SOME RIEMANNIAN PROPERTIES OF $\mathrm{SU_n}$ ENDOWED WITH A BI-INVARIANT METRIC

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Abstract. We study some properties of SU_n endowed with the Frobenius metric ϕ, which is, up to a positive constant multiple, the unique bi-invariant Riemannian metric on SU_n. In particular we express the distance between $P, Q \in SU_n$ in terms of eigenvalues of P^*Q; we compute the diameter of (SU_n, ϕ) and we determine its diametral pairs; we prove that the set of all minimizing geodesic segments with endpoints P, Q can be parametrized by means of a compact connected submanifold of $\mathfrak{su}_n$, diffeomorphic to a suitable complex Grassmannian depending on P and Q.

Contents

Keywords. Special unitary group, Frobenius metric, bi-invariant metrics on a compact Lie group, minimizing geodesics, homogeneous space, complex Grassmannian, diameter, diametral pairs, generalized principal logarithm;

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Introduction

Let G be a closed subgroup of the unitary group U_n and let $\mathfrak{g} \subseteq \mathfrak{u}_n$ be its Lie algebra. It is natural to consider on G the *Frobenius* (or *Hilbert-Schmidt*) *metric* ϕ induced on G by the Euclidean metric of $\mathbb{C}^{n^2}$ (identified with $\mathfrak{gl}_n(\mathbb{C})$: the Lie algebra of complex square matrices of order n). Since $G \subseteq U_n$, the metric ϕ is bi-invariant on G. Furthermore, if

G is also absolutely simple and connected (e.g. the classical groups SO_n for $n \geq 3$ and $n \neq 4$, SU_n for $n \geq 2$ and Sp_n for $n \geq 1$), every bi-invariant $(0,2)$-tensor on G (such as the Killing metric) is a constant multiple of ϕ (see, for instance, [Dolcetti-Pertici 2023, Prop. 1.8]). In particular, all bi-invariant $(0,2)$-tensors on such groups are Riemannnian or anti-Riemannian metrics and have the same geodesics and the same isometries.

In [Dolcetti-Pertici 2023, Thm. 2.3] we determined all isometries of any absolutely simple, compact, connected, real Lie group endowed with any bi-invariant metric; from that result, we deduced the list of all isometries of SO_n ($n \geq 3$, $n \neq 4$), SU_n ($n \geq 2$) and Sp_n ($n \geq 1$), endowed with any bi-invariant metric ([Dolcetti-Pertici 2023, Thm. 2.5 (c), (d)]). Furthermore, by means of suitable different methods, we determined the list of all isometries also of $SO(4)$ and of $U(n)$ ($n \geq 2$), all endowed with the Frobenius metric ([Dolcetti-Pertici 2023, Thm. 3.5, Thm. 4.7]).

On the other hand, in [Pertici-Dolcetti 2022], we studied the Riemannian properties of many suitable subgroups G of the unitary group U_n (we called them SVD-closed) endowed with the Frobenius metric ϕ. In particular, we expressed the distance between points and we parametrized the set of all minimizing geodesic segments of (G, ϕ) with arbitrary endpoints P, Q, by means of a set, denoted by $\mathfrak{g}$–$plog(P^*Q)$, consisting of generalized principal logarithms of P^*Q belonging to $\mathfrak{g}$; we proved that this last set is non-empty, it is a union of finitely many compact submanifolds of the Lie algebra $\mathfrak{u}_n$ and that each of these submanifolds is diffeomorphic to a suitable homogeneous space ([Pertici-Dolcetti 2022, Thm. 5.7 and Thm. 6.5]). For a lot of families of SVD-closed subgroups of U_n (among them U_n, SO_n, Sp_n and others) we also determined the (Frobenius) diameter of each subgroup and the structure of each connected component of $\mathfrak{g}$–$plog(P^*Q)$ as a homogeneous space ([Pertici-Dolcetti 2022, Prop. 6.7 and § 7, § 8]).

Most classical subgroups of U_n are SVD-closed with the exception of SU_n, when $n \geq 3$. For this reason, in this paper we try to obtain similar results for (SU_n, ϕ). In particular, we are able to express the distance between points, to compute the diameter of (SU_n, ϕ), to determine its diametral pairs (i.e. the pairs of points whose distance is equal to the diameter) and to parametrize the set of all minimizing geodesic segments with endpoints P, Q , by means of a compact connected submanifold of $\mathfrak{su}_n$, which in general is not equal to $\mathfrak{su}_n$–$plog(P^*Q)$), but it is always diffeomorphic to a suitable complex Grassmannian depending on P and Q (Theorem 5.1 and Theorem 5.2). Since we already studied generalized principal logarithms of some different types of matrices in [Dolcetti-Pertici 2018], [Pertici-Dolcetti 2022], [Pertici 2023], we also determine the conditions on $Q \in SU_n$, so that the set $\mathfrak{su}_n$–$plog(Q)$ is non-empty (Proposition 3.3).

The euclidean metric on $\mathfrak{gl}_n(\mathbb{C})$ can be also restricted to the Lie group $GL_n(\mathbb{C})$ of invertible complex matrices; this restriction is a Riemannian metric on $GL_n(\mathbb{C})$, but it is not bi-invariant. On the other hand $GL_n(\mathbb{C})$ can be endowed, in a natural way, with a semi-Riemannian metric, which, on the contrary, is bi-invariant: the so-called *trace metric* g, defined, at the identity matrix, by $g(X,Y) := Re(Tr(XY))$ (the real part of the trace of the product XY) for every $X, Y \in \mathfrak{gl}_n(\mathbb{C})$. We remark that, for any closed subgroup G of U_n, the restriction to G of the trace metric g agrees with the anti-Riemannian metric $-\phi$ (the opposite of the Frobenius metric of G) and hence it determines on G the same Riemannian structure of ϕ.

The restrictions of the trace metric define Riemannian structures on the space of positive definite hermitian matrices and on the space of positive definite symmetric real matrices. Such manifolds have a remarkable interest in many frameworks of both pure and applied mathematics (see for instance, among many others, [Bridson-Haefliger 1999], [Moakher 2005], [Bhatia 2007], [Barbaresco 2008], [Moakher-Zéraï 2011], [Nielsen-Bhatia 2013], [Nielsen-Barbaresco 2019], [Dolcetti-Pertici 2019], [Dolcetti-Pertici 2021], [Cruceru-Becigneul-Ganea 2021], [Boumal 2023], [Criscitiello-MartinezRubio-Boumal 2023], [Nielsen 2023a], [Nielsen 2023b], [Zhang-Zhang-Sra 2023], [Nieuwboer 2024]). We have also studied the restrictions of the trace metric on other submanifolds of $GL_n(\mathbb{C})$, where they induce semi-Riemannian (but not Riemannian) structures (see [Dolcetti-Pertici 2015], [Dolcetti-Pertici 2019], [Dolcetti-Pertici 2020]).

The present paper is organized as follows.

In Section 1, we recall the main notations and the basic facts used in this paper. We introduce the Frobenius metric ϕ on any closed connected subgroup G of U_n, having $\mathfrak{g} \subseteq \mathfrak{u}_n$ as Lie algebra. In particular, we observe that (Proposition 1.5):

(i) for every $P, Q \in G$, the (Frobenius) distance $d(P,Q)$ is equal to the minimum of the (Frobenius) norm $\|X\|_\phi$, where $X \in \mathfrak{g}$ is a logarithms of P^*Q;

(ii) the map: $X \mapsto \gamma(t) := P\exp(tX) \;\; (0 \leq t \leq 1)$ is a bijection from the set $\{X \in \mathfrak{g} \; : \; \exp(X) = P^*Q \text{ and } \|X\|_\phi = d(P,Q)\}$ onto the set of minimizing geodesic segments of (G,ϕ) with endpoints P and Q.

Next Sections are devoted to study some Riemannian properties of the group SU_n endowed with the Frobenius metric ϕ.

Aim of Section 2 is to find an explicit expression of the function $m(Q) := \min\{\|X\|_\phi^2 \; : \; X \in \mathfrak{su}_n \text{ and } \exp(X) = Q\} = d(I_n, Q)^2$ for any matrix $Q \in SU_n$ (I_n is the identity matrix); the value of $m(Q)$ is obtained in terms of the eigenvalues of Q (Proposition 2.8).

In Section 3, for every $Q \in SU_n$, we analyse the set

$\Theta(Q) := \{X \in \mathfrak{su}_n : \exp(X) = Q\,,\ \|X\|_\phi^2 = m(Q) = d(I_n, Q)^2\}$, which (as stated above) parametrizes the set of minimizing geodesic segments of (SU_n, ϕ) with endpoints I_n and Q. We prove that $\Theta(Q)$ is a compact connected submanifold of $\mathfrak{su}_n$, diffeomorphic to a suitable complex Grassmannian (Proposition 3.2). Furthermore we also determine the geometric structure of $\mathfrak{su}_n$–$plog(Q)$ (Proposition 3.3).
In Section 4, we obtain some fundamental results concerning the diameter of (SU_n, ϕ) and its diametral pairs (Proposition 4.1).
In the last Section 5, by means of the Propositions proved in Sections 2, 3, 4, we collect and prove the main results of this paper: the already mentioned Theorems 5.1 and 5.2.

