Binary words avoiding a pattern and marked succession rule

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Abstract

1 Introduction

Binary words excluding a given pattern $\mathfrak{p} = p_0...p_{h-1} \in \{0,1\}^h$ constitute a regular language and can be enumerated in terms of the number of bits 1 and 0 by using classical results (see, e.g., [9, 10, 16]). Recently, in [2, 13], this subject has been studied in relation to the concept of proper Riordan arrays. In particular, these papers prove necessary and sufficient conditions under which the number of words counted with respect to the number of zeroes and ones are related to proper Riordan arrays. This problem is interesting in the context of the Riordan arrays theory because the matrices which here arise are naturally defined by recurrence relations following the characterization given in [12] (see formula (1.3) below). In particular, if $F^{[\mathfrak{p}]}_{n,k}$ denotes the number of words excluding the pattern and having n bits 1 and k bits 0, then by using the results in [2] we have

$$F^{[\mathfrak{p}]}(x,y) = \sum_{n,k\geq 0} F^{[\mathfrak{p}]}_{n,k} x^n y^k = \frac{C^{[\mathfrak{p}]}(x,y)}{(1-x-y)C^{[\mathfrak{p}]}(x,y) + x^{n_1^{\mathfrak{p}}} y^{n_0^{\mathfrak{p}}}}, \tag{1.1}$$

where $n_1^{[\mathfrak{p}]}$ and $n_0^{[\mathfrak{p}]}$ correspond to the number of ones and zeroes in the pattern and $C^{[\mathfrak{p}]}(x,y)$ is the autocorrelation polynomial with coefficients given by the autocorrelation vector (see also [9, 10, 16]). For a given $\mathfrak{p}$, this vector of bits $c = (c_0, \ldots, c_{h-1})$ can be defined in terms of Iverson's bracket notation (for a predicate P, the expression $[\![P]\!]$ has value 1 if P is true and 0 otherwise) as follows: $c_i = [\![p_0 p_1 \cdots p_{h-1-i} = p_i p_{i+1} \cdots p_{h-1}]\!]$. In other words, the bit c_i is determined by shifting $\mathfrak{p}$ right by i positions and setting $c_i = 1$ iff the remaining letters match the original. For example, when $\mathfrak{p} = 101010$ the autocorrelation vector is $c = (1,0,1,0,1,0)$, as illustrated in Table 1.1, and $C^{[\mathfrak{p}]}(x,y) = 1 + xy + x^2y^2$, that is, we mark with $x^j y^i$ the tails of the pattern with j bits 1, i bits 0 and $c_{j+i} = 1$. Therefore, in this case we have:

$$F^{[\mathfrak{p}]}(x,y) = \frac{1 + xy + x^2y^2}{(1-x-y)(1+xy+x^2y^2) + x^3y^3}.$$

As another example, when $\mathfrak{p} = 11100$ then $C^{[\mathfrak{p}]}(x,y) = 1$ and $F^{[\mathfrak{p}]}(x,y) = 1/(1-x-y+x^3y^2)$.

In order to study the binary words avoiding a pattern in terms of Riordan arrays, we consider the array $\mathcal{R}^{[\mathfrak{p}]} = (R^{[\mathfrak{p}]}_{n,k})$ given by the lower triangular part of the array $\mathcal{F}^{[\mathfrak{p}]} = (F^{[\mathfrak{p}]}_{n,k})$, that is, $R^{[\mathfrak{p}]}_{n,k} = F^{[\mathfrak{p}]}_{n,n-k}$ with $k \leq n$. More precisely, $R^{[\mathfrak{p}]}_{n,k}$ counts the number of words avoiding $\mathfrak{p}$ and

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1	0	1	0	1	0	Tails					
1	0	1	0	1	0						1
	1	0	1	0	1	0					0
		1	0	1	0	1	0				1
			1	0	1	0	1	0			0
				1	0	1	0	1	0		1
					1	0	1	0	1	0	0

Table 1.1: The autocorrelation vector for $\mathfrak{p} = 101010$.

n/k	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28	36
3	1	4	9	18	32	52	79	114
4	1	5	13	29	58	106	180	288
5	1	6	18	44	96	192	357	624
6	1	7	24	64	151	325	650	1222
7	1	8	31	90	228	524	1116	2232

Table 1.2: The matrix $\mathcal{F}^{[\mathfrak{p}]}$ for $\mathfrak{p} = 11100$

having length $2n-k$, n bits one and $n-k$ bits zero. Given a pattern $\mathfrak{p} = p_0 \dots p_{h-1} \in \{0,1\}^h$, let $\bar{\mathfrak{p}} = \bar{p}_0 \dots \bar{p}_{h-1}$ be the pattern with $\bar{p}_i = 1 - p_i, \forall i = 0, \cdots, h-1$. We obviously have $R^{[\bar{\mathfrak{p}}]}_{n,k} = F^{[\bar{\mathfrak{p}}]}_{n,n-k} = F^{[\mathfrak{p}]}_{n-k,n}$, therefore, the matrices $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ represent the lower and upper triangular part of the array $\mathcal{F}^{[\mathfrak{p}]}$, respectively. Moreover, we have $R^{[\mathfrak{p}]}_{n,0} = R^{[\bar{\mathfrak{p}}]}_{n,0} = F^{[\mathfrak{p}]}_{n,n}$, $\forall n \in \mathbb{N}$, that is, columns zero of $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ correspond to the main diagonal of $\mathcal{F}^{[\mathfrak{p}]}$. Tables 1.2, 1.3 and 1.4 illustrate some rows for the matrices $\mathcal{F}^{[\mathfrak{p}]}$, $\mathcal{R}^{[\mathfrak{p}]}$ and $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ when $\mathfrak{p} = 11100$.

We briefly recall that a Riordan array is an infinite lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$, defined by a pair of formal power series $(d(t), h(t))$, such that $d(0) \neq 0, h(0) = 0, h'(0) \neq 0$ and the generic element $d_{n,k}$ is the n-th coefficient in the series $d(t)h(t)^k$, i.e.:

$$d_{n,k} = [t^n]d(t)h(t)^k, \qquad n, k \geq 0.$$

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	6	3	1					
3	18	9	4	1				
4	58	29	13	5	1			
5	192	96	44	18	6	1		
6	650	325	151	64	24	7	1	
7	2232	1116	524	228	90	31	8	1

Table 1.3: The triangle $\mathcal{R}^{[\mathfrak{p}]}$ for $\mathfrak{p} = 11100$

n/k	0	1	2	3	4	5	6	7
0	1							
1	2	1						
2	6	3	1					
3	18	10	4	1				
4	58	32	15	5	1			
5	192	106	52	21	6	1		
6	650	357	180	79	28	7	1	
7	2232	1222	624	288	114	36	8	1

Table 1.4: The triangle $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ for $\bar{\mathfrak{p}} = 00011$

From this definition we have $d_{n,k} = 0$ for $k > n$. An alternative definition is in terms of the so-called A-sequence and Z-sequence, with generating functions $A(t)$ and $Z(t)$ satisfying the relations:

$$h(t) = tA(h(t)), \quad d(t) = \frac{d_0}{1 - tZ(h(t))} \quad \text{with} \quad d_0 = d(0).$$

In other words, Riordan arrays correspond to matrices where each element $d_{n,k}$ is described by a linear combination of the elements in the previous row, starting from the previous column, with coefficients in A:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots \tag{1.2}$$

