The covariogram problem

THE COVARIOGRAM PROBLEM

GABRIELE BIANCHI

1. Introduction

Let X be a Lebesgue-measurable set of finite measure in the Euclidean space $\mathbb{R}^n$. The *covariogram* g_X of X is the function on $\mathbb{R}^n$ defined by

$$g_X(x) = \mathcal{H}^n(X \cap (X + x)), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. This function was introduced by Matheron in his book [52, Section 4.3] on random sets and it coincides with the *autocorrelation* of the characteristic function 1_X of X; that is,

$$g_X = 1_X * 1_{(-X)}. \tag{1.2}$$

The covariogram g_X is clearly unchanged with respect to translations and reflections of X, where, throughout the paper, *reflection* means reflection in a point. In 1986 Matheron [51, p. 20] asked the following question and conjectured a positive answer for the case $n = 2$ (Matheron's Conjecture).

Covariogram Problem. Does the covariogram determine a convex body in $\mathbb{R}^n$, among all convex bodies, up to translations and reflections?

We recall that a *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior, and we refer to the next section for all unexplained definitions. More generally, what information about a set, not necessarily convex, can be obtained from its covariogram? The covariogram appears in very different contexts, and the covariogram problem can be rephrased in different terms. Indeed, it is equivalent to any of the following problems.

Problem 1.1. Determine a convex body K from the knowledge, for each unit vector u in $\mathbb{R}^n$, of the distribution of the lengths of the chords of K parallel to u.

Problem 1.2. Determine a convex body K from the distribution of $W - Z$, where W and Z are independent random variables uniformly distributed over K.

Problem 1.3. Determine the characteristic function 1_K of a convex body K from the modulus of its Fourier transform $\widehat{1_K}$.

In Problem 1.1 a random chord parallel to u is obtained by taking the intersection of K with a random line L_u parallel to u, conditioned on $K \cap L_u \neq \emptyset$. Matheron [52, p. 86] explained the relation between Problem 1.1 and the covariogram of a set; see also Nagel [55]. Remark 2.3 in the next section explains this equivalence in detail. Blaschke [62, Section 4.2] asked whether the distribution of the lengths of *all* chords (that is, not separated direction by direction) of a planar convex body determines that body, up to isometries in $\mathbb{R}^2$. Mallows and Clark [50] constructed polygonal examples that show that the answer is negative in general. Gardner, Gronchi and Zong [33] observed that the distribution of the lengths of the chords of K parallel to u coincides, up to a multiplicative factor, with the *rearrangement* of the X-ray of K in the direction u, and rephrased Problem 1.1 in these terms.

Problem 1.2 was asked by Adler and Pyke [1] in 1991. By (1.2) we see that the distribution of $W-Z$ coincides with $g_K/\mathcal{H}^n(K)^2$. Since $g_K(o)=\mathcal{H}^n(K)$, knowing the covariogram is equivalent to knowing the distribution of $W-Z$. In stochastic geometry, integrals of functions of $W-Z$ can be written in terms of the covariogram, as in the formula (called Borel's overlap formula in [21])

$$\int_{X\times X} f(x-y)\,dx\,dy=\int_{\mathbb{R}^n} f(z)\,g_X(z)\,dz,$$

valid, say, when $X\in\mathcal{L}^n$ is bounded and $f\in L^1_{loc}(\mathbb{R}^n)$. Recent studies showing connections between the covariogram and stereology and stochastic geometry are [22], [37], [57] and [58].

Problem 1.3 is a special case of the *Phase Retrieval Problem*, where 1_K is replaced by a function with compact support. The equivalence of the problems comes by applying the Fourier transform to (1.2). One obtains, for $x\in\mathbb{R}^n$,

$$\widehat{g_K}(x)=\widehat{1_K}(x)\widehat{1_{-K}}(x)=\widehat{1_K}(x)\overline{\widehat{1_K}(x)}=|\widehat{1_K}(x)|^2. \tag{1.3}$$

The Phase Retrieval Problem arises naturally and frequently in various areas of science, such as X-ray crystallography, electron microscopy, optics, astronomy and remote sensing, in which only the magnitude of the Fourier transform can be measured and the phase is lost. The literature is vast; see the surveys [42], [44], [48], [54] and [59], as well as the references given there. Phase retrieval is fundamentally under-determined without additional constraints, and our problem is the Phase Retrieval Problem with the constraint $f=1_K$, the characteristic function of a convex body K in $\mathbb{R}^n$.

Baake and Grimm [8] have observed that the Covariogram Problem is relevant for the inverse problem of finding the atomic structure of a quasicrystal from its X-ray diffraction image. It turns out that quasicrystals can often be described by means of the so-called cut-and-project scheme. In this scheme a quasiperiodic discrete subset S of $\mathbb{R}^m$, $m\in\mathbb{N}$, which models the atomic structure of a quasicrystal, is described as the orthogonal projection of $Z\cap(\mathbb{R}^m\times W)$ onto $\mathbb{R}^m$, where W is a subset of $\mathbb{R}^n$ and Z is a lattice in $\mathbb{R}^m\times\mathbb{R}^n$. For many quasicrystals, the lattice Z can be recovered from the diffraction image of S. Thus, in order to determine S, it is necessary to know W. The Covariogram Problem enters at this point, since g_W can be obtained from the diffraction image of S. Note that the set W is in many cases a convex body.

The first partial solution of Matheron's Conjecture was given by Nagel [55] in 1993, who confirmed it for all convex polygons. A complete positive answer in the plane is contained in the combined works of Bianchi [14] and Averkov and Bianchi [4]. In higher dimensions, the first result was a positive one valid for a class of convex polytopes by Goodey, Schneider and Weil [38]. This result implies that most (in the sense of Baire category) convex bodies are determined by their covariograms. It is easy to see that centrally symmetric convex bodies are determined. Bianchi [15] proved a positive answer for convex polytopes in $\mathbb{R}^3$, and in [14] found counterexamples in $\mathbb{R}^n$, for any $n\geq 4$, which may be chosen to be polytopes. Whether arbitrary convex bodies in $\mathbb{R}^3$ are determined is not yet known. Regarding strictly convex bodies $K\subset\mathbb{R}^n$, $n\geq 3$, of class C^m_+, Bianchi [18] proves a positive answer when m is higher than a threshold which depends on n.

Methods of Fourier analysis are relevant in attacking the Covariogram Problem. Studies of the asymptotic behaviour of $\widehat{1_K}(\xi)$, where $\xi \in \mathbb{R}^n$ and $|\xi| \to \infty$, initiated by Haviland and Wintner [39], provide information about curvatures, for strictly convex and sufficiently smooth convex bodies, at points of the boundary with opposite outer normals (see Section 3.1 for the planar case). For the same class of convex bodies, studies of the zero set of $\widehat{1_K}(\zeta)$, seen as a function of $\zeta \in \mathbb{C}^n$, by Kobayashi [45] play a fundamental role in resolving some ambiguities in their determination (see Section 3.5). Some results on the Phase Retrieval Problem applied to our situation show a connection between the determination of K and the irreducibility of $\widehat{1_K}$. In Section 7 we explain this and also why this property of of $\widehat{1_K}$ is related to the Pompeiu Problem in integral geometry.

Section 3 is devoted to the Covariogram Problem and is divided into subsections. Section 3.1 explains the ideas behind the proof of the positive result in the plane; Section 3.4 does the same for the result on polytopes in $\mathbb{R}^3$; Sections 3.2 and 3.3 treat, respectively, the examples of nondetermination in dimension $n \geq 4$ and the results for convex polytopes in dimension $n \geq 3$. Section 3.6 presents the associated problem of determination from the cross-covariogram. The *cross-covariogram* $g_{K,L}$ of two measurable sets $K, L \subset \mathbb{R}^n$ is the function that associates to $x \in \mathbb{R}^n$ the volume $\mathcal{H}^n(K \cap (L + x))$. It appears naturally in our study, and it is also natural to ask whether $g_{K,L}$ determines both K and L, up to the inherent ambiguities. Surprisingly, in certain classes of sets it does. One such class is that of C^8_+-regular planar convex bodies. The case of convex polygons is also completely solved, with the understanding of exactly which pairs are determined and which pairs are not. Section 4 is devoted to algorithms for reconstruction. Section 5 presents examples of information that can be obtained from the covariogram of sets that are not necessarily convex. In particular, it deals with the possibility of recognizing whether a set is convex, whether it is centrally symmetric, and whether it is radial. Section 6 is devoted to the counterpart of the covariogram in the discrete case. We explain the relation between the continuous and the discrete covariogram and how this was used in [33] to construct a pair of noncongruent nonconvex polygons with equal covariograms. Later a whole family of such pairs was found; each set in these pairs is a horizontally and vertically convex union of lattice squares. These examples show that the convexity assumption in the Covariogram Problem cannot be significantly weakened. Baake and Grimm [8] use the pair in [33] to construct two different quasicrystal model sets in $\mathbb{R}^2$ with equal diffraction images.

Some aspects of the Covariogram Problem have been neglected in this survey. We briefly mention two.

In many situations where we have a positive answer to the Covariogram Problem, knowledge of the covariogram on a proper subset of its domain still suffices. For instance, the results in Bianchi, Gardner and Kiderlen [19] show that if a convex body is determined, up to translations and reflections, by its covariogram, then it is also so determined by its values at certain countable sets of points, even, almost surely, when these values are contaminated with noise. The recent paper by Engel and Laasch [25] proves that if $P, P' \subset \mathbb{R}^3$ are convex polytopes and $E \subset \mathbb{R}^3$ is the nonempty intersection of an open set with a sphere and $|\widehat{1_P}(\xi)| = |\widehat{1_{P'}}(\xi)|$, for each $\xi \in E$, then $P = P'$, up to translations and reflections. Averkov and Bianchi [3] also find some results where restricted information about the covariogram is sufficient for determination.

Substituting, in the definition of the covariogram, the volume with a different functional, like the surface area or other valuations, one defines different covariograms, and for each of them there is a corresponding covariogram problem. Averkov and Bianchi [5] study, for planar convex bodies, the problems associated to the perimeter covariogram and to the width covariogram.

2. Definitions, notations and preliminaries

As usual, S^{n-1} denotes the unit sphere, B^n the unit ball and o the origin in the Euclidean n-space $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then $\langle x, y\rangle$ is the scalar product of x and y, while $|x|$ is the norm of x. If $\zeta \in \mathbb{C}^n$ and $\zeta = x + \mathrm{i}\, y$, with $x, y \in \mathbb{R}^n$, then $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ denote respectively x and y. Moreover $|\zeta| = (|\operatorname{Re}\zeta|^2 + |\operatorname{Im}\zeta|^2)^{1/2}$ denotes the norm of ζ. For $\delta > 0$, $B(x, \delta)$ denotes $\{y \in \mathbb{R}^n : |y - x| < \delta\}$.

If X and Y are sets in $\mathbb{R}^n$, we denote by $\operatorname{lin} X$, $\operatorname{aff} X$, $\operatorname{conv} X$, $\operatorname{cl} X$, $\operatorname{int} X$ and 1_X the *linear hull*, *affine hull convex hull*, *closure*, *interior* and *characteristic function* of X, respectively. Also, $\operatorname{relint} X$ is the *relative interior* of X, that is, the interior of X relative to $\operatorname{aff} X$. The symbol $|X|$ denotes the *cardinality* of X. If $t \in \mathbb{R}$, then $tX = \{tx : x \in X\}$, $X + Y = \{x + y : x \in X, y \in Y\}$ denotes the *Minkowski sum* of X and Y, and

$$DX = X + (-X)$$

the *difference set* of X. A set is *o-symmetric* if it is centrally symmetric, with center at the origin.

A *lattice set* is a finite subset of $\mathbb{Z}^n$ and a *lattice body* is a subset P of $\mathbb{R}^n$ which can be written as $P = A + [0,1]^n$, where A is a lattice set. We call P the lattice body associated to A and A the lattice set associated to P. A lattice set whose associated lattice body has connected interior is called a *polyomino*. Lattice bodies associated to polyominoes are called *lattice animals* (or polyominoes, by many authors). A lattice set A is *convex* if $A = (\operatorname{conv} A) \cap \mathbb{Z}^n$.

If $u \in S^{n-1}$ then $u^\perp = \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$. If E is a linear subspace of $\mathbb{R}^n$, then $X|E$ is the orthogonal projection of X on E and $x|E$ is the projection of a vector $x \in \mathbb{R}^n$ on E. The symbol $\mathcal{R}_{\frac{\pi}{2}}$ denotes a counterclockwise rotation in $\mathbb{R}^2$ by $\pi/2$.

For $i \in \{0, 1, \dots, n\}$, we write $\mathcal{H}^i$ for i-dimensional Hausdorff measure in $\mathbb{R}^n$. We write ω_n for the surface area of the unit ball in $\mathbb{R}^n$. We denote by $\mathcal{C}^n$, $\mathcal{M}^n$ and $\mathcal{L}^n$ the class of nonempty compact sets, $\mathcal{H}^n$-measurable sets and $\mathcal{H}^n$-measurable sets of finite measure, respectively, in $\mathbb{R}^n$. A compact set X is *regular* if $X = \operatorname{cl}\ \operatorname{int} X$. The *Hausdorff distance* $\delta(C, D)$ between two sets $C, D \in \mathcal{C}^n$ is defined as

$$\delta(C, D) = \min\{\varepsilon \geq 0 : C \subset D + \varepsilon B^n, D \subset C + \varepsilon B^n\}.$$

Let $\mathcal{K}^n$ be the class of nonempty compact convex subsets of $\mathbb{R}^n$ and let $\mathcal{K}^n_n$ be the class of *convex bodies*, i.e., members of $\mathcal{K}^n$ with interior points. The treatise of Schneider [66] is an excellent general reference for convex geometry. For $K \in \mathcal{K}^n$, the function

$$h_K(u) = \max\{\langle u, y\rangle : y \in K\},$$

for $u \in \mathbb{R}^n$, is the *support function* of K,

$$w_K(u) = h_K(u) + h_K(-u),$$

its *width function* and

$$b_K(u) = \mathcal{H}^{n\text{-}1}(K|u^\perp),$$

for $u \in S^{n-1}$, its *brightness function*. Note that

$$w_{DK} = 2\, w_K\,.$$

Any $K \in \mathcal{K}^n$ is uniquely determined by its support function. Given $u \in S^{n-1}$, the *support set of* K *in direction* u is

$$K_u = \{x \in K : \langle x, u\rangle = h_K(u)\}.$$

The support sets are also called *exposed faces of* K. Note that [66, Theorem 1.7.5(c)]

$$(DK)_u = K_u + (-K)_u. \tag{2.1}$$

The *Blaschke body* ∇K of $K \in \mathcal{K}^n_n$ is the unique o-symmetric convex body such that

$$b_{\nabla K} = b_K.$$

We say that a convex body K is in the class C^m, for $m \in \mathbb{N}$, if ∂K is an m-differentiable manifold, and write $K \in C^m_+$, for $m \geq 2$, if $K \in C^m$ and the Gauss curvature of ∂K is positive everywhere. We say that $K \in C^\infty_+$ if $K \in C^m_+$ for each $m \in \mathbb{N}$. When $K \in C^2_+$, $\nu_K : \partial K \to S^{n-1}$ denotes the *Gauss map* and $\tau_K(u)$ denotes the *Gauss curvature* of ∂K at the point $\nu_K^{-1}(u)$ in ∂K with outer normal $u \in S^{n-1}$.

Given a face F of a convex polytope $P \subset \mathbb{R}^n$, the *normal cone* of P at F is denoted by $N(P,F)$ and is the set of all outer normal vectors to P at x, where $x \in \operatorname{relint} F$, together with o. The *support cone of* P *at* F is the set

$$\operatorname{cone}(P,F) = \{a(y-x)\ :\ y \in P\ ,\ a \geq 0\},$$

where $x \in \operatorname{relint} F$. Neither definition depends on the choice of x. Two faces F and G of P are *antipodal* if $\operatorname{relint} N(P,F) \cap (-\operatorname{relint} N(P,G)) \neq \emptyset$.

If $Y \in \mathcal{M}^n$,

$$\Theta(Y,x) = \lim_{r\to 0^+} \frac{\mathcal{H}^n\big(Y \cap (rB^n + x)\big)}{\mathcal{H}^n(rB^n + x)}, \tag{2.2}$$

is the *density of* Y *at* x, provided the limit exists. For $t \in [0,1]$, define $Y^t = \{x \in \mathbb{R}^n : \Theta(Y,x) = t\}$. The *essential boundary* $\partial^e Y$ *of* Y is $\partial^e Y = \mathbb{R}^n \setminus (Y^0 \cup Y^1)$. The *perimeter* $\operatorname{Per} Y$ *of* Y is $\operatorname{Per} Y = \mathcal{H}^{n-1}(\partial^e Y)$, while the *directional variation* $V_u(Y)$ *of* Y *in the direction* $u \in S^{n-1}$ is

$$V_u(Y) = \int_{u^\perp} \mathcal{H}^0\big(\partial^e Y \cap (l_u + x)\big)\, d\mathcal{H}^{n-1}(x),$$

where l_u is the line through o parallel to u. If $K \in \mathcal{K}^n_n$, then $\partial^e K = \partial K$, perimeter coincides with surface area, and $V_u(K) = 2b_K(u)$.

The *X-ray of* $Y \in \mathcal{L}^n$ *in the direction* $u \in S^{n-1}$ is defined for $\mathcal{H}^{n-1}$-almost all $x \in u^\perp$ by

$$X_u Y(x) = \mathcal{H}^1\big(Y \cap (l_u + x)\big).$$

Given a function f defined on a subset of $\mathbb{R}^n$, $\operatorname{supp} f$, ∇f and $D^2 f$ denote its support, its gradient and its Hessian, respectively. We say that $f \in C^\infty_0(\mathbb{R}^n)$ if f is m-times differentiable for each $m \in \mathbb{N}$ and $\operatorname{supp} f$ is compact.