1. Preliminary facts

1.1. **Notations-Definitions.** a) We denote by $\mathbb{C}$ the field of complex numbers, by $\mathbf{i}$ its imaginary unit, by $\mathbb{R}$ the field of real numbers and by $\mathbb{Z}$ the ring of integers. The integer part of any $x \in \mathbb{R}$ is denoted by $\lfloor x \rfloor$. We denote by $|w|$ and $e^w := \sum_{i=0}^{+\infty} \dfrac{w^i}{i!}$, respectively, the modulus and the exponential of an arbitrary complex number w, while, for any $z \in \mathbb{C} \setminus \{0\}$, we denote by $\arg(z) \in (-\pi, \pi]$ the *principal value of the argument* of z and by $\log(z) := \ln|z| + \arg(z)\mathbf{i}$ the *principal value of the logarithm* of z. Clearly $\log(z)$ is the unique complex logarithm of z, whose imaginary part lies in the interval $(-\pi, \pi]$. Note that, when $|z| = 1$, we have $\log(z) = \arg(z)\mathbf{i}$, so, in particular, $\log(-1) = \pi\mathbf{i}$. Of course, if $|z| = |w| = 1$, we have $z = w$ if and only if $\arg(z) = \arg(w)$ (i.e. $\log(z) = \log(w)$).

b) We denote by $GL_n(\mathbb{C})$ (with $n \geq 1$) the Lie group of invertible complex square matrices of order n, by $\mathfrak{gl}_n(\mathbb{C})$ its Lie algebra consisting of all $n \times n$ complex matrices and by I_n the identity matrix of $GL_n(\mathbb{C})$. For every $A \in \mathfrak{gl}_n(\mathbb{C})$, A^T, $\overline{A}$, $A^* := \overline{A}^T$ and A^{-1} (provided that A is invertible) are respectively transpose, conjugate, adjoint and inverse of the matrix A, $\operatorname{tr}(A)$ is its trace, $\det(A)$ is its determinant, while $\exp(A) := \sum_{i=0}^{+\infty} \dfrac{A^i}{i!} \in GL_n(\mathbb{C})$ denotes the exponential of A. It is clear that, if $\lambda_1, \cdots, \lambda_n$ are the n eigenvalues of $A \in \mathfrak{gl}_n(\mathbb{C})$, then $e^{\lambda_1}, \cdots, e^{\lambda_n}$ are the n eigenvalues of $\exp(A)$. For convenience, we agree that a complex number η is an *eigenvalue of multiplicity* 0 of a matrix $A \in \mathfrak{gl}_n(\mathbb{C})$ if η is not a usual eigenvalue of A, i.e. if η is not a root of the characteristic polynomial of A.
For any integer $n \geq 1$, we denote by $U_n := \{A \in \mathfrak{gl}_n(\mathbb{C}) : AA^* = I_n\}$ the unitary group of degree n and by $\mathfrak{u}_n := \{X \in \mathfrak{gl}_n(\mathbb{C}) : X = -X^*\}$ its Lie algebra of skew-hermitian matrices; we also denote by $SU_n := \{A \in U_n : \det(A) = 1\}$ the special unitary group of degree n and by $\mathfrak{su}_n := \{X \in \mathfrak{u}_n : \operatorname{tr}(X) = 0\}$ its Lie algebra. Finally, for any $U \in U_n$, we denote by $Ad_U : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ the map defined by $Ad_U(X) := UXU^*$ (for every $X \in \mathfrak{gl}_n(\mathbb{C})$) and we still denote by Ad_U the restriction of this map to any subset of $\mathfrak{gl}_n(\mathbb{C})$; so we can write both $Ad_U : SU_n \to SU_n$ and $Ad_U : \mathfrak{su}_n \to \mathfrak{su}_n$. Note that Ad_U commutes with the exponential map.

c) Let G be a closed connected subgroup of $GL_n(\mathbb{C})$ having $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ as its Lie algebra. If $B \in G,\ A \in \mathfrak{g}$ and $\exp(A) = B$, we say that A is a $\mathfrak{g}$–*logarithm* of B. Moreover, if A is a $\mathfrak{g}$–logarithm of B such that $-\pi \leq Im(\lambda) \leq \pi$ for every eigenvalue λ of A, we say that A is a *generalized principal* $\mathfrak{g}$*–logarithm* of B.
We denote by $\mathfrak{g}$–*plog*(B) the set of all generalized principal $\mathfrak{g}$–logarithms of $B \in G$.
Recall that, if G is also compact, the exponential map $\exp : \mathfrak{g} \to G$ is surjective (see, for instance, [Sepanski 2007, Thm. 5.12, p. 102]).

d) If $B_1, \cdots, B_t$ are square matrices (of various orders), then $\bigoplus_{j=1}^{t} B_j = B_1 \oplus \cdots \oplus B_t$ denotes the block diagonal square matrix with $B_1, \cdots, B_t$ on its diagonal.
Clearly we have $\exp(\bigoplus_{j=1}^{t} B_j) = \bigoplus_{j=1}^{t} \exp(B_j)$, for every $B_1, \cdots, B_t$.
If $\mathcal{S}_1, \ldots, \mathcal{S}_t$ are sets of square matrices, then $\bigoplus_{j=1}^{t} \mathcal{S}_j = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_t$ denotes the set of all matrices $\bigoplus_{j=1}^{t} B_j$, with $B_j \in \mathcal{S}_j$ for every j.

1.2. **Lemma.** *Let $n_1, \cdots, n_t$ be positive integers such that $n = \sum_{j=1}^{t} n_j$ and let $\lambda_1, \cdots, \lambda_t$ be distinct complex numbers. Set $A := \bigoplus_{j=1}^{t} \lambda_j I_{n_j}$. Then a matrix $B \in U_n$ commutes with A if and only if $B \in \bigoplus_{j=1}^{t} U_{n_j}$.*

For a proof of the Lemma, see for instance [Pertici 2023, Lemma 1.3].

1.3. **Remark-Definition.** We denote by ϕ the *Frobenius* (or *Hilbert-Schmidt*) *positive definite real scalar product* on $\mathfrak{gl}_n(\mathbb{C})$ defined by $\phi(A, B) := \mathrm{Re}(\mathrm{tr}(AB^*))$, and we denote by $\|A\|_\phi := \sqrt{\phi(A, A)} = \sqrt{\mathrm{tr}(AA^*)}$ the related *Frobenius norm.* Note that, if $A \in \mathfrak{u}_n$, then $\|A\|_\phi^2 = -\mathrm{tr}(A^2)$. Since the eigenvalues of the skew-hermitian matrix A are purely imaginary, we also get $\|A\|_\phi = \sqrt{-\mathrm{tr}(A^2)} = \sqrt{\sum_{j=1}^{n} |\lambda_j|^2}$, where $\lambda_1, \cdots, \lambda_n$ are the n (possibly repeated) eigenvalues of A. For any arbitrary closed subgroup G of U_n, we still denote by ϕ the Riemannian metric on G, obtained by restriction of the Frobenius scalar product of $\mathfrak{gl}_n(\mathbb{C})$. It is easy to check that the metric ϕ (called the *Frobenius metric* of G) is bi-invariant on G and we have $\phi_A(X, Y) = -\mathrm{tr}(A^* X A^* Y)$, for any pair X, Y of tangent vectors to G at A and for any $A \in G$. We denote by $d := d_{(G,\phi)}$ the distance on G induced by ϕ and by $\delta(G, \phi) := \sup\{d(P, Q) \ : \ P, Q \in G\}$ the *diameter* of G with respect to d. Of course $\delta(G, \phi) = d(A, B) < +\infty$, for some $A, B \in G$, because G is compact.
If $A, B \in G$ and $d(A, B) = \delta(G, \phi)$, we say that A and B are *diametral in* (G, ϕ) or, equivalently, that they form a *diametral pair of* (G, ϕ). Since ϕ is bi-invariant on the compact Lie group $G \subset U_n$, it is clear that every point of G belongs at least to a diametral pair of (G, ϕ). For some results on diametral pairs in real orthogonal groups, see [Dolcetti-Pertici 2018, Prop. 4.18].

1.4. **Proposition.** *Let G be a closed subgroup of U_n and let $\mathfrak{g} \subseteq \mathfrak{u}_n$ be its Lie algebra. Then (G, ϕ) is a globally symmetric Riemannian manifold with non-negative sectional curvature, whose Levi-Civita connection agrees with the 0-connection of Cartan-Schouten of G. The geodesics of (G, ϕ) are the curves $t \mapsto P\exp(tX)$ (with $t \in \mathbb{R}$), for every $X \in \mathfrak{g}$ and $P \in G$; furthermore (G, ϕ) is a totally geodesic submanifold of (U_n, ϕ).*

For a proof of Proposition 1.4, we refer, for instance, to [Alexandrino-Bettiol 2015, § 2.2].

1.5. **Proposition.** *Let G be a closed connected subgroup of U_n and let $\mathfrak{g} \subseteq \mathfrak{u}_n$ be its Lie algebra. Then, for every $P, Q \in G$, the (Frobenius) distance $d(P, Q)$ is equal to the minimum of the set $\{ \|X\|_\phi \ : \ X$ is a $\mathfrak{g}$–logarithm of $P^*Q \}$.*
Furthermore, the map: $X \mapsto \gamma(t) := P\exp(tX) \quad (0 \le t \le 1)$ is a bijection from the set $\{X \ : \ X$ is a $\mathfrak{g}$–logarithm of P^*Q with $\|X\|_\phi = d(P, Q)\}$ onto the set of minimizing geodesic segments of (G, ϕ) with endpoints P and Q.

