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A unified notion of regularity in one hypercomplex variable

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ABSTRACT

We define a very general notion of regularity for functions taking values in an alternative real $*$-algebra. Over Clifford numbers, this notion subsumes the well-established notions of monogenic function and slice-monogenic function. Over quaternions, in addition to subsuming the notions of Fueter-regular function and of slice-regular function, it gives rise to an entirely new theory, which we develop in some detail.

1. Introduction

The aim of this work is two-fold. On the one hand, it aims at defining a unified notion of regularity for functions taking values in an alternative real $*$-algebra A, general enough to subsume some of the best-known notions of regularity in one hypercomplex variable. On the other hand, it aims at finding, within the scope of this new notion, some interesting new function theory.

The first aim is addressed in Section 2, which sets up appropriate domains Ω within the $*$-algebra A, and in Section 3, which defines *T-regular functions* $f : \Omega \to A$. When A is the real Clifford algebra $C\ell(0, n)$, T-regularity subsumes both the notions of monogenic function (see, e.g., [1,2,16]) and of slice-monogenic function ([3,4]). When A is the real algebra of quaternions $\mathbb{H}$, T-regularity not only subsumes the theories of Fueter (see [7,8,20]), of Gentili-Struppa (see [10,11] and [9]) and of Moisil-Teodorescu (see [18]), but also includes the brand new theory of $(1, 3)$-*regular* quaternionic functions. When A is the real algebra of octonions $\mathbb{O}$, the concept of T-regular function subsumes the notions of octonionic monogenic function (based on [5] and considered in the recent work [17]) and of slice-regular function (see [12]).

Section 4 studies in some detail the new theory of $(1, 3)$-regular quaternionic functions, which turns out to be very interesting and motivates a forthcoming work on general T-regular functions.

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2. Hypercomplex subspaces

Let $(A, +, \cdot, ^c)$ be a real $*$-algebra of finite dimension, i.e., a finite-dimensional $\mathbb{R}$-vector space endowed with an $\mathbb{R}$-bilinear multiplicative operation and with an involutive $\mathbb{R}$-linear antiautomorphism $x \mapsto x^c$ (called $*$-involution). Assume A to be alternative, i.e., assume $x(xy) = x^2y$, $(xy)y = xy^2$ for all $x, y \in A$, which is automatically true if A is associative. Set $t(x) := x + x^c$ and $n(x) := xx^c$ for all $x \in A$. The *quadratic cone* was defined in [13] by means of the equality $Q_A := \mathbb{R} \cup \{x \in A \setminus \mathbb{R} : t(x) \in \mathbb{R}, n(x) \in \mathbb{R}, 4n(x) > t(x)^2\}$ and has the property

$$Q_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J \,,$$

where $\mathbb{S}_A = \{x \in A : t(x) = 0, n(x) = 1\}$ and where $\mathbb{C}_J := \mathbb{R} + J\mathbb{R}$ for all $J \in \mathbb{S}_A$ is $*$-isomorphic to $\mathbb{C}$. In particular, if $x = \alpha + \beta J \in Q_A$ (with $\alpha, \beta \in \mathbb{R}$, $J \in \mathbb{S}_A$) then: the conjugate $x^c = \alpha - \beta J$ belongs to Q_A; $t(x) = 2\alpha \in \mathbb{R}$; $n(x) = n(x^c) = \alpha^2 + \beta^2 \in \mathbb{R}$; provided $x \neq 0$, the element x has a multiplicative inverse, namely $x^{-1} = n(x)^{-1}x^c = x^c n(x)^{-1}$, which still belongs to Q_A. We assume $\mathbb{S}_A \neq \emptyset$, whence $\mathbb{R} \subsetneq Q_A$. Some examples follow, see [6,16] for full details.

Examples 2.1. The division algebras $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ of complex numbers, $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ of real quaternions and $\mathbb{O} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R} + l\mathbb{R} + li\mathbb{R} + lj\mathbb{R} + lk\mathbb{R}$ of real octonions are alternative real $*$-algebras of dimensions $2, 4, 8$, respectively. The equalities $Q_{\mathbb{C}} = \mathbb{C}$, $Q_{\mathbb{H}} = \mathbb{H}$, $Q_{\mathbb{O}} = \mathbb{O}$ hold, while $\mathbb{S}_{\mathbb{C}}, \mathbb{S}_{\mathbb{H}}, \mathbb{S}_{\mathbb{O}}$ are, respectively, the $0, 2, 6$-dimensional unit spheres in the respective subspaces $t(x) = 0$.

For any $n \in \mathbb{N}^*$, let $\mathscr{P}(n)$ denote the power set of $\{1, \ldots, n\}$.

