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Non-solvable groups whose character degree graph has a cut-vertex. III

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Abstract

Let G be a finite group. Denoting by $\mathrm{cd}(G)$ the set of the degrees of the irreducible complex characters of G, we consider the *character degree graph* of G: this, is the (simple, undirected) graph whose vertices are the prime divisors of the numbers in $\mathrm{cd}(G)$, and two distinct vertices p, q are adjacent if and only if pq divides some number in $\mathrm{cd}(G)$. This paper completes the classification, started in Dolfi et al. (Non-solvable groups whose character degree graph has a cut-vertex. II, 2022. https://doi.org/10.1007/s10231-022-01299-3) and Dolfi et al. (Non-solvable groups whose character degree graph has a cut-vertex. I, 2022. https://doi.org/10.48550/arXiv.2207.10119), of the finite non-solvable groups whose character degree graph has a *cut-vertex*, i.e., a vertex whose removal increases the number of connected components of the graph. More specifically, it was proved in Dolfi et al. (Non-solvable groups whose character degree graph has a cut-vertex. I, 2022. https://doi.org/10.48550/arXiv.2207.10119 that these groups have a unique non-solvable composition factor S, and that S is isomorphic to a group belonging to a restricted list of non-abelian simple groups. In Dolfi et al. (Non-solvable groups whose character degree graph has a cut-vertex. II, 2022. https://doi.org/10.1007/s10231-022-01299-3) and Dolfi et al. (Non-solvable groups whose character degree graph has a cut-vertex. I, 2022. https://doi.org/10.48550/arXiv.2207.10119) all isomorphism types for S were treated, except the case $S \cong \mathrm{PSL}_2(2^a)$ for some integer $a \geq 2$; the remaining case is addressed in the present paper.

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1 Introduction

The *character degree graph* $\Delta(G)$ of a finite group G is a very useful tool for studying the arithmetical structure of the set $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$, i.e., the set of the irreducible (complex) character degrees of G. As many results in the literature show, there is a profound interaction between the group structure of G and certain graph-theoretical properties (in particular, connectivity properties) of $\Delta(G)$.

In the papers [5, 6] we considered the problem of classifying the finite non-solvable groups G such that $\Delta(G)$ has a *cut-vertex*, which is a vertex whose removal (together with all the edges incident to it) produces a graph having more connected components than the original. Among the various properties of such a group G, it is proved in [6] that G has a unique non-solvable composition factor S, and that S is isomorphic to one of the simple groups in the following list: the projective special linear group $\mathrm{PSL}_2(t^a)$ (where t^a is a prime power greater than 3), the Suzuki group $\mathrm{Sz}(2^a)$ (where $2^a - 1$ is a prime number), $\mathrm{PSL}_3(4)$, the Mathieu group M_{11}, and the first Janko group J_1. The aforementioned papers carry out an analysis (and provide a complete classification) of all the possibilities, except for the case $S \cong \mathrm{PSL}_2(2^a)$ when $\Delta(G)$ is connected; the present work addresses the remaining case, thus completing the classification of these groups. We refer the reader to [5, 6] for a thorough description of the problem and, in particular, for the full statements of the relevant theorems (see the introductions of [5, 6], and Section 2 of [6]).

The situation that remains to be studied is treated in the following Theorems 1 and 2, which deal with the cases $2^a > 4$ and $2^a = 4$, respectively (see [6, Theorem A, Case (f)], and [6, Theorem B]), and which are the main results of this paper.

In order to clarify the statements we mention that, for $H = \mathrm{SL}_2(t^a)$ (where t^a is a prime power), an H-module V over the field $\mathbb{F}_t$ of order t is called *the natural module for H* if V is isomorphic to the standard module for $\mathrm{SL}_2(t^a)$, or any of its Galois conjugates, seen as an $\mathbb{F}_t[H]$-module. We will freely use this terminology also referred to the conjugation action of a group on a suitable elementary abelian normal subgroup. For our purposes, it is important to recall that the standard module for $\mathrm{SL}_2(t^a)$ is self-dual.

Also, given a finite group G, we denote by $R = R(G)$ the *solvable radical* (i.e., the largest solvable normal subgroup), and by $K = K(G)$ the *solvable residual* (i.e., the smallest normal subgroup with a solvable factor group) of G. Equivalently, $K(G)$ is the last term of the derived series of G.

Theorem 1 *Let R and K be, respectively, the solvable radical and the solvable residual of the finite group G and assume that G has a composition factor $S \cong \mathrm{SL}_2(2^a)$, with $a \geq 3$. Then, $\Delta(G)$ is a connected graph and has a cut-vertex p if and only if G/R is an almost simple group with socle isomorphic to S, $V(G) = \pi(G/R) \cup \{p\}$ and one of the following holds.*

(a) *$K \cong S$ is a minimal normal subgroup of G; also, either $p = 2$ and $V(G/K) \cup \pi(G/KR) = \{2\}$, or $p \neq 2$, $V(G/K) = \{p\}$, and G/KR has odd order.*
(b) *K contains a minimal normal subgroup L of G such that $K/L \cong S$ and L is the natural module for K/L; also, $p \neq 2$, $V(G/K) = \{p\}$, G/KR has odd order and, for a Sylow 2-subgroup T of G, we have $T' = (T \cap K)'$.*

In all cases, p is a complete vertex and the unique cut-vertex of $\Delta(G)$.

Theorem 2 *Let R and K be, respectively, the solvable radical and the solvable residual of the finite group G and assume that G has a composition factor $S \cong \mathrm{SL}_2(4)$. Then, $\Delta(G)$ is a connected graph and has a cut-vertex p if and only if G/R is an almost simple group with socle isomorphic to S, $V(G) = \{2, 3, 5\} \cup \{p\}$ and one of the following holds.*

(a) *K is isomorphic either to $\mathrm{SL}_2(4)$ or to $\mathrm{SL}_2(5)$, and $V(G/K) = \{p\}$; if $p = 5$, then $K \cong \mathrm{SL}_2(4)$ and $G = K \times R$.*
(b) *K contains a minimal normal subgroup L of G with $|L| = 2^4$. Moreover, $G = KR$ and*
  (i) *either L is the natural module for K/L, $p \neq 2$ and $V(G/K) = \{p\}$,*
  (ii) *or L is isomorphic to the restriction to K/L, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$. Moreover $p = 5$, $G = K \times R_0$ where $R_0 = \mathbf{C}_G(K)$, and $V(R_0) = V(G/K) \subseteq \{5\}$.*
(c) *K contains a minimal normal subgroup L of G such that K/L is isomorphic to $\mathrm{SL}_2(5)$, and*
  (i) *either L is the natural module for K/L, $p \neq 5$ and $V(G/K) = \{p\}$,*
  (ii) *or L is isomorphic to the restriction to K/L, embedded in $\mathrm{SL}_4(3)$, of the standard module of $\mathrm{SL}_4(3)$, $p = 2$ and $V(G/K) \subseteq \{2\}$.*

In all cases, p is a complete vertex and the unique cut-vertex of $\Delta(G)$.

To conclude this introduction, we display in Table 1 the graphs related to the groups as in Theorems 1 and 2, so, all the possible connected graphs having a cut-vertex p, of the form $\Delta(G)$ where G is a finite group with a composition factor isomorphic to $\mathrm{SL}_2(2^a)$, $a \geq 2$. The first row of the table shows the graphs arising from Theorem 1, whereas the second row shows the graphs arising from Theorem 2 *in the case when p is larger than* 5. As regards the remaining graphs coming from Theorem 2, they are displayed in the third row of the table, and they are all the paths of length 2 with vertex set $\{2, 3, 5\}$. Each of them actually occurs for groups as in Theorem 2(a) (it is enough to consider the direct product $\mathrm{SL}_2(4) \times R$ where R is a non-abelian q-group, for $q \in \{2, 3, 5\}$). Also, case (b)(ii) is associated to the path $2 - 5 - 3$, and case (c)(ii) to the path $3 - 2 - 5$.

All the groups considered in the following discussion will be tacitly assumed to be finite groups.

2 Preliminaries

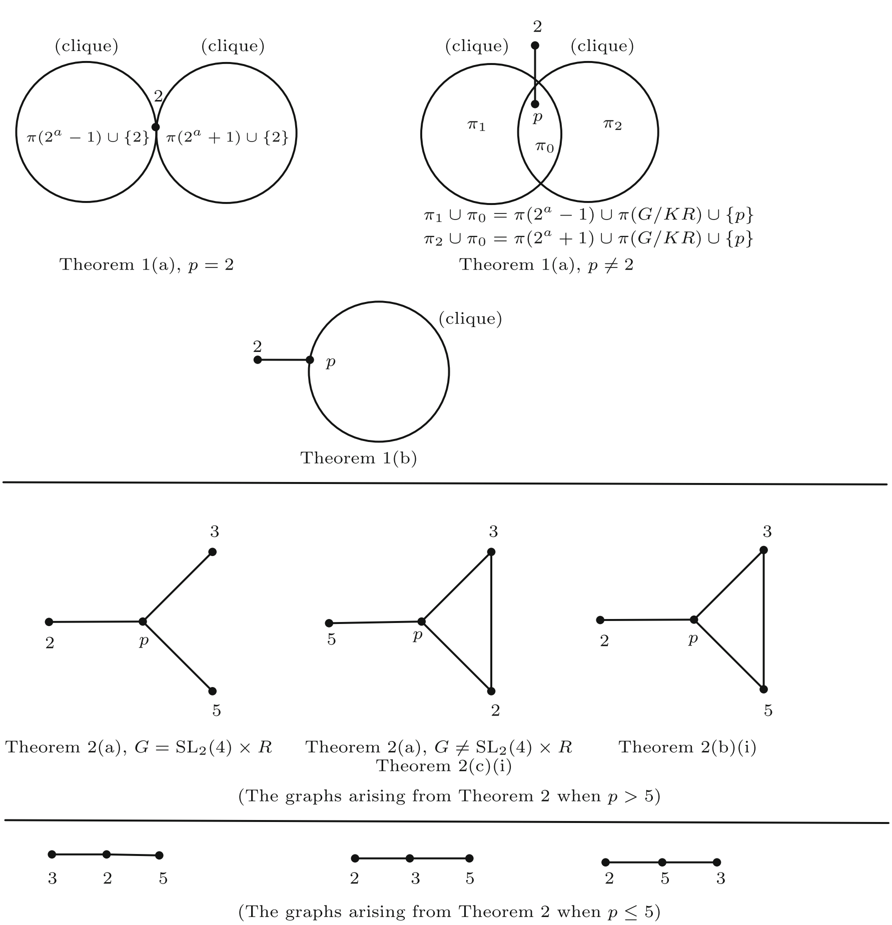
Given a group G, we denote by $\Delta(G)$ the character degree graph (or *degree graph* for short) of G as defined in the Introduction. Our notation concerning character theory is standard, and we will freely use basic facts and concepts such as Ito-Michler's theorem, Clifford's theory, Gallagher's theorem, character triples and results about extension of characters (see [8]).

For a positive integer n, the set of prime divisors of n will be denoted by $\pi(n)$, and we simply write $\pi(G)$ for $\pi(|G|)$. If q is a prime power, then the symbol $\mathbb{F}_q$ will denote the field of order q.

We start by recalling some structural properties of the groups $\mathrm{SL}_2(2^a)$.

Remark 2.1 The group $\mathrm{SL}_2(2^a) = \mathrm{PSL}_2(2^a)$ has order $2^a(2^a - 1)(2^a + 1)$, and the proper subgroups of this group are of the following types ([7, II.8.27]):

Table 1 The graphs related to the groups of Theorem 1 and Theorem 2



(i_+) dihedral groups of order $2(2^a+1)$ and their subgroups;
(i_-) dihedral groups of order $2(2^a-1)$ and their subgroups;
(ii) Frobenius groups with elementary abelian kernel of order 2^a and cyclic complements of order 2^a-1, and their subgroups;
(iii) A_4 when a is even or A_5 when 5 divides $|\mathrm{SL}_2(2^a)|$;
(iv) $\mathrm{SL}_2(2^b)$, where b is a proper divisor of a.

When dealing with subgroups of $\mathrm{SL}_2(2^a)$, we will refer to the above labels to identify the type of these subgroups. By a subgroup of type (i) we will mean a subgroup that is either of type (i_-) or of type (i_+).

Lemma 2.2 *Let $G\cong\mathrm{SL}_2(2^a)$, where $a\geq 2$. Let u be a prime divisor of 2^a-1, and let U be a subgroup of G with $|U|=u^b$ for a suitable $b\in\mathbb{N}-\{0\}$. Then U lies in the normalizer in G of precisely two Sylow 2-subgroups of G.*

Proof See [5, Lemma 2.2]. □

Next, some properties of the degree graph of simple and almost-simple groups.