An *entire function* is a complex-valued function that is holomorphic over the whole $\mathbb{C}^n$. An entire function f is of *exponential type* if there exist $a, b \in \mathbb{R}$ and $m \in \mathbb{Z}$ such that $|f(\zeta)| \leq a(1+|\zeta|)^m e^{b|\operatorname{Im}\zeta|}$, for each $\zeta \in \mathbb{C}^n$.

The *Fourier transform* of a function $f \in L^2(\mathbb{R}^n)$ with compact support is defined for $\zeta \in \mathbb{C}^n$ as

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} e^{\mathrm{i}\langle x,\zeta\rangle} f(x)\, dx.$$

By the Paley–Wiener Theorem, $\widehat{f}$ is an entire function of exponential type whose restriction to $\mathbb{R}^n$ belongs to L^2. The version of this theorem for distributions asserts that $\widehat{f}$ is an entire function of exponential type if and only if f is a distribution with compact support. See [60, Theorem 7.23]. Distributions will enter this paper only very marginally and we refer to Rudin [60] for their definition.

Taking Fourier transforms in (1.2) for $\zeta \in \mathbb{C}^n$ gives the relation

$$\widehat{g_K}(\zeta) = \widehat{1_K}(\zeta)\,\overline{\widehat{1_K}\,(\overline{\zeta})}. \tag{2.3}$$

2.1. **Properties of the covariogram.** Given X, $Y \in \mathcal{M}^n$, the *cross covariogram* of X and Y is the function

$$g_{X,Y}(x) = \mathcal{H}^n(X \cap (Y + x)),$$

where $x \in \mathbb{R}^n$ is such that $\mathcal{H}^n(X \cap (Y + x))$ is finite. Clearly, $g_{X,X} = g_X$.

The translation of X and Y by the same vector, and the substitution of X with $-Y$ and of Y with $-X$, leave $g_{X,Y}$ unchanged. Let X' and Y' be in $\mathcal{M}^n$. We call (X, Y) and (X', Y') *trivial associates* when one pair is obtained from the other one via a combination of the two operations above, that is, when either $(X, Y) = (X' + x, Y' + x)$ or $(X, Y) = (-Y' + x, -X' + x)$, for some $x \in \mathbb{R}^n$. When dealing with the ordinary covariogram the previous definition simplifies to the following one: X and X' are called *trivial associates* if $X' = X + x$ or $X' = -X + x$ for some $x \in \mathbb{R}^n$.

The following propositions list some properties of the covariogram.

Proposition 2.1. *Let $X \in \mathcal{L}^n$.*

a) For all $x \in \mathbb{R}^n$, $0 \le g_X(x) \le g_X(o) = \mathcal{H}^n(X)$.

b) The function g_X is even.

c) We have $\int_{\mathbb{R}^n} g_X(x)\, dx = \mathcal{H}^n(X)^2$.

d) The function g_X is uniformly continuous in $\mathbb{R}^n$ and $\lim_{|x|\to\infty} g_X(x) = 0$. Moreover, for all $x, y \in \mathbb{R}^n$,

$$|g_X(x) - g_X(y)| \le g_X(o) - g_X(x - y).$$

e) The right directional derivative of g_X at o in direction $u \in S^{n-1}$ can be expressed as

$$\frac{\partial^+ g_X}{\partial u}(o) = -\frac{1}{2} V_u(A). \tag{2.4}$$

f) The covariogram g_X is Lipschitz if and only if X has finite perimeter $\operatorname{Per} X$. Moreover, the Lipschitz constant of g_X equals $(1/2)\sup_{u\in S^{n-1}} V_u(X)$, and

$$\operatorname{Per} X = -\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \frac{\partial^+ g_X}{\partial u}(o)\, d\mathcal{H}^{n-1}(u).$$

These properties, in the generality of measurable sets, are proved in Galerne [29]. Some of them are immediate, like Items a), b) and c), while others were already known in the case of convex bodies [52] and of full-dimensional compact $\mathcal{U}_{PR}$ sets [56] (a family of sets that consists of certain unions of sets of positive reach).

Proposition 2.2. *Let $K, L \in \mathcal{K}_n^n$ and let $C, D \in \mathcal{C}^n$ be regular.*

a) We have $\operatorname{supp} g_C = DC$ *and* $\operatorname{supp} g_{C,D} = C + (-D)$.

b) The function $g_{K,L}^{1/n}$ *is concave on its support. In particular,* g_K *is log-concave.*

c) The following inequalities hold:

$$2^n g_C(o) \le \mathcal{H}^n(\operatorname{supp} g_C); \tag{2.5}$$

$$\mathcal{H}^n(\operatorname{supp} g_K) \le \binom{2n}{n} g_K(o). \tag{2.6}$$

Equality in (2.5) *holds precisely when* C *is convex and centrally symmetric, while equality in* (2.6) *holds precisely when* K *is a simplex.*

d) If $\mathcal{H}^{n-1}(\partial K \cap (\partial K + x)) = 0$ *then* $\nabla g_K(x)$ *exists and*

$$\nabla g_K(x) = -\int_{\partial K \cap (K+x)} \nu(y)\, d\mathcal{H}^{n-1}(y), \tag{2.7}$$

where $\nu(y)$ *denotes the unit outer normal vector to* $y \in \partial K$*, defined* $\mathcal{H}^{n-1}$*-almost everywhere. If* $u \in S^{n-1}$, $r > 0$ *and* $ru \in \operatorname{int} \operatorname{supp} g_K$*, then*

$$\frac{\partial g_K}{\partial u}(ru) = -\mathcal{H}^{n\text{-}1}\left(\big(K \cap (K + ru)\big)|u^\perp\right). \tag{2.8}$$

Moreover

$$\frac{\partial^+ g_K}{\partial u}(o) = -b_K(u). \tag{2.9}$$

For arbitrary sets $C, D \in \mathcal{L}^n$, Item a) is not valid, even in the case of the ordinary covariogram. The property $x \notin DC$ is equivalent to $C \cap (C + x) = \emptyset$ and therefore the inclusion $\operatorname{supp} g_C \subset DC$ is still valid. However, the other inclusion may be false. For instance, if C is the Cantor ternary set in $[0, 1]$ then $\operatorname{supp} g_C = \emptyset$, since $\mathcal{H}^1(C) = 0$, while $DC = [-1, 1]$. We give a proof of Item a) below, since we could not find one in the literature valid in the class of regular sets in $\mathcal{C}^n$.

Item b) was first observed by Gardner and Zhang [35] and we give their proof below. The set

$$K(\delta) = \{x \in \mathbb{R}^n : g_K(x) \ge \delta\}$$

is called the *convolution body* of K. This notion is due to Kiener [43] who noted that $K(\delta)$ is convex, as Item b) implies.

Formulas (2.5) and (2.6), together with their equality cases, are an immediate consequence, respectively, of the general Brunn–Minkowski inequality [30, p. 362] and of the Rogers–Shepard inequality [66, p. 530] together with Item a) and fact that the value of the covariogram at o equals the volume of the set.

Formula (2.9) shows that g_K provides the brightness function b_K of K. Formula (2.8) and the interpretation of the right-hand side of (2.8) as

$$-\mathcal{H}^{n\text{-}1}\left(\{x \in u^\perp : X_u K(x) \ge r\}\right), \tag{2.10}$$

proves a connection between the covariogram and the X-rays of a convex body first observed by Gardner, Gronchi and Zhong [33]: knowing g_K is equivalent to knowing the rearrangement of $X_u K$ for each $u \in S^{n-1}$.

Formulas (2.7) and (2.9) are present in Matheron [51]. Regarding the existence of ∇g_K, [51] simply writes that this happens almost everywhere, due to the concavity property in Item b). The existence of the derivatives in (2.7) and (2.8) is proved by Meyer, Reisner and Schmuckenschläger [53], who also deal with the second-order derivatives of g_K.

Proof. a). The property $x \notin C + (-D)$ is equivalent to $C \cap (D + x) = \emptyset$ and therefore $\operatorname{supp} g_{C,D} \subset C + (-D)$. If $x \in C + (-D)$, and $x = c - d$ with $c \in C$ and $d \in D$, then $c = d + x \in C \cap (D + x)$. For any $\varepsilon > o$ there exist $c' \in B(c, \varepsilon) \cap \operatorname{int} C$ and $d' \in B(d, \varepsilon) \cap \operatorname{int} D$, due to C and D being regular. If $y = c' - (d' + x)$ then $|y| \leq 2\varepsilon$ and $C \cap (D + x + y)$ has nonempty interior, since it contains $c' = d' + x + y$. Therefore $g_{C,D}(x + y) > 0$. Thus $B(x, 2\varepsilon)$ contains points where $g_{C,D}$ is positive, which proves $C + (-D) \subset \operatorname{supp} g_{C,D}$.

b). For $x, y \in \mathbb{R}^n$ and $a \in [0, 1]$, we have

$$\begin{aligned} K \cap \big(L + (1-a)x + ay)\big) &= K \cap \big((1-a)(L + x) + a(L + y)\big) \\ &\supset (1-a)\big(K \cap (L + x)\big) + a\big(K \cap (L + y)\big). \end{aligned}$$

Using the Brunn–Minkowski inequality we obtain

$$\begin{aligned} g_{K,L}((1-a)x + ay)^{1/n} &\geq \mathcal{H}^n\left((1-a)\big(K \cap (L + x)\big) + a\big(K \cap (L + y)\big)\right)^{1/n} \\ &\geq (1-a)\,\mathcal{H}^n\left(K \cap (L + x)\right)^{1/n} + a\,\mathcal{H}^n\left(K \cap (L + y)\right)^{1/n} \\ &= (1-a) g_{K,L}(x)^{1/n} + a g_{K,L}(y)^{1/n}. \end{aligned}$$

□

Remark 2.3. Some formulas in Proposition 2.1 and 2.2 explain the equivalence between the Covariogram Problem and Problem 1.1. Let Z_u be the length of the chord $L_u \cap K$, where L_u is a random line parallel to $u \in S^{n-1}$ conditioned on $L_u \cap K \neq \emptyset$. Formula (2.8), with the right-hand side interpreted as in (2.10), shows that the probability of the event $\{Z_u \geq r\}$, for $r > 0$, is equal to $-\left(\partial g_K/\partial u\right)(ru)/b_K(u)$. Integrating the latter expression with respect to r we determine $f(ru) = g_K(ru)/b_K(u)$. Consequently, the distribution of Z_u for each $u \in S^{n-1}$ determines $f(ru)/f(0u) = g_K(ru)/\mathcal{H}^n(K)$, for every $r > 0$ and every $u \in S^{n-1}$. The latter is equivalent to the determination of $g_K(x)/\mathcal{H}^n(K)$ for every $x \in \mathbb{R}^n$. Integration of $g_K/\mathcal{H}^n(K)$ over $\mathbb{R}^n$ yields $\mathcal{H}^n(K)$, by Proposition 2.1 c), and we determine g_K. Conversely, g_K determines the distribution of Z_u, by (2.9).

In the plane, ∇g_K has a simple geometric interpretation. Consider an arbitrary $x \in \operatorname{int} \operatorname{supp} g_K$ and assume that $\partial K \cap (\partial K + x)$ consists of two points. Then there exist points $p_i(x)$, $i \in \{1, \dots, 4\}$, in counterclockwise order on ∂K, such that $x = p_1(x) - p_2(x) = p_4(x) - p_3(x)$; see Figure 1. These points define a parallelogram

$$P(x) = \operatorname{conv}\{p_1(x), \dots, p_4(x)\} \tag{2.11}$$

inscribed in K, whose edges are translates of $[o, x]$ and $[o, D(x)]$, where

$$D(x) = p_1(x) - p_4(x).$$

Proposition 2.4. *Let $K, L \in \mathcal{K}_2^2$ and $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ be such that $\partial K \cap (\partial K + x)$ consists of two points. Then $\nabla g_K(x) = \mathcal{R}_{\frac{\pi}{2}}\, D(x)$.*

If $g_L = g_K$, then any parallelogram inscribed in K has a translate inscribed in L.

The representation of $\nabla g_K(x)$ in terms of $D(x)$ was already observed in [51]. The second claim is proved in [14] and is related to the first one. This is easily understood when K and L are strictly convex. In this case any parallelogram P inscribed in K coincides with $P(x)$ when x is chosen so that $[o, x]$ is a translate of an edge of P. The parallelograms $P(K, x)$ and $P(L, x)$ are translates of each

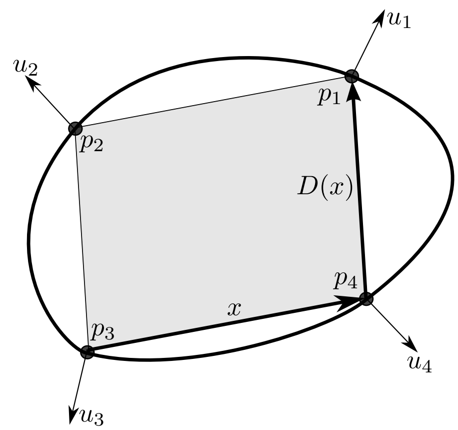

FIGURE 1. $P(x)$ and the vector $D(x)$, a rotation of $\nabla g_K(x)$.

other, since both have two edges that are translates of $[o, x]$ and two edges that are translates of $D(K, x) = D(L, x)$, where these vectors coincide due to the first claim.

3. The Covariogram Problem

The focus on covariograms of *convex* bodies is natural. There exist noncongruent nonconvex polygons, even (see Figure 7) horizontally and vertically convex ones, with the same covariogram, indicating that the convexity assumption cannot be significantly weakened. Sections 5 and 6 contains a discussion of this example and information on the determination of nonconvex sets (including the case of discrete sets) by the covariogram (or by its discrete version).

Another preliminary observation is that objects which are centrally symmetric with respect to some point are easy to determine, up to translations, in the class of centrally symmetric objects. If K_1 and K_2 are convex bodies which are centrally symmetric with respect to a_1 and a_2, respectively, and $g_{K_1} = g_{K_2}$, this follows from the formula

$$2(K_1 - a_1) = D(K_1 - a_1) = \operatorname{supp} g_{K_1} = \operatorname{supp} g_{K_2} = D(K_2 - a_2) = 2(K_2 - a_2).$$

An analogous result holds in a much larger class. Cabo and Janssen [23] prove that that if f and h are even functions in $L^1(\mathbb{R}^n)$ with compact support and with the same autocorrelation (i.e. $f(\cdot) * f(-\cdot) = h(\cdot) * h(-\cdot)$), then $f = \pm h$ almost everywhere. Note that the autocorrelation of the characteristic function of a set is its covariogram. This implies the following result, of which the elegant proof is taken from [33].

Theorem 3.1 (Cabo and Janssen [23]). *A centrally symmetric regular compact subset C of $\mathbb{R}^n$ is determined by g_C, up to translations, in the class of centrally symmetric regular compact sets.*

Proof. If D is a centrally symmetric regular compact set with $g_C = g_D$, then (1.3) implies $|\widehat{1_C}|^2 = |\widehat{1_D}|^2$. Up to translations, we may assume that C and D are o-symmetric. Fourier transforms of even functions are real valued and the previous condition becomes $(\widehat{1_C})^2 = (\widehat{1_D})^2$. Therefore $\widehat{1_C} = \pm\widehat{1_D}$. Since Fourier transforms of functions with compact supports are analytic and any analytic function is determined by its value on a set with a limit point we conclude that $\widehat{1_C} = \pm\widehat{1_D}$. Fourier inversion yields $1_C = 1_D$ almost everywhere and since C and D are regular we have $C = D$. □

We remark that here we are not asking whether a symmetric object is determined in the class of *all* objects. The answer to this last question is more subtle. It is known that a centrally symmetric convex body is determined in the class of all regular compact sets, but the same question is already open for the determination of a centrally symmetric regular compact set in the class of all regular compact sets. This will be explained in Section 5.2, when speaking of the possibility of recognizing the central symmetry of a set from its covariogram.

A great deal of effort has been spent on the determination of convex bodies from the combined information in their width and brightness functions. Since the covariogram of a convex body determines both functions, by Proposition 2.2 a) and (2.9), the question is directly related to the Covariogram Problem. The many known results do not add to what we know about the latter, but the interested reader can consult [31, Chapter 3 and Notes 3.3 and 3.6] and the update for [31] available on its author's website.

3.1. **Complete answer in the plane.** The first answer to the Covariogram Problem in the plane was a positive one for convex polygons proved by Nagel [55] in 1993. Bianchi [14] and Bianchi and Averkov [4] prove the following theorem, which confirms Matheron's Conjecture.

Theorem 3.2. *Every planar convex body is determined among all planar convex bodies by its covariogram, up to translations and reflections.*

This is the combination of the following two results.

Theorem 3.3 (Bianchi [14])**.** *Let K and L be planar convex bodies with equal covariograms, one of which is not strictly convex or not in C^1. Then $K = L$, up to translations and reflections.*

Theorem 3.4 (Averkov and Bianchi [4])**.** *Let K and L be planar strictly convex bodies in C^1 with equal covariograms. Then $K = L$, up to translations and reflections.*

In the rest of this section K and L will denote planar convex bodies with $g_K = g_L$.

A unified proof of Theorem 3.2 would be very welcome but it is still missing. The proofs of Theorems 3.3 and 3.4 both rely on two ingredients. One is the following.

Proposition 3.5 (Bianchi [14])**.** *If K and L have a common nondegenerate boundary arc, then $K = L$, up to translations and reflections.*

The other ingredient is the proof that K and L do have a boundary arc in common, up to translations and reflections. The proof of Proposition 3.5 is different according to whether the common arc is strictly convex or not, even though the structures of the proofs for the two cases are similar. The proofs of the other step differ very much from Theorem 3.3 to Theorem 3.4, with smaller differences present even between the different cases handled by Theorem 3.3.

