EXTENDED ABSTRACT

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Preliminaries

Let σ be a permutation in S_n. We say that σ has a *descent* at position i whenever $\sigma(i) > \sigma(i+1)$. Analogously, we say that σ has an *ascent* at position i whenever $\sigma(i) < \sigma(i+1)$. It is well known that the permutations of length n with k descents are enumerated by the Eulerian numbers $\mathcal{E}(n,k)$ satisfying:

$$\mathcal{E}(n,k) = (n-k) \cdot \mathcal{E}(n-1,k-1) + (k+1) \cdot \mathcal{E}(n-1,k) \tag{1}$$

anchored by $\mathcal{E}(n,0) = 1$ and $\mathcal{E}(n,n-1) = 1$ for $n \geq 1$.

In [1] *minimal* permutations are defined as those permutations with k descents that contain no pattern with k descents, except themselves. Indeed, it is not very easy to explain this definition completely and therefore to use it. So, a more exhaustive characterization of minimal permutations is given in [1] and it can be summarized in the following theorem giving a local characterization of minimal permutations with k descents.

Theorem 1. *A permutation $\sigma \in S_n$ is minimal with k descents if and only if it has exactly k descents and its ascents $\sigma_i\sigma_{i+1}$ are such that $2 \leq i \leq n-2$ and $\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_{i+2}$ forms an occurrence of either the pattern 2143 or the pattern 3142.*

In this paper we study some properties of *non minimal* permutations of length n with $n-2$ descents.

From (1) with $k = n-2$ we can prove that:

Proposition 1. *The total number $\mathcal{N}$ of permutations of length n with $n-2$ descents is*

$$\mathcal{N} = 2^n - n - 1$$

In [1] it is proved that the number of *minimal* permutations of length n with $n-2$ descents is $2^n - n(n-1) - 2$. So, the number $\mathcal{NM}$ of *non minimal* permutations of length n with $n-2$ descents is:

$$\begin{aligned} \mathcal{NM} &= \mathcal{N} - (2^n - n(n-1) - 2) \\ &= 2^n - n - 1 - (2^n - n^2 + n - 2) \\ &= n^2 - 2n + 1 \end{aligned}$$

that is

$$\mathcal{NM} = (n-1)^2$$

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Generating non minimal permutations of length n with $n-2$ descents

Here we give a combinatoric proof of the following theorem.

Theorem 2. *The number of non minimal permutations of length n with $n-2$ descents is $(n-1)^2$.*

Proof. A permutation of length n with $n-2$ descents has only one ascent at position i, $i = 1, \dots n-1$. Therefore, if it is *non minimal*, it does not contain four elements which form the pattern 2143 or the pattern 3142, [1].
For $1 \leq i \leq (n-1)$, we generate the *non minimal* permutations of length n with the unique ascent at position i by the following procedure.

1. Set $\pi = n(n-1)\dots(n-i+1)(n-i)\dots 321$;

2. the first permutation π_0 is obtained swapping $\pi(i)$ with $\pi(i+1)$;

3. π_0 generates $(n-i-1)$ permutations π_j such that π_j is obtained from π_{j-1} swapping $\pi_{j-1}(i)$ with $\pi_{j-1}(i+j+1)$, $(1 \leq j \leq n-i-1)$:

$$\begin{aligned}
\pi_0 &= n(n-1)\dots(n-i+2)(n-i)(n-i+1)(n-i-1)\dots 1 \\
\pi_1 &= n(n-1)\dots(n-i+2)(n-i-1)(n-i+1)(n-i)\dots 1 \\
&\vdots \\
\pi_{n-i-1} &= n(n-1)\dots(n-i+2)1(=\pi_n)(n-i+1)(n-i)\dots 2
\end{aligned}$$

4. the first $(i-1)$ entries of π_0 are in decreasing order and they are greater than both $(n-i)$ and $(n-i+1)$. So, π_0 generates $(i-1)$ more permutations $\bar{\pi}_k$ such that $\bar{\pi}_k$ is obtained from $\bar{\pi}_{k-1}$ swapping $\bar{\pi}_{k-1}(i-k)$ with $\bar{\pi}_{k-1}(i+1)$ $(\bar{\pi}_0 = \pi_0)$:

$$\begin{aligned}
\bar{\pi}_1 &= n(n-1)\dots(n-i+3)(n-i+1)(n-i)(n-i+2)(n-i-1)\dots 1 \\
\bar{\pi}_2 &= n(n-1)\dots(n-i+2)(n-i+1)(n-i)(n-i+3)(n-i-1)\dots 1 \\
&\vdots \\
\bar{\pi}_{i-1} &= (n-1)\dots(n-i+2)(n-i+1)(n-i)n(n-i-1)\dots 1.
\end{aligned}$$

Then for any i such that $1 \leq i \leq (n-1)$ we generate $1+(n-i-1)+(i-1) = n-1$ permutations.

