allows tackling the problem via LMIs of lower dimension. It will be introduced a suitable notion of gain for linear systems in Definitions 6 and 7 and for nonlinear systems in Definitions 8 and 9. Then, Theorems 7 and 8 and Theorems 9 and 10 provide small-gain like conditions for linear and nonlinear systems, respectively. The material of the Chapter has been collected in the scientific paper [4]. The proof of the various results is reported since it is useful for a better understanding of the analysis and for the development of the results in the next Chapter.

3.1 Small-gain theorems for stability of $A^{[2]}$

Consider an interconnected linear system of the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.1}$$

where x_1 and x_2 are vectors of dimension $n_1, n_2 \geq 2$, and A_{11}, A_{12}, A_{21} and A_{22} are blocks of compatible dimensions, with A_{11} and A_{22} being square. We interpret equation (3.1) as the equation of a feedback interconnection of the x_1 and x_2 sub-systems. In particular, off-diagonal blocks may be of low rank, (corresponding to fewer input and output variables), but this is not needed for the results to follow. We are interested in modular conditions to guarantee asymptotic stability of the 2-additive compound matrix $A^{[2]}$, where A is the block matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.2}$$

Consider next a skew-symmetric matrix X, which is partitioned according to A as in (3.2), i.e.,

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \tag{3.3}$$

By skew-symmetry we have that $X_{11}^T = -X_{11}$ and $X_{22}^T = -X_{22}$, viz. diagonal blocks are themselves skew-symmetric. In addition, $X_{21}^T = -X_{12}$. The 2-additive compound $A^{[2]}$ characterizes the dynamics of the operator $\vec{X}$ according to (1.12), viz.

$$\dot{\vec{X}} = A^{[2]}\vec{X}. \tag{3.4}$$

Our goal is to decompose the dynamics of (3.4) by looking at the different state-components $\vec{X}_{11}$, $\vec{X}_{22}$ and $\mathrm{vec}(X_{12})$.

Proposition 13 *[4] Consider the matrix-valued differential equation*

$$\dot{X} = AX + XA^T.$$

and assume that its unknown X is a skew-symmetric matrix partitioned according to (3.3). Then, the vectors $\vec{X}_{11}$, $\vec{X}_{22}$ and $vec(X_{12})$ fulfill the following linear system of coupled differential equations:

$$\begin{cases} \dot{\vec{X}}_{11} = A_{11}^{[2]}\vec{X}_{11} + B_1 vec(X_{12}) \\ \dot{\vec{X}}_{22} = A_{22}^{[2]}\vec{X}_{22} + B_2 vec(X_{12}) \\ vec(\dot{X}_{12}) = (A_{11} \oplus A_{22}) vec(X_{12}) + G_1\vec{X}_{11} + G_2\vec{X}_{22} \end{cases} \tag{3.5}$$

where the matrices B_1, B_2, G_1 and G_2 are given by:

$$B_1 = L_{n_1}(I_{n_1} \otimes A_{12}) - L_{n_1}(A_{12} \otimes I_{n_1})H_{n_1,n_2} \tag{3.6}$$

$$B_2 = L_{n_2}(I_{n_2} \otimes A_{21}) - L_{n_2}(A_{21} \otimes I_{n_2})H_{n_2,n_1} \tag{3.7}$$

$$G_1 = (I_{n_1} \otimes A_{21})M_{n_1} \tag{3.8}$$

$$G_2 = (A_{12} \otimes I_{n_2})M_{n_2} \tag{3.9}$$

and the matrix H_{n_1,n_2} is defined as

$$H_{n_1,n_2} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} e_{[(j-1)n_1+i]}e^T_{[(i-1)n_2+j]} \ ,$$

converts row vectorisation to column vectorisation, viz. $vec(X_{12}^T) = H_{n_1,n_2}vec(X_{12})$.

Proof. To see the result, compute the block-partitioned expression of $\dot{X}$ according to:

$$\begin{aligned}
\dot{X} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\
&+ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T \\
&= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\
&+ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}
\end{aligned}$$

Then, $\dot{X}$ assumes the following formulation:

$$\dot{X} = \begin{bmatrix} A_{11}X_{11} + A_{12}X_{21} + X_{11}A_{11}^T + X_{12}A_{12}^T & A_{11}X_{12} + A_{12}X_{22} + X_{11}A_{21}^T + X_{12}A_{22}^T \\ A_{21}X_{11} + A_{22}X_{21} + X_{21}A_{11}^T + X_{22}A_{12}^T & A_{21}X_{12} + A_{22}X_{22} + X_{21}A_{21}^T + X_{22}A_{22}^T \end{bmatrix} \tag{3.10}$$

Recalling that $X_{21} = -X_{12}^T$, we may remark that:

$$\dot{X}_{11} = A_{11}X_{11} + X_{11}A_{11}^T + X_{12}A_{12}^T - A_{12}X_{12}^T$$

$$\dot{X}_{22} = A_{22}X_{22} + X_{22}A_{22}^T + A_{21}X_{12} - X_{12}^T A_{21}^T$$
$$\dot{X}_{12} = A_{11}X_{12} + A_{12}X_{22} + X_{11}A_{21}^T + X_{12}A_{22}^T$$

Taking vec$(\cdot)$ in both sides of the last equation and exploiting the row vectorisation identity $\text{vec}(AXB^T) = (A \otimes B)\text{vec}(X)$ yields:

$$\begin{aligned}
\text{vec}(\dot{X}_{12}) &= \text{vec}(A_{11}X_{12}) + \text{vec}(X_{12}A_{22}^T) \\
&+ \text{vec}(A_{12}X_{22}) + \text{vec}(X_{11}A_{21}^T) \\
&= (A_{11} \otimes I_{n_2})\text{vec}(X_{12}) + (I_{n_1} \otimes A_{22})\text{vec}(X_{12}) \\
&+ (A_{12} \otimes I_{n_2})\text{vec}(X_{22}) + (I_{n_1} \otimes A_{21})\text{vec}(X_{11}) \\
&= (A_{11} \oplus A_{22})\text{vec}(X_{12}) + (A_{12} \otimes I_{n_2})\text{vec}(X_{22}) \\
&+ (I_{n_1} \otimes A_{21})\text{vec}(X_{11}) \\
&= (A_{11} \oplus A_{22})\text{vec}(X_{12}) + (A_{12} \otimes I_{n_2})M_{n_2}\vec{X}_{22} \\
&+ (I_{n_1} \otimes A_{21})M_{n_1}\vec{X}_{11}.
\end{aligned}$$

Next, taking the $\vec{(\cdot)}$ operator in both sides of $\dot{X}_{11}$ and $\dot{X}_{22}$ equations yields:

$$\begin{aligned}
\dot{\vec{X}}_{11} &= \overrightarrow{(A_{11}X_{11} + X_{11}A_{11}^T)} + \overrightarrow{(X_{12}A_{12}^T - A_{12}X_{12}^T)} \\
&= A_{11}^{[2]}\vec{X}_{11} + L_{n_1}\text{vec}(X_{12}A_{12}^T - A_{12}X_{12}^T) \\
&= A_{11}^{[2]}\vec{X}_{11} + L_{n_1}\text{vec}(X_{12}A_{12}^T) - L_{n_1}\text{vec}(A_{12}X_{12}^T) \\
&= A_{11}^{[2]}\vec{X}_{11} + L_{n_1}(I_{n_1} \otimes A_{12})\text{vec}(X_{12}) \\
&- L_{n_1}(A_{12} \otimes I_{n_1})\text{vec}(X_{12}^T).
\end{aligned}$$

We next make use of matrix H_{n_1,n_2} which converts row vectorisation to column vectorisation, viz. $\text{vec}(X_{12}^T) = H_{n_1,n_2}\text{vec}(X_{12})$. Exploiting the latter identity in the previous equation we prove that:

$$\begin{aligned}
\dot{\vec{X}}_{11} &= A_{11}^{[2]}\vec{X}_{11} + L_{n_1}(I_{n_1} \otimes A_{12}) \\
&- L_{n_1}(A_{12} \otimes I_{n_1})H_{n_1,n_2}\text{vec}(X_{12}).
\end{aligned}$$

Hence, $B_1 = L_{n_1}(I_{n_1} \otimes A_{12}) - L_{n_1}(A_{12} \otimes I_{n_1})H_{n_1,n_2}$. A similar expression can be proved for $\dot{\vec{X}}_{22}$. ■

We introduce now the following notions of gain.

Definition 6 *For a system of equations:*

$$\dot{x} = Ax + Bw$$

we denote its $\mathcal{L}_2$ gain as the minimum $\gamma \geq 0$ such that the following LMI admits a positive definite solution $P > 0$:

$$\begin{bmatrix} A^T P + PA + I & PB \\ B^T P & -\gamma^2 I \end{bmatrix} \leq 0 \ .$$

Consider the $\mathcal{L}_2$ gains γ_1, γ_2 and γ_{12} for the $\vec{X}_{11}$, $\vec{X}_{22}$ and $\mathrm{vec}(X_{12})$ subsystems of (3.5). They fufill the following LMIs:

$$\begin{bmatrix} {A_{11}^{[2]}}^T P_1 + P_1 A_{11}^{[2]} + I & P_1 B_1 \\ B_1^T P_1 & -\gamma_1^2 I \end{bmatrix} \leq 0 \qquad (3.11)$$

$$\begin{bmatrix} {A_{22}^{[2]}}^T P_2 + P_2 A_{22}^{[2]} + I & P_2 B_2 \\ B_2^T P_2 & -\gamma_2^2 I \end{bmatrix} \leq 0 \qquad (3.12)$$

$$\begin{bmatrix} Q_{12} & P_{12}[G_1, G_2] \\ [G_1, G_2]^T P_{12} & -\gamma_{12}^2 I \end{bmatrix} \leq 0 \qquad (3.13)$$

where Q_{12} is defined as

$$Q_{12} = (A_{11} \oplus A_{22})^T P_{12} + P_{12}(A_{11} \oplus A_{22}) + I \ . \qquad (3.14)$$

In the case of subsystem $\mathrm{vec}(X_{12})$ in (3.5), an alternative notion of $\mathcal{L}_2$ gain can be introduced when the input vector is explicitly partitioned into two different signals.

Definition 7 *For a linear system:*

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2$$

whose input vector is decomposed according to $u = [u_1^T, u_2^T]^T$, the "partitioned" $\mathcal{L}_2$ gains $\eta_1 \geq 0$ and $\eta_2 \geq 0$ are defined with respect to the individual inputs components, provided that the following condition

$$\begin{bmatrix} A^T P + PA + I & PB_1 & PB_2 \\ B_1^T P & -\eta_1^2 I & 0 \\ B_2^T P & 0 & -\eta_2^2 I \end{bmatrix} \leq 0$$

holds for some symmetric matrix $P > 0$.

Definition 7 implies that the partitioned $\mathcal{L}_2$ gains η_1 and η_2 of subsystem $\mathrm{vec}(X_{12})$ in (3.5) fulfill for some $P_{12} = P_{12}^T > 0$ the following LMI:

$$\begin{bmatrix} Q_{12} & P_{12}G_1 & P_{12}G_2 \\ G_1^T P_{12} & -\eta_1^2 I & 0 \\ G_2^T P_{12} & 0 & -\eta_2^2 I \end{bmatrix} \leq 0 \tag{3.15}$$

where Q_{12} is as in (3.14).

Remark 11 *It is worth noting that in Definition 7, differently from Definition 6, we do not look for the minimal values η_1, η_2 of such "partitioned" gains, as a priori it is not obvious that one can simultaneously minimize η_1 and η_2. In particular, a non-trivial Pareto front of minimal values for η_1 and η_2 might occur, i.e., they may not be independent from each other.*

Theorem 7 *[4] Consider an interconnected system formulated as in (3.1), and let γ_1 and γ_2 denote the $\mathcal{L}_2$ gains of subsystems $A_{11}^{[2]}$ and $A_{22}^{[2]}$ computed according to Definition 6 and (3.11)-(3.12). Then, the related 2-additive compound matrix $A^{[2]}$ is Hurwitz, if $\exists\, \varepsilon > 0$ such that the following small-gain condition is satisfied*

$$1 > \min_{P_{12},\tilde{\eta}_1,\tilde{\eta}_2} \gamma_1^2\tilde{\eta}_1 + \gamma_2^2\tilde{\eta}_2 \tag{3.16}$$

$$\begin{bmatrix} Q_{12} & P_{12}G_1 & P_{12}G_2 \\ G_1^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_2^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0$$

$$P_{12} = P_{12}^T \geq \varepsilon I$$

$$\tilde{\eta}_1 \geq 0, \quad \tilde{\eta}_2 \geq 0$$

where Q_{12} is as in (3.14).

Proof. - As a preliminary observation, notice that the LMI in (3.16) is just a more convenient formulation of (3.15), where η_1^2 and η_2^2 have been replaced with $\tilde{\eta}_1$ and $\tilde{\eta}_2$ in order to keep the matrix inequality linear with respect to its unknowns. The proof is based on the construction of a block-diagonal quadratic Lyapunov function, exploiting the equivalent formulation of $A^{[2]}$ dynamics provided by equation (3.5). To this end, notice that, after a suitable reordering of state-variables, the matrix $A^{[2]}$ can be transformed as:

$$\mathcal{A} = \begin{bmatrix} A_{11}^{[2]} & B_1 & 0 \\ G_1 & A_{11} \oplus A_{22} & G_2 \\ 0 & B_2 & A_{22}^{[2]} \end{bmatrix}.$$

We consider a quadratic Lyapunov function defined by the following symmetric definite matrix:

$$\mathcal{P} = \begin{bmatrix} \lambda_1 P_1 & 0 & 0 \\ 0 & P_{12} & 0 \\ 0 & 0 & \lambda_2 P_2 \end{bmatrix}.$$

Matrices P_1 and P_2 are as in (3.11) and (3.12) and $\lambda_1 > 0$, $\lambda_2 > 0$ are to be chosen later. Direct calculations show that $\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A}$ gets the formulation reported in (3.17) and it satisfies the inequalities (3.18) and (3.19) by virtue of LMIs (3.11) and (3.12).

$$\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} = \begin{bmatrix} \lambda_1\big(A_{11}^{[2]^T} P_1 + P_1 A_{11}^{[2]}\big) & \lambda_1 P_1 B_1 + G_1^T P_{12} & 0 \\ \lambda_1 B_1^T P_1 + P_{12} G_1 & (A_{11} \oplus A_{22})^T P_{12} + P_{12}(A_{11} \oplus A_{22}) & \lambda_2 B_2^T P_2 + P_{12} G_2 \\ 0 & \lambda_2 P_2 B_2 + G_2^T P_{12} & \lambda_2\big(A_{22}^{[2]^T} P_2 + P_2 A_{22}^{[2]}\big) \end{bmatrix} \tag{3.17}$$

$$\leq \begin{bmatrix} -\lambda_1 I & G_1^T P_{12} & 0 \\ P_{12} G_1 & \lambda_1\gamma_1^2 I + (A_{11} \oplus A_{22})^T P_{12} + P_{12}(A_{11} \oplus A_{22}) & \lambda_2 B_2^T P_2 + P_{12} G_2 \\ 0 & \lambda_2 P_2 B_2 + G_2^T P_{12} & \lambda_2\big(A_{22}^{[2]^T} P_2 + P_2 A_{22}^{[2]}\big) \end{bmatrix} \tag{3.18}$$

$$\leq \begin{bmatrix} -\lambda_1 I & G_1^T P_{12} & 0 \\ P_{12} G_1 & (\lambda_1\gamma_1^2 + \lambda_2\gamma_2^2) I + (A_{11} \oplus A_{22})^T P_{12} + P_{12}(A_{11} \oplus A_{22}) & P_{12} G_2 \\ 0 & G_2^T P_{12} & -\lambda_2 I \end{bmatrix} \tag{3.19}$$

The inequality (3.19) can be rearranged via a suitable permutation matrix $\mathcal{S}$, so that the same inequality assumes the form

$$\mathcal{S}^T\left(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A}\right)\mathcal{S} \leq \begin{bmatrix} (\lambda_1\gamma_1^2 + \lambda_2\gamma_2^2) I + (A_{11} \oplus A_{22})^T P_{12} + P_{12}(A_{11} \oplus A_{22}) & P_{12} G_1 & P_{12} G_2 \\ G_1^T P_{12} & -\lambda_1 I & 0 \\ G_2^T P_{12} & 0 & -\lambda_2 I \end{bmatrix} \tag{3.20}$$

Finally, if we select $P_{12} = P_{12}^*$ and $\lambda_1 = \tilde{\eta}_1^* + \sigma$, $\lambda_2 = \tilde{\eta}_2^* + \sigma$ for some $\sigma > 0$ to be chosen later, where P_{12}^*, $\tilde{\eta}_1^*$, $\tilde{\eta}_2^*$ solve the small-gain condition (3.16), we get

$$\mathcal{S}(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A})\mathcal{S} \leq \begin{bmatrix} -\alpha I & 0 & 0 \\ 0 & -\sigma I & 0 \\ 0 & 0 & -\sigma I \end{bmatrix} \tag{3.21}$$

with

$$\alpha = 1 - (\tilde{\eta}_1^*\gamma_1^2 + \tilde{\eta}_2^*\gamma_2^2) - \sigma(\gamma_1^2 + \gamma_2^2).$$

If we choose σ such that

$$\sigma < \frac{1 - (\tilde{\eta}^*\gamma_1^2 + \tilde{\eta}^*\gamma_2^2)}{\gamma_1^2 + \gamma_2^2},$$

we get $\alpha > 0$, which by virtue of (3.21) implies that $\mathcal{A}$ is Hurwitz, thus proving 2-contraction of the matrix $A^{[2]}$. ■

A similar result can be formulated without involving the partitioned $\mathcal{L}_2$ gains of Definition 7, as follows.

Theorem 8 *[4] Consider an interconnected system formulated as in (3.1). The related 2-additive compound matrix $A^{[2]}$ is Hurwitz if the $\mathcal{L}_2$ gains γ_1, γ_2 and γ_{12}, computed according to Definition 6 and (3.11)-(3.13), fulfill the small-gain condition:*

$$\gamma_{12} \cdot \sqrt{\gamma_1^2 + \gamma_2^2} < 1. \tag{3.22}$$

Proof. - The argument proceeds along the same lines as in the proof of Theorem 7 by constructing a quadratic Lyapunov function defined by the following symmetric definite matrix:

$$\mathcal{P} = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & \lambda P_{12} & 0 \\ 0 & 0 & P_2 \end{bmatrix},$$

where P_1, P_2, P_{12} are as in (3.11), (3.12), (3.13) and $\lambda > 0$ is to be chosen later. A direct calculation leads to the formulation of $\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A}$ as reported in (3.23), which satisfies condition (3.24) and (3.25) thanks to the LMIs

(3.11) and (3.12).

$$\begin{aligned}
&\mathcal{A}^T\mathcal{P}+\mathcal{P}\mathcal{A}= \\
&\begin{bmatrix} {A_{11}^{[2]}}^T P_1 + P_1 A_{11}^{[2]} & P_1 B_1 + \lambda G_1^T P_{12} & 0 \\ B_1^T P_1 + \lambda P_{12} G_1 & \lambda\big((A_{11}\oplus A_{22})^T P_{12} + P_{12}(A_{11}\oplus A_{22})\big) & B_2^T P_2 + \lambda P_{12} G_2 \\ 0 & P_2 B_2 + \lambda G_2^T P_{12} & {A_{22}^{[2]}}^T P_2 + P_2 A_{22}^{[2]} \end{bmatrix} \qquad (3.23) \\
&\leq \begin{bmatrix} -I & \lambda G_1^T P_{12} & 0 \\ \lambda P_{12} G_1 & \gamma_1^2 I + \lambda\big((A_{11}\oplus A_{22})^T P_{12} + P_{12}(A_{11}\oplus A_{22})\big) & B_2^T P_2 + \lambda P_{12} G_2 \\ 0 & P_2 B_2 + \lambda G_2^T P_{12} & {A_{22}^{[2]}}^T P_2 + P_2 A_{22}^{[2]} \end{bmatrix} \qquad (3.24) \\
&\leq \begin{bmatrix} -I & \lambda G_1^T P_{12} & 0 \\ \lambda P_{12} G_1 & (\gamma_1^2+\gamma_2^2) I + \lambda\big((A_{11}\oplus A_{22})^T P_{12} + P_{12}(A_{11}\oplus A_{22})\big) & \lambda P_{12} G_2 \\ 0 & \lambda G_2^T P_{12} & -I \end{bmatrix} \qquad (3.25)
\end{aligned}$$

The last matrix in (3.25) can be properly rearranged through suitable permutations of the state variables via a permutation matrix $\mathcal{S}$ to get the formulation

$$\begin{aligned}
&\mathcal{S}^T\left(\mathcal{A}^T\mathcal{P}+\mathcal{P}\mathcal{A}\right)\mathcal{S} \leq \\
&\begin{bmatrix} (\gamma_1^2+\gamma_2^2) I + \lambda\big((A_{11}\oplus A_{22})^T P_{12} + P_{12}(A_{11}\oplus A_{22})\big) & \lambda P_{12}[G_1, G_2] \\ \lambda [G_1, G_2]^T P_{12} & -I \end{bmatrix},
\end{aligned} \qquad (3.26)$$

which finally leads to

$$\mathcal{S}^T\left(\mathcal{A}^T\mathcal{P}+\mathcal{P}\mathcal{A}\right)\mathcal{S} \leq \begin{bmatrix} (\gamma_1^2+\gamma_2^2-\lambda) I & 0 \\ 0 & (\lambda\gamma_{12}^2 - 1) I \end{bmatrix} \qquad (3.27)$$

by exploiting LMI (3.13). Hence, from

$$\mathcal{S}^T(\mathcal{A}^T\mathcal{P}+\mathcal{P}\mathcal{A})\mathcal{S} \leq \begin{bmatrix} (\gamma_1^2+\gamma_2^2) I - \lambda I & 0 \\ 0 & \lambda\gamma_{12}^2 I - I \end{bmatrix}$$

one gets that $\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} < 0$, if λ is chosen so that:

$$\gamma_1^2 + \gamma_2^2 < \lambda < \frac{1}{\gamma_{12}^2}.$$

■

Remark 12 *It is interesting to remark that $A_{11}^{[2]}$, $A_{22}^{[2]}$ and $A_{11} \oplus A_{22}$ may be Hurwitz matrices even if A_{11} or A_{22} are not. In particular, asymptotic stability of the individual subsystems is not a necessary condition for the application of the proposed small-gain conditions to the stability of $A^{[2]}$.*

Remark 13 *In the case of systems in cascade, i. e. when A is block-triangular, the traditional small-gain condition for asymptotic stability of interconnected systems is automatically satisfied, as soon as one of the gains of the two subsystems is zero, because the loop gain becomes $\gamma_1\gamma_2 = 0 < 1$. One would expect something similar to hold in small-gain conditions for 2-contraction. In this regard, observe that when matrix A is block-triangular, so is $A^{[2]}$, albeit up to permutation of variables. For instance, $A_{12} = 0$ implies $B_1 = 0$ and $G_2 = 0$ in equation (3.5). As a result, the evolution of X_{11} describes an autonomous system, which feeds into subsystem X_{12}, that in turn forces subsystem X_{22}. Therefore, if matrices $A_{11}^{[2]}$, $A_{11} \oplus A_{22}$ and $A_{22}^{[2]}$ are Hurwitz, so is matrix $A^{[2]}$ and the overall system is 2-contracting. Then, consider the small-gain condition (3.16) of Theorem 7 and let us denote by P_{12}^*, η_1^* and η_2^* the optimal values of P_{12}, η_1 and η_2, respectively. In the case of cascaded systems this condition is tight, because $B_1 = 0$ and $G_2 = 0$ allows for computing $\gamma_1 = 0$ and selecting $\eta_2^* = 0$, so that $\gamma_1^2\eta_1^* + \gamma_2^2\eta_2^* = 0 < 1$ is guaranteed. Conversely, in Theorem 8, where a unique gain γ_{12} is adopted to characterize the amplification introduced by subsystem X_{12}, the small-gain condition (3.22) is not automatically fulfilled. In the case exemplified, for instance, $\gamma_1 = 0$, but this still requires the fulfillment of the condition $\gamma_{12} \cdot \gamma_2 < 1$ to guarantee 2-contraction of the cascade according to Theorem 8. This situation is unideal as it hints at some conservatism in this latter formulation.*

3.2 Modular 2-contraction of nonlinear systems

We consider next the case of interconnected nonlinear systems, defined by $\mathcal{C}^1$ equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} =: f(x), \tag{3.28}$$

where x_1 and x_2 are vectors of dimension $n_1, n_2 \geq 2$. Due to the smoothness of f_1 and f_2 we may define the block-partitioned Jacobian matrix J given below:

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix}. \tag{3.29}$$

It was shown in [36] that suitable contraction conditions (expressed through matrix norms) of the 2-additive compound of the Jacobian $J^{[2]}(x)$, can be used to rule out periodic solutions in nonlinear dynamical systems. Such conditions were reformulated in [2,34] through the use of Lyapunov functions or LMIs and extended to rule out oscillatory behaviors of periodic, almost periodic and chaotic nature. The goal of this section is to exploit/extend the modular criteria proposed in Section 3.1 to the case of interconnected nonlinear systems as given by (3.28) in order to rule out oscillatory behaviors.

