Properties of unique degree sequences of 3-uniform hypergraphs

M. Ascolese[1], A. Frosini[1], W.L. Kocay[2], and L. Tarsissi[3]

[1] Università di Firenze, Dipartimento di Sistemi e Informatica, Viale Morgagni 65, 50134 Firenze, Italy *corresponding author:* `andrea.frosini@unifi.it`
[2] Department of Computer Science and St. Pauls College, University of Manitoba, Winnipeg, Manitoba, CANADA
[3] LIGM, Univ Gustave Eiffel, CNRS, ESIEE Paris, F-77454 Marne-la-Vallée, France

Abstract. In 2018 Deza et al. proved the NP-completeness of deciding wether there exists a 3-uniform hypergraph compatible with a given degree sequence. A well known result of Erdös and Gallai (1960) shows that the same problem related to graphs can be solved in polynomial time. So, it becomes relevant to detect classes of uniform hypergraphs that are reconstructible in polynomial time. In particular, our study concerns 3-uniform hypergraphs that are defined in the NP-completeness proof of Deza et al. Those hypergraphs are constructed starting from a non-increasing sequence s of integers and have very interesting properties. In particular, they are unique, i.e., there do not exist two non isomorphic 3-uniform hypergraphs having the same degree sequence d_s. This property makes us conjecture that the reconstruction of these hypergraphs from their degree sequences can be done in polynomial time. So, we first generalize the computation of the d_s degree sequences by Deza et al., and we show their uniqueness. We proceed by defining the equivalence classes of the integer sequences determining the same d_s and we define a (minimal) representative. Then, we find the asymptotic growth rate of the maximal element of the representatives in terms of the length of the sequence, with the aim of generating and then reconstructing them. Finally, we show an example of a unique 3-uniform hypergraph similar to those defined by Deza et al. that does not admit a generating integer sequence s. The existence of this hypergraph makes us conjecture an extended generating algorithm for the sequences of Deza et al. to include a much wider class of unique 3-uniform hypergraphs. Further studies could also include strategies for the identification and reconstruction of those new sequences and hypergraphs.
AMS classification: 05C65, 05C60, 05C99

Keywords: Hypergraph · Graphic sequence · Uniqueness problem · Analysis of algorithms

1 Introduction

The notion of hypergraph naturally extends that of graphs, where each edge is defined to be a subset of the vertices (see [2] for basic definitions and results on hypergraphs).

In this paper, we consider *simple* hypergraphs, i.e., hypergraphs that are loopless and with distinct edges. Here a loop is considered to be an edge consisting of just one vertex. A hypergraph is *h-uniform* if every edge is a subset of cardinality h.

The degree sequence of a simple hypergraph is the sequence of the degrees of its vertices arranged in non-increasing order. A degree sequence usually does not characterize the hypergraph it is related to, but however it reveals interesting properties. For this reason, degree sequences are widely studied in (hyper)graph theory. We say that a *reconstruction* of a degree sequence d is a (hyper)graph whose degree sequence is d. One of the most challenging problems related to hypergraph degree sequences is the reconstruction of a compatible (hyper)graph, if any [4]. For graphs, this problem has been solved in a milestone paper by Erdös and Gallai [7], in 1960, providing an efficiently computable characterization of them. And an algorithm to construct a graph with a given degree sequence was given by Havel [10] and Hakimi [9]. On the other hand, the same problem for hypergraphs remained unsolved till 2018, when Deza et al. in [6] proved its NP-completeness, even for the simplest case of 3-uniform hypergraphs.

As a consequence, the study of wide classes of uniform hypergraphs whose reconstruction from a degree sequences can be performed in polynomial time has acquired more and more relevance, in order to spot the hard core of the generic reconstruction problem.

Recently, but still before the result in [6], some necessary conditions had been given for a sequence to be the degree sequence of an h-uniform hypergraph. Such a sequence is called an *h-sequence*. Most of these generalized the Erdös and Gallai theorem, or were based on two well known theorems by Havel [10] and Hakimi [9]. On the other hand, few necessary conditions were known. Among them, one of prominent interest is provided in [1], which uses Dewdney's theorem and sets a lower bound on the length of a sequence related to an h-uniform hypergraph. This result has been algorithmically rephrased in [3, 8].