Example 1. *Let be $n = 6$. The $(n-1)^2 = 25$ permutations obtained by applying the above procedure are listed in Figure 1.* □

In order to prove that the obtained permutations are *non minimal*, we will show that they belong to the set of avoiding permutations $S_n(2143, 3142)$.

□

Some statistics on non minimal permutations of length n with $n-2$ descents

Proposition 2. *We will prove that (see Table 1.a):*

1. *the permutations with the first entry equal to $1, 2, \dots, (n-2)$ are 1, for each value;*

2. *the permutations with the first entry equal to $(n-1)$ are $(n-1)$;*

3. *the permutations with the first entry equal to n are $(n-2)^2$.* □

Proposition 3. *Every column in Table 1.b is opposite versus the corresponding column in Table 1.a, that is, it is read from bottom to top.* □

$$
i=1 \begin{cases} \pi_0 & = & 564321 \\ \pi_1 & = & 465321 \\ \pi_2 & = & 365421 \\ \pi_3 & = & 265431 \\ \pi_4 & = & 165432 \end{cases}
\quad
i=2 \begin{cases} \pi_0 & = & 645321 \\ \pi_1 & = & 635421 \\ \pi_2 & = & 625431 \\ \pi_3 & = & 615432 \\ \bar{\pi}_1 & = & 546321 \end{cases}
\quad
i=3 \begin{cases} \pi_0 & = & 653421 \\ \pi_1 & = & 652431 \\ \pi_2 & = & 651432 \\ \bar{\pi}_1 & = & 643521 \\ \bar{\pi}_2 & = & 543621 \end{cases}
$$

$$
i=4 \begin{cases} \pi_0 & = & 654231 \\ \pi_1 & = & 654132 \\ \bar{\pi}_1 & = & 653241 \\ \bar{\pi}_2 & = & 643251 \\ \bar{\pi}_3 & = & 543261 \end{cases}
\quad
i=5 \begin{cases} \pi_0 & = & 654312 \\ \bar{\pi}_1 & = & 654213 \\ \bar{\pi}_2 & = & 653214 \\ \bar{\pi}_3 & = & 643215 \\ \bar{\pi}_4 & = & 543216 \end{cases}
$$

Figure 1: *Non minimal* permutations of length 6 with 4 descents

n / $\pi(1)$	3	4	5	6	7	8	n / $\pi(n)$	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	4	9	16	25	36
2	2	1	1	1	1	1	2	2	3	4	5	6	7
3	1	3	1	1	1	1	3	1	1	1	1	1	1
4		4	4	1	1	1	4		1	1	1	1	1
5			9	5	1	1	5			1	1	1	1
6				16	6	1	6				1	1	1
7					25	7	7					1	1
8						36	8						1

a) **b)**

Table 1: a): number of permutations of length $n = 3, 4 \ldots, 8$ with *first* entry $1, 2 \ldots, 8$; b): number of permutations of length $n = 3, 4 \ldots, 8$ with *last* entry $1, 2 \ldots, 8$

A first bijection

Given $\pi \in S_n$, the *reverse* and *complement* permutations π^r and π^c are defined as follows: $\pi^r(i) = \pi(n+1-i)$ and $\pi^c(i) = n+1-\pi(i)$, for $i = 1, \ldots, n$.
Let Σ^1 be the set of *non minimal* permutations σ of length n with $n-2$ descents and with $\sigma(1) = (n-1)$; let Σ^2 be the set of *non minimal* permutations τ of length n with $n-2$ descents and with $\tau(n) = 2$. We define the bijection

$$\varphi : \Sigma^1 \longmapsto \Sigma^2$$

such that

$$\varphi(\sigma) = (\sigma^r)^c$$

Proposition 4. *$\varphi(\sigma)$ is a permutation in Σ^2, that is: $\varphi(\sigma)$ has $n-2$ descents, it is non minimal and its last entry is* 2. □

Therefore, let $\tau \in \Sigma^2$ be a permutation with ascent at position i, that is $\tau(i) = 1$. Then the shape of τ is

$$n\,(n-1)\,(n-2)(\,n-3)\,\ldots\,(n-i+2)\,1\,(n-i+1)\,\ldots\,2$$

and there is a permutation $\sigma \in \Sigma^1$ with ascent in position $n-i$ whose shape is

$$(n-1)\,(n-2)\,\ldots\,i\,n\,(i-1)\,\ldots\,2\,1$$

such that $(\sigma^r)^c = \tau$.
Let Σ_n be the set of all the *non minimal* permutations of length n with $n-2$ descents.