Another characterization (see [12]) states that a lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$ is Riordan if and only if there exists another array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and a sequence $(\rho_j)_{j\in\mathbb{N}}$ such that:

$$d_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0}\alpha_{i,j} d_{n-i,k+j} + \sum_{j\geq 0}\rho_j d_{n+1,k+j+2}. \tag{1.3}$$

Matrix $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ is called the A-matrix of the Riordan array. If $P^{[0]}(t), P^{[1]}(t), P^{[2]}(t), \ldots$ denote the generating functions of rows $0, 1, 2, \ldots$ in the A-matrix, i.e.:

$$P^{[i]}(t) = \alpha_{i,0} + \alpha_{i,1}t + \alpha_{i,2}t^2 + \alpha_{i,3}t^3 + \ldots$$

and $Q(t)$ is the generating function for the sequence $(\rho_j)_{j\in\mathbb{N}}$, then we have:

$$\frac{h(t)}{t} = \sum_{i\geq 0} t^i P^{[i]}(h(t)) + \frac{h(t)^2}{t} Q(h(t)), \tag{1.4}$$

$$A(t) = \sum_{i\geq 0} t^i A(t)^{-i} P^{[i]}(t) + tA(t)Q(t). \tag{1.5}$$

The theory of Riordan arrays and the proofs of their properties can be found in [11, 12]. The Riordan arrays which arise in the context of pattern avoidance (see [2, 13]) have the nice property to be defined by a quite simple recurrence relation following the characterization (1.3), while the relation induced by the A-sequence is, in general, more complex. From a combinatorial point of view, this means that it is very challenging to find a construction allowing to obtain objects of size $n + 1$ from objects of size n. Instead, the existence of a simple A-matrix corresponds to a possible construction from objects of different sizes less than $n+1$. On the other hand, as we will see in Section 3, the recurrence following characterization (1.3) contains negative coefficients and therefore gives rise to interesting non trivial combinatorial problems. In this paper we examine in particular the family of patterns $\mathfrak{p} = 1^{j+1}0^j$ and show that the corresponding recurrence relation can be combinatorially interpreted. To this purpose, we translate the recurrence into a succession rule, as it is typically done from problems related to Riordan arrays (see, e.g., [3, 15]), and give a construction for the class of binary words avoiding the pattern $\mathfrak{p}$.

2 Basic definitions and notations

A *succession rule* Ω is a system constituted by an *axiom* (a), with $a \in \mathbb{N}$, and a set of *productions* of the form:

$$(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)), \qquad k \in \mathbb{N},\ e_i : \mathbb{N} \to \mathbb{N}.$$

A production rule explains how to derive, for any given label (k), its *successors* $(e_1(k)), (e_2(k)), \dots, (e_k(k))$. In most of the cases, for a succession rule Ω, we use the more compact notation:

$$\begin{cases} (a) & \\ (k) & \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k)) \end{cases} \tag{2.6}$$

The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (a) is the label of the root and each node labelled (k) produces k sons labelled $(e_1(k)), (e_2(k)), \dots, (e_k(k))$, respectively. It is evident from this definition that we have a notion of *level* on generating tree, by saying that the root lies at level 0, and a node lies at level n when its parent lies at level $n-1$. In case the succession rule describes the growth of a class of combinatorial object, then a given object can be coded by a sequence of labels met from the root of the generating tree to the object itself. We refer to [4] for details and examples.
The concept of succession rules was first introduced in [6] by Chung et al. to study reduced Baxter permutations, and was later applied to the enumeration of permutations with forbidden subsequences [5, 18].
We remark that, from the above definition, a node labelled (k) has precisely k sons. In [1], a succession rule having this property is said to be *consistent*. However, we can also consider succession rules, introduced in [7], in which the value of a label does not necessarily represent the number of its sons and this will be frequently done in the sequel.
A slight generalization of the notion of succession rule is provided by the concept of *jumping succession rules*. Roughly speaking, the idea is to consider a set of succession rules acting on the objects of a class and producing sons at different levels.
The usual notation to indicate a jumping succession rule is the following:

$$\begin{cases} (a) & \\ (k) & \overset{1}{\rightsquigarrow} (e_1(k))(e_2(k))\dots(e_k(k)), \\ (k) & \overset{j}{\rightsquigarrow} (d_1(k))(d_2(k))\dots(d_k(k)). \end{cases} \tag{2.7}$$

The generating tree associated with (2.7) has the property that each node labelled (k) lying at level n produces two sets of sons, the first set at level $n+1$ and having labels $(e_1(k)), (e_2(k)), \dots, (e_k(k))$ respectively and the second one at level $n+j$, with $j > 1$, and having labels $(d_1(k)), (d_2(k)), \dots, (d_k(k))$ respectively. For more details about these topics, see [8].
Another generalization concerning succession rule is introduced in [14], where the authors deal

with *marked succession rules*. In this case the labels appearing in succession rule can be marked or not, therefore the concept of a *marked label* has to be considered together with usual labels. In this way the concept of generating tree can be extended to deal with negative value if we consider a node labelled (k) as opposed to a node labelled $(\overline{k})$ lying on the same level.
A *marked generating tree* is a rooted labelled tree where appear marked or non-marked label developped according to the corresponding succession rule. The main property is that, on the same level, marked labels kill or annihilate the non-marked labels with the same label value, in particular the enumeration of the combinatorial objects of a class at a given level is the difference between the number of non-marked and marked labels.
For any label (k), we introduce the following notations for generating tree specifications:

$$(\overline{\overline{k}}) = (k);$$

$$(k)^n = \underbrace{(k)\dots(k)}_{n}, \quad n > 0.$$

Each succession rules (2.6) can be trivially rewritten as (2.8)

$$\begin{cases} (a) & \\ (k) & \rightsquigarrow (e_1(k))(e_2(k))\dots(e_k(k))(k), \\ (k) & \rightsquigarrow (\overline{k}). \end{cases} \tag{2.8}$$

For example, the classical succession rule for Catalan numbers can be rewritten in the form (2.9) and Figure 2.1 shows the associated generating tree.

$$\begin{cases} (2) & \\ (k) & \rightsquigarrow (2)(3)\dots(k)(k+1)(k), \\ (k) & \rightsquigarrow (\overline{k}). \end{cases} \tag{2.9}$$

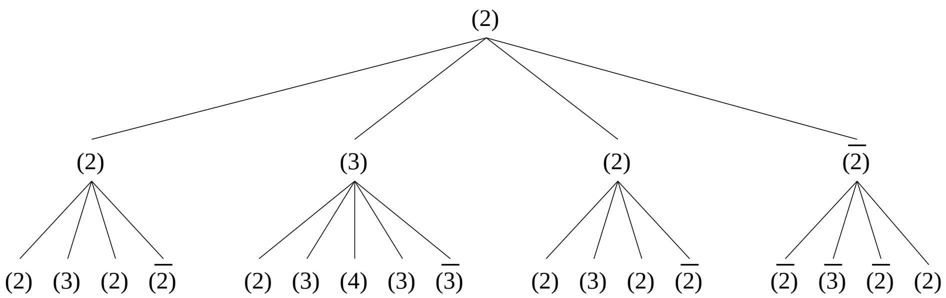

Figure 2.1: Some levels of the generating tree associated with the succession rule (1.4)

The concept of marked labels has been implicity used for the first time in [15], the same concept has been used in [7] in relation with the definition of the ECO-systemes signes. In Section 3, we show how marked succession rules appears in the enumeration of a class of particular binary words according to the number of ones. Let $F \subset \{0,1\}^*$ be the class of binary words w such that $|w|_0 \leq |w|_1$ for any $w \in F$, $|w|_0$ and $|w|_1$ corresponding to the number of zeroes and ones in the word w, respectively. In this paper we are interested in studying the subclass $F^{[\mathfrak{p}]}$ of F of binary words excluding a given pattern $\mathfrak{p} = p_0 \dots p_{h-1} \in \{0,1\}^h$, i.e. the word $w \in F^{[\mathfrak{p}]}$ that does not admit a sequence of consecutive indices $i, i+1, \dots, i+h-1$ such that $w_i w_{i+1} \dots w_{i+h-1} = p_0 p_1 \dots p_{h-1}$.