Examples 2.2. For each $n \in \mathbb{N}^*$, the Clifford algebra $C\ell(0, n)$, whose elements have the form $\sum_{K \in \mathscr{P}(n)} x_K e_K$ with $x_K \in \mathbb{R}$ for all $K \in \mathscr{P}(n)$, is an associative real $*$-algebra of dimension 2^n. The sets $\mathbb{S}_{C\ell(0,n)}$ and $Q_{C\ell(0,n)}$ are nested proper real algebraic subsets of $C\ell(0, n)$.

We now focus on specific subsets of the quadratic cone Q_A.

Definition 2.3. Let M be a real vector subspace of A. An ordered real vector basis $(v_0, v_1, \ldots, v_m)$ of M is called a *hypercomplex basis* of M if: $m \geq 1$; $v_0 = 1$; $v_s \in \mathbb{S}_A$ and $v_s v_t = -v_t v_s$ for all distinct $s, t \in \{1, \ldots, m\}$. The subspace M is called a *hypercomplex subspace* of A if $\mathbb{R} \subsetneq M \subseteq Q_A$.

A basis $(v_0, v_1, \ldots, v_m)$ is a hypercomplex basis if, and only if, $t(v_s) = 0, n(v_s) = 1$ and $t(v_s v_t^c) = 0$ for all distinct $s, t \in \{1, \ldots, m\}$. We point out that, for any $\ell \in \{1, \ldots, m\}$, the shortened ordered set $(v_0, v_1, \ldots, v_\ell)$ is a hypercomplex basis of its span. The concept of hypercomplex subspace was defined in [19, §3], which also proved the first statement in the next theorem (cf. [14, Lemma 1.4]).

Theorem 2.4. *Every hypercomplex subspace of A admits a hypercomplex basis.*

Conversely, if a real vector subspace M of A has a hypercomplex basis $\mathcal{B} = (v_0, v_1, \ldots, v_m)$, then M is a hypercomplex subspace of A. Moreover, if we complete $\mathcal{B}$ to a real vector basis $\mathcal{B}' = (v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_d)$ of A and if we endow A with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ associated to $\mathcal{B}'$, then $t(xy^c) = t(yx^c) = 2\langle x, y \rangle$ and $n(x) = n(x^c) = \|x\|^2$ for all $x, y \in M$.

Proof. The first statement was proven in [19, §3]. Let $(v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_d)$ be a real vector basis of A. If $x = \sum_{s=0}^d x_s v_s$ and $y = \sum_{t=0}^d y_t v_t$, then

$$t(xy^c) = \sum_{s=0}^d x_s y_s t(n(v_s)) + \sum_{s=1}^d (x_0 y_s + y_0 x_s) t(v_s) + \sum_{1 \leq s < t \leq d} (x_s y_t + x_t y_s) t(v_s v_t^c) \,,$$

$$n(x) = \sum_{s=0}^d x_s^2 n(v_s) + \sum_{s=1}^d x_0 x_s t(v_s) + \sum_{1 \leq s < t \leq d} x_s x_t t(v_s v_t^c) \,.$$

If $(v_0, v_1, \ldots, v_m)$ is a hypercomplex basis of M, then $n(v_s) = 1, t(n(v_s)) = 2$ and $t(v_s) = 0 = t(v_s v_t^c)$ for all distinct $s, t \in \{1, \ldots, m\}$. If $x, y \in M$ (whence $x_s = 0 = y_t$ for all $s, t \in \{m+1, \ldots, d\}$), we find that $t(xy^c) = 2\sum_{s=0}^m x_s y_s = 2\langle x, y \rangle$ and that $n(x) = \sum_{s=0}^m x_s^2 = \|x\|^2$, i.e., the last statement in the theorem. We are left with proving the second statement: namely, that $M = \mathrm{Span}(v_0, v_1, \ldots, v_m)$ is a hypercomplex subspace of A. Since $v_0 = 1$, $v_1 \notin \mathbb{R}$, the proper inclusion $\mathbb{R} \subsetneq M$ immediately follows. The inclusion $M \subseteq Q_A$ can be proven as follows. If $x \in M \setminus \mathbb{R}$, then $4n(x) = 4\sum_{s=0}^m x_s^2 > 4x_0^2 = (2x_0)^2 = (x + x^c)^2 = t(x)^2$, whence $x \in Q_A$. □

Example 2.5. The subspace of paravectors $\mathbb{R}^{n+1}$ is a hypercomplex subspace of the Clifford algebra $C\ell(0,n)$, having $\mathcal{B} = (e_\emptyset, e_1, \dots, e_n)$ as a hypercomplex basis. If we complete $\mathcal{B}$ to the standard basis $\mathcal{B}' = (e_K)_{K\in\mathscr{P}(n)}$ of $C\ell(0,n)$ and consider the standard scalar product and norm on $C\ell(0,n)$, then the equalities appearing in Theorem 2.4 are well-known properties of paravectors.