We first present a sketch of the proof of Proposition 3.5 assuming, for simplicity, that K and L are strictly convex and C^1-regular. Under this assumption the proof of Proposition 3.5 was substantially already present in Bianchi, Segala and Volčič [20], and the version that we present is taken from there.

Sketch of the proof of Proposition 3.5. Let E be a maximal (with respect to inclusion) arc in $\partial K \cap \partial L$. The portion of ∂K antipodal to E is contained in ∂L too. By this we mean that, if U is the subset of S^1 consisting of the vectors u such that

$$K_u = L_u \subset E,$$

then, for each $u \in U$,

$$K_{-u} = L_{-u}.$$

This comes from $DK = \operatorname{supp} g_K = \operatorname{supp} g_L = DL$ and (2.1). Thus $\partial K \cap \partial L$ also contains the boundary arc $F = \{K_{-u} : u \in U\}$ antipodal to E. The arc F is maximal too.

The crucial point in the proof of Proposition 3.5 is the next lemma. We shall only give its proof, though further work is required to prove Proposition 3.5.

Lemma 3.6. *The arcs E and F are reflections of each other.*

Proof. Suppose, on the contrary, that E and F are not reflections of each other. We aim to obtain a contradiction by showing that E (and hence F) is not maximal, i.e. that $\partial K \cap \partial L$ contains an arc strictly larger than E. It follows from the discussion above that K and L have the same tangents at the endpoints of E and F, and that these tangents are pairwise parallel. Moreover, U has length less than π and thus E can be represented as the graph of a convex function.

We need a definition. Suppose that X and Y are arcs of ∂K corresponding to opposite arcs V and $-V$ of S^1. Let z be one of the endpoints of Y. We denote by $\bar{Y}$ the convex curve formed by Y and the appropriate half of the tangent to Y at z. We say that *the point z can be captured by the arc X*, if an appropriate translation of X intersects $\bar{Y}$ in two points determining an arc of $\bar{Y}$ containing z in its relative interior.

Claim 1. *Let E and F be disjoint arcs in the boundary of a planar strictly convex body K corresponding to $U \subset S^1$ and $-U \subset S^1$, respectively, which are not reflections of each other. Then one arc has an endpoint which can be captured by the other.*

To see this, let u denote an endpoint of U. The point K_u is an endpoint of E and K_{-u} is an endpoint of F. Changing, if necessary, the coordinate system, we may assume that $K_u = (0,0)$, that $u = (0,-1)$ and that locally the arc E is represented by the graph of a convex function defined in a right neighborhood of 0. Let $\widetilde{F} = -(F + K_u - K_{-u})$. Then $\widetilde{F}$ is tangent to E at K_u. Either $\widetilde{F} \subset E$, or $E \subset \widetilde{F}$ or there is a point (x,y) on one arc such that the other contains a point (x',y), with $x' > x$. The first two alternatives are impossible, since E and $\widetilde{F}$ are strictly convex arcs with the same set of outer normals, so if $\widetilde{F} \subset E$ or $E \subset \widetilde{F}$, then $E = \widetilde{F}$, contradicting our assumption.

We may assume that $(x,y) \in E$. Then the map $z \to -z + (x,y)$ takes $\widetilde{F}$ to a translate of F with one endpoint at (x,y) and with a point (the image of (x',y)) on the negative $x-$axis. The origin is thus an endpoint of E which is captured by F. This proves Claim 1.

By Claim 1, we may assume that there is an endpoint z of E which can be captured by F via a translation by a vector p. As in the proof of Claim 1, we may assume that z is the origin, and that the arc E is represented by the graph of a convex function g defined in a right neighborhood of 0, and we may also assume that $g(0) = 0$ and $g'(0)$ is finite. It is possible to extend the definition of g to a left neighborhood of 0 so that it represents a portion of ∂K adjacent to E. Let f be the concave function whose graph is $F+p$. The arc $F+p$ intersects E in a point (b,c) with $b>0$ and moreover, possibly by changing the translation, we may assume that $F+p$ also intersects the graph of g in a point with a negative abscissa a.

If we show that the covariogram determines the boundary of $K\cap(K+p)$, we are done, since this means that the arc E is not maximal in $\partial K\cap\partial L$.

The covariogram gives the area of $K\cap(K+p-(0,t))$ for every $t>0$. If we denote by $[a_t,b_t]$ the interval where $f(x)-t\geq g(x)$, then

$$g_K(p-(0,t))=\int_{a_t}^{0}(f(x)-t-g(x))\,dx+\int_{0}^{b_t}(f(x)-t-g(x))\,dx.$$

The latter integral is known for any $t\in[0,f(0)]$, since by assumption, f and g are known on $[0,b]$. Therefore we can deduce from the covariogram the value of

$$\int_{a_t}^{0}(f(x)-t-g(x))\,dx,$$

for any $t\in[0,f(0)]$. By assumption, f is known on $[a,0]$. We now claim that this information is sufficient to determine g on $[a,0]$.

Claim 2. *Suppose that f is a continuous strictly increasing function on $[a,0]$, with $f(0)>0$. If g is continuous and strictly decreasing on $[a,0]$ such that $g(a)>f(a)$ and $g(0)=0$, then g is determined in a left neighborhood of 0 by the areas*

$$A_t=\mathcal{H}^2\left(\{(x,y):\,x\in[a,0],\,g(x)\leq y\leq f(x)-t\}\right), \tag{3.1}$$

for $0\leq t\leq f(0)$.

Indeed, let a_t be the point where $g(a_t)=f(a_t)-t$. Then $a_0<0$. The function $h(t)=a_t$ is continuous since h is the inverse of the increasing and continuous function $f-g$ restricted to $[a_0,0]$. An elementary calculation shows that for every $\delta>0$,

$$\delta a_{t+\delta}\leq A(t)-A(t+\delta)\leq\delta a_t.$$

It follows that $(A(t)-A(t+\delta))/\delta\to a_t=h(t)$, as $\delta\to 0$, because h is continuous.

We see from this that h is determined on its natural domain $[0,f(0)]$, and so therefore is its inverse $f-g$, defined on $[a_0,0]$. But f is determined by assumption, so g is determined on $[a_0,0]$. This proves Claim 2 and hence Lemma 3.6. □

Determination of an arc of the boundary. We will describe this step in two cases, that of convex bodies in C^2_+, where it is particularly simple, and that of strictly convex C^1-regular bodies, historically the last case to be solved.

Determination of an arc of the boundary: convex bodies in C^2_+. We recall that, for $u\in S^1$, $\tau_K(u)$ denotes the Gauss curvature of ∂K at the point K_u with outer normal u.

Proposition 3.7 (Bianchi, Segala and Volčič [20])**.** *If the planar convex body K is in C^2_+, then g_K determines the nonordered pair $\{\tau_K(u),\tau_K(-u)\}$, for $u\in S^1$.*

The analogous result is valid also for C^2_+ convex bodies in $\mathbb{R}^n$ [18, Proposition 3.1].

In the planar case the information about the couple above is contained in the asymptotic behaviour of g_K near the point $p=(\operatorname{supp}g_K)_u$. Since $\operatorname{supp}g_K=DK$, $p=K_u-K_{-u}$, by (2.1). Studying the behavior of g_K near p is equivalent to studying the behavior of the area of $K\cap(K+x)$ for x such that $K\cap(K+x)$ is contained in a small neighborhood of K_u. For these x, the boundary of $K \cap (K+x)$ consists of a portion of ∂K near K_u and a translation of a portion of ∂K near K_{-u}. The next formula expresses this area in terms of the curvatures and gives a proof of Proposition 3.7. Choose a reference system so that $u = (0, 1)$ and let $p = (p_1, p_2)$. For brevity, let $a = \tau_K(u)$ and $b = \tau_K(-u)$. Then, for (x_1, x_2) in a neighborhood of o such that $(p_1 + x_1, p_2 + x_2) \in \operatorname{supp} g_K$,

$$g_K(p_1 + x_1, p_2 + x_2) = \frac{2}{3} \frac{\big(-2(a+b)x_2 - abx_1^2 \big)^{\frac{3}{2}}}{(a+b)^2} \big(1 + \mathrm{o}(|(x_1, x_2)|) \big).$$

An alternative proof of Proposition 3.7, valid under the stronger assumption $K \in C^4_+$, derives from (1.3) and the study of the asymptotic behaviour at infinity of $\widehat{1_K}$ by Haviland and Wintner [39]. The result in [39] yields the following asymptotic expansion, as $|t| \to \infty$, for $\widehat{g_K}$:

$$\widehat{g_K}(tu) = \frac{2\pi}{t^3} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{\sqrt{ab}} \sin(|t|\, w_K(u)) + \mathrm{O}\left(\frac{1}{\sqrt{|t|}} \right) \right).$$

It remains to prove that Proposition 3.7 implies the conclusion of this step. Assume both K and L are C^2_+-regular.

If K is centrally symmetric the curvatures at antipodal points are equal and Proposition 3.7 implies $\tau_K(u) = \tau_L(u)$, for each u. The curvature determines the point $K_u = (x(u), y(u))$ of ∂K via the parametric representation (see for instance [28, p. 79])

$$(3.2)\quad x(u) = x(v) + \int_{\theta(v)}^{\theta(u)} \frac{-\sin t}{\tau_K(\cos t, \sin t)}\, dt,\ \ y(u) = y(v) + \int_{\theta(v)}^{\theta(u)} \frac{\cos t}{\tau_K(\cos t, \sin t)}\, dt,$$

where $\theta(u)$ denotes the angular coordinate of $u \in S^1$. If $v \in S^1$ is fixed and L is translated so that $K_v = L_v$, then $K = L$ follows from (3.2).

If K is not centrally symmetric, the continuity of the curvature implies that given any component U of $\{u \in S^1 : \tau_K(u) \neq \tau_K(-u)\}$, we have, possibly after a reflection of L,

$$\tau_K(u) = \tau_L(u), \quad \text{for each } u \in U.$$

If $v \in U$ is fixed and L is translated so that $K_v = L_v$, then (3.2) implies that $K_u = L_u$, for $u \in U$.

Determination of an arc of the boundary: strictly convex C^1-regular bodies. We recall some notation introduced at the end of Section 2. For $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$, let $p_i(x)$, $i \in \{1, \dots, 4\}$, be points in counterclockwise order on ∂K such that $x = p_1(x) - p_2(x) = p_4(x) - p_3(x)$; see Figure 1. Let $u_i(x)$ be the unit outer normal vector to ∂K at $p_i(x)$, and let $P(x)$ be the parallelogram $\operatorname{conv}\{p_1(x), \dots, p_4(x)\}$. The crucial point is that the outer normals of K are determined by g_K, up to the ambiguities arising from reflections of the body.

Proposition 3.8 (Averkov and Bianchi [4])**.** *Let K be a strictly convex C^1-regular body. Then, for every $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ with $\det G(x) \neq -1$, the set $\{u_1(x), -u_3(x)\}$ is determined by g_K.*

Here

$$G(x) = G(K, x) = \left(\frac{\partial^2 g_K}{\partial x_i \partial x_j}(x) \right)$$

is the Hessian matrix of g_K at x. The existence of the second-order derivatives at $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ is proved for strictly convex C^1-regular bodies in $\mathbb{R}^n$ in [53, Theorem 2.5], while the planar case was already treated in [52, pp. 12-18]. The Hessian $G(x)$ contains information about the vectors $u_1(x), \dots, u_4(x)$, as expressed in the next proposition. For $x, y \in \mathbb{R}^2$, denote by $\det(x, y)$ the determinant of the matrix whose first column is x and the second is y.

The next goal is to outline the proof of Proposition 3.8, based on the following two lemmas.

Lemma 3.9. *Let K be a planar strictly convex C^1-regular body. Then $g_K(x)$ is twice continuously differentiable at every $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$. Furthermore, for every $x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$, the following statements hold.*

a) The Hessian $G(x)$ is given by

$$G(x) = \frac{u_2 u_1^\top}{\det(u_2, u_1)} - \frac{u_3 u_4^\top}{\det(u_3, u_4)} = \frac{u_1 u_2^\top}{\det(u_2, u_1)} - \frac{u_4 u_3^\top}{\det(u_3, u_4)}.$$

b) The determinant of $G(x)$ depends continuously on x and satisfies

$$1 + \det G = \frac{\det(u_2, u_4) \det(u_1, u_3)}{\det(u_3, u_4) \det(u_1, u_2)}. \tag{3.3}$$

c) The vectors u_1, u_3 and $G(x)$ are related by

$$u_1^\top G(x)^{-1} u_3 = 0. \tag{3.4}$$

One can tell from $G(x)$ whether K is centrally symmetric. We say that a chord of K is an *affine diameter* if the normal vectors at ∂K at the endpoints of the chord are parallel.

Lemma 3.10. *Let K be a planar strictly convex C^1-regular body. The following are equivalent.*

a) K is centrally symmetric.

b) At least one diagonal of each parallelogram inscribed in K is an affine diameter of K.

c) The covariogram g_K is a solution of the Monge–Ampère differential equation

$$\det G(x) = -1, \quad \text{for } x \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}.$$

If one diagonal of the parallelogram $P(x)$ is an affine diameter, then u_1 is parallel to u_3 or u_2 is parallel to u_4 and (3.3) implies that $\det G(x) = -1$.

We are now ready to sketch the proof of Proposition 3.8. Due to the assumptions of Proposition 3.8 and (3.3) we have $u_1(x) \neq -u_3(x)$. We prove that there is a $y \neq x$ such that the parallelograms $P(x)$ and $P(y)$ satisfy $p_1(x) = p_1(y)$ and $p_3(x) = p_3(y)$. This clearly implies that $u_1(x) = u_1(y)$ and $u_3(x) = u_3(y)$. Thus, $u_1(x)$ and $u_3(x)$ satisfy the system given by the two equations obtained by evaluating (3.4) at both x and y. [4, Lemma 5.3] expresses the vectors $u_1(x)$ and $u_3(x)$ in terms of the eigenvectors of $G(x)G(y)^{-1}$. In order to make this expression of $u_1(x)$ and $u_3(x)$ dependent only on the covariogram, it remains to prove that the property that $P(x)$ and $P(y)$ have a diagonal in common is shared by convex bodies with equal covariograms. The latter is done in [4, Proposition 5.4].

We now sketch how Proposition 3.8 is used to prove that if K and L are strictly convex C^1-regular bodies with $g_K = g_L$, then ∂K and ∂L have an arc in common, up to translations and reflections. Choose $x_0 \in \operatorname{int} \operatorname{supp} g_K \setminus \{o\}$ such that $\det G(x_0) \neq$

-1. We claim that if x belongs to a suitable neighborhood U of x_0, and if $p_3(K,x) = p_3(K,x_0)$, then $p_3(L,x) = p_3(L,x_0)$. Indeed, Proposition 3.8 together with a continuity argument allows us to prove that when x is close to x_0 and $u_3(K,x) = u_3(K,x_0)$, we have $u_3(L,x) = u_3(L,x_0)$. In view of the strict convexity of K and L, this proves the claim.

Now let $x(t)$, for $t \in [0,1]$, be a parametrization of a curve contained in U with the property that, for each $t \in [0,1]$, the parallelograms $P(K,x_0)$ and $P(K,x(t))$ are such that $p_3(K,x_0) = p_3(K,x(t))$. The previous claim implies that the arc of ∂K formed by the locus of $p_4(K,x(t))$, as t varies in $[0,1]$, is a translate of the arc of ∂L formed by the locus of $p_4(L,x(t))$. Therefore, up to translations, ∂K and ∂L have an arc in common.

3.2. **Counterexamples in $\mathbb{R}^n$ for any $n \geq 4$.** We explain in Section 6 that the Covariogram Problem has a discrete counterpart which asks whether a finite set $A \subset \mathbb{R}^n$ is determined by its discrete covariogram (see (6.1) for the definition), and that there exists a construction that produces different sets (possibly multisets, sets with repetitions allowed) with equal discrete covariograms. If $B, C \subset \mathbb{R}^n$ are finite sets, then the multisets $B + C$ and $B + (-C)$ have the same discrete covariogram. When one tries to construct counterexamples to Matheron's Conjecture via a similar procedure one immediately encounters two problems: the requirement that the corresponding Minkowski sums are sets, not multisets, and the requirement that they are convex. Choosing convex bodies in linear subspaces that intersect only in o solves both of these problems.

Theorem 3.11 (Bianchi [14])**.** *Let $\mathbb{R}^n = E \oplus F$ be the direct sum of the linear subspaces E and F, and let $H \subset E$ and $K \subset F$ be convex bodies. Then the convex bodies $H + K$ and $H + (-K)$ in $\mathbb{R}^n$ have the same covariogram.*

If neither H nor K are centrally symmetric, then $H + K$ and $H + (-K)$ are not equal up to translations and reflections.

Proof. The property of having equal covariograms is invariant under linear maps, and the same is true for the property of being equal up to translations and reflections, since

$$g_{\mathcal{L}K}(x) = (\det \mathcal{L})\, g_K(\mathcal{L}^{-1}x) \quad \text{and}$$
$$\mathcal{L}K = \mathcal{L}(\pm L) + y \qquad \Longleftrightarrow \qquad K = \pm L + \mathcal{L}^{-1}y,$$

for any invertible linear map $\mathcal{L}$, $x, y \in \mathbb{R}^n$ and $K, L \in \mathcal{K}^n$. We may therefore assume that E and F are orthogonal subspaces.

