

FORMALITY CRITERIA FOR ALGEBRAS OVER OPERADS

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ABSTRACT. We study some formality criteria for differential graded algebras over differential graded operads. This unifies and generalizes other known approaches like the ones contained in [Ka] and [Ma]. In particular, we construct general operadic Kaledin classes and show that they provide obstructions to formality. Moreover, we show that an algebra A is formal if and only if its operadic cohomology spectral sequence degenerates at E_2 .

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INTRODUCTION

The notion of formality of differential graded algebras is a very important one. One says that a differential graded algebra A is *formal* if it is quasi-isomorphic to its cohomology graded algebra $H(A)$. The most famous instance of formality in the mathematical literature is probably the paper [DGMS] by Deligne, Griffiths, Morgan and Sullivan, in which the authors showed that if X is a

compact Kähler manifold, then the de Rham algebra $DR(X)$ of X is formal. This has important consequences in the study of the topology of the manifold X .

The case of formality for differential graded Lie algebras is also a well studied subject. It is well known that differential graded Lie algebras are fundamental objects in the realm of deformation theory, and that to every dg Lie algebra L one can associate a natural deformation functor. Moreover, if L is assumed to be formal, the deformation functor is simplified considerably. In [GM] Goldman and Millson used the same approach of [DGMS] to prove that the dg Lie algebra of exterior differential forms taking values in certain flat vector bundles is formal. A geometric consequence of this result is that the moduli space of certain representations of the fundamental group of a compact Kähler manifold has at most quadratic singularities.

Another very famous manifestation of formality is found in the seminal paper [Ko] of Kontsevich. Consider the algebra A of smooth functions on a differentiable manifold, then Kontsevich proves that the dg Lie algebra of the Hochschild cochain complex of A (with Hochschild differential and Gerstenhaber bracket) is formal. A corollary of this result shows that every finite-dimensional Poisson manifold can be canonically quantized in the sense of deformation quantization.

More recently, Kaledin wrote the short and dense paper [Ka], where he addressed the question of finding cohomological obstructions to formalities for *families* of associative dg algebras. A concise exposition of Kaledin's work can be found in the paper [Lu], in which Lunts provides proper foundations and complete proofs for the main results of [Ka]. Kaledin's formality criteria were later used in [KL] and in [Zh] to attack some particular cases of a conjecture of Kaledin and Lehn about the singularities of the moduli space of semistable sheaves on K3 surfaces.

Given an associative dg algebra A , Kaledin's arguments are based on the construction of a certain Hochschild cohomology class of A (called *Kaledin class* in [Lu]). Kaledin is then able to prove that this class measures the obstruction to formality of A . In fact, one of the main results in [Ka] states that the algebra A is formal if and only if its Kaledin class vanishes. It is worth noting that similar results were obtained for commutative dg algebras by Sullivan in [Su] and by Halperin-Stasheff in [HS].

Inspired by [Ka] and [Lu], Manetti introduced a different flavour of formality criteria for dg Lie algebras in [Ma]. Manetti's approach differs from Kaledin's in that the main tool in [Ma] is the Chevalley-Eilenberg spectral sequence of a given dg Lie algebra L , which by definition is simply the natural spectral sequence computing the classical Chevalley-Eilenberg cohomology of L . It was already known that homotopy abelianity of a dg Lie algebra L is equivalent to the degeneration at E_1 of the Chevalley-Eilenberg spectral sequence of L (see [Ba]). The main result of [Ma] extends this by proving that formality of L is equivalent to the degeneration at E_2 of the Chevalley-Eilenberg spectral sequence.

In view of these formality criteria for different cases, the purpose of the present paper is to generalize formality criteria of both [Ka] and [Ma] to the general case of algebras over a sufficiently nice Koszul operad \mathcal{P} . In particular, we want \mathcal{P} to be weight graded and thus we will always assume \mathcal{P} to be reduced. For example, \mathcal{P} can be any of the usual suspects *Ass*, *Lie*, *Com*, *PreLie*, *Pois* etc. In the same fashion as [Lu] and [Ma], we found it more convenient to use the language of homotopy \mathcal{P} -algebras, or \mathcal{P}_∞ -algebras. For this purpose, we use the Koszul resolutions described in [GiK] and [LV].

For every \mathcal{P} -algebra A , we give an elementary construction of a natural class K_A in the \mathcal{P} -operadic cohomology of A , which we call *operadic Kaledin class*, in analogy with the choice of terminology of [Lu]. We remark that our construction avoids some of the technicalities of [Ka] and [Lu], as we don't need to pass through deformations to the normal cone. As expected, the operadic Kaledin class controls formality of A . Our first main result can therefore be stated as follows.

Theorem (see Theorem 3.10). *The \mathcal{P} -algebra A is formal if and only if its operadic Kaledin class K_A vanishes.*

Notice that in the case where \mathcal{P} is the operad Ass of associative algebras, our result gives back the Kaledin formality criterion.

We then pass to the study of the natural spectral sequence computing \mathcal{P} -operadic cohomology of the algebra A . Generalizing ideas from [Ma], we construct a distinguished element e_A in the second page E_2 of the operadic spectral sequence of the \mathcal{P} -algebra A , which we call *operadic Euler class* following the notations in [Ma]. We are then able to link the vanishing of the operadic Kaledin class K_A to the operadic Euler class e_A . More precisely, we show the following statement.

Theorem (see Theorem 5.1). *The Kaledin class K_A vanishes if and only if $d_r(e_A) = 0$ for every $r \geq 2$, where d_r denotes the differential in the r -th page of the operadic spectral sequence on A .*

As an immediate corollary of both theorems, we find the following formality criteria for the \mathcal{P} -algebra A .

Corollary. *Let A be a \mathcal{P} -algebra. The following are equivalent:*

- (1) A is formal;
- (2) the operadic Kaledin class K_A is zero;
- (3) $d_r(e_A) = 0$ for every $r \geq 2$;
- (4) the operadic spectral sequence of A degenerates at E_2 .

Moreover, if we take \mathcal{P} to be the operad Lie of Lie algebras, then the equivalence of items (1) – (3) – (4) is the main result of [Ma]. We remark however that the formality criterion of item (2) appears to be new even in the case of dg Lie algebras.

The paper is structured as follows. The first section is devoted to briefly recalling the operadic (and cooperadic) notions that we use in the paper. Following the exposition in [LV], we introduce quadratic (co)operads, and the basic construction of Koszul duality.

Section 2 deals with homotopy \mathcal{P} -algebras, and more precisely the description of \mathcal{P}_∞ -structures in terms of coderivations on cofree coalgebras.

In Section 3, we define the fundamental notions of minimal and formal \mathcal{P}_∞ -algebras. Moreover, we construct our main tool for the study of formality, which is the operadic Kaledin class. We then obtain our first formality criterion, which is Theorem 3.10. As mentioned, this generalizes results of [Ka] and [Lu].

The goal of Section 4 is to extend some constructions and ideas from [Ma] to our more general setting. We start by defining a natural operadic cohomology spectral sequence, which specializes to the Chevalley-Eilenberg spectral sequence whenever $\mathcal{P} = Lie$. Then, we define the operadic Euler class: a canonical element of the second page of the operadic spectral sequence. With these new tools, we are able to formulate a different formality criterion, generalizing a theorem of [Ma]. We summarize our formality criteria in Corollary 5.3.

In the final section, we treat in more detail the cases where \mathcal{P} is the operad Ass or Lie . We explicitly show how one can get back known results of Kaledin and Manetti from Corollary 5.3.

Future directions. We list here some of the topics we chose to not treat in the present version of the paper. We plan to come back to these question in future works.

As already explained, Manetti studies formality criteria of dg Lie algebra using degeneration of the Chevalley-Eilenberg spectral sequence in [Ma]. This is much in the spirit of [Ba], where Bandiera shows that the Chevalley-Eilenberg spectral sequence also detects homotopy abelianity. We certainly feel like the arguments of [Ba] can be smoothly extended to prove a homotopy abelianity criterion for algebras over a nice enough operad \mathcal{P} .

Apart from giving formality criteria for dg Lie algebras, [Ma] also contains very interesting results about formality transfers. It seems likely that similar statements can be proved for algebras over different operads.

Another direction arises from the work of Kaledin [Ka], which deals with formality for families of associative dg algebras. Algebraically, the use of families corresponds to working over a non-trivial base, as it is explained in [Lu]. In our opinion, it would be interesting to extend Kaledin's criterion for formality in families to algebras over general operads. We hope that the methods developed in this paper can be useful for this purpose.

Finally, formality has important consequences in deformation theory. From a more geometric point of view, the formality of a dg Lie algebra implies that the associated formal moduli problem has at most quadratic singularities. It would be interesting to study how different formalities impact the associated geometrical object. For example, pre-Lie formality could possibly have nice consequences in the context of pre-Lie deformation theory, as developed in [DSV].