Each word $w \in F$ can be naturally represented as a lattice path on the Cartesian plane by associating a *rise step*, defined by $(1,1)$ and denoted by x, to each 1's in F, and a *fall step*, defined by $(1,-1)$ and denoted by $\overline{x}$, to each 0's in F.
In the sequel, we will refer indistinctly to words or their graphical representations on the Cartesian plane, that is paths.

3 The Riordan array for the pattern $\mathfrak{p} = 1^{j+1}0^j$

Let us consider the family of patterns $\mathfrak{p} = 1^{j+1}0^j$ and let $F^{[\mathfrak{p}]}_{n,k}$ denote the number of words excluding the pattern and having n bits 1 and k bits 0; from (1.1) we have

$$F^{[\mathfrak{p}]}(x,y) = \sum_{n,k\geq 0} F^{[\mathfrak{p}]}_{n,k} x^n y^k = \frac{1}{1-x-y+x^{j+1}y^j}. \tag{3.10}$$

Now let $R^{[\mathfrak{p}]}_{n,k}$ counts the number of words avoiding $\mathfrak{p}$ and having n bits one and $n-k$ bits zero. Obviously we have $R^{[\mathfrak{p}]}_{n,k} = F^{[\mathfrak{p}]}_{n,n-k}$ with $k \leq n$. By extracting the coefficients from (3.10) we have:

$$[x^{n+1}y^{k+1}](1-x-y+x^{j+1}y^j)F^{[\mathfrak{p}]}(x,y) = F^{[\mathfrak{p}]}_{n+1,k+1} - F^{[\mathfrak{p}]}_{n,k+1} - F^{[\mathfrak{p}]}_{n+1,k} + F^{[\mathfrak{p}]}_{n-j,k+1-j} = 0$$

and therefore:

$$R^{[\mathfrak{p}]}_{n+1,k+1} = R^{[\mathfrak{p}]}_{n,k} + R^{[\mathfrak{p}]}_{n+1,k+2} - R^{[\mathfrak{p}]}_{n-j,k}. \tag{3.11}$$

This is a recurrence relation of type (1.3) and therefore $\mathcal{R}^{[\mathfrak{p}]} = (R^{[\mathfrak{p}]}_{n,k})$ is a Riordan array. In particular, the coefficients of the relation correspond to $P^{[j]}(t) = -1$, $P^{[0]}(t) = 1$, and $Q(t) = 1$, therefore we have

$$\frac{h^{[\mathfrak{p}]}(t)}{t} = \sum_{i\geq 0} t^i P^{[i]}(h^{[\mathfrak{p}]}(t)) + \frac{h^{[\mathfrak{p}]}(t)^2}{t} Q(h(t)) = 1 - t^j + \frac{h^{[\mathfrak{p}]}(t)^2}{t}$$

that is,

$$h^{[\mathfrak{p}]}(t)^2 - h^{[\mathfrak{p}]}(t) + t - t^{j+1} = 0, \quad h^{[\mathfrak{p}]}(t) = \frac{1-\sqrt{1-4t+4t^{j+1}}}{2}.$$

We explicitly observe that from formula (1.5) the generating function $A(t)$ of the A-sequence is the solution of a $j+1$ degree equation $(1-t)A(t)^{j+1} - A(t)^j + t^j = 0$. For example, when $\mathfrak{p} = 11100$ by developing into series we find:

$$A(t) = 1 + t + 2t^3 - t^4 + 7t^5 - 12t^6 + 38t^7 - 99t^8 + 281t^9 + O(t^{10})$$

and this result excludes that there might exist a simple dependence of the elements in row $n+1$ from the elements in row n. For what concerns $d^{[\mathfrak{p}]}(t)$, we simply use the Cauchy formula for finding the main diagonal of matrix $\mathcal{F}^{[\mathfrak{p}]}$ (see, e.g., [17, Cap. 6, p. 182]):

$$d^{[\mathfrak{p}]}(t) = [x^0]F^{[\mathfrak{p}]}(x, \frac{t}{x}) = \frac{1}{2\pi i} \oint F^{[\mathfrak{p}]}(x, \frac{t}{x}) \frac{dx}{x}.$$

We have:

$$\frac{1}{x} F^{[\mathfrak{p}]}(x, \frac{t}{x}) = \frac{-1}{x^2(1-t^j) - x + t}$$

and in order to compute the integral, it is necessary to find the singularities $x(t)$ such that $x(t) \to 0$ with $t \to 0$ and apply the Residue theorem. In this case the right singularity is:

$$x(t) = \frac{1 - \sqrt{1 - 4t(1 - t^j)}}{2(1 - t^j)}$$

and finally we have:

$$d^{[\mathfrak{p}]}(t) = \lim_{x \to x(t)} \frac{-1}{x^2(1 - t^j) - x + t}(x - x(t)) = \frac{1}{\sqrt{1 - 4t + 4t^{j+1}}}.$$

Observe also that:

$$\frac{d^{[\mathfrak{p}]}(t) - 1}{d^{[\mathfrak{p}]}(t)h^{[\mathfrak{p}]}(t)} = 2$$

and therefore $R^{[\mathfrak{p}]}_{n+1,0} = 2R^{[\mathfrak{p}]}_{n+1,1}$. Recurrence (3.11) is quite simple, however, the presence of negative coefficients leads to a possible non trivial combinatorial interpretation. In order to study this problem we proceed as follows. The dependence of $R^{[\mathfrak{p}]}_{n+1,k+1}$ from the same row $n+1$ can be simply eliminated and we have:

$$R^{[\mathfrak{p}]}_{n+1,k+1} = R^{[\mathfrak{p}]}_{n,k} - R^{[\mathfrak{p}]}_{n-j,k} + R^{[\mathfrak{p}]}_{n+1,k+2} =$$

$$= R^{[\mathfrak{p}]}_{n,k} - R^{[\mathfrak{p}]}_{n-j,k} + R^{[\mathfrak{p}]}_{n,k+1} - R^{[\mathfrak{p}]}_{n-j,k+1} + R^{[\mathfrak{p}]}_{n+1,k+3} = \cdots =$$

$$= (R^{[\mathfrak{p}]}_{n,k} + R^{[\mathfrak{p}]}_{n,k+1} + R^{[\mathfrak{p}]}_{n,k+2} + \cdots) - (R^{[\mathfrak{p}]}_{n-j,k} + R^{[\mathfrak{p}]}_{n-j,k+1} + R^{[\mathfrak{p}]}_{n-j,k+2} + \cdots) \tag{3.12}$$

Similarly we have:

$$R^{[\mathfrak{p}]}_{n+1,0} = 2(R^{[\mathfrak{p}]}_{n,0} + R^{[\mathfrak{p}]}_{n,1} + R^{[\mathfrak{p}]}_{n,2} + \cdots) - 2(R^{[\mathfrak{p}]}_{n-j,0} + R^{[\mathfrak{p}]}_{n-j,1} + R^{[\mathfrak{p}]}_{n-j,2} + \cdots) \tag{3.13}$$