The present study focuses on investigating the properties of the 3-sequences that originate from a generalization of the gadget used by Deza et al. in [6] for their NP-completeness proof. These are denoted by $\mathcal{D}_n$, according to their length n. The relevance of those sequences is mainly due to their uniqueness property, i.e., there exists a unique 3-hypergraph compatible with them, up to isomorphism. It is known that there exists an operator called a *trade* that allows one to travel amongst all hypergraphs having the same degree sequence [11]; here we prove that the 3-hypergraphs related to the elements of each $\mathcal{D}_n$ act as a sort of fixed point for this operator.

Furthermore, since each element of $\mathcal{D}_n$ can be related to an infinite number of integer sequences, we group them into equivalence classes and we choose a representative for each class. We compute a lower bound to the asymptotic growth of the representative's elements gaining information about the cardinality of $\mathcal{D}_n$ and obtaining clues for a strategy for reconstruction.

From this preliminary study, a series of open problems results, with the long term aim of characterizing and of reconstructing wider classes of unique degree sequences. An example is given in Example 2.

So, in the next section, we will provide definitions and the results useful for our study. Section 3 will be devoted to presenting the most relevant properties of the degree sequences in $\mathcal{D}_n$ and the asymptotic growth of the elements of their representatives. In Section 4, we consider the reconstruction problem of a simple subclass of $\mathcal{D}_n$ and the isomorphism problem on unique sequences. We conclude our work by pointing out some open questions in Section 5.

2 Basic Notions and Definitions

We recall the basic definition of hypergraphs and fix the notation used in the rest of the paper. A hypergraph H is defined as a pair $H = (V, E)$ such that V is the set of vertices and $E \subset \mathcal{P}(V) \setminus \{\emptyset\}$ is the set of hyperedges, or briefly edges when no ambiguities occur, where $\mathcal{P}(V)$ is the power set of V.

A hypergraph is *simple* if it is loopless and has distinct edges. In other words, it does not allow singleton edges or edges that are contained in or equal to other edges. The *degree* of a vertex $v \in V$ is the number of edges containing v. The degree sequence $d = (d_1, d_2, \dots, d_n)$ of a hypergraph H is the list of its vertex degrees usually arranged in non-increasing order. Let us denote by $\sigma(d)$ the sum of the elements of d. When $\mathcal{H}$ is k-uniform, the sequence d is called k-graphic. Notice that the case $k = 2$ corresponds to graphs, and a 2-graphic sequence is simply called graphic.

The seminal book by Berge [2] contains some essential results about hypergraphs and also information about their applications.

The problem of characterizing the graphic sequences of simple graphs was solved by Erdös and Gallai [7], in 1960, and algorithmically by Havel [10] and Hakimi [9], while only recently Deza et al. in [6] have shown the NP-completeness of the characterization of k-graphic sequences, with $k \geq 3$. The *3-Hypergraph Degree Sequence Problem* is the question of determining whether a given sequence d is the degree sequence of a 3-hypergraph.

In their proof, the authors mapped the instances of the known NP-complete problem *3-Partition* into instances of the 3-Hypergraph Degree Sequence problem. The mapping has the property that there is a 1-to-1 correspondence between the solutions of the instances I of *3-partition* and the 3-hypergraphs having the prescribed degree sequence d, computed from I.

To compute d, Deza et al. used an intermediate step in which they constructed a "gadget", and computed its degree sequence d' from I. This sequence turns out to have very interesting properties, as shown in the next sections, and it constitutes the focus of our research.

Hereafter, we define the procedure *Gen-pi*(s) that generalizes the gadget computation presented in [6], regardless the specific characteristics of the length and element sum of the instance of *3-Partition.* The input of this computation is an integer sequence $s = (s_1, \dots, s_n)$, and it returns a degree sequence, denoted

d_s, to emphasize its dependence on s, of a 3-hypergraph. Following the notation in [6], let $\{0,1\}^n_3$ be the set of all the binary sequences of length n having exactly three elements equal to 1 and let s^T be the transpose of the vector s.

Algorithm 1 *Gen-pi*(s)

set $E \leftarrow \emptyset$
for all $e \in \{0,1\}^n_3$ **do**
  if $e \cdot s^T > 0$ **then**
    $E \leftarrow E \cup \{e\}$
  end if
end for
return $d_s = \Sigma E$

The class of degree sequences of length n generated by the algorithm *Gen-pi* from an input sequence s is indicated as $\mathcal{D}_n$.