We work under the assumption that a compact forward invariant set $\mathcal{X} \subseteq \mathbb{R}^n$ for the dynamics of (3.28) is available or that solutions are a priori known to be bounded. Then, oscillatory behaviors may be ruled out provided a symmetric and positive definite x-dependent matrix $P(x) \in \mathbb{R}^{\binom{n}{2}}$ is known to satisfy both $\alpha_1 I \leq P(x) \leq \alpha_2 I$, for positive α_1, α_2, and

$$J^{[2]}(x)^T P(x) + P(x) J^{[2]}(x) + \dot{P}(x) \leq -\varepsilon I \tag{3.30}$$

for some $\varepsilon > 0$ and $\forall x \in \mathcal{X}$ [34].
Similarly to the linear case, the variational equation associated to the 2-additive compound matrix, i.e.

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\delta}^{(2)} &= J^{[2]}(x)\, \delta^{(2)} \end{aligned} \tag{3.31}$$

can be rearranged according to equation (3.5) as

$$\begin{cases} \dot{x} = f(x) \\ \dot{\delta}_1 = J_{11}^{[2]}(x)\,\delta_1 + B_1(x)\delta_{12} \\ \dot{\delta}_{12} = (J_{11}(x) \oplus J_{22}(x))\delta_{12} + G_1(x)\delta_1 + G_2(x)\delta_2 \\ \dot{\delta}_2 = J_{22}^{[2]}(x)\,\delta_2 + B_2(x)\delta_{12}. \end{cases} \tag{3.32}$$

Condition (3.30) ensures exponential convergence of $\delta^{[2]}(t)$ for any initial condition in $\mathcal{X}$ and any initial value of $\delta^{[2]}(0) \in \mathbb{R}^{\binom{n}{2}}$. Our goal is the formulation of small-gain conditions analogous to (3.16) and (3.22) to ensure (3.30). To this end we introduce the notion of $\mathcal{L}_2$ gain for state dependent matrices according to the following LMIs.

Definition 8 *For a system of equations:*

$$\dot{\delta} = A(x)\delta + B(x)w$$

we define its $\mathcal{L}_2$ gain as any value $\gamma \geq 0$ such that the LMI:

$$\begin{bmatrix} A(x)^T P(x) + P(x)A(x) + \dot{P}(x) + I & P(x)B(x) \\ B(x)^T P(x) & -\gamma^2 I \end{bmatrix} \leq 0.$$

is fulfilled for all $x \in \mathcal{X}$ and for some positive definite symmetric matrix function $P(x)$ of class $\mathcal{C}^1$.

It is worth pointing out that $\dot{P}(x)$ is the matrix of entries $[L_f P_{ij}(x)]$ with $i, j \in 1, \ldots n$ and L_f denotes the Lie derivative along solutions of $\dot{x} = f(x)$.

We can now define the gains of the δ_1, δ_2 and δ_{12} subsystems in (3.32). In particular, we say that γ_1 is the gain of the δ_1 subsystem if for some $P_1(x)$ of class $\mathcal{C}^1$ and all $x \in \mathcal{X}$ it fulfills

$$\begin{bmatrix} Q_1(x) & P_1(x)B_1(x) \\ B_1(x)^T P_1(x) & -\gamma_1^2 I \end{bmatrix} \leq 0 \tag{3.33}$$

where

$$Q_1(x) = J_{11}^{[2]}(x)P_1(x) + P_1(x)J_{11}^{[2]}(x) + \dot{P}_1(x) + I\ .$$

Similarly, for subsystem δ_2 the gain γ_2 is computed according to the following LMI condition:

$$\begin{bmatrix} Q_2(x) & P_2(x)B_2(x) \\ B_2(x)^T P_2(x) & -\gamma_2^2 I \end{bmatrix} \leq 0 \tag{3.34}$$

where

$$Q_2(x) = J_{22}^{[2]}(x)P_2(x) + P_2(x)J_{22}^{[2]}(x) + \dot{P}_2(x) + I \ .$$

Finally, the gain γ_{12} of the component δ_{12} of the variational equation (3.32) is given by the fulfillment of

$$\begin{bmatrix} Q_{12}(x) & P_{12}(x)[G_1(x), G_2(x)] \\ [G_1(x), G_2(x)]^T P_{12}(x) & -\gamma_{12}^2 I \end{bmatrix} \leq 0 \tag{3.35}$$

where

$$\begin{aligned} Q_{12}(x) =&(J_{11}(x) \oplus J_{22}(x))^T P_{12}(x) + P_{12}(x)(J_{11}(x) \oplus J_{22}(x)) + \\ &\dot{P}_{12}(x) + I \ . \end{aligned}$$

As for the Definition 7, when a system has the same formulation of subsystem δ_{12} in (3.32), an alternative notion of $\mathcal{L}_2$ gain can be introduced.

Definition 9 *For a system of equations:*

$$\dot{\delta} = A(x)\delta + B_1(x)u_1 + B_2(x)u_2$$

whose input vector is decomposed according to $u = [u_1^T, u_2^T]^T$, the "partitioned" $\mathcal{L}_2$ gains η_1 and η_2 are defined with respect to those individual input components, provided that the LMI condition:

$$\begin{bmatrix} Q(x) & P(x)B_1(x) & P(x)B_2(x) \\ B_1(x)^T P(x) & -\eta_1^2 I & 0 \\ B_2(x)^T P(x) & 0 & -\eta_2^2 I \end{bmatrix} \leq 0 \ ,$$

where

$$Q(x) = A(x)^T P(x) + P(x)A(x) + I + \dot{P}(x) \ ,$$

holds for some positive definite symmetric matrix function $P(x)$ of class $\mathcal{C}^1$ and for all the $x \in \mathcal{X}$.

Definition 9 implies that the partitioned $\mathcal{L}_2$ gains η_1 and η_2 of subsystem δ_{12} in (3.32) fulfill the following LMI

$$\begin{bmatrix} Q_{12}(x) & P_{12}(x)G_1(x) & P_{12}(x)G_2(x) \\ G_1^T(x)P_{12}(x) & -\eta_1^2 I & 0 \\ G_2^T(x)P_{12}(x) & 0 & -\eta_2^2 I \end{bmatrix} \leq 0, \tag{3.36}$$

where

$$Q_{12}(x) = (J_{11}(x) \oplus J_{22}(x))^T P_{12}(x) + P_{12}(x)(J_{11}(x) \oplus J_{22}(x)) + \dot{P}_{12}(x) + I, \tag{3.37}$$

for some positive definite matrix function $P_{12}(x)$ of class $\mathcal{C}^1$ and for all $x \in \mathcal{X}$.

Remark 14 *While it is in principle possible to use state-dependent matrices $P_1(x)$, $P_2(x)$ and $P_{12}(x)$ for the definition of the gains, as in (3.33), (3.34), (3.35), and (3.36), the computation of the derivatives $\dot{P}_1(x)$, $\dot{P}_{12}(x)$, and $\dot{P}_2(x)$ cannot be done in a decoupled fashion. In this respect, a noteworthy simplification occurs when dealing with constant matrices, as the gains can be computed independently of each other. Namely, changing $f_2(x_1, x_2)$ for $\dot{x}_2$ will not affect the gain γ_1 of the δ_1 subsystem and vice-versa. On the other hand the γ_{12} gain is affected both by $\dot{x}_1$ and $\dot{x}_2$. An intermediate situation can be pursued by choosing $P_1(x_1)$, $P_2(x_2)$ and P_{12} constant, so as to still retain some decoupling in the computation of gains and allow the flexibility of state-dependent matrices.*

Theorem 9 *[4] Consider the interconnected system (3.28) and assume that everywhere in some forward invariant set its Jacobian matrix (3.29) belongs to the convex hull of the set of matrices $\mathcal{V} = \{V_i\}_{i=1,\dots,m}$, i.e., $J(x) \in \mathrm{conv}(\mathcal{V})$ for all $x \in \mathcal{X}$. Let γ_1 and γ_2 be computed according to Definition 8 and (3.33)-(3.34). Then, the 2-additive compound matrix of the Jacobian $J^{[2]}(x)$ fulfills the contraction property (3.30), if $\exists\, \varepsilon > 0$ such that the following small-gain condition is satisfied for all the matrices V_i*

$$1 > \min_{P_{12}, \tilde{\eta}_1, \tilde{\eta}_2} \gamma_1^2 \tilde{\eta}_1 + \gamma_2^2 \tilde{\eta}_2 \tag{3.38}$$

$$\begin{bmatrix} Q_{12,i}(x) & P_{12}G_{1,i} & P_{12}G_{2,i} \\ G_{1,i}^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_{2,i}^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0 \ , \ i = 1, \dots, m \tag{3.39}$$

$$P_{12} = P_{12}^T \geq \varepsilon I$$

$$\tilde{\eta}_1 \geq 0, \ \tilde{\eta}_2 \geq 0$$

where $Q_{12,i}(x)$ has the same form as in (3.37) but P_{12} does not depend on x and $J(x)$ is played by V_i.

Proof. - To see the result, notice that the variational equation (3.31) can be rearranged through suitable permutations according to (3.32). In particular,

$$\dot{\delta} = \mathcal{A}(x)\delta,$$

for the block matrix

$$\mathcal{A}(x) = \begin{bmatrix} J_{11}^{[2]}(x) & B_1(x) & 0 \\ G_1(x) & J_{11}(x) \oplus J_{22}(x) & G_2(x) \\ 0 & B_2(x) & J_{22}^{[2]}(x) \end{bmatrix}.$$

We adopt a candidate solution for (3.30) of the following form:

$$\mathcal{P}(x) = \begin{bmatrix} \lambda_1 P_1(x) & 0 & 0 \\ 0 & P_{12} & 0 \\ 0 & 0 & \lambda_2 P_2(x) \end{bmatrix}.$$

A direct computation shows that $\mathcal{A}^T(x)\mathcal{P} + \mathcal{P}\mathcal{A}(x) + \dot{\mathcal{P}}(x)$ assumes the form

$$\mathcal{A}^T(x)\mathcal{P}(x) + \mathcal{P}(x)\mathcal{A}(x) + \dot{\mathcal{P}}(x)$$

$$= \left[\begin{array}{cc} \lambda_1\big({J_{11}^{[2]}}^T(x)P_1(x) + P_1(x)J_{11}^{[2]}(x)\big) + \lambda_1\dot{P}_1(x) & \lambda_1 P_1(x)B_1(x) + G_1^T(x)P_{12} \\ \lambda_1 B_1^T(x)P_1(x) + P_{12}G_1(x) & \big(J_{11}(x) \oplus J_{22}(x)\big)^T P_{12} + P_{12}\big(J_{11}(x) \oplus J_{22}(x)\big) \\ 0 & \lambda_2 P_2(x)B_2(x) + G_2^T(x)P_{12} \end{array} \right. \tag{3.40}$$

$$\left. \begin{array}{c} 0 \\ \lambda_2 B_2^T(x)P_2(x) + P_{12}G_2(x) \\ \lambda_2\big({A_{22}^{[2]}}^T(x)P_2(x) + P_2(x)A_{22}^{[2]}(x)\big) + \lambda_2\dot{P}_2(x) \end{array} \right] \tag{3.41}$$

Then, the proof follows along similar lines as the proof of Theorem 7 by applying the inequalities considered in (3.33), (3.34) and (3.39), this latter evaluated in the vertexes of the hull. ■

Theorem 10 *[4] Consider the interconnected system (3.28). The* 2*-additive compound matrix of its Jacobian* $J^{[2]}(x)$ *fulfills the contraction property (3.30) if the* $\mathcal{L}_2$ *gains* γ_1*,* γ_2 *and* γ_{12}*, computed according to Definition 8 and (3.33)-(3.35), satisfy the small-gain condition:*

$$\gamma_{12} \cdot \sqrt{\gamma_1^2 + \gamma_2^2} < 1. \tag{3.42}$$

Proof. - The argument proceeds along the same lines as in the proof of Theorem 8, by constructing a quadratic Lyapunov function defined by the following symmetric definite matrix:

$$\mathcal{P}(x) = \begin{bmatrix} P_1(x) & 0 & 0 \\ 0 & \lambda P_{12}(x) & 0 \\ 0 & 0 & P_2(x) \end{bmatrix}. \tag{3.43}$$

Direct computations lead $\mathcal{A}^T(x)\mathcal{P}(x) + \mathcal{P}(x)\mathcal{A}(x) + \dot{\mathcal{P}}(x)$ to the formulation

$$\mathcal{A}^T(x)\mathcal{P}(x) + \mathcal{P}(x)\mathcal{A}(x) + \dot{\mathcal{P}}(x)$$

$$= \begin{bmatrix} \begin{matrix} J_{11}^{[2]^T}(x)P_1(x) + P_1(x)J_{11}^{[2]}(x) \\ + \dot{P}_1(x) \end{matrix} & P_1(x)B_1(x) + \lambda G_1^T(x)P_{12}(x) & 0 \\ B_1^T(x)P_1(x) + \lambda P_{12}(x)G_1(x) & \begin{matrix} \lambda(J_{11}(x) \oplus J_{22}(x))^T P_{12}(x) \\ + \lambda P_{12}(x)(J_{11}(x) \oplus J_{22}(x)) + \lambda \dot{P}_{12}(x) \end{matrix} & B_2^T(x)P_2(x) + \lambda P_{12}(x)G_2(x) \\ 0 & P_2(x)B_2(x) + \lambda G_2^T(x)P_{12}(x) & \begin{matrix} A_{22}^{[2]^T}(x)P_2(x) + P_2(x)A_{22}^{[2]}(x) \\ + \dot{P}_2(x) \end{matrix} \end{bmatrix} \tag{3.44}$$

The proof follows along similar lines as the proof of Theorem 8 by applying the inequalities considered in (3.33), (3.34) and (3.35). ■

Remark 15 *It is worth noting that in Section 3.1 the dimension n_1 and n_2 of the subsystems' states are limited to be greater or equal than two. However, the approach can be applied in the case of systems of dimension $n = 3$ as well. In such case, one of the two subsystems is empty and one is scalar, for which the gain γ_1 can be readily computed. Therefore, conditions (3.22) and (3.42) become:*

$$\gamma_{12} \cdot \gamma_1 < 1. \tag{3.45}$$

Chapter 4

Controlling Chaos

In this Chapter, the 2-additive compound approach is utilized to synthesize a feedback control law making the closed loop system 2-contractive. Some general notions of 2-contraction stabilizability are provided in Definition 10 and 11 for linear systems, together with the main result reported in Theorem 11. Consequently, the extension to nonlinear systems is presented, introducing a notion of 2-contraction stabilizability with derivative feedback in Definition 12, in order to design a feedback control law that allows removing chaos while preserving equilibria (Propositions 20 and 21).

4.1 Stabilizability and 2-contraction

Consider the finite dimensional linear control system:

$$\dot{x} = Ax + Bu, \tag{4.1}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_i}$ is the input vector matrix and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_i}$ are given matrices. We assume, for simplicity, that the state of the system is measured and instantaneously available, and that the following feedback law is implemented

$$u = Kx, \tag{4.2}$$

where $K \in \mathbb{R}^{n_i \times n}$ is the matrix gain. Therefore, the closed loop control system assumes the form

$$\dot{x} = (A + BK)\, x. \tag{4.3}$$

Our aim is to find conditions that allow concluding existence of a linear feedback as in (4.2) such that the closed-loop system is 2-contractive. To this end, the following definitions are needed.

Definition 10 *We say that the pair (A, B) is 2-contraction stabilizable if there exists a matrix K, of suitable dimension, such that $(A+BK)^{[2]}$ is Hurwitz stable.*

Recalling the reachability property of a linear system, a spectral signature of Definition 10 can be defined as follows.

Definition 11 *We say that the pair (A, B) has a pairwise stabilizable spectrum, if the following condition*

$$\mathrm{Re}(\lambda_1 + \lambda_2) < 0$$

holds for all pairs (λ_1, λ_2) of distinct unreachable eigenvalues of A.

Recall, an eigenvalue is unreachable if the associated eigenvector does not belong to the image of the controllability matrix. Notice that the condition is always true if there exist less than two unreachable eigenvalues.
The next Proposition provides a useful formula for the 2-additive compound matrix of a product [3].

Proposition 14 *[3] Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m \times n}$. Then, it holds*

$$(AB)^{[2]} = \overline{L}^{A} \underline{L}^{B}\ , \tag{4.4}$$

where the matrix $\overline{L}^{A} \in \mathbb{R}^{\binom{n}{2}\times(mn)}$ depends on A and the matrix $\underline{L}^{B} \in \mathbb{R}^{(mn)\times\binom{n}{2}}$ depends on B.

The next result completely characterizes 2-contraction stabilizability of (A, B).

Theorem 11 *The following facts are equivalent:*

1. *The pair (A, B) is 2-contraction stabilizable;*
2. *The pair (A, B) has pairwise stabilizable spectrum;*

3. *The pair* $(A^{[2]}, \overline{L}^B)$ *is stabilizable;*

Before proving Theorem 11, the following facts are needed.

Lemma 3 *The pair* (A, B) *is completely reachable if and only if for all* $M \in \mathbb{R}$ *there exists a matrix* K *of compatible dimension such that*

$$\text{spec}(A + BK) \subset \{s : \text{Re}[s] \leq M\}. \tag{4.5}$$

Proof. *If the pair* (A, B) *is completely reachable, then the eigenvalues of* $A + BK$ *can be located arbitrarily in the complex plane via some suitable gain matrix* K *and hence condition (4.5) can be satisfied for all* $M \in \mathbb{R}$. *Conversely, we proceed by contradiction supposing (4.5) holds for all* $M \in \mathbb{R}$ *and the pair* (A, B) *is not completely reachable. Hence, there exists at least one eigenvalue, say* λ_u*, such that* $\lambda_u \in \text{spec}(A + BK)$ *for all* K*, which implies that (4.5) does not hold for* $M < \text{Re}\,(\lambda_u)$. ■

Corollary 3 *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n_i}$ *decomposed in the following form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ O_2 \end{bmatrix}, \tag{4.6}$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, O_{21} *is a* $n_2 \times n_1$ *matrix of zeros,* $B_1 \in \mathbb{R}^{n_1\times n_i}$, O_2 *is a* $n_2 \times n_i$ *matrix of zeros and* $n = n_1 + n_2$. *The pair* (A, B) *is decomposed in the reachability canonical form, with* n_1 *and* n_2 *the dimensions of the reachable and unreachable part, respectively, if and only if for all* $M \in \mathbb{R}$ *there exists a matrix* $K_1 \in \mathbb{R}^{n_i\times n_1}$ *such that*

$$\text{spec}(A_{11} + B_1K_1) \subset \{s : \text{Re}[s] \leq M\}. \tag{4.7}$$

Proof. Consider the pair (A, B) decomposed in the reachability canonical form as in (4.6). The pair (A_{11}, B_1) is the completely reachable part, while A_{22} represents the unreachable part. From Lemma 3 it follows that there exists K_1 of suitable dimensions such that the spectrum of $(A_{11} + B_1K_1)$ can be placed to the left of any real abscissa M.
Conversely, if the spectrum of $(A_{11} + B_1K_1)$ can be placed to the left of any real abscissa M, it follows from Lemma 3 that the pair (A_{11}, B_1) is completely reachable. Therefore, the matrix A in (4.6) is decomposed in the reachability canonical form. ■

Lemma 4 *Consider a matrix $A \in \mathbb{R}^{n\times n}$ decomposed as*

$$A = \begin{bmatrix} A_{11} & * \\ O_{21} & A_{22} \end{bmatrix}, \tag{4.8}$$

where $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, O_{21} is a $n_2 \times n_1$ matrix of zeros and the symbol $*$ denotes an arbitrary matrix whose value is irrelevant to the subsequent discussion. Furthermore, let X be a skew-symmetric matrix partitioned according to A in (4.8), as in Section 3.1, Chapter 3. Consider further the state vector provided by the coordinates transformation

$$\begin{bmatrix} \vec{X}_{11} \\ vec(X_{12}) \\ \vec{X}_{22} \end{bmatrix} = P\vec{X}, \tag{4.9}$$

where P is a permutation matrix. The 2-additive compound of A in the new coordinates can be written as:

$$\mathcal{A} = PA^{[2]}P^\top = \begin{bmatrix} A_{11}^{[2]} & * & 0 \\ 0 & A_{11} \oplus A_{22} & * \\ 0 & 0 & A_{22}^{[2]} \end{bmatrix} \tag{4.10}$$

$$= \begin{bmatrix} \mathcal{A}_1 & * \\ 0 & A_{22}^{[2]} \end{bmatrix}. \tag{4.11}$$

Moreover, the permutation matrix P enjoys the following structure:

$$P = \begin{bmatrix} \tilde{P} & O_1 \\ O_2 & I_{\binom{n_2}{2}} \end{bmatrix}, \tag{4.12}$$

where $\tilde{P} \in \mathbb{R}^{(\binom{n}{2}-\binom{n_2}{2})\times(\binom{n}{2}-\binom{n_2}{2})}$, O_1 and O_2 are matrices of zeros of dimension $\left(\binom{n}{2} - \binom{n_2}{2}\right) \times \binom{n_2}{2}$ and $\binom{n_2}{2} \times \left(\binom{n}{2} - \binom{n_2}{2}\right)$, respectively, and $I_{\binom{n_2}{2}}$ is the $\binom{n_2}{2} \times \binom{n_2}{2}$ identity matrix.