Proof. Any geodesic segment γ, joining P and Q, can be parametrized by $\gamma(t) = P\exp(tX)$ $(t \in [0, 1])$, with $X \in \mathfrak{g}$, $\exp(X) = P^*Q$, and its length is $\sqrt{-tr(X^2)} = \|X\|_\phi$; so we conclude by means of the classical Hopf-Rinow theorem. □

1.6. **Remark.** Let G be as in Proposition 1.5. We resume some results obtained in [Pertici-Dolcetti 2022].
In Thm. 6.5, we proved that, if G is SVD-closed too (for definition see § 4 of that paper) and $P, Q \in G$, then
a) $d(P, Q) = \sqrt{\sum_{j=1}^{n} |\log(\mu_j)|^2}$, where $\mu_1, \cdots, \mu_n$ are the n eigenvalues of P^*Q;
b) $\mathfrak{g}$–*plog*$(P^*Q) = \{X : X$ is a $\mathfrak{g}$–logarithm of P^*Q with $\|X\|_\phi = d(P, Q)\}$ and so $\mathfrak{g}$–*plog*(P^*Q) parametrizes the set of minimizing geodesic segments of (G, ϕ) with endpoints P and Q.
In Prop. 6.7 and Rem. 6.8, we computed the diameter of suitable families of SVD-closed subgroups of U_n, among them all classical subgroups of U_n with the exception of SU_n.
For the same families of SVD-closed subgroups G of U_n, in Sections 7 and 8, we proved that, for any matrix $M \in G$, the set $\mathfrak{g}$–*plog*(M) is a disjoint union of finitely many simply connected compact submanifolds of $\mathfrak{u}_n$, each of them is diffeomorphic to a symmetric (homogeneous) space, we explicitly determined.
As remarked in [Pertici-Dolcetti 2022, Rem. 3.7], SU_n is not SVD-closed as soon as $n \ge 3$ and therefore the Riemannian manifold (SU_n, ϕ) was excluded from that study. Hence, the goal of the present paper is to research for similar results in case of (SU_n, ϕ).

2. About the distance in (SU_n, ϕ) of any matrix from I_n

2.1. **Remarks-Definitions.** a) Fix a matrix $Q \in SU_n$ $(n \ge 2)$, and let $\mu_1 \cdots, \mu_n$ be the (possibly repeated) eigenvalues of Q, so we have $\mu_1\mu_2 \cdots \mu_n = 1$; it is also known that

$|\mu_1| = |\mu_2| = \cdots |\mu_n| = 1$, so $\log(\mu_j) = \arg(\mu_j)\mathbf{i}$, for $j = 1, \cdots, n$. Up to reordering the eigenvalues, we can assume $\quad -\pi < \arg(\mu_1) \le \arg(\mu_2) \le \cdots \le \arg(\mu_n) \le \pi$.
Since $1 = \prod_{j=1}^{n} e^{\log(\mu_j)} = e^{(\arg(\mu_1)+\cdots+\arg(\mu_n))\mathbf{i}}$, we also have $\sum_{j=1}^{n} \arg(\mu_j) = 2k\pi$, with $k \in \mathbb{Z}$.
We also denote by $s = s(Q) \ge 0$ the multiplicity of -1 as an eigenvalue of Q.
When $s \ge 1$ we have $\arg(\mu_j) = \pi$ if and only if $n - s + 1 \le j \le n$, while if $s = 0$ we have $\arg(\mu_j) \ne \pi$ for every $j = 1, \cdots, n$.

b) With the same notations as above, the eigenvalues of the inverse Q^* of Q are $\overline{\mu}_1, \cdots, \overline{\mu}_n$. Hence, if $s = 0$ we have $\arg(\overline{\mu}_j) = -\arg(\mu_j)$ for every $j = 1, \cdots, n$, while if $s \ge 1$ we have $\arg(\overline{\mu}_j) = -\arg(\mu_j)$ for every $j = 1, \cdots, n - s$ and $\arg(\overline{\mu}_j) = \arg(\mu_j) = \pi$ for every $j = n - s + 1, \cdots, n$. In particular, in any case we have $s(Q^*) = s(Q) = s$ and $-\pi < \arg(\overline{\mu}_{n-s}) \le \cdots \le \arg(\overline{\mu}_1) < \pi$; moreover, if $s \ge 1$ we have $\arg(\overline{\mu}_j) = \pi$ for every $j = n - s + 1, \cdots, n$.
c) We set $\zeta(Q) := \frac{1}{2\pi} \sum_{j=1}^{n} \arg(\mu_j)$.
From (a), $\zeta(Q)$ is an integer dependent only on the principal arguments of the eigenvalues of Q, while, from (b), it is easy to check that we have $\zeta(Q^*) = s(Q) - \zeta(Q)$.
From this last equality it is easy to deduce that $s(Q) - \lfloor \frac{n}{2} \rfloor \le \zeta(Q) \le \lfloor \frac{n}{2} \rfloor$ and also that the assumption $\zeta(Q) \ge \zeta(Q^*)$ implies $\zeta(Q) \ge 0$.

2.2. **Remark-Definition.** Consider any integer ζ such that $-\lfloor \frac{n}{2} \rfloor \le \zeta \le \lfloor \frac{n}{2} \rfloor \quad (n \ge 2)$; we say that an n-tuple $(\alpha_1, \cdots, \alpha_n)$ of real numbers is ζ-*admissible* if
$-\pi < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \pi$ and $\sum_{j=1}^{n} \alpha_j = 2\pi\zeta$.
As seen in Remarks-Definitions 2.1 (a) and (c), the eigenvalues $\mu_1, \cdots, \mu_n$ of any matrix Q of SU_n can be reordered so that the n-tuple $(\arg(\mu_1), \cdots, \arg(\mu_n))$ is $\zeta(Q)$-admissible, being $\zeta(Q)$ the integer belonging to $\left[-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\right]$ defined in Remarks-Definitions 2.1 (c). Conversely, it is clear that, if $(\alpha_1, \cdots, \alpha_n)$ is any ζ-admissible n-tuple of real numbers (with $\zeta \in \left[-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\right] \cap \mathbb{Z}$), there is at least one matrix $Q \in SU_n$ whose n eigenvalues are $\mu_1 = e^{\alpha_1 \mathbf{i}}, \cdots, \mu_n = e^{\alpha_n \mathbf{i}}$; so we have $\alpha_1 = \arg(\mu_1), \cdots, \alpha_n = \arg(\mu_n)$ and $\zeta = \zeta(Q)$.

2.3. **Remarks-Definitions.** a) Let $m : SU_n \to \mathbb{R}$ be the function defined by
$m(Q) := \inf\{\|X\|_\phi^2 \; : \; X \in \mathfrak{su}_n \text{ and } \exp(X) = Q\}$
and, for any $Q \in SU_n$, let
$\Theta(Q) := \{X \in \mathfrak{su}_n : \exp(X) = Q \, , \; \|X\|_\phi^2 = m(Q)\}$.
From Proposition 1.5, it follows that, for every $Q \in SU_n$, the map $X \mapsto \|X\|_\phi^2$ has an absolute minimum on the set of $\mathfrak{su}_n$–logarithms of Q; so, for any $Q \in SU_n$, we can write $m(Q) = \min\{\|X\|_\phi^2 \; : \; X \in \mathfrak{su}_n \text{ and } \exp(X) = Q\} = d(I_n, Q)^2$ (where d is the distance induced by the Frobenius metric of SU_n), so that the set $\Theta(Q)$ is non-empty.
Now, fix $Q \in SU_n$ $(n \ge 2)$ and denote $\mu_1 \cdots, \mu_n$ the n eigenvalues of Q, so that the n-tuple $(\arg(\mu_1) \cdots, \arg(\mu_n))$ is $\zeta(Q)$-admissible, with $\zeta(Q) \in \left[-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\right] \cap \mathbb{Z}$. We have
(*) $\qquad m(Q) \ge \sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2$.

Indeed by definition we have: $m(Q) \geq \min\{\|X\|_\phi^2 \ : \ X \in \mathfrak{u}_n \text{ and } \exp(X) = Q\}$ and this last is equal to $\sum_{j=1}^n \big(\arg(\mu_j)\big)^2$ (by [Pertici-Dolcetti 2022, Rem. 5.4 and Prop. 5.5]).
Note that, if X is a $\mathfrak{su}_n$–logarithm of Q, then the n eigenvalues of X are of the form $(\arg(\mu_1) + 2k_1\pi)\mathbf{i}, \cdots, (\arg(\mu_n) + 2k_n\pi)\mathbf{i}$, for some integers $k_1, \cdots, k_n$ such that $\sum_{j=1}^n (\arg(\mu_j) + 2k_j\pi) = 2\pi\big(\zeta(Q) + \sum_{j=1}^n k_j\big) = 0$, i.e. $\sum_{j=1}^n k_j = -\zeta(Q)$.
Conversely, if $h_1, \cdots, h_n \in \mathbb{Z}$ and $\sum_{j=1}^n h_j = -\zeta(Q)$ (i.e. $\sum_{j=1}^n (\arg(\mu_j) + 2h_j\pi) = 0$), it is easy to determine a matrix $X_0 \in \mathfrak{su}_n$ such that $\exp(X_0) = Q$, whose n eigenvalues are $(\arg(\mu_1) + 2h_1\pi)\mathbf{i}, \cdots, (\arg(\mu_n) + 2h_n\pi)\mathbf{i}$, and therefore $\|X_0\|_\phi^2 = \sum_{j=1}^n \big(\arg(\mu_j) + 2h_j\pi\big)^2$.
Hence we get $m(Q) = \min\{\sum_{j=1}^n \big(\arg(\mu_j) + 2k_j\pi\big)^2 \ : \ k_1, \cdots k_n \in \mathbb{Z} \text{ and } \sum_{j=1}^n k_j = -\zeta(Q)\}$.
Let $W_Q := \{\underline{k} := (k_1, \cdots, k_n) \in \mathbb{Z}^n \ : \ \sum_{j=1}^n k_j = -\zeta(Q)\}$ and denote by $\psi_Q : W_Q \to \mathbb{R}$ the map defined by $\psi_Q(\underline{k}) = \psi_Q(k_1, \cdots, k_n) = \sum_{j=1}^n \big(\arg(\mu_j) + 2k_j\pi\big)^2$; then we can write $m(Q) = \min\{\psi_Q(\underline{k}) \ : \ \underline{k} \in W_Q\}$.
Next, it will be also useful to consider the map $\Delta_Q : W_Q \to \mathbb{Z}$ defined by $\Delta_Q(\underline{k}) := \max\{k_1, \cdots, k_n\} - \min\{k_1, \cdots, k_n\}$, with $\underline{k} = (k_1, \cdots, k_n) \in W_Q$, and the set $Z_Q := \{\underline{k} \in W_Q : \Delta_Q(\underline{k}) \leq 1\}$.
Note that $Z_Q \neq \emptyset$. Indeed, if $\zeta = \zeta(Q) \geq 0$, then $(\underbrace{0, \cdots, 0}_{n-\zeta}, \underbrace{-1, \cdots, -1}_{\zeta}) \in Z_Q$, while if $\zeta = \zeta(Q) < 0$, then $(\underbrace{1, \cdots, 1}_{-\zeta}, \underbrace{0, \cdots, 0}_{n+\zeta}) \in Z_Q$.