Finally, by using the results in [2, 3], recurrences (3.12) and (3.13) translate into the following succession rule:

$$\begin{cases} (0) & \\ (k) & \overset{1}{\rightsquigarrow} (0)^2(1)\cdots(k+1) \\ (k) & \overset{j+1}{\rightsquigarrow} (\overline{0})^2(\overline{1})\cdots(\overline{k+1}) \end{cases} \tag{3.14}$$

This rule can be represented as a tree having its root labeled 0 and where each node with label k at a given level n has $k+3$ sons at level $n+1$ labeled $(0), (0), (1), \cdots, (k+1)$ and $k+3$ sons at level $n+j+1$ with labels $(\overline{0}), (\overline{0}), (\overline{1}), \cdots, (\overline{k+1})$ (this kind of trees are called *level generating trees* in [3]). If we denote by $d_{n,k}$ the number of nodes having label k at level n in the tree and count as negative the marked nodes then we obtain matrix $\mathcal{R}^{[\mathfrak{p}]} = (R^{[\mathfrak{p}]}_{n,k})_{n,k \in \mathbb{N}}$, that is, $\mathcal{R}^{[\mathfrak{p}]}$ corresponds to the matrix associated to the rule (3.14). The relations between Riordan arrays and succession rules has been widely studied and we refer the reader to [3, 14, 15] for more details. We just conclude this section by observing that by using the results in [2, 13] it can be proved that the matrix $\mathcal{R}^{[\bar{\mathfrak{p}}]}$ corresponding to the pattern $\bar{\mathfrak{p}} = 0^{j+1}1^j$ is also a Riordan array.

4 A construction for the class $F^{[\mathfrak{p}]}$

In this section we define an algorithm associates a lattice path in $F^{[\mathfrak{p}]}$, where $\mathfrak{p} = x^{j+1}\overline{x}^j = 1^{j+1}0^j$, to a sequence of labels obtained by means of the succession rule (3.14). This give a construction for the set $F^{[\mathfrak{p}]}$ according to the number of rise steps or equivalently the number of ones.

The axiom (0) is associated to the empty path ε.
A lattice path $\omega \in F$, with n rise steps and such that its endpoint has ordinate k, provides $k+3$ lattice paths, with $n+1$ rise steps, according to the first rule of (3.14) having $0, 0, 1, \ldots, k+1$ as endpoint ordinate, respectively. The last $k+2$ labels are obtained by adding to ω a sequence of steps made up of one rise step followed by $k+1-h$, $0 \leq h \leq k+1$, fall steps (see Figure 4.2). Each lattice path so obtained has the property that its rightmost suffix beginning from the x-axis, either remains strictly above the x-axis itself or ends on the x-axis by a fall step. Note that in this way the paths ending on the x-axis and having a rise step as last step are never obtained. These paths are lied to the first label (0) of the first rule in (3.14) and the way to obtain them will be described later in the paper.

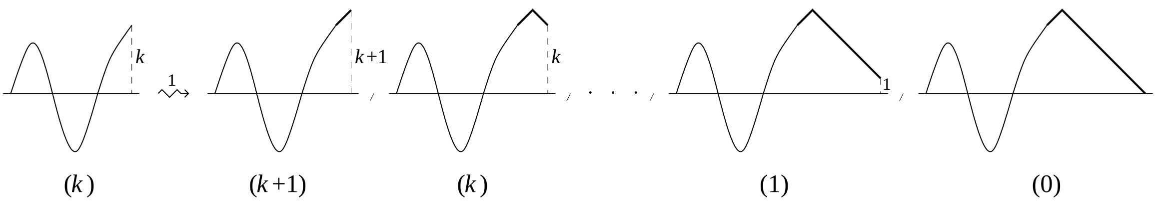

Figure 4.2: The mapping associated to $(k) \overset{1}{\rightsquigarrow} (0)(1)\ldots(k+1)$ of (3.14)

We define a *marked forbidden pattern* $\mathfrak{p}$ as a pattern $\mathfrak{p} = x^{j+1}\overline{x}^j$ whose steps are not separable, that is they can not be divided so they must lay always in that defined sequence. We denote a marked forbidden pattern by marking its peak. We say that a point is strictly contained in a marked forbidden pattern if it is between two steps of the pattern itself.
A lattice path $\omega \in F$, with n rise steps and such that its endpoint has ordinate k, provides $k+3$ lattice paths, with $n+j+1$ rise steps, according to the second rule of (3.14) having $0, 0, 1, \ldots, k+1$ as endpoint ordinate, respectively. The last $k+2$ labels are obtained by adding to ω a sequence of steps made up of the marked forbidden pattern $\mathfrak{p} = x^{j+1}\overline{x}^j$ followed by $k+1-h$, $0 \leq h \leq k+1$, fall steps (see Figure 4.3). Each lattice path so obtained has the property that its rightmost suffix beginning from the x-axis, either remains strictly above the x-axis itself or ends on the x-axis by a fall step. At this point the first label (0) due to the first and the second rule of (3.14) must give lattice paths which either do not contain marked forbidden pattern in its rightmost suffix and end on the x-axis by a rise step or having the rightmost marked point with ordinate less than or equal to j.

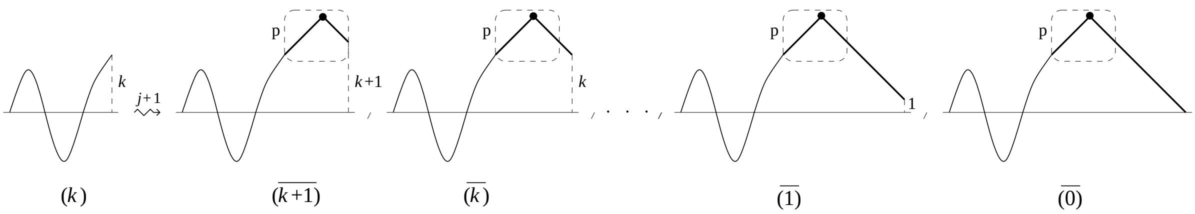

Figure 4.3: The mapping associated to $(k) \overset{j+1}{\rightsquigarrow} (\overline{0})(\overline{1})\ldots(\overline{k+1})$ of (3.14)

In order to obtain the first label (0) according to the first succession rule of (3.14), we consider the lattice path ω' obtained from ω having the ordinate of its endpoint equal to k, by adding a sequence of steps made up of one rise step followed by k fall steps, while in order to obtain the first label (0) according to the second succession rule of (3.14), we consider the lattice path ω' obtained from ω by adding a sequence of steps made up of the marked forbidden pattern $\mathfrak{p} = x^{j+1}\overline{x}^j$ followed by k fall steps. A path ω' by applying the previous actions can be written as $\omega' = v\varphi$, where φ is the rightmost suffix in ω' beginning from the x-axis and strictly remaining above the x-axis (see Figure 4.4).

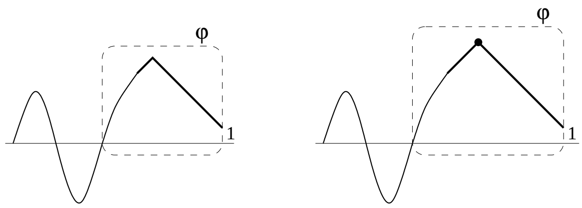

Figure 4.4: Graphical representation of the suffix φ in ω'

We distinguish two cases: in the first one φ contains no marked points and in the second one φ contains at least a marked point.

If the suffix φ does not contain any marked point, the desired label (0) is associated to the path $v\varphi^c x$, being φ^c the path obtained from φ by switching rise and fall steps (see Figure 4.5).