Theorem 2.3 ([15, Theorem 5.2]) *Let $S \cong \mathrm{PSL}_2(t^a)$ or $S \cong \mathrm{SL}_2(t^a)$, with t prime and $a \geq 1$. Let $\rho_+ = \pi(t^a+1)$ and $\rho_- = \pi(t^a-1)$. For a subset ρ of vertices of $\Delta(S)$, we denote by Δ_ρ the subgraph of $\Delta = \Delta(S)$ induced by the subset ρ. Then*

(a) *if $t = 2$ and $a \geq 2$, then $\Delta(S)$ has three connected components, $\{t\}$, Δ_{ρ_+} and Δ_{ρ_-}, and each of them is a complete graph.*
(b) *if $t > 2$ and $t^a > 5$, then $\Delta(S)$ has two connected components, $\{t\}$ and $\Delta_{\rho_+ \cup \rho_-}$; moreover, both Δ_{ρ_+} and Δ_{ρ_-} are complete graphs, no vertex in $\rho_+ - \{2\}$ is adjacent to any vertex in $\rho_- - \{2\}$ and 2 is a complete vertex of $\Delta_{\rho_+ \cup \rho_-}$.*

Theorem 2.4 *Let G be an almost-simple group with socle S, and let $\delta = \pi(G) - \pi(S)$. If $\delta \neq \emptyset$, then S is a simple group of Lie type, and every vertex in δ is adjacent to every other vertex of $\Delta(G)$ that is not the characteristic of S. Moreover, if $S \cong \mathrm{SL}_2(2^a)$ and $a \geq 3$, then any prime in $\pi(G/S)$ is adjacent to every other vertex of $\Delta(G)$, except possibly to 2.*

Proof The first claim is Theorem 3.9 of [6]. As for the second claim, by Theorem A of [16] we see that both $(2^a-1)|G/S|$ and $(2^a+1)|G/S|$ are irreducible character degrees of G. □

Lemma 2.5 *Let G be a group and let R be its solvable radical. Assume that G/R is an almost-simple group with socle isomorphic to $\mathrm{PSL}_2(t^a)$, for a prime t with $t^a > 4$ and $t^a \neq 9$. Then, denoting by K the solvable residual of G, one of the following conclusions holds.*

(a) *K is isomorphic to $\mathrm{PSL}_2(t^a)$ or to $\mathrm{SL}_2(t^a)$;*
(b) *K has a non-trivial normal subgroup L such that K/L is isomorphic to $\mathrm{PSL}_2(t^a)$ or to $\mathrm{SL}_2(t^a)$, and every non-principal irreducible character of L/L' is not invariant in K.*

Proof See [5, Lemma 2.5]. □

Lemma 2.6 *Let G be a group, let R be its solvable radical and K its solvable residual. Assume that L is a normal subgroup of G, contained in K, such that $K/L \cong \mathrm{SL}_2(2^a)$ with $a \geq 2$, and L is isomorphic to the natural module for K/L. Let T be a Sylow 2-subgroup of KR, let $T_0 = T \cap R$ and $T_1 = T \cap K$. Then $L \leq \mathbf{Z}(T_0)$. Furthermore, every non-principal T-invariant character $\lambda \in \mathrm{Irr}(L)$ extends to T_1 and, assuming that T_0/L is abelian, λ extends to T if and only if $T' = T_1'$.*

Proof Observe that $L = K \cap R$ is an elementary abelian 2-group of order 2^{2a}, T_0 is normalized by K and $T = T_0T_1$. As $\mathbf{Z}(T_0) \cap L$ is non-trivial and normal in K, by the irreducibility of L as a K-module it follows that $L \leq \mathbf{Z}(T_0)$.

It is well known that $\mathbf{N}_K(T_1)/L = \mathbf{N}_{K/L}(T_1/L)$ is a subgroup of type (ii) of K/L whose order is $2^a \cdot (2^a-1)$ (in fact, $\mathbf{N}_{K/L}(T_1/L)$ can be identified with the subgroup of lower-triangular matrices of $\mathrm{SL}_2(2^a)$); thus we write $\mathbf{N}_K(T_1) = T_1D$, where D is cyclic of order 2^a-1. By looking at the action of T_1 on the natural module L, we see that $Z = \mathbf{Z}(T_1) = (T_1)'$ is a normal subgroup of order 2^a of T_1D. Since L is a self-dual K-module, we have $|\mathbf{C}_{\widehat{L}}(T_1)| = |\mathbf{C}_L(T_1)| = 2^a = |\widehat{L/Z}|$ and hence, as certainly the characters of L/Z are T_1-invariant, we conclude that the T_1-invariant characters of L are precisely the elements of $\widehat{L/Z}$. They are clearly T-invariant and they extend to T_1, because T_1/Z is abelian.

Let $\lambda \in \mathrm{Irr}(L)$ be a non-principal T-invariant character and assume that λ has an extension $\tau \in \mathrm{Irr}(T)$. Since $\tau(1) = \lambda(1) = 1$, we have $T' \leq \ker\tau$. Assuming that T_0/L is abelian, then $T/L = T_0/L \times T_1/L$ is abelian and $T' \leq L$. So, $T' = T' \cap L \leq \ker\tau_L = \ker\lambda$ and, as $\lambda \neq 1_L$, hence $T' < L$. Observing that T' is normalized by D and that D acts irreducibly on L/Z, we conclude that $T' \leq Z$ and, since $Z = T_0' \leq T'$, that $T' = T_0'$.

Conversely, if $\lambda \in \mathrm{Irr}(L)$ is T-invariant, then as observed above $Z \leq \ker\lambda$ and, assuming $Z = T'$, clearly λ, seen as a character of L/Z, extends to the abelian group T/Z. □

Remark 2.7 Let K be a group having a normal subgroup L with $K/L \cong \mathrm{SL}_2(2^a)$ (for $a \geq 2$), and such that L is isomorphic to the natural K/L-module. Then, as a consequence of the previous lemma, we can see that the graph $\Delta(K)$ is disconnected with two connected components, whose vertex sets are $\{2\}$ and $\pi(K) - \{2\}$ respectively, and which are both complete subgraphs of $\Delta(K)$.

In fact, by Theorem 2.3, $\Delta(K/L)$ has three connected components with vertex sets $\pi(2^a - 1)$, $\pi(2^a + 1)$ and $\{2\}$ respectively, which are all complete subgraphs of $\Delta(K/L)$. On the other hand, if λ is any non-principal character in $\mathrm{Irr}(L)$, then $I_K(\lambda)$ is a Sylow 2-subgroup of K, and Lemma 2.6 guarantees that λ extends to $I_K(\lambda)$; our claim then easily follows by Clifford's theory.

Theorem 2.8 *Let G be a non-solvable group such that $\Delta(G)$ is connected and it has a cut-vertex p. Then, denoting by R the solvable radical of G, we have that G/R is an almost-simple group such that $V(G) = \pi(G/R) \cup \{p\}$.*

Proof See [6, Theorem 3.15]. □

To conclude this preliminary section, we recall the statements of three crucial results proved in [5, 6], concerning certain module actions of $\mathrm{SL}_2(t^a)$).

Let H and V be finite groups, and assume that H acts by automorphisms on V. Given a prime number q, we say that *the pair* (H, V) *satisfies the condition* $\mathcal{N}_q$ if q divides $|H : \mathbf{C}_H(V)|$ and, for every non-trivial $v \in V$, there exists a Sylow q-subgroup Q of H such that $Q \trianglelefteq \mathbf{C}_H(v)$ (see [2]).

If (H, V) satisfies $\mathcal{N}_q$ then V turns out to be an elementary abelian r-group for a suitable prime r, and V is in fact an *irreducible* module for H over the field $\mathbb{F}_r$ (see Lemma 4 of [17]).

Lemma 2.9 ([6, Lemma 3.10]) *Let t, q, r be prime numbers, let $H = \mathrm{SL}_2(t^a)$ (with $t^a \geq 4$) and let V be an $\mathbb{F}_r[H]$-module. Then (H, V) satisfies $\mathcal{N}_q$ if and only if either $t^a = 5$ and V is the natural module for $H/\mathbf{C}_H(V) \cong \mathrm{SL}_2(4)$ or V is faithful and one of the following holds.*

(1) *$t = q = r$ and V is the natural $\mathbb{F}_r[H]$-module (so $|V| = t^{2a}$);*
(2) *$q = r = 3$ and $(t^a, |V|) \in \{(5, 3^4), (13, 3^6)\}$.*

Theorem 2.10 ([5, Theorem 3.3]) *Let V be a non-trivial irreducible module for $G = \mathrm{SL}_2(t^a)$ over the field $\mathbb{F}_q$, where $t^a \geq 4$ and q is a prime number, $q \neq t$. For odd primes $r \in \pi(t^a - 1)$ and $s \in \pi(t^a + 1)$ (possibly $r = q$ or $s = q$) let R, S be respectively a Sylow r-subgroup and a Sylow s-subgroup of G, and let T be a Sylow t-subgroup of G. Then, considering the sets*

$$V_{I_-} = \{v \in V \mid \text{there exists } z \in G \text{ such that } R^z \trianglelefteq \mathbf{C}_G(v)\},$$
$$V_{I_+} = \{v \in V \mid \text{there exists } z \in G \text{ such that } S^z \trianglelefteq \mathbf{C}_G(v)\},$$
$$V_{II} = \{v \in V \mid \text{there exists } z \in G \text{ such that } T^z \trianglelefteq \mathbf{C}_G(v)\},$$

we have that $V - \{0\}$ strictly contains $V_{I_-} \cup V_{II}$, $V_{I_+} \cup V_{II}$, and $V_{I_-} \cup V_{I_+}$, unless one of the following holds.

(a) $G \cong \mathrm{SL}_2(5)$, $s = 3$, $|V| = 3^4$ *and* $V\backslash\{0\} = V_{I_+}$,
(b) $G \cong \mathrm{SL}_2(13)$, $r = 3$, $|V| = 3^6$ *and* $V\backslash\{0\} = V_{I_-}$.

Theorem 2.11 ([5, Theorem 3.4]) *Let T be a Sylow t-subgroup of $G \cong \mathrm{SL}_2(t^a)$ (where $t^a \geq 4$) and, for a given odd prime divisor r of $t^{2a} - 1$, let R be a Sylow r-subgroup of G. Assuming that V is a t-group such that G acts by automorphisms (not necessarily faithfully) on V and $\mathbf{C}_V(G) = 1$, consider the sets*

$$V_I = \{v \in V \mid \text{there exists } x \in G \text{ such that } R^x \trianglelefteq \mathbf{C}_G(v)\}, \text{ and}$$
$$V_{II} = \{v \in V \mid \text{there exists } x \in G \text{ such that } T^x \trianglelefteq \mathbf{C}_G(v)\}.$$

Then, the following conditions are equivalent.

(a) *V_I and V_{II} are both non-empty and $V - \{1\} = V_I \cup V_{II}$.*
(b) *$G \cong \mathrm{SL}_2(4)$, and V is an irreducible G-module of dimension 4 over $\mathbb{F}_2$. More precisely, V is the restriction to G, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$.*

3 The structure of the solvable residual

Let G be a group having a composition factor isomorphic to $\mathrm{SL}_2(2^a)$ (with $a \geq 2$), such that $\Delta(G)$ is connected and has a cut-vertex: as the first step in our analysis, our purpose is to describe the structure of the solvable residual K of G. In particular we will see that, except for two sporadic cases, either we have $K \cong \mathrm{SL}_2(2^a)$, or $K \cong \mathrm{SL}_2(5)$, or K contains a minimal normal subgroup L of G such that either $K/L \cong \mathrm{SL}_2(2^a)$ or $K/L \cong \mathrm{SL}_2(5)$ and L is the natural module for K/L.

We collect the main results of this section in the following single statement (which is the counterpart in characteristic 2 of [5, Theorem 4.1]). This will be proved by treating separately the case $a > 2$ and the case $a = 2$, in Theorems 3.2 and 3.4, respectively.

Theorem 3.1 *Assume that the group G has a composition factor isomorphic to $\mathrm{SL}_2(2^a)$ with $a \geq 2$, and let p be a prime number. Assume also that $\Delta(G)$ is connected with cut-vertex p. Then, denoting by K the solvable residual of G, one of the following conclusions holds.*

(a) *K is isomorphic to $\mathrm{SL}_2(2^a)$ or to $\mathrm{SL}_2(5)$;*
(b) *K contains a minimal normal subgroup L of G such that K/L is isomorphic either to $\mathrm{SL}_2(2^a)$ or to $\mathrm{SL}_2(5)$ and L is the natural module for K/L.*
(c) *$a = 2$, and K contains a minimal normal subgroup L of G such that K/L is isomorphic to $\mathrm{SL}_2(4)$. Moreover, L is isomorphic to the restriction to K/L, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$.*
(d) *$a = 2$, and K contains a minimal normal subgroup L of G such that K/L is isomorphic to $\mathrm{SL}_2(5)$. Moreover, L is isomorphic to the restriction to K/L, embedded in $\mathrm{SL}_4(3)$, of the standard module of $\mathrm{SL}_4(3)$.*

We will then start by treating the case $a > 2$. Before stating the next theorem we recall that, for m and n integers larger than 1, a prime divisor q of $m^n - 1$ is called a *primitive prime divisor* if q does not divide $m^b - 1$ for all $1 \leq b < n$. In this case, n is the order of m modulo q, so n divides $q - 1$. In view of [10, Theorem 6.2], $m^n - 1$ always has primitive prime divisors except when $n = 2$ and $m = 2^c - 1$ for some integer c (i.e., m is a Mersenne number), or when $n = 6$ and $m = 2$.

In the following, for a normal subgroup N of a group G, and a character $\theta \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G|\theta)$ the set of all irreducible characters of G that lie over θ.