In this case, if we write $x = (x_1, x_2) \in E \oplus F$ and if $\dim E = n_1$, $\dim F = n_2$ and $H \subset E$ and $K \subset F$ are convex bodies, then

$$\begin{aligned} g_{H+K}(x_1,x_2) &= \mathcal{H}^n\left((H+K) \cap ((H+K) + (x_1,x_2)\right) \\ &= \mathcal{H}^n\left((H \cap (H+x_1)) + (K \cap (K+x_2))\right) \\ &= \mathcal{H}^{n_1}\big(H \cap (H+x_1)\big)\mathcal{H}^{n_2}\big(K \cap (K+x_2)\big) \\ &= g_H(x_1)\, g_K(x_2). \end{aligned} \tag{3.5}$$

A similar formula holds for $H + (-K)$ and the invariance of the covariogram with respect to reflections implies that $g_{H+K} = g_{H+(-K)}$.

To prove the second claim, suppose on the contrary that

$$H + K = H + (-K) + y \quad \text{or} \quad H + K = -\left(H + (-K)\right) + y, \tag{3.6}$$

for some $y \in \mathbb{R}^n$. We may assume that both H and K have centroid at the origin, since translations of H and K result in translations of $H + K$ and $H + (-K)$. Since the centroids of $H + K$ and $H + (-K)$ are at the origin, we have $y = 0$. Then by the cancellation law for Minkowski addition [66, p. 48], the first equality in (3.6) implies that $K = -K$, that is, K is centrally symmetric, a contradiction. Similarly, the second equality in (3.6) implies that H is centrally symmetric, again a contradiction. □

Convex sets which are not centrally symmetric exist when the dimension of the ambient space is at least two, yielding the following corollary.

Corollary 3.12 (Bianchi [14])**.** *If $n \geq 4$ there exist convex bodies in $\mathcal{K}_n^n$ with the same covariogram which are not equal up to translations and reflections.*

This example is better understood if seen in the context of the decomposition of a convex body into direct summands. For $K \in \mathcal{K}_n^n$, we write

$$K = K_1 \oplus \cdots \oplus K_s \tag{3.7}$$

if $K = K_1 + \cdots + K_s$ for suitable convex bodies K_i lying in linear subspaces E_i of $\mathbb{R}^n$ such that $E_1 \oplus \cdots \oplus E_s = \mathbb{R}^n$. If a representation $K = L \oplus M$ is only possible when $\dim L = 0$ or $\dim M = 0$, then K is *directly indecomposable.* Each K with $\dim K \geq 1$ has a representation, unique up to the order of the summands, as in (3.7), with $\dim K_i \geq 1$ and K_i directly indecomposable.

If at least two of the summands of K, say K_1 and K_2, are not centrally symmetric, then $(-K_1) \oplus K_2 \oplus \cdots \oplus K_s$ has the same covariogram as K and is not equal to K up to translations and reflections. Two questions arise naturally.

a) If $H \in \mathcal{K}_n^n$ and $g_H = g_K$, does H have a similar structure to K?
b) If each directly indecomposable summand K_i is determined, up to translations and reflections, among the convex bodies in E_i by g_{K_i}, considered as a function defined in E_i, can the structure of H be understood?

The next theorem gives a positive answer to these questions.

Theorem 3.13 (Bianchi [15])**.** *Let $K \in \mathcal{K}_n^n$ and let E_i and K_i, $i = 1, \dots, n$, be as in* (3.7)*. If $H \in \mathcal{K}_n^n$ and $g_H = g_K$, then $H = H_1 \oplus \cdots \oplus H_s$, where, for each i, H_i is a directly indecomposable convex body contained in E_i and $g_{H_i} = g_{K_i}$.*

If in addition for each i, $g_{K_i} : E_i \to \mathbb{R}$ determines K_i among the convex bodies in E_i, up to translations and reflections, then H is a translation of $\sigma_1 K_1 \oplus \cdots \oplus \sigma_s K_s$, for suitable $\sigma_1, \dots, \sigma_s \in \{-1, 1\}$.

In view of this, to understand the Covariogram Problem for general convex bodies it suffices to study it for indecomposable bodies. This last problem is however widely open, and the examples of nondetermination described above are the only ones known.

For the proof of Theorem 3.13 we refer to [15]. Briefly, the first claim follows from the uniqueness of the decomposition into direct summands, the equality $DH = DK$ and a lemma that proves that a convex body is directly indecomposable if and only if its difference body is directly indecomposable. The second claim is a direct consequence of the first and the fact that the covariogram can be written in terms of the covariograms of the direct summands, with a formula similar to (3.5).

Before we conclude this section we prove, for later use, that if E, F, H and K are as in the statement of Theorem 3.11, then $H + K$ is not in the class C^1.

This property is invariant under linear maps and we may therefore assume that E and F are orthogonal subspaces. For x in the boundary of H relative to E, let $N_E(H,x)$ (or $N(H,x)$) be the normal cone of H at x relative to E (or relative to $\mathbb{R}^n$, respectively). For y in the boundary of K relative to F, let $N_F(K,y)$ and $N(K,y)$ be defined in the analogous way. Schneider [66, (2.4) and Theorem 2.2.1(a)] proves that

$$\begin{aligned} N(H+K,x+y) &= N(H,x)\cap N(K,y) \\ &= (N_E(H,x)+F)\cap(E+N_F(K,y)) = N_E(H,x)+N_F(K,y). \end{aligned}$$

This implies that the dimension of $N(H+K,x+y)$ is larger than 1. Thus, $\partial(H+K)$ is not C^1 regular at $x+y$.

3.3. **Polytopes in $\mathbb{R}^n$, $n \geq 3$.** In dimensions higher than two the Covariogram Problem has only partial results. The situation is better understood in the case of polytopes.

Theorem 3.14 (Goodey, Schneider and Weil [38])**.** *If $P \in \mathcal{K}^n_n$, $n \geq 3$, is a polytope such that P and $-P$ are in general relative position and all its two-dimensional faces are triangles then P is determined by g_P, up to translations and reflections, in the class $\mathcal{K}^n_n$.*

The polytopes P and $-P$ are said to be in *general relative position* if for any two faces F and G of P lying in antipodal parallel supporting hyperplanes of P, $F \cap (G+x)$ contains at most one point, for any $x \in \mathbb{R}^n$. In $\mathbb{R}^3$, for instance, this means that P does not have pairs of parallel antipodal facets, or pairs of parallel antipodal edges, or an edge antipodal and parallel to a facet.

The proof of Theorem 3.14 is based on the Brunn–Minkowski inequality, together with its equality cases, and on a result about the decomposition of convex bodies in terms of sums of other convex bodies. Schneider [65] proves that the assumptions on P imply that every summand of DP is of the form $aP+(1-a)(-P)+x$ with $a \in [0,1]$ and $x \in \mathbb{R}^n$. If $K \in \mathcal{K}^n$ satisfies $g_K = g_P$, then $\mathcal{H}^n(K) = \mathcal{H}^n(P)$ and $DK = DP$, by Propositions 2.1 a) and 2.2 a). The formula $K+(-K) = DK = DP$ says that K is a summand of DP and therefore

$$K = aP + (1-a)(-P) + x.$$

If $a=0$ or $a=1$, then $K=-P+x$ or $K=P+x$, and we are done. If $a \in (0,1)$, then

$$\begin{aligned} \mathcal{H}^n(K)^{1/n} &= \mathcal{H}^n(aP+(1-a)(-P)+x)^{1/n} = \mathcal{H}^n(aP+(1-a)(-P))^{1/n} \\ &\geq a\,\mathcal{H}^n(P)^{1/n} + (1-a)\,\mathcal{H}^n(-P)^{1/n} \\ &= \mathcal{H}^n(P)^{1/n} = \mathcal{H}^n(K)^{1/n}. \end{aligned}$$

Thus the Brunn-Minkowski inequality holds with equality and this implies that P and $-P$ are homothetic, i.e. P is centrally symmetric with respect to some point. Therefore $K = aP+(1-a)(-P)+x$ is a translate of P.

Theorem 3.15 (Bianchi [15])**.** *Let $P \in \mathcal{K}^3_3$ be a polytope. Then g_P determines P, in the class $\mathcal{K}^3_3$, up to translations and reflections.*

The proof of Theorem 3.15 is described in Section 3.4.

Theorem 3.11, when H and K are convex polytopes, has the following corollary.

Corollary 3.16. *For each $n \geq 4$, there exist pairs of polytopes in $\mathcal{K}_n^n$ with the same covariogram which are not equal up to translations and reflections.*

3.4. **Some problems and ideas from the proof of Theorem 3.15.** The proof of Theorem 3.15 is done in three steps and is contained in [15], [16] and [17]. We first briefly describe all three steps and then each step in more detail. Let P and P' be convex polytopes in $\mathbb{R}^3$ with nonempty interior such that $g_P = g_{P'}$.

The *first step* consists in proving that ∂P and $\partial P'$ coincide locally up to translations and reflections. What this means is expressed by the next proposition.

Proposition 3.17. *Let P and P' be convex polytopes in $\mathbb{R}^3$ with nonempty interior such that $g_P = g_{P'}$. If $w \in S^2$, then there exists $\sigma = \sigma(w) \in \{-1, 1\}$ and $x = x(\sigma) \in \mathbb{R}^3$ such that*

$$\begin{aligned} P_w &= (\sigma P')_w + x \qquad \textit{and} \qquad \operatorname{cone}(P, P_w) = \operatorname{cone}(\sigma P', (\sigma P')_w); \\ P_{-w} &= (\sigma P')_{-w} + x \qquad \textit{and} \qquad \operatorname{cone}(P, P_{-w}) = \operatorname{cone}(\sigma P', (\sigma P')_{-w}). \end{aligned} \tag{3.8}$$

We recall that P_w is the face of P with outer normal w and that $\operatorname{cone}(P, P_w)$ is the support cone of P at P_w. Condition (3.8) implies that, for each proper face of P, be it a facet, an edge or a vertex, after possibly a reflection and a translation, P and P' coincide in a neighborhood of that face and of the antipodal one.

One may wonder if the validity, for each w, of the local conditions (3.8) is sufficient to guarantee that $P' = P$, up to translations and reflections, but this is not the case, as Example 3.23 below shows. It can be proved that when (3.8) holds with $\sigma = 1$ (or with $\sigma = -1$) for each $w \in S^2$ then P' (or $-P'$, respectively) is a translate of P. A priori, however, (3.8) may hold with $\sigma = 1$ for some w, with $\sigma = -1$ for some w, and, possibly, both with $\sigma = 1$ and with $\sigma = -1$ for other w.

The set $\operatorname{int}\big(\partial P \cap \partial(\sigma(w)P' + x(\sigma))\big)$ (in this section the terms boundary, interior and neighborhood of a subset of ∂P refer to the relative topology induced on ∂P by the Euclidean topology in $\mathbb{R}^3$) may have multiple components which depend on w. The *second step* of the proof consists in a study of these components and of their boundaries that leads to a choice of w such that the corresponding components satisfy certain convenient properties.

In the *third step* we use the setting prepared in the second step to conclude the proof by contradiction, by identifying some $y \in \mathbb{R}^3$ such that $g_P(y) \neq g_{P'}(y)$.

First step. In order to prove Proposition 3.17 we investigate two related problems. The first problem helps in proving the equalities in the first column in (3.8). In this paper we explain only how to prove them when both P_w and P_{-w} are facets.

Assume that P_w and P_{-w} are facets, and let $F = P_w|w^\perp$ and $G = P_{-w}|w^\perp$. We consider $P \cap (P + x)$ for x such that $P \cap (P + x) \neq \emptyset$ and the plane aff $P_{-w} + x$ has a small distance, say ε, from the plane aff P_w. In this situation, $P \cap (P + x)$ is approximately equal to a parallelepiped, with height ε and base a translate of $F \cap (G + y)$, where $y = x|w^\perp$. Thus

$$\mathcal{H}^3(P \cap (P + x)) = \varepsilon \mathcal{H}^2\big(F \cap (G + y)\big) + \mathrm{o}(\varepsilon) = \varepsilon g_{F,G}(y) + \mathrm{o}(\varepsilon).$$

This formula proves that g_P determines the cross covariogram $g_{F,G}(y)$, for each $y \in w^\perp$. We thus encounter a first problem.

Problem 3.18 (Cross Covariogram Problem for polygons). Does the cross covariogram of the convex polygons F, $G \subset \mathbb{R}^2$ determine the pair (F, G), among all pairs of convex polygons, up to trivial associates?

A detailed description of its solution is presented in Section 3.6. Here we just anticipate that, for each choice of some real parameters, there exist four different pairs of parallelograms $(H_1, K_1), \dots, (H_4, K_4)$ such that, for $i = 1, 3$, $g_{H_i,K_i} = g_{H_{i+1},K_{i+1}}$, but (H_i, K_i) is not a trivial associate of (H_{i+1}, K_{i+1}), and that, up to an affine transformations, the previous counterexamples are the only ones.

Thus, $g_{F,G}$ alone is not sufficient to determine F and G, and we have to get from g_P other information that eliminates the ambiguities due to the presence of these pairs of parallelograms. Rufibach [61, p. 14] was the first to observe the possibility of determining $g_F + g_G$ from g_P. His idea led to the next proposition.

Proposition 3.19. *Let $P \subset \mathbb{R}^n$ be a convex polytope with nonempty interior, let $w \in S^{n-1}$, $F = P_w|w^\perp$ and $G = P_{-w}|w^\perp$. The covariogram g_P determines both $(g_F + g_G)(y)$ and $g_{F,G}(y)$, for each $y \in w^\perp$.*

[15] presents a proof of Proposition 3.19 based on the expression of the second-order distributional derivative of g_P given in the next lemma.

Lemma 3.20. *Let $P \subset \mathbb{R}^n$, $n \geq 2$, be a convex polytope with nonempty interior. Let $F_1, \dots, F_m$ be its facets, ν_i be the unit outer normal of P at F_i, for $i = 1, \dots, m$, let $w \in S^{n-1}$ and let $I_p = \{(i,j) : F_i$ is parallel to $F_j\}$ and $I_{np} = \{(i,j) : F_i$ is not parallel to $F_j\}$. Then, for $\phi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$(3.9)\qquad -\frac{\partial^2 g_P}{\partial w^2}(\phi) = \sum_{(i,j)\in I_{np}} \frac{\langle w, \nu_i\rangle\langle w, \nu_j\rangle}{\sqrt{1-\langle \nu_i, \nu_j\rangle^2}} \int_{\mathbb{R}^n} \mathcal{H}^{n\text{-}2}(F_i \cap (F_j + z))\, \phi(z)\, dz$$
$$+ \sum_{(i,j)\in I_p} \langle w, \nu_i\rangle\langle w, \nu_j\rangle \int_{F_i+(-F_j)} \mathcal{H}^{n\text{-}1}(F_i \cap (F_j + z))\, \phi(z)\, d\,\mathcal{H}^{n\text{-}1}(z).$$

Both terms on the right-hand side of (3.9) *are determined by g_P.*

Proof of Proposition 3.19. The distribution defined by the second term on the right-hand side in (3.9) determines its support

$$S(P, w) = \bigcup \{F_i + (-F_j) : (i,j) \in I_p, \langle \nu_i, w\rangle \neq 0\}$$

and, for $\mathcal{H}^{n\text{-}1}$-almost each $x \in S(P, w)$, the expression

$$(3.10)\qquad \sum_{(i,j)\in I_p} \langle w, \nu_i\rangle\langle w, \nu_j\rangle\ \ \mathcal{H}^{n\text{-}1}(F_i \cap (F_j + x)).$$

The set $S(P, w)$ is contained in DP, and consists of differences $F_i + (-F_j)$ of distinct parallel facets and of differences $F_i + (-F_i)$, with i such that $\langle \nu_i, w\rangle \neq 0$. The difference $F_i + (-F_j)$ is the facet $(DP)_{\nu_i}$ of DP. The difference $F_i + (-F_i)$ is contained in $\nu_i^\perp$.

Given $w \in S^2$, this information tells us whether P has zero, one or two facets orthogonal to w. The polytope P has at least one facet orthogonal to w if and only if $(DP)_w$ is a facet of DP. It has two facets orthogonal to w if and only if $(DP)_w$ is a facet of DP and $(DP)_w \subset S(P, w)$. If it has two facets, then the part of the distribution supported in $w^\perp$ determines $g_F + g_G$ while the part supported in $(DP)_w$ determines $g_{F,G}$. If P has only one facet orthogonal to w and this facet is P_w, say, then the same holds, with the difference that now $g_F + g_G = g_F$ and $g_{F,G} = 0$. If P has no facet orthogonal to w, then $g_F + g_G = 0$ and $g_{F,G} = 0$. □

The next result decouples the information given by Proposition 3.19.

Lemma 3.21. *Let F, F', G and G' be convex bodies in $\mathbb{R}^n$. If*

$$\left\{\begin{array}{l} g_F + g_G = g_{F'} + g_{G'}, \\ g_{F,G} = g_{F',G'} \end{array}\right. \tag{3.11}$$

then either $g_F = g_{F'}$ and $g_G = g_{G'}$, or else $g_F = g_{G'}$ and $g_G = g_{F'}$.