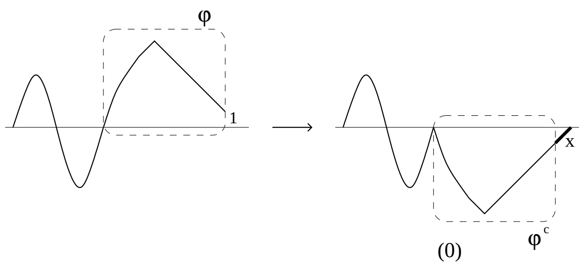

Figure 4.5: A graphical representation of the actions giving the first label (0) in case of no marked points in φ

If the suffix φ contains marked points, let r be the rightmost marked point in φ having highest ordinate and t be the nearest point on the right of the marked forbidden pattern containing r with highest ordinate and which is not strictly within a marked forbidden pattern. We consider the straight line s through the point t and the leftmost point z in φ with highest ordinate, which lies above or on the line s and which is not strictly within a marked forbidden pattern (see the left side of Figure 4.6.a)). Obviously if the straight line s does not intersect any points on the left of t (see the left side of Figure 4.6.b)) or intersects only points lying strictly within a marked forbidden patterns (see the left side of Figure 4.6.c)), then $z \equiv t$.

The desired label (0) is associated to the path obtained by concatenating a fall step $\overline{x}$ with the path in φ running from z to the endpoint of the path, say α, and the path running from the initial point in φ to z, say β (see Figure 4.6 and 4.7).

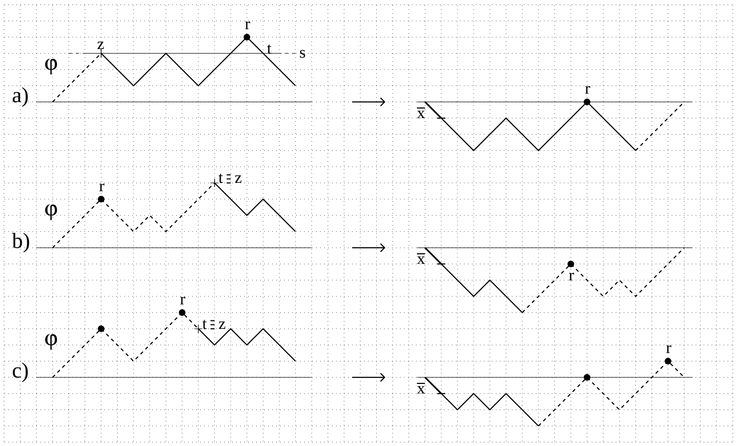

Figure 4.6: Some examples of the actions giving the first label (0) in the case of marked points in φ, $\mathfrak{p} = x^2\overline{x}$

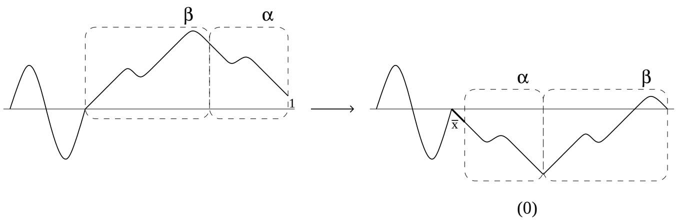

Figure 4.7: A graphical representation of the cut and paste actions giving the first label (0) in case of marked points in φ

This last mapping can be inverted as follows. Let d be the rightmost fall step in a path ω' labeled (0) such that it begins from the x-axis and each marked point, on its right, has ordinate less than or equal to j. Let $\omega' = \omega d \varphi'$ and l the rightmost point in φ' with lowest ordinate. The inverted lattice path of ω' is given by $\omega\beta\alpha$, where β is the path in φ' running from l to the endpoint of the path and α is the path running from the initial point in φ' to l (see Figure 4.8).

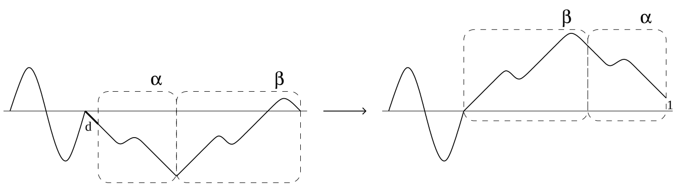

Figure 4.8: A graphical representation of the lattice path obtained by means of the inverted mapping related to the first label (0) in case of marked points in φ

Figure 4.9 shows the cut and paste actions related to the just described inverted mapping on an example, being $\mathfrak{p} = x^2\overline{x}$.

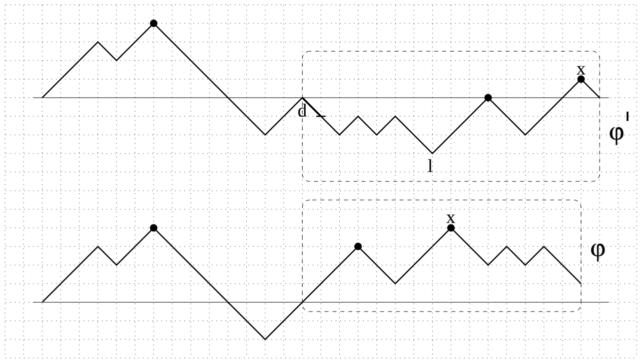

Figure 4.9: The inverted mapping related to the first label (0) in case of marked points in φ sketched on an example

At this point, we can describe the complete mapping defined by the succession rule (3.14), in particular Figure 4.10 shows this complete mapping on an example being $\mathfrak{p} = x^2\overline{x}$ and Figure 4.11 sketches some levels of the generating tree for the paths in $F^{[\mathfrak{p}]}$, $\mathfrak{p} = x^2\overline{x}$, enumerated according to the number of the rise steps.

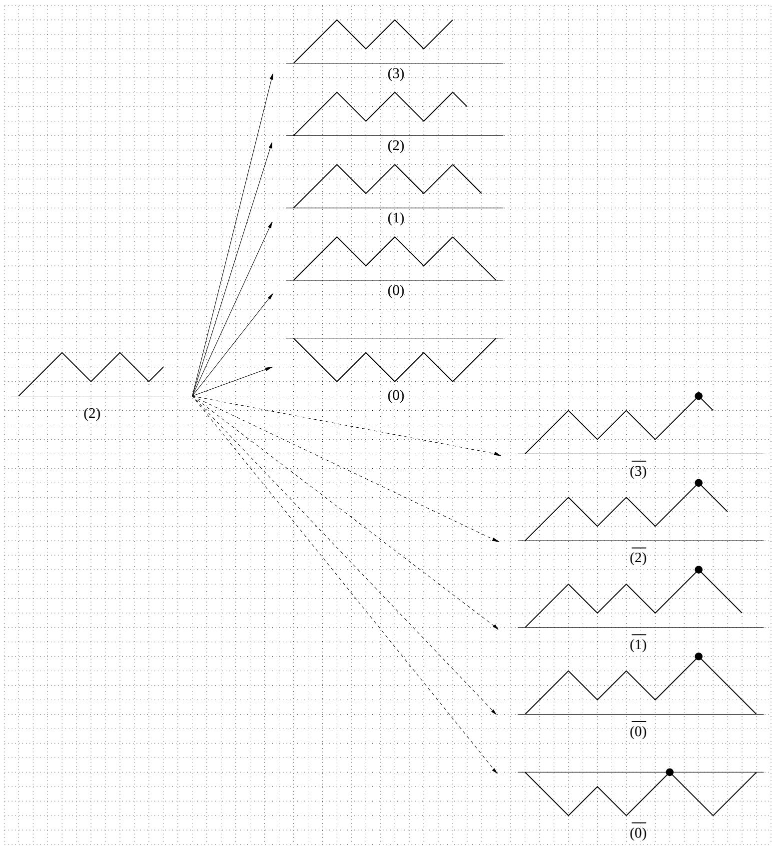

Figure 4.10: The set of lattice paths obtained from a given (k), by means of the succession rule (3.14)