The action of *Gen-pi* on s will be clearer after introducing the notion of incidence matrix of a hypergraph. Given a hypergraph $H = (V, E)$ such that $|V_\mathcal{H}| = n$ and $|E_\mathcal{H}| = m$, its *incidence matrix* is a $m \times n$ binary matrix where $a_{i,j} = 1$ if and only if the edge e_i contains the vertex v_j, otherwise $a_{i,j} = 0$. So $\Sigma^m_{i=1} a_{i,j} = d_j$ is the degree of the vertex v_j, and when $\mathcal{H}$ is $k-$uniform we have $\Sigma^n_{j=1} a_{i,j} = k$ for each edge e_i.

We observe that, for k-hypergraphs, the property of being simple means that all the rows of the incidence matrix are different.

Example 1 *Let us consider the integer sequence $s = (3, 2, 0, -1, -2)$. The output of Gen-pi(s) is the degree sequence $d_s = (5, 4, 4, 3, 2)$. Figure 2 shows a 3-hypergraph $H(d_s)$ having d_s as degree sequence, together with its related incidence matrix $M(d_s)$.*

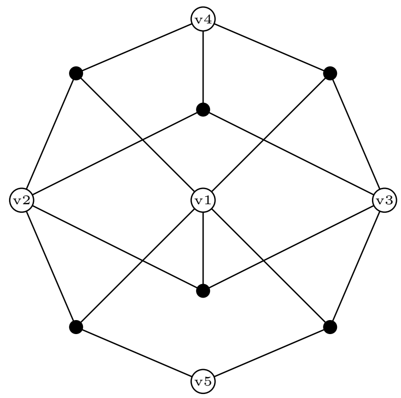
The matrix representation of the hypergraph $H(d_s)$ in Fig. 2 provides an immediate idea of the action of *Gen-pi* on s. In the sequel, we will consider only degree sequences obtained from sequences s such that $s_1 + s_2 > s_n$, i.e., without null columns or, equivalently, hypergraphs without isolated vertices.

3 Properties of the elements of $\mathcal{D}_n$

A relevant property of a degree sequence d_s directly follows from the action of *Gen-pi* on a generic integer sequence s:

Theorem 1 *There exists one only 3-hypergraph (up to isomorphism) having degree sequence d_s.*

Proof. The result can be obtained by contradiction. Let $d_s = (d_1, \ldots, d_n)$ and $s = (s_1, \ldots, s_n)$ be as in the algorithm *Gen-pi*. We observe that the number Max$= \Sigma^n_{i=1} s_i d_i$ is the maximum that can be realized by a sequence of



$$M(d_s) = \begin{bmatrix} 1\,1\,1\,0\,0 \\ 1\,1\,0\,1\,0 \\ 1\,1\,0\,0\,1 \\ 1\,0\,1\,1\,0 \\ 1\,0\,1\,0\,1 \\ 0\,1\,1\,1\,0 \end{bmatrix}$$

Fig. 1. The 3-hypergraph $H(d_s)$ and its related incidence matrix $M(d_s)$. The black nodes represent the edges of the 3-hypergraph and the white circles represent its vertices.

$m = \Sigma_{i=1}^{n} d_s/3$ different triplets $(s_{i_1}, s_{j_1}, s_{k_1}), \ldots, (s_{i_m}, s_{j_m}, s_{k_m})$ of elements of s, containing all the edges $(i_1, j_1, k_1), \ldots, (i_m, j_m, k_m)$ of the 3-hypergraph $H(d_s)$. Any other 3-hypergraph $H(d_s)'$ having the same degree sequence would involve at least an edge $e = (i, j, k)$ not in $H(d_s)$, so that $s_i + s_j + s_k = t \leq 0$. As a consequence, the remaining $m-1$ different edges $(i'_1, j'_1, k'_1), \ldots, (i'_{m-1}, j'_{m-1}, k'_{m-1})$ of $H(d_s)'$ must satisfy $\Sigma_{z=1}^{m-1} s_{i'_z} + s_{j'_z} + s_{k'_z} = \text{Max} + t$, and we reach a contradiction. □

We say that the sequence d_s is *unique*, as well as the corresponding 3-hypergraph.

Note that the number of elements of $\mathcal{D}_n$ is finite, since each d_s has $\binom{n}{3}\frac{3}{n}$ as a maximum entry. So, by cardinality reasons, there exist an infinite number of non-increasing integer sequences s that generate at least one degree sequence d_s. An easy check reveals that for each degree sequence d_s there exists an infinite number of generating integer sequences.