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1. PRELIMINARIES

1.1. Notations and conventions. We will work over a field \mathbb{K} of characteristic 0.

The category of complexes of \mathbb{K} -modules will be denoted \mathbf{dgMod} , and its objects will be simply be called dg modules. We will adopt the cohomological point of view, and say that the differential of a dg module $V \in \mathbf{dgMod}$ is of degree +1. If $V \in \mathbf{dgMod}$ is a dg module, and $x \in V$ is a homogeneous element, we will denote by \bar{x} the cohomological degree of x .

1.2. Operads and cooperads. In this section we follow Chapter 5 to 7 of [LV] to recall the basic definitions of (co)operads. We start by introducing the monoidal category $\mathbf{dg S-Mod}$ of dg \mathbb{S} -modules.

Definition 1.1. A dg \mathbb{S} -module (M, d) is a collection $\{M(n)\}_{n \geq 0}$ of dg modules over the symmetric group \mathbb{S}_n , equipped with a differential of degree +1. A *morphism* f of dg \mathbb{S} -modules is a collection $\{f(n) : M(n) \rightarrow N(n)\}_{n \geq 0}$ of morphisms of graded \mathbb{S}_n -modules of degree 0 which commutes with the differentials, i.e.

$$d_N \circ f = f \circ d_M.$$

Note that we can also extend the tensor product of \mathbb{S} -modules to $\mathbf{dg S-Mod}$ by the following formula:

$$(M \otimes N)_p(n) := \bigoplus_{\substack{i+j=n \\ q+r=p}} \text{Ind}_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} (M_q(i) \otimes N_r(j)).$$

We now equip $\mathbf{dg S-Mod}$ with a monoidal product and a unit to see that it is indeed a monoidal category. We define a monoidal product of two graded \mathbb{S} -modules by the formula:

$$(M \circ N)(n) := \bigoplus_{k \geq 0} M(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1 + \dots + i_k = n} \text{Ind}_{\mathbb{S}_{i_1} \times \dots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} (N(i_1) \otimes \dots \otimes N(i_k)) \right).$$

The \mathbb{S} -module $I := (0, \mathbb{K}, 0, 0, \dots)$, considered as a graded \mathbb{S} -module concentrated in degree 0, is the unit for the monoidal product. We define a grading in $M \circ N$ by

$$(M \circ N)_g(n) := \bigoplus_{\substack{k \geq 0 \\ e+g_1+g_2+\dots+g_k=g}} M_e(k) \otimes_{\mathbb{S}_k} \left(\bigoplus_{i_1+\dots+i_k=n} \text{Ind}_{\mathbb{S}_{i_1} \times \dots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} (N_{g_1}(i_1) \otimes \dots \otimes N_{g_k}(i_k)) \right),$$

with differential $d_{M \circ N} := d_M \circ \text{id}_N + \text{id}_M \bullet d_N$. Note that the second summand is an infinitesimal composite of morphisms. More explicitly,

$$d_{M \circ N}(\mu; \nu_1, \dots, \nu_k) = (d_M(\mu); \mu_1, \dots, \mu_k) + \sum_{i=1}^k (-1)^{|\mu| + \sum_{j=1}^{i-1} |\nu_j|} (\mu; \nu_1, \dots, d_N(\nu_i), \dots, \nu_k).$$

Moreover, for any dg \mathbb{S} -modules M and N , we denote by $M \circ_{(1)} N$ the dg \mathbb{S} -module $M \circ (I, N)$. And whenever $f : M \rightarrow M'$ and $g : N \rightarrow N'$, the map $f \circ (\text{id}_I, g) : M \circ_{(1)} N \rightarrow M' \circ_{(1)} N'$ is denoted by $f \circ_{(1)} g$.

With the definitions above, the category $(\text{dg } \mathbb{S}\text{-Mod}, \circ, I)$ is a monoidal category [LV].

Before jumping into the theory of operads we state the Künneth formula in this setting, which will be used later on.

Proposition 1.2. [LV] *Let M and N be two dg \mathbb{S} -modules. Then, we have the following isomorphism of graded \mathbb{S} -modules*

$$H(M \circ N) \cong H(M) \circ H(N).$$

Proof. Since \mathbb{K} is a field of characteristic zero, by Maschke's theorem the ring $\mathbb{K}[\mathbb{S}_n]$ is semi-simple. Therefore, every $\mathbb{K}[\mathbb{S}_n]$ -module is projective (see [We, Section 4.2]). Applying this result to the explicit formula of the composite product \circ concludes the proof. \square

Definition 1.3. An *operad* $\mathcal{P} = (\mathcal{P}, \gamma, \iota)$ is a monoid in the monoidal category $(\text{dg } \mathbb{S}\text{-Mod}, \circ, I)$. In particular, it is an \mathbb{S} -module $\{\mathcal{P}(n)\}_{n \geq 0}$ equipped with a composition map

$$\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$$

and a unit

$$\iota : I \rightarrow \mathcal{P}$$

which satisfy the axioms of monoids, i.e. associativity and unitality. Moreover, when (\mathcal{P}, γ) is an operad, we define the *infinitesimal composition map* $\gamma_{\mathcal{P}}$ by

$$\mathcal{P} \circ_{(1)} \mathcal{P} \hookrightarrow \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P}.$$

A *morphism of operads* is a morphism of dg \mathbb{S} -modules compatible with the operads structure.

For example, let V be a dg \mathbb{K} -module. Then, $\text{End}_V = \{\text{End}_V(n) = \text{Hom}_{\mathbb{K}}(V^{\otimes n}, V)\}_{n \geq 0}$ is the endomorphism operad of V .

Definition 1.4. A \mathcal{P} -*algebra*, or an *algebra over an operad* \mathcal{P} , is a dg \mathbb{K} -module V endowed with a morphism of operads $\alpha : \mathcal{P} \rightarrow \text{End}(V)$. A *morphism of \mathcal{P} -algebras* $\varphi : (V, \alpha) \rightarrow (W, \beta)$ is a \mathbb{K} -linear map $\varphi : V \rightarrow W$ such that

$$\varphi(\alpha(\mu)(v_1, \dots, v_n)) = \beta(\mu)(\varphi(v_1), \dots, \varphi(v_n))$$

holds for any $v_1, \dots, v_n \in V$ and $\mu \in \mathcal{P}(n)$.

Note that a \mathcal{P} -algebra structure can also be given by a map $\gamma_V : \mathcal{P}(V) \rightarrow V$ compatible with respect to the composition product and the unit, where

$$\mathcal{P}(V) := \mathcal{P} \circ (V, 0, 0, \dots) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}.$$

We can also give similar definitions in the dual sense.

Definition 1.5. A *cooperad* $\mathcal{C} = (\mathcal{C}, \Delta, \eta)$ is a comonoid in the monoidal category $(\text{dg } \mathbb{S}\text{-Mod}, \bar{\circ}, I)$, where $\bar{\circ}$ denotes the invariants for the diagonal action rather than the coinvariants. In particular, it is an \mathbb{S} -module \mathcal{C} equipped with a decomposition map

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \bar{\circ} \mathcal{C}$$

and a counit

$$\epsilon : \mathcal{C} \longrightarrow I$$

which satisfy the axioms of comonoids, i.e. coassociativity and counitality. A *morphism of cooperads* is thus a morphism of dg \mathbb{S} -modules compatible with the cooperads structure. Moreover, when (\mathcal{C}, Δ) is an operad, we define the *infinitesimal decomposition map* $\Delta_{\mathcal{C}}$ by

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \rightarrow \mathcal{C} \circ_{(1)} \mathcal{C}$$

A *cofree cooperad* on an \mathbb{S} -module M is the cooperad $\mathcal{F}^c(M)$, which is cofree in the category of conilpotent cooperads.

Definition 1.6. A *\mathcal{C} -coalgebra*, or a *coalgebra over a cooperad \mathcal{C}* , is a dg \mathbb{K} -module V endowed with a morphism

$$\delta : V \longrightarrow \mathcal{C}(V) = \bigoplus_{n \geq 0} (\mathcal{C}(n) \otimes V^{\otimes n})^{\mathbb{S}_n},$$

satisfying compatibility properties. Here $(-)^{\mathbb{S}_n}$ denotes the space of invariant elements.

Definition 1.7. In order to use spectral sequence arguments for (co)operads, one needs to introduce an additional grading different from that of the cohomological degree, this will be the *weight grading*. In particular, a weight grading on a (co)operad \mathcal{P} (\mathcal{C}) is a decomposition

$$\mathcal{P} = \mathbb{K}\text{id} \oplus \mathcal{P}^{(1)} \oplus \dots \oplus \mathcal{P}^{(w)} \oplus \dots,$$

where both the differential and the (de)composition map are compatible with the total weight. Furthermore, morphisms between such (co)operads also preserve the weight.