Proof. Following the same calculations as in Section 3.1, Chapter 3, by noticing that in this case we have $A_{21} = O_{21}$ and $A_{12} = *$. From equation (3.6), we have that $\tilde{B}_1 = *$, $\tilde{B}_2 = 0$, $\tilde{G}_1 = 0$ and $\tilde{G}_2 = *$. Therefore, the matrix $A^{[2]}$ assumes the form as in (4.10).
Since the last part of the vector $\vec{X}$ is exactly $\vec{X}_{22}$, the final block of the permutation matrix P is equal to $\left[O_2,\, I_{\binom{n_u}{2}}\right]$, where O_2 is a matrix of zeros

of dimension $\binom{n_u}{2} \times \left(\binom{n}{2} - \binom{n_u}{2}\right)$ and $I_{\binom{n_u}{2}}$ is a $\binom{n_u}{2} \times \binom{n_u}{2}$ identity matrix. Furthermore, it has also a $\left(\binom{n}{2} - \binom{n_2}{2}\right) \times \binom{n_2}{2}$ block of zeros in position $(1,2)$ since when the matrix P selects the components $\vec{X}_{11}$ and $\mathrm{vec}(X_{12})$ from the vector $\vec{X}$, the component $\vec{X}_{22}$ has not to be selected. Therefore, P can be decomposed as

$$P = \begin{bmatrix} \tilde{P} & O_1 \\ O_2 & I_{\binom{n_u}{2}} \end{bmatrix},$$

where $\tilde{P} \in \mathbb{R}^{\left(\binom{n}{2}-\binom{n_2}{2}\right)\times\left(\binom{n}{2}-\binom{n_2}{2}\right)}$. ■

Lemma 5 *Consider a matrix $B \in \mathbb{R}^{n\times n_i}$ as follows:*

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_{n_r} \\ b_{n_r+1} \\ \vdots \\ b_n \end{bmatrix}, \tag{4.13}$$

where $b_1, \dots, b_n \in \mathbb{R}^{1\times n_i}$ and $b_{n_r+1}, \dots, b_n \equiv 0_{1\times n_i}$. The matrix $\overline{L}^B$ in (4.4) can be decomposed as

$$\overline{L}^B = \begin{bmatrix} L_r \\ O_u \end{bmatrix}, \tag{4.14}$$

where $L_r \in \mathbb{R}^{\left(\binom{n}{2}-\binom{n_u}{2}\right)\times(n\, n_i)}$, O_u is a matrix of zeros of dimension $\binom{n_u}{2} \times (n\, n_i)$ and $n_u = n - n_r$.

Proof. - Recalling the form of the matrix $\overline{L}_B$ in the case of $B = (b_1, b_2, b_3, b_4)^\top$ [3]:

$$\overline{L}_B = \begin{array}{c} \\ (1,2) \\ (1,3) \\ (1,4) \\ (2,3) \\ (2,4) \\ (3,4) \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} b_2 & -b_1 & 0 & 0 \\ b_3 & 0 & -b_1 & 0 \\ b_4 & 0 & 0 & -b_1 \\ 0 & b_3 & -b_2 & 0 \\ 0 & b_4 & 0 & -b_2 \\ 0 & 0 & b_4 & -b_3 \end{bmatrix} \end{array}, \tag{4.15}$$

and consider a matrix B as in (4.13). It can be understood that the matrix $\overline{L}^B$ can be constructed in blocks, where the first block is $(n-1) \times n\,n_i$, the second block is $(n-2) \times n\,n_i$ and so on until the final vector block of size $1 \times n\,n_i$, as it is shown in (4.16).

$$
\overline{L}_B =
\left[\begin{array}{cccccccccc}
\bar{b}_1 & -\bar{b}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & 0 & -\bar{b}_2 & 0 & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\
\bar{b}_r & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots \\
\bar{b}_{r+1} & \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & 0 \\
\bar{b}_n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\bar{b}_2 \\
0 & \bar{b}_2 & -\bar{b}_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\
\vdots & \bar{b}_r & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \bar{b}_{r+1} & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \bar{b}_n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\bar{b}_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \bar{b}_{r+1} & -\bar{b}_r & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \bar{b}_n & \vdots & \cdots & \cdots & -\bar{b}_r \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \bar{b}_n & -\bar{b}_{n-1}
\end{array}\right]
\begin{array}{c}
\\ \\ \\ \\ n-1 \\ \\ \\ \\ \\ \hline \\ \\ \\ \\ n-2 \\ \\ \\ \\ \hline \\ \hline n_u \\ \\ \\ \\ \hline \\ \hline 1 \\ \hline
\end{array}
\tag{4.16}
$$

Since the rows $b_{r+1}, \ldots, b_n$ are equal to zero, the block with dimension

$n_u \times n\,n_i$ in (4.16) is the last block with not all the components equal to zero. After that, the matrix has all the blocks equal to zero, corresponding to $1 + 2 + 3 + \cdots + (n_u - 2) + (n_u - 1) = \binom{n_u}{2}$ rows of $n\,n_i$ zero elements. Therefore, the matrix $\overline{L}^B$ can be decomposed in

$$\overline{L}^B = \begin{bmatrix} L_r \\ O_u \end{bmatrix}, \tag{4.17}$$

where $L_r \in \mathbb{R}^{(\binom{n}{2} - \binom{n_u}{2}) \times (n\,n_i)}$ and O_u is a matrix of zeros of dimension $\binom{n_u}{2} \times n\,n_i$.

Proof. of Theorem 11 - Without loss of generality, we consider the matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n_i}$ in Kalman's canonical reachability decomposition, viz.

$$A = \begin{bmatrix} A_r & * \\ 0 & A_u \end{bmatrix}, \qquad B = \begin{bmatrix} B_r \\ 0 \end{bmatrix}, \tag{4.18}$$

where $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times n_i}$, (A_r, B_r) is a completely reachable pair, and A_r, A_u represent the reachable and unreachable dynamics.
We show first the implications (1) $\iff$ (2). Considering the matrix of the closed loop system $A_{cl} = (A + BK)$. Partitioning the matrix K as $[K_r, K_u]$, where $K_r \in \mathbb{R}^{n_i \times n_r}$, the matrix A_{cl} can be written as

$$A_{cl} = \begin{bmatrix} A_r + B_r K_r & * \\ 0 & A_u \end{bmatrix}.$$

By Lemma 4, after the application of the permutation matrix P in (4.12), the 2-additive compound $A_{cl}^{[2]}$ can be written as

$$\mathcal{A}_{cl} = P A_{cl}^{[2]} P^\top = \begin{bmatrix} (A_r + B_r K_r)^{[2]} & * & 0 \\ 0 & (A_r + B_r K_r) \oplus A_u & * \\ 0 & 0 & A_u^{[2]} \end{bmatrix},$$

since $A_{11} = (A_r + B_r K_r)$, $A_{22} = A_u$. Therefore, it can be readily shown that the pair (A, B) is 2-contraction stabilizable if and only if the matrix $A_u^{[2]}$ is Hurwitz. This is indeed necessary, since $A_u^{[2]}$ is a block along the diagonal of the upper triangular matrix $\mathcal{A}_{cl}$. It is also sufficient, since by complete reachability of the pair (A_r, B_r), the eigenvalues of $A_r + B_r K_r$ can be allocated arbitrarily to the left of any real abscissas, and in particular of $\min\{0, -\max \mathrm{Re}(\mathrm{spec}(A_u))\}$, so that matrices $(A_r + B_r K_r)^{[2]}$ and $(A_r + B_r K_r) \oplus A_u$ are simultaneously stabilized.

We show now the implication (1) ⇒ (3). Assume that (1) holds and let K be any matrix that makes $(A+BK)^{[2]}$ Hurwitz. By Proposition 14:

$$(A+BK)^{[2]} = A^{[2]} + (BK)^{[2]} = A^{[2]} + \overline{L}^{B}\underline{L}^{K}.$$

Since $(A+BK)^{[2]}$ is Hurwitz, there exists a gain $\underline{L}^{K}$ which makes $A^{[2]}+\overline{L}^{B}\underline{L}^{K}$ Hurwitz. Hence, the pair $\left(A^{[2]}, \overline{L}^{B}\right)$ is stabilizable, and (3) holds.
Finally, the implication (3) ⇒ (1). Since $A_{11} = A_r$, $A_{22} = A_u$, by Lemma 4 we have:

$$\begin{aligned} \mathcal{A} = PA^{[2]}P^{\top} &= \begin{bmatrix} A_r^{[2]} & * & 0 \\ 0 & A_r \oplus A_u & * \\ 0 & 0 & A_u^{[2]} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A}_r & * \\ 0 & A_u^{[2]} \end{bmatrix}, \end{aligned}$$

where P is as in (4.12). Moreover, according with Lemma 4 and 5, the matrix $\overline{L}^{B}$ can be written as

$$\overline{\mathcal{L}}^{B} = P\overline{L}^{B} = \begin{bmatrix} \tilde{L}_r \\ O_u \end{bmatrix}.$$

Since (A_r, B_r) is completely reachable, all the eigenvalues of the matrix $(A_r + B_rK_r)$ can be placed to the left of any real abscissas M. Accordingly, this is also true for $(\mathcal{A}_r, \tilde{L}_r)$, through the choice of a feedback in the form $\underline{L}^{K_r}$ by noticing that, for negative M:

$$\begin{aligned} &\text{Re}\left(\text{spec}\left(\mathcal{A}_r + \tilde{L}_r\underline{L}^{K_r}\right)\right) \leq \\ &\quad \max\left\{\max\left\{\text{Re}\left(\text{spec}(A_r + B_rK_r)\right)\right\} + \max\left\{\text{Re}\left(\text{spec}(A_u)\right)\right\}, 2\,M\right\}. \end{aligned}$$

Therefore, $(\mathcal{A}, \overline{L}^{B})$ is a canonical reachability decomposition and by (3) $A_u^{[2]}$ is Hurwitz and contains all the unreachable eigenvalues. Overall then, the feedback $K = (K_r, 0)$ achieves 2-contraction for $(A+BK)$ provided eigenvalues of $(A_r + B_rK_r)$ are to the left of $-\max\{0, \text{Re}\left(\text{spec}(A_u)\right)\}$. Then, (1) follows. ■

4.1.1 LMIs for checking 2-contraction stabilizability

In order to checking if a linear system is 2-contraction stabilizable, some well-known LMIs can be used for a given A, B matrix pair, or even for a polytope of matrices.

Proposition 15 *The following LMI is equivalent to* (A, B) *being* 2*-contraction stabilizable:*

$$A^{[2]}P + PA^{[2]^T} < \overline{L}^B(\overline{L}^B)^T \qquad P > 0 \tag{4.19}$$

Proof. This is seen by pre and post multiplying the LMI by P^{-1}:

$$P^{-1}A^{[2]} + A^{[2]^T}P^{-1} < P^{-1}\overline{L}^B(\overline{L}^B)^T P^{-1}.$$

Hence, there exists a positive matrix Q, such that:

$$A^{[2]^T}P^{-1} + P^{-1}A^{[2]} - P^{-1}\overline{L}^B(\overline{L}^B)^T P^{-1} = -Q$$

and P^{-1} is the solution of a Riccati equation for the design of LQ optimal stabilizing feedback for the pair $(A^{[2]}, \overline{L}^B)$. ■

Proposition 16 *For unknowns matrices* $P > 0$ *and* $\mathcal{K}$*, the LMI in equation (4.19) can be written as the following Bilinear Matrix Inequality (BMI), in order to find a stabilizing feedback making the system (4.3)* 2*-contractive stable.*

$$A^{[2]}P + PA^{[2]^T} + \overline{L}^B\mathcal{K} + \mathcal{K}^T(\overline{L}^B)^T < 0 \qquad P > 0. \tag{4.20}$$

Proof. Pre and post multiplication of the LMI by P^{-1} yields:

$$P^{-1}A^{[2]} + A^{[2]^T}P^{-1} + P^{-1}\overline{L}^B\mathcal{K}P^{-1} + P^{-1}\mathcal{K}^T(\overline{L}^B)^T P^{-1} < 0 \qquad P > 0.$$

Defining the feedback gain as $\tilde{\mathcal{K}} := \mathcal{K}P^{-1}$, we see that the previous LMI proves that $A^{[2]} + \overline{L}^B\tilde{\mathcal{K}}$ is Hurwitz as it can be rearranged according to:

$$(A^{[2]} + \overline{L}^B\tilde{\mathcal{K}})^T P^{-1} + P^{-1}(A^{[2]} + \overline{L}^B\tilde{\mathcal{K}}) < 0, \qquad P^{-1} > 0\ .$$

■

Proposition 17 *Consider the system (4.3). If the matrix* A *belongs to the convex hull*

$$A \in \mathrm{conv}(\{A_1, \ldots, A_N\}),$$

the problem (4.20) boils down to the following BMI with unknowns $\mathcal{K}$ *and* $P > 0$*:*

$$A_i^{[2]}P + PA_i^{[2]^T} + \overline{L}^B\mathcal{K} + \mathcal{K}^T(\overline{L}^B)^T < 0 \qquad P > 0, i = 1, \ldots N. \tag{4.21}$$

While Theorem 11 clarifies that stabilizability (in the classical sense) of $(A^{[2]}, \overline{L}_B)$ ensures 2-contraction stabilizability of (A, B), it does not provide a constructive mechanism to derive a suitable gain K to achieve 2-contraction for $A + BK$. We propose below an algorithm to design $\underline{L}^K$, initialising the constant matrix as $P = \varepsilon I_{\binom{n}{2}}$, with $\varepsilon \geq 1$, $\varepsilon \in \mathbb{R}$.

1. Let $\varepsilon = 1$ and $P^{(0)} := \varepsilon I_{\binom{n}{2}}$. Let $h = 0$.
2. Repeat:
3. Solve the following LMI optimisation:
$$(L^{(h)}, *, *) := \arg\min_{L,K,q} \{q : Q_P^{(h)} \leq qI_{\binom{n}{2}},\ L = \underline{L}^K\}$$
where
$$Q_P^{(h)} = (A^{[2]} + \overline{L}^B L)^T P^{(h)} + P^{(h)}(A^{[2]} + \overline{L}^B L)$$
4. Solve the following LMI optimisation:
$$(P^{(h+1)}, q^*) = \arg\min_{P,q} \{q : Q_L^{(h)} \leq qI_{\binom{n}{2}},\ P \geq \varepsilon I_{\binom{n}{2}}\}$$
where
$$Q_L^{(h)} = (A^{[2]} + \overline{L}^B L^{(h)})^T P + P(A^{[2]} + \overline{L}^B L^{(h)})$$
5. $h = h + 1$;
6. Until $q^* < 0$

If the algorithm stops, at step h, we may choose the feedback K, such that $\underline{L}^K = L^{(h)}$. By construction this is a stabilizing gain for $A^{[2]} + \overline{L}^B \underline{L}^K$ and it corresponds to a 2-contraction stabilizing feedback K for matrix $(A + BK)$.

4.2 Removing chaos while preserving equilibria through 2-contraction

Consider a nonlinear control system of the following form:

$$\dot{x} = f(x) + Bu. \tag{4.22}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector, $f : X \to \mathbb{R}^n$ is at least of class $\mathcal{C}^1$, $B \in \mathbb{R}^{n \times n_i}$ is the input matrix and $u \in \mathbb{R}_i^n$ is the input vector. Our aim is to design a feedback control law for system (2.1) that allows removing chaotic behaviours while altering the system dynamics as little as possible. Specifically, we will follow the approach presented in the previous Chapters to study the behaviour of a second order perturbation η along solutions of a system confined within a certain forward invariant subset $\mathcal{D}$ of state-space. In particular, we are looking for conditions that guarantee 2-contraction by means of a feedback control law that preserves equilibria, thus becoming inactive in steady-state. We assume, as in the previous Section, that the state of the system is measured and instantaneously available. Furthermore, the feedback control law

$$u = Kf(x), \tag{4.23}$$

where $K \in \mathbb{R}^{n_i \times n}$ is the matrix gain, is considered. Note that (4.23) amounts to a derivative feedback control since $f(x) = \dot{x}$. Therefore, the closed loop control system assumes the form

$$\dot{x} = f(x) + BKf(x) = (I + BK)f(x), \tag{4.24}$$

while, the Jacobian of the closed loop system, named $J_{cl}(x)$, becomes:

$$J_{cl}(x) = (I + BK)J(x), \tag{4.25}$$

where $J(x) = \frac{\partial f}{\partial x}(x)$ is the Jacobian of system (2.1).
Consider the case of non-state dependent Jacobian, i.e. $J(x) \equiv A,\ \forall x \in X$. Similarly to Definition 10, we may consider a related notion of stabilizability while adopting a feedback of the form

$$u = KAx, \tag{4.26}$$

which preserves the Kernel of A.

Definition 12 *We say that the pair (A, B) is 2-contraction stabilizable with derivative feedback if there exists a matrix K of suitable dimension, such that $[(I + BK)A]^{[2]}$ is Hurwitz stable.*

Proposition 18 *The following facts are equivalent for a pair (A, B) with A invertible:*

1. *(A, B) is 2-contraction stabilizable.*

2. *(A, B) is 2-contraction stabilizable with derivative feedback.*

Proof. For the proof it is enough to show that (1) $\implies$ (2), since the implication (2) $\implies$ (1) is obvious.
Since (A, B) is 2-contraction stabilizable, there exists K such that $(A + BK)^{[2]}$ is Hurwitz. If A is non-singular, the following equalities

$$(A + BK)^{[2]} = \left(A + BKA^{-1}A\right)^{[2]} = \left[\left(I_n + B\bar{K}\right) A\right]^{[2]}$$

hold. Hence, according to Definition 12, the gain matrix K makes the pair (A, B) 2-contraction stable with derivative feedback, thus completing the proof. ■

The next results hold true.

Proposition 19 *Consider a closed loop control system as in (4.24). Then, the feedback control law does not modify the location of open loop equilibria.*

Proof. The proof directly follows from observing that equilibria of system (4.24) trivially fulfill:

$$\{x : f(x) \in \mathrm{Ker}(I + BK)\} \supset \{x : f(x) = 0\}.$$

■

Proposition 20 *The following matrix inequality, fulfilled for all $x \in X \subset \mathbb{R}^n$, is a sufficient condition to ensure non positive Lyapunov exponents for the system (4.24):*

$$\begin{aligned} &\left([(I + BK)J(x)]^{[2]}\right)^T P + P\left([(I + BK)J(x)]^{[2]}\right) \leq 0, \\ &P > 0\ , \end{aligned} \tag{4.27}$$

where $P = P^T$ and K is the gain matrix of the derivative feedback (4.23) *for system (4.22).*

Proof. The proof directly follows from Theorem 5. ■

Proposition 21 *Consider the control system in equation (4.24) and let $\mathcal{D}$ be a globally attractive and forward invariant set for system (2.1). Assume that:*

$$J(x) \in \mathrm{conv}(J_1, \ldots, J_N), \qquad \forall\, x \in \mathcal{D}$$

for fixed constant matrices J_i of suitable dimension. Then, the following matrix inequality for $i = 1, \ldots, N$

$$\left([(I_n + BK)J_i]^{[2]}\right)^T P + P\left([(I_n + BK)J_i]^{[2]}\right) \leq 0 \tag{4.28}$$

is a sufficient condition to ensure that the controlled system (4.24) has no positive Lyapunov exponents.

Chapter 5

Case studies and examples of application

This Chapter is devoted to some examples of application and case studies, in order to integrate and clarify the results presented in the previous Chapters. The material can be found in [35], [4].

5.1 Examples of application of Theorem 5

5.1.1 Case study: the Lorenz system

The Lorenz system is described by the following equations

$$\begin{array}{rcl} \dot{x}_1 & = & -\sigma\,(x_1 - x_2) \\ \dot{x}_2 & = & \rho\,x_1 - x_2 - x_1\,x_3 \\ \dot{x}_3 & = & x_1\,x_2 - b\,x_3 \end{array} \quad , \tag{5.1}$$

where $x = (x_1, x_2, x_3)^\top$ is the state vector and σ, b and ρ are positive parameters. In the classical analysis $\sigma = 10$, $b = 8/3$ and $\rho \in \mathbb{R}^+ \setminus 0$ is used as bifurcation parameter.
Several aspects of the dynamics of system (5.1) have been investigated since long time. A powerful approach based on the Lyapunov's direct method was introduced by G. A. Leonov for estimating the dimension of the Lorenz attractors (see [23] and references therein). Notably, exploiting some known a-priori bounds on the system solutions and suitably selecting some Lyapunov functions, this approach permits to obtain conditions on the param-

eters σ, b, ρ under which (5.1) displays convergence towards its equilibrium points [24, 25]. It is known that (5.1) has a unique equilibrium point at $x = 0$ for $0 < \rho < 1$ which undergoes to a (supercritical) pitchfork bifurcation at $\rho = 1$ with the birth of two additional equilibrium points $x_{\pm} = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho-1)^{\top}$. If $\sigma > b+1$ then both these equilibrium points undergo to a (subcritical) Hopf bifurcation at $\rho = \rho_M$, where $\rho_M = \sigma(\sigma+b+3)/(\sigma-b-1)$. For $\sigma = 10$, $b = 8/3$ we get $\rho_M = 24.74$. It is also known that for all initial conditions the solutions are eventually confined in some invariant sets [27]. For instance, if $b \geq 2$ and $\sigma \geq 1$ then

$$\mathcal{D} = \left\{ x \in \mathbb{R}^3 : x_2^2 + (\rho - x_3)^2 \leq \frac{\rho^2}{\rho_b^2} \right\} , \tag{5.2}$$

where

$$\rho_b = \frac{2\sqrt{b-1}}{b} , \tag{5.3}$$

is an invariant set of (5.1). Note that $\rho_b \leq 1$ for $b \geq 2$ and hence the equilibrium point $x = 0$ belongs to $\mathcal{D}$. It can be checked that $x_{\pm} \in \mathcal{D}$, too. The Jacobian of the system (5.1) reads

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - x_3 & -1 & -x_1 \\ x_2 & x_1 & -b \end{pmatrix} =: J(x) \tag{5.4}$$

and its 2-additive compound has the following form

$$J^{[2]}(x) = \begin{pmatrix} -(\sigma+1) & -x_1 & 0 \\ x_1 & -(b+\sigma) & \sigma \\ -x_2 & \rho - x_3 & -(b+1) \end{pmatrix} . \tag{5.5}$$

It can be readily checked that $J^{[2]}(0)$ is marginally stable if and only if

$$\rho \leq \rho_s = \frac{(b+\sigma)(b+1)}{\sigma} , \tag{5.6}$$

while $J^{[2]}(x_{\pm})$ is marginally stable if and only if $\rho \leq \rho_M$. Since $\rho_s < \rho_M$ [1], from Proposition 11 it follows that $(0, \rho_s]$ is the largest positive interval $\mathcal{I}$ such that (2.14) can be solved for some $P(x)$, possibly state-dependent, for all $\rho \in \mathcal{I}$.