Proposition 5. *The bijection φ can be applied to the set Σ_n:*

$$\varphi : \Sigma_n \longmapsto \Sigma_n$$

and $\forall \sigma \in \Sigma_n$, such that $\sigma(1) = \ell$, then the last entry of $\varphi(\sigma)$ is $(n + 1 - \ell)$. $\square$

"Expansion" generation of the non minimal permutations of length n with $n - 2$ descents

Remark 1. *From Table 1.a it follows that the non minimal permutations σ_i of length $(n + 1)$ (with $n - 1$ descents) such that $\sigma_i(1) = (n + 1)$ are as many as the non minimal permutations π_j of length n with $(n - 2)$ descents.* $\square$

On the base of Remark 1, we define Algorithm A to generate the n^2 *non minimal* permutations of length $(n + 1)$ with $(n - 1)$ descents from the ones of length n.
Algorithm A

1. From every permutation $\pi_j \in \Sigma_n$ generate a permutation $\sigma_i \in \Sigma_{n+1}$ inserting $(n + 1)$ on the left of $\pi_j(1)$;

2. let $\pi_j \in \Sigma_n$ be a permutation such that $\pi_j(1) < n$; from π_j generate a permutation $\sigma_i \in \Sigma_{n+1}$ increasing every element of π_j by 1 and adding 1 on its right;

3. insert in the set Σ_{n+1} the permutation $n\,(n - 2) \ldots 2\,1\,(n + 1)$, that is, the permutation ending with $(n + 1)$ and with the ascent at position n;

4. insert in the set Σ_{n+1} the permutation $1\,(n + 1)\,n \ldots 2$, that is, the permutation having 1 as first entry and with the ascent at position 1. $\square$

The permutations generated at step 1. are $(n - 1)^2$, since the cardinality of Σ_n is $(n - 1)^2$. The permutations in Σ_n with n as first entry are $(n - 2)^2$, so the ones with the first entry less than n are:

$$(n - 1)^2 - (n - 2)^2 = 2n - 3$$

Then, the number of the permutations of length $(n + 1)$ generated by Algorithm A is:

$$(n - 1)^2 + (2n - 3) + 1 + 1 = n^2 - 2n + 1 + 2n - 3 + 2 = n^2$$

Table 2 shows the permutations generated by applying Algorithm A for $n = 2, 3, 4, 5$; the permutations generated at step 1. are **bold** typed, while the *italic* ones are the permutations inserted in Σ_{n+1} at steps 3. and 4.

						45321	35421	25431	*15432*
			3421	2431	*1432*	**53421**	**52431**	**51432**	*43215*
	231	*132*	**4231**	**4132**	*3214*	**54231**	**54132**	**53214**	43251
12	**312**	*213*	**4312**	**4213**	3241	**54312**	**54213**	**53241**	43521
$n = 2$	$n = 3$		$n = 4$			$n = 5$			

Table 2: "Expansion" generation of Σ_n

Random generation of the non minimal permutations of length n with $n-2$ descents

We can think the $(n-1)^2$ *non minimal* permutations of length n with $n-2$ descents as the nodes of a square lattice of size $(n-1) \times (n-1)$ numbered as in the following schema, related to $n = 5$.

$$\begin{array}{cccc} 10 & 11 & 12 & 16 \\ 5 & 6 & 9 & 15 \\ 2 & 4 & 8 & 14 \\ 1 & 3 & 7 & 13 \end{array}$$

The $(n-2)^2$ permutations on the left bottom square (see Table 2), that is the numbers from 1 to 9 in the example, are generated at step 1. of Algorithm A; the $(n-2)$ permutations on the leftmost columns of the top row and the $(n-3)$ ones on the lowest rows of the rightmost column (numbers from 10 to 14, in the example) are generated at step 2. of Algorithm A. Finally, the two permutations on the last rows of the rightmost column (numbers 15 and 16) are generated at steps 3. and 4. of Algorithm A.
Let