Remark 1.8. An operad \mathcal{P} such that $\mathcal{P}(0) = 0$ is called *reduced*. Such operads have the following canonical weight grading:

$$\mathcal{P}^{(k)} := \mathcal{P}(k+1).$$

This weight grading can be carried over to \mathcal{P} -algebras. In particular, $\mathcal{P}(V)$ from Definition 1.4 can be written as a decomposition $\mathcal{P}(V)^{(w)} := \mathcal{P}^{(w)} \circ (V, 0, 0, \dots)$. The same holds for reduced cooperads and their coalgebras.

1.3. Quadratic (co)operads.

Definition 1.9. The *free operad* over the \mathbb{S} -module M is an operad $\mathcal{F}(M)$ equipped with an \mathbb{S} -module morphism $\eta(M) : M \rightarrow \mathcal{F}(M)$ satisfying the following universal condition:

Let \mathcal{P} be an operad. Any \mathbb{S} -module morphism $f : M \rightarrow \mathcal{P}$, extends uniquely into an operad morphism $\tilde{f} : \mathcal{F}(M) \rightarrow \mathcal{P}$

$$\begin{array}{ccc}
 M & \xrightarrow{\eta(M)} & \mathcal{F}(M) \\
 & \searrow f & \downarrow \bar{f} \\
 & & \mathcal{P}.
 \end{array}$$

Definition 1.10. An *operadic ideal* of an operad \mathcal{P} is a sub- \mathbb{S} -module \mathcal{I} of \mathcal{P} such that the operad structure on \mathcal{P} transfers to the quotient \mathcal{P}/\mathcal{I} .

Let E be an \mathbb{S} -module and R a sub- \mathbb{S} -module such that $R \subseteq \mathcal{F}(E)^{(2)}$, where $\mathcal{F}(E)^{(2)}$ is the graded sub- \mathbb{S} -module of the free operad $\mathcal{F}(E)$, which is spanned by the composites of two elements of E ; i.e. the set of trees with two vertices. Such a pair (E, R) is called a *quadratic data*.

Definition 1.11. Given a quadratic data (E, R) , its associated *quadratic operad* $\mathcal{P}(E, R)$ is the quotient of the free operad $\mathcal{F}(E)$ over E by the operadic ideal (R) generated by R ,

$$\mathcal{P}(E, R) = \mathcal{F}(E)/(R).$$

We can give $\mathcal{P}(E, R)$ a weight grading by endowing $\mathcal{F}(E)$ with a weight grading, which differs from the cohomological degree; this is given by the number of vertices.

Definition 1.12. Given a quadratic data (E, R) , its associated *quadratic cooperad* $\mathcal{P}(E, R)$ is the sub-cooperad of the cofree cooperad $\mathcal{F}^c(E)$, which is universal among the sub-cooperads of $\mathcal{F}^c(E)$ such that the following composite is zero:

$$\mathcal{C} \hookrightarrow \mathcal{F}^c(E) \twoheadrightarrow \mathcal{F}^c(E)^{(2)}/(R).$$

Note that since (R) is a homogeneous ideal with respect to the weight, we get a weight grading in $\mathcal{F}^c(E)$. Explicitly, for a given $E = (0, E(1), E(2), \dots)$ we have

$$\mathcal{F}^c(E) = \bigoplus_k \mathcal{F}^c(E)^{(k)}.$$

Definition 1.13. The *Koszul dual cooperad* of the quadratic operad $\mathcal{P}(E, R)$ is the quadratic cooperad

$$\mathcal{P}^i := \mathcal{C}(sE, s^2R),$$

where sE denotes the \mathbb{S} -module E whose degree is shifted by 1.

Here, we equip the sub- \mathbb{S} -module $\mathcal{C}(E, R)$ of $\mathcal{F}^c(E)$ with a weight grading such that

$$\begin{aligned}
 \mathcal{C}(E, R)^{(0)} &= I \\
 \mathcal{C}(E, R)^{(1)} &= E \\
 \mathcal{C}(E, R)^{(2)} &= (0, R(1), R(2), \dots).
 \end{aligned}$$

Thus, the cooperad structure on $\mathcal{F}^c(E)$ induces the cooperad structure on $\mathcal{C}(E, R)$.

1.4. Koszulity. Let us first introduce the *bar construction*, that is a functor

$$\Omega : \{\text{augmented dg operads}\} \longrightarrow \{\text{conilpotent dg operads}\}.$$

Let \mathcal{P} be an augmented operad and recall that its augmentation ideal is $\bar{\mathcal{P}} := \ker(\epsilon : \mathcal{P} \rightarrow I)$. Then, the bar construction $\mathbf{B}\mathcal{P}$ of \mathcal{P} is a dg cooperad defined on the cofree cooperad $\mathcal{F}^c(s\bar{\mathcal{P}})$ on the

suspension of $\bar{\mathcal{P}}$ with differential $d := d_1 + d_2$. Here, d_2 is the map on $s\bar{\mathcal{P}}$ defined as

$$\begin{aligned} d_2 : \mathcal{F}^c(s\bar{\mathcal{P}}) &\rightarrow \mathcal{F}^c(s\bar{\mathcal{P}})^{(2)} \cong \bigoplus_{\text{2-vertices trees}} (\mathbb{K}s \otimes \bar{\mathcal{C}}) \otimes (\mathbb{K}s \otimes \bar{\mathcal{C}}) \\ &\xrightarrow{\text{id} \otimes \tau \otimes \text{id}} \bigoplus_{\text{2-vertices trees}} (\mathbb{K}s \otimes \mathbb{K}s) \otimes (\bar{\mathcal{C}} \otimes \bar{\mathcal{C}}) \\ &\xrightarrow{\Delta_s \otimes \gamma_{\mathcal{P}}} \mathbb{K}s \otimes \bar{\mathcal{C}}, \end{aligned}$$

where $\Delta_s(s \otimes s) := s$ is the degree -1 diagonal map, and $\tau : \bar{\mathcal{P}} \otimes \mathbb{K}s \rightarrow \mathbb{K}s \otimes \bar{\mathcal{P}}$ is the symmetry isomorphism given by the Koszul sign of two elements.

Note that $(\mathcal{F}^c(s\bar{\mathcal{P}}), d_2)$ is a conilpotent dg cooperad that can already be regarded as the bar construction of the augmented operad \mathcal{P} . We extend this construction to dg operads $(\mathcal{P}, d_{\mathcal{P}})$ by adding to d_2 the internal differential d_1 on $\mathcal{F}^c(s\bar{\mathcal{P}})$ induced by the differential $d_{\mathcal{P}}$. Adding these two differentials gives a well-defined bicomplex because d_1 and d_2 anticommute, i.e. $d_1 \circ d_2 + d_2 \circ d_1 = 0$. Finally, the total complex of this bicomplex is called the bar construction

$$\Omega\mathcal{C} := (\mathcal{F}(s^{-1}\bar{\mathcal{C}}), d)$$

of the augmented dg operad $(\mathcal{P}, d_{\mathcal{P}})$.

Remark. Assume that \mathcal{P} is weight graded. Then, the bar construction is bigraded by the number of non-trivial indexed vertices w and by the total weight ρ :

$$\mathbf{B}_{(w)}\mathcal{P} := \bigoplus_{\rho \in \mathbb{N}} \mathbf{B}_{(w)}\mathcal{P}^{(\rho)}.$$

Let us now introduce the dual of this construction, the *cobar construction*

$$\Omega : \{\text{coaugmented dg operads}\} \longrightarrow \{\text{augmented dg operads}\}.$$

Let \mathcal{C} be a cooperad and recall that the coaugmentation coideal of \mathcal{C} is $\bar{\mathcal{C}} := \text{Coker} \eta : \mathbf{I} \rightarrow \mathcal{C}$. Then, the cobar construction $\Omega\mathcal{C}$ of \mathcal{C} is an augmented dg operad defined on the free operad $\mathcal{F}(s^{-1}\bar{\mathcal{C}})$ over the desuspension of $\bar{\mathcal{C}}$ with differential $d := d_1 + d_2$. Here, d_2 is the map on $s^{-1}\bar{\mathcal{C}} = \mathbb{K}s^{-1} \otimes \bar{\mathcal{C}}$ defined as

$$\begin{aligned} d_2 : \mathbb{K}s^{-1} \otimes \bar{\mathcal{C}} &\xrightarrow{\Delta_s \otimes \Delta_{(1)}} \bigoplus_{\text{2-vertices trees}} (\mathbb{K}s^{-1} \otimes \mathbb{K}s^{-1}) \otimes (\bar{\mathcal{C}} \otimes \bar{\mathcal{C}}) \\ &\xrightarrow{\text{id} \otimes \tau \otimes \text{id}} \bigoplus_{\text{2-vertices trees}} (\mathbb{K}s^{-1} \otimes \bar{\mathcal{C}}) \otimes (\mathbb{K}s^{-1} \otimes \bar{\mathcal{C}}) \cong \mathcal{F}(s^{-1}\bar{\mathcal{C}})^{(2)} \hookrightarrow \mathcal{F}(s^{-1}\bar{\mathcal{C}}), \end{aligned}$$

where Δ_s is degree -1 diagonal map, and τ is the symmetry isomorphism given by the Koszul sign of two elements. This map is indeed a differential because $\Delta_{(1)}$ shares dual relations with the infinitesimal composition map of an operad.