b) Note that the map $m : Q \mapsto m(Q)$ depends only on the $\zeta(Q)$-admissible n-tuple $(\arg(\mu_1), \cdots, \arg(\mu_n))$, so we can unambiguously write $m(\arg(\mu_1), \cdots, \arg(\mu_n)) := m(Q)$. From this fact and from Remark-Definition 2.2, in the sequel we will also be able to consider m as a function only of an arbitrary ζ-admissible n-tuple $(\alpha_1, \cdots, \alpha_n)$ of real numbers such that $\zeta \in \big[-\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}$.

c) Keeping in mind Remarks 2.1 (b), it is easy to check that we have $m(Q^*) = m(Q)$, for any $Q \in SU_n$, or equivalently, $m(\alpha_1, \cdots, \alpha_{n-s}, \underbrace{\pi, \cdots, \pi}_{s}) = m(-\alpha_{n-s}, \cdots, -\alpha_1, \underbrace{\pi, \cdots, \pi}_{s})$, for any ζ-admissible n-tuple $(\alpha_1, \cdots, \alpha_{n-s}, \underbrace{\pi, \cdots, \pi}_{s})$, with $\alpha_{n-s} < \pi$, $s = 0, 1, \cdots, n$ and $\zeta \in \big[-\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}$. Also note that, if Ψ is the automorphism of the vector space $\mathfrak{su}_n$ defined by $\Psi(X) := -X$ (for any $X \in \mathfrak{su}_n$), we have $\Psi\big(\Theta(Q^*)\big) = \Theta(Q)$.

d) The map $Q \mapsto m(Q) = d(Q, I_n)^2$ is continuous on the compact Lie group SU_n, so it has a maximum δ_n on SU_n. Since (SU_n, ϕ) is a homogeneous Riemannian manifold, it is clear that $\delta_n = \max\{m(Q) : Q \in SU_n\}$ agrees with the square of the diameter of SU_n with respect to d, i.e. $\delta(SU_n, \phi) = \sqrt{\delta_n}$. Keeping in mind (b), (c) above, Remarks 2.1 (c) and Remark-Definition 2.2, we deduce that δ_n is the maximum of the set $\big\{m(\alpha_1, \cdots, \alpha_n) : (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ is any ζ-admissible n-tuple, with $\zeta \in \big[0, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}\big\}$.

2.4. **Remark.** For any $Q \in SU_n$ and $U \in U_n$, we have $m\big(Ad_U(Q)\big) = m(Q)$ and $\Theta\big(Ad_U(Q)\big) = Ad_U\big(\Theta(Q)\big)$.

The equality $m\big(Ad_U(Q)\big) = m(Q)$ holds because the function $m(Q)$ depends only on the eigenvalues of Q, while the second equality $\Theta\big(Ad_U(Q)\big) = Ad_U\big(\Theta(Q)\big)$ can be easily deduced from the first one, remembering that the map $Ad_U : (SU_n, \phi) \to (SU_n, \phi)$ is an isometry that commutes with the exponential map.

2.5. **Lemma.** *Let Q be a matrix of SU_n $(n \geq 2)$, whose n eigenvalues are $\mu_1, \cdots, \mu_n$. Denote by $s(Q)$ the multiplicity of -1 as an eigenvalue of Q and let*
$\zeta(Q) = \frac{1}{2\pi} \sum_{j=1}^{n} \arg(\mu_j)$, $\quad m(Q) = \min\{\|X\|_\phi^2 \ : \ X \in \mathfrak{su}_n$ *and* $\exp(X) = Q\}$,
$\Theta(Q) = \{X \in \mathfrak{su}_n : \exp(X) = Q,\ \|X\|_\phi^2 = m(Q)\}$.
Then the following facts are equivalent:
i) $\mathfrak{su}_n$*–plog*$(Q) \neq \emptyset$;
ii) $\zeta(Q) \in \{0, 1, \cdots, s(Q)\}$;
iii) $m(Q) = \sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2$.
Moreover, if any of the above conditions holds, then we have $\Theta(Q) = \mathfrak{su}_n$*–plog*$(Q)$.

Proof. The equivalence between (i) and (iii) and the final assertion follow directly from the proof of [Pertici-Dolcetti 2022, Prop. 5.5 (b)].
Set $s := s(Q)$ and $\zeta := \zeta(Q)$. We can assume $\arg(\mu_1) \leq \arg(\mu_2) \leq \cdots \leq \arg(\mu_n)$; so we have $\mu_{n-s+1} = \mu_{n-s+2} = \cdots = \mu_n = -1$, while $\mu_1, \cdots, \mu_{n-s}$ are all different from -1.
Assume now (i) and fix a matrix $X \in \mathfrak{su}_n$–*plog*(Q) so that the n eigenvalues of X are: $\arg(\mu_1)\mathbf{i}, \cdots, \arg(\mu_{n-s})\mathbf{i}$, $\ -\pi\mathbf{i}$ with multiplicity $k \geq 0$ and $\pi\mathbf{i}$ with multiplicity $s - k \geq 0$, for some $k \in \{0, \cdots, s\}$. The condition $\operatorname{tr}(X) = 0$ implies $\sum_{j=1}^{n-s} \arg(\mu_j) + (s - 2k)\pi = 0$. Since $\zeta = \frac{1}{2\pi} \sum_{j=1}^{n} \arg(\mu_j) = \frac{1}{2\pi}\big(\sum_{j=1}^{n-s} \arg(\mu_j) + s\pi\big)$, we obtain $\zeta = k \in \{0, 1, \cdots, s\}$ and so (i) implies (ii).
Assume now (ii) (i.e. $\zeta \in \{0, 1, \cdots, s\}$). By Remarks-Definition 2.3 (a), we have
$m(Q) = \min\{\sum_{j=1}^{n} \big(\arg(\mu_j) + 2k_j\pi\big)^2 \ : \ k_1, \cdots k_n \in \mathbb{Z}$ and $\sum_{j=1}^{n} k_j = -\zeta\}$.
Choose $h_1 = \cdots h_{n-\zeta} = 0$ and $h_{n-\zeta+1} = h_{n-\zeta+2} = \cdots = h_n = -1$, so that $\sum_{j=1}^{n} h_j = -\zeta$.
Since $\zeta \leq s$, we get $\sum_{j=1}^{n} \big(\arg(\mu_j) + 2h_j\pi\big)^2 = \sum_{j=1}^{n-\zeta} \big(\arg(\mu_j)\big)^2 + \zeta(-\pi)^2 = \sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2$. This gives $m(Q) \leq \sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2$. The equality follows from the inequality (*) of Remarks-Definitions 2.3 (a). Thus (ii) implies (iii) and the proof is complete. □

2.6. **Lemma.** *With the same notations as in Remarks-Definitions 2.3 (a), we have* $\psi_Q(\underline{k}) > m(Q)$ *for every* $\underline{k} \in W_Q \setminus Z_Q$. *Therefore* $m(Q) = \min\{\psi_Q(\underline{k}) \ : \ \underline{k} \in Z_Q\}$.

Proof. Remembering that $m(Q) = \min\{\psi_Q(\underline{k}) \ : \ \underline{k} \in W_Q\}$, it suffices to prove that, if $\underline{k} \in W_Q \setminus Z_Q$, then there exists $\underline{l} \in W_Q$ such that $\psi_Q(\underline{k}) > \psi_Q(\underline{l})$.