[1] For $\sigma > b+1$ we have $\rho_M > \sigma + b + 3 > \sigma + b > \rho_s$.

We first consider the case when $P(x)$ is a constant matrix. Specifically, we assume that the matrix P has the following form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} , \tag{5.7}$$

where ε is a positive parameter to be designed later.
According to condition (2.16) of Proposition 12, a sufficient condition to rule out positive Lyapunov exponents for system (5.1) is that the matrix

$$\begin{aligned} Q(x) &= -\left(\left(J^{[2]}\right)^\top (x)\, P + P\, J^{[2]}(x)\right) \\ &= \begin{pmatrix} 2\,(\sigma+1) & 0 & \varepsilon\, x_2 \\ 0 & 2\,(b+\sigma) & -\sigma - \varepsilon\,(\rho - x_3) \\ \varepsilon\, x_2 & -\sigma\, - \varepsilon\,(\rho - x_3) & 2\,\varepsilon\,(b+1) \end{pmatrix} , \end{aligned} \tag{5.8}$$

is positive semidefinite within the invariant set $\mathcal{D}$ in (5.2).
The next result holds true.

Proposition 22 *Suppose that the system parameters satisfy the following conditions*

$$b \geq 2 \ , \qquad \sigma > b - 2 \ , \qquad \rho \leq \rho_b \rho_s \ , \tag{5.9}$$

where ρ_b and ρ_s are as in (5.3) and (5.6), respectively. Then, selecting $\varepsilon = \dfrac{\sigma}{\rho_s} := \varepsilon^*$ we have $Q(x) \geq 0$ for all $x \in \mathcal{D}$.

Proof. It can be readily verified that the matrix $Q(x)$ is positive semidefinite once its determinant is greater or equal than zero. This condition leads to the following inequality:

$$2(\sigma+1)(4(b+\sigma+1)(b+1)\varepsilon - (\sigma + \varepsilon(\rho - x_3)^2) - 2(b+\sigma)\varepsilon^2 x_2^2 \geq 0 \ , \quad \forall x \in \mathcal{D} \ . \tag{5.10}$$

Defining the new variable $\bar{x}_3 = \rho - x_3$, the invariant set $\mathcal{D}$ becomes the following cylinder in the $(x_1, x_2, \bar{x}_3)$ coordinates:

$$\mathcal{D} = \left\{ x \in \mathbb{R}^3 : x_2^2 + \bar{x}_3^2 \leq \frac{\rho^2}{\rho_b^2} \right\} \ . \tag{5.11}$$

Similarly, by setting $\varepsilon = \varepsilon^*$ the inequality in (5.10) can be rewritten as

$$\frac{\sigma+\beta}{\sigma+1} x_2^2 + (\bar{x}_3 + \rho_s)^2 \leq 4\rho_s^2 \ . \tag{5.12}$$

For any fixed b and σ, on the $(x_2, \bar{x}_3)$-plane (5.12) defines the closed region bounded by an ellipse centered at $x_2 = 0$, $\bar{x}_3 = -\rho_s$ and symmetric with respect to the $\bar{x}_3$-axis, while equation (5.11) is a disk centered at $x_2 = 0$, $\bar{x}_3 = 0$, whose radius grows as ρ increases. Therefore, since $x_2 = 0$, $\bar{x}_3 = 0$ satisfies (5.11), the matrix $Q(x)$ is positive semidefinite on the invariant set $\mathcal{D}$ if the ellipse (5.12) contains the disk. In particular, the maximum achievable ρ is obtained when the circle bounding the disk is tangent to the ellipse. Hence, let $\rho = \rho_s\rho_b$ and denote by $\bar{x}_{3_C}$ the values of $\bar{x}_3$ on the circle with $\rho = \rho_s\rho_b$, which implies $x_2^2 = \rho_s^2 - \bar{x}_{3_C}^2$. Then, condition (5.12) can be written as:

$$4\rho_s^2 - \frac{\sigma+b}{\sigma+1}(\rho_s^2 - \bar{x}_{3_C}^2) - (\bar{x}_{3_C} + \rho_s)^2 \geq 0 \ . \tag{5.13}$$

The left-hand side of (5.13) is a convex quadratic function of $\bar{x}_{3_C}$ and it achieves its global minimum value at:

$$\bar{x}_{3_C}^* = \frac{\sigma+1}{b-1}\rho_s \ . \tag{5.14}$$

Notice that, for $\sigma + 2 - b > 0$ we get:

$$\frac{\sigma+1}{b-1}\rho_s > \rho_s \ , \tag{5.15}$$

which implies that the point $x = (0, 0, \bar{x}_{3_C}^*)^\top \notin \mathcal{D}$ with $\rho = \rho_b\rho_s$. Hence, the left-hand side of (5.13) assumes its minimum value within the interval of variation of $\bar{x}_{3_C}$ exactly on its boundary, i.e. for $\bar{x}_{3_C} = \dfrac{\rho_s\rho_b}{\rho_b} = \rho_s$. Setting this value in (5.13), we obtain:

$$4\rho_s^2 - \frac{\sigma+b}{\sigma+1}(\rho_s^2 - \rho_s^2) - (\rho_s + \rho_s)^2 = 0 \ , \tag{5.16}$$

thus completing the proof. ■

Remark 16 *It is worth noting that the matrix* $P^* := \mathrm{diag}(1, 1, \varepsilon^*)$ *ensures that* $Q(x)$ *is positive semidefinite for all* $x \in \mathcal{D}$ *if* $\rho = \rho_b\rho_s$ *and positive definite for all* $x \in \mathcal{D}$ *if* $\rho < \rho_b\rho_s$*. Moreover, for* $\rho = \rho_b\rho_s$*,* $Q(x)$ *is positive definite for all* $x \in \mathcal{D}$ *except for* $x = x^* = (0, 0, x_3^*)^\top$*,* $x_3^* = -\left(\dfrac{\sigma+1}{b-1} - \rho_b\right)\rho_s$*, where it vanishes.*

For $b = 8/3$ and $\sigma = 10$, we get that $\rho_b\rho_s \approx 4.4970$ and hence positive Lyapunov exponents are ruled out for ρ less than such number. Figure 5.1

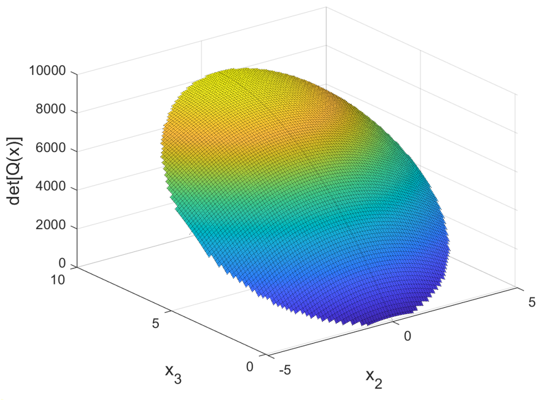


Figure 5.1: Determinant of $Q(x)$ for $x \in \mathcal{D}$ ($b = 8/3$, $\sigma = 10$, $\rho = 4.4969$).

displays the determinant of $Q(x)$ for $\varepsilon = \varepsilon^* = 2.1531$ and $\rho = 4.4969$ showing that it is always positive inside the relative set $\mathcal{D}$, which implies that $Q(x) > 0$ since it is positive definite at $x = (0, 0, \rho)$.
We note that for $b = 2$ we have $\rho_b = 1$, while for $b > 2$ the upper bound ρ_s of ρ is no longer achieved, since $\rho_b < 1$. In order to obtain a larger bound for the value of ρ, a state-dependent matrix $P(x) = P \exp(V(x))$ is considered where P is as in (5.7) and $V(x)$ is the following quadratic function

$$V(x) = \kappa(x_2^2 + x_3(x_3 - 2\rho)) \ , \tag{5.17}$$

with κ being a positive parameter to be designed later. In this case, to rule out the existence of positive Lyapunov exponents condition (2.17) should be satisfied. This amounts to require that the matrix

$$S(x) := Q(x) - P\dot{V}(x) = \begin{pmatrix} 2(\sigma+1) - \dot{V}(x) & 0 & \varepsilon\, x_2 \\ 0 & 2(b+\sigma) - \dot{V}(x) & -\sigma - \varepsilon\,(\rho - x_3) \\ \varepsilon\, x_2 & -\sigma - \varepsilon\,(\rho - x_3) & \varepsilon\left(2(b+1) - \dot{V}(x)\right) \end{pmatrix}, \tag{5.18}$$

where the time-derivative of $V(x)$ reads

$$\dot{V}(x) = -2\,\kappa\left({x_2}^2 + b x_3(x_3 - \rho)\right) \ , \tag{5.19}$$

should be positive semidefinite within the set $\mathcal{D}$ in (5.2). Since from Remark 16 we have that $Q(x)$ vanishes at $x = x^*$, where $x_2^* = 0$ and $x_3^* < 0$, it turns out that $\dot{V}(x^*)$ is negative for all $\kappa > 0$. This ensures that $S(x)$ is positive definite at $x = x^*$ thus making it possible to enlarge the value of ρ with respect to the bound (5.9) ensured by Proposition 22. Indeed, we have the next result where the functions $\sigma_1(b)$ and $\sigma_2(b)$ are defined as

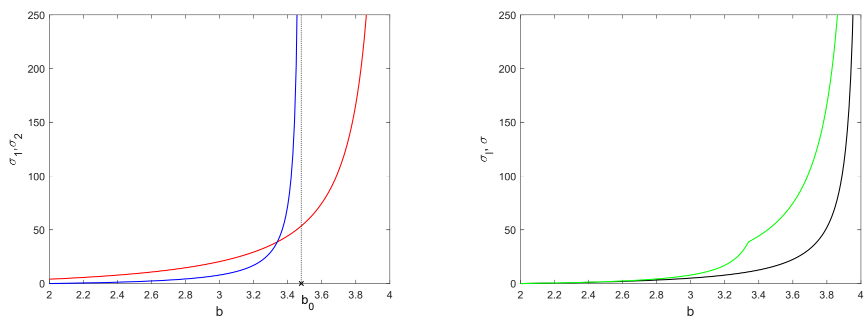
$$\sigma_1(b) := \frac{(4b^2 - 5b - 7) + \sqrt{(4b^2 - 5b - 7)^2 - 4(4 - b)(6b + 4 - 5b^2 - 4b^3)}}{2(4 - b)} \tag{5.20}$$

$b \in [2, 4)$ and

$$\sigma_2(b) := \begin{cases} \dfrac{b^2 - 4 + 2\left(\dfrac{1}{\rho_b} - 1\right)(b^2 - 1)}{4 - b - 2\left(\dfrac{1}{\rho_b} - 1\right)(b - 1)} & \text{if } \ b \in [2, b_0) \\ +\infty & \text{if } \ b \in [b_0, 4) \end{cases} , \tag{5.21}$$

with $b_0 = 3.48$ being the unique positive solution of the scalar equation

$$4 - b - 2\left(\frac{1}{\rho_b} - 1\right)(b - 1) = 0 \ . \tag{5.22}$$



a) b)

Figure 5.2: a) Functions $\sigma_1(b)$ (red) and $\sigma_2(b)$ (blue) for $b \in [2, 4)$; b) Functions $\sigma(b)$ (green) and $\sigma_l(b)$ (black) for $b \in [2, 4)$.

Proposition 23 *Suppose that the system parameters satisfy the following conditions*

$$b \in [2,4)\ , \qquad \sigma \geq \sigma(b)\ , \qquad \rho \leq \rho_s\ , \tag{5.23}$$

where

$$\sigma(b) := \min\{\sigma_1(b), \sigma_2(b)\} \tag{5.24}$$

and ρ_s is as in (5.6). Then, choosing $\varepsilon = \dfrac{\sigma}{\rho}$ and $\kappa = \dfrac{\gamma}{\rho^2}$, with $\gamma = \dfrac{(b+\sigma)(b+1)}{b(2b+\sigma+1)}$, we have $S(x) \geq 0$ for all $x \in \mathcal{D}$.

Before proving Proposition 23, the following Lemma is needed.

Lemma 6 *Consider the matrix*

$$S_{\rho_s}(z) := \begin{pmatrix} 2\,(\sigma+1) - \dot{\bar{V}}(z) & 0 & \sigma z_2 \\ 0 & 2\,(b+\sigma) - \dot{\bar{V}}(z) & \sigma(z_3-2) \\ \sigma z_2 & \sigma(z_3-2) & \dfrac{\sigma}{\rho_s}\left(2\,(b+1) - \dot{\bar{V}}(z)\right) \end{pmatrix}, \tag{5.25}$$

where $\dot{\bar{V}}(z) = -2\gamma(z_2^2 + bz_3(z_3-1))$ with $\gamma = \dfrac{(b+\sigma)(b+1)}{b(2b+\sigma+1)}$. If $b \in [2, b_0)$ and $\sigma > \sigma(b)$, then

$$\min_{z\in\bar{\mathcal{D}}} \det S_{\rho_s}(z) = \min_{z_3\in[1-1/\rho_b,2]} \det S_{\rho_s}(z)_{|z_2=0}\ . \tag{5.26}$$

Proof. Since $\bar{S}(z)$ depends only on z_2 and z_3, $\det \bar{S}(z)$ reduces to a function $H(z_2, z_3)$ which for $\rho = \rho_s$ can be written as

$$H(z_2,z_3) = \lambda_0(z_2,z_3) + \lambda_1(z_2,z_3)R(z_2,z_3) + \lambda_2 R^2(z_2,z_3) + \lambda_3 R^3(z_2,z_3)\ , \tag{5.27}$$

where

$$
\begin{aligned}
R(z_2, z_3) &= \frac{z_2^2}{b} + \left(z_3 - \frac{1}{2}\right)^2 - \frac{1}{4} \\
\lambda_0(z_2, z_3) &= 2\sigma^2(\sigma+1)\left(4 - \frac{b+\sigma}{\sigma+1}z_2^2 - (z_3-2)^2\right) \\
\lambda_1(z_2, z_3) &= 2\gamma b\sigma^2\left(4\left(1 + \frac{(\sigma+1)(2b+\sigma+1)}{(b+1)(b+\sigma)}\right) - z_2^2 - (z_3-2)^2\right) \\
\lambda_2 &= 4\gamma^2 b^2 \frac{\sigma^2(b+\sigma+1)}{(b+1)(b+\sigma)} \\
\lambda_3 &= 8\gamma^3 b^3 \frac{\sigma^2}{(b+1)(b+\sigma)} \ .
\end{aligned}
\tag{5.28}
$$

On the (z_2, z_3)-plane the cylinder $\bar{\mathcal{D}}$ reduces to the disk

$$
\bar{\mathcal{D}}_\circ = \left\{(z_2, z_3)^\top \in \mathbb{R}^2 : \ z_2^2 + (z_3-1)^2 \leq \frac{1}{\rho_b^2}\right\} \ , \tag{5.29}
$$

and hence z_3 is confined to lie in the interval $[1 - 1/\rho_b, 1 + 1/\rho_b]$. Observe that the function $\lambda_0(z_2, z_3)$ is constant over the ellipse

$$
\mathcal{E}_\mu = \left\{(z_2, z_3)^\top \in \mathbb{R}^2 : \ \frac{b+\sigma}{\sigma+1}z_2^2 + (z_3-2)^2 = \mu^2\right\} \ , \tag{5.30}
$$

parameterized by $\mu \geq 0$ and centered at $z_2 = 0$, $z_3 = 2$.
Consider the intersection of this ellipse with the disk (5.29), i.e., the set

$$
\Gamma_\mu = \mathcal{E}_\mu \cap \bar{\mathcal{D}}_\circ \ , \tag{5.31}
$$

which reduces to the point $z_2 = 0$, $z_3 = 2$ for $\mu = 0$, it is the entire ellipse for small positive μ, a single arc or a couple of arcs symmetric with respect to the z_3-axis for sufficiently large μ, and the empty set for much larger μ. Figure 5.3 illustrates this scenario, also showing that in general Γ_μ may not intersect the z_3-axis. Specifically, this happens when $\mathcal{E}_\mu$ and the circle bounding $\bar{\mathcal{D}}_\circ$ have more than one common point with $z_2 > 0$. From (5.29) and (5.30) we have that a point of the circle bounding $\bar{\mathcal{D}}_\circ$ belongs to Γ_μ only if its coordinate z_3 is such that

$$
\frac{b+\sigma}{\sigma+1}\left(\frac{1}{\rho_b^2} - (z_3-1)^2\right) + (z_3-2)^2 = \mu^2 \ ,
$$

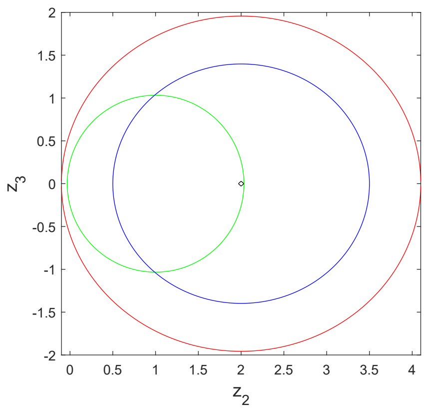


Figure 5.3: Plot of the disk $\mathcal{D}_\circ$ and the ellipse $\mathcal{E}_\mu$ for a different values of μ.

which can be rewritten as

$$-\frac{b-1}{\sigma+1}z_3^2 - 2\frac{\sigma+2-b}{\sigma+1}z_3 + \frac{b+\sigma}{\sigma+1}\frac{1}{\rho_b^2} + \frac{4-b+3\sigma}{\sigma+1} = \mu^2 \ . \qquad (5.32)$$

The left term of is a concave parabola which is positive for $z_3 = 0$ and whose maximum value is achieved at $z_3 = -(\sigma+2-b)/(b-1)$. Since for all $b \in [2, b_0)$ we have that

$$\sigma(b) > b - 2 + (\frac{1}{\rho_b} - 1)(b-1)$$

and, hence,

$$-\frac{\sigma+2-b}{b-1} < 1 - \frac{1}{\rho_b} \ ,$$

for $\sigma > \sigma(b)$. This implies that there is a unique solution of (5.32) z_3 lying in the interval $[1-1/\rho_b, 1+1/\rho_b]$, thus yielding that each set Γ_μ intersects the z_3-axis at $z_3 = z_{3_\mu}$ with $z_{3_\mu} = 2-\mu$. Specifically, we have that Γ_μ is the entire ellipse for $0 < \mu \leq 1/\rho_b - 1$, a single arc for $1/\rho_b - 1 < \mu \geq 1 + 1/\rho_b$, while it reduces to the point $z_2 = 0$, $z_3 = 2$ (resp. $z_2 = 0$, $z_3 = 1 - 1/\rho_b$) for $\mu = 0$ (resp. for $\mu = 1 + 1/\rho_b$).
Observe that $\lambda_1(z_2, z_3)$ is constant along circles centered at $z_2 = 0$, $z_3 = 2$ and it decreases as the distance from the center increases. Since $b + \sigma >$

$\sigma + 1$ for all $b \in [2, b_0)$ and $\sigma > 0$, it can be readily checked that for any $\mu \in [0, 1 + 1/\rho_b]$ we have:

$$\lambda_1(z_2, z_3) \geq \lambda_1(0, z_{3_\mu}) = \lambda_1(0, 2 - \mu) \ , \quad \forall \ (z_2, z_3)^\top \in \Gamma_\mu \tag{5.33}$$

i.e., the minimum of $\lambda_1(z_2, z_3)$ on any set Γ_μ is achieved at $(z_2, z_3)^\top = (0, z_{3_\mu})^\top$.