In fact, $(\mathcal{F}(s^{-1}\bar{\mathcal{C}}), d_2)$ is a dg operad that can already be regarded as the cobar construction of the cooperad \mathcal{C} , but we extend this construction to coaugmented dg operads $(\mathcal{C}, d_{\mathcal{C}})$ by adding to d_2 the internal differential d_1 induced by the differential $d_{\mathcal{C}}$. Adding these two differentials gives a well-defined bicomplex because d_1 and d_2 anticommute. Finally, the total complex of this bicomplex is called the cobar construction

$$\Omega\mathcal{C} := (\mathcal{F}(s^{-1}\bar{\mathcal{C}}), d)$$

of the coaugmented dg cooperad $(\mathcal{C}, d_{\mathcal{C}})$.

Given a quadratic data (E, R) we can associate to it the twisting morphism

$$\kappa : \mathcal{P}^i \rightarrow sE \xrightarrow{s^{-1}} E \hookrightarrow \mathcal{P},$$

which induces the Koszul complex of the operad \mathcal{P} . This is a chain complex of \mathbb{S} -modules

$$\mathcal{P}^i \circ_{\kappa} \mathcal{P} := \left(\mathcal{P}^i \circ \mathcal{P}, d_{\kappa} \right),$$

i.e. for any $n \geq 0$ we have a chain complex of \mathbb{S}_n -modules $((\mathcal{P}^i \circ \mathcal{P})(n), d_{\kappa})$, called the Koszul complex in arity n .

Definition 1.14. A quadratic operad \mathcal{P} is *Koszul* if its associated Koszul complex $(\mathcal{P}^i \circ \mathcal{P}, d_{\kappa})$ is acyclic.

Proposition 1.15. [LV] *Let \mathcal{P} be a quadratic operad. If \mathcal{P} is Koszul, then $\Omega\mathcal{P}^i$ is the minimal model of \mathcal{P} .*

1.5. Spectral sequences. In this section we recall the classical construction of the cohomology spectral sequence associated to a filtered cochain complex. We mainly use this section to fix our notation with spectral sequences. A standard reference is the book [We].

Let (C, d) be a cochain complex, equipped with a decreasing filtration indexed by $p \in \mathbb{Z}$

$$\dots \subset F^{p-1}C \subset F^pC \subset F^{p+1}C \subset \dots$$

where we are assuming that the differential d satisfies $d(F^pC) \subset F^pC$.

If we define

$$Z_r^{p,q} := \{x \in F^pC^{p+q} \mid dx \in F^{p+r}C^{p+q+1}\}$$

to be the set of r -almost (p, q) -cocycles, then the associated cohomology spectral sequence $E(C)_r^{p,q}$ is defined to be

$$E_r^{p,q} := \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}}.$$

The differential

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$$

on the r -th page of the spectral sequence is induced by the differential d of the complex C . For notational convenience, we will sometimes drop the index q , and use the notation

$$Z_r^p = \bigoplus_q Z_r^{p,q}, \quad E_r^p = \bigoplus_q E_r^{p,q}.$$

2. RECOLLECTIONS ON HOMOTOPY ALGEBRAS

Throughout this paper, we let \mathcal{P} be *binary quadratic operad*, which is an operad generated by operations of arity 2 and which is quadratic (see [LV, Section 7.1.3]). Notice that these are also called simply *quadratic operads* in other texts (for example, in [GeK]). Moreover, we will suppose for simplicity that the generating operations of \mathcal{P} are of cohomological degree 0, as this simplifies some of our arguments. Let us also assume that \mathcal{P} is a Koszul dg operad, in the sense of Definition 1.14, or see equivalently [LV, Section 7.4.3]. Consider the Koszul dual cooperad $\mathcal{C} = \mathcal{P}^i$ of \mathcal{P} , as in Definition 1.13. The cooperad \mathcal{C} plays a fundamental role in the theory of homotopy algebras.

Definition 2.1. A *homotopy \mathcal{P} -algebra* is an algebra over the Koszul dual operad $\Omega\mathcal{C}$.

Alternatively, homotopy \mathcal{P} -algebras will also be called \mathcal{P}_{∞} -algebras, where \mathcal{P}_{∞} stands for the canonical resolution $\Omega\mathcal{C}$ of \mathcal{P} .

Let now $A \in \mathbf{dgMod}$ be a dg module. Then a \mathcal{P}_{∞} -structure on A is the same as a codifferential of degree +1 on the cofree \mathcal{C} -coalgebra $\mathcal{C}(A)$ (see for example [LV, Section 10.1.17]). In other terms, a \mathcal{P}_{∞} -structure on A is by definition a coderivation on $\mathcal{C}(A)$ of cohomological degree 1, which moreover squares to zero. More precisely, we can consider the graded Lie algebra of coderivations $\mathbf{Coder}(\mathcal{C}(A))$, where the Lie bracket is simply given the graded commutator of coderivations.

Remark 2.2. There is also a slightly more general construction: if C is a graded \mathcal{C} -coalgebra and M is a graded C -comodule, one has the notion of coderivations $M \rightarrow C$. These again form a graded Lie algebra $\text{Coder}(M, C)$. For example, if we have a morphism of \mathcal{C} -coalgebras $f : C \rightarrow D$, we can view C as a D -comodule. In this case, we use the notation $\text{Coder}(C, D, f)$.

Notice that since $\mathcal{C}(A)$ is cofree as a \mathcal{C} -coalgebra, a coderivation $f : \mathcal{C}(A) \rightarrow \mathcal{C}(A)$ is uniquely determined by the composition

$$\mathcal{C}(A) \xrightarrow{f} \mathcal{C}(A) \longrightarrow A,$$

where the last map is the canonical projection to A . It follows that we have an isomorphism

$$\text{Coder}(\mathcal{C}(A)) \simeq \text{Hom}(\mathcal{C}(A), A),$$

where the right hand side is the internal hom of complexes. We can thus use this isomorphism to get an induced graded Lie bracket on $\text{Hom}(\mathcal{C}(A), A)$. In the case of $\mathcal{P} = \text{Lie}$, this is commonly known as Nijenhuis-Richardson bracket.

An important observation here is that $\mathcal{C}(A)$ is a *graded complex*: by this we mean that it has an *internal* grading, coming from the fact that A has a cohomological grading itself, but also an *external* grading, coming from the fact that it is a cofree coalgebra. To avoid confusion, we will refer to this external grading by calling it *weight*, while the internal grading will be called *degree*. Finally, the weight grading on $\mathcal{C}(A)$ automatically gives an additional grading on $\text{Hom}(\mathcal{C}(A), A)$.

An alternative way to understand the weight grading is the following. By [LV, Proposition 6.3.17], we have an isomorphism

$$\text{Coder}(\mathcal{C}(A)) \simeq \text{Hom}_{\mathbb{S}}(\mathcal{C}, \text{End}_A).$$

Recall from Section 1 that the cooperad $\mathcal{C} = \mathcal{P}^i$ comes equipped with a natural weight grading decomposition

$$\mathcal{C} = \mathcal{P}^i = \bigoplus_{n \geq 0} \mathcal{C}^{(n)},$$

which in turn induces a weight grading

$$\text{Hom}_{\mathbb{S}}(\mathcal{C}, \text{End}_A) = \bigoplus_{n \geq 0} \text{Hom}_{\mathbb{S}}(\mathcal{C}, \text{End}_A)^{(n)},$$

where we set

$$\text{Hom}_{\mathbb{S}}(\mathcal{C}, \text{End}_A)^{(n)} = \text{Hom}_{\mathbb{S}}(\mathcal{C}^{(n)}, \text{End}_A).$$

The induced weight grading on $\text{Coder}(\mathcal{C}(A))$ is compatible with the Lie bracket of coderivations. More specifically, the Lie bracket can be seen as a morphism of graded complexes

$$\text{Coder}(\mathcal{C}(A)) \otimes \text{Coder}(\mathcal{C}(A)) \rightarrow \text{Coder}(\mathcal{C}(A)).$$

Consider now a codifferential $Q \in \text{Coder}(\mathcal{C}(A))$, that is to say a coderivation of cohomological degree 1 which squares to zero. With such Q , the dg module A becomes a \mathcal{P}_{∞} -algebra, which will be denoted by (A, Q) . Since $\mathcal{C}^{(0)} = I$, the weight zero component of Q must be zero itself, and thus Q can be thought as an infinite sum

$$Q = q_1 + q_2 + \dots$$

where q_i has weight i .