As an example, all the length n sequences of positive integers generate the same constant degree sequence $d_s \in \mathcal{D}_n$ whose elements are $\binom{n}{3}\frac{3}{n} = \frac{(n-1)(n-2)}{2}$. Obviously, this sequence is maximal in $\mathcal{D}_n$ w.r.t the lexicographical order.

An easy check reveals that if a sequence s has two equal elements s_i and s_{i+1}, then also the elements d_i and d_{i+1} of the related d_s degree sequence are equal. The reverse is also true.

Property 1 *If there exists an index $i < n$ of d_s such that $d_i = d_{i+1}$, then there exists a sequence s' such that $s'_i = s'_{i+1}$ and $d_s = d_{s'}$.*

Proof. Let us construct s' from s: if $s_i = s_{i+1}$, then $s' = s$. So, let us consider the case $s_i \neq s_{i+1}$. Since $s_i > s_{i+1}$, then each triplet $s_{i+1} + s_j + s_k > 0$ implies the triplet $s_i + s_j + s_k > 0$, and each triplet $s_i + s_{j'} + s_{k'} \leq 0$ implies the triplet $s_{i+1} + s_{j'} + s_{k'} \leq 0$, with $j, j', k, k' \leq n$. Since $d_i = d_{i+1}$, then the reverse of both implications also holds, and furthermore, these inequalities also hold if s_i is replaced by an integer $s_{i+1} \leq \overline{s} \leq s_i$. For the inequalities that involve both s_i and s_{i+1}, i.e., those of the forms $s_i + s_{i+1} + s_k > 0$ or $s_i + s_{i+1} + s_{k'} \leq 0$, they are preserved when s_i and s_{i+1} are both replaced by the value $\overline{s} = \frac{s_i + s_{i+1}}{2}$. Unfortunately, $\overline{s}$ is not always an integer, and a final observation is required: the inequalities used in the *Gen-pi* procedure are preserved when all the elements of s are doubled, i.e., by abuse of notation, it holds $d_s = d_{2*s}$.

Putting things together, from s we can define the sequence s' as follows: first we initialize $s' = 2s$, then we set $s'_i = s'_{i+1} = \frac{s'_i + s'_{i+1}}{2} = \overline{s}'$. Doubling the elements of s, we find that $\overline{s}'$ is integer, and since $s'_i \leq \overline{s}' \leq s'_{i+1}$, all the inequalities that define d_s are preserved, as required. □

Property 2 *Let $s_i = s_{i+1}$ be two elements of an integer sequence s. The i^{th} column and the $(i+1)^{\text{st}}$ column of the incidence matrix M_s of the hypergraph H_s generated by procedure* Gen-pi(s) *are equal.*

The proof directly follows from the fact that the elements d_i and d_{i+1} satisfy the same inequalities, and so they are present in the edges involving the same vertices.

Returning to the equivalence classes of the elements in $\mathcal{D}_n$, we propose as representative of the (non-void) class $[d]$, the sequence s_d that is minimal in lexicographic order. We stress the fact that s_d has the minimal first element among all the elements of $[d]$.

Unfortunately, equal elements of d do not always correspond to equal elements in the representative of s_d. As an example, exhaustive computation reveals that the representative of $d = (6, 5, 5, 4, 4)$ is $s_d = (2, 1, 1, 0, -1)$.

In order to compute and characterize the representative s_d of each equivalence class $[d]$, it is useful to understand the growth rate of the elements inside s_d according to length, with special attention to its first (and maximal) one.

So, for each length n and each $d \in \mathcal{D}_n$, we denote

$$M_n = \max_{s_d}\{s_1 : s_d = (s_1, s_2, \ldots, s_n)\}.$$

We call the sequences s_d where such a maximal first element is present *maximal* sequences. The first elements of the sequence $\{M_n\}_{n>2}$ up to $n = 8$, obtained by exhaustive computation are $1, 1, 2, 4, 6, 10$. For each n, there are several degree sequences whose representatives have the same maximal first element. The following table shows some of them according to the length parameter n:

Let us investigate the asymptotic growth of the sequence $\{M_n\}_{n>3}$ by constructing a class of degree sequences $\mathcal{C}$ that provides a lower bound to it.