Proof. Applying the Fourier transform to the equalities in (3.11) we arrive, with the help of (1.3), to the system

$$\left\{\begin{array}{l} |\widehat{1_F}|^2 + |\widehat{1_G}|^2 = |\widehat{1_{F'}}|^2 + |\widehat{1_{G'}}|^2 \\ |\widehat{1_F}|^2|\widehat{1_G}|^2 = |\widehat{1_{F'}}|^2|\widehat{1_{G'}}|^2. \end{array}\right.$$

For each $\xi \in \mathbb{R}^n$, the previous system implies that either we have $|\widehat{1_F}(\xi)| = |\widehat{1_{F'}}(\xi)|$ and $|\widehat{1_G}(\xi)| = |\widehat{1_{G'}}(\xi)|$ or else we have $|\widehat{1_F}(\xi)| = |\widehat{1_{G'}}(\xi)|$ and $|\widehat{1_G}(\xi)| = |\widehat{1_{F'}}(\xi)|$. A priori, the alternative may depend on ξ. The Fourier transform of a function with compact support is analytic and therefore the squared moduli of the previous transforms are analytic. Since any analytic function is determined by its values on a set with a limit point, we conclude that the previous alternative does not depend on ξ. Going back to covariograms via Fourier inversion, this means that either $g_F = g_{F'}$ and $g_G = g_{G'}$, or else $g_F = g_{G'}$ and $g_G = g_{F'}$. □

We are now ready to prove the equalities in the first column of (3.8) when both P_w and P_{-w} are facets. Let F and G be as above, and let $F' = P'_w|w^\perp$ and $G' = P'_{-w}|w^\perp$. The faces P'_w and P'_{-w} are facets too, because otherwise $g_{F',G'} \equiv 0 \neq g_{F,G}$. If (F,G) and (F',G') are trivial associates, then, up to a reflection and/or translation of P', the equalities in the first column of (3.8) hold .

Now assume that (F,G) and (F',G') are not trivial associates. Theorem 3.30 states that (F,G) and (F',G') are, respectively, trivial associates of $(\mathcal{T}H_i, \mathcal{T}K_i)$ and $(\mathcal{T}H_j, \mathcal{T}K_j)$, for some affine transformation $\mathcal{T}$ and different indices i,j, with either $i,j \in \{1,2\}$ or $i,j \in \{3,4\}$.

Proposition 3.19 implies $g_{\mathcal{T}H_i}+g_{\mathcal{T}K_i} = g_{\mathcal{T}H_j}+g_{\mathcal{T}K_j}$. Lemma 3.21 and the positive answer to the Covariogram Problem in the plane imply that, up to translations and reflections, either $H_i = H_j$ and $K_i = K_j$, or else $H_i = K_j$ and $K_i = H_j$. In view of the definition of these sets (see Figure 2) this is false.

It remains to prove the formulas in (3.8) regarding the support cones. For this purpose, P. Mani-Levitska, in a message to the author, suggested studying the following problem.

Problem 3.22 (Cross Covariogram Problem for polyhedral cones). Let A and B be convex polyhedral cones in $\mathbb{R}^n$, $n \geq 2$, with apex o and $A \cap B = \{o\}$. Does the cross covariogram of A and B determine the pair (A,B), among all pairs of convex polyhedral cones, up to trivial associates?

To see the relevance of this problem, suppose that P_w and P_{-w} are vertices and $x \in \mathbb{R}^3$ is chosen so that $P_{-w} + x$ is close to P_w. Then

$$P \cap (P + x) = A \cap (B + y),$$

where $A = \operatorname{cone}(P, P_w)$, $B = \operatorname{cone}(P, P_{-w})$ and $y = P_{-w} - P_w + x$. Thus, g_P determines $g_{A,B}(y)$ for each y in a neighborhood of o (and also for each $y \in \mathbb{R}^3$, since $g_{A,B}$ is 3-homogeneous). If we were able to determine A and B from $g_{A,B}$, we would be able to determine cone(P, P_w) and cone(P, P_{-w}) from g_P, at least when P_w and P_{-w} are vertices.

A partial answer to Problem 3.22 in $\mathbb{R}^3$, which is sufficient for the purpose of proving Theorem 3.15, is given in Bianchi [18, Proposition 5.1]. Bianchi [16, Theorem 1.3] completely solves the problem in the plane, also describing some situations of nondetermination.

Second step. The next example show how to construct polytopes P and P' which satisfy (3.8) for every $w \in S^2$ and such that $P \neq P'$, up to translations and reflections.

Example 3.23. Let $P \subset \mathbb{R}^3$ be a convex polytope such that $\Gamma \cup (-\Gamma) \subset \partial P$, where Γ is a simple closed curve such that $\Gamma \cap (-\Gamma) = \emptyset$. The union $\Gamma \cup (-\Gamma)$ disconnects ∂P into three components Σ_j, $j = 1, 2, 3$. Let $\partial\Sigma_1 = \Gamma$, $\partial\Sigma_2 = -\Gamma$ and $\partial\Sigma_3 = \Gamma \cup -\Gamma$. Choose P in such a way that $\Sigma_1 \neq -\Sigma_2$, $\Sigma_3 \neq -\Sigma_3$, and there exists a neighborhood W of Γ in ∂P which contains all faces of P intersecting Γ and $-W$ contains all faces of P intersecting $-\Gamma$.

Define P' as the polytope whose boundary is $\Sigma_1 \cup \Sigma_2 \cup (-\Sigma_3)$. The polytope P can be chosen so that $P' \neq P$, up to translations and reflections. We claim that (3.8) holds for each w. Indeed, if w is such that $P_w \cap \Gamma \neq \emptyset$, then $P_w \subset W$ and (3.8) holds both with $\sigma = -1$ and $x = o$ and with $\sigma = 1$ and $x = o$. If $P_w \cap (-\Gamma) \neq \emptyset$, then the same holds. If $P_w \subset (\operatorname{int}\Sigma_1) \cup (\operatorname{int}\Sigma_2)$, then (3.8) holds with $\sigma = 1$ and $x = o$. If $P_w \subset \operatorname{int}\Sigma_3$, then (3.8) holds with $\sigma = -1$ and $x = o$.

The construction above can be iterated and made more complex, by considering other pairs of curves in ∂P which are reflections of each other, possibly with respect to a point different from o, not intersecting Γ and $-\Gamma$, and substituting one of the components of ∂P less all these curves with its reflection.

In the second step we study the components of

$$\operatorname{int}\Big(\partial P \cap \partial\big(\sigma(w)P' + x(\sigma)\big)\Big) \tag{3.12}$$

and their boundaries when w varies in S^2. When P and P' are as in Example 3.23 and w is such that $P_w \subset \Sigma_1$, the set in (3.12) is $\operatorname{int}\big(\partial P \cap \partial P'\big)$. This set has a component Σ_+ containing Σ_1 and a different "antipodal" component Σ_- containing Σ_2 (we assume here that P has been chosen so that $\Sigma_+ \neq \Sigma_-$). They satisfy

$$\Sigma_+ \neq -\Sigma_- \quad \text{and} \quad \partial\Sigma_+ = -\partial\Sigma_-.$$

The first formula holds because $\Sigma_1 \neq -\Sigma_2$, and the second one holds because both boundaries are contained in $\Sigma_3 \cap (-\Sigma_3)$, which is o-symmetric.

If we leave Example 3.23 and pass to the general case, there may exist w such that, if one defines Σ_+ and Σ_- as the components of the set in (3.12) containing P_w and P_{-w}, respectively, then $\partial\Sigma_+$ is not a reflection, with respect to some point, of $\partial\Sigma_-$. This can be seen if one modifies Example 3.23 as follows. Assume that ∂P contains, besides Γ and $-\Gamma$, a closed simple curve Λ and its reflection $-\Lambda + 2z$ with respect to $z \neq o$, with $\Lambda \subset \operatorname{int}\Sigma_1$ and $-\Lambda + 2z \subset \operatorname{int}\Sigma_2$. In this case one can define P' starting from P, and not only substituting Σ_3 with $-\Sigma_3$, but also exchanging the component Σ_4 of $\Sigma_1 \setminus \Lambda$ bounded by Λ with the reflection with respect to z of the component of $\Sigma_2 \setminus (-\Lambda + 2z)$ bounded by $-\Lambda + 2z$. If $P_w \subset \Sigma_1 \setminus \Sigma_4$, then it is not true that $\partial\Sigma_+$ is a reflection, with respect to some point, of $\partial\Sigma_-$. Indeed, in this case $\partial\Sigma_+$ has two components, and to obtain the two components of $\partial\Sigma_-$ one has to reflect one component of $\partial\Sigma_+$ with respect to o and the other component of $\partial\Sigma_+$ with respect to z.

The second step in the proof consists in proving in the general case that if $P \neq P'$, up to translations and reflections, then it is always possible to choose $w \in S^2$ so that there exist $z \in \mathbb{R}^3$ and two antipodal components $\Sigma_+ \supset P_w$ and Σ_- of the set in (3.12) such that

$$\Sigma_+ \neq -\Sigma_- + 2z \quad \text{and} \quad \partial\Sigma_+ = \partial(-\Sigma_- + 2z). \tag{3.13}$$

Third step. In this step we use the structure discovered in Step 2 to conclude and prove that, if $P \neq P'$, up to translations and reflections, then we can find $y \in \mathbb{R}^3$ such that $g_P(y) \neq g_{P'}(y)$. For the details we refer to [15, p. 1804].

3.5. **Smooth convex bodies in $\mathbb{R}^n$, $n \geq 3$.** In this section we deal with convex bodies that are at least C^2_+-regular. Every such body is directly indecomposable (we have proved at the end of Section 3.2 that the direct sum of two lower dimensional convex bodies is not C^1-regular) and we do not have to worry about the examples in Section 3.2.

The Covariogram Problem for C^2_+ bodies is still open, even in $\mathbb{R}^3$, and the only results available are positive ones for bodies with higher regularity. To prove these results it has been useful to connect the Covariogram Problem to some studies regarding the zero set $\mathcal{Z}(K) = \{\zeta \in \mathbb{C}^n : \widehat{1_K}(\zeta) = 0\}$ of the Fourier transform $\widehat{1_K}$ seen as a function on $\mathbb{C}^n$. This set plays a role in attempts to solve the famous Pompeiu Problem, a long-standing open problem in integral geometry (see, for instance, Garofalo and Segala [36] and Machado and Robins [49]), which we describe in details in Section 7. Here we focus on the work of Kobayashi [45, 46] regarding the geometric information about K contained in $\mathcal{Z}(K)$. In 1986 Kobayashi [45] posed the following problem.

Problem 3.24. Does the zero set $\mathcal{Z}(K) = \{\zeta \in \mathbb{C}^n : \widehat{1_K}(\zeta) = 0\}$ determine the convex body K, among all convex bodies, up to translations?

(Note that a translation of K leaves $\mathcal{Z}(K)$ unchanged.) In the class of C^∞_+ convex bodies, Problem 3.24 has been solved by Kobayashi [45] in the planar case, but it is still open for $n \geq 3$. In connection with Problem 3.24, Kobayashi studies the asymptotic behavior at infinity of $\mathcal{Z}(K)$, in any dimension but only in the case of C^∞_+ convex bodies. It turns out that this asymptotic behaviour contains information about the width function of K and the ratio of the Gauss curvatures of ∂K at antipodal points (see Proposition 3.26 and Problem 3.27 below).

In Bianchi [18] Kobayashi's result regarding the asymptotics of $\mathcal{Z}(K)$ is proved under weaker regularity assumptions, replacing $K \in C^\infty_+$ by $K \in C^{r(n)}_+$, where $r(n)$ is as in Theorem 3.25. This is a key tool in the following positive answer to the Covariogram Problem for $C^{r(n)}_+$ convex bodies in $\mathbb{R}^n$, $n \geq 2$.

Theorem 3.25 (Bianchi [18])**.** *Let $n \geq 2$ and define $r(n) = 8$ when $n = 2, 4, 6$, $r(n) = 9$ when $n = 3, 5, 7$ and $r(n) = [(n-1)/2] + 5$ when $n \geq 8$. Let H and K be convex bodies in $\mathbb{R}^n$ of class $C^{r(n)}_+$. Then $g_H = g_K$ implies $H = K$, up to translations and reflections.*

Note that Theorem 3.25 only proves that the covariogram determines a $C_+^{r(n)}$ body among $C_+^{r(n)}$ bodies, and it is not known whether the determination holds among all convex bodies.

We have explained in Section 3.1 that if K is C_+^2 regular, then for each $u \in S^{n-1}$, g_K provides the nonordered pair $\{\tau_K(u), \tau_K(-u)\}$. Thus if H is of class C_+^2 and $g_H = g_K$, the continuity of the curvature implies that given any component U of $\{u \in S^{n-1} : \tau_K(u) \neq \tau_K(-u)\}$, we have, possibly after a reflection of H,

$$\tau_H(u) = \tau_K(u), \quad \text{for each } u \in U. \tag{3.14}$$

If (3.14) were true for each $u \in S^{n-1}$ then H and K would coincide, up to a translation, by the uniqueness part of Minkowski's Theorem [66, Theorem 7.2.1]. However, a priori the reflection that makes (3.14) valid may depend on the component U. The key ingredient in resolving this ambiguity, when the body is $C_+^{r(n)}$ regular, is the fact that the maps $F_{m,K}$ appearing in the statement of next proposition are analytic.

Proposition 3.26 (Kobayashi [45], Bianchi [18])**.** *Let $S = \{\zeta \in \mathbb{C}^n : \zeta = zu,\ \text{with } z \in \mathbb{C}, u \in S^{n-1}\}$, where we identify zu and $(-z)(-u)$, for each $z \in \mathbb{C}$ and $u \in S^{n-1}$. Let K be a convex body in $\mathbb{R}^n$ of class $C_+^{r(n)}$, where $r(n)$ is as in Theorem 3.25. Then there exists a positive integer $m(K)$ such that*

$$\mathcal{Z}(K) \cap S = \left(C(K) \cup \cup_{m=m(K)}^{\infty} \mathcal{Z}_m(K) \right),$$

where $C(K)$ is a bounded set and the union is disjoint. Moreover, for each integer $m \geq m(K)$, there exists an analytic map $F_{m,K} : S^{n-1} \to \mathbb{C}$ such that

$$\mathcal{Z}_m(K) = \{F_{m,K}(u)\, u \ : u \in S^{n-1}\}, \tag{3.15}$$

where

$$F_{m,K}(u) = \frac{\pi(4m+n-1)}{2\, w_K(u)} + i\ \frac{\ln \tau_K(-u) - \ln \tau_K(u)}{2\, w_K(u)} + \mathrm{O}\left(\frac{1}{m}\right) \tag{3.16}$$

and $\mathrm{O}(1/m) \to 0$ as $m \to \infty$, uniformly in $u \in S^{n-1}$.

Indeed, formula (2.3) implies that

$$\{\zeta \in \mathbb{C}^n : \widehat{g_K}(\zeta) = 0\} = \mathcal{Z}(K) \cup \overline{\mathcal{Z}(K)}.$$

Thus g_K gives the real part of $\mathcal{Z}(K)$ and hence, in view of (3.16), the width function of K. This is nothing new, since DK, the support of g_K, already determines w_K. But g_K also determines the imaginary part of $\mathcal{Z}(K)$, up to conjugation. In view of (3.16), it determines the imaginary part of $\mathcal{Z}(K)$, up to reflections of K. Thus if H and K are as in Theorem 3.25 and, possibly, we have reflected H so that (3.14) holds for $u \in U$, then

$$F_{m,H}(u) = F_{m,K}(u) \quad \text{for } m \text{ large enough and } u \in U. \tag{3.17}$$

We can then use the analyticity of these maps to deduce that (3.17) holds for $u \in S^{n-1}$, which implies that (3.14) holds for $u \in S^{n-1}$. This concludes the sketch of the proof of Theorem 3.25.

We restate here [45, Problem 1.13] in the class $C_+^{r(n)}$.

Problem 3.27. If $H, K \in \mathcal{K}_n^n$ are in $C_+^{r(n)}$ and, for $u \in S^{n-1}$,

$$w_H(u) = w_K(u) \quad \text{and} \quad \frac{\tau_H(-u)}{\tau_H(u)} = \frac{\tau_K(-u)}{\tau_K(u)},$$

is $H = K$, up to translations?

A positive answer implies, due to Proposition 3.26, a positive answer to Problem 3.24 in $C_+^{r(n)}$. An answer is known only for $n = 2$, and is positive in that case [45, Corollary 2.3.10].

3.6. **Determination from cross covariogram.** We restate the Cross Covariogram Problem in greater generality.

Problem 3.28 (Cross Covariogram Problem). Does $g_{H,K}$ determine the pair (H, K) of closed convex sets among all pairs of closed convex sets, up to trivial associates?

When H and K are convex polygons, and also when they are convex cones, a complete answer is given in Bianchi [16]. When H and K are sufficiently regular planar convex bodies, the solution can be found in Bianchi [18]. In the case of polygons (and of cones as well) there are examples of nondetermination.

Example 3.29. Let α, β, γ, δ, α', β', γ' and δ' be positive real numbers, $m \in \mathbb{R}$, $y, y' \in \mathbb{R}^2$, $I_1 = [(-1,0),(1,0)]$, $I_2 = 1/\sqrt{2}\,[(-1,-1),(1,1)]$, $I_3 = [(0,-1),(0,1)]$, $I_4 = 1/\sqrt{2}\,[(1,-1),(-1,1)]$ and $I_5 = (1/\sqrt{1+m^2})\,[(-m,-1),(m,1)]$. Assume that either $m = 0$, $\alpha' \neq \gamma'$ and $\beta' \neq \delta'$ or else $m \neq 0$ and $\alpha' \neq \gamma'$. We define four pairs of parallelograms as follows (see Figure 2):

$$\begin{aligned} H_1 &= \alpha I_1 + \beta I_2, & K_1 &= \gamma I_3 + \delta I_4 + y;\\ H_2 &= \alpha I_1 + \delta I_4, & K_2 &= \beta I_2 + \gamma I_3 + y;\\ H_3 &= \alpha' I_1 + \beta' I_3, & K_3 &= \gamma' I_1 + \delta' I_5 + y';\\ H_4 &= \gamma' I_1 + \beta' I_3, & K_4 &= \alpha' I_1 + \delta' I_5 + y'. \end{aligned}$$

For $i = 1, 3$, we have $g_{H_i,K_i} = g_{H_{i+1},K_{i+1}}$ but (H_i, K_i) is not a trivial associate of (H_{i+1}, K_{i+1}).