The weight grading on $\text{Coder}(\mathcal{C}(A))$ will be very important in what follows. In particular, we will use the following result.

Proposition 2.3. *Let (A, Q) be a \mathcal{P}_{∞} -algebra. Then the following are equivalent:*

- (1) *the algebra (A, Q) is a strict \mathcal{P} -algebra;*

(2) the codifferential Q is concentrated in weight 1.

Proof. This is an immediate consequence of [LV, Proposition 10.1.7]. \square

The interpretation of \mathcal{P}_∞ -structures in terms of codifferentials is particularly nicely suited to define \mathcal{P}_∞ -morphisms.

Definition 2.4. A morphism of \mathcal{P}_∞ -algebras $f : (A, Q) \rightarrow (B, R)$ is a map of \mathcal{C} -coalgebras $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$ commuting with the differentials Q and R .

Notice that \mathcal{P}_∞ -morphisms have also a weight grading decomposition. More specifically, every morphism of \mathcal{C} -coalgebras $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$ is completely determined by the composite

$$\mathcal{C}(A) \rightarrow \mathcal{C}(B) \rightarrow B,$$

as $\mathcal{C}(B)$ is cofree. Let us denote by End_B^A the \mathbb{S} -module defined by

$$\text{End}_B^A(n) := \text{Hom}(A^{\otimes n}, B).$$

Then a linear map $\mathcal{C}(A) \rightarrow B$ is equivalent to an element of $\text{Hom}_{\mathbb{S}}(\mathcal{C}, \text{End}_B^A)$, and as before the weight grading on \mathcal{C} induces a decomposition

$$f = (f_0, f_1, \dots).$$

In particular, the component f_0 gives a map of complexes $A \rightarrow B$.

Definition 2.5. We say that a map f of \mathcal{P}_∞ -algebras is a quasi-isomorphism if the induced map $f_0 : A \rightarrow B$ is a quasi-isomorphism of complexes in the usual sense.

2.1. Filtrations. Let A be a \mathcal{P}_∞ -algebra. In what follows, we will use the natural filtration on $\text{Coder}(\mathcal{C}(A))$ induced by the weight grading. It is actually convenient to construct it in the slightly more general context of \mathcal{P}_∞ -morphisms, as follows.

Take a \mathcal{P}_∞ -morphism $f : (A, Q) \rightarrow (B, R)$. This gives a map $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$ of \mathcal{C} -coalgebras, and following Remark 2.2 we can construct the complex

$$\text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f).$$

Notice that the differential is induced by the two differentials Q and R on $\mathcal{C}(A)$ and $\mathcal{C}(B)$ respectively. In the case of $\mathcal{P} = \text{Lie}$, this is precisely the Chevalley-Eilenberg complex of a L_∞ -map, as constructed in [Ma, Definition 5.2].

Since $\mathcal{C}(B)$ is cofree, every coderivation $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$ is completely determined by the composition

$$\mathcal{C}(A) \rightarrow \mathcal{C}(B) \rightarrow B.$$

Moreover, as mentioned there is a weight grading decomposition

$$\mathcal{C}(A) = \bigoplus_{i \geq 0} \mathcal{C}(A)^{(i)}$$

on the cofree coalgebra $\mathcal{C}(A)$, which in turn induces a decomposition

$$\text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f) = \prod_{i \geq 0} \text{Coder}(\mathcal{C}(A), \mathcal{C}(B), f)^{(i)}.$$

Therefore, we define a filtration

$$F^p \text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f) := \prod_{i \geq p} \text{Coder}(\mathcal{C}(A), \mathcal{C}(B), f)^{(i)}.$$

In the special case where f is the identity of a \mathcal{P}_∞ -algebra (A, Q) , the corresponding filtration on $\text{Coder}(\mathcal{C}(A))$ will be denoted by $F^p \text{Coder}(\mathcal{C}(A))$.

3. FORMALITY OF \mathcal{P}_∞ -ALGEBRAS

In this section we introduce the important notion of *formal* \mathcal{P}_∞ -algebras. Our first main result is that formality is controlled by a certain cohomology class, which we call *operadic Kaledin class*.

3.1. Minimal and formal homotopy algebras. We start by the simplest notion of minimality for algebras over operads.

Definition 3.1. Let \mathcal{P} be any operad, and let A be a \mathcal{P} -algebra. We say that A is *minimal* if its differential is zero.

Notice that if (A, Q) is a minimal \mathcal{P}_∞ -algebra, and $Q = q_1 + q_2 + \dots$ is the weight decomposition of Q , then (A, q_1) is a minimal \mathcal{P} -algebra.

Lemma 3.2. *Every \mathcal{P}_∞ -algebra (A, Q) is quasi-isomorphic to a minimal \mathcal{P}_∞ -algebra.*

Proof. This is an immediate consequence of [LV, Theorem 10.3.15]. Alternatively, we know that A is quasi-isomorphic to its cohomology $H(A)$ as dg modules over \mathbb{K} . It follows that there is an induced \mathcal{P}_∞ -structure Q' on $H(A)$ such that $(H(A), Q') \simeq (A, Q)$ as \mathcal{P}_∞ -algebras. \square

In view of Lemma 3.2, we deduce following guiding principle: if we are only interested in constructions and properties of \mathcal{P}_∞ -algebras which are invariant under quasi-isomorphisms, then we can safely restrict ourselves to minimal \mathcal{P}_∞ -algebras.

Definition 3.3. A \mathcal{P}_∞ -algebra A is said to be *formal* if it is quasi isomorphic to a strict \mathcal{P} -algebra, which is moreover minimal.

Remark 3.4. Since we are working over a field, Lemma 3.2 tells us that any \mathcal{P}_∞ -algebra (A, Q) is quasi-isomorphic to the \mathcal{P}_∞ -algebra $(H(A), Q')$. On the other hand, the differential on $H(A)$ is of course zero, and thus $H(A)$ is also a strict \mathcal{P} -algebra. However, the quasi-isomorphism between A and $H(A)$ is not compatible with this strict \mathcal{P} -structure.

We now address the question of finding a cohomological obstruction to formality for \mathcal{P}_∞ -algebras. As a consequence of Lemma 3.2, we can restrict our attention to minimal \mathcal{P}_∞ -algebras.

Let thus (A, Q) be a minimal \mathcal{P}_∞ -algebra. In other words, let $Q = q_1 + q_2 + \dots$ be a Maurer-Cartan element in the graded Lie algebra $\text{Coder}(\mathcal{C}(A))$. Then in particular we can use Q to define a differential $d_Q := [Q, -]$ on $\text{Coder}(\mathcal{C}(A))$. The cohomology of the complex $(\text{Coder}(\mathcal{C}(A)), d_Q)$ is the *operadic cohomology* of A .

Remark 3.5. If for example $\mathcal{P} = \text{Lie}$, we get back the usual Chevalley-Eilenberg cohomology. In other interesting cases such as $\mathcal{P} = \text{Ass}$, $\mathcal{P} = \text{Comm}$ or $\mathcal{P} = \text{Pois}$ we obtain the standard notion of Hochschild cohomology, Harrison cohomology and Poisson cohomology respectively.

3.2. Operadic Kaledin class and formality. Again, let (A, Q) be a minimal \mathcal{P}_∞ -algebra, and consider the weight decomposition $Q = q_1 + q_2 + \dots$ of Q inside $\text{Coder}(\mathcal{C}(A))$. If we define

$$\tilde{Q} := q_2 + 2q_3 + 3q_4 + \dots,$$

then a simple check gives us that $[Q, \tilde{Q}] = 0$. Moreover, clearly \tilde{Q} is a degree 1 element of $F^1\text{Coder}(\mathcal{C}(A))$, and thus \tilde{Q} defines a class in $H^1(F^1\text{Coder}(\mathcal{C}(A)))$, where the cohomology is computed with respect to the differential d_Q .

Remark 3.6. The additional grading on the complex $\text{Coder}(\mathcal{C}(A))$ can be interpreted as a structure of a $k[h, h^{-1}]$ -comodule, where h is a formal variable. In this sense, \tilde{Q} can be thought as a sort of formal derivative of Q with respect to the formal variable h , which controls the weight grading. This is essentially the approach taken in [Ka] and [Lu].

Definition 3.7. The class $K_A \in H^1(F^1(\text{Coder}(\mathcal{C}(A))))$ defined by \tilde{Q} is the *operadic Kaledin class* of the \mathcal{P}_∞ -algebra A .

Remark 3.8. Notice that \tilde{Q} also clearly defines a class in the cohomology of the whole complex $\text{Coder}(\mathcal{C}(A))$, which includes the weight 0 component. However, this class turns out to be identically zero, as we will show in Section 4.1.