$n=5$		$n=7$	
s	d_s	s	d_s
$[2,1,1,0,-1]$	$[6,5,5,4,4]$	$[6,3,2,0,-1,-2,-3]$	$[15,11,10,9,8,7,6]$
$[2,1,1,-1,-2]$	$[5,4,4,3,2]$	$[6,3,2,0,-2,-3,-4]$	$[13,10,9,7,6,5,4]$
$[2,1,0,0,-1]$	$[6,4,4,4,3]$	$[6,3,1,1,0,-2,-3]$	$[15,13,10,10,9,8,7]$
$[2,1,0,-1,-2]$	$[4,3,2,2,1]$	$[6,2,1,1,-1,-2,-3]$	$[15,10,9,9,8,7,5]$
$[2,0,0,-1,-1]$	$[5,3,3,2,2]$	$[6,2,1,0,-1,-2,-3]$	$[15,9,8,7,7,6,5]$

$n=6$		$n=8$	
s	d_s	s	d_s
$[4,2,1,0,-1,-2]$	$[10,8,7,6,6,5]$	$[10,4,3,2,0,-1,-3,-6]$	$[21,16,15,14,13,12,10,7]$
$[4,1,1,-1,-1,-2]$	$[10,6,6,5,5,4]$	$[10,4,2,2,1,-2,-4,-5]$	$[21,17,14,14,13,12,9,8]$

Table 1. Table of the representatives s of the equivalence classes $[d_s]$ having maximal first element for the sequence lengths $n = 5, 6, 7$ and 8.

First we define the operator *Extend* that allow us to compute from a sequence $s = (s_1, \ldots, s_n)$ a sequence $s' = (s'_1, \ldots, s'_{n+2}) = \mathit{Extend}(s)$ as follows:

$$s'_i = \begin{cases} s_{i-1}, & \text{for } 2 \leq i \leq n+1 \\ s_1 + s_2 - s_n + 1, & \text{for } i = 1 \\ -(s_1 + s_2), & \text{for } i = n+2 \end{cases}$$

The following property holds

Property 3 *If s is the representative of the class $[d] \in D_n$, then $s' = \mathit{Extend}(s)$ is the representative of the respective class in D_{n+2}.*

Proof. Let $d_{s'}$ be the degree sequence generated by s'. We proceed by contradiction assuming there exists a representative $t = (t_1, \ldots, t_{n+2})$ of $[d_{s'}]$ different from s' and having $t_1 < s'_1$. We notice that the sequence $\tilde{t} = (t_2, \ldots, t_{n+1})$ generates d, since the elements $s'_2 + s'_3 \leq s'_{n+2}$, and such inequality must hold also in t, so also the elements $t_2, \ldots, t_{n+1}$ are not involved in any edge including t_{n+2}. On the other hand, by construction, $s'_2 + s'_3 = -s'_{n+2}$, so it holds $t_{n+2} \leq s'_{n+2}$.

Finally, we notice that the inequality $s'_1 + s'_{n+1} + s'_{n+2} > 0$ also holds since $s'_1 + s'_{n+1} + s'_{n+2} = 1$. In order to preserve the same inequality in t, i.e., $t_1 + t_{n+1} + t_{n+2} > 0$, having $t_{n+2} \leq s'_{n+2}$, we need $t_1 \geq s'_1$, and we reach a contradiction. □

We underline that by iterating the application of the procedure *Extend* to a sequence s of length n, it produces longer sequences having the same length parity as s.

To illustrate the action of *Extend*, we depict its behaviour on the incidence matrices H_s and $H_{s'}$, with $s' = \mathit{Extend}(s)$. The basic idea is to extend H_s by adding an initial and a final column, as well as a set of starting rows, maximizing their number while leaving those in H_s unchanged:

$$H_{s'} = \begin{pmatrix} 1 & 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & 0 & 1 & \dots & \dots & 0 \\ \vdots & & & & & & \\ 1 & \dots & \dots & \dots & \dots & 1 & 1 \\ & & & & & & \\ 0 & & & & & & 0 \\ \vdots & \Big[& & H_s & & \Big] & \vdots \\ 0 & & & & & & 0 \end{pmatrix}$$

Let $s_0 = (s_{0,1}, \dots, s_{0,n})$ be an integer sequence, and $s_k = (s_{k,1}, \dots, s_{k,n+2k}) = \mathit{Extend}^k(s_0)$ be the sequence obtained by recursively applying the procedure *Extend* k-times to s_0. The following result holds

Theorem 2 *The integer sequence $\{s_{k,1}\}_k$ satisfies the recurrence relation:*

$$s_{k,1} = s_{k-1,1} + 2s_{k-2,1} + s_{k-3,1} + 1 \qquad k \geq 3 \tag{1}$$

with $s_{1,1}$ and $s_{2,1}$ being the first elements of $s_1 = \mathit{Extend}(s_0)$, and $s_2 = \mathit{Extend}^2(s_0)$, respectively.