Theorem 3.2 *Assume that the group G has a composition factor isomorphic to $\mathrm{SL}_2(2^a)$ with $a > 2$, and let p be a prime number. Assume also that $\Delta(G)$ is connected with cut-vertex p. Then, denoting by K the solvable residual of G, one of the following conclusions holds.*

(a) *K is isomorphic to $\mathrm{SL}_2(2^a)$;*
(b) *K contains a minimal normal subgroup L of G such that K/L is isomorphic to $\mathrm{SL}_2(2^a)$ and L is the natural module for K/L.*

Proof Let R be the solvable radical of G. By Theorem 2.8, we have that G/R is an almost-simple group with socle isomorphic to $\mathrm{SL}_2(2^a)$, and $V(G) = \pi(G/R) \cup \{p\}$. Note that, since $a > 2$, Lemma 2.5 applies here; so either we get conclusion (a), or K has a non-trivial normal subgroup L such that K/L is isomorphic to $\mathrm{SL}_2(2^a)$, and every non-principal irreducible character of L/L' is not invariant in K. Therefore, we can assume that the latter condition holds.

Consider then a non-principal ξ in $\mathrm{Irr}(L/L')$: as $I_K(\xi)/L$ is a proper subgroup of $K/L \cong \mathrm{SL}_2(2^a)$, its possible structures are described in Remark 2.1. In particular, if 2 is not a divisor of $|K : I_K(\xi)|$, then $I_K(\xi)/L$ contains a Sylow 2-subgroup of K/L as a normal subgroup. Assuming for the moment that this happens for every non-principal $\xi \in \mathrm{Irr}(L/L')$, Lemma 2.9 (together with the paragraph preceding it) yields that the dual group $\widehat{L/L'}$ is the natural module for K/L, and the same holds for L/L' by self-duality; so, in order to get the desired conclusion, we only have to show that L' is trivial (note that, once this is proved, $L = \mathbf{O}_2(K)$ is a minimal normal subgroup of G), and this is what we do next.

For a proof by contradiction assume $L' \neq 1$, and consider a chief factor L'/Z of K. As observed in Remark 2.7, the graph $\Delta(K/L')$ has two connected components having vertex sets $\{2\}$ and $\pi(K/L') - \{2\}$, respectively; since the vertex set of $\Delta(G)$ is $\pi(G/R) \cup \{p\}$ and, also in view of Theorem 2.4, $\pi(G/R) - \{2\}$ is now a clique of $\Delta(G)$, we see that the cut-vertex p of $\Delta(G)$ cannot be 2, and that p is the unique vertex adjacent to 2 in $\Delta(G)$.

Now, let λ be a non-principal irreducible character of L'/Z, and let $\chi \in \mathrm{Irr}(K/Z \mid \lambda)$. If ψ is an irreducible constituent of $\chi_{L/Z}$ lying over λ, then clearly $\psi(1) \neq 1$, and since L'/Z is an abelian normal subgroup of L/Z whose index is a 2-power, we conclude that $\psi(1)$ (whence $\chi(1)$) is a multiple of 2. As a consequence, we get $\pi(|K : I_K(\lambda)|) \subseteq \{2, p\}$. Observe that $I_K(\psi)$ is a proper subgroup of K, as otherwise (the Schur multiplier of K/L being trivial) ψ would extend to K yielding a contradiction via Gallagher's theorem; of course $I_K(\lambda)$ is a proper subgroup of K as well, unless L'/Z lies in $\mathbf{Z}(K/Z)$.

We conclude this part of the proof by considering three situations that are exhaustive, and that all lead to a contradiction.

(i) $L'/Z \nsubseteq \mathbf{Z}(L/Z)$.
Consider the normal subgroup $\mathbf{C}_{L'/Z}(L/L')$ of K/Z; since L'/Z is a chief factor of K and it is not centralized by L/L', we deduce that $\mathbf{C}_{L'/Z}(L/L')$ is trivial. Thus we can apply the proposition appearing in the Introduction of [3], which ensures that the second cohomology group $\mathrm{H}^2(K/L', L'/Z)$ is trivial, and therefore K/Z is a split extension of L'/Z; in particular, every irreducible character of L'/Z extends to its inertia subgroup in K/Z. Now, let λ be any non-principal character in $\mathrm{Irr}(L'/Z)$: since $\pi(|K : I_K(\lambda)|) \subseteq \{2, p\}$, Gallagher's theorem implies that $I_K(\lambda)/L'$ contains a unique Sylow q-subgroup of K/L' for every prime $q \in \pi(2^{2a} - 1) - \{p\}$. But this yields a contradiction via, for example, Proposition 3.13 of [6]; in fact, according to that result, K/L' should have a cyclic solvable radical (whereas $L/L' = \mathbf{O}_2(K/L')$ is non-cyclic).

(ii) $L'/Z \subseteq \mathbf{Z}(L/Z)$, but $L'/Z \nsubseteq \mathbf{Z}(K/Z)$.
First, we note that L'/Z is a 2-group in this case, as otherwise L/Z would be isomorphic to the direct product $(L'/Z) \times (L/L')$ and it would then be abelian, a clear contradiction. Also, for a non-principal λ in $\mathrm{Irr}(L'/Z)$, we already observed that $I_K(\lambda)$ is a proper subgroup of K such that $\pi(|K : I_K(\lambda)|) \subseteq \{2, p\}$.

We claim that $I_K(\lambda)/L$ cannot be a subgroup of type (iv) of K/L unless it is also of type (iii). In fact, assume $I_K(\lambda)/L \cong \mathrm{SL}_2(2^b)$ where $b > 2$ and $a = bc$ for some $c > 1$. If c is an odd number, then $2^b + 1$ is a divisor of $2^a + 1$ and it is easy to see that $\pi(|K : I_K(\lambda)|)$ contains at least two odd primes, not our case. On the other hand, if c is even, then $2^{2b} - 1$ divides $2^a - 1$ and again (recalling [10, Proposition 3.1]) we reach a contradiction unless $c = 2$ and $p = 2^a + 1$ (note that p is neither 3 nor 5). Now we look at $I_K(\psi)$, where ψ lies in $\mathrm{Irr}(L/Z \mid \lambda)$ (recall that $\psi(1)$ is a multiple of 2, and that $I_K(\psi)$ is contained in $I_K(\lambda)$ because L'/Z is central in L/Z): we have $\pi(|I_K(\lambda) : I_K(\psi)|) \subseteq \{2\}$, and therefore $I_K(\psi)/L$ is either the whole $I_K(\lambda)/L$ or it is necessarily isomorphic to A_5. In any case we get the adjacency of 2 with odd primes different from p, a contradiction.

So, assume that $I_K(\lambda)/L$ is of type (iii) isomorphic to A_4: then there must be a prime in $\pi(|K : I_K(\lambda)|) - \{2, 3\}$, and this prime is necessarily p. This forces the 3-part of $|K/L|$ to be 3, yielding the contradiction that either $2^a - 1 = 3$ or $2^a + 1 = 3$. On the other hand, let $I_K(\lambda)/L$ be of type (iii) isomorphic to A_5. If $\pi(|K : I_K(\lambda)|) - \{3, 5\} \subseteq \{2\}$, then either the 3-part or the 5-part of $|K/L|$ is forced to be 3 or 5 respectively, and we get a contradiction from the fact that one among 3 and 5 is $2^a - 1$ or $2^a + 1$; if $\pi(|K : I_K(\lambda)|) - \{3, 5\}$ contains an odd prime (which is p), then the 3-part and the 5-part of $|K/L|$ are 3 and 5, respectively, and we get the same contradiction as before unless $3 \cdot 5 = 2^a - 1$, i.e., $K/L \cong \mathrm{SL}_2(2^4)$ and $p = 17$ is the only vertex adjacent to 2 in $\Delta(G)$. But in the latter case, taking $\psi \in \mathrm{Irr}(L/Z \mid \lambda)$, we see that $I_K(\psi)$ cannot be a proper subgroup of $I_K(\lambda)$ (otherwise $|I_K(\lambda) : I_K(\psi)|$ would be divisible by 3 or 5 and we would get the adjacency between one of these primes and 2); thus, recalling that $I_K(\psi) \subseteq I_K(\lambda)$, we get $I_K(\psi)/L \cong A_5$. Working with character triples we now get the adjacency between 2 and 3, again a contradiction. Our conclusion so far is that, for every non-principal $\lambda \in \mathrm{Irr}(L'/Z)$, the subgroup $I_K(\lambda)/L$ of K/L is either of type (i) or of type (ii).

Next, assume that $I_K(\lambda)/L$ is a subgroup of type (i_+). Then we get $p = 2^a - 1$ and, since 2 cannot be adjacent in $\Delta(G)$ to any prime in $\pi(2^a + 1)$, for every non-principal $\nu \in \mathrm{Irr}(L'/Z)$ the subgroup $I_K(\nu)/L$ must be either of type (i_+) containing a unique Hall $\pi(2^a + 1)$-subgroup of K/L, or of type (ii) containing a unique Sylow 2-subgroup of K/L. Now, the former situation cannot occur for every ν, by Lemma 2.9; on the other hand, if the latter situation occurs for some non-principal $\nu \in \mathrm{Irr}(L'/Z)$, then we reach a contradiction via Theorem 2.11 (recall that L'/Z is a 2-group).

If $I_K(\lambda)/L$ is a subgroup of type (i_-) then, as above, for every non-principal $\nu \in \mathrm{Irr}(L'/Z)$, the subgroup $I_K(\nu)/L$ must be either of type (i_-) containing a unique Hall $\pi(2^a-1)$-subgroup of K/L or of type (ii). Observe that if, in the latter case, $|K : I_K(\nu)|$ is divisible by 2, then $I_K(\nu)/L$ must contain a Hall $\pi(2^a - 1)$-subgroup of K/L; hence, by the structure of the subgroups of type (ii), $I_K(\nu)/L$ should contain a full Sylow 2-subgroup of K/L as well, against the fact that $|K : I_K(\lambda)|$ is even. Therefore $I_K(\nu)/L$ actually contains a (unique) Sylow 2-subgroup of K/L whenever it is a subgroup of type (ii), and now we reach a contradiction as in the previous paragraph.

We conclude that, for every non-principal $\lambda \in \mathrm{Irr}(L'/Z)$, the subgroup $I_K(\lambda)/L$ of K/L is of type (ii), and the same argument as in the paragraph above shows that it must contain a full Sylow 2-subgroup of K/L. This yields (via Lemma 2.9) that L'/Z is the natural module for K/L, so that $I_K(\lambda)/L$ is a Sylow 2-subgroup of K/L for every non-principal $\lambda \in \mathrm{Irr}(L'/Z)$.

Considering $\psi \in \mathrm{Irr}(L/Z)$ lying over such a λ, and recalling once again that $\psi(1)$ is even and $I_K(\psi) \subseteq I_K(\lambda)$, Clifford's theory yields that the primes in $\pi(K/L)$ are pairwise adjacent in $\Delta(G)$ and, also in view of Theorem 2.4, every odd prime divisor of $|K/L|$ is a complete vertex of $\Delta(G)$. This is clearly not compatible with the existence of a cut-vertex of $\Delta(G)$.

(iii) $L'/Z \subseteq \mathbf{Z}(K/Z)$.
As in case (ii), we have that L'/Z is a 2-group. If λ is a non-principal irreducible character of L'/Z, then λ is fully ramified with respect to the K/Z-chief factor L/L' (see Exercise 6.12 of [8]); therefore, the unique ψ in $\mathrm{Irr}(L/Z \mid \lambda)$ is such that $I_K(\psi) = I_K(\lambda) = K$. The fact that the Schur multiplier of K/L is trivial implies that ψ extends to K, yielding a clear contradiction via Gallagher's theorem.

To conclude the proof, we will show that $I_K(\xi)/L$ contains a unique Sylow 2-subgroup of K/L for every non-principal ξ in $\mathrm{Irr}(L/L')$. To this end, we will proceed through a number of steps.

(a) For every non-principal $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ of K/L cannot be of type (iv), unless it is also of type (iii).
For a proof by contradiction, let $\xi \in \mathrm{Irr}(L/L')$ be such that $I_K(\xi)/L \cong \mathrm{SL}_2(2^b)$ for some $b > 2$ properly dividing a. Thus, 2 is a divisor of $|K : I_K(\xi)|$. Since the Schur multiplier of $I_K(\xi)/L$ is trivial, ξ extends to $I_K(\xi)$ and this yields (via Clifford's correspondence and Gallagher's theorem) that 2 is adjacent in $\Delta(G)$ to every prime in $\pi(K/L) - \{2\}$. Moreover, taking into account Theorem 2.4 (which, together with Theorem 2.3, will be freely used from now on and should be kept in mind), also each prime in $\pi(G/R) - \pi(K/L)$ is adjacent to every prime in $\pi(K/L) - \{2\}$. Finally, $2^{2a} - 1$ has a primitive prime divisor q because $a \neq 3$; this prime q, which clearly belongs to $\pi(2^a + 1)$, is a divisor of $|K : I_K(\xi)|$, so every prime in $\pi(2^b - 1)$ is adjacent to q in $\Delta(G)$. As easily seen, this setting is not compatible with the existence of a cut-vertex of $\Delta(G)$.