The n -truncation $K_A^{\leq n}$ of K_A is obtained by considering only weight components of weights $\leq n$. More explicitly, the element

$$\tilde{Q}^{\leq n} = q_2 + 2q_3 + \cdots + (n-1)q_n$$

is a cocycle in the complex $F^1\text{Coder}(\mathcal{C}(A))/F^{n+1}$, and we set

$$K_A^{\leq n} = [q_2 + 2q_3 + \cdots + (n-1)q_n] \in H^1(F^1\text{Coder}(\mathcal{C}(A))/F^{n+1}).$$

The importance of the operadic Kaledin class is that it controls formality of A .

Proposition 3.9. *Let (A, Q) be a minimal \mathcal{P}_∞ -algebra, and let K_A be its Kaledin class. The following are equivalent:*

- (1) *there exists an isomorphism of minimal \mathcal{P}_∞ -algebras $(A, Q) \rightarrow (A, R)$, where $r_1 = q_1$ and $r_2 = r_3 = \cdots = r_n = 0$;*
- (2) *the truncated class $K_A^{\leq n}$ is zero.*

Proof. The fact that (1) implies (2) is clear, since the n -truncation of the Kaledin class of (A, R) is zero, and the Kaledin class is invariant under isomorphisms. In order to prove (2) \Rightarrow (1), we can work by induction. The case $n = 1$ is clear. Suppose now that the proposition holds for $n - 1$ and that $K_A^{\leq n} = 0$. In particular, also $K_A^{\leq (n-1)} = 0$, and by induction hypothesis, without loss of generality we can safely suppose that $q_2 = \cdots = q_{n-1} = 0$. In this case we have

$$Q = q_1 + q_n + q_{n+1} + \cdots$$

and therefore

$$\tilde{Q} = (n-1)q_n + nq_{n+1} + \cdots$$

Since \tilde{Q} must be zero in cohomology in weights $\leq n$, there exists a $T = t_1 + t_2 + \cdots \in F^1\text{Coder}(\mathcal{C}(A))$ such that

$$[Q, T] = \tilde{Q}$$

in weights $\leq n$. By looking at what happens in weight n , we deduce that we have

$$[q_1, t_{n-1}] = (n-1)q_n.$$

Putting $\tau = \frac{t_{n-1}}{n-1}$, we get $[q_1, \tau] = q_n$. Consider $R = e^{-\tau}Qe^\tau \in F^1\text{Coder}(\mathcal{C}(A))$. Then we get that $[R, R] = 0$, and thus (A, R) is a \mathcal{P}_∞ -algebra.

Moreover, by definition e^τ gives a \mathcal{P}_∞ -isomorphism $(A, R) \rightarrow (A, Q)$. Finally, it is straightforward to check that indeed $r_i = q_i$ for $i < n$, and that $r_n = q_n - [q_2, \tau] = 0$, which concludes the proof. \square

From the above Proposition 3.9 we can deduce our first main result.

Theorem 3.10. *A minimal \mathcal{P}_∞ -algebra (A, Q) is formal if and only if its Kaledin class is K_A is zero.*

Proof. One direction is straightforward: if A is formal, then there exists an isomorphisms

$$(A, q_2) \longrightarrow (A, Q)$$

and by Proposition 3.9 we deduce that K_A is zero, since all its truncations are zero. For the other direction, suppose that $K_A = 0$. Let i be the smallest integer $i \geq 2$ such that $q_i \neq 0$. Then by Proposition 3.9 we know that there is an isomorphism

$$f_i = e^{\tau_i} : (A, R_i) \rightarrow (A, Q)$$

where $r_2 = \dots = r_i = 0$, and $r_1 = q_1$. Repeating the procedure, we can get elements τ_j for every $j \geq i$. Putting

$$f := \dots f_{i+1} f_i = \dots e^{\tau_{i+1}} e^{\tau_i}$$

we obtain that f gives the desired isomorphism

$$(A, q_2) \longrightarrow (A, Q).$$

□

4. OPERADIC EULER CLASSES

The goal of this section is to present an alternative approach to formality. In particular we will prove a reformulation of the formality criteria for a minimal \mathcal{P}_∞ -algebra A of Theorem 3.10, expressed in terms of its operadic cohomology spectral sequence.

4.1. Euler derivations. In this section we expand Remark 3.8. More specifically, we introduce the important notion of *operadic Euler class*, generalizing the Euler derivation of [Ma, Section 5].

Definition 4.1. The *operadic Euler derivation* e_f of a \mathcal{P}_∞ -morphism $f : (A, Q) \rightarrow (B, R)$ is the map of graded \mathbb{K} -modules $A \rightarrow B$ defined by

$$e_f(a) = (\bar{a} + 1)f_0(a),$$

where $a \in A$ is homogeneous, and f_0 is the induced map of complexes

$$f_0 : A \rightarrow B.$$

Notice that in general e_f is not compatible with the differentials of A and B . However, if both A and B are minimal \mathcal{P}_∞ -algebras, then we can consider e_f as a map of complexes.

Definition 4.2. The Euler derivation e_A of a \mathcal{P}_∞ -algebra (A, Q) is the Euler derivation of the identity of A .

Suppose now that A is a minimal \mathcal{P}_∞ -algebra. The map e_A is a \mathbb{K} -linear map $A \rightarrow A$, and as such can be regarded as a weight 0 element (which we still denote by e_A) in $\text{Coder}(\mathcal{C}(A))$. For our purposes, the main property of e_A is given by the following important lemma.

Lemma 4.3. *Let $\beta \in \text{Coder}(\mathcal{C}(A))^{(p)}$ be a weight p coderivation. Then we have*

$$[\beta, e_A] = (p - \bar{\beta})\beta,$$

where as always $\bar{\beta}$ is the cohomological degree of β .

Proof. This is a straightforward verification, using the explicit definition of the bracket on the complex $\text{Hom}(\mathcal{C}(A), A) \simeq \text{Coder}(\mathcal{C}(A))$, as given for example in [LV, Proposition 6.4.5]. □

As an immediate consequence of Lemma 4.3, we find that in $\text{Coder}(\mathcal{C}(A))$ we have

$$[q, e_A] = [q_1 + q_2 + \dots + e_A] = \sum_{i \geq 1} [q_i, e_A] = \sum_{i \geq 1} (i - 1)q_i = \tilde{Q}.$$

In particular, the cohomology class of \tilde{Q} in $\text{Coder}(\mathcal{C}(A))$ is always zero. However, e_A lives in weight 0, and hence doesn't tell us much about K_A , which is a cohomology class of $F^1 \text{Coder}(\mathcal{C}(A))$. Nevertheless, the Euler class e_A can be used to formulate an alternative formality criterion based on spectral sequences.

4.2. Operadic cohomology spectral sequences. Let $f : A \rightarrow B$ be a \mathcal{P}_∞ -map. We saw in Section 2.1 that the complex $\text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f)$ carries a natural filtration, induced by the weight grading. In this section we study the associated spectral sequence.

Definition 4.4. The operadic cohomology spectral sequence of a \mathcal{P}_∞ -map $f : A \rightarrow B$ is the spectral sequence $(E(A, B; f)_r^{p,q}, d_r)$ associated to the filtered complex $\text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f)$.

Our first goal is to show that these spectral sequences are functorial, in the sense of the following result.

Lemma 4.5. *Let $f : (A, Q) \rightarrow (B, R)$ and $g : (B, R) \rightarrow (C, S)$ be \mathcal{P}_∞ -morphisms. Then the composition maps produces morphisms*

$$\begin{aligned} g_* : \text{Coder}(\mathcal{C}(A), \mathcal{C}(B); f) &\rightarrow \text{Coder}(\mathcal{C}(A), \mathcal{C}(C); gf) \\ f^* : \text{Coder}(\mathcal{C}(B), \mathcal{C}(C); g) &\rightarrow \text{Coder}(\mathcal{C}(A), \mathcal{C}(C); gf) \end{aligned}$$

of filtered complexes.

Proof. This is a straightforward check. In fact, it suffice to verify that both g_* and f^* are compatible with the filtrations, and that they commute with differentials. \square

The next lemma asserts that the spectral sequence $E(A, B; f)_r^{p,q}$ has a nice homotopy invariance property with respect to weak equivalences.

Proposition 4.6. *Let again $f : (A, Q) \rightarrow (B, R)$ and $g : (B, R) \rightarrow (C, S)$ be \mathcal{P}_∞ -morphisms, which by Lemma 4.5 induce morphisms*

$$E(A, B; f)_r^{p,q} \xrightarrow{g_*} E(A, C; gf)_r^{p,q} \xleftarrow{f^*} E(B, C; g)_r^{p,q}$$

of spectral sequences. If g is a weak equivalence, then g^* is an isomorphism for every $r \geq 1$. Similarly, if f is a weak equivalence, then f_* is an isomorphism for every $r \geq 1$.