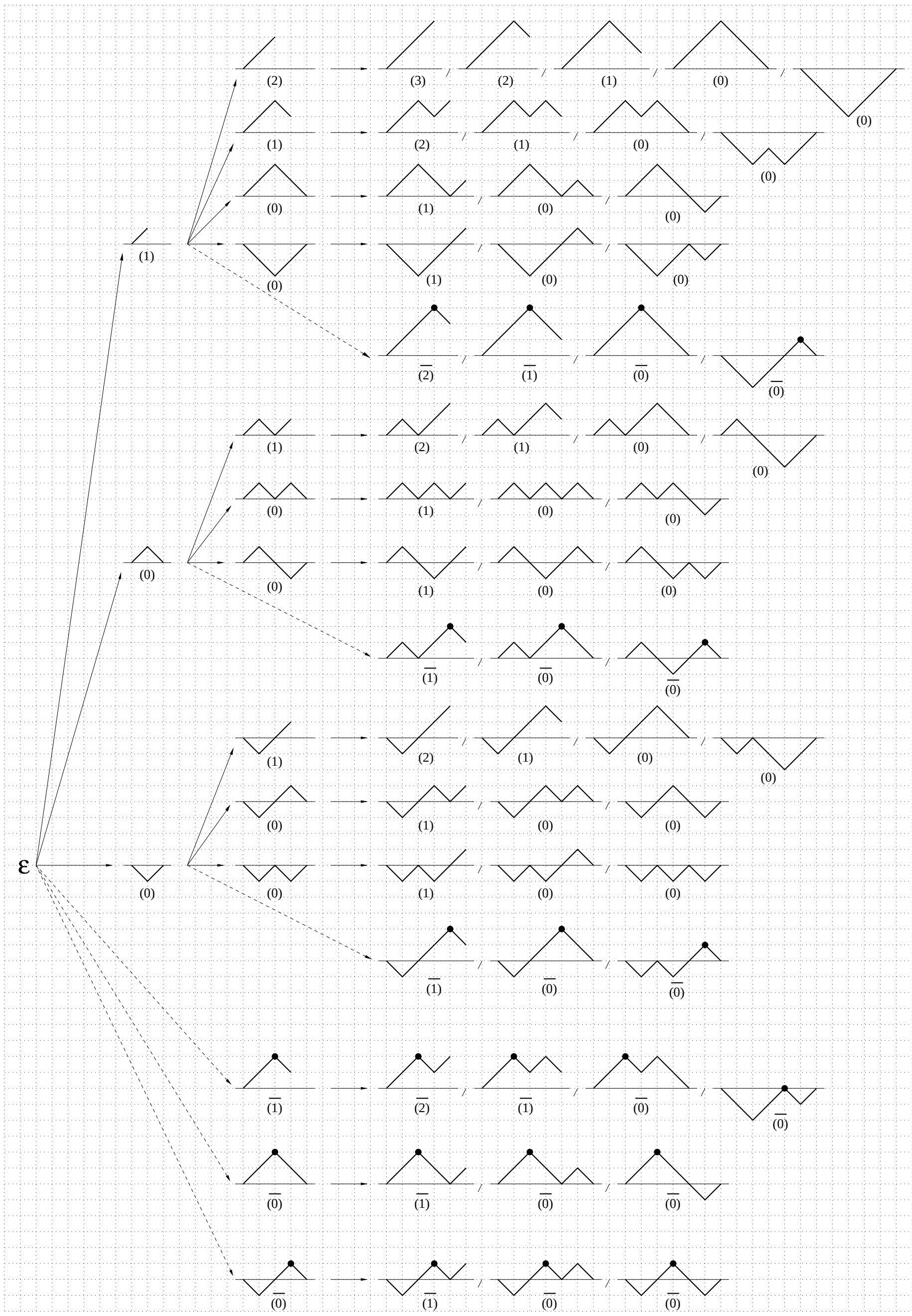

Figure 4.11: Some levels of the generating tree associated with the succession rule (3.14) for the path in $F^{[\mathfrak{p}]}$, being $\mathfrak{p} = x^2\overline{x}$

The just described construction generates 2^C copies of each path having C forbidden patterns such that 2^{C-1} are coded by a sequence of labels ending by a marked label, say $(\overline{k})$, and contain an odd number of marked forbidden pattern, and 2^{C-1} are coded by a sequence of labels ending by a no marked label, say (k), and contain an even number of marked forbidden pattern. For example, Figure 4.12 shows the 4 copies of a given path having 2 forbidden patterns, being $\mathfrak{p} = x^2\overline{x}$.

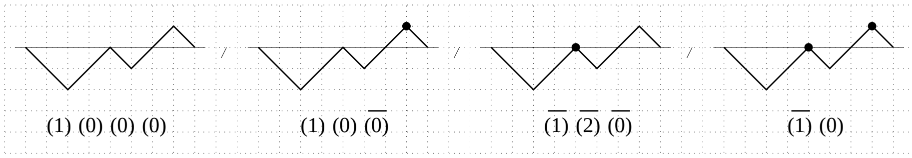

Figure 4.12: The 4 copies of a given path having 2 forbidden patterns

This observation is due to the fact that when a path is obtained according to the first rule of (3.14) then no marked forbidden pattern is added, moreover when a path is obtained according to the second rule of (3.14) exactly one marked forbidden pattern is added. In any case, the actions performed to obtain the first label (0) do not change the number of marked forbidden patterns in the path.

Theorem 4.1 *The generating tree of the lattice paths in* $F^{[\mathfrak{p}]}$*, where* $\mathfrak{p} = x^{j+1}\overline{x}^j$*, according to the number of rise steps, is isomorphic to the tree having its root labelled* (0) *and recursively defined by the following succession rule (3.14):*

$$\begin{cases} (k) \overset{1}{\rightsquigarrow} (0)^2(1)\dots(k+1), \\ (k) \overset{j+1}{\rightsquigarrow} (\overline{0})^2(\overline{1})\dots(\overline{k+1}). \end{cases}$$

Proof. In order to prove the theorem we have to show that the algorithm described in the previous pages is a construction for the set $F^{[\mathfrak{p}]}$ according to the number of rise steps. This means that all the paths in F with n rise steps are obtained. Moreover, for each obtained path ω in $F\backslash F^{[\mathfrak{p}]}$, having C forbidden patterns, with n rise steps and (k) as last label of the associated code, a path ω' in $F\backslash F^{[\mathfrak{p}]}$ with n rise steps, C forbidden patterns and $(\overline{k})$ as last label of the associated code is also generated having the same form of ω but such that the last forbidden pattern is marked if it is not in ω and vice-versa.
The first assertion is an immediate consequence of the described construction according to the first rule of (3.14).
In order to prove the second assertion we have to distinguish two cases: in the first one we consider marked the last forbidden pattern, in the second one the last forbidden pattern is not marked. We denote by h be the ordinate of the peak of the last forbidden pattern.

First case: the last forbidden pattern in ω is a marked forbidden pattern.
We consider the following subcases: $h > j$, $h = j$, $0 < h < j$, $h = 0$ and $h < 0$.