The next theorem proves that, up to affine transformations, the previous counterexamples are the only ones.

Theorem 3.30 (Bianchi [16])**.** *Let H and K be convex polygons and let H' and K' be planar convex bodies with $g_{H,K} = g_{H',K'}$. If (H, K) is a not a trivial associate of (H', K'), then there is an affine transformation $\mathcal{T}$ and different indices i, j, with either $i, j \in \{1, 2\}$ or $i, j \in \{3, 4\}$, such that $(\mathcal{T}H, \mathcal{T}K)$ and $(\mathcal{T}H', \mathcal{T}K')$ are trivial associates of (H_i, K_i) and of (H_j, K_j), respectively.*

Contrary to the situation for polygons, no counterexample exists among pairs of sufficiently regular planar convex bodies.

Theorem 3.31 (Bianchi [18])**.** *Let H, K, H' and K' be planar convex bodies of class C_+^8. Then $g_{H,K} = g_{H',K'}$ implies that (H, K) and (H', K') are trivial associates.*

In summary, the information provided by the cross covariogram of convex polygons or of sufficiently smooth planar convex bodies is rich enough to determine not only one unknown body, as required by Matheron's Conjecture, but two bodies, with a few exceptions.

Problem 3.27 is also relevant in trying to extend Theorem 3.31 to $\mathbb{R}^n$, $n > 2$.

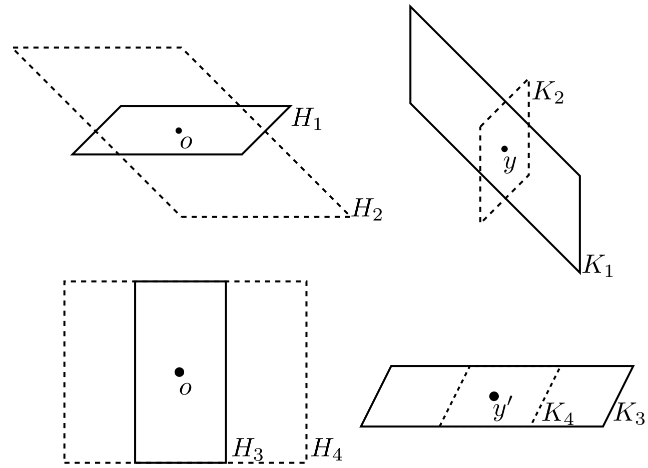

FIGURE 2. Here, $g_{H_1,K_1} = g_{H_2,K_2}$ and $g_{H_3,K_3} = g_{H_4,K_4}$. Moreover, up to affine transformations, these are the only pairs of planar convex polygons with equal cross covariograms.

4. Algorithms for reconstruction

None of the uniqueness proofs provide a method for actually reconstructing a convex body from its covariogram. For the phase retrieval problems, many algorithms have been developed, motivated by the diverse applications. We refer the interested reader to [10], [26] and [27] for a description. We are aware of only three papers dealing specifically with the reconstruction from the covariogram. Schmitt [64] gives an explicit reconstruction procedure for a convex polygon when no pair of its edges are parallel, an assumption removed in an algorithm due to Benassi and D'Ercole [12]. Bianchi, Kiderlen and Gardner [19] solve the following three problems. In each, K is a convex body in $\mathbb{R}^n$.

Problem 4.1 (Reconstruction from covariograms). Construct an approximation to K from a finite number of noisy (i.e., taken with error) measurements of g_K.

Problem 4.2 (Phase retrieval for characteristic functions of convex bodies: squared modulus). Construct an approximation to K (or, equivalently, to 1_K) from a finite number of noisy measurements of $|\widehat{1_K}|^2$.

Problem 4.3 (Phase retrieval for characteristic functions of convex bodies: modulus). Construct an approximation to K from a finite number of noisy measurements of $|\widehat{1_K}|$.

In both [64] and [12], all the exact values of the covariogram are supposed to be available. In contrast, the set of algorithms in [19] for Problem 4.1 take as input only a finite number of values of the covariogram of K. Moreover, these measurements are corrupted by errors, modeled by zero mean random variables with uniformly bounded pth moments, where p is at most six and usually four. It is assumed that K is determined by its covariogram, has its centroid at the origin, and is contained in a known bounded region of $\mathbb{R}^n$, which for convenience is taken to be the unit cube $[-1/2, 1/2]^n$. [19] provides two different methods for reconstructing, for each suitable $k \in \mathbb{N}$, a convex polytope P_k that approximates K or its reflection $-K$. Each method involves two algorithms, an initial algorithm that produces suitable outer unit normals to the facets of P_k, and a common main algorithm that goes on to actually construct P_k.

In the first method, the covariogram of K is measured, *multiple times*, at the origin and at vectors $(1/k)u_i$, $i = 1, \dots, k$, where the u_i's are mutually nonparallel unit vectors that span $\mathbb{R}^n$. From these measurements, the initial Algorithm NoisyCovBlaschke constructs an o-symmetric convex polytope Q_k that approximates ∇K, the Blaschke body of K. The crucial property of ∇K is that when K is a convex polytope, each of its facets is parallel to some facet of ∇K. It follows that the outer unit normals to the facets of P_k can be taken to be among those of Q_k. Algorithm NoisyCovBlaschke utilizes (2.9), i.e., the fact that $(\partial^+ g_K)/(\partial u)(o)$ equals the brightness function $b_K(u)$. This connection allows most of the work to be done by an algorithm, designed earlier by Gardner and Milanfar (see [34] and the references given there) for reconstructing a o-symmetric convex body from finitely many noisy measurements of its brightness function.

The second method achieves the same goal with a quite different approach. This time the covariogram of K is measured once at each point in a cubic array in $[-1, 1]^n$ of side length $1/k$. From these measurements, the initial Algorithm NoisyCovDiff(φ) constructs an o-symmetric convex polytope Q_k that approximates the difference body DK. The set DK has precisely the same property as ∇K, that when K is a convex polytope, each of its facets is parallel to some facet of DK. Furthermore, DK is just the support of g_K. The known property that $g_K^{1/n}$ is concave can therefore be combined with techniques from multiple regression. Algorithm NoisyCovDiff(φ) employs a Gasser–Müller type kernel estimator for g_K, with suitable kernel function φ, bandwidth and threshold parameter.

The output Q_k of either initial algorithm forms part of the input to the main common Algorithm NoisyCovLSQ. The covariogram of K is now measured *again*, once at each point in a cubic array in $[-1, 1]^n$ of side length $1/k$. Using these measurements, Algorithm NoisyCovLSQ finds a convex polytope P_k, each of whose facets is parallel to some facet of Q_k, whose covariogram fits best the measurements in the least squares sense.

These algorithms are strongly consistent. Whenever K is determined among convex bodies, up to translations and reflections, by its covariogram, [19] shows that, almost surely,

$$\min\{\delta(K, P_k), \delta(-K, P_k)\} \to 0$$

as $k \to \infty$. (If K is not so determined, the algorithms still construct a sequence (P_k) whose accumulation points exist and have the same covariogram as K.) From a theoretical point of view, this completely solves Problem 1.

The basic idea in [19] to solve Problem 4.2 is simple enough: Use (1.3) and the measurements of $|\widehat{1_K}|^2$ at points in a suitable cubic array to approximate g_K via its Fourier series, and feed the resulting values into the algorithms for Problem 4.1. Two major technical obstacles arise. The new estimates of g_K are corrupted by noise that now involves dependent random variables, and a new deterministic error appears as well. A substitute for the Strong Law of Large Numbers must be proved, and the deterministic error controlled using Fourier analysis and the fortunate fact that g_K is Lipschitz. In the end the basic idea works, assuming that for suitable $1/2 < \gamma < 1$, measurements of $|\widehat{1_K}|^2$ are taken at the points in $(1/k^\gamma)\mathbb{Z}^n$ contained

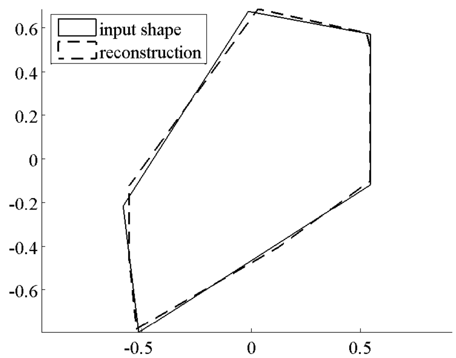

FIGURE 3. Pentagon, no noise.

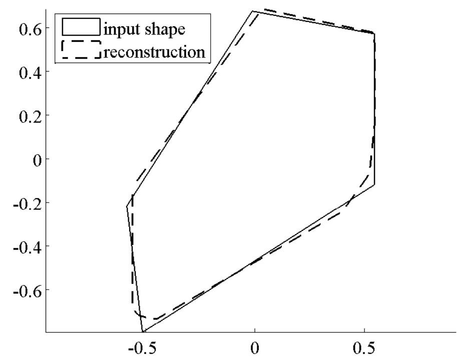

FIGURE 4. Pentagon, $\sigma = 0.01$.

in the cubic window $[-k^{1-\gamma}, k^{1-\gamma}]^n$, whose size increases with k at a rate depending on the parameter γ. The three resulting algorithms, Algorithm NoisyMod^{2}LSQ, Algorithm NoisyMod^{2}Blaschke and Algorithm NoisyMod^{2}Diff(φ), are stated in detail and, with suitable restrictions on γ, proved to be strongly consistent under the same hypotheses as for Problem 4.1.

[19] also constructs three algorithms for Problem 4.3. Again there is a basic simple idea, namely, to take two independent measurements at each of the points in the same cubic array as in the previous paragraph, multiply the two, and feed the resulting values into the algorithms for Problem 4.2. No serious extra technical difficulties arise, and it is proved that the three new algorithms are strongly consistent under the same hypotheses as for Problem 4.2. This provides a complete theoretical solution to the Phase Retrieval Problem for characteristic functions of convex bodies.

The study in [19] is a theoretical one. Convergence rates are given for Algorithm NoisyCovDiff(ϕ), and hence for the two related algorithms for phase retrieval, but are missing for the other algorithms. In particular, proving convergence rates for Algorithm NoisyCovLSQ would need suitable stability versions of the uniqueness results for the Covariogram Problem, which are not available. Figures 3, 4, 5 and 6, taken from [19], present the reconstructions obtained in some experiments of a rudimentary implementation of Algorithm NoisyCovBlaschke and NoisyCovLSQ in the planar case. They are based on Gaussian $N(0, \sigma^2)$ noise, $k = 60$ equally spaced directions in Algorithms NoisyCovBlaschke and $k = 8$ in Algorithm NoisyCovLSQ.

The website Geometric Tomography [32], a project of R. J. Gardner, offers a GUI to access a basic implementation of the algorithms for Problem 1 in the plane.

5. WHAT INFORMATION ABOUT A SET CAN BE OBTAINED FROM ITS COVARIOGRAM?

A natural question is to understand what information about a general regular compact set C, not necessarily convex, can be obtained from g_C. Only a few results are known.

5.1. **Recognizing convexity.** There are some properties of the covariogram of a convex set that may help in distinguishing it from a nonconvex one. For instance, the concavity of $g_C^{1/n}$ on its support, the convexity of $\operatorname{supp} g_C$ and the inequality (2.6) between $g_C(o)$ and the volume of $\operatorname{supp} g_C$ coming from the Rogers–Shepard inequality.

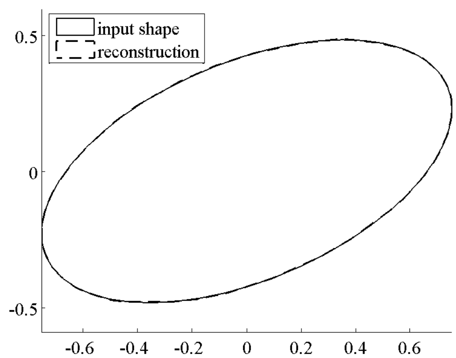

FIGURE 5. Ellipse, no noise.

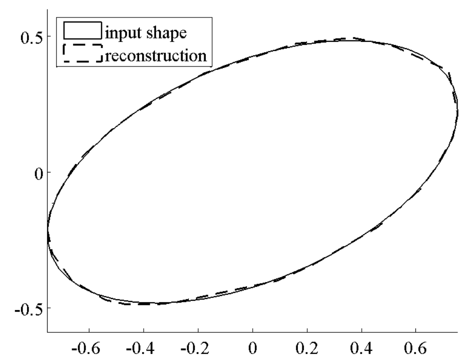

FIGURE 6. Ellipse, $\sigma = 0.01$.

Benassi, Bianchi and D'Ercole [11] give some other answers to this problem, mostly in the case of planar sets. Their main result is synthesized in the following theorem.

Theorem 5.1. *Let $\mathcal{D}$ be the class of regular compact sets in $\mathbb{R}^2$ whose interiors have at most two components and let $\mathcal{P}$ denote the class of regular compact sets in $\mathbb{R}^2$ whose boundaries consists of a finite number of closed disjoint polygonal curves, each with finitely many edges.*

If $C \in \mathcal{D}$ then the information provided by $\operatorname{supp} g_C$ and by $(\partial^+ g_C/\partial u)(o)$, for all $u \in S^1$, determines whether C is convex. If $C \in \mathcal{P}$ then the information provided by $\operatorname{supp} g_C$ and by the discontinuities of ∇g_C determines whether C is convex.

We explain the result regarding the class $\mathcal{D}$. If $C \subset \mathbb{R}^2$ is a convex body and $u \in S^1$, the formula

$$b_C(u) = w_C(\mathcal{R}_{\frac{\pi}{2}}\, u) = \frac{1}{2}\, w_{DC}(\mathcal{R}_{\frac{\pi}{2}}\, u),$$

allows (2.9) to be written as

$$-\frac{\partial^+ g_C}{\partial u}(o) = \frac{1}{2}\, w_{\operatorname{supp} g_C}(\mathcal{R}_{\frac{\pi}{2}}\, u). \tag{5.1}$$

If $C \in \mathcal{D}$ is not convex, then there exists $u \in S^1$ such that $C|u^\perp$ is an interval and $C \cap (l_u + x)$ has at least two components for a set of $x \in u^\perp$ of positive $\mathcal{H}^1$-measure. For such u one can prove that either $(\partial^+ g_C/\partial u)(o)$ does not exist or

$$-\frac{\partial^+ g_C}{\partial u}(o) > \frac{1}{2}\, w_{\operatorname{supp} g_C}(\mathcal{R}_{\frac{\pi}{2}}\, u), \tag{5.2}$$

violating (5.1). To see in a particular example why strict inequality holds in (5.2), imagine what happens when C is the union of two disjoint convex bodies C_1 and C_2 and u is as above. In this case, $-(\partial^+ g_C/\partial u)(o) = b_{C_1}(u) + b_{C_2}(u) = \mathcal{H}^1(C_1|u^\perp) + \mathcal{H}^1(C_2|u^\perp)$, while the term on the right-hand side in (5.2) is

$$\begin{aligned} \frac{1}{2}\, w_{DC}(\mathcal{R}_{\frac{\pi}{2}}\, u) &= \frac{1}{2}\, w_{D(\operatorname{conv} C)}(\mathcal{R}_{\frac{\pi}{2}}\, u) = w_{\operatorname{conv} C}(\mathcal{R}_{\frac{\pi}{2}}\, u) = \\ &= \mathcal{H}^1(\operatorname{conv}(C_1 \cup C_2)|u^\perp) < \mathcal{H}^1(C_1|u^\perp) + \mathcal{H}^1(C_2|u^\perp). \end{aligned} \tag{5.3}$$

The first equality in (5.3) is due to the fact that $C|u^\perp$ is an interval, while the last inequality holds because $C_1|u^\perp$ and $C_2|u^\perp$ overlap, a consequence of the assumption about $C \cap (l_u + x)$.

When $C \in \mathcal{P}$, the result rests ultimately on Lemma 3.20, which remains valid for nonconvex elements of $\mathcal{P}$. This lemma expresses the discontinuities of ∇g_C through the singular part of the distributional derivative $\partial^2 g_C/\partial w^2$, for $w \in S^1$, i.e. through the distribution defined by the second sum in (3.9). This distribution is supported in the set formed by the differences of any two parallel edges of C, including the differences of an edge with itself. This is a finite union of segments. Moreover, if $x \neq o$ belongs to this set, it provides the length of $\partial C \cap (\partial C + x)$. [11] proves that this information and the knowledge of $\operatorname{supp} g_C$ distinguishes between convex and nonconvex sets in $\mathcal{P}$.

A consequence of Theorem 5.1 is a strengthening of Theorem 3.2.

Corollary 5.2. *Every planar convex body is determined within the class $\mathcal{D} \cup \mathcal{P}$ by its covariogram, up to translations and reflections.*

5.2. **Recognizing symmetry properties.** The covariogram g_C is an even function, independently of any symmetry property of C. When C is convex, recognizing from g_C whether C is centrally symmetric is possible.

Theorem 5.3. *a) Let $C \in \mathcal{C}^n$ be regular. The set C is convex and centrally symmetric if and only if $2^n g_C(o) = \mathcal{H}^n(\operatorname{supp} g_C)$. If this equality holds, then $C = (1/2) \operatorname{supp} g_C$, up to translations.*
b) A centrally symmetric convex body is determined by its covariogram, up to translations, in the class of all regular compact sets.

Proof. Item a) is a consequence of the equality condition in the inequality (2.5) in Proposition 2.2, and of $\operatorname{supp} g_C = DC = 2C$. Item b) follows from Item a). □

In contrast to this, we do not know of a way of recognizing from g_C the central symmetry of a nonconvex set C.

Is it possible to recognize the radial symmetry of the set from its covariogram. A result of Lawton [47, Corollary 1] yields the following theorem.