Let $\underline{k} = (k_1, \cdots, k_n) \in W_Q \setminus Z_Q$, i.e. $k_1, \cdots, k_n \in \mathbb{Z}$, $\sum_{j=1}^{n} k_j = -\zeta(Q)$ and $\Delta_Q(\underline{k}) \geq 2$. Fix two indices $r, t \in \{1, \cdots, n\}$ such that $k_t = \max\{k_1, \cdots, k_n\}$ and $k_r = \min\{k_1, \cdots, k_n\}$. From the definition we have $k_t - k_r = \Delta_Q(\underline{k}) \geq 2$.
We define $l_j := k_j$ for every $j \in \{1, \cdots, n\} \setminus \{r, t\}$, $l_r := k_r + 1$, $l_t := k_t - 1$ and $\underline{l} := (l_1, \cdots, l_n) \in \mathbb{Z}^n$. Since $\sum_{j=1}^{n} l_j = \sum_{j=1}^{n} k_j = -\zeta(Q)$, we have $\underline{l} \in W_Q$.
The inequality $\psi_Q(\underline{k}) > \psi_Q(\underline{l})$ is equivalent to
$(\arg(\mu_r)+2k_r\pi)^2+(\arg(\mu_t)+2k_t\pi)^2 > (\arg(\mu_r)+2l_r\pi)^2+(\arg(\mu_t)+2l_t\pi)^2$ and the latter is equivalent to the inequality $2(k_t - k_r)\pi > 2\pi + \arg(\mu_r) - \arg(\mu_t)$, which is satisfied since $2(k_t - k_r)\pi \geq 4\pi$ and $\arg(\mu_r) - \arg(\mu_t) < 2\pi$. Hence we have $\psi_Q(\underline{k}) > \psi_Q(\underline{l})$ and this concludes the proof. □

2.7. **Lemma.** *With the same notations as in Remarks-Definitions 2.3 (a), let* $\underline{k} = (k_1, \cdots, k_n) \in Z_Q$.
If $\zeta(Q) \geq 0$, *then* $\underline{k}$ *has* $\zeta(Q)$ *entries equal to* -1, *while the remaining ones are equal to* 0. *In particular, if* $\zeta(Q) = 0$, *then* Z_Q *consists of the unique element* $(0, \cdots, 0)$.

Proof. Set $\zeta := \zeta(Q)$. Since $\Delta_Q(\underline{k}) \leq 1$, there exists an integer H and a non-empty set $J \subseteq \{1, \cdots, n\}$ such that $k_j = H$ for every $j \in J$ and $k_i = H-1$ for every $i \in \{1, \cdots, n\} \setminus J$. Let $\chi \geq 1$ be the cardinality of J. Then $-\zeta = \sum_{j=1}^{n} k_j = \chi H + (n-\chi)(H-1) = n(H-1)+\chi$.
Remembering Remarks 2.1 (c), we get $n(H-1) = -\zeta - \chi \geq -\lfloor \frac{n}{2} \rfloor - n > -2n$: so $H \geq 0$. Since $\zeta \geq 0$, we have $n(H-1) = -\zeta - \chi \leq -1$, hence $H < 1$ and so $H = 0$. Therefore $\underline{k}$ necessarily has ζ entries equal to -1, while the remaining ones are equal to 0. □

2.8. **Proposition.** *Let* Q *be a matrix of* SU_n $(n \geq 2)$ *whose* n *eigenvalues are* $\mu_1, \cdots, \mu_n$ *ordered so that we have* $\arg(\mu_1) \leq \arg(\mu_2) \leq \cdots \leq \arg(\mu_n)$ *and assume that* $\zeta(Q) = \frac{1}{2\pi} \sum_{j=1}^{n} \arg(\mu_j) \geq 0$. *Let* $m(Q) = \min\{\|X\|_\phi^2 \ : \ X \in \mathfrak{su}_n$ *and* $\exp(X) = Q\}$ *and* $\Theta(Q) = \{X \in \mathfrak{su}_n : \exp(X) = Q, \ \|X\|_\phi^2 = m(Q)\}$.
Finally, let X_0 *be any* $\mathfrak{su}_n$*–logarithm of* Q *whose* n *eigenvalues are :*
$(\arg(\mu_1) + 2h_1\pi)\mathbf{i}, \cdots, (\arg(\mu_n) + 2h_n\pi)\mathbf{i}$ (*with* $h_1, \cdots, h_n \in \mathbb{Z}$ *and* $\sum_{j=1}^{n} h_j = -\zeta(Q)$).
a) If $\zeta(Q) = 0$, *then* $m(Q) = \sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2$;
furthermore, $X_0 \in \Theta(Q)$ *if and only if* $h_j = 0$ *for every* $j \in \{1, \cdots, n\}$.
b) If $\zeta(Q) \geq 1$, *then* $m(Q) = \sum_{j=1}^{n-\zeta(Q)} \big(\arg(\mu_j)\big)^2 + \sum_{j=n-\zeta(Q)+1}^{n} \big(2\pi - \arg(\mu_j)\big)^2$;
furthermore, if $\mu_{n-\zeta(Q)} \neq \mu_{n-\zeta(Q)+1}$, *we have* $X_0 \in \Theta(Q)$ *if and only if* $h_j = 0$ *for every* $j \in \{1, \cdots, n-\zeta(Q)\}$ *and* $h_l = -1$ *for every* $l \in \{n-\zeta(Q)+1, \cdots, n\}$;
while, if $\mu_{n-\zeta(Q)} = \mu_{n-\zeta(Q)+1}$, *we have* $X_0 \in \Theta(Q)$ *if and only if*
$h_r = 0$ *for every* $r \in \{1, \cdots, n - \zeta(Q) - 1\}$ *such that* $\mu_r \neq \mu_{n-\zeta(Q)}$,
$h_t = -1$ *for every* $t \in \{n - \zeta(Q) + 2, \cdots, n\}$ *such that* $\mu_t \neq \mu_{n-\zeta(Q)}$,
and $h_m = 0$ *or* $h_m = -1$ *for every index* m *such that* $\mu_m = \mu_{n-\zeta(Q)}$

(*with the constraint that the equality* $\sum_{j=1}^{n} h_j = -\zeta(Q)$ *is satisfied*).

Proof. Set $\zeta := \zeta(Q)$.

If $\zeta = 0$, the formula for $m(Q)$ follows directly from Lemma 2.5 . Clearly, if $h_1 = \cdots = h_n = 0$ we have $\|X_0\|^2_\phi = m(Q)$, and so $X_0 \in \Theta(Q)$. Conversely, if $X_0 \in \Theta(Q)$ then, by Lemma 2.6 , $(h_1, \cdots, h_n) \in Z_Q$; hence, by Lemma 2.7 , we obtain $h_j = 0$ for every $j \in \{1, \cdots, n\}$ and the proof of (a) is complete.

Now let $\zeta \geq 1$, and denote $\underline{k}_0 := (\underbrace{0, \cdots, 0}_{n-\zeta}, \underbrace{-1, \cdots, -1}_{\zeta}) \in Z_Q \subset W_Q$. Remembering Remarks-Definitions 2.3 (a), we want to prove that the minimum of the map ψ_Q on W_Q is reached at $\underline{k}_0$. By Lemma 2.6, this minimum is reached only on Z_Q, while, by Lemma 2.7, every $\underline{k} \in Z_Q$ has ζ entries equal to -1 and the remaining ones equal to 0. Let $\underline{k} = (k_1, \cdots, k_n) \in Z_Q$ and assume that there are two indices $1 \leq i < r \leq n$ such that $k_i = -1$ and $k_r = 0$; then we define a new element $\underline{l} = (l_1, \cdots, l_n) \in Z_Q$ with $l_j = k_j$ for any $j \neq i, r$, $l_i = 0$, $l_r = -1$, and we obtain $\psi_Q(\underline{k}) \geq \psi_Q(\underline{l})$. Indeed this inequality is equivalent to $\big(\arg(\mu_i) - 2\pi\big)^2 + \big(\arg(\mu_r)\big)^2 \geq \big(\arg(\mu_i)\big)^2 + \big(\arg(\mu_r) - 2\pi\big)^2$, and the latter inequality is equivalent to $\arg(\mu_i) \leq \arg(\mu_r)$, which is satisfied since $i < r$. Also note that we have $\psi_Q(\underline{k}) = \psi_Q(\underline{l})$ if and only if $\arg(\mu_i) = \arg(\mu_r)$, i.e. if and only if $\mu_i = \mu_r$. Repeating, if necessary, on $\underline{l}$ the same operation performed on $\underline{k}$, after a finite number of steps we obtain $\psi_Q(\underline{k}) \geq \psi_Q(\underline{k}_0)$, for any $\underline{k} \in Z_Q$. It follows that we have $m(Q) = \psi_Q(\underline{k}_0) = \sum_{j=1}^{n-\zeta} \big(\arg(\mu_j)\big)^2 + \sum_{j=n-\zeta+1}^{n} \big(2\pi - \arg(\mu_j)\big)^2$. We also obtain that, if $\mu_{n-\zeta} \neq \mu_{n-\zeta+1}$, the equality $\psi_Q(\underline{k}) = \psi_Q(\underline{k}_0)$ holds if and only if $\underline{k} = \underline{k}_0$; while, if $\mu_{n-\zeta} = \mu_{n-\zeta+1}$, we have $\psi_Q(\underline{k}) = \psi_Q(\underline{k}_0)$ if and only if $k_r = 0$ for every $r \in \{1, \cdots, n - \zeta - 1\}$ such that $\mu_r \neq \mu_{n-\zeta}$, $k_t = -1$ for every $t \in \{n - \zeta + 2, \cdots, n\}$ such that $\mu_t \neq \mu_{n-\zeta}$, and $k_m = 0$ or $k_m = -1$ for every m such that $\mu_m = \mu_{n-\zeta}$. Thus the second part of statement (b) has also been proved. □

2.9. **Remark.** Note that, if $0 \leq \zeta(Q) \leq s(Q)$, the formulas in part (a) and in part (b) of Proposition 2.8, reduce to formula (iii) of Lemma 2.5.