(b) For every non-principal $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ of K/L cannot be isomorphic to A_5.
Assume the contrary, and take $\xi \in \mathrm{Irr}(L/L')$ such that $I_K(\xi)/L \cong A_5$. Working with character triples, we observe that $\mathrm{Irr}(K \mid \xi)$ contains characters whose degrees are divisible by every prime in $\pi(|K : I_K(\xi)|) \cup \{3\}$, which contains $\pi(K/L) - \{5\}$ (note that 2 divides $|K : I_K(\xi)|$ because $2^a > 4$); thus the 5-part of $|K/L|$ is 5, otherwise the primes in $\pi(K/L)$ would be pairwise adjacent in $\Delta(G)$, easily contradicting the existence of a cut-vertex of $\Delta(G)$. Observe also that, since neither $2^a - 1$ nor $2^a + 1$ can be 5, there exists an odd prime q in $\pi(K/L) - \{5\}$ that is adjacent to 5 in $\Delta(K/L)$; as q is now a complete vertex in the subgraph of $\Delta(G)$ induced by $\pi(G/R)$, we get $q = p$, and it is readily seen that no other prime divisor of $|K/L|$ can be adjacent to 5 in $\Delta(G)$. This implies on one hand that ξ does not have an extension to $I_K(\xi)$ (otherwise, by Gallagher's theorem, we would get the adjacency between 5 and 2 in $\Delta(G)$), which in turn yields (via [8, 8.16, 11.22, 11.31]) that the order of L/L' is divisible by 2; on the other hand, one among the sets $\pi(2^a - 1)$ and $\pi(2^a + 1)$ is in fact $\{5, p\}$.

Now, since $2^{2a} - 1$ is divisible by 5, we see that a must be even, so $2^2 - 1 = 3$ divides $2^a - 1$. Assuming for the moment $\pi(2^a - 1) = \{5, p\}$, we then get $p = 3$, and we also note that $2^a - 1$ has a primitive prime divisor (otherwise a would be 6, but $2^6 - 1 = 63$ is not divisible by 5). Certainly 3 is not such a divisor, as 3 divides $2^2 - 1$ and $a > 2$; hence 5 is a primitive prime divisor for $2^a - 1$, so we get $a = 4$ and $K/L \cong \mathrm{SL}_2(16)$. But in this case, since $|L/L'|$ is even, we can consider a chief factor L/X of K whose order is a 2-power: the dual group of V of L/X is then an irreducible module for $\mathrm{SL}_2(16)$ over $\mathbb{F}_2$. It is well known (see [1], for instance) that such modules all have a dimension belonging to $\{8, 16, 32\}$; if V is the natural module (of dimension 8) for $K/L \cong \mathrm{SL}_2(16)$, then the centralizer in K/L of every non-trivial element of V is a Sylow 2-subgroup of K/L, yielding the contradiction that 5 is adjacent to 17 in $\Delta(G)$. Also, a direct computation with GAP [14] shows that in the modules of dimensions 16 and 32 there are elements lying in regular orbits for the action of K/L, thus the primes in $\Delta(K/L)$ would be pairwise adjacent in $\Delta(G)$. Only one module is left, which has dimension 8 and is not the natural module: to handle this, we can see via GAP [14] that in all possible isomorphism types of extensions of V by $\mathrm{SL}_2(16)$ the set of irreducible character degrees is $\{1, 15, 16, 17, 51, 68, 204, 255, 272, 340\}$, so $\pi(K/L)$ would again be a set of pairwise adjacent vertices of $\Delta(G)$.

It remains to consider the case when 5 divides $2^a + 1$, hence $\pi(2^a + 1) = \{5, p\}$, and again we choose a chief factor L/X of K that is a 2-group. Now, the dual group of L/X can be viewed as a (non-trivial) irreducible K/L-module over $\mathbb{F}_2$, and if T/L is a Sylow 2-subgroup of K/L, then clearly there exists a non-principal μ in $\mathrm{Irr}(L/X)$ which is fixed by T/L (so, such that $I_K(\mu)/L$ contains T/L); as $I_K(\mu)/L$ is a proper subgroup of K/L, the only possibility for $I_K(\mu)/L$ is to be of type (ii). Moreover, since 5 is only adjacent to p in $\Delta(G)$, no prime divisor of 2^a-1 lies in $\pi(|K : I_K(\mu)|)$, thus in fact $I_K(\mu)/L = \mathbf{N}_{K/L}(T/L)$ has irreducible characters of degree $2^a - 1$. Now, μ does not extend to $I_K(\mu)$, as otherwise (by Gallagher's theorem and Clifford correspondence) we would get adjacencies in $\Delta(G)$ between 5 and all the primes in $\pi(2^a-1)$; but then, for $\psi \in \mathrm{Irr}(I_K(\mu) \mid \mu)$ and θ an irreducible constituent of $\psi_{T/X}$ lying over μ, we have that $\theta(1)$ is a 2-power larger than 1 (otherwise ψ would be an extension of μ to T, and μ would then extend to the whole $I_K(\mu)$). As a consequence, 2 divides $\psi(1)$ and we get the adjacency in $\Delta(G)$ between 5 and 2. This is the final contradiction that rules out the case $I_K(\xi)/L \cong A_5$.

(c) For every non-principal $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ of K/L cannot be isomorphic to A_4.
Assume $I_K(\xi)/L \cong A_4$ for some $\xi \in \mathrm{Irr}(L/L')$. Then we see at once that the primes in $\pi(K/L) - \{3\}$ are pairwise adjacent in $\Delta(G)$, thus the 3-part of $|K/L|$ is 3 and we get the same conclusions as in the first paragraph of (b) with 3 in place of 5: the cut-vertex p is an odd prime and it is the unique neighbor of 3 in $\Delta(G)$ among the primes in $\pi(K/L)$, and one among the sets $\pi(2^a - 1)$ and $\pi(2^a + 1)$ is $\{3, p\}$.

Assuming first $\pi(2^a - 1) = \{3, p\}$, we see that $a \neq 6$ because the 3-part of $2^6 - 1$ is 3^2. Hence $2^a - 1$ has a primitive prime divisor, which is necessarily p. Note that a cannot be a prime number, as otherwise it would be odd and K/L would not have subgroups isomorphic to A_4; moreover, if k is a divisor of a such that $1 < k < a$, then $2^k - 1$ divides $2^a - 1$ and is coprime to p, so $2^k - 1$ must be 3 and k is 2. We conclude that a is 4, so $K/L \cong \mathrm{SL}_2(16)$, and we reach a contradiction as in the second paragraph of (b).

As regards the case $\pi(2^a + 1) = \{3, p\}$, the same argument as in the last paragraph of (b) (replacing 5 with 3) completes the proof.

(d) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (ii) and of ever order, unless each of them contains a (unique) Sylow 2-subgroup of K/L.
Let us assume that all the subgroups $I_K(\xi)/L$ of K/L (for ξ non-principal in $\mathrm{Irr}(L/L')$) are of type (ii) and of even order, but there exists $\xi_0 \in \mathrm{Irr}(L/L')$ such that 2 divides $|K : I_K(\xi_0)|$. In this setting we observe that $2^a - 1$ does not divide $|I_K(\xi_0)/L|$, because $I_K(\xi_0)/L$ is a Frobenius group whose kernel is its unique Sylow 2-subgroup T_0/L, and we are assuming $|T_0/L| = 2^f < 2^a$. Therefore there exists $r \in \pi(2^a - 1) \cap \pi(|K : I_K(\xi_0)|)$, and Clifford's correspondence yields that $\{2, r\} \cup \pi(2^a + 1)$ is a set of pairwise adjacent vertices of $\Delta(G)$. It follows that r is adjacent in $\Delta(G)$ to every prime in $\pi(G/R) - \{r\}$, thus r is the cut-vertex p, and no other prime in $\pi(2^a - 1)$ can have any neighbor in $\{2\} \cup \pi(2^a + 1)$; in particular, no prime in $\pi(2^a - 1) - \{p\}$ shows up as a divisor of $|K : I_K(\xi)|$ for any $\xi \in \mathrm{Irr}(L/L')$.

Note also that a primitive prime divisor of $2^a - 1$ cannot lie in $\pi(I_K(\xi_0)/L)$, as otherwise it would divide $2^f - 1$ (and $f < a$); so, if $a \neq 6$, p is forced to be the unique primitive prime divisor of $2^a - 1$. Observe finally that the p'-part of $2^a - 1$ is not 1, otherwise p would not be a cut-vertex of $\Delta(G)$. Thus there exists a prime $q \in \pi(2^a - 1) - \{p\}$ such that, for every $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ contains a Sylow q-subgroup of K/L.

Furthermore, the character ξ_0 does not extend to $I_K(\xi_0)$, as otherwise we would get characters in $\mathrm{Irr}(K \mid \xi_0)$ whose degree is divisible by q and every prime in $\{2\} \cup \pi(2^a + 1)$, not our case; so $|L/L'|$ is even, and there exists a chief factor L/X of K whose order is a 2-power. Note that, by the conclusion in the paragraph above, the subgroups of the kind $I_K(\xi)/L$ for ξ non-principal in $\mathrm{Irr}(L/X)$ are not Sylow 2-subgroups of K/L, thus L/X is not the natural module for K/L; this in turn implies (via Lemma 2.9) that, for some non-principal $\xi \in \mathrm{Irr}(L/X)$, $I_K(\xi)/L$ does not contain a full Sylow 2-subgroup of K/L. In other words, we can assume that ξ_0 is in fact an irreducible character of L/L' whose kernel has index 2 in L.

Assume for the moment that a is an even number different from 6 (say, $a = 2b$): as $2^b - 1$ is coprime to p, we get that $2^b - 1$ divides the order of (a Frobenius complement of) $I_K(\xi_0)/L$, and so $|T_0/L| - 1 = 2^f - 1$ is a multiple of $2^b - 1$. This forces f to be a multiple of b and, since $f < a = 2b$, the only possibility is $f = b$; note that T_0/L is then a minimal normal subgroup of $I_K(\xi_0)/L$. The fact that ξ_0 does not extend to its inertia subgroup in K also implies that $T_0/\ker\xi_0$ is a non-abelian 2-group; thus $L/\ker\xi_0$, which has order 2, is in fact the derived subgroup of $T_0/\ker\xi_0$. Moreover, the normal subgroup $Z/\ker\xi_0 = \mathbf{Z}(T_0/\ker\xi_0)$ of $I_K(\xi_0)/\ker\xi_0$ cannot be larger than $L/\ker\xi_0$, because T_0/L is a minimal normal subgroup of $I_K(\xi_0)/L$ and clearly Z/L is not the whole T_0/L. We deduce that $T_0/\ker\xi_0$ is an extraspecial 2-group, so (b is even and) an application of [7, II, Satz 9.23] yields the contradiction that $2^b - 1$ divides $2^{b/2} + 1$.

If $a = 6$, then p can be either 3 or 7. In the former case, 7 divides $|I_K(\xi_0)/L|$ and so T_0/L has order 2^3; the same argument as above shows that $T_0/\ker\xi_0$ is an extraspecial 2-group, a clear contradiction. On the other hand, if $p = 7$, then $|I_K(\xi_0)/L|$ should be a multiple of 9, but 9 is not a divisor of $2^f - 1$ for any $f < 6$, contradicting the fact that $I_K(\xi_0)/L$ is a Frobenius group with kernel T_0/L.

It remains to treat the case when a is odd. In this case, we start by fixing a Sylow q-subgroup Q of K/L: if a non-principal $\xi \in \mathrm{Irr}(L/X)$ is stabilized both by Q and by another $Q_1 \in \mathrm{Syl}_q(K/L)$, then Q and Q_1 are contained in the same subgroup of type (ii) of K/L, whence in the normalizer of a suitable Sylow 2-subgroup of K/L. By Lemma 2.2, Q normalizes precisely two Sylow 2-subgroups of K/L, and since these normalizers contain a total number of 2^a Sylow q-subgroups each, there are at most $2(2^a - 1)$ choices for Q_1. On the other hand, the total number of Sylow q-subgroups of K/L is $2^{a-1}(2^a + 1)$, so there certainly exists an element $h \in K/L$ such that no non-trivial element in the dual group $\widehat{L/X}$ of L/X is centralized by both Q and Q^h. As a consequence, setting $|L/X| = 2^d$, we get $|\mathbf{C}_{\widehat{L/X}}(Q)| \leq 2^{d/2}$, and then

$$2^d - 1 \leq (2^{d/2} - 1) \cdot 2^{a-1} \cdot (2^a + 1).$$

It is easily checked that the above inequality yields $d < 4a$ and, since a is odd, Lemma 3.12 in [12] (whose hypotheses require $d \leq 3a$, but whose proof works assuming $d < 4a$ as well) leaves only one possibility for the isomorphism type of the K/L-module $\widehat{L/X}$ over $\mathbb{F}_2$. First of all, a is a multiple of 3 (say $a = 3c$) and $d = 8c$; then, denoting by $R(1)$ the natural module for K/L over $\mathbb{F}_{2^a}$ and by ω an automorphism of order 3 of $\mathbb{F}_{2^a}$, we have that $\widehat{L/X}$ is a "triality module", which can be described as follows. Start from the K/L-module

$V = R(1) \otimes R(1)^{\omega} \otimes R(1)^{\omega^2}$ over $\mathbb{F}_{2^a}$ (or one of its Galois twists), and observe that the field of values of (the character of) V is $\mathbb{F}_{2^c}$; now, restricting the scalars to $\mathbb{F}_{2^c}$, V is a homogeneous K/L-module and we take an irreducible constituent of it. This irreducible constituent remains irreducible if the scalars are restricted further to $\mathbb{F}_2$, and this is the $\mathbb{F}_2[K/L]$-module we are considering. In order to finish the proof for this case, it will be enough to show that there exist non-trivial elements of V whose centralizer in K/L is not a subgroup of type (ii).