The function $R(z_2, z_3)$ is constant along ellipses centered at $z_2 = 0$, $z_3 = 1/2$ and it decreases as the distance from the center decreases. In particular, consider the ellipse

$$\mathcal{R}_\mu = \left\{ (z_2, z_3)^\top \in \mathbb{R}^2 : \ \frac{z_2^2}{b} + \left(z_3 - \frac{1}{2} \right)^2 = \left(\frac{3}{2} - \mu \right)^2 \right\} \tag{5.34}$$

and assume that $\mu \in [0, 3/2]$. It can be readily checked that Γ_μ is tangent to $\mathcal{R}_\mu$ at $(z_2, z_3)^\top = (0, z_{3_\mu})^\top$, while all other points are outside the bounded region defined by $\mathcal{R}_\mu$. This implies that

$$R(z_2, z_3) \geq R(0, z_{3_\mu}) = R(0, 2 - \mu) \ , \quad \forall \ (z_2, z_3)^\top \in \Gamma_\mu \ , \ \ \mu \in [0, 3/2] \ . \tag{5.35}$$

By construction, $\mathcal{R}_\mu$ intersects Γ_μ, and hence $\mathcal{E}_\mu$, at $(z_2, z_3)^\top = (0, z_{3_\mu})^\top$ also for $\mu \in (3/2, 1 + 1/\rho_b]$. From (5.30) and (5.34) it follows that $\mathcal{E}_\mu$ and $\mathcal{R}_\mu$ have another intersection if and only if the relation

$$\frac{b + \sigma}{\sigma + 1} b \left(\left(\frac{3}{2} - \mu \right)^2 - (z_3 - \frac{1}{2})^2 \right) + (z_3 - 2)^2 = \mu^2 \ , \tag{5.36}$$

is solved for some $z_3 \in (2 - \mu, 2 + \mu]$. Since (5.36) can be equivalently expressed as

$$-\frac{(b - 1)(b + \sigma + 1)}{\sigma + 1} (z_3 - 2 + \mu) \left(z_3 + 2 - \mu + \frac{(4 - b)\sigma - b^2 + 4}{(b - 1)(b + \sigma + 1)} \right) = 0 \ ,$$

it follows that the condition for the existence of such additional intersection boils down to

$$\frac{(4 - b)\sigma - b^2 + 4}{(b - 1)(b + \sigma + 1)} < 2(\mu - 2) \ .$$

However, since $\sigma > \sigma(b)$ implies

$$\sigma > \frac{b^2 - 4 + 2(\frac{1}{\rho_b} - 1)(b^2 - 1)}{4 - b - 2(\frac{1}{\rho_b} - 1)(b - 1)}$$

which can be rewritten as

$$\frac{(4-b)\sigma - b^2 + 4}{(b-1)(b+\sigma+1)} > 2(\frac{1}{\rho_b} - 1) \ ,$$

it follows that for any $\mu \in [3/2, 1+1/\rho_b]$ the ellipses $\mathcal{R}_\mu$ and $\mathcal{E}_\mu$, and hence Γ_μ, do not have the additional intersection. This means that all the points of the Γ_μ with $z_2 \neq 0$ are outside the bounded region defined by $\mathcal{R}_\mu$, implying that condition (5.35) indeed holds for all $\mu \in [0, 1+1/\rho_b]$.
Conditions (5.33) and (5.35) ensure that for each $\mu \in [0, 1+1/\rho_b]$ the relation

$$\begin{aligned} H(z_2, z_3) \geq & \lambda_0(0, z_{3_\mu}) + \lambda_1(0, z_{3_\mu}) R(0, z_{3_\mu}) + \lambda_2 R^2(0, z_{3_\mu}) + \lambda_3 R^3(0, z_{3_\mu}) \\ & = H(0, z_{3_\mu}) \ , \quad \forall \ (z_2, z_3)^\top \in \Gamma_\mu \ , \end{aligned}$$

holds. Taking into account that $z_{3_\mu} = 2 - \mu$ and $\bar{D}_0 \equiv \bigcup_\mu \Gamma_\mu$, $\mu \in [0, 1+1/\rho_b]$, we can conclude that

$$\min_{(z_2,z_3)^\top \in \bar{\mathcal{D}}_0} H(z_2, z_3) = \min_{z_3 \in [1-1/\rho_b, 2]} H(0, z_3) \ ,$$

thus completing the proof. ■

Proof. of Proposition 23 - By introducing the scaled variables $z_1 = \dfrac{x_1}{\rho}$, $z_2 = \dfrac{x_2}{\rho}$, $z_3 = \dfrac{x_3}{\rho}$ and taking into account the expressions of ε and κ, it turns out that $S(x)$ is semidefinite positive for all $x \in \mathcal{D}$ if and only if the matrix

$$S_\rho(z) := \begin{pmatrix} 2\,(\sigma+1) - \dot{V}(z) & 0 & \sigma z_2 \\ 0 & 2\,(b+\sigma) - \dot{V}(z) & \sigma(z_3 - 2) \\ \sigma z_2 & \sigma(z_3-2) & \dfrac{\sigma}{\rho}\left(2\,(b+1) - \dot{V}(z)\right) \end{pmatrix}, \tag{5.37}$$

where $\dot{V}(z) = -2\,\gamma\left({z_2}^2 + b z_3(z_3 - 1)\right)$, is positive semidefinite for all z belonging to the following cylinder

$$\bar{\mathcal{D}} = \left\{ z \in \mathbb{R}^3 : z_2^2 + (z_3 - 1)^2 \leq \frac{1}{\rho_b^2} \right\} \ , \tag{5.38}$$

which depends only on the system parameter b. Since $\dot{V}(z)$ does not depend on ρ, the structure of $S_\rho(z)$ yields that if $S_\rho(z)$ is positive semidefinite on

$\bar{\mathcal{D}}$ for $\rho = \rho_s$ then it is also positive semidefinite for all $\rho < \rho_s$. Hence, it is enough to consider the case $\rho = \rho_s$ and hence the matrix $S_{\rho_s}(z)$.
The first step is to show that the determinant of $S_{\rho_s}(z)$ has a local minimum at $z = 0$ where it vanishes. Tedious though straightforward computations lead to:

$$\det S_{\rho_s}(z)\Big|_{z=0} = 0 \tag{5.39}$$

$$\begin{aligned} \frac{\partial \det S_{\rho_s}(z)}{\partial z_2}\Big|_{z=0} &= 0 \\ \frac{\partial \det S_{\rho_s}(z)}{\partial z_3}\Big|_{z=0} &= 0 \end{aligned} \tag{5.40}$$

$$\begin{aligned} \frac{\partial^2 \det S_{\rho_s}(z)}{\partial z_2^2}\Big|_{z=0} &= 4\frac{\sigma^2}{b}\left(4(\sigma+1) - b(b+\sigma)\right) \\ \frac{\partial^2 \det S_{\rho_s}(z)}{\partial z_2 \partial z_3}\Big|_{z=0} &= 0 \\ \frac{\partial^2 \det\ S_{\rho_s}(z)}{\partial z_3^2}\Big|_{z=0} &= 4\sigma^2(\sigma+1)\left(3 + 4\frac{(b+1)(b+\sigma)}{(2b+\sigma+1)^2}\right) \end{aligned} \quad . \tag{5.41}$$

Clearly, $z = 0$ is a local minimum if and only if $(4-b)\sigma + 4 - b^2 > 0$, which in turn implies $b \in [2, 4)$ and

$$\sigma > \frac{b^2-4}{4-b} =: \sigma_l(b) \ . \tag{5.42}$$

Since $\sigma(b) \geq \sigma_l(b)$ for $b \in [2, 4)$ (see also Fig. 5.2), we have that $\det S_{\rho_s}(z)$ has a local minimum at $z = 0$.
The second step is to show that this is indeed a global minimum if $\sigma \geq \sigma(b)$. Taking into account (5.24) and (5.20)-(5.21), this is accomplished by considering two cases: I) $\sigma > \sigma_1(b)$, $b \in [2, 4)$; II) $\sigma > \sigma_2(b)$, $b \in [2, b_0)$.
Case I). It can be observed that the inequality $S_{\rho_s}(z) \geq \bar{S}_{\rho_s}(z)$ holds for all $z \in \bar{\mathcal{D}}$, where

$$\bar{S}_{\rho_s}(z) = \begin{pmatrix} 2(\sigma+1+G(z_3)) & 0 & \sigma z_2 \\ 0 & 2(b+\sigma+G(z_3)) & \sigma(z_3-2) \\ \sigma z_2 & \sigma(z_3-2) & \frac{2\sigma^2}{(b+1)(b+\sigma)}\left(1 + \frac{(b+1)(b+\sigma)}{b(2b+\sigma+1)}z_2^2 + G(z_3)\right) \end{pmatrix},$$

and

$$G(z_3) = \frac{1}{2}\left(\dot{\bar{V}}(z) - \frac{(b+1)(b+\sigma)}{b(2b+\sigma+1)}z_2^2\right) = \frac{(b+1)(b+\sigma)}{2b+\sigma+1}z_3(z_3-1)\ .$$

The determinant of $\bar{S}_{\rho_s}(z)$ can be written as

$$\det \bar{S}_{\rho_s}(z) = M(z_3) + N(z_3)z_2^2$$

where

$$\begin{aligned} M(z_3) &= 2\sigma^2\left(\sigma+1+G(z_3)\right)\left(3+4\frac{(b+1)(b+\sigma)}{2b+\sigma+1}(z_3-1)^2\right)z_3^2 \\ N(z_3) &= \frac{2\sigma^2}{b(2b+\sigma+1)}\left(b+\sigma+G(z_3)\right)\left(4(\sigma+1+G(z_3)) - b(2b+\sigma+1)\right) \end{aligned} .$$

Taking into account that $b \in [2,4)$, we have that $M(z_3)$ and $N(z_3)$ are both non-negative if the following condition

$$\min_{1-1/\rho_b \le z_3 \le 1+1/\rho_b} 4(\sigma+1+G(z_3)) = 4(\sigma+1) - \frac{(b+1)(b+\sigma)}{2b+\sigma+1} \ge b(2b+\sigma+1),$$

which can be rewritten as

$$(4-b)\sigma^2 - (7+5b-4b^2)\sigma + 4 + 6b - 5b^2 - 4b^3 \ge 0\ ,$$

holds. Since such a condition is satisfied if and only if $\sigma \ge \sigma_1(b)$, it follows that $S_{\rho_s}(z) \ge \bar{S}_{\rho_s}(z) \ge 0$ for all $z \in S_{\rho_s}(z)$, which proves that the local minimum is indeed a global one.
Case II). To show that $z = 0$ is a global minimum when $\sigma > \sigma_2(b)$, $b \in [2, b_0)$, we exploit condition (5.26) of Lemma 6 in the appendix which states that it is enough to prove that $z_3 = 0$ is the global minimum of $\det S_{\rho_s}(z)_{|z_2=0}$. It can be verified that $\det S_{\rho_s}(z)_{|z_2=0}$ can be written as

$$\begin{aligned} \det S_{\rho_s}(z)_{|z_2=0} = 2\sigma^2 &\left(\sigma+1+\frac{(b+1)(b+\sigma)}{(2b+\sigma+1)}z_3(z_3-1)\right)\cdot \\ &\left(3+4\frac{(b+1)(b+\sigma)}{(2b+\sigma+1)^2}(z_3-1)^2\right)z_3^2\ , \end{aligned} \tag{5.43}$$

thus proving that $z = 0$ is a global minimum since

$$\min_{z_3\in[1-1/\rho_b,2]} \sigma+1+\frac{(b+1)(b+\sigma)}{(2b+\sigma+1)}z_3(z_3-1) = \sigma+1-\frac{(b+1)(b+\sigma)}{4(2b+\sigma+1)} > 0. \quad \blacksquare$$

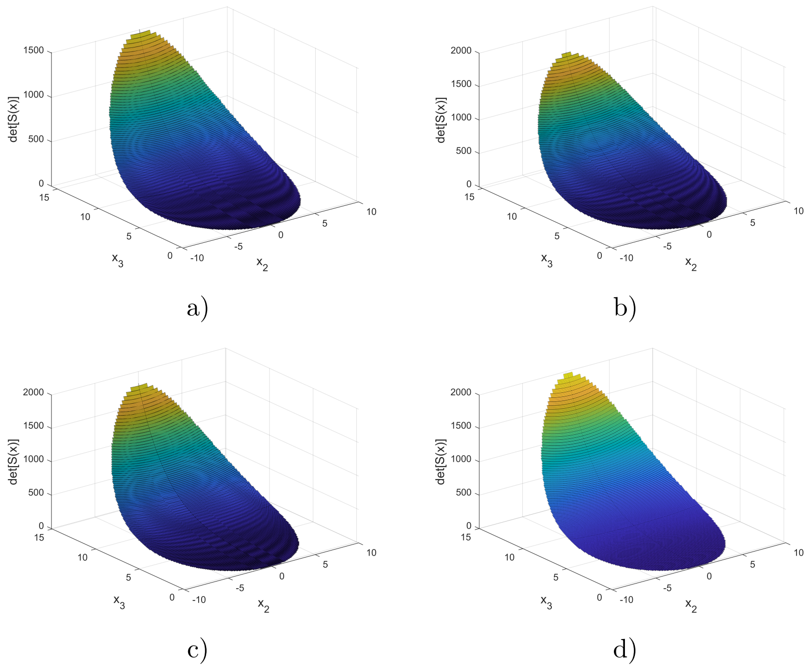


Figure 5.4: Determinant of $S(x)$ for $x \in \mathcal{D}$: a) $\sigma = 2.4$, $\rho = 7.73$; b) $\sigma = 2.5$, $\rho = 7.57$; c) $\sigma = 2.6$, $\rho = 7.42$; d) $\sigma = 2.7$, $\rho = 7.28$.

Comments and final remarks *It is worth noting that Proposition 23 applies to the standard values $b = 8/3$ and $\sigma = 10$ since $\sigma(8/3) = 2.8692$. Also, as shown in Fig. 5.2, there is a gap between $\sigma(b)$ and $\sigma_l(b)$, i.e., if $\sigma \in (\sigma_l(b), \sigma(b))$ it is proven that $x = 0$ is a local minimum of $\det S(x)$ for $\rho = \rho_s$ but not that it is a global one. Since condition (5.26) of Lemma 6 in the appendix is somewhat conservative, it can be conjectured that Proposition 23 holds even if conditions (5.23) are replaced with*

$$b \in [2, 4) \ , \qquad \sigma > \frac{b^2 - 4}{4 - b} \ , \qquad \rho \leq \rho_s \ .$$

Indeed, extensive numerical simulations confirm the validity of the conjecture. As an example, Fig. 5.4 displays the determinant of $S(x)$ for $b = 8/3$, $\rho = \rho_s(b, \sigma) - \delta$, where $\delta = 0.01$ and some values of $\sigma \in [\sigma_l(8/3), \sigma(8/3)] =$

$(2.3333, 2.8692)$. *According to Remark 5, we can conclude that if the system parameters satisfy either condition (5.9), with* $\rho < \rho_b\rho_s$, *or condition (5.23), with* $\rho < \rho_s$, *all the solutions of (5.1) converge towards one of two asymptotically stable equilibrium points* $x_\pm = (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho-1)^\top$. *As an example, this convergent scenario is illustrated in Fig. 5.5 for the case of Lorenz system with parameters* $b = 8/3$, $\sigma = 10$, *and* $\rho = 4.6$.

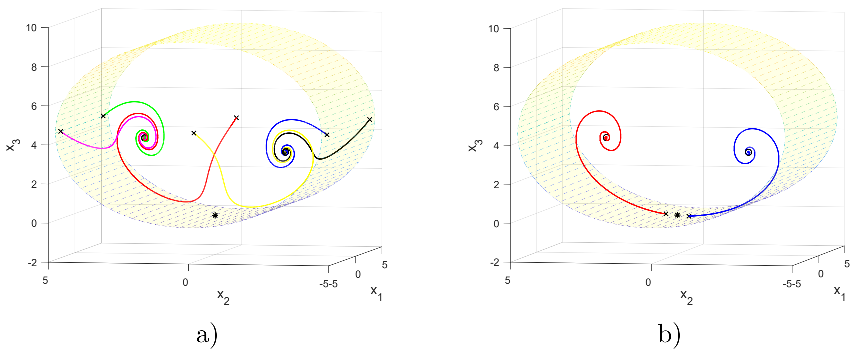


Figure 5.5: Trajectories of Lorenz system with parameters $b = 8/3$, $\sigma = 10$, $\rho = 4.6$. Stable equilibrium points (marked by $\circ$): $x_\pm = (\pm 3.099, \pm 3.099, 3.6)$; unstable equilibrium point at $x = 0$ (marked by $*$). The initial conditions $(x_1(0), x_2(0), x_3(0))$ (marked by $\times$) are: a) $(4, -4.75, 4.6)$, $(4, 0, 4.6)$, $(4, 4.75, 4.6)$, $(-4, -4.75, 4.6)$, $(-4, 0, 4.6)$, $(-4, 4.75, 4.6)$; b) $(-0.5, -0.5, 0)$, $(0.5, 0.5, 0)$.

5.1.2 Thomas system

The three-dimensional Thomas system is described by the following equations

$$\begin{array}{rcl} \dot{x}_1 & = & \sin(x_2) - b\,x_1 \\ \dot{x}_2 & = & \sin(x_3) - b\,x_2 \\ \dot{x}_3 & = & \sin(x_1) - b\,x_3 \end{array}, \qquad (5.44)$$

where $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$ is the state vector and b is a positive parameter. This simple system displays quite a rich dynamics including the so-called 'labyrinth chaos' [49]. It is known that

$$\mathcal{D} := \{x \in \mathbb{R}^3 : b\|x\|_\infty \le 1\} \qquad (5.45)$$

is an invariant set of (5.44).
A detailed analysis of the route to chaos for decreasing values of b can be found in [47], where the behavior of the maximum Lyapunov exponent is also reported. For all $b > 1$ there is a unique equilibrium point at $x = 0$ inside $\mathcal{D}$. At $b = 1$ this equilibrium point undergoes to a (supercritical) pitchfork bifurcation with the birth of two additional symmetrical equilibrium points at $x = \pm(\eta, \eta, \eta)^\top$, with η such that $\eta = \sin\eta/b$. Both these equilibrium points undergo to a (supercritical) Hopf bifurcation when η is such that $\tan(\eta) = -0.5\eta$ which yields $\eta = \eta_h = 2.2889$ and corresponds to $b_{\mathcal{H}} = 0.3290$. As b decreases within the range between $0.11 < b < b_{\mathcal{H}}$, the system displays quite a rich dynamical scenario characterized by a succession of period-doubling bifurcations and the presence of strange attractors
To obtain conditions ruling out positive Lyapunov exponents, we compute the Jacobian of the system (5.44)

$$J(x) = \begin{pmatrix} -b & \cos(x_2) & 0 \\ 0 & -b & \cos(x_3) \\ \cos(x_1) & 0 & -b \end{pmatrix} \tag{5.46}$$

and its 2-additive compound

$$J^{[2]}(x) = \begin{pmatrix} -2\,b & \cos(x_3) & 0 \\ 0 & -2\,b & \cos(x_2) \\ -\cos(x_1) & 0 & -2\,b \end{pmatrix} . \tag{5.47}$$

It can be verified that $J^{[2]}(0)$ is marginally stable if and only if $b \geq 0.25$, while $J^{[2]}(\pm(\eta_h, \eta_h, \eta_h))$ is marginally stable if and only if $b \geq 0.3290$. Hence, according to Proposition 11, condition (2.14) can admit a solution $P(x)$ only if

$$b \geq b_l = 0.3290 . \tag{5.48}$$

It is worth noting that for any fixed $b \geq b_l$ the set of compound matrices $J^{[2]}(x)$ for $x \in \mathcal{D}$ can be equivalently described by the set of matrices obtained by introducing the new variables $z_1 = \cos(x_1)$, $z_2 = \cos(x_2)$, $z_3 = \cos(x_3)$. Specifically, the matrices in (5.117) are replaced with

$$\bar{J}^{[2]}(z) = \begin{pmatrix} -2\,b & z_3 & 0 \\ 0 & -2\,b & z_2 \\ -z_1 & 0 & -2\,b \end{pmatrix} , \tag{5.49}$$

and the set $\mathcal{D}$ is transformed into

$$\bar{\mathcal{D}} := \{z \in \mathbb{R}^3 : \cos\left(\frac{1}{b}\right) \leq z_i \leq 1 \ , \quad i = 1, 2, 3\} \ . \tag{5.50}$$

According to condition (2.16) of Proposition 12, system (5.44) has no attractors with positive Lyapunov exponents if the following condition

$$\left(\bar{J}^{[2]}\right)^\top (z)\, P + P\, \bar{J}^{[2]}(z) \leq 0 \tag{5.51}$$

holds for all $z \in \bar{\mathcal{D}}$. Observe that $\bar{J}^{[2]}(z)$ depends affine linearly on z and $\bar{\mathcal{D}}$ is the convex hull of the following 8 vertices:

$$\begin{aligned}\mathcal{V} = \{ \ & (1,1,1)^\top, (\cos(1/b),1,1)^\top, (1,\cos(1/b),1)^\top, (1,1,\cos(1/b))^\top, \\ & (\cos(1/b),\cos(1/b),1)^\top, (1,\cos(1/b),\cos(1/b))^\top, \\ & (\cos(1/b),1,\cos(1/b))^\top, (\cos(1/b),\cos(1/b),\cos(1/b))^\top \} \ .\end{aligned} \tag{5.52}$$

Hence, we can exploit Remark 8 which ensures that there are no positive Lyapunov exponents if there is $P > 0$ such that the LMI (5.51) holds on these vertices, a feasibility problem that can solved numerically in an efficient way.
However, due to the strong symmetry enjoyed by $\bar{J}^{[2]}(z)$ and $\bar{\mathcal{D}}$, an analytic bound can be readily obtained by choosing P as the identity matrix, which makes the problem reducing to verify that the matrix

$$Q(z) = \begin{pmatrix} 4\,b & -z_3 & z_1 \\ -z_3 & 4\,b & -z_2 \\ z_1 & -z_2 & 4\,b \end{pmatrix} \tag{5.53}$$

is positive semidefinite for all $z \in \mathcal{V}$. The next result holds true.

Proposition 24 *Let b_i^0, $i = 1, \ldots, 4$, be the largest positive solution for $b > 0$ of the scalar equation $F_i(b) = 0$, $i = 1, \ldots, 4$, where*

$$\begin{aligned} F_1(b) &= 64\,b^3 - 2 + 12\,b \\ F_2(b) &= 64\,b^3 + 2\,\cos\left(\frac{1}{b}\right) - 4\,b\left(2 + \cos^2\left(\frac{1}{b}\right)\right) \\ F_3(b) &= 64\,b^3 + 2\,\cos^2\left(\frac{1}{b}\right) - 4\,b\left(1 + 2\,\cos^2\left(\frac{1}{b}\right)\right) \\ F_4(b) &= 64\,b^3 + 2\,\cos^3\left(\frac{1}{b}\right) - 12\,b\,\cos^2\left(\frac{1}{b}\right) \end{aligned} \tag{5.54}$$

Then, $Q(z) \geq 0$ *for all* $z \in \mathcal{V}$ *if and only if*

$$b \geq b^* = \max_{i=1,\ldots,4} b_i^0 \ . \tag{5.55}$$

Proof. It can be readily checked that for any fixed $b \geq b_l$ the first two leading principal minors of (5.53) are positive, thus implying that it is enough to prove that $\det Q(z) \geq 0$ for all $z \in \mathcal{V}$. Since

$$\det Q(z) = 64\, b^3 + 2\, z_1\, z_2\, z_3 - 4\, b\, \left(z_1^2 + z_2^2 + z_3^2\right) \ , \tag{5.56}$$

setting z as any vertex (5.52) makes $\det Q(z)$ being equal to one the four scalar equations $F_i(b)$, $i = 1, \ldots, 4$, in (5.54). By definition of b_i^0, $i = 1, \ldots, 4$, we have that $F_i(b) \geq 0$ for all $b \geq b_i^0$ and hence all the four functions $F_i(b)$ are non-negative for all $b \geq b^*$, which implies that $\det Q(z) \geq 0$ for all $z \in \mathcal{V}$. ■

Figure 5.6 reports the functions $F_i(b)$, $i = 1, \ldots, 4$. Note that $F_2(b)$,

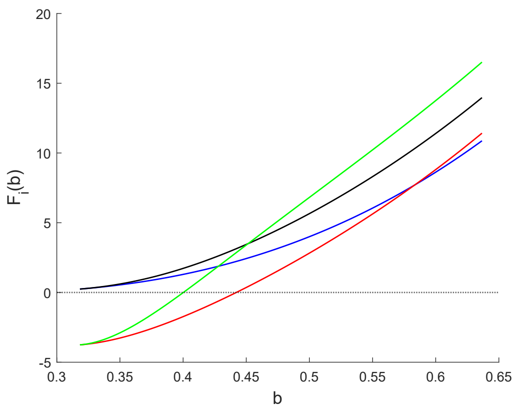


Figure 5.6: Functions F_1 (blue), F_2 (red), F_3 (black) and F_4 (green) as a function of the parameter b.

which corresponds to the determinant of $Q(z)$ for $z \in \{1, (\cos(1/b), 1))^\top$, $(\cos(1/b), 1, 1))^\top, (1, 1, \cos(1/b)))^\top\}$ and hence for $x \in \{(1/b, 0, 0))^\top$, $(0, 1/b, 0))^\top, (0, 0, 1/b))^\top\}$, is the function having the largest positive solution. Specifically, we get $b^* = b_2^0 = 0.442$. Clearly, b^* is relatively close to the lower bound b_l in (5.48), suggesting that a different choice for P might work

better. However, from Remark 7 we have that $J^{[2]}(x)$ must be marginally stable for all $x \in \mathcal{D}$. In particular, it turns out that $J^{[2]}((1/b, 0, 0))$ is marginally stable if and only if $b \geq 0.436$, thus showing the identity matrix is indeed a good choice for P. This is confirmed by numerically solving the LMI problem (5.51) for $z \in \mathcal{V}$.
According to Remark 5, if $b > b^*$ the solutions of (5.44) converge towards one of two asymptotically stable equilibrium points $x_\pm = \pm(\eta, \eta, \eta)^\top$. This convergent scenario is confirmed by numerical simulations, as illustrated in Fig. 5.7 where some state space trajectories are reported in the case $b = 0.44$. This result complements that derived in [6, 53] where it is shown that the Thomas system is 2-contracting for $b > 1/2$.