3. The set $\Theta(Q)$ of minimizing logarithms of any special unitary matrix Q

3.1. **Remark.** From Proposition 2.8 and with the same notations and hypotheses, we get:

i) if $\zeta(Q) = 0$, then $\Theta(Q)$ agrees with the set of all $\mathfrak{su}_n$–logarithms of Q whose n eigenvalues are : $\arg(\mu_1)\mathbf{i}$, $\arg(\mu_2)\mathbf{i}$, $\cdots$, $\arg(\mu_{n-1})\mathbf{i}$, $\arg(\mu_n)\mathbf{i}$;

ii) if $\zeta(Q) \geq 1$, then $\Theta(Q)$ agrees with the set of all $\mathfrak{su}_n$–logarithms of Q whose n eigenvalues are the following:

$\arg(\mu_1)\mathbf{i}$, $\arg(\mu_2)\mathbf{i}$, $\cdots$, $\arg(\mu_{n-\zeta(Q)-1})\mathbf{i}$, $\arg(\mu_{n-\zeta(Q)})\mathbf{i}$, $\big(\arg(\mu_{n-\zeta(Q)+1}) - 2\pi\big)\mathbf{i}$, $\big(\arg(\mu_{n-\zeta(Q)+2}) - 2\pi\big)\mathbf{i}$, $\cdots$, $\big(\arg(\mu_{n-1}) - 2\pi\big)\mathbf{i}$, $\big(\arg(\mu_n) - 2\pi\big)\mathbf{i}$.

3.2. **Proposition.** *Let Q be a matrix of SU_n ($n \geq 2$) whose n eigenvalues are $\mu_1, \cdots, \mu_n$ ordered so that we have $\arg(\mu_1) \leq \arg(\mu_2) \leq \cdots \leq \arg(\mu_n)$ and assume that $\zeta(Q) = \frac{1}{2\pi} \sum_{j=1}^{n} \arg(\mu_j) \geq 0$. Let $m(Q) = \min\{\|X\|_\phi^2 \ : \ X \in \mathfrak{su}_n$ and $\exp(X) = Q\}$ and $\Theta(Q) = \{X \in \mathfrak{su}_n : \exp(X) = Q\,,\ \|X\|_\phi^2 = m(Q)\}$.*

If $\zeta(Q) \geq 1$, we also set $\nu_1 := max\{j \in \{1, \cdots, n - \zeta(Q)\} : \mu_{n-\zeta(Q)+1-j} = \mu_{n-\zeta(Q)}\}$ and $\nu_2 := max\{j \in \{1, \cdots, \zeta(Q)\} : \mu_{n-\zeta(Q)+j} = \mu_{n-\zeta(Q)+1}\}$.

a) If $\zeta(Q) = 0$, then $\Theta(Q)$ is a single point.

b) If $\zeta(Q) \geq 1$ and $\mu_{n-\zeta(Q)} \neq \mu_{n-\zeta(Q)+1}$, then $\Theta(Q)$ is a single point.

c) If $\zeta(Q) \geq 1$ and $\mu_{n-\zeta(Q)} = \mu_{n-\zeta(Q)+1}$, then $\Theta(Q)$ is a compact submanifold of $\mathfrak{su}_n$ diffeomorphic to the symmetric space $\dfrac{U_{(\nu_1+\nu_2)}}{U_{\nu_1} \oplus U_{\nu_2}}$ and therefore also to the complex Grassmannian $\mathbf{Gr}(\nu_2; \mathbb{C}^{(\nu_1+\nu_2)})$.

Proof. Set $\beta := \mu_{n-\zeta(Q)}$. We denote by S_1 the set consisting of all distinct eigenvalues λ of Q such that $\arg(\lambda) < \arg(\beta)$ and by S_2 the set consisting of all distinct eigenvalues φ of Q such that $\arg(\varphi) > \arg(\beta)$ (S_1 and S_2 can also be empty); for every eigenvalue ϵ of Q, we also denote by $m(\epsilon) \geq 1$ the multiplicity of ϵ as an eigenvalue of Q. If we set $J := \big(\bigoplus_{\lambda \in S_1} \lambda I_{m(\lambda)}\big) \oplus \big(\beta I_{m(\beta)}\big) \oplus \big(\bigoplus_{\varphi \in S_2} \varphi I_{m(\varphi)}\big)$, it is a well-known fact that there exists $U \in U_n$ such that $Q = Ad_U(J)$; so, by Remark 2.4 , we can assume that

$$Q = J = \big(\bigoplus_{\lambda \in S_1} \lambda I_{m(\lambda)}\big) \oplus \big(\beta I_{m(\beta)}\big) \oplus \big(\bigoplus_{\varphi \in S_2} \varphi I_{m(\varphi)}\big).$$

Now, if $\zeta(Q) = 0$ as in (a) or if $\zeta(Q) \geq 1$ and $\beta = \mu_{n-\zeta(Q)} \neq \mu_{n-\zeta(Q)+1}$ as in (b), we set

$$\widehat{J} := \big(\bigoplus_{\lambda \in S_1} \arg(\lambda)\mathbf{i}\ I_{m(\lambda)}\big) \oplus \big(\arg(\beta)\mathbf{i}\ I_{m(\beta)}\big) \oplus \big(\bigoplus_{\varphi \in S_2} (\arg(\varphi) - 2\pi)\mathbf{i}\ I_{m(\varphi)}\big).$$

(Here we agree that, if S_1 or S_2 are empty, the terms corresponding to them do not appear in all previous direct sums.)

Note that $\exp(\widehat{J}) = J = Q$. By Remark 3.1, a matrix $X \in \mathfrak{su}_n$ belongs to $\Theta(Q)$ if and only if there is $R \in U_n$ such that $X = Ad_R(\widehat{J})$ and $Q = \exp(X) = Ad_R(\exp(\widehat{J})) = Ad_R(Q)$, i. e. if and only if $X = Ad_R(\widehat{J})$ for some $R \in U_n$ such that $RQ = QR$. By Lemma 1.2, a matrix $R \in U_n$ commutes with Q if and only if $R \in \big(\bigoplus_{\lambda \in S_1} U_{m(\lambda)}\big) \oplus U_{m(\beta)} \oplus \big(\bigoplus_{\varphi \in S_2} U_{m(\varphi)}\big)$; this implies that R also commutes with $\widehat{J}$, and so $X = Ad_R(\widehat{J}) = \widehat{J}$. Hence $\widehat{J}$ is the unique element of $\Theta(Q)$. This proves (a) and (b).

Now, assume as in (c): $\zeta(Q) \geq 1$ and $\beta = \mu_{n-\zeta(Q)} = \mu_{n-\zeta(Q)+1}$, so that $m(\beta) = \nu_1 + \nu_2$. If we set $\widetilde{J} :=$

$$\big(\bigoplus_{\lambda \in S_1} \arg(\lambda)\,\mathbf{i}\ I_{m(\lambda)}\big) \oplus \big(\arg(\beta)\,\mathbf{i}\ I_{\nu_1}\big) \oplus \big((\arg(\beta) - 2\pi)\,\mathbf{i}\ I_{\nu_2}\big) \oplus \big(\bigoplus_{\varphi \in S_2} (\arg(\varphi) - 2\pi)\,\mathbf{i}\ I_{m(\varphi)}\big),$$

we have $\exp(\widetilde{J}) = J = Q$; as in the proof of parts (a) and (b) and using Lemma 1.2 and Remark 3.1 again, we can prove that a matrix $X \in \mathfrak{su}_n$ belongs to $\Theta(Q)$ if and only if $X = Ad_R(\widetilde{J})$ for some $R \in U_n$ commuting with Q, i.e. if and only if $X = Ad_R(\widetilde{J})$ for some $R \in G := \big(\bigoplus_{\lambda \in S_1} U_{m(\lambda)}\big) \oplus U_{(\nu_1+\nu_2)} \oplus \big(\bigoplus_{\varphi \in S_2} U_{m(\varphi)}\big)$. Since the map $(R, X) \mapsto Ad_R(X)$ defines a left action of the compact Lie group G on $\mathfrak{su}_n$, we conclude that $\Theta(Q)$ agrees with the orbit of $\widetilde{J}$ with respect to this action, while the corresponding isotropy subgroup at $\widetilde{J}$ is $\widehat{G} := \big(\bigoplus_{\lambda\in S_1} U_{m(\lambda)}\big) \oplus U_{\nu_1} \oplus U_{\nu_2} \oplus \big(\bigoplus_{\varphi\in S_2} U_{m(\varphi)}\big)$. Hence we conclude that $\Theta(Q)$ is a compact submanifold of $\mathfrak{su}_n$ diffeomorphic to $\dfrac{G}{\widehat{G}} \cong \dfrac{U_{(\nu_1+\nu_2)}}{U_{\nu_1}\oplus U_{\nu_2}}$ (see, for instance, [EoM-Orbit]). Since it is known that $\dfrac{U_{(\nu_1+\nu_2)}}{U_{\nu_1}\oplus U_{\nu_2}}$ is a symmetric space diffeomorphic to the complex Grassmannian $\mathbf{Gr}(\nu_2;\mathbb{C}^{(\nu_1+\nu_2)})$, the proof of the Proposition is complete. □

3.3. **Proposition.** *Let Q be a matrix of SU_n $(n\geq 2)$ whose n eigenvalues are $\mu_1,\cdots,\mu_n$. Denote by $s(Q)$ the multiplicity of -1 as an eigenvalue of Q and let* $\zeta(Q) = \dfrac{1}{2\pi}\sum_{j=1}^{n}\arg(\mu_j)$. *Then*
a) if $\zeta(Q) > s(Q)$ or $\zeta(Q) < 0$, the set $\mathfrak{su}_n$–plog(Q) is empty;
b) if $0\leq\zeta(Q)\leq s(Q)$, the set $\mathfrak{su}_n$–plog(Q) is a compact submanifold of $\mathfrak{su}_n$ diffeomorphic to the complex Grassmannian $\mathbf{Gr}(\zeta(Q);\mathbb{C}^{s(Q)})$; in particular, $\mathfrak{su}_n$–plog(Q) is a single point if and only if $\zeta(Q)=0$ or $\zeta(Q)=s(Q)$.