Recall that the elements of $\mathrm{SL}_2(2^a)$ whose order is a divisor of $2^a + 1$ are conjugate to elements of the form $x = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}$, where $\lambda = \mu + \mu^{2^a}$ for $\mu \in \mathbb{F}_{2^{2a}} - \{1\}$ such that $\mu^{2^a+1} = 1$. The action of such an x on V is of course given by the Kronecker product

$$\begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & \lambda^{\omega} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & \lambda^{\omega^2} \end{pmatrix}.$$

Now, setting $\mathbb{K} = \mathbb{F}_{2^{2a}}$ and $V^{\mathbb{K}} = V \otimes \mathbb{K}$, we have $\dim_{\mathbb{K}} \mathbf{C}_{V^{\mathbb{K}}}(x) = \dim_{\mathbb{F}_{2^a}} \mathbf{C}_V(x)$; moreover, $V^{\mathbb{K}} = R(1)^{\mathbb{K}} \otimes (R(1)^{\mathbb{K}})^{\omega} \otimes (R(1)^{\mathbb{K}})^{\omega^2}$, so the action of x on $V^{\mathbb{K}}$ is expressed by the same Kronecker product as above. But x is conjugate to $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ in $\mathrm{SL}_2(2^{2a})$, so our aim is in fact to find μ such that the matrix

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \otimes \begin{pmatrix} \mu^{2^c} & 0 \\ 0 & \mu^{-2^c} \end{pmatrix} \otimes \begin{pmatrix} \mu^{2^{2c}} & 0 \\ 0 & \mu^{-2^{2c}} \end{pmatrix}$$

has a nonzero eigenspace for the eigenvalue 1. A direct calculation shows that it is enough to choose μ of order $2^{2c} - 2^c + 1$.

(e) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (ii) of even order or of type (i$_-$) (both types occurring).
If, assuming the contrary, there exists a non-principal $\xi_0 \in \mathrm{Irr}(L/L')$ such that $I_K(\xi_0)/L$ is of type (ii) and of even order, but not containing a full Sylow 2-subgroup of K/L then, as in (d), we get the following conditions: there exists a prime $q \in \pi(2^a - 1)$ such that, for every non-principal $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ contains a Sylow q-subgroup of K/L, and ξ_0 does not extend to $I_K(\xi_0)$. Similarly, no power of ξ_0 can extend to its inertia subgroup I in K, if I/L does not contain a full Sylow 2-subgroup of K/L. Since all Sylow q-subgroups of K/L are cyclic for $q \neq 2$, by [8, Theorem 6.26] we can actually assume that the order $o(\xi_0)$ in the dual group of L/L' is a power of 2. Hence ξ_0 is in $\mathrm{Irr}(L/X)$, for a chief factor L/X of K that is a 2-group, and the rest of the argument in (d) goes through.

On the other hand, if all the inertia subgroups of type (ii) and of even order contain a full Sylow 2-subgroup of K/L, and there is an inertia subgroup of type (i$_-$) whose index in K/L is divisible by a prime in $\pi(2^a - 1)$ (that is necessarily p), then again we are in the same situation as in (d): for every $\xi \in \mathrm{Irr}(L/L')$, the subgroup $I_K(\xi)/L$ contains a Sylow q-subgroup of K/L for a suitable prime $q \in \pi(2^a - 1) - \{p\}$, and L/L' has even order. Taking a chief factor L/X of K that is a 2-group, we are in a position to apply Theorem 2.11 together with Lemma 2.9, and we get a contradiction. Finally, if all the inertia subgroups of type (ii) and even order contain a Sylow 2-subgroup of K/L, and all those of type (i$_-$) have order divisible by $2^a - 1$, then Theorem 2.10 (applied to the action of K/L on any chief factor $V = L/X$ of K of odd order) yields that L/L' is a 2-group, and now Theorem 2.11 yields a contradiction.

(f) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (ii) of even order or of type (i$_+$) (both types occurring).

If, assuming the contrary, there exists a non-principal $\xi_0 \in \mathrm{Irr}(L/L')$ such that $I_K(\xi_0)/L$ is of type (ii) but not containing a full Sylow 2-subgroup of K/L, then 2 is adjacent in $\Delta(G)$ to every prime in $\pi(K/L) - \{2\}$; however, there also exists a prime $r \in \pi(2^a - 1)$ such that r divides $|K : I_K(\xi_0)|$, and this r is adjacent in $\Delta(G)$ to every other prime in $\pi(G/R)$; this is incompatible with the existence of a cut-vertex of $\Delta(G)$. The case when all the inertia subgroups of type (ii) contain a full Sylow 2-subgroup of K/L, and there is an inertia subgroup of type (i_+) whose index in K/L is divisible by a prime in $\pi(2^a + 1)$ (which must be p), yields the following situation: every inertia subgroup of type (i_+) contains a Sylow q-subgroup of K/L for a suitable prime $q \in \pi(2^a + 1) - \{p\}$, and every inertia subgroup of type (ii) is a full normalizer of a Sylow 2-subgroup of K/L. Moreover, $|L/L'|$ is even, and again we reach a contradiction via Theorem 2.11. Finally, if all the inertia subgroups of type (ii) contain a Sylow 2-subgroup of K/L, and all those of type (i_+) have order divisible by $2^a + 1$, then Theorem 2.10 (applied to the action of K/L on any chief factor $V = L/X$ of K of odd order) yields that L/L' is a 2-group, and again Theorem 2.11 yields a contradiction.

(g) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (ii) with even order, of type (i_+), or of type (i_-) (all types occurring).
Let us assume the contrary. Then 2 is adjacent in $\Delta(G)$ to all the primes in $\pi(K/L) - \{2\}$, and any inertia subgroup $I_K(\xi_0)/L$ of type (ii) is the full normalizer of a Sylow 2-subgroup T_0/L of K/L, i.e., a Frobenius group of order $2^a \cdot (2^a - 1)$.

Note that T_0/L is then a minimal normal subgroup of $I_K(\xi_0)/L$. Moreover, ξ_0 does not extend to $I_K(\xi_0)$, as otherwise we would get adjacencies between primes in $\pi(2^a - 1)$ and primes in $\pi(2^a + 1)$; hence L/L' has even order and (as already observed) for a chief factor L/X of K having 2-power order, there exists a character in $\mathrm{Irr}(L/X)$ whose stabilizer in K/L contains a Sylow 2-subgroup of K/L. In other words, we can assume that ξ_0 lies in $\mathrm{Irr}(L/X)$, so $|L/\ker \xi_0| = 2$. Now, $T_0/\ker \xi_0$ is a non-abelian 2-group, thus $L/\ker \xi_0$ is the derived subgroup of $T_0/\ker \xi_0$. Moreover, the normal subgroup $Z/\ker \xi_0 = \mathbf{Z}(T_0/\ker \xi_0)$ of $I_K(\xi_0)/\ker \xi_0$ cannot be larger than $L/\ker \xi_0$, because Z/L is not the whole T_0/L. We deduce that $T_0/\ker \xi_0$ is an extraspecial 2-group, so (a is even and) an application of [7, II, Satz 9.23] yields the contradiction that $2^a - 1$ divides $2^{a/2} + 1$.

(h) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (i_+) or of type (i_-) (both types occurring).
Assuming the contrary, as in the previous case we see that 2 is adjacent in $\Delta(G)$ to all the primes in $\pi(K/L) - \{2\}$; moreover, all the inertia subgroups are forced to contain either a subgroup of order $2^a - 1$ or a subgroup of order $2^a + 1$. Now, let L/X be a chief factor of K; by Lemma 2.9, both types (i_+) and (i_-) occur for the inertia subgroups even if we only consider the characters in $\mathrm{Irr}(L/X)$, but then Theorem 2.10 yields that L/X is a 2-group, which is impossible because no non-trivial element of $\widehat{L/X}$ is centralized by a Sylow 2-subgroup of K/L.

(i) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (i_+).
Let us assume the contrary, and let L/X be a chief factor of K. If there exists $\xi_0 \in \mathrm{Irr}(L/X)$ such that $I_K(\xi_0)/L$ does not contain a subgroup of order $2^a + 1$, which means that there exists $r \in \pi(2^a + 1)$ dividing $|K : I_K(\xi_0)|$, then r is a complete vertex of $\Delta(G)$ and it is in fact p. Now $\pi(2^a + 1) - \{p\}$ is forced to contain at least one prime q, and this q cannot show up in the index of any inertia subgroup $I_K(\xi)$ in K. In other words, for every non-principal ξ in $\mathrm{Irr}(L/X)$, the inertia subgroup $I_K(\xi)/L$ contains a Sylow q-subgroup of K/L (as a normal subgroup).

Of course the conclusion of the previous paragraph holds if $I_K(\xi)/L$ does contain a subgroup of order $2^a + 1$ for every $\xi \in \mathrm{Irr}(L/X)$. Thus, in any case, Lemma 2.9 applies and we get a contradiction.

(j) The subgroups $I_K(\xi)/L$ of K/L, for ξ non-principal in $\mathrm{Irr}(L/L')$, cannot be all of type (i$_-$).
This is totally analogous to (i).

As we saw, the only possibility that is left is the desired one: $I_K(\xi)/L$ contains a unique Sylow 2-subgroup of K/L for every non-principal ξ in $\mathrm{Irr}(L/L')$. The proof is complete. □

Next, we conclude the proof of Theorem 3.1 addressing the remaining case, i.e., when $a = 2$. We start by introducing some notation and a few facts concerning a relevant set of modules.

- We denote by V_0 the natural module for $S = \mathrm{SL}_2(4)$. We have $|V_0| = 2^4$, $|\mathbf{C}_S(v)| = 2^2$ for all non-trivial $v \in V_0$, and the cohomology group $\mathrm{H}^2(S, V_0)$ is trivial (whereas $\mathrm{H}^1(S, V_0) \neq 0$).
- We denote by V_1 the restriction to $S = \mathrm{SL}_2(4)$, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$. We have $|V_1| = 2^4$; moreover, S has two orbits O_1 and O_2 on $V_1 - \{0\}$, and $\mathbf{C}_S(v) \cong S_3$ for $v \in O_1$, while $\mathbf{C}_S(v) \cong A_4$ for $v \in O_2$. As for the relevant cohomology groups, we have $\mathrm{H}^1(S, V_1) = 0 = \mathrm{H}^2(S, V_1)$.
- We denote by W the restriction to $S_1 = \mathrm{SL}_2(5)$, seen as a subgroup of $\mathrm{SL}_4(3)$, of the standard module of $\mathrm{SL}_4(3)$. We have $|W| = 3^4$ and $|\mathbf{C}_{S_1}(v)| = 3$ for all non-trivial $v \in W$; moreover, $\mathrm{H}^2(S_1, W) = 0$.
- We denote by U the natural module for $S_1 = \mathrm{SL}_2(5)$. We have $|U| = 5^2$ and $|\mathbf{C}_{S_1}(v)| = 5$ for all non-trivial $v \in U$; moreover, $\mathrm{H}^2(S_1, U) = 0$.

Note that all the above modules are self-dual: this follows from [12, Lemma 3.10] for V_0, V_1 and U, and for W by observing that $\mathrm{GL}_4(3)$ has a unique conjugacy class of subgroups isomorphic to $\mathrm{SL}_2(5)$.

Finally, let B be an abelian group and A a group acting on B via automorphisms: we will denote by $\Delta_{orb}(B)$ the graph whose vertex set is the set of the prime divisors of the set of orbit sizes $\{|A : \mathbf{C}_A(b)| : b \in B\}$ of the action of A on B, and such that two (distinct) vertices p and q are adjacent if and only if there exists $b \in B$ such that the product pq divides $|A : \mathbf{C}_A(b)|$.

Lemma 3.3 *Let q be a prime number and V an elementary abelian q-group.*

(a) *If V is a non-trivial irreducible $\mathrm{SL}_2(4)$-module and the graph $\Delta_{orb}(V)$ is not a clique with vertex set $\{2, 3, 5\}$, then $q = 2$ and V is isomorphic either to V_0 or to V_1;*

(b) *If V is a faithful irreducible $\mathrm{SL}_2(5)$-module and the graph $\Delta_{orb}(V)$ is not a clique with vertex set $\{2, 3, 5\}$, then either $q = 3$ and V is isomorphic to W or $q = 5$ and V is isomorphic to U.*

Proof By Theorem 2.3 of [9], both $\mathrm{SL}_2(4)$ and $\mathrm{SL}_2(5)$ always have regular orbits on a faithful module of characteristic $p \geq 7$. The remaining cases, of characteristic $p \in \{2, 3, 5\}$, can be settled by direct computation using GAP [14]. □

Theorem 3.4 *Assume that the group G has a composition factor isomorphic to $\mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$, and let p be a prime number. Assume also that $\Delta(G)$ is connected and that it has a cut-vertex p. Then, denoting by K the solvable residual of G, one of the following conclusions holds.*

(a) *K is isomorphic to* $\mathrm{SL}_2(4)$ *or to* $\mathrm{SL}_2(5)$.
(b) *K contains a minimal normal subgroup L of G such that K/L is isomorphic either to* $\mathrm{SL}_2(4)$ *or to* $\mathrm{SL}_2(5)$ *and L is the natural module for K/L.*
(c) *K contains a minimal normal subgroup L of G such that K/L is isomorphic to* $\mathrm{SL}_2(4)$. *Moreover, L is isomorphic to the restriction to K/L, embedded as $\Omega_4^-(2)$ into* $\mathrm{SL}_4(2)$, *of the standard module of* $\mathrm{SL}_4(2)$.
(d) *K contains a minimal normal subgroup L of G such that K/L is isomorphic to* $\mathrm{SL}_2(5)$. *Moreover, L is isomorphic to the restriction to K/L, embedded in* $\mathrm{SL}_4(3)$, *of the standard module of* $\mathrm{SL}_4(3)$.