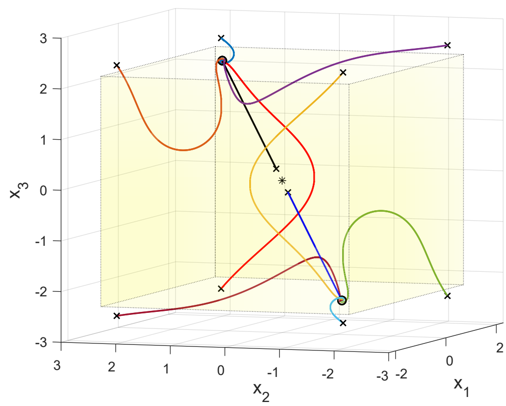


Figure 5.7: Trajectories of Thomas system with $b = 0.44$. Stable equilibrium points at $x = \pm(2.034, 2.034, 2.034)$ (marked by $\circ$); unstable equilibrium point at $x = 0$ (marked by $*$). The initial conditions $(x_1(0), x_2(0), x_3(0))$ (marked by $\times$) are: $(-1.8, 1.8, 2.2)$, $(-1.8, -1.8, 2.2)$, $(1.8, -1.8, 2.2)$, $(1.8, 1.8, 2.2)$, $(-1.8, 1.8, -2.2)$, $(-1.8, -1.8, -2.2)$, $(1.8, -1.8, -2.2)$, $(1.8, 1.8, -2.2)$, $(-0.2, -0.2, -0.2)$, $(0.2, 0.2, 0.2)$.

5.1.3 System with a first intregral of motion: the Chua's memristor circuit

In the last decade memristor circuits have gained a prominent role as a viable approach to develop new computing paradigms to potentially overcome some of the limitations of conventional digital computer architectures [22, 51, 54]. The memristor (a shorthand for memory resistor) is the fourth basic passive circuit element theoretically introduced by L.O. Chua in 1971 [11], whose first electronic implementation was given in 2008 [48]. Later, it became clear that circuits containing (ideal) memristor are systems with a first integral of motion, i.e., their state space can be decomposed into a continuum of invariant manifolds (see [12, 19] and references therein).
In this section we consider the memristor Chua's circuit introduced in [20], where the nonlinear resistor of the classical Chua's circuit is replaced with a flux-controlled memristor. According to [14], such a memristor circuit admits the following state space representation

$$\begin{array}{rcl} \dot{\xi}(t) & = & A\xi(t) - BN'(\varphi(t))C\xi(t) \\ \dot{\varphi}(t) & = & C\xi(t) \end{array} , \tag{5.57}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^\top$ is the state vector of the linear part of the circuit, φ is the memristor flux, $N : \mathbb{R} \longrightarrow \mathbb{R}$ is the memristor flux-charge nonlinear characteristic and $N'(\varphi)$ denotes its first-derivative with respect to φ, and A, B, C read:

$$A = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & \gamma \end{pmatrix} \qquad B = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} , \tag{5.58}$$

with $\alpha > 0$, $\beta > 0$, and $\gamma \geq 0$ being constant parameters. In the sequel, we consider the case $\gamma = 0$ and assume that the memristor nonlinear characteristic has the following form

$$N(\varphi) = m_0\varphi + m_1\varphi^3 , \tag{5.59}$$

where m_0 and m_1 are constant parameters. It is worth noting that system (5.57)-(5.59) has an equilibrium point at $x = (0, 0, 0, \varphi_e)^\top$ for all $\varphi_e \in \mathbb{R}$, i.e. there exist infinitely many non-isolated equilibrium points.
It can be readily verified that the memristor Chua's circuit admits a first integral of motion. In fact, since A is non-singular, from the first equation

of (5.57) we get

$$\xi(t) = A^{-1}\left(\dot{\xi}(t) + B\dot{N}(\varphi(t))\right) ,$$

which yields

$$\dot{\varphi}(t) - CA^{-1}\dot{\xi}(t) - CA^{-1}B\dot{N}(\varphi(t)) = 0 ,$$

thus implying that

$$\varphi(t) - CA^{-1}\xi(t) - CA^{-1}BN(\varphi(t)) = \text{constant} = \varphi_0 - CA^{-1}\xi_0 - CA^{-1}BN(\varphi_0) . \tag{5.60}$$

The 4th-order system (5.57) can be represented in the form (2.1) once the state $x = (x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4$ is defined as $x := (\xi^\top, \varphi)^\top$ and

$$f(x) = \begin{pmatrix} A & 0_{(n-1)\times 1} \\ C & 0 \end{pmatrix} x - BN'(e_4^\top x)\begin{pmatrix} C & 0 \end{pmatrix} x , \tag{5.61}$$

where $0_{(n-1)\times 1}$ is the column vector with $n-1$ zeros and $e_4 = (0,0,0,1)^\top$. The invariant manifold (2.22) can be derived from (5.60), taking into account that $CA^{-1} = \left(-\frac{1}{\alpha}\ ,0\ ,-\frac{1}{\beta}\right)$ and $CA^{-1}B = -1$. Indeed, we get

$$\begin{aligned} g(x) &= \frac{1}{\alpha}x_1 + \frac{1}{\beta}x_3 + (1+m_0)x_4 + m_1x_4^3 \\ &= \frac{1}{\alpha}x_1(0) + \frac{1}{\beta}x_3(0) + (1+m_0)x_4(0) + m_1x_4^3(0) =: \mathcal{I}, \end{aligned} \tag{5.62}$$

where $\mathcal{I}$ is a constant parameter which can assume any real value as the initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0))^\top$ are varied. This implies that the system state space is composed by a continuum of invariant manifolds. It is known that the system (5.57)-(5.59) is capable to display onto its invariant manifolds either convergent or oscillatory and more complex behaviors for suitable values of the parameters α, β, m_0, m_1 [12, 14].
To apply Theorem 6 to system (5.57)-(5.59) we need to compute $\tilde{J}^{[2]}(x)$. From (5.62) it follows that

$$\frac{\partial g}{\partial x} = \left(\frac{1}{\alpha}\ ,0\ ,\frac{1}{\beta}\ ,1 + m_0 + 3m_1x_4^2\right) . \tag{5.63}$$

The transformation matrix $T(x)$ in (2.26) is chosen as

$$T(x) = \begin{pmatrix} -3\,m_1\,x_4^2 - m_0 - 1 & -\frac{1}{\beta} & 0 & \frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\alpha} & 0 & \frac{1}{\beta} \\ \frac{1}{\alpha} & 0 & 0 & 3\,m_1\,x_4^2 + m_0 + 1 \end{pmatrix}, \tag{5.64}$$

which clearly satisfies the perpendicularity condition between the first $n-1$ columns and the last one. From the Jacobian of system (5.57)-(5.59), which is given by

$$J(x) = \begin{pmatrix} -3\,\alpha\,m_1\,x_4^2 - \alpha - \alpha\,m_0 & \alpha & 0 & -6\,\alpha\,m_1\,x_1\,x_4 \\ 1 & -1 & 1 & 0 \\ 0 & -\beta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{5.65}$$

we can derive the matrix $H(x)$ in (2.30) as

$$H(x) = \begin{pmatrix} -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & -\frac{\alpha}{\beta} & 0 & * \\ 0 & 0 & -\beta & * \\ -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & -\frac{\alpha-\beta}{\beta} & -1 & * \\ 0 & 0 & 0 & * \end{pmatrix}. \tag{5.66}$$

It turns out that $\tilde{J}(x)$ and its 2-additive compound $\tilde{J}^{[2]}(x)$ read:

$$\tilde{J}(x) = \begin{pmatrix} -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & -\frac{\alpha}{\beta} & 0 \\ 0 & 0 & -\beta \\ -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & -\frac{\alpha-\beta}{\beta} & -1 \end{pmatrix} \tag{5.67}$$

$$\tilde{J}^{[2]}(x) = \begin{pmatrix} -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & -\beta & 0 \\ -\frac{\alpha-\beta}{\beta} & -\alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) - 1 & -\frac{\alpha}{\beta} \\ \alpha\,\left(3\,m_1\,x_4^2 + m_0 + 1\right) & 0 & -1 \end{pmatrix}. \tag{5.68}$$

Let us now apply Theorem 6 in the case of $P(x)$ being a constant positive definite matrix $P = P^\top \in \mathbb{R}^{3\times 3}$ and setting $\mathcal{D} = \mathbb{R}^4$. First, we find convenient to introduce the new variable $z := x_4^2$ and to rewrite the 2-additive compound (5.68) as:

$$\tilde{J}^{[2]}(z) = \tilde{J}_0^{[2]} + 3\alpha m_1 z \tilde{J}_1^{[2]} \ , \tag{5.69}$$

where $\tilde{J}_0^{[2]}$ ans $\tilde{J}_1^{[2]}$ are the following constant matrices

$$\tilde{J}_0^{[2]} := \begin{pmatrix} -\alpha\,(m_0+1) & -\beta & 0 \\ -\dfrac{\alpha-\beta}{\beta} & -\alpha\,(m_0+1)-1 & -\dfrac{\alpha}{\beta} \\ \alpha\,(m_0+1) & 0 & -1 \end{pmatrix} ,$$

$$\tilde{J}_1^{[2]} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{5.70}$$

It can be readily verified that condition (2.32) of Theorem 6 boils down to

$$Q(z) := Q_0 + 3\alpha m_1 z Q_1 \geq 0 \ , \quad \forall z \in [0, +\infty) \ , \tag{5.71}$$

where

$$Q_i = -\left(\tilde{J}_i^{[2]}\right)^\top P - P\tilde{J}_i^{[2]} \ , \quad i = 0, 1 \ . \tag{5.72}$$

To show that $Q(z)$ is positive semidefinite for all $z \in [0, +\infty)$, we can resort to an argument similar to that in Remark 8, since $Q(z)$ depends affine linearly on z. Indeed, the next result holds true.

Proposition 25 *Let $m_1 \geq 0$ and suppose that there exists $P = P^\top > 0$ such that $Q_i \geq 0$, $i = 0, 1$. Then, $Q(z) \geq 0$ for all $z \in [0, +\infty)$.*

Proof. For any $z \in [0, +\infty)$ and $m_1 \geq 0$ we have that the following equality

$$\zeta^\top Q(z)\zeta = \zeta^\top Q_0 \zeta + 3\alpha m_1 z \zeta^\top Q_1 \zeta$$

holds for all $\zeta \in \mathbb{R}^3$. Hence, the proof follows by observing that $\zeta^\top Q_i \zeta \geq 0$, $i = 0, 1$, for all $\zeta \in \mathbb{R}^3$. ■

To illustrate Proposition 25, we consider the case when $\alpha = 0.25$, $\beta = 3$, $m_0 = -1.05$. The matrices A and B in (5.57) become

$$A = \begin{pmatrix} -0.25 & 0.25 & 0 \\ 1 & -1 & 1 \\ 0 & -3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0.25 \\ 0 \\ 0 \end{pmatrix} , \tag{5.73}$$

while (5.62) boils down to

$$4x_1 + \frac{1}{3}x_3 - 0.05x_4 + m_1 x_4^3 = \mathcal{I} \ . \tag{5.74}$$

It can be checked that $\tilde{J}_1^{[2]}$ is marginally stable and

$$\tilde{J}_0^{[2]} = \begin{pmatrix} 0.013 & -3 & 0 \\ 0.917 & -0.988 & -0.083 \\ -0.013 & 0 & -1 \end{pmatrix} \tag{5.75}$$

is Hurwitz. Hence, the necessary condition for the solution of the LMI problem

$$\begin{cases} P = P^\top > 0 \\ \left(\tilde{J}_0^{[2]}\right)^\top P + P\tilde{J}_0^{[2]} \leq 0 \\ \left(\tilde{J}_1^{[2]}\right)^\top P + P\tilde{J}_1^{[2]} \leq 0 \end{cases} \tag{5.76}$$

are satisfied. By numerically solving (5.76), we get

$$P = \begin{pmatrix} 3.85 & -0.7 & 1.34 \\ -0.7 & 8.54 & 0 \\ 1.34 & 0 & 1.34 \end{pmatrix} ,$$

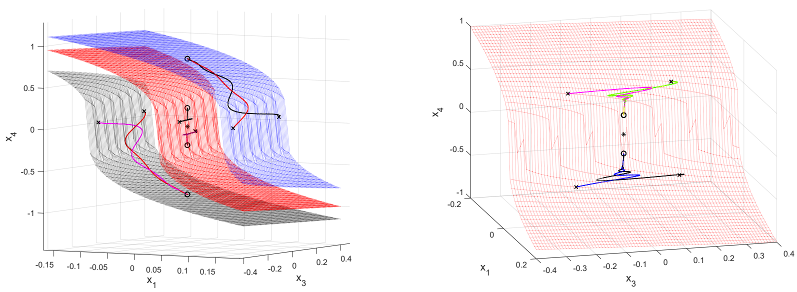
while the matrix Q_0 and Q_1 are equal to

$$Q_0 = \begin{pmatrix} 1.22 & 3.08 & 1.28 \\ 3.08 & 12.59 & 4.73 \\ 1.28 & 4.73 & 2.68 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 5.02 & -1.4 & 0 \\ -1.4 & 17.0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

It can be readily verified that P is positive definite and Q_i, $i = 0, 1$, are positive semidefinite. Hence, we can conclude that for $\alpha = 0.25$, $\beta = 3$, $m_0 = -1.05$ and all $m_1 \geq 0$ the Lyapunov exponents of all the attractors of the memristor Chua's circuit (5.57)-(5.59), whose ω-limit set does not contain equilibrium points, are non-positive.
Figure 5.8 illustrates the dynamics in the case $m_1 = 1$. The invariant manifolds (5.74) with $\mathcal{I} = 0$ (red), $\mathcal{I} = 0.5$ (blue), and $\mathcal{I} = -0.5$ (black) are reported in Fig. 5.8 a). It is worth noting that in the case $\mathcal{I} = \pm 0.5$ there exists a unique equilibrium at $x = \pm(0, 0, 0, 0.815)^\top$ to which all the trajectories converge. As highlighted in Fig. 5.8 b), for $\mathcal{I} = 0$ the invariant manifolds have two stable and one unstable equilibrium points at $x = \pm(0, 0, 0, 0.224)^\top$

and $x = (0, 0, 0, 0)^\top$, respectively, and all the (non-trivial) trajectories converge towards the two stable ones. Extensive numerical simulations show that similar convergent behaviors are displayed onto the other manifolds and also for different values of m_1.

Comments and final remarks *It is worth noting that in the case of the Chua's circuit just discussed, the* 2*-additive compound approach depends on the choice of the matrix* $T(x)$ *in (2.26). In particular, different choices of the matrix* $T(x)$ *may simplify the treatment.*



a) b)

Figure 5.8: System (5.57)-(5.59) with $\alpha = 0.25$, $\beta = 3$, $\gamma = 0$, $m_0 = -1.05$, $m_1 = 1$. The stable and unstable equilibrium points are marked with $\circ$ and $*$, respectively, the initial conditions $x(0)$ are marked with $\times$. a) Trajectories onto the invariant manifolds with $x(0) = (0.01, -0.05, -0.12, 0.08)^\top$ and $x(0) = (-0.01, -0.05, 0.12, -0.08)^\top$ for $\mathcal{I} = 0$ (red), $x(0) = (0.15, -0.05, -0.3, 0.08)^\top$ and $x(0) = (0.1, -0.05, 0.25, 0.08)^\top$ for $\mathcal{I} = 0.5$ (blue), $x(0) = (0.01, -0.05, -0.12, 0.08)^\top$ and $x(0) = (-0.15, -0.05, 0.32, 0.08)^\top$ for $\mathcal{I} = -0.5$ (black). b) Trajectories onto the invariant manifolds with $\mathcal{I} = 0$ (red). The initial conditions are $x(0) = (-0.01, -0.05, -0.17, 0.5)^\top$, $x(0) = (-0.04, -0.05, 0.19, 0.5)^\top$, $x(0) = (0.01, -0.05, 0.17, -0.5)^\top$, $x(0) = (0.04, -0.05, -0.19, -0.5)^\top$.

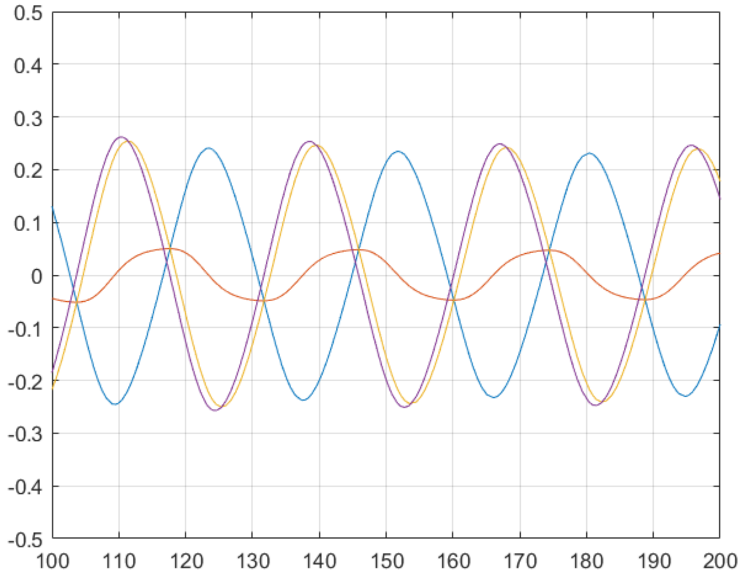


Figure 5.9: Oscillatory solution of system (5.77), when the configuration is set at $k = 1.1$, a value for which the model can be regarded as the feedback interconnection of the two subsystems illustrated in Section 5.2.1.

5.2 Examples of application of the results in Chapter 3

5.2.1 Transition from multistability to limit cycles

Consider the system of equations:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + \text{atan}(2x_1) - 2x_2 + x_3 \\ \dot{x}_3 & = & -x_3 + x_4 \\ \dot{x}_4 & = & -kx_1 - x_4 \end{array} . \tag{5.77}$$

The system can be regarded as the feedback interconnection of the (x_1, x_2) and (x_3, x_4) subsystems through the linking signals x_1 and x_3. Notably, for $k = 0$ the system boils down to the cascade (series) interconnection of the asymptotically stable linear subsystem (x_3, x_4), forced with vanishing intensity by the multistable bidimensional subsystem (x_1, x_2). In this latter case, nonoscillatory behaviors of the multistable system for $x_3(t) \equiv 0$ can be shown by considering the Lyapunov functional

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} [\xi - \text{atan}(2\xi)] d\xi.$$

Hence, for $k = 0$ this system is multistable and it has two asymptotically stable equilibria, and a third, unstable, saddle in 0. Our goal is to find sufficient conditions that guarantee non-oscillatory behaviors (2-contraction) of the system also for some range of $k > 0$.

It is easy to see, through simulations, that for k sufficiently large the system admits oscillatory solutions, as shown in Fig. 5.9. In fact, this occurs for all $k > 1$. The Jacobian $J(x)$ is given as:

$$J(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 + \frac{2}{1+4x_1^2} & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix}$$

Notice that, no matter what x_1 is, the Jacobian $J(x)$ belongs to the interval matrix

$$J(x) \in \begin{bmatrix} 0 & 1 & 0 & 0 \\ [-1, 1] & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix} = \text{conv}\left(J_1(k), J_2(k)\right)$$

where $J_1(k)$ and $J_2(k)$ read

$$J_1(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix},$$

$$J_2(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -k & 0 & 0 & -1 \end{bmatrix}.$$

Rather than considering the full $J(x)$, and the corresponding $J^{[2]}$ matrix (of dimension 6×6), we decompose the system into its (x_1, x_2) and (x_3, x_4)

components, respectively. Notice that standard small-gain results do not apply, as $J(0)$, even for $k = 0$, has a positive eigenvalue in $\frac{-1+\sqrt{2}}{2}$. The modular version of the 2-additive compound variational equation looks like:

$$
\begin{aligned}
\dot{\delta}_1 &= -2\delta_1 + [1,0,0,0]\,\delta_{12} \\
\dot{\delta}_{12} &= \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 + \frac{2}{1+(2x_1)^2} & 0 & -3 & 1 \\ 0 & -1 + \frac{2}{1+(2x_1)^2} & 0 & -3 \end{bmatrix} \delta_{12} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \end{bmatrix} \delta_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \delta_2 \\
\dot{\delta}_2 &= -2\delta_2 + [k,0,0,0]\delta_{12}
\end{aligned}
\tag{5.78}
$$

It is easy to see that $\gamma_1 = 1/2$ and $\gamma_2 = k/2$. To apply Theorem 9 we need to solve the following minimization problem:

$$
\begin{aligned}
&\min_{P_{12},\tilde{\eta}_1,\tilde{\eta}_2} \gamma_1^2\tilde{\eta}_1 + \gamma_2^2\tilde{\eta}_2 < 1 \qquad (5.79)\\
&\text{subject to} \\
&\begin{bmatrix} A_1^T P_{12} + P_{12}A_1 + I & P_{12}G_1 & P_{12}G_2 \\ G_1^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_2^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} A_2^T P_{12} + P_{12}A_2 + I & P_{12}G_1 & P_{12}G_2 \\ G_1^T P_{12} & -\tilde{\eta}_1 I & 0 \\ G_2^T P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0 \\
&P_{12} = P_{12}^T \geq \varepsilon I \\
&\tilde{\eta}_1 \geq 0 \\
&\tilde{\eta}_2 \geq 0
\end{aligned}
$$

where the matrices A_1 and A_2 are given by

$$A_1 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & 0 & -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix},$$

and $\varepsilon = 0.01$. The maximum value of the parameter k for which the minimization problem (5.79) turns out to be feasible is $k^* = 0.755$, and it is obtained for

$$P_{12} = \begin{bmatrix} 1.24 & 0.68 & 0.14 & 0.19 \\ 0.68 & 4.98 & 0.18 & 1.48 \\ 0.14 & 0.18 & 0.85 & 0.19 \\ 0.19 & 1.48 & 0.19 & 1.45 \end{bmatrix}$$

Instead, exploiting Theorem 10, the maximum gain $\gamma_{12}(k)$ allowed by the small-gain condition (3.42) as a function of parameter k is given by $\gamma_{12}(k) = 1/\sqrt{(1/2)^2 + (k/2)^2}$. In this case, we have to solve the following maximization problem:

$$\max_{k \geq 0, P_{12} = P_{12}^T} k \tag{5.80}$$

subject to

$$\begin{bmatrix} A_1^T P_{12} + P_{12} A_1 + I & P_{12}[G_1, G_2] \\ [G_1 G_2]^T P_{12} & -\gamma_{12}^2(k) I \end{bmatrix} \leq 0$$

$$\begin{bmatrix} A_2^T P_{12} + P_{12} A_2 + I & P_{12}[G_1, G_2] \\ [G_1 G_2]^T P_{12} & -\gamma_{12}^2(k) I \end{bmatrix} \leq 0$$

$$P_{12} \geq 0$$

The maximum value of the parameter k for which the maximization problem (5.80) is feasible is $k^* = 0.715$, which is obtained for

$$P_{12} = \begin{bmatrix} 1.25 & 0.68 & 0.13 & 0.18 \\ 0.68 & 5.0 & 0.17 & 1.47 \\ 0.13 & 0.17 & 0.89 & 0.19 \\ 0.18 & 1.47 & 0.19 & 1.49 \end{bmatrix}.$$

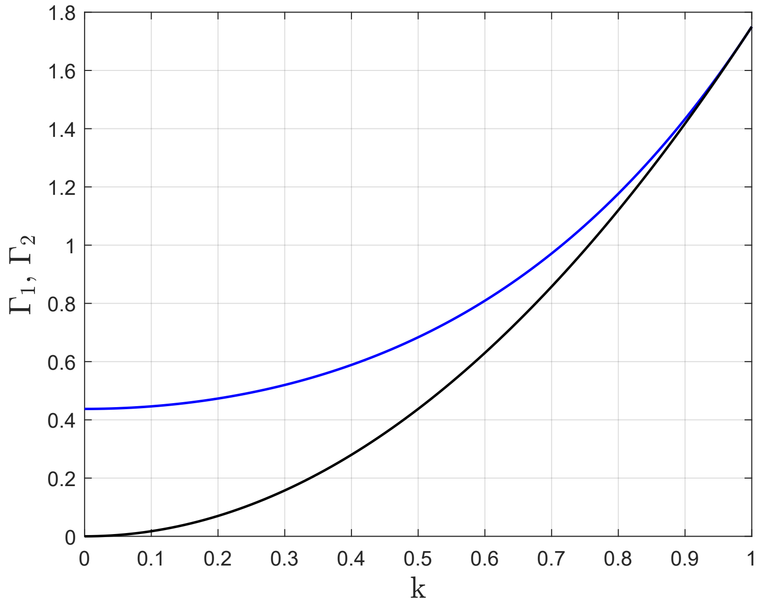


Figure 5.10: Functions $\Gamma_1(k) \doteq \gamma_1^2(k)\tilde{\eta}_1(k) + \gamma_2^2(k)\tilde{\eta}_2(k)$ (black) and $\Gamma_2(k) \doteq \gamma_{12}^2(k)(\gamma_1^2(k) + \gamma_2^2(k)$ (blue) as a function of the parameter $k \in [0, 1]$.