Proof. The proof follows directly from Lemma 2.5 and Proposition 3.2 . □

4. About diameter and diametral pairs of (SU_n,ϕ)

4.1. **Proposition.** *Let $m: SU_n\to\mathbb{R}$ be the map defined in Remarks-Definitions 2.3 (a) and let $\delta_n = \max\{m(Q): Q\in SU_n\}$ $(n\geq 2)$. Then we have*

$$\delta_n = \begin{cases} n\pi^2 & \text{if } n \text{ is even} \\ (n-\frac{1}{n})\pi^2 & \text{if } n \text{ is odd} \end{cases} ;$$

furthermore a matrix $Q\in SU_n$ satisfies $m(Q)=\delta_n$ if and only if

$$\begin{cases} Q = -I_n & \text{if } n \text{ is even} \\ Q = e^{\frac{(n-1)\pi\mathrm{i}}{n}} I_n \ \ \text{or} \ \ Q = e^{\frac{-(n-1)\pi\mathrm{i}}{n}} I_n & \text{if } n \text{ is odd} \end{cases} .$$

Proof. As noted in Remarks-Definitions 2.3 (d), δ_n agrees with the absolute maximum of the map $(\alpha_1,\cdots,\alpha_n)\mapsto m(\alpha_1,\cdots,\alpha_n)$, where $(\alpha_1,\cdots,\alpha_n)$ varies over the set of all ζ-admissible n-tuples of real numbers, with $\zeta\in\big[0,\lfloor\frac{n}{2}\rfloor\big]\cap\mathbb{Z}$. Now, fix any ζ-admissible n-tuple $(\beta_1,\cdots,\beta_n)$ such that $m(\beta_1,\cdots,\beta_n)=\delta_n$ (where $\frac{1}{2\pi}\sum_{j=1}^{n}\beta_j = \zeta\in\big[0,\lfloor\frac{n}{2}\rfloor\big]\cap\mathbb{Z}$) . First of all, we will prove that we necessarily have $(\beta_1,\cdots,\beta_n) = (\underbrace{\pi,\cdots,\pi}_{n})$ when n is even and $(\beta_1,\cdots,\beta_n) = (\underbrace{\frac{(n-1)\pi}{n},\cdots,\frac{(n-1)\pi}{n}}_{n})$ when n is odd, and then we will prove all the statements of the Proposition. The proof will be done in several steps.

i) If $\zeta\geq 1$, at least one of the following two conditions is necessarily true:
(ia) $\beta_1=\beta_2=\cdots=\beta_{n-\zeta}=\beta_{n-\zeta+1}$;
(ib) $\beta_{n-\zeta}=\beta_{n-\zeta+1}=\cdots=\beta_{n-1}=\beta_n$;
while, if $\zeta=0$, condition (ib) is obviously always satisfied.
In the case $\zeta\geq 1$, if both conditions (ia) and (ib) are false, then the sets $J_1 := \big\{i\in\{1,\cdots,n-\zeta\}:\beta_i<\beta_{n-\zeta+1}\big\}$, $J_2 := \big\{j\in\{n-\zeta+1,\cdots,n\}:\beta_j>\beta_{n-\zeta}\big\}$ are both non-empty, and therefore we can define $h:=\max J_1$, $k:=\min J_2$. Note that, from

definitions of h and k, we have $\beta_h < \beta_{h+1}$ and $\beta_{k-1} < \beta_k$; so we can choose $\epsilon > 0$ such that $\beta_h + \epsilon < \beta_{h+1}$ and $\beta_{k-1} < \beta_k - \epsilon$. Now we consider the new n-tuple $(\gamma_1, \cdots, \gamma_n)$, where $\gamma_r := \beta_r$ when $r \neq h, k$, $\gamma_h := \beta_h + \epsilon$, $\gamma_k := \beta_k - \epsilon$; we have $\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \beta_j$, so the n-tuple $(\gamma_1, \cdots, \gamma_n)$ is also $\zeta-$admissible. Since $\beta_k - \beta_h < 2\pi$, by Proposition 2.8 (b), we get $m(\gamma_1, \cdots, \gamma_n) - \delta_n = \gamma_h^2 + (2\pi - \gamma_k)^2 - \big[\beta_h^2 + (2\pi - \beta_k)^2\big] = 2\epsilon\big(\epsilon + 2\pi - (\beta_k - \beta_h)\big) > 0$, and this is contrary to the definition of δ_n . So we conclude that (ia) or (ib) is true.

ii) If condition (ia) holds (with $\zeta \geq 1$), then $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = \beta_n$.

In fact, otherwise, there is an index $r \in \big\{n - \zeta + 1,\, n - \zeta + 2, \cdots, n - 1, n\big\}$, such that $\beta_{r+1} > \beta_r = \beta_{r-1} = \cdots = \beta_{n-\zeta+1} = \beta_{n-\zeta} = \cdots = \beta_1$. Then, after setting $w := \beta_{r+1}$ and $z := \beta_r = \cdots = \beta_1$ (so that we have $r\, z = 2\pi\zeta - w - \sum_{j=r+2}^{n} \beta_j$), we choose $\epsilon > 0$ such that $w - r \cdot \epsilon > z + \epsilon$, and we consider the ζ-admissible n-tuple $(\varphi_1, \cdots, \varphi_n)$ defined by $\varphi_j := z + \epsilon$ for $j = 1, \cdots, r$, $\varphi_{r+1} := w - r \cdot \epsilon$ and $\varphi_h := \beta_h$ for $h = r + 2, \cdots, n$.

Taking into account Proposition 2.8 (b), we obtain $m(\varphi_1, \cdots, \varphi_n) - \delta_n =$
$(n - \zeta)\big[(z+\epsilon)^2 - z^2\big] + (r - n + \zeta)\big[(2\pi - z - \epsilon)^2 - (2\pi - z)^2\big] + (2\pi - w + r\,\epsilon)^2 - (2\pi - w)^2 =$
$\epsilon^2(r+r^2) + 2\epsilon\big[(n-\zeta)z - (r-n+\zeta)(2\pi - z) + r(2\pi - w)\big] = \epsilon^2(r+r^2) + 2\epsilon\big[r\,z + 2\pi(n-\zeta) - r\,w\big] =$
$\epsilon^2(r + r^2) + 2\epsilon\big[2\pi n - (r + 1)w - \sum_{j=r+2}^{n} \beta_j\big]$. Since $(r + 1)w + \sum_{j=r+2}^{n} \beta_j \leq n\pi$, then we get $m(\varphi_1, \cdots, \varphi_n) - \delta_n \geq \epsilon^2(r + r^2) + 2\epsilon n\pi > 0$, and this is impossible, bearing in mind the definition of δ_n . So we conclude that (ii) holds.

iii) If condition (ib) holds (now also with $\zeta \geq 0$), then $\beta_2 = \beta_3 = \cdots = \beta_{n-1} = \beta_n$.

Otherwise, we have $K := \big\{j \in \{2, 3, \cdots, n - \zeta - 1\} : \beta_j < \beta_{n-\zeta}\big\} \neq \emptyset$, and so, called $t := \max K$, we fix a positive real number ϵ such that $-\pi < \beta_1 - \epsilon$, $\beta_t + \epsilon < \beta_{t+1} = \beta_{n-\zeta}$ and we consider the ζ-admissible n-tuple $(\sigma_1, \cdots, \sigma_n)$ defined by $\sigma_i := \beta_i$ for $i \neq 1, t$, $\sigma_1 := \beta_1 - \epsilon$ and $\sigma_t := \beta_t + \epsilon$. From Proposition 2.8, we obtain $m(\sigma_1, \cdots, \sigma_n) - \delta_n =$ $(\beta_1 - \epsilon)^2 - \beta_1^2 + (\beta_t + \epsilon)^2 - \beta_t^2 = 2\epsilon(\epsilon + \beta_t - \beta_1) > 0$, since $\beta_1 \leq \beta_t$. Again, this contradicts the definition of δ_n, so we conclude that (iii) is true.

iv) From (i), (ii), (iii), we deduce that, whatever the value of $\zeta \in \big[0, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}$, we necessarily have $-\pi < \beta_1 \leq \beta_2 = \beta_3 = \cdots = \beta_n \leq \pi$.

Setting $x_0 := \beta_2 = \cdots = \beta_n$, we have $\beta_1 = 2\pi\zeta - (n - 1)x_0$, where $\zeta \in \big[0, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}$, $-\pi < 2\pi\zeta - (n - 1)x_0 \leq x_0 \leq \pi$; these inequalities imply $\dfrac{2\zeta\pi}{n} \leq x_0 < \dfrac{(2\zeta + 1)\pi}{n - 1}$ if $0 \leq \zeta \leq \lfloor\frac{n}{2}\rfloor - 1$ and $\dfrac{2\zeta\pi}{n} \leq x_0 \leq \pi$ if $\zeta = \lfloor\frac{n}{2}\rfloor$.