$$\Delta : \Sigma_n \rightarrow \Sigma_{n+1}$$

be the operator such that $\tau = \Delta(\sigma)$ is the permutation obtained increasing every entry of σ by 1 and adding 1 on its right (see step 2. of Algorithm A) and let k be an integer such that $1 \leq k \leq (n-1)^2$.
The procedure *gen*(n, k) in Figure 2 generates the k-th permutation of length n. The operation '$\cdot$' stands for concatenation.

```
procedure gen(n, k)
begin
if n = 2
then σ := 1 2;
      return σ
else  if k ≤ (n − 2)^2
      then σ := n·gen(n − 1, k);
            return σ
      else
            if k < (n − 1)^2 − 1
            then k := k − (n − 2)^2;
                  if k < (n − 2)
                  then σ := Δ(gen(n − 1, (n − 3)^2 + k));
                        return σ
                  else
                        if k = (n − 2)
                        then σ := Δ(gen(n − 1, k^2));
                              return σ
                        else σ := Δ(gen(n − 1, (n − 3)^2 + k − 1));
                             return σ
            else
                  if k = (n − 1)^2 − 1
                  then σ := (n − 1) (n − 2) ... 1 n;
                        return σ
                  else
                        σ := 1 n (n − 1) ... 2;
                        return σ
end
```

Figure 2: Procedure *gen*(n, k)

Another bijection

Let Π_{n-1} be the set of permutations of 2 distinct letters, `a,b`, each with $n-1$ copies and with 2 fixed points. A *fixed point* is the occurrence of a letter `a` in a position $i \in [1\,..\,n-1]$ or the occurrence of a letter `b` in a position $j \in [n\,..\,2n-2]$. The cardinality of Π_{n-1} is $(n-1)^2$ (see sequence A000290 of [2]).
We define the following bijection ϕ between the sets Σ_n and Π_{n-1}. Let $\sigma \in \Sigma_n$ be a permutation with the ascent at position i; then $\tau = \phi(\sigma)$ is obtained in the following way:

place a letter `b` at position $n-1-i$ (the position of the ascent in σ determines the position of the last `b` in τ);

let k be the number of consecutive elements greater than $\sigma(i)$ and that are on the left of $\sigma(i)$; place the first `a` at position $j = \sigma(i) + k$;

complete τ so as to respect the properties of the permutations in Π_{n-1}. $\square$

Example 2. *Let* $\sigma = 643251$ *be a permutation of* Σ_6. *The ascent of* σ *is at position* 4, *then the last* `b` *in the permutation* τ *of length* 10 *is at position* 9. $\sigma(4) = 2$ *and there are two consecutive elements greater than* 2 *on the left of* $\sigma(4)$, *then the first* `a` *is at position* 4. *Finally, we have* $\tau = \phi(\sigma) =$ `bbbabaaaba`. $\square$

"Expansion" generation of the permutations of Π_{m+1}

The permutations in Π_{m+1} can be generated from the ones in Π_m by way of the following Algorithm B.
Algoritmo B

1. From every permutation $\pi_j \in \Pi_m$ generate a permutation $\sigma_i \in \Pi_{m+1}$ inserting the pair '`ba`' in the "middle", that is `b` at position $m+1$ and `a` at position $m+2$;

2. let $\pi_j \in \Pi_m$ be a permutation that does not contain the pair '`ba`' in the middle; from π_j generate a permutation $\sigma_i \in \Pi_{m+1}$ inserting `b` on the left of $\pi_j(1)$ and `a` on the right of $\pi_j(2m)$;

3. insert in the set Π_{m+1} the permutation $\underbrace{\texttt{b}\ldots\texttt{b}}_{m}\underbrace{\texttt{aa}\ldots\texttt{a}}_{m+1}\texttt{b}$;

4. insert in the set Π_{m+1} the permutation $\texttt{a}\underbrace{\texttt{bb}\ldots\texttt{b}}_{m+1}\underbrace{\texttt{aa}\ldots\texttt{a}}_{m}$. $\square$

In Table 3 are shown the permutations of Π_m for $m = 1, 2, 3$ and 4.

						bbbaabaa	bbabbaaa	babbbaaa	*abbbbaaa*
			bbaaba	babbaa	*abbbaa*	**bbabaaba**	**babbabaa**	**abbbabaa**	*bbbaaaab*
	baba	*abba*	**bababa**	**abbaba**	*bbaaab*	**babbaaba**	**abbbaaba**	**bbabaaab**	bbbaaaba
ab	**abab**	*baab*	**abbaab**	**babaab**	bbaaba	**abbbaaab**	**babbaaab**	**bbabaaba**	bbbaabaa
$m=1$	$m=2$			$m=3$				$m=4$	

Table 3: "Expansion" generation of Π_m

References

[1] M. Bouvel, E. Pergola, Posets and Permutations in the Duplicated-Loss Model: Minimal Permutations with d Descents, arXiv:0806.1494vl (2008).

[2] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/∼ njas/sequences/index.html