Proof. The result immediately follows from the definition of the *Extend* operator:

$$\begin{aligned} s_{k,1} &= s_{k-1,1} + s_{k-1,2} - s_{k-1,n+2(k-1)} + 1 \\ s_{k-1,2} &= s_{k-2,1} \\ s_{k-1,n+2(k-1)} &= -s_{k-2,1} - s_{k-2,2} \end{aligned}$$

Replacing $s_{k-1,2}$ and $s_{k-1,n+2(k-1)}$ in the first equation we obtain the recurrence relation. □

We observe that, starting from two representative sequences $s_e = (1, 1, -1, -1)$ and $s_o = (2, 1, 0, -1, -2)$, having even and odd length, respectively, that are maximal w.r.t. the first element, the procedure *Extend* produces the two sequences $\mathit{Extend}(s_e) = (4, 1, 1, -1, -1, -2)$ and $\mathit{Extend}(s_o) = (6, 2, 1, 0, -1, -2, -3)$ that turn out to be two representatives with maximal first element of length 6 and 7, respectively. This property is not maintained when considering $\mathit{Extend}^2(s_e) = (8, 4, 1, 1, -1, -1, -2, -5)$, since $M_8 = 10$.

From this simple observation, we realize that the action of *Extend* on representatives with maximal first (and second) element deserves a deeper investigation in order to find an operator that allows us to pass from maximal representatives to maximal representatives. However, the *Extend* operator provides a lower bound to the growth rate of the M_n sequence.

Theorem 3 *The growth constant λ of the sequence $\{s_{k,1}\}_k$ is $2.147 < \lambda < 2.148$.*

Proof. The result follows from constructing the generating function for the recurrence relation (1) whose denominator is $q(x) = (1 - x - 2x^2 - x^3)(1 - x)$. The denominator has a unique minimal real root which can be computed numerically, $\rho = 0.466$. Therefore the asymptotic behaviour is controlled by λ that is the inverse of ρ, namely $f_{3,k} \sim (\lambda)^k$ □

Corollary 1 *The growth constant λ' of $\{M_n\}_{n>3}$ is $\lambda' \geq \lambda$.*

4 Further Results on Unique Sequences

The property that each element d of $\mathcal{D}_n$ is unique, has a great relevance in the reconstruction of the related 3-hypergraph H_d. In particular, it is remarkable that, once one detects some edges of H_d, then those edges will not be modified till the end of the reconstruction or, equivalently, they will not be involved in possible backtracking steps, so limiting the complexity of the process.

As an example, this is the case of one or more elements of $d = (d_1, \dots, d_n)$ that are maximal or minimal. Focusing on the first element d_1, if its value is $\binom{n-1}{2}$ this means that all edges involving the vertex v_1 are present, so they can be added and the process can proceed recursively on the next elements of d. Equivalently if $d_1 = n - 2$, then, by the definition of d, the edges involving v_1 are $(v_1, v_2, v_3), (v_1, v_2, v_4), \dots, (v_1, v_2, v_n)$, that can be added to H_d.

Apart from those simple cases, some more are possible that may involve all the elements of d in a complex pattern of entanglements. In the intent of discovering a polynomial time way of managing these fixed patterns, a first and natural reconstruction algorithm is provided to deal with those simple cases that we address as *maximal instances*. This name is due to the fact that the sequence is maximal w.r.t. the lexicographical order among those having the same sum of elements.

The algorithm *Rec-max*, here not fully detailed for brevity sake, accepts a degree sequence d as input and produces the incidence matrix of a 3-hypergraph compatible with d, if d is a maximal sequence, otherwise it fails. We use the notation $M \oplus h$ to append the row h to the matrix M and $\deg(M)$ to denote the vector of column sums of the matrix M, i.e., the degree sequence of the related hypergraph.

The proof of the correctness of the algorithm is straightforward.