Proof By Lemma 2.5 (applied with $t^a = 5$) either (a) holds, or K has a non-trivial normal subgroup L such that K/L is isomorphic to $\mathrm{SL}_2(4)$ or to $\mathrm{SL}_2(5)$ and every non-principal irreducible character of L/L' is not invariant in K. In the latter case, consider a chief factor L/X of K and set V to be its dual group; then, taking into account that $V(G) = V(K) \cup \{p\}$, the hypothesis of p being a cut-vertex for $\Delta(G)$ implies that the subgraph of $\Delta(G)$ induced by the set of vertices $\{2, 3, 5\}$ is not a clique. Moreover, V is a non-trivial irreducible module for K/L, and Clifford's theory yields that $\Delta_{orb}(V)$ is not a clique as well. Therefore Lemma 3.3 applies, and the K/L-module V is isomorphic to V_0 or to V_1 if $K/L \cong \mathrm{SL}_2(4)$ whereas it is isomorphic to W or to U if $K/L \cong \mathrm{SL}_2(5)$. Note that $L/X = \mathbf{F}(K/X)$ is a chief factor of G as well, and our proof is complete if $X = 1$.

Working by contradiction, we assume $X \neq 1$ and we consider a chief factor X/Y of K: in this situation, we first show that X/Y is the unique minimal normal subgroup of K/Y. In fact, let M/Y be another minimal normal subgroup of K/Y. Setting $N/L = \mathbf{Z}(K/L)$ (and observing that N is contained in the solvable radical R of G), we have that K/N is the unique non-solvable chief factor of K; so, if M/Y is non-solvable, then we get $M/Y \cong K/N$ and hence $K/Y = M/Y \times N/Y$, contradicting the fact that K is perfect. Therefore, M/Y is abelian, so the normal subgroup MX/X of K/X lies in $\mathbf{F}(K/X) = L/X$, and we conclude that M/Y is contained in L/Y. As a consequence, the K/L-module M/Y is isomorphic to L/X, i.e., to one of the K/L-modules V_0, V_1, W and U. Now $L/Y \cong M/Y \times X/Y$ can be regarded as a K/L-module which is the direct sum of two modules in $\{V_0, V_1\}$ or two modules in $\{W, U\}$ (depending on whether $K/L \cong \mathrm{SL}_2(4)$ or $K/L \cong \mathrm{SL}_2(5)$, respectively); but it is easy to see that K/L has regular orbits on (the duals of) such modules, and this leads via Clifford's theory to the contradiction that $\{2, 3, 5\}$ is a clique of $\Delta(G)$.

Next, suppose that L/Y is nilpotent. Since K/Y has a unique minimal normal subgroup, clearly L/Y must be a group of prime-power order and, since $|L/X|$ is a q-power for $q \in \{2, 3, 5\}$, the same holds for $|L/Y|$. Furthermore, we have $X/Y \leq \mathbf{Z}(L/Y)$ and, in particular, $I_K(\lambda) \subseteq I_K(\mu)$ for every $\mu \in \mathrm{Irr}(X/Y)$ and $\lambda \in \mathrm{Irr}(L/Y \mid \mu)$.

If $q \neq 2$, then $|N/L| = 2$ and we claim that X/Y is a non-trivial K/L-module. In fact, assuming the contrary, we get $X/Y \subseteq \mathbf{Z}(K/Y)$ and $|X/Y| = q$. Observe that $\mathbf{C}_{L/Y}(N/L)$ is a normal subgroup of K/Y which contains X/Y but is not the whole L/Y, so, as L/X is a chief factor of K, we have $\mathbf{C}_{L/Y}(N/L) = X/Y$. Now if L is abelian, then by coprime action we get $L = X/Y \times [L/Y, N/L]$, contradicting the uniqueness of X/Y as a minimal normal subgroup of K/Y. On the other hand, if L is non-abelian, then $X/Y = (L/Y)' = \mathbf{Z}(L/Y)$ and L/Y is an extraspecial q-group.

So, every nonlinear irreducible character of L/Y is K-invariant and, since K/L has cyclic Sylow q-subgroups, it extends to K. It easily follows that $\{2, 3, 5\}$ is a clique of $\Delta(G)$, a contradiction. Thus the claim is proved, and Lemma 3.3 applies: our assumption that q is not 2 yields then $X/Y \cong L/X \cong U$, or $X/Y \cong L/X \cong W$, as K/L-modules. By the fact that $\{2, 3, 5\}$ cannot be a clique and by the observation in the last sentence of the previous paragraph, it follows that $I_{K/L}(\lambda)$ is a Sylow q-subgroup of K/L for every $\lambda \in \mathrm{Irr}(L/Y)$, a contradiction by the paragraph preceding Lemma 2.9.

So we can assume $q = 2$ and $L = N$. One can check with GAP [14] that the perfect groups of order $2^5 \cdot |\mathrm{SL}_2(4)|$ always have irreducible characters whose degrees are multiple, respectively, of 6, 10 and 15: it follows that X/Y is not the trivial K/L-module. Hence by Clifford's theory, together with the fact that $I_K(\lambda) \subseteq I_K(\mu)$ for every $\mu \in \mathrm{Irr}(X/Y)$ and $\lambda \in \mathrm{Irr}(L/Y \mid \mu)$, the assumptions of Lemma 3.3 are satisfied for the action of K/L on X/Y. As a result, X/Y is isomorphic either to V_0 or to V_1 as a K/L-module and, in particular, we get $|X/Y| = 2^4 = |L/X|$. But again, a direct check via GAP [14]shows that the perfect groups of order $2^8 \cdot |\mathrm{SL}_2(4)|$ all have irreducible characters whose degrees are multiples of 6, 10, 15, yielding the same contradiction as above.

Finally, we assume that L/Y is non-nilpotent. Thus we have $X/Y = \mathbf{F}(L/Y) = \mathbf{F}(K/Y)$, and $|X/Y|$ is coprime to $|L/X|$. Observe that $\mathbf{\Phi}(K/Y) \leq \mathbf{F}(K/Y) = X/Y$ and that $\mathbf{\Phi}(K/Y) \neq X/Y$, because otherwise K/Y modulo its Frattini subgroup would be isomorphic to K/X and would have a trivial Fitting subgroup, not our case. Since X/Y is a minimal normal subgroup of K/Y, we deduce that $\mathbf{\Phi}(K/Y)$ is trivial and hence X/Y has a complement K_0/Y in K/Y; in particular, every $\mu \in \mathrm{Irr}(X/Y)$ extends to its inertia subgroup $I_K(\mu)$. Let Z/Y be an irreducible L/Y-submodule of X/Y (i.e., a minimal normal subgroup of L/Y contained in X/Y). Set $C/Y = \mathbf{C}_{L/Y}(Z/Y)$: as L/X is an elementary abelian q-group (where q is a suitable prime in $\{2, 3, 5\}$), the factor group L/C is a cyclic group of order q acting fixed-point freely on Z/Y. Writing the completely reducible L/Y-module X/Y as $(Z/Y) \times (Z_1/Y)$ for a suitable L/Y-module Z_1/Y, we consider the character $\mu = \mu_0 \times 1_{Z_1/Y} \in \mathrm{Irr}(X/Y)$, where μ_0 is a non-principal irreducible character of Z/Y. We observe that $I_{L/Y}(\mu) = C/Y$ and that every $\chi \in \mathrm{Irr}(K/Y|\mu)$ has a degree divisible by q. We also remark that, setting $L_0/Y = (L/Y) \cap (K_0/Y)$, if $L_0/Y \cong L/X$ is isomorphic (as a K/L-module) either to V_0, V_1 or W, then $|I_{L_0/Y}(\mu)| = |C/X| = |L_0/Y|/q > |L_0/Y|^{1/2}$. We claim that, as a consequence, for every prime divisor $r \neq q$ of $|K/L|$, either r divides $|K : I_K(\mu)|$ or r divides the degree of some irreducible character of $I_{K/X}(\mu)$ that lies over μ. In fact, fixing $R_0/Y \in \mathrm{Syl}_r(K_0/Y)$, it is not difficult to see that there exists another Sylow r-subgroup R_1/Y of K_0/Y with $\langle R_0L_0/L_0, R_1L_0/L_0\rangle = K_0/L_0$ and, since no non-trivial element of L_0/Y is centralized by the whole K_0/L_0, the dimension over $\mathbb{F}_q$ of the vector space $\mathbf{C}_{L_0/Y}(R_0L_0/L_0)$ cannot be larger than a half of $\dim_{\mathbb{F}_q}(L_0/Y)$. Now, if $I_{K_0/Y}(\mu)$ (which is isomorphic to $I_{K/X}(\mu)$) contains a Sylow r-subgroup R_0/Y of K_0/Y as a normal subgroup, then R_0/Y centralizes $I_{L_0/Y}(\mu)$ because $I_{L_0/Y}(\mu)$ and R_0/Y are normal subgroups of coprime order of $I_{K_0/Y}(\mu)$, and this is not possible as $|I_{L_0/Y}(\mu)| > |L_0/Y|^{1/2}$. By Gallagher's theorem, it hence follows that $\{2, 3, 5\}$ is a clique of $\Delta(G)$, a contradiction.

It only remains the case $L_0/Y \cong U$ (as K_0/L_0-module); but in this case $q = 5$ divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(K \mid \mu)$, and the Sylow 2-subgroups and 3-subgroups of K_0/L_0 act fixed point freely on L_0/Y. Recalling that $I_{L_0/Y}(\mu)$ is normal in $I_{K_0/Y}(\mu)$ and that $|I_{L_0/Y}(\mu)| = 5$, we hence see that 6 divides $[K_0 : I_{K_0}(\mu)]$, and again $\{2, 3, 5\}$ is a clique of $\Delta(G)$, a contradiction. □

4 Proof of Theorem 1

We are ready to prove Theorem 1, that was stated in the Introduction and that is stated again here, for the convenience of the reader, as Theorem 4.2.

Lemma 4.1 *Let K be a normal subgroup of the group G with $K \cong \mathrm{SL}_2(2^a)$, $a \geq 2$. Let R be the solvable radical of G and assume that $V(G) = \pi(G/R) \cup \{p\}$ for a suitable prime p. Then*

(a) *The primes in $V(R)$ (if any) are complete vertices of $\Delta(G)$.*
(b) *If $a \geq 3$ and $2 \in \pi(G/KR)$, then 2 is a complete vertex of $\Delta(G)$.*
(c) *If $2 \notin \pi(G/KR) \cup V(R)$, then 2 is adjacent in $\Delta(G)$ to a vertex q if and only if $q \in V(G/K)$.*

Proof We start by proving claim (a). Let $q \in V(R)$; as $KR = K \times R$, q is adjacent in $\Delta(G)$ to all vertices $\neq q$ in $V(K) = \pi(K) = \pi(KR/R)$. For $t \in \pi(G/R) - \pi(KR/R)$, by part (a) of Proposition 2.10 of [4] there exists a character $\theta \in \mathrm{Irr}(K)$ such that t divides $|G : I_G(\theta)|$. Take $\varphi \in \mathrm{Irr}(R)$ such that q divides $\varphi(1)$ and let $\psi = \theta \times \varphi \in \mathrm{Irr}(KR)$. Since $I_G(\psi) \leq I_G(\theta)$, tq divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ that lies over θ. Finally, if $p \in V(G)$ but $p \notin \pi(G/R)$, then $p \in V(R)$; so if $q \neq p$, then $q \in \pi(G/R)$ by the assumption on $V(G)$, and hence by what we have just proved q is adjacent to p as well. So, q is a complete vertex of $\Delta(G)$.

We now move to claim (b). Assuming $2 \in \pi(G/KR)$ and $a \geq 3$, by Theorem 2.4 we get that 2 is adjacent in $\Delta(G)$ to all primes $\neq 2$ of $\pi(G/R)$; so to p as well if $p \neq 2$ and $p \in \pi(G/R)$. On the other hand, if $p \in V(G) - \pi(G/R)$, so $p \neq 2$, then $p \in V(R)$ and p is adjacent to 2 in $\Delta(G)$ by part (a). Hence, 2 is a complete vertex of $\Delta(G)$.