In Fig. 5.10 are reported the conditions (3.38) and (3.42) as a function of the parameter k. From the figure it can be observed that the condition in Theorem 10 is more conservative than the condition in Theorem 9, as observed in Remark 13.

To measure the conservativeness of the small-gain conditions (3.38) and (3.42), we compare the value k^* with the one achievable by means of the following maximization problem

$$\begin{aligned} &\max_{k \geq 0, P = P^T} k \\ &\text{subject to} \\ &J_1^{[2]}(k)^T P + P J_1^{[2]}(k) \leq 0 \\ &J_2^{[2]}(k)^T P + P J_2^{[2]}(k) \leq 0 \\ &P \geq I \end{aligned} \tag{5.81}$$

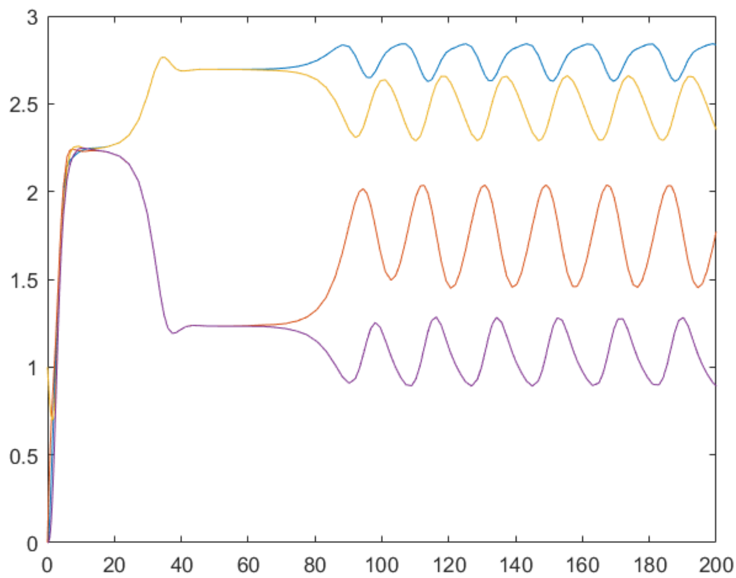


Figure 5.11: The fourth order Thomas' system (5.82) of Section 5.2.2 shows periodic solutions when it is configured with $b = 0.35$.

which directly involves the additive compound matrix and a proper matrix P of dimension 6×6. It turns out that problem (5.81) is feasible for all $k \in [0, 1]$ and that the maximum value is achieved for

$$P = \begin{bmatrix} 11.45 & 0 & 2.43 & -0.209 & 1.99 & -1.01 \\ 0 & 14.09 & 10.88 & 1.37 & 3.51 & 0 \\ 2.43 & 10.88 & 37.17 & 2.14 & 7.31 & 2.43 \\ -0.209 & 1.37 & 2.14 & 8.49 & 1.70 & -0.21 \\ 1.99 & 3.51 & 7.31 & 1.70 & 9.29 & 1.99 \\ -1.01 & 0 & 2.43 & -0.21 & 1.99 & 11.45 \end{bmatrix}.$$

It is worth nothing that for values of k greater than 1 the system starts to display periodic motions (see Fig. 5.9).

5.2.2 Thomas' example of dimension 4

As a further example, let us consider the Thomas' system (see [49]) of the fourth order, described by the following set of first order differential equations

$$\begin{array}{rcl} \dot{x}_1 & = & -bx_1 + \sin(x_2) \\ \dot{x}_2 & = & -bx_2 + \sin(x_3) \\ \dot{x}_3 & = & -bx_3 + \sin(x_4) \\ \dot{x}_4 & = & -bx_4 + \sin(x_1) \end{array}, \tag{5.82}$$

where b is a positive scalar parameter. For $b > 1$ the system has a unique asymptotically stable equilibrium point at $x = 0$, which undergoes a (supercritical) pitchfork bifurcation at $b = 1$. For $b < 1$ the system is multistable and it exhibits quite a rich dynamic behavior as b is decreased towards 0. For $b = 0.35$, periodic solutions arise, as seen from numerical simulations depicted in Fig. 5.11. Moreover, the hypercube $[-1/b, 1/b]^4$ is a forward invariant set for system (5.82).
Our aim is to find conditions, similar to conditions (5.80)-(5.81), to rule out oscillatory behaviors for some range of $b < 1$.

The Jacobian $J(x)$ of the system has the following form

$$J(x) = \begin{bmatrix} -b & c_2 & 0 & 0 \\ 0 & -b & c_3 & 0 \\ 0 & 0 & -b & c_4 \\ c_1 & 0 & 0 & -b \end{bmatrix},$$

where $c_i = \cos(x_i)$. It is worth noting that, since $c_i \in [-1, 1]$, then $J(x) \in \text{conv}(\mathcal{V})$, where $\mathcal{V} = \{V_i\}_{i=1,\dots,16}$ collects the Jacobian matrices at the vertexes of the hypercube, i.e.,

$$\text{conv}(\mathcal{V}) = \begin{bmatrix} -b & [-1,1] & 0 & 0 \\ 0 & -b & [-1,1] & 0 \\ 0 & 0 & -b & [-1,1] \\ [-1,1] & 0 & 0 & -b \end{bmatrix}.$$

Its 2-additive compound reads

$$J^{[2]}(x) = \begin{bmatrix} -2b & c_3 & 0 & 0 & 0 & 0 \\ 0 & -2b & c_4 & c_2 & 0 & 0 \\ 0 & 0 & -2b & 0 & c_2 & 0 \\ 0 & 0 & 0 & -2b & c_4 & 0 \\ -c_1 & 0 & 0 & 0 & -2b & c_3 \\ 0 & -c_1 & 0 & 0 & 0 & -2b \end{bmatrix}.$$

We choose to partition the state-space according to (x_1, x_3) and (x_2, x_4). Therefore, the modular version of the 2-additive compound variational equation of the fourth order Thomas' system assumes the following form:

$$
\begin{aligned}
\dot{\delta}_1 &= -2b\delta_1 + [0, c_4, -c_3, 0]\,\delta_{12} \\
\dot{\delta}_{12} &= \begin{bmatrix} -2b & 0 & 0 & 0 \\ 0 & -2b & 0 & 0 \\ 0 & 0 & -2b & 0 \\ 0 & 0 & 0 & -2b \end{bmatrix} \delta_{12} \\
&\quad + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ -c_1 \end{bmatrix} \delta_1 + \begin{bmatrix} 0 \\ c_3 \\ -c_4 \\ 0 \end{bmatrix} \delta_2 \\
\dot{\delta}_2 &= -2b\delta_2 + [c_1, 0, 0, c_2]\delta_{12}
\end{aligned}
\tag{5.83}
$$

The gains γ_1 and γ_2 can be readily computed, obtaining $\gamma_1 = \gamma_2 = 1/\sqrt{2}b$. Our aim is to solve the minimization problem

$$
\min_{P_{12},\tilde{\eta}_1,\tilde{\eta}_2} \gamma_1^2\tilde{\eta}_1 + \gamma_2^2\tilde{\eta}_2 < 1 \tag{5.84}
$$

subject to

$$
\begin{bmatrix} A^T P_{12} + P_{12}A + I & P_{12}G_1^{(h)} & P_{12}G_2^{(h)} \\ G_1^{(h)T} P_{12} & -\tilde{\eta}_1 I & 0 \\ G_2^{(h)T} P_{12} & 0 & -\tilde{\eta}_2 I \end{bmatrix} \leq 0, \; h = 1, \dots, 16
$$

$$P_{12} = P_{12}^T \geq \varepsilon I$$

$$\tilde{\eta}_1 \geq 0$$

$$\tilde{\eta}_2 \geq 0$$

where $\varepsilon = 0.01$,

$$
A = \begin{bmatrix} -2b & 0 & 0 & 0 \\ 0 & -2b & 0 & 0 \\ 0 & 0 & -2b & 0 \\ 0 & 0 & 0 & -2b \end{bmatrix},
$$

$$
G^{(h)} = \begin{bmatrix} v_2^{(h)} & 0 \\ 0 & v_3^{(h)} \\ 0 & -v_4^{(h)} \\ -v_1^{(h)} & 0 \end{bmatrix}
$$

and $v^{(h)}$, $h = 1, \ldots, 16$, are the vertices of the hypercube $[-1, 1]^4$. It turns out that the minimization problem (5.84) is feasible up to $b = b^* \approx 0.841$, which is obtained for

$$P_{12} = \begin{bmatrix} 0.595 & 0 & 0 & 0 \\ 0 & 0.595 & 0 & 0 \\ 0 & 0 & 0.595 & 0 \\ 0 & 0 & 0 & 0.595 \end{bmatrix}. \tag{5.85}$$

In the case of Theorem 10, since $\gamma_1 = \gamma_2 = 1/\sqrt{2}b$, the maximum gain $\gamma_{12}(b)$ allowed by the small-gain condition (3.42) as a function of the parameter b is given by $\gamma_{12}(b) = b$. Therefore, we want to solve the following minimization problem

$$\begin{aligned} &\min_{b \geq 0, P_{12} = P_{12}^T} b \\ &\text{subject to} \\ &\begin{bmatrix} A^T P_{12} + P_{12} A + I & P_{12} G^{(h)} \\ G^{(h)T} P_{12} & -\gamma_{12}^2(b) I \end{bmatrix} \leq 0, \; h = 1, \ldots, 16 \\ &P_{12} \geq 0 \end{aligned} \tag{5.86}$$

It turns out that also problem (5.86) is feasible up to $b = b^* \approx 0.841$, and it is achieved for the same P_{12} in (5.85). It is worth notice that, differently from the previous example, in the case of the Thomas' system of the fourth dimension conditions (3.38) and (3.42) provide the same results.

As in the previous case, we compare the value of b^* obtained with the small-gain conditions with the one provided by means of direct optimization. This latter minimization problem assumes the following form:

$$\begin{aligned} &\min_{b \geq 0, P = P^T} b \\ &\text{subject to} \\ &J_h^{[2]}(b)^T P + P J_h^{[2]}(b) \leq 0, \; h = 1, \ldots, 16 \\ &P \geq I \end{aligned} \tag{5.87}$$

where

$$J_h^{[2]}(b) = \begin{bmatrix} -2\,b & v_3^{(h)} & 0 & 0 & 0 & 0 \\ 0 & -2\,b & v_4^{(h)} & v_2^{(h)} & 0 & 0 \\ 0 & 0 & -2\,b & 0 & v_2^{(h)} & 0 \\ 0 & 0 & 0 & -2\,b & v_4^{(h)} & 0 \\ -v_1^{(h)} & 0 & 0 & 0 & -2\,b & v_3^{(h)} \\ 0 & -v_1^{(h)} & 0 & 0 & 0 & -2\,b \end{bmatrix}.$$

It turns out that problem (5.87) is feasible for all $b > 0.5$, and that the minimum value of b is achieved for the following matrix:

$$P = \begin{bmatrix} 7.27 & 0 & 0 & 0 & 0 & 0 \\ 0 & 140.26 & 0 & 0 & 0 & 0 \\ 0 & 0 & 128.39 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.31 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.53 & 0 \\ 0 & 0 & 0 & 0 & 0 & 130.33 \end{bmatrix}.$$

5.2.3 Thomas' example of dimension 3

In order to clarify Remark 15, we consider the Thomas' system of the third order. Its equations read

$$\begin{array}{rcl} \dot{x}_1 & = & -bx_1 + \sin(x_2) \\ \dot{x}_2 & = & -bx_2 + \sin(x_3) \\ \dot{x}_3 & = & -bx_3 + \sin(x_1) \end{array},$$

the cube $[-1/b, 1/b]^3$ is a forward invariant set, and linearization yields a 2-additive compound of the Jacobian of the following form:

$$J^{[2]}(x) = \begin{bmatrix} -2\,b & \cos(x_3) & 0 \\ 0 & -2\,b & \cos(x_2) \\ -\cos(x_1) & 0 & -2\,b \end{bmatrix}. \tag{5.88}$$

Since $\cos(x_i) \in [-1, 1]$, then $J^{[2]}(x) \in \text{conv}(\mathcal{V})$, where $\text{conv}(\mathcal{V})$ is the interval matrix

$$\text{conv}(\mathcal{V}) = \begin{bmatrix} -2\,b & [-1,1] & 0 \\ 0 & -2\,b & [-1,1] \\ [-1,1] & 0 & -2\,b \end{bmatrix}. \tag{5.89}$$

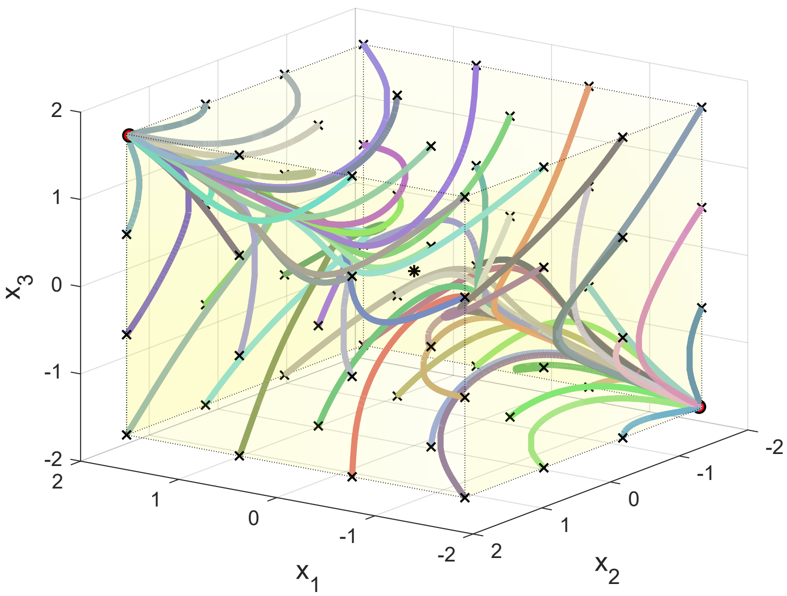


Figure 5.12: Evolution of trajectories started within the forward invariant set represented by the cube $[-1/b, 1/b]^3$ for $b = 0.58$. The crosses denote the initial conditions, while the asterisk is the unstable equilibrium point. All the other solutions converges to one of the the two stable fixed points contained in the cube.

The modular version looks like:

$$\begin{aligned} \dot{\delta}_1 &= -2b\delta_1 + [\cos(x_3), 0]\,\delta_{12} \\ \dot{\delta}_{12} &= \begin{bmatrix} -2b & \cos(x_2) \\ 0 & -2b \end{bmatrix} \delta_{12} + \begin{bmatrix} 0 \\ -\cos(x_1) \end{bmatrix} \delta_1 \end{aligned} \quad . \tag{5.90}$$

The gain γ_1 can be easily computed, obtaining $\gamma_1 = 1/(2b)$. Then, the minimum value of b allowed by the small-gain condition (3.38) can be found by solving the minimization problem

$$\begin{aligned}
&\min_{\tilde{\eta}_1, P=P^T} \gamma_1^2 \tilde{\eta}_1 < 1 \\
&\text{subject to} \\
&\begin{bmatrix} A_h^T P_{12} + P A_h + I & P_{12} G_h \\ G_h^T P_{12} & -\tilde{\eta}_1 I \end{bmatrix} \leq 0, \;\; h = 1, \dots, 4 \\
&P_{12} = P_{12}^T \geq \varepsilon I \\
&\tilde{\eta}_1 \geq 0
\end{aligned} \tag{5.91}$$

where $\varepsilon = 0.01$,

$$A_h = \begin{bmatrix} -2b & v_2^{(h)} \\ 0 & -2b \end{bmatrix}, \qquad G_h = \begin{bmatrix} 0 \\ -v_1^{(h)} \end{bmatrix}$$

and $v^{(h)}$, $h = 1, \dots, 4$, are the vertices of the square $[-1, 1]^2$, which is the projection of the forward invariant set in the space of the chosen subsystem. It turns out that problem (5.91) is feasible for all $b > 0.575$, and that the optimal matrix P providing the minimum value of b is

$$P_{12} = \begin{bmatrix} 0.87 & 0 \\ 0 & 1.52 \end{bmatrix}.$$

As in the case of the fourth order Thomas' system, it can be proved that conditions (3.38) and (3.42) provide the same results. In Fig. 5.12 the convergence to two distinct fixed points of the trajectories started within the forward invariant set is illustrated by numerical simulations obtained for $b = 0.58$.

It is interesting also in this case to compare the value of b^* provided by the small-gain condition with the one achievable by considering the 2-additive compound $J^{[2]}(x)$. From Section 2, we have that the condition is $b \geq 0.442$, with P equal to the identity matrix.

Comments and final remarks *Table 5.1 shows the comparison between the results obtained with the application of Theorem 5 and with the small gain condition. As it can be notice, the small gain approach shows some conservatism, although the differences between the two approach is not so marked as one may expect.*

Table 5.1: Summary of the results of this section.

Examples	Analysis with small-gain conditions	Analysis with Theorem 5
Example in Section 5.2.1	$k \leq k^* = 0.715$	$k \leq k^* = 1$
Thomas of fourth order	$b \geq b^* \approx 0.841$	$b > 0.5$
Thomas of third order	$b > 0.575$	$b \geq b^* = 0.442$

5.3 Examples of controlling chaos

5.3.1 Lorenz system

The Lorenz system is described by the following system of equations

$$\begin{array}{rcl} \dot{x}_1 & = & -\sigma\,(x_1 - x_2) \\ \dot{x}_2 & = & \rho\,x_1 - x_2 - x_1\,x_3 \\ \dot{x}_3 & = & x_1\,x_2 - b\,x_3 \end{array} \quad , \tag{5.92}$$

where $x = (x_1, x_2, x_3)^\top$ is the state vector and σ, b and ρ are positive parameters. In the classical analysis $\sigma = 10$, $b = 8/3$ and $\rho \in \mathbb{R}^+ \setminus 0$ is used as bifurcation parameter.
Recalling that for all initial conditions, solutions are eventually confined in some invariant sets [27], which depends on the parameters value. For instance, if $b \geq 2$ and $\sigma \geq 1$ then

$$\mathcal{D} = \left\{ x \in \mathbb{R}^3 : x_2^2 + (\rho - x_3)^2 \leq \frac{\rho^2}{\rho_b^2} \right\} \ , \tag{5.93}$$

where

$$\rho_b = \frac{2\,\sqrt{b-1}}{b} \ , \tag{5.94}$$

is an invariant set of (5.92).
Considering a control law as introducing in (4.24) with $u = K f(x)$, where

the input and the control vectors are chosen as

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \tag{5.95}$$

$$K = \begin{bmatrix} 0 & 0 & k \end{bmatrix}$$

we have that the third equation of (5.92) becomes:

$$\dot{x}_3 = (k+1)x_1 x_2 - b(k+1)x_3 \ . \tag{5.96}$$

The next result holds true.

Proposition 26 *Suppose that the system parameters satisfy the conditions*

$$b \geq 2 \ , \qquad \sigma \geq 1 \ , \tag{5.97}$$

and let consider a control law of the form (4.24) with vectors B and K as in (5.95). Then

$$\mathcal{D}_k = \left\{ x \in \mathbb{R}^3 : x_2^2 + \frac{(x_3-\rho)^2}{(1+k)} = \frac{\rho^2\, b^2}{4\left(b - \dfrac{1}{1+k}\right)} \right\} \ , \tag{5.98}$$

is an invariant set for the closed loop system.