$h > j$: Each path ω in $F\backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu x^{j+1}\overline{x}^f\nu$, being $\mu \in F$, $\nu \in F^{[\mathfrak{p}]}$ and $j \leq f \leq d + j + 1$ where $d \geq 0$ is the ordinate of the endpoint of μ (see Figure 4.13).

The path ω' which kills ω is obtained by applying to the path μ the following consecutive actions: add the path x^j by applying j times the mapping associated to $(k) \overset{1}{\rightsquigarrow} (k+1)$ of the first rule of (3.14), add the path $x\overline{x}^f$ by applying the mapping associated to $(k) \overset{1}{\rightsquigarrow} (d + j + 1 - f)$ of the first rule of (3.14). The path ν in ω' is obtained as in ω.

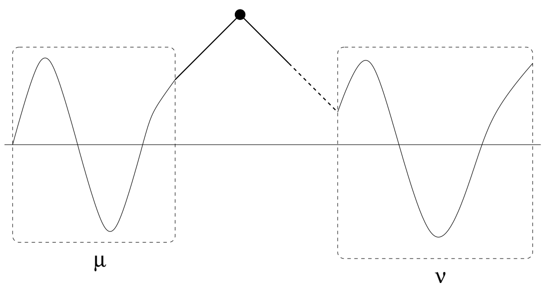

Figure 4.13: A graphical representation of the path ω in the case $h > j$

$h = j$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu\overline{x}\gamma x^{j+1}\overline{x}^{j}\nu$, being $\mu, \gamma \in F$ and $\nu \in F^{[\mathfrak{p}]}$ (see Figure 4.19). We observe that the path γ can contain marked points, with ordinate $b < j$, or not. If the path γ contains no marked points then remains strictly under the x-axis, otherwise the marked forbidden patterns intersect the x-axis when $0 \leq b < j$. In the following cases we consider a path γ having the same property.

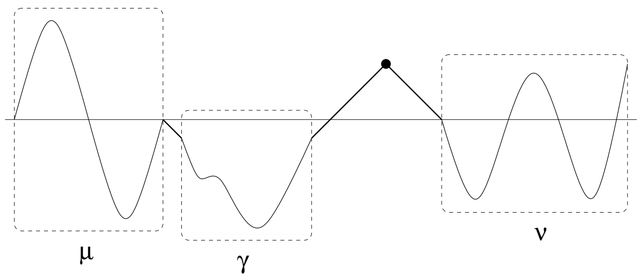

Figure 4.14: A graphical representation of the path ω in the case $h = j$

The path ω' which kills ω is obtained by applying to the path $\mu\overline{x}\gamma x$ the following consecutive actions: add the path x^{j-1} by applying $j-1$ times the mapping associated to $(k) \overset{1}{\rightsquigarrow} (k+1)$ of the first rule of (3.14), add the path $x\overline{x}^{j}$ by applying the mapping associated to $(k) \overset{1}{\rightsquigarrow} (0)$ of the first rule of (3.14) for the second label (0). The path ν in ω' is obtained as in ω.

$0 < h < j$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu\overline{x}\gamma x^{j+1}\overline{x}^{j}\eta x\nu$, being $\mu, \gamma \in F$ and $\eta, \nu \in F^{[\mathfrak{p}]}$ (see Figure 4.15). We observe that the path η remains strictly under the x-axis. In the following cases we consider a path η having the same property.

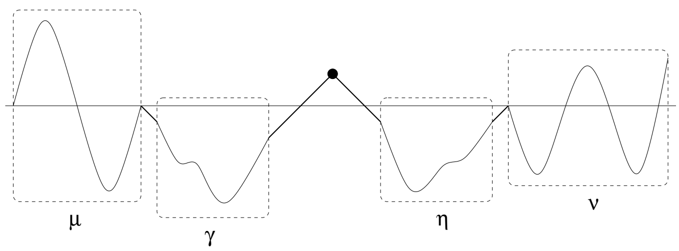

Figure 4.15: A graphical representation of the path ω in the case $0 < h < j$

The path ω' which kills ω is obtained by applying to the path $\mu\overline{x}\gamma x^{j+1-h}$ the following consecutive actions: add the path x^{h-1} by applying $h-1$ times the mapping associated to $(k) \overset{1}{\rightsquigarrow} (k+1)$ of the first rule of (3.14), add the path $x\overline{x}^h$ by applying the mapping associated to $(k) \overset{1}{\rightsquigarrow} (0)$ of the first rule of (3.14) for the second label (0), add the path $\overline{x}^{j-h}\eta x$ by applying consecutive and appropriate mappings of the first rule of (3.14) and this mappings must be completed applying the actions giving the first label (0) in case of no marked points. The path ν in ω' is obtained as in ω.

$h = 0$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu\overline{x}\gamma x^{j+1}\overline{x}^j\eta x\nu$, being $\mu, \gamma \in F$ and $\eta, \nu \in F^{[\mathfrak{p}]}$ (see Figure 4.16).

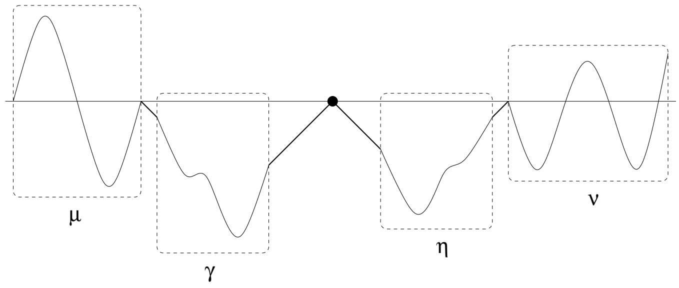

Figure 4.16: A graphical representation of the path ω in the case $h = 0$

The path ω' which kills ω is obtained by applying to the path $\mu\overline{x}\gamma x^{j+1}$ the following action: add the path $\overline{x}^j\eta x$ by applying consecutive and appropriate mappings of the first rule of (3.14), apply the actions giving the first label (0) in case of no marked points. The path ν in ω' is obtained as in ω.

$h < 0$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu\overline{x}\gamma x^{j+1}\overline{x}^j\eta x\nu$, being $\mu, \gamma \in F$ and $\eta, \nu \in F^{[\mathfrak{p}]}$ (see Figure 4.17).

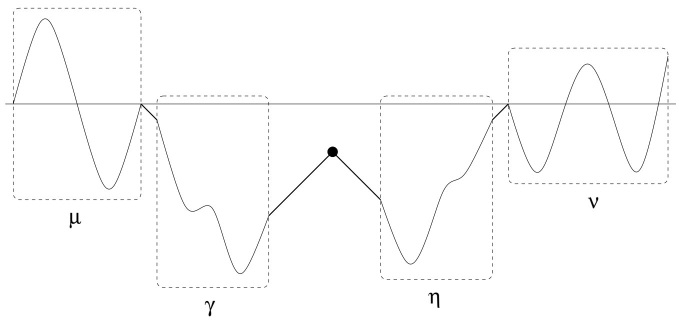

Figure 4.17: A graphical representation of the path ω in the case $h < 0$

We distinguish two subcases: in the first one the path γ contains no marked points and remains strictly under the x-axis and in the second one the path γ contains at least a marked point.
In the first subcase, the path ω' which kills ω is obtained by applying to the path μ the following action: add the path $\overline{x}\gamma x^{j+1}\overline{x}^j\eta x$ by applying consecutive and appropriate mappings of the first rule of (3.14), apply the actions giving the first label (0) in case of no marked points. The path ν in ω' is obtained as in ω.
In the second subcase, we consider the rightmost point l of the path $\overline{x}\gamma x^{j+1}\overline{x}^j\eta x$ with lowest ordinate. The path ω' which kills ω is obtained by applying to the path μ the following consecutive actions: add the path in $\gamma x^{j+1}\overline{x}^j\eta x$ running from l to the endpoint

of the path, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), add the path in $\gamma x^{j+1}\overline{x}^j \eta x$ running from its initial point to l, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), apply the cut and paste actions giving the first label (0) in case of marked points. Obviously the last forbidden pattern in the path must be generated by applying consecutive and appropriate mappings of the first rule of (3.14). The path ν in ω' is obtained as in ω.