4.1 On the isomorphism properties of $\mathcal{D}_n$

The remarkable properties of the elements of $\mathcal{D}_n$ have an interesting consequence on the isomorphism problem restricted to the class. We recall that two (hyper)graphs H_1 and H_2 are isomorphic if and only if one can pass from the incidence matrix of H_1 to that of H_2 by a first round of column shifts, that corresponds to a mapping φ of the vertices of H_1 into those of H_2, followed by a second round of row shifts that allows to check the exact correspondence of the (hyper)edges.

Algorithm 2 *Rec-max*(d)

set $d' = d$, $M_d = \emptyset$ and $i = 1$;
Step 1: create the maximal d'_i (w.r.t. the lexicographical order) length n binary sequences $h_1, \ldots, h_{d'_i}$ having exactly three elements 1, and such that the first of them lies in position i;
Step 2: set $M_d = M_d \oplus h_1 \oplus \cdots \oplus h_{d'_i}$;
Step 3: set $d' = d - \deg(M_d)$;
Step 4: **if** d' is the null vector **then**
RETURN M_d;
else if d' has an element less than 0 or $i = n - 2$ **then**
RETURN failure
else GoTo Step 1 updating $i = i + 1$.

Obviously, to check the isomorphism, φ has to preserve the vertices' degrees in H_1 and H_2, so in case of degree sequences with no equal elements φ equals Id, the identity mapping.

Actually, in case of equal elements in the degree sequence of H_1 and H_2, φ has, in general, to inspect the isomorphism for each possible permutations of the related columns (at present, no better strategy is available).

On the other hand, Property 2 states that, by construction, in each element $H \in \mathcal{D}_n$, the columns related to vertices having the same degree are equal, so again $\varphi = Id$ is a suitable mapping. This equality does not hold, in general, for each unique degree sequence (e.g., the degree sequence $d = (1, 1, 1, 1, 1, 1)$ is unique w.r.t. 3-hypergraphs, but the check of the isomorphism of two related 3-hypergraphs H_1 and H_2 needs, in general, φ to be different from Id).

The following example reveals a new potential research line concerning the characterization and reconstruction of unique sequences that include, but are not restricted to, those generated by *Gen-pi*:

Example 2 Consider the degree sequence $\hat{d} = (25, 19, 19, 16, 16, 12, 10, 10, 5)$. Its uniqueness is witnessed by the related matrix $M_{\hat{d}}$ in Fig. 2, and whose construction seems to be not so far from that performed by *Gen-pi*. It remains an open problem to find a suitable meaning to the words "not so far" that could lead to a generalization of *Gen-pi*.

The following computations show that there does not exist an integer sequence s such that $\hat{d} = \hat{d}_s$. Recall that, by Property 1, among all the sequences in $[\hat{d}]$, if any, there exists one s such that $s_2 = s_3$, $s_4 = s_5$ and $s_7 = s_8$:
By the inequalities of Table 2 we obtain the result, in particular: by (1) and (3) we obtain $-2s_8 + s_6 + s_9 > 0$, by this last and (2) we obtain $2s_2 + 2s_5 + s_6 + s_9 > 0$ and finally using (5) we reach $2s_2 + s_9 > 0$, against (4).

v1	v2	v3	v4	v5	v6	v7	v8	v9	v1	v2	v3	v4	v5	v6	v7	v8	v9
1	1	1	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0
				...									...				
1	1	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	0
1	0	1	1	0	0	0	0	0	0	1	0	1	1	0	0	0	0
				...									...				
1	0	1	0	0	0	0	0	1	0	1	0	1	0	0	0	1	0
1	0	0	1	1	0	0	0	0	0	1	0	0	1	1	0	0	0
				...									...				
1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	1	0
1	0	0	0	1	1	0	0	0	0	0	1	1	1	0	0	0	0
				...									...				
1	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	1	0
1	0	0	0	0	1	1	0	0	0	0	1	0	1	1	0	0	0
				...									...				
1	0	0	0	0	0	1	0	1	0	0	1	0	1	0	0	1	0

Fig. 2. The incidence matrix $M(\hat{d})$ of a 3-hypergraph having degree sequence $\hat{d}$. On the left the part of the matrix related to the edges involving v_1, while on the right the remaining ones. The 3-hypergraph is unique as can be seen by the construction of the matrix.