Proof. It is enough to prove the claim for $r = 1$. By definition, we have

$$E(A, B; f)_0^{p,q} = \text{Hom}^{p+q}(\mathcal{C}(A)^{(p)}, B),$$

where the right hand side denotes the degree $p + q$ maps between $\mathcal{C}(A)^{(p)}$ and B . Moreover, the differential d_0 on the 0-th page of the spectral sequence is precisely the one induced by the differentials on $\mathcal{C}(A)^{(p)}$ and B . By the Künneth formula, we deduce that the first page has the form

$$E(A, B; f)_1^{p,q} = \text{Hom}^{p+q}(\mathcal{C}(H(A))^{(p)}, H(B)),$$

and thus similarly we have

$$E(A, C; gf)_1^{p,q} = \text{Hom}^{p+q}(\mathcal{C}(H(A))^{(p)}, H(C)),$$

It is now clear that if g is a weak equivalence, then g_* provides an isomorphisms between the two spectral sequences. The case of f being a weak equivalence is also completely analogous. \square

4.3. Operadic Euler classes. Let again $f : A \rightarrow B$ be a \mathcal{P}_∞ -morphism. Notice that the Euler derivation of Section 4.1 can be regarded as living in the first page of the spectral sequence $E(A, B; f)$. More specifically, we have

$$e_f \in E(A, B; f)_1^{0,0} = \text{Hom}^0(H(A), H(B)).$$

The following Lemma is a generalization of a similar result of Manetti, namely [Ma, Lemma 5.8].

Lemma 4.7. *The Euler derivation of any \mathcal{P}_∞ -map $f : (A, Q) \rightarrow (B, R)$ satisfies $d_1(e_f) = 0$.*

Proof. Using Proposition 4.6, we can safely assume that both A and B are in fact minimal \mathcal{P}_∞ -algebras.

If $\phi : A \rightarrow B$ is any k -linear map, then we can look at ϕ as an element in

$$E(A, B; f)_1^{0,0} = \text{Hom}^0(A, B).$$

In other terms, ϕ induces a well-defined coderivation $\hat{\phi} : \mathcal{C}(A) \rightarrow \mathcal{C}(B)$. By definition, the value of $d_1(\phi)$ is the weight 1 component of the coderivation given by

$$R \circ \hat{\phi} - \hat{\phi} \circ Q,$$

where R and Q are the (co)differentials on $\mathcal{C}(B)$ and $\mathcal{C}(A)$ respectively. Recall that the weight 1 component of a coderivation $\mathcal{C}(A) \rightarrow \mathcal{C}(B)$ is given by a linear map $\mathcal{C}(A)^{(1)} \rightarrow B$, or equivalently by a map of \mathbb{S} -modules

$$\mathcal{C}^{(1)} \rightarrow \text{End}_B^A.$$

Since $\mathcal{P} = \mathcal{P}(E, R)$, we have by definition that $\mathcal{C} = \mathcal{P}^i = \mathcal{C}(sE, s^2R)$, and thus $\mathcal{C}^{(1)}$ can be identified with E .

Let $c \in E$ be a generating (co)operation of \mathcal{C} (which is of co-arity 2 and cohomological degree 0), and take $a \in A$.

Then in the special case of ϕ being the Euler derivation e_f , the weight 1 component of $\hat{e}_f \circ Q$ is simply given by

$$\begin{array}{ccc} E \otimes A^{\otimes 2} & \longrightarrow & B \\ c \otimes a_1 \otimes a_2 & \longmapsto & e_f(q_1(c \otimes a_1 \otimes a_2)), \end{array}$$

where we are identifying q_1 with the corresponding map $E \otimes A^{\otimes 2} \rightarrow A$. Moreover, we have

$$e_f(q_1(c \otimes a_1 \otimes a_2)) = (\bar{a}_1 + \bar{a}_2 + 2)f_0(q_1(c \otimes a_1 \otimes a_2)),$$

where we used that q_1 has cohomological degree 1.

On the other hand, \hat{e}_f is the coderivation extending e_f , and thus we have that the weight 1 component of $R \circ \hat{e}_f$ can also be computed explicitly as

$$\begin{array}{ccc} E \otimes A^{\otimes 2} & \longrightarrow & B \\ c \otimes a_1 \otimes a_2 & \longmapsto & r_1(c \otimes e_f(a_1) \otimes f_0(a_2)) + r_1(c \otimes f_0(a_1) \otimes e_f(a_2)). \end{array}$$

Now a straightforward check shows that in this case $R \circ \hat{\phi}$ and $\hat{\phi} \circ Q$ have the same weight 1 component, and this concludes the proof. \square

Remark 4.8. In the special case of f being the identity of a minimal \mathcal{P}_∞ -algebra (A, Q) , the fact that $d_1(e_A) = 0$ was already implicit in Lemma 4.3.

It follows in particular that e_f defines an element in $E(A, B; f)_2^{0,0}$.

Definition 4.9. The *Euler class* of a \mathcal{P}_∞ -morphism $f : (A, Q) \rightarrow (B, R)$ is the class of the Euler derivation in $E(A, B; f)_2^{0,0}$. The Euler class e_A of a single \mathcal{P}_∞ -algebra (A, Q) is simply the Euler class of the identity.

Notice that, by definition, the Euler class of a \mathcal{P}_∞ -algebra is invariant under quasi-isomorphisms.

5. AN ALTERNATIVE FORMALITY CRITERION FOR \mathcal{P}_∞ -ALGEBRAS

Recall that Theorem 3.10 relates formality of a minimal \mathcal{P}_∞ -algebra (A, Q) with the vanishing of a certain cohomology class $K_A \in H^1(F^1\text{Coder}(\mathcal{C}(A)), [Q, -])$. Also, recall from Section 2.1 that the complex $\text{Coder}(\mathcal{C}(A))$ carries a natural filtration, for which we considered the associated cohomology spectral sequence $E(A)_r^{p,q}$. As usual, the differential in the r -th page of the spectral sequence $E(A)_r^{p,q}$ will be denoted by d_r .

Theorem 5.1. *Let (A, Q) be a minimal \mathcal{P}_∞ -algebra. Then the following are equivalent:*

- (1) *The truncated Kaledin class $K_A^{<n}$ is zero;*
- (2) *$d_r(e_A) = 0$ for $r = 2, \dots, n$.*

Remark 5.2. Item (2) in Theorem 5.1 contains a slight abuse of notation. In fact, its content is first of all that $d_2(e_A) = 0$, and thus e_A defines an element in the third page E_3 of the operadic cohomology spectral sequence. If we keep denoting it with e_A , we are asking that $d_3(e_A)$ also vanishes, so that e_A in turn defines an element in the fourth page E_4 , and so on and so forth.

Proof. Suppose that $K_A^{<n} = 0$. By Proposition 3.9 we can assume that $q_2 = q_3 = \dots = q_n = 0$.

Let us write $Q = q_1 + q'$, where $q' \in F^{n+1}\text{Coder}(\mathcal{C}(A))$. Suppose $1 < r \leq n$, and take $x \in \text{Coder}(\mathcal{C}(A))^{(p)}$ such that $[Q, x] \in F^{p+r}\text{Coder}(\mathcal{C}(A))$. Since $r > 1$, we get $[q_1, x] = 0$ by weight reasons, and hence

$$[Q, x] = [q', x] \in F^{p+n+1}\text{Coder}(\mathcal{C}(A)) \subset F^{p+r+1}\text{Coder}(\mathcal{C}(A)).$$

This yields $d_r = 0$, and in particular proves that (1) \Rightarrow (2).

Let us prove that (2) \Rightarrow (1). Recall that the differential of the r -th page of the spectral sequence

$$d_r : \frac{Z_r^0}{Z_{r-1}^1 + dZ_{r-1}^1} \longrightarrow \frac{Z_r^r}{Z_{r-1}^{r+1} + dZ_{r-1}^1}$$

is induced by the differential $d_Q = [Q, -]$ on $\text{Coder}(\mathcal{C}(A))$. The fact that $d_2(e_A) = 0$ means that

$$d_Q(e_A) = [Q, e_A] \equiv 0 \pmod{Z_1^2 + dZ_1^1},$$

or equivalently that the weight 2 component of $[Q, e_A]$ belongs to the image of $[q_1, -]$. In other words, there exists an element $t_1^1 \in \text{Coder}(\mathcal{C}(A))^{(1)}$ of cohomological degree 0 such that

$$[q_1, t_1^1] = [q_2, e_A] = q_2.$$

Similarly, $d_3(e_A) = 0$ implies that we can find $t^2 = t_1^2 + t_2^2 \in F^1\text{Coder}(\mathcal{C}(A))/F^3$ such that

$$[q_1, t_1^2] = 0, \quad [q_1, t_2^2] + [q_2, t_1^2] = [q_3, e_A] = 2q_3.$$

The same argument shows that for $1 \leq i < n$, we can find

$$t^i = t_1^i + t_2^i + \dots + t_i^i \in F^1\text{Coder}(\mathcal{C}(A))/F^{i+1}$$

such that

$$\sum_{j=1}^{p-1} [q_j, t_{p-j}^i] = \begin{cases} 0 & \text{if } 1 < p \leq i \\ iq_{i+1} & \text{if } p = i + 1. \end{cases}$$

In particular, setting

$$r_i := \sum_{j=1}^i t_i^j, \quad R := r_1 + r_2 + \dots + r_{n-1},$$

we get that $[Q, R] \equiv \tilde{Q} \pmod{F^{n+1}}$, showing that $K_A^{<n} = 0$ and thus finishing the proof. \square

We summarize our results in the following corollary of Theorem 5.1.