Theorem 5.4. *Let $n \geq 2$ and let $C \subset \mathbb{R}^n$ be a regular compact set such that g_C is radially symmetric. Then a translation of C is radially symmetric and C is determined by g_C, up to translations and reflections, in the class of regular compact sets.*

Lawton proves the corresponding result for real-valued $L^2(\mathbb{R}^n)$ functions with compact support using techniques from the theory of functions of several complex variables. More precisely, the result is a consequence of a representation formula for entire functions of exponential type such that the modulus of their restriction to $\mathbb{R}^n$ is radially symmetric and in $L^2(\mathbb{R}^n)$.

6. The discrete covariogram

There is a counterpart to the covariogram in the discrete case. The *discrete covariogram* g_A of a finite subset A of $\mathbb{R}^n$ is defined by

$$g_A(x) = |A \cap (A + x)|, \tag{6.1}$$

for $x \in \mathbb{R}^n$. When no confusion can arise, we shall refer to the discrete covariogram of a finite set simply as its covariogram. As in the case of the ordinary (continuous) covariogram, it is unchanged by a translation or a reflection, and its support is DA. Note that

$$g_A(x) = |\{y \in A : y - x \in A\}|,$$

i.e., $g_A(x)$ is the number of "chords" of A that are translates of the line segment $[o,x]$. Thus the covariogram can be identified with the *multiset* $A+(-A)$, that is, the set DA where each element is repeated with multiplicity. In particular, $g_A = g_B$ if and only if A and B have the same set of chords, each repeated with multiplicity, and this is true if and only if $A+(-A)$ and $B+(-B)$ are equal as multisets.

Finite sets with equal covariograms are sometimes called *homometric*. Also multisets A and B such that the multisets $A+(-A)$ and $B+(-B)$ are equal are called *homometric*. When two sets are homometric and not equal up to translations and reflections, we say that they are *nontrivially homometric*. We refer to the survey paper of Senechal [67] for an introduction to homometric sets. Here we mention only a few facts.

If A and B are multisets, then the multisets

$$A+B \quad \text{and} \quad A+(-B) \tag{6.2}$$

are homometric. Indeed,

$$(A+B)+\big(-(A+B)\big) = A+(-A)+B+(-B) = \big(A+(-B)\big)+\big(-(A+(-B))\big).$$

If $|A|=2$ or $|B|=2$, then A or B is centrally symmetric and the two sets in (6.2) are equal up to translations and reflections. Thus one cannot construct four-, six- and eight-point nontrivially homometric pairs this way, but nine-point pairs abound. If A and B are sets, not multisets, and each point of $A+B$ (and of $A+(-B)$) can be written in an unique way as sum of a point of A and a point of B (or of $-B$, respectively), then $A+B$ and $A+(-B)$ are sets with equal covariograms. Thus, for example, $\{0,1,3,8,9,11,12,13,15\}$ and $\{0,1,3,4,5,7,12,13,15\}$ in $\mathbb{R}$ have equal covariograms and arise from the above construction by taking $A=\{6,7,9\}$ and $B=\{-6,2,6\}$. Not every homometric set can be constructed by this procedure. For example, $\{0,1,2,5,7,9,12\}$ and $\{0,1,5,7,8,10,12\}$ have equal covariograms, but do not arise from the above construction. Indeed, if they did, we would have $|A||B| = |A+B| = 7$ and hence either $|A|=1$ or $|B|=1$, an impossibility.

Gardner, Gronchi and Zong [33, Theorem 4.5] establish the following connection between the discrete and the continuous covariogram.

Theorem 6.1. *Let A and B be finite subsets of $\mathbb{R}^n$ with equal discrete covariograms. If X is a bounded Lebesgue-measurable set such that*

$$\mathcal{H}^n(A+X) = |A|\,\mathcal{H}^n(X) \quad \textit{and} \quad \mathcal{H}^n(B+X) = |B|\,\mathcal{H}^n(X),$$

then $A+X$ and $B+X$ have equal continuous covariograms.

The assumption $\mathcal{H}^n(A+X) = |A|\,\mathcal{H}^n(X)$ says that there are no overlaps in the sum $A+X$, i.e. in the union $\bigcup_{a\in A}(X+a)$, except for sets of measure zero. A consequence of this theorem is that if A and B are lattice sets with equal discrete covariograms, then the associated lattice bodies $A+[0,1]^n$ and $B+[0,1]^n$ have equal continuous covariograms.

Gardner, Gronchi and Zong [33] present a pair of noncongruent nonconvex polygons with equal covariograms; see Figure 7. They are the lattice bodies associated to nontrivially homometric planar polyominoes which can be written as $A+B$ and $A+(-B)$, where A and B are the lattice sets in Figure 8.

Another pair of planar lattice bodies with equal covariograms, made of nine squares and not equal up to translations and reflections, appeared in [24, Figure 1]. The polyominoes in the examples in [33] and in [24] are convex, and the associated

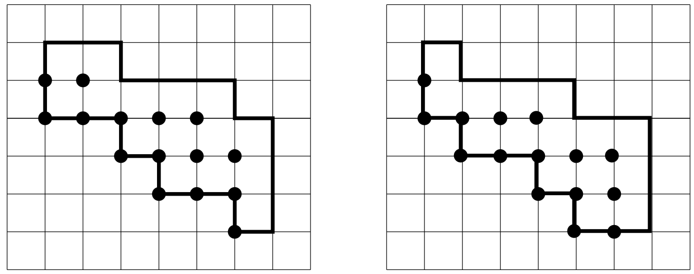

FIGURE 7. Two noncongruent nonconvex polygons with equal covariograms. They arise as lattice bodies of two homometric convex polyominoes.

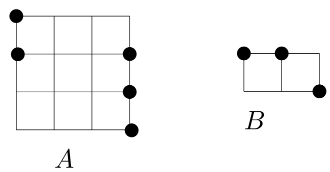

FIGURE 8. The polyominoes in Figure 7 are equal to $A + B$ and $A + (-B)$.

lattice bodies are both horizontally and vertically convex. In the example in [24] one lattice body is the reflection with respect to a line of the other one.

Gardner, Gronchi and Zhong [33] prove that a centrally symmetric finite set A is determined by g_A, up to translations, in the class of centrally symmetric finite sets, thus extending Theorem 3.1 to the discrete case. Averkov [2] considerably strengthens this result by proving that the determination holds in the class of all finite sets.

Theorem 6.2 (Averkov [2])**.** *A centrally symmetric finite subset A of $\mathbb{R}^n$ is determined by g_A, up to translations, in the class of all finite sets.*

Proof. Let B be a finite subset of $\mathbb{R}^n$ with $g_A = g_B$.

Suppose that $n = 1$; this case contains the heart of the proof. Let $a = \max \operatorname{supp} g_A$. Then

$$DA = \operatorname{supp} g_A \subset [-a, a] \quad \text{and} \quad a \in DA, \tag{6.3}$$

and analogous formulas hold for B. We may assume that $a > 0$, because otherwise A is a singleton, the same is true for B, and therefore B is a translate of A. The first formula in (6.3) implies that a translate of A is contained in $[0, a]$. We may thus assume, up to translations, that

$$A \subset [0, a] \quad \text{and} \quad B \subset [0, a].$$

The second formula in (6.3) implies $0, a \in A$. Similarly, $0, a \in B$. The set A is symmetric about $a/2$. Let $A \cap [a/2, a] = \{y_1, y_2, \ldots, y_m\}$, with $m \in \mathbb{N}$ and appropriate $y_1 < y_2 < \cdots < y_m = a$. We show by (reverse) induction that the sets

$$A_k = ([0, a - y_k] \cup [y_k, a]) \cap A \quad \text{and} \quad B_k = ([0, a - y_k] \cup [y_k, a]) \cap B$$

coincide, for every $k = 1, \ldots, m$. For $k = m$ this follows from $0, a \in A \cap B$.

Suppose that $A_{k+1} = B_{k+1}$.

We claim that $B \cap (y_k, y_{k+1}) = \emptyset$. Assume the contrary and let $x \in B \cap (y_k, y_{k+1})$. All pairs (x_0, x_1) with $x_0, x_1 \in A$ and $x_1 - x_0 = x$ satisfy $x_0, x_1 \in A_{k+1} = B_{k+1}$. Indeed, $x_1 = x_0 + x \geq x > y_k$ and $x_0 = x_1 - x \leq a - x < a - y_k$. Furthermore, B possesses at least one further pair (x_0, x_1) with $x_0, x_1 \in B$ and $x_1 - x_0 = x$, since we may set $x_0 = 0$ and $x_1 = x$. By the definition of the points y_k, we have $A \cap (y_k, y_{k+1}) = \emptyset$ and therefore $x \notin A$. Hence $g_B(x) > g_A(x)$, a contradiction. An analogous argument proves that $B \cap (a - y_{k+1}, a - y_k) = \emptyset$.

Next we show that $\{a - y_k, y_k\} \subset B$. We look at the pairs (x_0, x_1) with $x_0, x_1 \in A$ and $x_1 - x_0 = y_k$. Since $x_1 = x_0 + y_k \geq y_k$ and equality holds if and only if $x_0 = 0$, either $(x_0, x_1) = (0, y_k)$ or $x_1 \in A_{k+1}$. Analogously, either $(x_0, x_1) = (a - y_k, a)$ or $x_0 \in A_{k+1}$. Thus,

$$\begin{aligned} g_A(y_k) &= |\{(x_0, x_1) : x_0, x_1 \in A, x_1 - x_0 = y_k\}| \\ &= |\{(x_0, x_1) : x_0, x_1 \in A_{k+1}, x_1 - x_0 = y_k\}| + |\{(0, y_k), (a - y_k, a)\}| \\ &= |\{(x_0, x_1) : x_0, x_1 \in B_{k+1}, x_1 - x_0 = y_k\}| + |\{(0, y_k), (a - y_k, a)\}| \\ &\geq g_B(x) \end{aligned}$$

and equality holds if and only if $(0, y_k), (a - y_k, a) \in B \times B$, i.e., $a - y_k, y_k \in B$. This concludes the proof that $A_k = B_k$ and, by induction, that $A_1 = B_1$. To prove that $A = B$ it remains to prove that $B \cap (a - y_1, y_1) = \emptyset$, and this can be done via analogous arguments.

Now assume that $n > 1$. We argue by induction on n and assume the claim true for dimension $n - 1 \geq 1$. Without loss of generality, we assume that o is the centroid both of A and of B. Let

$$U = S^{n-1} \setminus \big\{(x_1 - x_2)/|x_1 - x_2| : x_1, x_2 \in \operatorname{supp} g_A, x_1 \neq x_2\big\}.$$

If $u \in U$, then the orthogonal projection of $\mathbb{R}^n$ onto $u^\perp$ is injective on the sets A and B. It is not difficult to prove that this projection maps sets with equal covariograms to sets with equal covariograms. The inductive hypothesis, the central symmetry of $A|u^\perp$, and the assumption about the centroids of A and B prove that $A|u^\perp = B|u^\perp$. This is true for every $u \in U$ and hence for infinitely many $u \in S^{n-1}$. Heppes [40] proves that a finite set with cardinality k is determined by its orthogonal projections in $k + 1$ mutually nonparallel directions. This implies $A = B$. □

This result applies also to lattice sets and, due to Theorem 6.1, it implies the following corollary.

Corollary 6.3 (Averkov [2]). *A centrally symmetric lattice body $A \subset \mathbb{R}^n$ is determined by g_A, up to translations, in the class of all lattice bodies.*

Averkov and Langfeld [6, 7] study the problem of determination from the covariogram in the class of *convex* lattice sets in $\mathbb{Z}^2$. The polyominoes associated to the examples of nondetermination in [33] and [24] discussed above are convex and therefore we cannot expect a global positive answer. In [6] it is shown that if a planar convex lattice set A samples $\operatorname{conv} A$ well enough (that is, if, in a certain sense, A is close enough to a convex polygon) then the determination from g_A is similar to the determination in the case of convex polygons.

We need some terminology. For $u \in \mathbb{R}^2 \setminus \{o\}$, let $A_u = \{x \in A : \langle x, u\rangle = h_A(u)\}$ denote the support set of A in the direction u. A support set A_u which contains more than one element will be called an *edge* of A with outer normal u. Let

$$U(A) = \{(u_1, u_2) \in \mathbb{Z}^2 \setminus \{o\} : (u_1, u_2) \text{ is an outer normal to an edge of } A \text{ and } u_1 \text{ and } u_2 \text{ are relatively prime}\}.$$

To measure the number of lattice points on the edges of A and the difference between the number of points on one edge and the number of points on the antipodal parallel edge, we introduce the following functions:

$$m'(A) = \min\{|A_u|, u \in U(A)\},$$
$$m''(A) = \min\{|A_u| - |A_{-u}| + 1 : u \in \mathbb{Z}^2 \setminus \{o\} \text{ and } |A_u| > |A_{-u}| > 1\},$$
$$m(A) = \min\{m'(A), m''(A)\},$$

where we use the convention that $\min \emptyset = \infty$.

Theorem 6.4 (Averkov and Langfeld [6])**.** *Let A be a convex lattice set in $\mathbb{Z}^2$. Then $m'(A)$, $m''(A)$, $m(A)$ and $U(A) \cup U(-A)$ are determined by g_A. Let $l \in \mathbb{N}$ be such that $U(A) \cup U(-A) \subset \{-l, \dots, l\}^2$. If $m(A) > 4l^4 + 2l^2 + 1$, then A is determined by g_A in the class of convex lattice sets in $\mathbb{Z}^2$.*

Thus, for a given collection of prescribed edge normals, A is determined if all its edges have sufficiently large cardinality and the difference between cardinalities of parallel edges is either zero or sufficiently large.

Averkov and Langfeld [6, 7] also make substantial progress towards understanding the structure of nontrivially homometric pairs of convex lattice sets in $\mathbb{Z}^n$.

Example 6.5. ([6]) Let $k \in \mathbb{N} \setminus \{0\}$, $w_1 = (-k-1, 1)$, $w_2 = (k, 1)$ and

$$L = \mathbb{Z}w_1 + \mathbb{Z}w_2.$$

Choose A to be any finite subset of L which is convex with respect to L (i.e., $A = (\operatorname{conv} A) \cap L$) and such that each edge of the polygon $\operatorname{conv} A$ is parallel either to w_1 or to w_2 or to $w_2 - w_1 = (-1, 2)$. Let

$$B = (\{0, \dots, k\} \times \{0\}) \cup (\{0, \dots, k-1\} \times \{1\}).$$

The lattice sets $A + B$ and $A + (-B)$ in $\mathbb{Z}^2$ are convex (with respect to $\mathbb{Z}^2$) and have the same covariogram. If A is not centrally symmetric, $A + B$ and $A + (-B)$ are not equal up to translations and reflections. See Figure 9.

Up to linear transformations of $\mathbb{Z}^2$ and up to translations of K and L, the nontrivially homometric pairs H and K from [24, 33] are members of the family presented in Example 6.5. The example in [24] is obtained by taking $k = 1$ and $A = \{(0,0), (2,-1), (1,-2)\}$, and that in [33] is obtained by taking $k = 1$ and $A = \{(0,0), (1,1), (1,-2), (2,-1), (3,0)\}$, and applying to H and K the linear transformation $(x, y) \to (x - y, y)$.

The lattice L in Example 6.5 is 2-dimensional and B is convex with respect to $\mathbb{Z}^2$. Moreover $\mathbb{Z}^2 = L + B$ and this is a *direct sum* (which, in this setting, means that each element of $\mathbb{Z}^2$ can be written in an unique way as sum of elements of L and B). This implies that the translations of B by vectors in L tile $\mathbb{Z}^2$. [7, Theorem 2.4] proves that if B is a convex subset of $\mathbb{Z}^2$ and $L \subset \mathbb{Z}^2$ is a 2-dimensional lattice such that $\mathbb{Z}^2$ is the direct sum $L + B$, then the sets $A + B$ and $A + (-B)$ are nontrivially homometric if and only if, up to linear transformations in $\mathbb{Z}^2$ and translations, A, B and L are those described in Example 6.5.

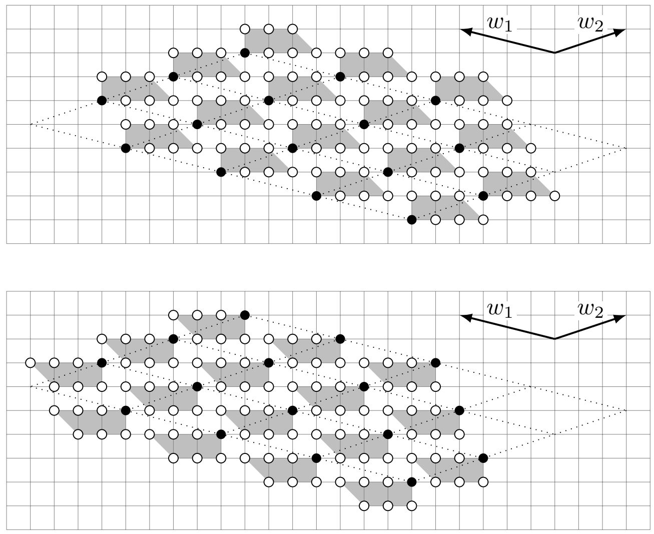

Figure 9. The sets $A+B$ (the union of the black and white points above) and $A+(-B)$ (below). The elements of A are drawn as black points and the convex hulls of the translates of B and $-B$ are indicated by gray polygons (from [6]).

One may wonder whether there are nontrivially homometric pairs of *convex* lattice sets in $\mathbb{Z}^2$ that do not arise from the construction $A+B$ and $A+(-B)$. Averkov and Langfeld [6] write that they performed an exhaustive computer search of such pairs among lattice sets which are contained in $\{1,\dots,6\}\times\{1,\dots,5\}$ without finding any.