Second case: the last forbidden pattern in ω is not a marked forbidden pattern.
We consider the following subcases: $h > j$, $h = j$, $h < j$.

$h > j$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu x^{j+1}\overline{x}^f \nu$, being $\mu \in F$, $\nu \in F^{[\mathfrak{p}]}$ and $j \leq f \leq d + j + 1$ where $d \geq 0$ is the ordinate of the endpoint of μ (Figure 4.18).

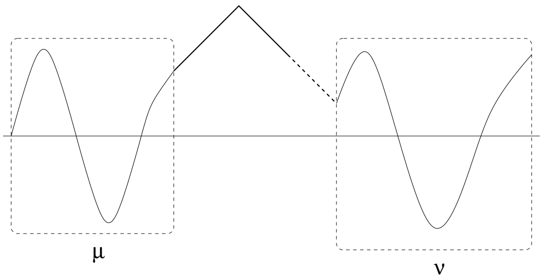

Figure 4.18: A graphical representation of the path ω in the case $h > j$

The path ω' which kills ω is obtained by applying to the path μ the following consecutive actions: add the path $x^{j+1}\overline{x}^f$ by applying the mapping associated to $(k) \overset{j+1}{\rightsquigarrow} (d+j+1-f)$ of the second rule of (3.14). The path ν in ω' is obtained as in ω.

$h = j$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu \overline{x} \gamma x^{j+1}\overline{x}^j \nu$, being $\mu, \gamma \in F$ and $\nu \in F^{[\mathfrak{p}]}$ (see Figure 4.19). We observe that the path γ can contains marked points, with ordinate $b < j$, or not. If the path γ contains no marked points then remains strictly under the x-axis, otherwise the marked forbidden patterns intersect the x-axis when $0 \leq b < j$. In the following case we consider a path γ having the same property.

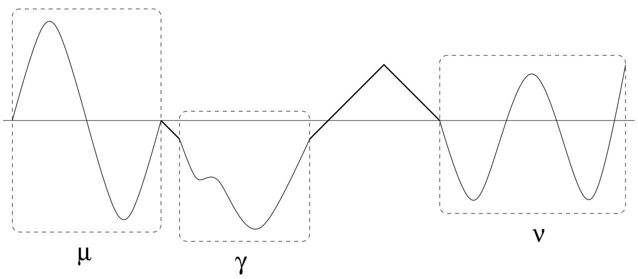

Figure 4.19: A graphical representation of the path ω in the case $h = j$

Let l be the rightmost point m of the path $\overline{x}\gamma x^{j+1}\overline{x}^j$ with lowest ordinate. The path ω' which kills ω is obtained by applying to the path μ the following consecutive actions: add the path in $\gamma x^{j+1}\overline{x}^j$ running from l to the endpoint of the path, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), add the path in $\gamma x^{j+1}\overline{x}^j$ running from its initial point to l, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), apply the cut and paste actions giving the first label (0) in case of marked points. Obviously the last forbidden pattern in the path must be generated by applying the mapping of the second rule of (3.14). The path ν in ω' is obtained as in ω.

$h < j$: Each path ω in $F \backslash F^{[\mathfrak{p}]}$ can be written as $\omega = \mu\overline{x}\gamma x^{j+1}\overline{x}^j\eta x\nu$, being $\mu, \gamma \in F$ and $\eta, \nu \in F^{[\mathfrak{p}]}$ (see Figure 4.20). We observe that the path η remains strictly under the x-axis.

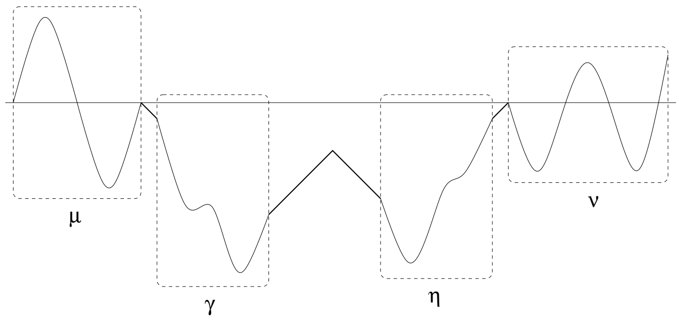

Figure 4.20: A graphical representation of the path ω in the case $h < j$

Let l be the rightmost point of the path $\overline{x}\gamma x^{j+1}\overline{x}^j\eta x$ with lowest ordinate. The path ω' which kills ω is obtained by applying to the path μ the following consecutive actions: add the path in $\gamma x^{j+1}\overline{x}^j\eta x$ running from l to the endpoint of the path, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), add the path in $\gamma x^{j+1}\overline{x}^j\eta x$ running from its initial point to l, apply consecutive and appropriate mappings of the first and of the second rule of (3.14), apply the cut and paste actions giving the first label (0) in case of marked points. Obviously the last forbidden pattern in the path must be generated by applying the mapping of the second rule of (3.14). The path ν in ω' is obtained as in ω.

We observe that for each path ω in $F \backslash F^{[\mathfrak{p}]}$, having C forbidden patterns, with n rise steps and last label (k), exists a unique path ω' in $F \backslash F^{[\mathfrak{p}]}$ with n rise steps, C forbidden patterns and last label $(\overline{k})$ having the same form of ω but such that the last forbidden pattern is marked if it is not in ω and vice-versa. This assertion is an immediate consequence of the constructions in the proof, since the described actions are uniquely determined. Therefore, it is not possible to obtain a path ω' which kills a given path ω applying two distinct procedures. ■

5 Conclusions and further developments

In this paper we have studied the enumeration, according to the number of ones, of particular binary words excluding a fixed pattern $\mathfrak{p} = 1^{j+1}0^j$, $j \geq 1$. Initially, we have solved the problem algebraically by means of Riordan arrays. This approach allows us to obtain a jumping and marked succession rule describing the growth of such words. By the way, it is not possible to associate to a word a walk in the generating tree obtained by the succession rule. This problem is solved by means of an algorithm constructing all the binary words having a fixed number of ones and eliminating the words which contain the forbidden pattern $\mathfrak{p} = 1^{j+1}0^j$, $j \geq 1$.

Another presumably fertile line of research concerns the generalization of the forbidden pattern $\mathfrak{p} = 1^{j+1}0^j$, $j \geq 1$.

In particular, is it possible to modify the previous algorithm to study the class F of binary words avoiding the pattern $\mathfrak{p} = 1^j0^i$, $j > i > 0$?
Using the same approach in the paper, is it possible to find other forbidden patterns $\mathfrak{p}$, which are not only formed by rise steps followed by fall steps, to study the enumeration of a new class of binary words avoiding the pattern $\mathfrak{p}$?
Moreover, is it possible to find a procedure which allows us to say that two classes of binary words avoiding distinct patterns have the same enumeration?

Finally, another line of research concerns the development of theory through the expansion of the alphabet $\{0, 1\}$. By the way, it is possible to introduce other classes of combinatorial structures and consider different parameter for the enumeration.

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