Corollary 5.3. *Let (A, Q) be a \mathcal{P}_∞ -algebra. Then the following are equivalent:*

- (1) A is formal;
- (2) the operadic Kaledin class K_A vanishes;
- (3) the spectral sequence $E(A)_r^{p,q}$ degenerates at E_2 ;
- (4) $d_r(e_A) = 0$ for every r .

Proof. The fact that (1) is equivalent to (2) is the content of Theorem 3.10. Moreover, an immediate consequence of Theorem 5.1 is that condition (2) is equivalent to condition (4). Also, the fact that (3) implies (4) is obvious. It is thus enough to show that (1) implies (3).

We can suppose without loss of generality that (A, Q) is minimal. If (A, Q) is formal, then we can suppose that the only non-zero component of Q is q_1 . In order to prove (3), we use the following lemma.

Lemma 5.4. *Let k be a positive integer, and let (A, Q) be a minimal \mathcal{P}_∞ -algebra, such that Q is concentrated in weight k . Then the spectral sequence $E(A)_r^{p,q}$ degenerates at E_{k+1} .*

Proof. This is very similar to the proof of [Ma, Lemma 6.1], but we nonetheless provide a proof here. The differential on $\text{Coder}(\mathcal{C}(A))$ has the form $d = [q_k, -]$. Consider an element $x \in \text{Coder}(\mathcal{C}(A))$, together with its weight decomposition $x = x_1 + x_2 \dots$. If x is such that $dx \in F^{p+k+1}\text{Coder}(\mathcal{C}(A))$, then $dx_i = 0$ for every $i \leq p$. In other words, if $r > k$, we have

$$dZ_r^p \subset dZ_{r-1}^{p+1}$$

which immediately implies that the differential d_r is zero. □

Using Lemma 5.4, we immediately see that if A is formal the spectral sequence $E(A)_r^{p,q}$ degenerates at E_2 , which concludes the proof. □

6. EXAMPLES

In this section, we look more closely at the special cases where the Koszul operad \mathcal{P} is the operad of Lie algebras or the operad of associative algebras. In particular, we show how our results are in fact generalizations of the main theorems of [Ka] and [Ma].

6.1. Associative algebras. Let $\mathcal{P} = \text{Ass}$ be the operad encoding associative algebras. In this case \mathcal{P}_∞ -algebras are called A_∞ -algebras. In the paper [Ka], Kaledin addresses the question of finding obstructions to formality of a given associative algebra A . Let us briefly recall, mainly following the exposition presented in [Lu], the content of Kaledin's paper.

Remark 6.1. As already mentioned in the introduction, it is important to remark that the arguments of Kaledin work for algebras over a possibly non-trivial base \mathbb{K} -algebra R . Geometrically, this corresponds to investigating formality for families of A_∞ -algebras. We did not attempt to reach that level of generality in the present text, but we certainly feel that our methods can be smoothly adapted to the case of families of \mathcal{P} -algebras, where \mathcal{P} is any sufficiently nice Koszul operad. We plan to come back to this question in a future work.

Let (A, Q) be a minimal A_∞ -algebra, and let h be a formal variable. With $Q = q_1 + q_2 + \dots$ we denote as usual the weight decomposition of the coderivation Q . Kaledin observes that one can construct another minimal A_∞ -algebra $(A[h], Q')$, where the weight components of Q' are given by

$$Q' = q_1 + hq_2 + h^2q_3 \dots$$

It can be checked that $\tilde{A} := (A[h], Q')$ is indeed a $\mathbb{K}[h]$ -linear A_∞ -algebra. The algebra \tilde{A} is an algebraic incarnation of the deformation of A to the normal cone, and it is easy to verify that A

is formal if and only if \tilde{A} is. Notice also that the quotient \tilde{A}/h^{n+1} is automatically a minimal $\mathbb{K}[h]/h^{n+1}$ -linear A_∞ -algebra.

To \tilde{A}/h^{n+1} , Kaledin associates a cohomology class as follows. The A_∞ -structure is encoded by a coderivation

$$Q'_{\tilde{A}/h^{n+1}} = q_1 + hq_2 + \dots h^n q_{n+1}$$

which squares to zero. In particular, if we denote by δ_h the formal derivative with respect to h , then $\delta_h(Q'_{\tilde{A}/h^{n+1}})$ defines a cocycle in the $\mathbb{K}[h]/h^n$ -linear Hochschild complex of \tilde{A}/h^n . The associated degree 1 element $[\delta_h(Q'_{\tilde{A}/h^{n+1}})]$ in the Hochschild cohomology is the Kaledin class of \tilde{A}/h^{n+1} .

One of the main results in [Ka] and in [Lu] states that the A_∞ -algebra A is formal if and only if all the cohomology classes defined by $\delta_h(Q'_{\tilde{A}/h^{n+1}})$ are zero for all $n \geq 1$. The following Proposition shows that the more general Definition 3.7 specializes to the notion of Kaledin-Lunts in the case $\mathcal{P} = Ass$.

Proposition 6.2. *Let A be a minimal A_∞ -algebra. Then the class $[\delta_h(Q'_{\tilde{A}/h^{n+1}})]$ is zero if and only if $K_A^{\leq n} = 0$.*

Proof. The Hochschild cohomology class defined by $\delta_h(Q'_{\tilde{A}/h^{n+1}})$ is zero if and only if there exists a degree zero element t in the Hochschild complex of \tilde{A}/h^n such that $[Q'_{\tilde{A}/h^n}, t] = \delta_h(Q'_{\tilde{A}/h^{n+1}})$. Let us write

$$t = t_1 + ht_2 + \dots h^{n-1}t_n$$

for the decomposition of t in terms of the various powers of h . Notice that all the t_i 's are degree zero elements in the Hochschild complex of A . Let now s be an integer such that $0 \leq s < n$; by looking at the coefficients of h^s , we find that the condition $[Q'_{\tilde{A}/h^n}, t] = \delta_h(Q'_{\tilde{A}/h^{n+1}})$ is equivalent to the equations

$$\sum_{j=0}^s [q_{j+1}, t_{s-j}] = (s+1)q_{s+2}$$

for every s . Moreover, by weight considerations we can assume that the weight of t_j is j .

But it is now immediate to verify that the conditions satisfied by the elements t_1, \dots, t_n are equivalent to the more compact equation

$$[Q^{\leq n}, t'] = \tilde{Q}^{\leq n},$$

where t' is the element of $\text{Coder}(\mathcal{C}(A))$ whose weight decomposition is given by

$$t' = t_1 + t_1 + \dots + t_n,$$

and $Q^{\leq n}$ and $\tilde{Q}^{\leq n}$ are as in Section 3.2. □

In this sense, Theorem 3.10 specializes to the formality criterion described by Kaledin and Lunts if the base ring is trivial. We remark however that the formality criteria of items (3) and (4) of Corollary 5.3 were not discussed in the paper [Ka] and [Lu].

6.2. Lie algebras. Let now $\mathcal{P} = Lie$ be the operad of dg Lie algebras. In this case, algebras over \mathcal{P}_∞ are referred to as L_∞ -algebras. In his paper [Ma], Manetti studies formality criteria for dg Lie algebras (and L_∞ -algebras). As we have mentioned, his results have a somewhat different flavour from those of Kaledin.

In fact, Manetti starts with studying the Lie-incarnation of the operadic cohomology spectral sequence of Section 4.2, called the Chevalley-Eilenberg spectral sequence. He defines an appropriate

Euler class in the Chevalley-Eilenberg spectral sequence, and goes to show that it controls formality of L_∞ -algebras. It is straightforward to check that Definition 4.4 of operadic cohomology spectral sequences and Definition 4.1 of operadic Euler derivations give back the analogous notions described by Manetti.

More specifically, one of the main results of [Ma] is the equivalence of items (1)-(3)-(4) of Corollary 5.3, in the particular case where $\mathcal{P} = Lie$. Similarly to what happened in the case of associative algebras, we find that Theorem 6.3 of [Ma] is a consequence of the more general Corollary 5.3.

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