Finally, we prove claim (c). Assume that $2 \notin \pi(G/KR) \cup V(R)$. Then every character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is even lies over a character $\psi \in \mathrm{Irr}(KR)$ with $\psi(1)$ even. Writing $\psi = \alpha \times \beta$ with $\alpha \in \mathrm{Irr}(K)$ and $\beta \in \mathrm{Irr}(R)$, since $2 \notin V(R)$ we deduce that α has even degree, and hence α is the Steinberg character of K. Thus α extends to G (see for instance [13]) and hence $\chi(1) = \alpha(1)\gamma(1) = 2^a\gamma(1)$ for a suitable $\gamma \in \mathrm{Irr}(G/K)$, concluding the proof. □

Theorem 4.2 *Let R and K be, respectively, the solvable radical and the solvable residual of the group G and assume that G has a composition factor $S \cong \mathrm{SL}_2(2^a)$, with $a \geq 3$. Then, $\Delta(G)$ is a connected graph and it has a cut-vertex p if and only if G/R is an almost simple group with socle isomorphic to S, $V(G) = \pi(G/R) \cup \{p\}$ and one of the following holds.*

(a) *K is a minimal normal subgroup of G, $K \cong S$ and either $p = 2$ and $V(G/K) \cup \pi(G/KR) = \{2\}$, or $p \neq 2$, $V(G/K) = \{p\}$ and G/KR has odd order.*
(b) *K contains a minimal normal subgroup L of G such that $K/L \cong S$, L is the natural module for K/L, $p \neq 2$, $V(G/K) = \{p\}$, G/KR has odd order and, for a Sylow 2-subgroup T of G, $T' = (T \cap K)'$.*

In all cases, p is is a complete vertex and the only cut-vertex of $\Delta(G)$.

Proof We start by proving the "only if" part of the statement, assuming that $\Delta(G)$ is connected and that has a cut-vertex p. Then, by Theorem 2.8 G/R is an almost-simple group and $V(G) = \pi(G/R) \cup \{p\}$. As a consequence, we have that the socle M/R of G/R is isomorphic to S. Let $L = K \cap R$; since $KR = M$, we see that $K/L \cong S$.

We observe that by Theorem 2.4 every prime in $\pi(G/KR)$ is adjacent in $\Delta(G)$ to every other vertex in $\Delta(G)$, except possibly 2 and p. Moreover, part (a) of Lemma 4.1 yields that $V(R) \subseteq \{p\}$.

We consider first the situation arising when $L = 1$. Assuming $p = 2$, then $V(G) = \pi(G/R)$ and by the above observation we deduce that G/KR is a 2-group and that $V(R) \subseteq \{2\}$. If $G = KR = K \times R$, then, as $\Delta(G)$ is connected and 2 is a cut-vertex of $\Delta(G)$, it immediately follows that $V(R) = \{2\}$. So, in any case, $V(G/K) \cup \pi(G/KR) = \{2\}$. Assuming instead $p \neq 2$, then (since no vertex in $V(G) - \{p\}$ can be complete in $\Delta(G)$) part (b) of Lemma 4.1 implies that $|G/KR|$ is odd and it only remains to show that $V(G/K) = \{p\}$. As $V(R) \subseteq \{p\}$ and $p \neq 2$, part (c) of Lemma 4.1 yields that 2 is adjacent in $\Delta(G)$ to all primes in $V(G/K)$, and to them only. As $\Delta(G)$ is connected, it follows that $V(G/K)$ is non-empty.

If $q \in V(G/K)$ and $q \neq p$, then q divides $|G/KR|$ (because $V(KR/K) = V(R) \subseteq \{p\}$) and hence, by Theorem 2.4, q (being adjacent also to 2) would be a complete vertex of $\Delta(G)$, a contradiction. Hence, $V(G/K) = \{p\}$.

We assume now $L \neq 1$. Then, by Theorem 3.2, L is a minimal normal subgroup of G and L is the natural module for $K/L \cong S$. By Remark 2.7, the subgraph of $\Delta(G)$ induced by the vertex set $V(G) - \{2, p\}$ is a complete graph. Hence, the assumptions on $\Delta(G)$ imply that $p \neq 2$ and that 2 is adjacent only to p in $\Delta(G)$. Moreover, recalling that $\Delta(G/L)$ is a subgraph of $\Delta(G)$, by part (a) and part (b) of Lemma 4.1 we deduce that $2 \notin V(R/L) \cup \pi(G/KR)$ and hence, by part (c) of the same lemma, that $V(G/K) = \{p\}$. Let now T be a Sylow 2-subgroup of G; as $|G/KR|$ is odd, then $T \leq KR$. Setting $T_0 = T \cap R$, we observe that T_0/L is an abelian normal Sylow 2-subgroup of R/L because $2 \notin V(R/L)$. Let $T_1 = T \cap K$ and assume, working by contradiction, that $T' \neq T_1'$. Let $\lambda \in \mathrm{Irr}(L)$ be a non-principal character; by Lemma 2.6 $L \leq \mathbf{Z}(T_0)$, so λ is T_0-invariant and, since L is a self-dual K/L-module, $I_K(\lambda)/L$ is a Sylow 2-subgroup of K/L. Hence, since $T = T_0T_1$, we can assume (up to conjugation) that λ is T-invariant. So, by Lemma 2.6 λ has no extension to T. As $I_K(\lambda)/L = T_1/L$ and $KR/L = K/L \times R/L$, T/L is a normal subgroup of $I_{KR}(\lambda)$ and hence 2 divides the degree of every irreducible character ψ of $I_{KR}(\lambda)$ that lies over λ. By Clifford correspondence, it follows that 2 is adjacent in $\Delta(G)$ to all primes in $\pi(2^{2a} - 1) = \pi(|K : I_K(\lambda)|)$, a contradiction. Hence, $T' = (T \cap K)'$.

We proceed now to prove the "if" part of the statement and we assume that G/R is an almost simple group and that $V(G) = \pi(G/R) \cup \{p\}$ for some prime p.

Suppose first that (a) holds, so K is a minimal normal subgroup of G and $K \cong S$. Hence, $KR = K \times R$. Assume that $p = 2$, so $V(G) = \pi(G/R)$, and that $V(G/K) \cup \pi(G/KR) = \{2\}$. If $KR < G$, then 2 is a complete vertex of $\Delta(G)$ by part (b) of Lemma 4.1, and if $G = KR$, then the same is true because in this case $V(R) = V(G/K) = \{2\}$. For $\chi \in \mathrm{Irr}(G)$ and an irreducible constituent ψ of χ_{KR}, the odd parts of $\chi(1)$ and of $\psi(1)$ coincide by [8, Corollary 11.29], so by part (a) of Theorem 2.3 the graph $\Delta(G) - 2$, obtained by deleting the vertex 2 and all incident edges, has two complete connected components, with vertex sets $\pi(2^a - 1)$ and $\pi(2^a + 1)$. So, 2 is a cut-vertex of $\Delta(G)$ and, being a complete vertex of $\Delta(G)$, it is the unique cut-vertex of $\Delta(G)$. If $p \neq 2$, $V(G/K) = \{p\}$ and G/KR has odd order, then (as $R \cong KR/K \trianglelefteq G/K$) $2 \notin V(R)$ and by part (c) of Lemma 4.1 the vertex 2 is adjacent only to p in $\Delta(G)$. Hence, p is a cut-vertex of $\Delta(G)$. We also observe that p is a complete vertex of $\Delta(G)$: this is a consequence of Theorem 2.4 if $p \in \pi(G/KR)$, while if $p \notin \pi(G/KR)$ the assumption $V(G/K) = \{p\}$ implies that $p \in V(R)$ and hence the claim follows by part (a) of Lemma 4.1. Thus, p is the unique cut-vertex of $\Delta(G)$.

We assume now that (b) holds, so K contains a minimal normal subgroup L of G such that $K/L \cong S$ and L is the natural module for K/L. Moreover, $p \neq 2$, $V(G/K) = \{p\}$, G/KR has odd order and, for any Sylow 2-subgroup T of G, $T' = (T \cap K)'$. For a non-principal $\lambda \in \mathrm{Irr}(L)$, the argument used in the fourth paragraph of this proof shows that $I = I_G(\lambda)$ contains a Sylow 2-subgroup T of G, and T/L is abelian and normal in I/L. By Lemma 2.6 λ extends to T and hence λ extends to $I_G(\lambda)$ by [8, Theorem 6.26]. So, Gallagher's theorem implies that every irreducible character of G that lies over λ has odd degree. We hence deduce that if $\chi \in \mathrm{Irr}(G)$ has even degree, then $\chi \in \mathrm{Irr}(G/L)$. Then, by part (c) of Lemma 4.1, 2 is adjacent only to p in $\Delta(G)$. So, by Remark 2.7, the graph obtained by removing the vertex p from $\Delta(G)$ has two connected components: the single vertex 2 and the complete graph with vertex set $V(G) - \{2, p\}$. By the discussion of case (a), we know that p is a complete vertex of $\Delta(G/L)$, hence of $\Delta(G)$; thus, p is the only cut-vertex of $\Delta(G)$. □

5 Proof of Theorem 2

The last section of this paper is devoted to the proof of Theorem 2, that we state again (in a slightly different form, for technical reasons) as Theorem 5.3.

Lemma 5.1 *Let K be a normal subgroup of the group G with $K \cong \mathrm{SL}_2(4)$ or $K \cong \mathrm{SL}_2(5)$. Let R be the solvable radical of G, $N = K \cap R$ and assume that $V(G) = \{2, 3, 5, p\}$ for a suitable prime p. Then*

(a) *The primes in $V(G/K)$ (if any) are complete vertices of $\Delta(G)$.*
(b) *If $N \neq 1$ or $KR \neq G$, then 2 is adjacent to 3 in $\Delta(G)$.*
(c) *If $5 \notin V(G/K)$, then 5 is adjacent in $\Delta(G)$ exactly to the primes in $V(G/K)$.*

Proof (a): By part (a) of Lemma 4.1 the primes in $V(KR/K) = V(R/N)$ are complete vertices of $\Delta(G)$. Let $q \in V(G/K) - V(KR/K)$; then for $\chi \in \mathrm{Irr}(G/K)$ such that q divides $\chi(1)$ and an irreducible constituent θ of $\chi_{KR/K}$, q divides $\chi(1)/\theta(1)$ by Clifford's theorem and $\chi(1)/\theta(1)$ divides $|G/KR|$ by [8, Corollary 11.29]. As $|G/KR| \leq 2$, we have $q = |G/KR| = 2$ and $\chi = \theta^{G/K}$. Seeing by inflation $\theta \in \mathrm{Irr}(KR/N)$ with $K/N \leq \ker\theta$, we write $\theta = 1_{K/N} \times \psi$, with $\psi \in \mathrm{Irr}(R/N)$ and $I_{G/N}(\psi) = I_{G/N}(\theta) = KR/N$. So, for every $\varphi \in \mathrm{Irr}(K/N)$, $\varphi \times \psi \in \mathrm{Irr}(KR/N)$ and $I_{G/N}(\varphi \times \psi) = KR/N$, hence 2 is adjacent to both 3 and 5 in $\Delta(G)$. If $p \notin \{2, 3, 5\}$, then (since $|N| \leq 2$ and $p \in V(G)$) $R/N \cong KR/K$ cannot have a normal abelian Sylow p-subgroup, so $p \in V(KR/K)$ is adjacent to 2 in $\Delta(G)$ and $q = 2$ is a complete vertex of $\Delta(G)$.

Part (b) is clear, as both $\mathrm{SL}_2(5)$ and $\mathrm{Aut}(\mathrm{SL}_2(4)) \cong \mathrm{S}_5$ have an irreducible character of degree 6.

(c): Since $|G/KR| \leq 2$, KR contains every Sylow 5-subgroup of G and, as $5 \notin V(R) \subseteq V(G/K)$, if $\chi \in \mathrm{Irr}(G)$ has degree divisible by 5, then χ lies (both if $K \cong \mathrm{SL}_2(4)$, as well as if $K \cong \mathrm{SL}_2(5)$) over the unique character $\alpha \in \mathrm{Irr}(K)$ such that 5 divides $\alpha(1)$. It is easily seen that α extends to G. By Gallagher's theorem, we conclude that 5 is adjacent only to the vertices of $V(G/K)$ in $\Delta(G)$. □

Lemma 5.2 *Let R and K be, respectively, the solvable radical and the solvable residual of the group G, and let $N = R \cap K$.*

(a) *If $2 \notin V(G/K)$, $G = KR$ and N is the natural module for $K/N \cong \mathrm{SL}_2(4)$, then $N \leq \ker\chi$ for every $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is even.*
(b) *Let $L \trianglelefteq G$, $L \leq N$, be such that $K/L \cong \mathrm{SL}_2(5)$ and L is the natural module for K/L. If $5 \notin V(G/K)$, then 5 is adjacent in $\Delta(G)$ exactly to the primes in $V(G/K)$.*

Proof (a): Assume that $2 \notin V(G/K)$, $G = KR$ and that N is the natural module for $K/N \cong \mathrm{SL}_2(4)$. Let $\lambda \in \mathrm{Irr}(N)$ be a non-principal character and let $I = I_G(\lambda)$, T a Sylow 2-subgroup of I, $T_0 = T \cap R$ and $T_1 = T \cap K$. Since, by Lemma 2.6, I contains a Sylow 2-subgroup of R, we see that $T_0 \in \mathrm{Syl}_2(R)$; moreover, as $2 \notin V(G/K) = V(R/N)$, T_0/N is abelian and $T_0 \trianglelefteq R$. For $B/N \in \mathrm{Syl}_3(K/N)$, as $N \leq \mathbf{Z}(T_0)$ and $[B/N, T_0/N] = 1$ by coprimality we get $T_0 = N\mathbf{C}_{T_0}(B) = N \times \mathbf{C}_{T_0}(B)$, because $\mathbf{C}_N(B) = 1$; in particular, T_0 is abelian. Write $C = \mathbf{C}_{T_0}(B)$ and $D = \mathbf{C}_{T_0}(K)$; so $D \trianglelefteq C$. Since $I \cap K$ is a Sylow 2-subgroup of K, we have $T \in \mathrm{Syl}_2(G)$. As $T = T_1T_0$, we have $T' = T_1'[T_1, T_0]T_0' = T_1'[T_1, T_0]$. We claim that $[T_1, T_0] \leq T_1'$. Observing that $[T_1, T_0] = [T_1, N][T_1, C]$, it is enough to prove that $[T_1, C/D] \leq T_1'$. Identifying C/D with a normal subgroup of $\mathrm{Out}(K)$, one can check (for instance by GAP [14], as $K =$ SmallGroup(960,11357)) that

$$[T_1, C/D] \leq [T_1, \mathbf{O}_2(\mathrm{Out}(K))] \leq T_1',$$

so the claim follows. Hence, $T' = T_1'$ and by Lemma 2.6 λ extends to T. Thus, by [8, Theorem 6.26] λ extends to I. As I/N has odd index in G/N and has a normal abelian Sylow 2-subgroup, it follows that every irreducible character of G lying over λ, where λ is any non-principal character of N, has odd degree.