[7, Example 5.5] presents the first examples of nontrivially homometric convex sets in $\mathbb{Z}^n$, for any $n\geq 2$, which are intrinsically n-dimensional, in the sense that they are not lifted from $\mathbb{Z}^2$ by taking Cartesian products.

7. Connections to Fourier analysis

In the previous sections we have already seen applications of results from Fourier analysis in studying the problem of determination from the covariogram. Here we present some other connections. They come from the literature on the phase retrieval problem and deal with the irreducibility of $\widehat{1_K}$. This connection also shows a link between the Covariogram Problem and the Pompeiu Problem in integral geometry.

We say that an entire function g is *irreducible* if g cannot be written as the product of entire functions g_1 and g_2 with $g_1\neq ag_2$, for each $a\in\mathbb{C}$, and with both $\{\zeta\in\mathbb{C}^n : g_1(\zeta)=0\}$ and $\{\zeta\in\mathbb{C}^n : g_2(\zeta)=0\}$ nonempty.

Let $f\in L^2(\mathbb{R}^n)$ have compact support. Sanz and Huang [63] prove that if $\widehat{f}$ is irreducible, then f is determined, up to trivial associates, by the knowledge of $|\widehat{f}(x)|$ for all $x\in\mathbb{R}^n$. Barakat and Newsam [9] and Stefanescu [68] prove that if f_1 and f_2 belong to $L^2(\mathbb{R}^2)$, have compact support, are not trivial associates and $|\widehat{f}_1(x)|=|\widehat{f}_2(x)|$ for all $x\in\mathbb{R}^2$, then there exist entire functions g_1 and g_2 such that

$\{\zeta \in \mathbb{C}^2 : g_1(\zeta) = 0\}$ and $\{\zeta \in \mathbb{C}^2 : g_2(\zeta) = 0\}$ are both nonempty and

$$\widehat{f_1}(\zeta) = g_1(\zeta)g_2(\zeta) \quad \text{and} \quad \widehat{f_2}(\zeta) = e^{\mathrm{i}(c+\langle d,\zeta\rangle)}g_1(\zeta)\overline{g_2\left(\overline{\zeta}\right)}, \tag{7.1}$$

for $\zeta \in \mathbb{C}^2$ and suitable $c \in \mathbb{R}$ and $d \in \mathbb{R}^2$. I. S. Stefanescu, in a letter to the author, has expressed the opinion that a similar result holds in any dimension $n \geq 2$. It is not known whether the property that $\widehat{f}$ is not irreducible implies that f is not determined by $|\widehat{f}|$.

What is the significance of these results for the Covariogram Problem? All the examples of nondetermination presented in Section 3.2 arise from a factorization of $\widehat{1_K}$ as in (7.1). Indeed if E, F, H and K are as in Theorem 3.11, and E and F are orthogonal subspaces, then

$$1_{H+K} = \delta_H * \delta_K \quad \text{and} \quad 1_{H+(-K)} = \delta_H * \delta_{-K},$$

where δ_H and δ_K are the distributions defined for $\phi \in C_0^\infty(\mathbb{R}^n)$ by

$$\delta_H(\phi) = \int_H \phi(x,0)\,dx, \quad \delta_K(\phi) = \int_K \phi(0,y)\,dy$$

(here dx and dy indicate integration with respect to Lebesgue measure in E and in F, respectively), and δ_{-K} is defined similarly. By the Paley–Wiener Theorem, $\widehat{\delta_H}$, $\widehat{\delta_K}$ and $\widehat{\delta_{-K}}$ are entire functions in $\mathbb{C}^n$ of exponential type. Clearly $\widehat{\delta_{-K}}(\zeta) = \overline{\widehat{\delta_K}\left(\overline{\zeta}\right)}$ and we have

$$\widehat{1_{H+K}}(\zeta) = \widehat{\delta_H}(\zeta)\widehat{\delta_K}(\zeta) \quad \text{and} \quad (\widehat{1_{H+(-K)}})(\zeta) = \widehat{\delta_H}(\zeta)\overline{\widehat{\delta_K}\left(\overline{\zeta}\right)},$$

as in (7.1).

In view of these results it would be interesting to study the following problem.

Problem 7.1. Find explicit geometric conditions on a convex body K which guarantee that $\widehat{1_K}$ is irreducible.

To appreciate the difficulty in answering to this question, consider the following subproblem.

Understand for which convex bodies K the function $\widehat{1_K}$ is the product of a nontrivial polynomial and an entire function.

We need some notation. Given a polynomial $p(\zeta) = \sum_{|l|\leq m} c_l \zeta^l$, where $m \in \mathbb{N}$, $l = (l_1, \dots, l_n)$ denotes a multi-index, $c_l \in \mathbb{C}$, $|l| = l_i + \cdots + l_n$ and $\zeta^l = \zeta_1^{l_1} \cdots \zeta_n^{l_n}$, let $p(D)$ denote the differential operator

$$p(D) = \sum_{|l|\leq m} (\mathrm{i})^{-|l|} c_l \left(\partial^{l_1}/\partial x_1^{l_1}\right) \cdots \left(\partial^{l_n}/\partial x_n^{l_n}\right),$$

where $\partial^0/\partial x_i^0$ denotes the identity operator. [60, Theorem 8.4] states that

$$\widehat{1_K} = fp,$$

with f entire and p a polynomial, if and only if the equation

$$p(D)u = 1_K, \tag{7.2}$$

has a solution u in the class of distributions with support contained in K. Here $\widehat{u} = f$ and (7.2) has to be understood in the sense of distributions. The theorem of supports for convolutions [41, Theorem 4.3.3] and elementary considerations imply that if a solution u to (7.2) exists, then its support is K.

A particular instance of this problem has received much attention. When $p(\zeta) = \zeta_1^2 + \cdots + \zeta_n^2 - c$, for some $c > 0$, (7.2) becomes

$$\begin{cases} \Delta u + cu = -1 & \text{in } K \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial K, \end{cases} \tag{7.3}$$

where ν denotes the exterior normal to ∂K. Let $E \subset \mathbb{R}^n$ be a bounded simply connected Lipschitz domain. The Pompeiu Problem asks whether there exists a nonzero continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathcal{T}(E)} f\,dx = 0 \quad \text{for all rigid motions } \mathcal{T} \text{ in } \mathbb{R}^n$$

only when E is a ball. It is known that the Pompeiu Problem is equivalent to asking whether a solution to (7.3) (with K replaced by E) exists for some $c > 0$ only if E is a ball (see Berenstein [13]). As far as we know, these problems are still open.

The example of a ball implies that the irreducibility condition is not necessary for determination by covariogram. Indeed, when K is a ball a solution to (7.3) exists and $\widehat{1_K}$ factors. On the other hand, in any dimension a ball K is uniquely determined by g_K, as Theorem 5.4 implies.

References

[1] R. J. Adler and R. Pyke, *Problem 91–3*, Inst. Math. Statist. Bull., 20 (1991), p. 409.

[2] G. Averkov, *Detecting and reconstructing centrally symmetric sets from the autocorrelation: two discrete cases*, Appl. Math. Lett., 22 (2009), pp. 1476–1478.

[3] G. Averkov and G. Bianchi, *Retrieving convex bodies from restricted covariogram functions*, Adv. in Appl. Probab., 39 (2007), pp. 613–629.

[4] ———, *Confirmation of Matheron's conjecture on the covariogram of a planar convex body*, J. Eur. Math. Soc. (JEMS), 11 (2009), pp. 1187–1202.

[5] ———, *Covariograms generated by valuations*, Int. Math. Res. Not. IMRN, (2015), pp. 9277–9329.

[6] G. Averkov and B. Langfeld, *On the reconstruction of planar lattice-convex sets from the covariogram*, Discrete Comput. Geom., 48 (2012), pp. 216–238.

[7] ———, *Homometry and direct-sum decompositions of lattice-convex sets*, Discrete Comput. Geom., 56 (2016), pp. 216–249.

[8] M. Baake and U. Grimm, *Homometric model sets and window covariograms*, Z. Krist., 222 (2007), pp. 54–58.

[9] R. Barakat and G. Newsam, *Necessary conditions for a unique solution to two-dimensional phase recovery*, J. Math. Phys., 25 (1984), pp. 3190–3193.

[10] H. H. Bauschke, P. L. Combettes, and D. R. Luke, *Phase retrieval, error reduction algorithm, and Fienup variants: a view from convex optimization*, J. Opt. Soc. Amer. A, 19 (2002), pp. 1334–1345.

[11] C. Benassi, G. Bianchi, and G. D'Ercole, *Covariogram of non-convex sets*, Mathematika, 56 (2010), pp. 267–284.

[12] C. Benassi and G. D'Ercole, *An algorithm for reconstructing a convex polygon from its covariogram*, Rend. Istit. Mat. Univ. Trieste, 39 (2007), pp. 457–476.

[13] C. A. Berenstein, *An inverse spectral theorem and its relation to the Pompeiu problem*, J. Analyse Math., 37 (1980), pp. 128–144.

[14] G. Bianchi, *Matheron's conjecture for the covariogram problem*, J. London Math. Soc. (2), 71 (2005), pp. 203–220.

[15] ———, *The covariogram determines three-dimensional convex polytopes*, Adv. Math., 220 (2009), pp. 1771–1808.

[16] ———, *The cross covariogram of a pair of polygons determines both polygons, with a few exceptions*, Adv. in Appl. Math., 42 (2009), pp. 519–544.

[17] ———, *Geometric tomography of convex cones*, Discrete Comput. Geom., 41 (2009), pp. 61–76.

[18] ———, *The covariogram and Fourier-Laplace transform in* $\mathbb{C}^n$, Proc. Lond. Math. Soc. (3), 113 (2016), pp. 1–23.

[19] G. Bianchi, R. J. Gardner, and M. Kiderlen, *Phase retrieval for characteristic functions of convex bodies and reconstruction from covariograms*, J. Amer. Math. Soc., 24 (2011), pp. 293–343.

[20] G. Bianchi, F. Segala, and A. Volčič, *The solution of the covariogram problem for plane* C^2_+ *convex bodies*, J. Differential Geom., 60 (2002), pp. 177–198.

[21] A. J. Cabo and A. J. Baddeley, *Line transects, covariance functions and set convergence*, Adv. in Appl. Probab., 27 (1995), pp. 585–605.

[22] ———, *Estimation of mean particle volume using the set covariance function*, Adv. in Appl. Probab., 35 (2003), pp. 27–46.

[23] A. J. Cabo and R. H. P. Janssen, *Cross-covariance functions characterise bounded regular closed sets*, CWI Technical Report, Amsterdam, 1994.

[24] A. Daurat, Y. Gérard, and M. Nivat, *Some necessary clarifications about the chords' problem and the partial digest problem*, Theoret. Comput. Sci., 347 (2005), pp. 432–436.

[25] K. Engel and B. Laasch, *The modulus of the Fourier transform on a sphere determines 3-dimensional convex polytopes*, J. Inverse Ill-Posed Probl., 30 (2022), pp. 475–483.

[26] J. R. Fienup, *Phase retrieval algorithms: a comparison*, Appl. Optics, 21 (1982), pp. 2758–2769.

[27] J. R. Fienup, T. R. Crimmins, and W. Holsztyński, *Reconstruction of the support of an object from the support of its autocorrelation*, J. Opt. Soc. Amer., 72 (1982), pp. 610–624.

[28] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom., 23 (1986), pp. 69–96.

[29] B. Galerne, *Computation of the perimeter of measurable sets via their covariogram. Applications to random sets*, Image Anal. Stereol., 30 (2011), pp. 39–51.

[30] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.), 39 (2002), pp. 355–405.

[31] ———, *Geometric tomography*, vol. 58 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, New York, second ed., 2006.

[32] ———, *Geometric Tomography.* `http://www.geometrictomography.com`, 2013. Accessed on February 8, 2023.

[33] R. J. Gardner, P. Gronchi, and C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom., 34 (2005), pp. 391–409.

[34] R. J. Gardner, M. Kiderlen, and P. Milanfar, *Convergence of algorithms for reconstructing convex bodies and directional measures*, Ann. Statist., 34 (2006), pp. 1331–1374.

[35] R. J. Gardner and G. Zhang, *Affine inequalities and radial mean bodies*, Amer. J. Math., 120 (1998), pp. 505–528.

[36] N. Garofalo and F. Segala, *Asymptotic expansions for a class of Fourier integrals and applications to the Pompeiu problem*, J. Analyse Math., 56 (1991), pp. 1–28.

[37] A. Gasparyan and V. K. Oganyan, *Orientation-dependent distribution of the length of a random segment and covariogram*, Izv. Nats. Akad. Nauk Armenii Mat., 50 (2015), pp. 3–12.

[38] P. Goodey, R. Schneider, and W. Weil, *On the determination of convex bodies by projection functions*, Bull. London Math. Soc., 29 (1997), pp. 82–88.

[39] E. K. Haviland and A. Wintner, *On the Fourier transforms of distribution on convex curves*, Duke Math. J., 2 (1936), pp. 712–721.

[40] A. Heppes, *On the determination of probability distributions of more dimensions by their projections*, Acta Math. Acad. Sci. Hungar., 7 (1956), pp. 403–410.

[41] L. Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis*, Reprint of the second (1990) ed., Classics in Mathematics, Springer, Berlin, 2003.

[42] N. E. Hurt, *Phase retrieval and zero crossings*, Mathematical methods in image reconstruction, 52, Kluwer Academic Publishers Group, Dordrecht, 1989.

[43] K. Kiener, *Extremalität von Ellipsoiden und die Faltungsungleichung von Sobolev*, Arch. Math. (Basel), 46 (1986), pp. 162–168.

[44] M. V. Klibanov, P. E. Sacks, and A. V. Tikhonravov, *The phase retrieval problem*, Inverse Problems, 11 (1995), pp. 1–28.

[45] T. Kobayashi, *Asymptotic behaviour of the null variety for a convex domain in a non-positively curved space form*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 36 (1989), pp. 389–478.

[46] ———, *Convex domains and the Fourier transform on spaces of constant curvature*, in Lecture notes of the Unesco-Cimpa School on "Invariant differential operators on Lie groups and homogeneous spaces", P. Torasso, ed., WuHan University, China, 1991.

[47] W. Lawton, *Uniqueness results for the phase-retrieval problem for radial functions*, J. Opt. Soc. Amer., 71 (1981), pp. 1519–1522.

[48] D. R. Luke, J. V. Burke, and R. G. Lyon, *Optical wavefront reconstruction: theory and numerical methods*, SIAM Review, 44 (2002), pp. 169–224.

[49] F. C. Machado and S. Robins, *The null set of a polytope, and the Pompeiu property for polytopes,* arxiv:2104.01957, (2021).

[50] C. L. Mallows and J. M. C. Clark, *Linear-intercept distributions do not characterize plane sets*, J. Appl. Probability, 7 (1970), pp. 240–244.

[51] G. Matheron, *Le covariogramme géometrique des compacts convexes des R^2*, Technical report N-2/86/G, Centre de Géostatistique, Ecole Nationale Supérieure des Mines de Paris, 1986.

[52] ———, *Random sets and integral geometry*, John Wiley & Sons, New York-London-Sydney, 1975.

[53] M. Meyer, S. Reisner, and M. Schmuckenschläger, *The volume of the intersection of a convex body with its translates*, Mathematika, 40 (1993), pp. 278–289.

[54] R. P. Millane, *Recent advances in phase retrieval*, in Image Reconstruction from Incomplete Data IV, P. J. Bones, M. A. Fiddy, and R. P. Millane, eds., vol. 6316, Proc. SPIE, 2006.

[55] W. Nagel, *Orientation-dependent chord length distributions characterize convex polygons*, J. Appl. Probab., 30 (1993), pp. 730–736.

[56] J. Rataj, *On set covariance and three-point test sets*, Czechoslovak Math. J., 54(129) (2004), pp. 205–214.

[57] M. Reitzner, M. Schulte, and C. Thäle, *Limit theory for the Gilbert graph*, Adv. in Appl. Math., 88 (2017), pp. 26–61.

[58] M. Reitzner, E. Spodarev, and D. Zaporozhets, *Set reconstruction by Voronoi cells*, Adv. in Appl. Probab., 44 (2012), pp. 938–953.

[59] J. Rosenblatt, *Phase retrieval*, Comm. Math. Phys., 95 (1984), pp. 317–343.

[60] W. Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, second ed., 1991.

[61] K. Rufibach, *Phasenrekonstruktion von Gleichverteilungen in R^d*, Diploma Thesis, University of Bern, Switzerland, 2001.

[62] L. A. Santaló, *Integral geometry and geometric probability*, Cambridge University Press, Cambridge, second ed., 2004.

[63] J. L. C. Sanz and T. S. Huang, *Phase reconstruction from magnitude of band-limited multidimensional signals*, J. Math. Anal. Appl., 104 (1984), pp. 302–308.

[64] M. Schmitt, *On two inverse problems in mathematical morphology*, in Mathematical morphology in image processing, vol. 34 of Opt. Engrg., Dekker, New York, 1993, pp. 151–169.

[65] R. Schneider, *Polytopes and Brunn–Minkowski theory*, in Polytopes: abstract, convex and computational (Scarborough, ON, 1993), vol. 440 of NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1994, pp. 273–299.

[66] ———, *Convex bodies: the Brunn-Minkowski theory*, vol. 151 of Encycl. Math. Appl., Cambridge University Press, Cambridge, 2nd expanded ed., 2014.

[67] M. Senechal, *A point set puzzle revisited*, European J. Combin., 29 (2008), pp. 1933–1944.

[68] I. S. Stefanescu, *On the phase retrieval problem in two dimensions*, J. Math. Phys., 26 (1985), pp. 2141–2160.