$(1,6,9) \in H_{\hat{d}} \rightarrow s_1 + s_6 + s_9 > 0$	(1)
$(2,5,8) \in H_{\hat{d}} \rightarrow s_2 + s_5 + s_8 > 0$	(2)
$(1,7,8) \notin H_{\hat{d}} \rightarrow s_1 + 2s_8 \leq 0$	(3)
$(2,3,9) \notin H_{\hat{d}} \rightarrow 2s_2 + s_9 \leq 0$	(4)
$(4,5,6) \notin H_{\hat{d}} \rightarrow 2s_5 + s_6 \leq 0$	(5)

Table 2. Inequalities related to some rows of the matrix $H_{\hat{d}}$. Edges are represented by triplets of vertex indices.

5 Conclusions and Open Problems

In this article, we consider the class $\mathcal{D}_n$ of degree sequences of 3-hypergraphs on n vertices that extend those defined in [6] and that are computed starting from a given integer sequence. First, we prove that each degree sequence $d \in \mathcal{D}_n$ is unique, i.e., the related 3-hypergraph H_d is unique up to isomorphism, then we define the representative integer sequence s_d which leads to the reconstruction of H_d. Some properties of s_d are shown, in particular we determine a lower bound to the growth rate of their maximal elements according to the length n, related to the number of edges of the 3-hypergraph H_d. This result is useful to generate and enumerate the elements of $\mathcal{D}_n$, establishing the size of the class. In this context, we point out two open problems:

i) define a variant of the *Extend* operator that allows to maintain the maximality property of the representatives;

ii) find the growth rate of the sequence $\{M_n\}_{n>2}$ and characterize the maximal representatives of each $\mathcal{D}_n$.

Furthermore, a simple algorithm is defined to reconstruct the maximal (in lexicographic order) sequences of $\mathcal{D}_n$ having prescribed sum of elements. From its definition, we realize that a generalization to include all the elements of $\mathcal{D}_n$ would involve some backtracking when elements less than zero appear in d'. We propose the following research line:

iii) find some properties related to the computation of d_s from s that prevent or, at least, restrict the backtracking in *Rec-max*. This will lead to the solution of the reconstruction problem related to $\mathcal{D}_n$. As an alternative, prove that it cannot be done in polynomial time.

Finally, the degree sequence $\hat{d}$ in Example 2 deserves attention: it admits a unique 3-hypergraph whose *structure* is close to that of the hypergraphs related to the elements of $\mathcal{D}_n$, but without being generated by an integer sequence. A final open problem is proposed:

iv) define a notion of structure of the elements of $\mathcal{D}_n$ and expand the class including unique degree sequences with similar structure. Investigate the properties of the new class.

References

1. Behrens, S., Erbes, C., Ferrara, M., Hartke, S.G., Reiniger, B., Spinoza, H., Tomlinson, C.: *New Results on Degree Sequences of Uniform Hypergraphs*, Electron. J. Comb., 20(4), # $P14$ (2013).
2. Berge, C.: *Hypergraphs*, North-Holland, Amsterdam, (1989).
3. Brlek, S., Frosini, A.: *A Tomographical Interpretation of a Sufficient Condition for h-Graphical Sequences* DGCI 2016, Lect. Notes in Comput. Sc., 9647, pp.105 − 116 (2016).
4. Colbourne, C.J., Kocay, W.L. , Stinson, D.R.: *Some NP-complete problems for hypergraph degree sequences* Discrete Appl. Math., 14, pp.239 − 254 (1986).
5. Dewdney, A.K., *Degree sequences in complexes and hypergraphs*, Proc. Amer. Math. Soc., 53(2), pp.535 − 540 (1975).
6. Deza, A., Levin, A., Meesum, S.M., Onn, S.: *Optimization over degree sequences*, SIAM J. Disc. Math. 32(3), pp.2067 − 2079 (2018).
7. Erdös, P., Gallai, T.: *Graphs with prescribed degrees of vertices* (in Hungarian), Math. Lapok. 11 pp.264 − 274 (1960).
8. Frosini, A., Picouleau, C., Rinaldi, S.: *On the degree sequences of uniform hypergraphs* Lect. Notes in Comput. Sc., Vol. 7749, 300 − 311 (2013).
9. Hakimi, S.: *On realizability of a set of integers as degrees of the vertices of a lineargraph*, I.J. Soc. Indust. Appl. Math., 10, pp.496–506 (1962).
10. Havel, V.: *A remark on the existence of finite graphs* (in Czech), Časopis. Pěst. Mat., 80, pp.477–480 (1955).
11. Kocay, W., Li, P.C.:*On 3-Hypergraphs with Equal Degree Sequences*, Ars Combinatoria, 82, pp.145 − 157 (2006)