(b):We observe that G splits over L. In fact, if X is a Sylow 2-subgroup of N (so, $|X| = |N/L| = 2$), then by the Frattini argument $G = L\mathbf{C}_G(X)$ and, as X acts fixed-point-freely on L, $L \cap \mathbf{C}_G(X) = 1$.

Let $Q_0 \in \mathrm{Syl}_5(R)$; since $R/N \cong KR/K \trianglelefteq G/K$, $V(R/N) \subseteq V(G/K)$ and $5 \notin V(R/N)$, so Q_0N/N is abelian and normal in R/N. As $N/L \trianglelefteq R/L$ and $|N/L| = 2$, N/L is central in R/L and it follows that $Q_0/L \trianglelefteq G/L$, so $Q_0 \trianglelefteq G$. For a non-principal $\lambda \in \mathrm{Irr}(L)$, $I_K(\lambda) = Q_1 \in \mathrm{Syl}_5(K)$. So, as $|G/KR| \leq 2$, $Q = Q_0Q_1 \in \mathrm{Syl}_5()G$ and $Q \leq I = I_G(\lambda)$. Since G splits over L, λ extends to I and, as $Q/L = Q_1/L \times Q_0/L$ is abelian and normal in I/L, by Gallagher's theorem and Clifford correspondence it follows that 5 does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ that lies over λ. Thus, L is contained in the kernel of every irreducible character of G with degree divisible by 5, and part (c) of Lemma 5.1 applied to G/L yields that 5 is adjacent in $\Delta(G)$ exactly to the primes in $V(G/K)$. □

Theorem 5.3 *Let R and K be, respectively, the solvable radical and the solvable residual of the group G and assume that G has a composition factor $S \cong \mathrm{SL}_2(4)$. Let $N = K \cap R$. Then, $\Delta(G)$ is a connected graph and has a cut-vertex p if and only if G/R is an almost simple group with socle isomorphic to S, $V(G) = \{2, 3, 5\} \cup \{p\}$ and one of the following holds.*

(a) *K is isomorphic either to $\mathrm{SL}_2(4)$ or to $\mathrm{SL}_2(5)$ and $V(G/K) = \{p\}$; if $p = 5$, then $K \cong \mathrm{SL}_2(4)$ and $G = K \times R$.*
(b) *$K/N \cong \mathrm{SL}_2(4)$, $|N| = 2^4$, $G = KR$ and one of the following:*
   (i) *N is the natural module for K/N, $p \neq 2$, $V(G/K) = \{p\}$.*
   (ii) *N isomorphic to the restriction to K/L, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$. Moreover, $p = 5$, $G = K \times R_0$, where $R_0 = \mathbf{C}_G(K)$, and $V(R_0) = V(G/K) \subseteq \{5\}$;*
(c) *There exists $1 \neq L \leq N$, L normal in G, with $K/L \cong \mathrm{SL}_2(5)$ and one of the following:*
   (i) *$|L| = 5^2$, L is the natural module for $\mathrm{SL}_2(5)$, $p \neq 5$ and $V(G/K) = \{p\}$.*
   (ii) *$|L| = 3^4$, L is the natural module for $\mathrm{SL}_2(5)$ seen as a subgroup of $\mathrm{GL}_4(3)$, $p = 2$ and $V(G/K) \subseteq \{2\}$.*

In all cases, p is is a complete vertex and the only cut-vertex of $\Delta(G)$.

Proof We start by proving the "only if" part of the statement, assuming that $\Delta(G)$ is connected and that it has a cut-vertex p. Then, by Theorem 2.8 G/R is an almost-simple group and $V(G) = \pi(G/R) \cup \{p\}$. So, the socle M/R of G/R is isomorphic to $\mathrm{SL}_2(4)$, and $V(G) = \{2, 3, 5, p\}$. Hence, the subgraph of $\Delta(G)$ induced by the set of vertices $\{2, 3, 5\}$ cannot be a clique. As $N = K \cap R$ and $KR = M$, then $K/N \cong M/R \cong \mathrm{SL}_2(4)$ and $|G/KR| \leq 2$. Since no vertex of $\Delta(G)$ different from p can be complete, part (a) of Lemma 5.1 implies that $V(G/K) \subseteq \{p\}$.

We now apply Theorem 3.4, considering the possible structure types for the solvable residual K of G.

If K is isomorphic either to $\mathrm{SL}_2(4)$ or to $\mathrm{SL}_2(5)$ (i.e., $|N| \leq 2$), then part (c) of Lemma 5.1 implies (as 5 cannot be an isolated vertex of $\Delta(G)$) that $V(G/K)$ is non-empty, so $V(G/K) = \{p\}$. By part (b) of Lemma 5.1 $p \neq 5$ when $K \cong \mathrm{SL}_2(5)$ or $KR \neq G$; so we have case (a).

Assume now that $|N| > 2$, and that N is a minimal normal subgroup of G. Then, by Theorem 3.4 $K/N \cong \mathrm{SL}_2(4)$, $|N| = 2^4$ and we have two cases:

(x): N is the natural module for K/N: then 3 and 5 are adjacent in $\Delta(G)$ (see Remark 2.7), and hence $p \neq 2$, as othewise $\Delta(G)$ would be a complete graph. We show that $G = KR$: in fact, if this is not the case, then $G/R \cong S_5$ and the Sylow 2-subgroups of G/N are non-abelian. For a non-principal $\lambda \in \mathrm{Irr}(L)$ and $I = I_G(\lambda)$, 15 divides $|G : I|$. Hence, recalling Theorem A of [11], independently on the parity of $|G : I|$ there exists $\chi \in \mathrm{Irr}(G)$, lying above λ, that has degree 30, a contradiction. Finally, we observe that if $G/K \cong R/N$ is abelian, then 2 is an isolated vertex of $\Delta(G)$, because by part (a) of Lemma 5.2 every $\chi \in \mathrm{Irr}(G)$ of even degree is a character of $G/N = K/N \times R/N$. So, $V(G/K) = \{p\}$ and we have case (b)(i).

(xx): N is the restriction to K/L, embedded as $\Omega_4^-(2)$ into $\mathrm{SL}_4(2)$, of the standard module of $\mathrm{SL}_4(2)$. Then $\Delta(K)$ is the graph $2 - 5 - 3$ and hence necessarily $p = 5$.

Let $R_0 = \mathbf{C}_G(K)$ and $C = \mathbf{C}_G(N)$. So, $N \leq C \trianglelefteq G$ and $R_0 \leq C \leq R$, since K/N is the only non-solvable composition factor of G, and it acts non-trivially on N. As $\mathrm{H}^2(K/N, N) = 0$, K splits over N; let K_0 be a complement of N in K. Note that $R_0 = \mathbf{C}_C(K) = \mathbf{C}_C(K_0)$. We prove that $C = N \times R_0$. It is enough to show that $C = NR_0$, since $\mathbf{Z}(K) = 1$. As $[K, R] \leq N$, in particular $[K_0, C] \leq N$ and hence $K_0^c \leq K_0N = K$ for every $c \in C$. Since $\mathrm{H}^1(K_0, N) = 0$, all complements of N in K are conjugate in K. It follows that there exists an element $b \in N$ such that $K_0^c = K_0^b$, so $d = bc^{-1} \in \mathbf{N}_C(K_0)$ and hence $[K_0, d] \leq K_0 \cap C = 1$, as $K_0 \cong K/N$ acts faithfully on N. Thus, $d \in R_0$. So, $C = NR_0 = N \times R_0$.

The action of G on N gives an embedding ϕ of $\overline{G} = G/C$ in $\widehat{G} = \mathrm{GL}_4(2)$. One can check (for instance by GAP [14]) that $\mathbf{N}_{\widehat{G}}(\phi(\overline{K})) \cong S_5$, and that if $\phi(\overline{G}) \cong S_5$ then $\Delta(G/R_0)$, which is a subgraph of $\Delta(G)$, has a complete subgraph with vertex set $\{2, 3, 5\}$, a contradiction. So, $\phi(\overline{G}) = \phi(\overline{K})$, and hence $G = K \times R_0$, giving case (b)(ii).

As the final case, we assume that G has a minimal normal subgroup L, such that $L \leq N$ and $K/L \cong \mathrm{SL}_2(5)$. We have two possible cases:

(y): L is the natural module for K/L. Then $\Delta(K)$ is the graph with vertex set $\{2,3,5\}$ where 5 is an isolated vertex and 2, 3 are adjacent, so we deduce that $p \neq 5$. Moreover, part (b) of Lemma 5.2 yields that 5 is adjacent in $\Delta(G)$ only to the primes in $V(G/K)$. Thus, as $\Delta(G)$ is connected, $V(G/K) \neq \emptyset$, so $V(G/K) = \{p\}$ and we have case (c)(i).

(yy): L is the natural module for K/L seen as a subgroup of $\mathrm{GL}_4(3)$. So, $\Delta(K)$ is the graph $3 - 2 - 5$ and consequently $p = 2$ and we have case (c)(ii).

We now prove the "if" part of the statement, going through the various cases.

(a): If $G \cong \mathrm{SL}_2(4) \times R$ with $V(R) = V(G/K) = \{5\}$, then clearly $\Delta(G)$ is the graph $2 - 5 - 3$. If $p \neq 5$, then 5 is adjacent only to p in $\Delta(G)$ by part (c) of Lemma 5.1. By part (a) of Lemma 5.1, p is a complete vertex, and hence the only cut-vertex, of $\Delta(G)$.

(b): We assume that $G = KR$ and that $N = K \cap R$ is a normal in G of order 2^4.

In case (b)(i), since $G/N = K/N \times R/N$ and $V(R/N) = V(G/K) = \{p\}$ for some prime $p \neq 2$, part (a) of Lemma 5.2 and part (a) of Theorem 2.3 yield that the vertex 2 is adjacent only to p in $\Delta(G)$, so p is a cut-vertex of $\Delta(G)$. By part (a) of Lemma 5.1, p is a complete vertex, and hence the only cut-vertex, of $\Delta(G)$.

In case (b)(ii), it is clear that $\Delta(G) = \Delta(K)$ is the graph $2 - 5 - 3$.

(c): We assume that there exists $L \trianglelefteq G$, $L \leq K$, such that $K/L \cong \mathrm{SL}_2(5)$.

In case (c)(i), by part (b) of Lemma 5.2 the vertex 5 is adjacent only to p ($p \neq 5$) in $\Delta(G)$ and, by part (a) of the same lemma, p is a complete vertex of $\Delta(G)$.

In case (c)(ii), we prove that $\Delta(G) = \Delta(K)$, so $\Delta(G)$ is the graph $3 - 2 - 5$. To this end, it is enough to show that 3 and 5 are non-adjacent in $\Delta(G)$. Since $|G/KR| \leq 2$, KR contains a Sylow 3-subgroup Q of G; moreover, as $V(R/N) \subseteq V(G/K) \subseteq \{2\}$ and $|N/L| = 2$, it easily follows that, setting $Q_0 = Q \cap R$, Q_0/L is abelian and normal in R/L, and hence in G/L. Let $\lambda \in \mathrm{Irr}(L)$ be a non-principal character and let $I = I_G(\lambda)$. An application of the Frattini argument, as in the proof of part (b) of Lemma 5.2, proves that G splits over L, so λ extends to I. By [5, Lemma 2.6], $L \leq \mathbf{Z}(Q_0)$ and hence, since $I \cap K$ is a Sylow 3-subgroup of K, we can assume $Q \leq I$. So, Q/L is an abelian Sylow 3-subgroup of G/L and it is normal in I/L. Thus, by Gallagher's theorem we deduce that every $\chi \in \mathrm{Irr}(G)$ that lies over λ has degree not divisible by 3. Hence, if $\chi \in \mathrm{Irr}(G)$ and 3 divides $\chi(1)$, then $L \leq \ker \chi$ and $\chi \in \mathrm{Irr}(G/L)$. Now, an application of part (c) of Lemma 5.1 yields that 5 not adjacent to 3 in $\Delta(G/L)$, and hence 3 and 5 are not adjacent in $\Delta(G)$.

So, in every case, p is a cut-vertex of $\Delta(G)$ and, as p is also a complete vertex of $\Delta(G)$, there are no other cut-vertices in $\Delta(G)$. The proof is complete. □

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