In other words, necessarily the pair (x_0, ζ) belongs to the set Λ defined by

$$\Lambda := \Big\{(x, \vartheta) : \ x \in \Big[\frac{2\vartheta\pi}{n}, \frac{(2\vartheta + 1)\pi}{n - 1}\Big) ,\ \vartheta \in \big[0, \lfloor\tfrac{n}{2}\rfloor - 1\big] \cap \mathbb{Z} \quad \text{or} \quad x \in \Big[\frac{2\vartheta\pi}{n}, \pi\Big] ,\ \vartheta = \lfloor\tfrac{n}{2}\rfloor\Big\} .$$

Conversely, it is easy to check that, for any $(x, \vartheta) \in \Lambda$, the n-tuple $\big(2\pi\vartheta - (n-1)x, \underbrace{x, \cdots, x}_{n-1}\big)$ is ϑ-admissible, with $\vartheta \in \big[0, \lfloor\frac{n}{2}\rfloor\big] \cap \mathbb{Z}$. Hence, if we define $F_\vartheta(x) := m\big(2\pi\vartheta - (n-1)x, \underbrace{x, \cdots, x}_{n-1}\big)$, we have $\delta_n = F_\zeta(x_0) \geq F_\vartheta(x)$, for any $(x, \vartheta) \in \Lambda$.

From Proposition 2.8, by means of easy calculations we obtain :

$F_\vartheta(x) = n(n-1)x^2 - 4\pi\vartheta n x + 4\pi^2\vartheta(\vartheta+1)$, for every $(x,\vartheta) \in \Lambda$.
For any fixed $\vartheta \in \big[0, \lfloor \frac{n}{2} \rfloor - 1\big] \cap \mathbb{Z}$, the quadratic function F_ϑ reaches its absolute minimum at the point $\overline{x}_\vartheta = \frac{2\vartheta\pi}{n-1} \in [\frac{2\vartheta\pi}{n}, \frac{(2\vartheta+1)\pi}{n-1})$, and we have
$\overline{x}_\vartheta - \frac{2\vartheta\pi}{n} = \frac{2\vartheta\pi}{n(n-1)} < \frac{\pi}{n-1} = \frac{(2\vartheta+1)\pi}{n-1} - \overline{x}_\vartheta$; hence, for every $x \in [\frac{2\vartheta\pi}{n}, \frac{(2\vartheta+1)\pi}{n-1})$, there exists $\widehat{x} \in [\frac{2\vartheta\pi}{n}, \frac{(2\vartheta+1)\pi}{n-1})$ such that $F_\vartheta(\widehat{x}) > F_\vartheta(x)$; this implies that $\vartheta \neq \zeta$, for every $\vartheta \in \big[0, \lfloor \frac{n}{2} \rfloor - 1\big] \cap \mathbb{Z}$, and so we conclude that it must necessarily be $\zeta = \lfloor \frac{n}{2} \rfloor$. The function $F_\zeta(x)$ (with $\zeta = \lfloor \frac{n}{2} \rfloor$) is strictly decreasing on the interval $\big[\frac{2\lfloor \frac{n}{2} \rfloor}{n}\pi, \pi\big]$, so it has its unique maximum at the point $\frac{2\lfloor \frac{n}{2} \rfloor \pi}{n}$. We conclude that it must necessarily be $x_0 = \frac{(n-1)\pi}{n}$ when n is odd , and $x_0 = \pi$ when n is even.
So, with an easy calculation, we obtain that we necessarily have $\beta_1 = \beta_2 = \cdots = \beta_n = \pi$ when n is even, and $\beta_1 = \beta_2 = \cdots = \beta_n = \frac{(n-1)\pi}{n}$ when n is odd.
v) Since all matrices of SU_n are diagonalizable, any n-tuple $(\underbrace{\lambda, \cdots, \lambda}_{n})$ (with $\lambda \in [0,\pi]$ and $\frac{\lambda n}{2\pi} \in \big[0, \lfloor \frac{n}{2} \rfloor\big] \cap \mathbb{Z}$) corresponds only to the matrix $e^{\lambda \mathbf{i}} I_n \in SU_n$. Therefore, taking into account Remarks-Definitions 2.3 (c), from (iv) we get that we have $m(Q) = \delta_n$ if and only if $Q = -I_n$ when n is even, and $Q = e^{\frac{(n-1)\pi \mathbf{i}}{n}} I_n$ or $Q = e^{\frac{-(n-1)\pi \mathbf{i}}{n}} I_n$ when n is odd .
Hence, by means of Proposition 2.8, it is easy to check that $\delta_n = n\pi^2$ when n is even, and $\delta_n = (n - \frac{1}{n})\pi^2$ when n is odd; so the proof is complete. □

5. Some geometrical properties of the Riemannian manifold (SU_n, ϕ)

5.1. **Theorem.** *Let d be the distance induced on SU_n by the Frobenius metric ϕ and let $P, Q \in SU_n$. With the same notation of Remarks 2.1 (c), without loss of generality we can assume $\zeta(P^*Q) \geq \zeta(Q^*P)$ (so that $\zeta(P^*Q) \geq 0$ by Remarks 2.1 (c)); we denote by $\mu_1, \cdots, \mu_n$ the n eigenvalues of P^*Q ordered so that $\arg(\mu_1) \leq \arg(\mu_2) \leq \cdots \leq \arg(\mu_n)$, and we set $\zeta := \zeta(P^*Q) = \frac{1}{2\pi}\sum_{j=1}^{n} \arg(\mu_j) \geq 0$. Then the following statements hold:*

a) $$d(P;Q) = \begin{cases} \sqrt{\sum_{j=1}^{n} \big(\arg(\mu_j)\big)^2} & \text{if } \zeta = 0 , \\ \sqrt{\sum_{j=1}^{n-\zeta} \big(\arg(\mu_j)\big)^2 + \sum_{j=n-\zeta+1}^{n} \big(2\pi - \arg(\mu_j)\big)^2} & \text{if } \zeta \geq 1 \end{cases} ;$$

b) there exists a unique minimizing geodesic segment of (SU_n, ϕ) with endpoints P and Q if and only if either $\zeta = 0$ or $\zeta \geq 1$ and $\mu_{n-\zeta} \neq \mu_{n-\zeta+1}$;

c) if $\zeta \geq 1$ and $\mu_{n-\zeta} = \mu_{n-\zeta+1}$,
after setting $\Theta(P^*Q) = \big\{X \in \mathfrak{su}_n : \exp(X) = P^*Q \, , \ \|X\|_\phi = d(P,Q)\big\}$,
$\nu_1 := max\big\{j \in \{1, \cdots, n-\zeta\} : \mu_{n-\zeta+1-j} = \mu_{n-\zeta}\big\}$,
$\nu_2 := max\big\{j \in \{1, \cdots, \zeta\} : \mu_{n-\zeta+j} = \mu_{n-\zeta+1}\big\}$,

the map: $X \mapsto \gamma(t) := P\exp(tX)$ $(0 \le t \le 1)$ *is a bijection from* $\Theta(P^*Q)$ *onto the set of minimizing geodesic segments of* (SU_n, ϕ) *with endpoints* P *and* Q*, and* $\Theta(P^*Q)$ *is a compact submanifold of* $\mathfrak{su}_n$ *diffeomorphic to the complex Grassmannian* $\mathbf{Gr}(\nu_2; \mathbb{C}^{(\nu_1+\nu_2)})$.

Proof. Since ϕ is bi-invariant, we have $\sqrt{m(P^*Q)} = d(I_n, P^*Q) = d(P,Q)$; so part (a) follows from Proposition 2.8, while parts (b) and (c) follow from Propositions 1.5 and 3.2. □

5.2. **Theorem.** *a) The diameter of* (SU_n, ϕ) *is*

$$\delta(SU_n, \phi) = \begin{cases} \sqrt{n}\,\pi & \text{if } n \text{ is even} \\ \sqrt{n - \dfrac{1}{n}}\;\pi & \text{if } n \text{ is odd} \end{cases} .$$

b) If n *is even and* P *is any matrix of* SU_n*, then* $-P$ *is the unique diametral point of* P *in* (SU_n, ϕ) *and the set of minimizing geodesic segments joining* P *and* $-P$ *can be parametrized by the complex Grassmannian* $\mathbf{Gr}(\frac{n}{2}; \mathbb{C}^n)$.

c) If n *is odd and* P *is any matrix of* SU_n*, then* P *has precisely two diametral points* P^+, P^- *in* (SU_n, ϕ), *with* $P^+ = e^{\frac{(n-1)\pi \mathbf{i}}{n}} P$ *and* $P^- = e^{\frac{-(n-1)\pi \mathbf{i}}{n}} P$, *and the sets of minimizing geodesic segments joining* P *with* P^+ *and* P *with* P^- *can both be parametrized by the complex Grassmannian* $\mathbf{Gr}(\frac{n-1}{2}; \mathbb{C}^n)$.

Proof. The Theorem follows easily from Proposition 4.1 and Theorem 5.1, taking into account that, in case of diametral pairs, the values of the constants in part (c) of Theorem 5.1 are : $\zeta = \nu_2 = \lfloor \frac{n}{2} \rfloor$, $\nu_1 = n - \lfloor \frac{n}{2} \rfloor$. □

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