Proof. The proof follows the same line as in [27] by using a different Lyapunov function of the following form:

$$V(x) = x_2^2 + \frac{(x_3 - 2c)^2}{(1+k)} \ .$$

where $c := \dfrac{\rho}{2}$. The time derivative of $V(x)$ reads

$$\frac{1}{2}\dot{V}(x) = -x_2^2 - b\ (x_3 - c)^2 + bc^2 \ .$$

and hence $\dot{V}(x) \leq 0$ if and only if x satisfies the following condition:

$$x_2^2 + b\ (x_3 - c)^2 \geq b\, c^2. \tag{5.99}$$

This implies that any invariant set of the form $V(x) \leq \gamma$, $\gamma > 0$, should contain the ellipse defined by the equality sign in (5.99). Hence, the smallest invariant set $V(x) \leq \gamma^*$, $\gamma^* > 0$, can be computed by maximizing $V(x)$ in (**??**) for x belonging to this ellipse. It can be readily checked that this problem can be written as

$$\gamma^* = \max_{x_3 \in [0,2c]} W(x_3) \;, \tag{5.100}$$

where

$$W(x_3) = \frac{(x_3 - 2c)^2}{(1+k)} - b\,(x_3 - c)^2 + b\,c^2 \;. \tag{5.101}$$

Since $b > 2$, $W(x_3)$ is a concave parabola with the vertex located at $x_3 = \bar{x}_3$ where

$$\bar{x}_3 = \frac{\left(b - \dfrac{2}{(1+k)}\right) c}{\left(b - \dfrac{1}{(1+k)}\right)} \quad \in [0, 2c] \;, \quad \forall k \geq 0 \;. \tag{5.102}$$

This implies that

$$\gamma^* = W(\bar{x}_3) = \frac{c^2\, b^2}{\left(b - \dfrac{1}{(1+k)}\right)},$$

which shows that the smallest invariant set $V(x) \leq \gamma^*$ is indeed $\mathcal{D}_k$.
To prove that the equilibrium point at $x = 0$ belongs to $\mathcal{D}_k$ we have to show that the condition

$$\frac{1}{1+k} \leq \frac{b^2}{4\left(b - \dfrac{1}{1+k}\right)}$$

is satisfied for all $k \geq 0$. This can be readily verified since for $b \geq 2$ the following inequalities

$$\frac{1}{1+k} \leq 1 \leq \frac{b^2}{4\,(b-1)} \leq \frac{b^2}{4\left(b - \dfrac{1}{1+k}\right)}$$

hold true. ■

It is worth noticing that for $k > 0$ the invariant set (5.98) $\mathcal{D}_k$ lengthens significantly in the x_3 direction and shortens slightly in the x_2 direction.

The Jacobian of the system (5.92) reads

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - x_3 & -1 & -x_1 \\ x_2 & x_1 & -b \end{pmatrix} =: J(x) \tag{5.103}$$

and its 2-additive compound has the following form

$$J^{[2]}(x) = \begin{pmatrix} -(\sigma+1) & -x_1 & 0 \\ x_1 & -(b+\sigma) & \sigma \\ -x_2 & \rho - x_3 & -(b+1) \end{pmatrix} . \tag{5.104}$$

The 2-additive compound of the Jacobian of the closed loop system, named $J_{cl}(x)^{[2]}$, becomes:

$$J_{cl}^{[2]}(x) = \begin{pmatrix} -(\sigma+1) & -x_1 & 0 \\ x_1\ (k+1) & -(\sigma + b\ (k+1)) & \sigma \\ -x_2\ (k+1) & \rho - x_3 & -(b\ (k+1)+1) \end{pmatrix} . \tag{5.105}$$

Following the same approach as in Section 5.1 in Chapter 2, it can be easily verified that the matrix $J_{cl}^{[2]}(0)$ is marginally stable for all $\rho \le \rho_s(k)$, where

$$\rho_s(k) = \frac{(b\,(k+1)+\sigma)\,(b\,(k+1)+1)}{\sigma} . \tag{5.106}$$

Since whenever a matrix is Hurwitz also its 2-additive compound matrix is Hurwitz, it follows that $\rho_s(k)$ is the maximum reachable ρ by the approach. How it can be noticed by equation (5.106), the gain k operates in the closed loop to increase the maximum reachable ρ. In particular, for $k > 0$, $\rho_s(k)$ grows quadratically in k.
In order to remove the chaos from the Lorenz system preserving equilibria, we are interested in finding a matrix $P = P^T > 0$ and a vector K such that, for all x inside the invariant set (5.98) and for values of $\rho > \rho_M$, the following minimisation problem admits a solution

$$\begin{aligned} &\min_{P,K} \quad qI \\ &\text{subject to} \\ &\left([(I+BK)\,J_h(x)]^{[2]}\right)^T P + P\left([(I+BK)\,J_h(x)]^{[2]}\right) \le 0,\ h = 1,\dots,4 \\ &P = P^T \ge \varepsilon I \\ &q \ge -10 \\ &k <= k_{max} \\ &k_{max} = 8 \end{aligned} \tag{5.107}$$

where

$$J_h(x) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - v_3^h & -1 & -v_1^h \\ v_2^h & v_1^h & -b \end{pmatrix} \qquad (5.108)$$

and v^h, $h = 1, \ldots, 4$, are the vertices of the rectangle that contains the cylinder (5.98) per $k = k_{max}$. The constraint $q \geq -10$ has been imposed to ensure that the minimization problem is bounded, while the constraint $k \leq k_{max}$, where $k_{max} = 8$, is introduced to prevent an excessively high gain. Utilizing the software YALMIP as before, we obtain the control gain $k = 7.91$, while the matrix P has the following structure:

$$P = \begin{pmatrix} 3644.572 & 0 & 0 \\ 0 & 85.113 & 0 \\ 0 & 0 & 11.142 \end{pmatrix}. \qquad (5.109)$$

Fig. 5.14 reports some MATLAB simulations of the uncontrolled system with $\rho = 28$. While, in Fig. 5.13 is shown the simulation of the closed loop Lorenz system for a value of $\rho = 28$. As it can be noticed from the figure, the closed loop system does not present chaotic behaviours anymore, while it preserves its equilibrium points.

5.3.2 Thomas system

In the third-order case, the system dynamics is described by the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= \sin(x_2) - b\,x_1, \\ \dot{x}_2 &= \sin(x_3) - b\,x_2, \\ \dot{x}_3 &= \sin(x_1) - b\,x_3, \end{aligned} \qquad (5.110)$$

where $x = (x_1, x_2, x_3)^T$ is the space vector and b is a positive parameter. Recalling that for all initial conditions the solutions of (5.110) are eventually confined inside the invariant set

$$\mathcal{D} := \{x \in \mathbb{R}^3 : b\,||x||_\infty \leq 1\}. \qquad (5.111)$$

It is known that for $b = 0.1$ the system displays chaotic behaviors ([47]). To remove chaos, we choose an input vector B and a gain row vector K of the form

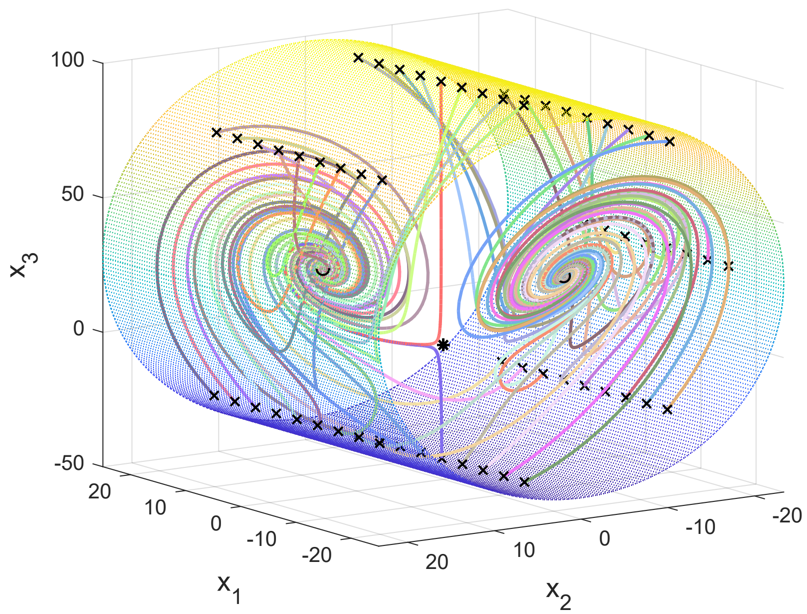


Figure 5.13: Trajectories of Lorenz system with parameters $b = 8/3$, $\sigma = 10$, $\rho = 28$ and $k = 7.91$. Stable equilibrium points (marked by $\circ$): $x_{\pm} = (\pm 8.485, \pm 8.485, 27)$; unstable equilibrium point at $x = 0$ (marked by $*$).

$$\begin{aligned} B &= (0, 0, 1)^T \\ K &= (0, 0, k) \end{aligned} \tag{5.112}$$

where $k \in \mathbb{R}$. Therefore, the closed-loop system assumes the form:

$$\dot{x} = f_{cl}(x) = \begin{pmatrix} \sin(x_2) - 0.1\, x_1 \\ \sin(x_3) - 0.1\, x_2 \\ (k+1)(\sin(x_1) - 0.1\, x_3) \end{pmatrix}. \tag{5.113}$$

The equilibrium points of system (5.113) are all the solutions of the system of equations provided by $f_{cl} = 0$. The closed-loop system has the same equilibria of the open loop-system for the same value of b, as it can be easily verified from equation (5.113). The coordinates of the equilibrium points are

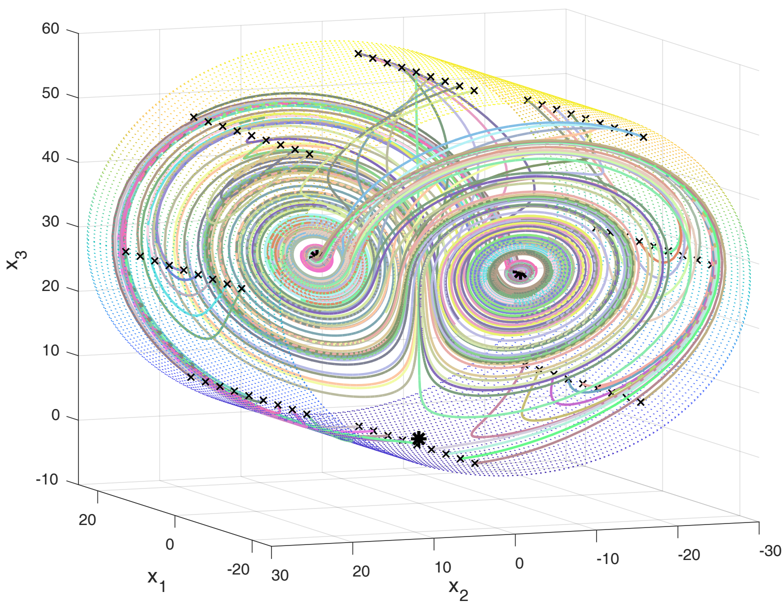


Figure 5.14: Trajectories of Lorenz system (5.92) with parameters $b = 8/3$, $\sigma = 10$, $\rho = 28$. Unstable equilibrium point at $x = 0$ and $x = (\pm 8.485, \pm 8.485, 27)$ (marked by $*$).

computed by first solving for x_1 the following equation

$$g_1(x_1) := bx_1 = \sin\left(\frac{\sin\left(\frac{\sin(x_1)}{b}\right)}{b}\right) =: g_2(x_1), \tag{5.114}$$

which amounts to find all the intersections between the functions $g_1(x_1)$ and $g_2(x_1)$, as it is shown in Fig. 5.15. The other two coordinates are automatically calculated as

$$\begin{aligned} x_3 &= \frac{\sin(x_1)}{b}, \\ x_2 &= \frac{\sin(x_3)}{b} = \frac{\sin\left(\frac{\sin(x_1)}{b}\right)}{b}. \end{aligned} \tag{5.115}$$

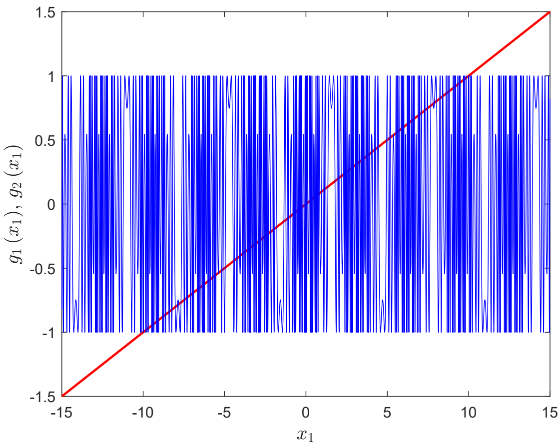


Figure 5.15: Functions $g_1(x_1)$ (red) and $g_2(x_2)$ (blue): their intersections provide the coordinate x_1 of the equilibrium points of (5.113).

It turns out that the system displays 283 equilibria, 141 of which are unstable (including the point $x = 0$) while 142 are stable. It is worth noting that all equilibria are unstable for the open-loop system and that only 7 equilibrium points follow the system's symmetry, i.e. the points $x_{e_i} = (\bar{x}_i, \bar{x}_i, \bar{x}_i)^T$ where $\bar{x}_i$ are all the solutions of the equation $b\bar{x} = \sin(\bar{x})$, meaning that most of them break the system's symmetry.
The Jacobian of the open-loop system (5.110) reads,

$$J(x) = \begin{pmatrix} -b & c_2 & 0 \\ 0 & -b & c_3 \\ c_1 & 0 & -b \end{pmatrix}, \tag{5.116}$$

where $c_i = \cos(x_i)$, while the 2-additive compound of the closed loop system assumes the form

$$J_{cl}^{[2]}(x) = \begin{pmatrix} -2b & c_3 & 0 \\ 0 & -b - b\,(k+1) & c_2 \\ -c_1\,(k+1) & 0 & -b - b\,(k+1) \end{pmatrix}. \tag{5.117}$$

In order to find the value of the control gain k, we wish to apply Proposition 21 inside the invariant set (5.111). Since the functions $\cos(x_i) \in [-1, 1]$, it follows that the matrix $J_{cl}^{[2]}(x)$ fulfills

$$J_{cl}^{[2]}(x) \in \begin{pmatrix} -2b & [-1,1] & 0 \\ 0 & -b-b\,(k+1) & [-1,1] \\ [-1,1]\,(k+1) & 0 & -b-b\,(k+1) \end{pmatrix}$$
$$\doteq \mathrm{conv}(\mathcal{V}) = \mathrm{conv}(J_1, J_2, \ldots, J_8). \tag{5.118}$$

Therefore, our aim is to find a matrix $P = P^T > 0$ and a control gain k such that the following minimization problem turns out to be feasible:

$$\begin{aligned} &\min_{P,K} \quad q \\ &\text{subject to} \\ &Q_h \leq qI, \qquad h = 1, \ldots, 8 \\ &P = P^T \geq \varepsilon I \\ &q \geq -10 \\ &k \leq k_{max} \\ &k_{max} = 500 \end{aligned} \tag{5.119}$$

where

$$Q_h = \left([(I + BK)\, J_h]^{[2]}\right)^T P + P\left([(I + BK)\, J_h]^{[2]}\right)$$

and

$$J_h = \begin{pmatrix} -b & v_2^{(h)} & 0 \\ 0 & -b & v_3^{(h)} \\ v_1^{(h)} & 0 & -b \end{pmatrix}$$

is the Jacobian calculated on the vertices $v^{(h)}$ of the cube $[-1, 1]^3$. The constraint $q \geq -10$ has been imposed to ensure that the minimization problem is bounded, while the constraint $k \leq k_{max}$, where $k_{max} = 500$, is introduced to prevent an excessively high gain. As it can be noted from (5.119), since we have to solve the problem with respect to both the control gain k and the matrix P, the problem turns out to be a Bilinear Matrix Inequality (BMI). To solve it, the software YALMIP ([31]) can be used. The optimiser provides the control gain $k = 499.313$, while the matrix P has the following form

$$P = \begin{pmatrix} 897018.86 & 0 & 0 \\ 0 & 88091.385 & 0 \\ 0 & 0 & 34.245 \end{pmatrix}. \tag{5.120}$$

Figure 5.16 reports some MATLAB simulations of the controlled system with $b = 0.1$ and $k = 499.313$, showing convergence of the trajectories towards the equilibrium points. Instead, in Fig. 5.17 is shown the simulation of the open loop Thomas system for a value of $b = 0.1$.

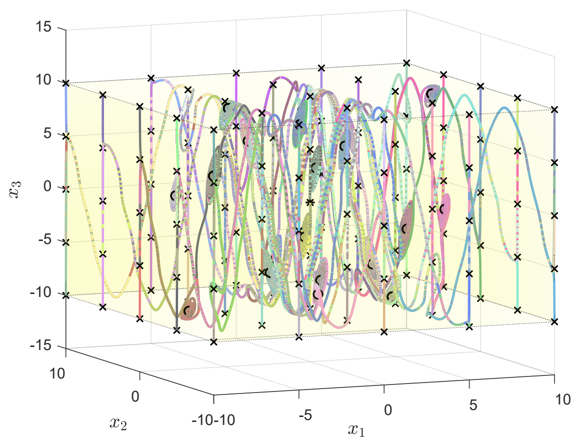


Figure 5.16: Trajectories of the controlled Thomas system for $b = 0.1$ and $k = 499.313$. The initial conditions are marked with $\times$. Stable equilibrium points are marked by $\circ$, while the unstable equilibrium point at $x = 0$ is marked by $*$.

Comments and final remarks *It is worth noting that in both Lorenz and Thomas system the minimum gains that allow removing chaos preserving equilibria depends on the value of the bifurcation parameter. In particular, in the case of the Lorenz systems, it grows as ρ increases. While, in the case of the Thomas system, it grows as b decreases. Furthermore, unlike the Thomas case, since for the Lorenz system the invariant set can be written as a function of the gain k, it is possible to impose the invariant manifold inside the BMI problem.*

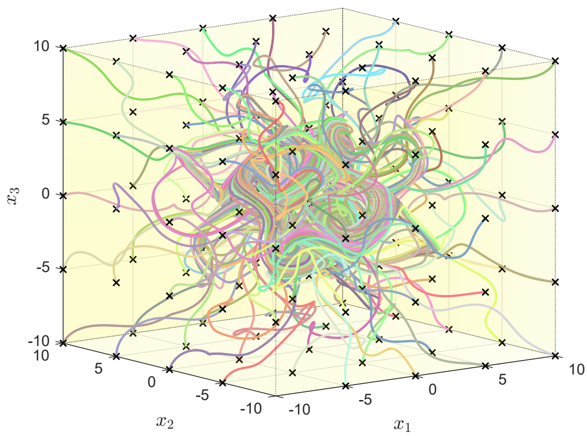


Figure 5.17: Trajectories of the uncontrolled Thomas system for $b = 0.1$. The initial conditions are marked with $\times$. The unstable equilibrium point at $x = 0$ is marked by $*$.

Conclusion

Lyapunov exponents are one of the most used tool to characterize the attractors of finite dimensional time-invariant nonlinear systems. A Lyapunov approach to the problem of ruling out the existence of attractors with positive Lyapunov exponents has been proposed. Specifically, it has been shown that sufficient conditions can be expressed in terms of Lyapunov equations involving the 2-additive compound of the system Jacobian matrix. Furthermore, more insights on the proposed approach by considering both state-dependent and constant quadratic Lyapunov functions for the resulting Lyapunov equations have been presented, showing that in some cases the problem reduces to solve a finite number of LMIs involving matrices of size $\binom{n}{2} \times \binom{n}{2}$. Then, the well-known Lorenz and Thomas systems have been investigated, showing that some regions of the parameters space where both the systems do not display attractors with positive Lyapunov exponents, can be analytically identified. These regions complement those derived by G.A. Leonov in his pioneering work on the estimation of the dimension of the Lorenz system attractors and those where it has been recently shown that the Thomas system is 2-contracting.
Furthermore, the approach has been extended to finite dimensional time-invariant nonlinear systems with a first integral of motion. Exploiting the fact that in these systems the state is confined to lie on a leaf of manifolds foliation, a Lyapunov equation involving 2-additive compound matrices of reduced dimensions is derived to rule out the existence of attractors with positive Lyapunov exponents. The solution of such a Lyapunov equation is addressed for the memristor Chua's circuit, which is known to possess infinitely many non-isolated equilibrium points. It is shown that simple LMI sufficient conditions can be derived by employing a constant quadratic Lyapunov function.
Exploiting a peculiar property of 2-additive compound matrices, it has been

shown how the problem can be solved considering the system as the interconnection of two subsystems, and, as a consequence, it allows tackling the problem via LMIs of lower dimension, which is quite useful when the dimension of the original system is large. The general criterion is based on the notion of 2-contraction and provides a modular approach for the stability analysis of 2-additive compound matrix variational equations, arising by considering virtual displacements for linear or nonlinear dynamical systems. The conditions are expressed as functionals that must be less than unity and which are computed once the system has been divided into interconnected subsystems. The proposed functionals account both for the individual gains of each subsystem's 2-additive variational equations, and for an "interconnection" gain, which arises from considerations on the Kronecker's sum of the Jacobians of the individual subsystems.
Finally, a general notion of 2-contraction via static state feedback control laws has been introduced. It has been shown how the 2-additive compound approach can be also exploited to design a feedback control law for a nonlinear system that allows removing chaotic behaviours while altering the system dynamics as little as possible. In particular, it has been derived a derivative feedback contro law based on 2-contraction concepts that allows removing the dense set of Unstable Periodic Orbits (UPOs) while preserving equilibria. Sufficient conditions to synthesize matrix gains in the form of matrix inequality is provided. Finally, the case in which the matrix inequality reduces to Bilinear Matrix Inequality (BMI) is also discussed.
It is worth noting that 2-contraction techniques presented in the thesis for addressing multistable nonlinear systems suffer of some limitations. In particular, as highlighted in Remark 1 and Remark 6, the technique fails when the Jacobian evaluated at any equilibrium point contained into the system invariant set has more than one eigenvalue with positive real part, since the 2-additive compound can not be a Hurwitz matrix. A similar limitation exists also for the synthesis of a feedback control law ensuring that the closed loop system is 2-contractive. In addition, all the conditions presented in this thesis are sufficient conditions and hence the presented techniques can be used to exclude the presence of chaotic or complex behaviours in some invariant set but they can not be employed to show the existence of chaos.
Some future research issues can be foreseen. The results reported in Chapter 3 could be extended to the case of large-scale nonlinear systems and also to neural networks. In particular, it is expected that some results ensuring

convergence towards the equilibrium points can be derived for neural networks whose interconnection matrix is not symmetric. Furthermore, it could be investigated if it is possible to derive conditions ensuring convergence towards the equilibrium points in the case of systems with first integral of motion, such as memristors and memelements. This appears to be a challenging problem since these systems possess infinitely many and non-isolated equilibrium points. Finally, it is worth noting that Propositions 20 and 21 provide conditions in terms of Matrix Inequality (MI) and Bilinear Matrix Inequality (BMI). It could be interesting to find a method to transform these conditions into some Linear Matrix Inequality (LMI), since the existing techniques to perform such a transformation do not work in this case, due to the particular structure of the matrix $\underline{L}^K$.

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