Examples 2.6 *([15, Example 1.15]).* For every $h \in \{1,\dots,n\}$ with $h \equiv 1 \bmod 4$,

$$V_h := \left\{ x_0 + \sum_{1\le s_1<\ldots<s_h\le n} x_{s_1\ldots s_h} e_{s_1\ldots s_h} : x_0, x_{s_1\ldots s_h} \in \mathbb{R} \right\}$$

is a hypercomplex subspace of $C\ell(0,n)$, with $\mathcal{B} = (e_{s_1\ldots s_h})_{1\le s_1<\ldots<s_h\le n}$ as a hypercomplex basis and $\dim V_h = \binom{n}{h} + 1 \ge \left(\frac{n}{h}\right)^h + 1$. If we set $h(n) := 4\lfloor \frac{n+2}{8} \rfloor + 1$ (whence $\frac{n}{2} - 2 < h(n) \le \frac{n}{2} + 2$), then $\dim V_{h(n)}$ grows exponentially with n.

More associative examples can be constructed by means of the next lemma.

Lemma 2.7. *Assume A is associative and M is a real vector subspace of A with a hypercomplex basis $\mathcal{B} = (v_0, v_1, \dots, v_m)$. Set $\widehat{v} := v_1 \cdots v_m$. Then $\widehat{\mathcal{B}} := (v_0, v_1, \dots, v_m, \widehat{v})$ is a hypercomplex basis of $\widehat{M} := \mathrm{Span}(\widehat{\mathcal{B}})$ if, and only if, $m \equiv 2 \bmod 4$. If so, not only M but also $\widehat{M}$ is a hypercomplex subspace of A.*

Proof. Using the hypothesis that $\mathcal{B}$ is a hypercomplex basis, a direct computation shows that $\widehat{v}^c = -\widehat{v}$ for $m \equiv 1, 2 \bmod 4$ and $\widehat{v}^c = \widehat{v}$ for $m \equiv 0, 3 \bmod 4$; that $n(\widehat{v}) = 1$; and that $v_\ell \widehat{v} = (-1)^{m-1} \widehat{v} v_\ell$ for all $\ell \in \{1,\dots,m\}$. Overall, the desired equalities $t(\widehat{v}) = 0, n(\widehat{v}) = 1$ and $t(v_\ell \widehat{v}^c) = 0$ for all $\ell \in \{1,\dots,m\}$ hold true if, and only if, $m \equiv 2 \bmod 4$. If this is the case then, by the first part of the proof of Theorem 2.4, any $x = \sum_{s=0}^n x_s v_s + \widehat{x}\widehat{v} \in \widehat{M}$ (with $x_0, \dots, x_n, \widehat{x} \in \mathbb{R}$) has $n(x) = \sum_{s=0}^n x_s^2 + \widehat{x}^2$. If $x = 0$, then $n(x) = 0$ and $x_0 = \ldots = x_n = \widehat{x} = 0$. We conclude that $v_0, v_1, \dots, v_m, \widehat{v}$ are linearly independent, as desired. □

Example 2.8. Examples of hypercomplex subspaces of the real algebra of quaternions $\mathbb{H} = C\ell(0,2)$ are not only the subspace of paravectors, or *reduced quaternions*, $\mathbb{R}^{2+1}$ with hypercomplex basis $(e_\emptyset, e_1, e_2)$, but also the whole algebra $\mathbb{H}$ with hypercomplex basis $(e_\emptyset, e_1, e_2, e_{12}) = (1, i, j, k)$. Unless otherwise stated, we shall assume $\mathcal{B} = \mathcal{B}' = (1, i, j, k)$ and endow $\mathbb{H}$ with standard scalar product and norm.

Example 2.9. An example of hypercomplex subspace of $C\ell(0,n)$, distinct from those considered in Examples 2.5 and 2.6, is $\mathrm{Span}(e_\emptyset, e_1, e_2, \dots, e_m, e_{12\ldots m})$, for any $m \le n$ with $m \equiv 2 \bmod 4$.

We also have a nonassociative example.

Example 2.10. $\mathbb{O}$ is a hypercomplex subspace of $\mathbb{O}$. We shall set $\mathcal{B} = \mathcal{B}' = (1, i, j, k, l, li, lj, lk)$ and endow $\mathbb{O}$ with its standard scalar product and norm.

3. Regularity in hypercomplex subspaces

In this section, we work on a fixed hypercomplex subspace of our alternative real $*$-algebra A, having a hypercomplex basis $\mathcal{B} = (v_0, v_1, \dots, v_n)$ for some $n \in \mathbb{N}^*$. After completing $\mathcal{B}$ to a real vector basis $\mathcal{B}' = (v_0, v_1, \dots, v_n, v_{n+1}, \dots, v_d)$ of A, we endow A with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ associated to $\mathcal{B}'$. We denote our fixed hypercomplex subspace simply by $\mathbb{R}^{n+1}$, thinking of the paravector subspace of $C\ell(0,n)$, Example 2.5, as a guiding example for this section. We need a few further notations: for $0 \le \ell < m \le n$, we consider the $(m-\ell+1)$-dimensional subspace

$$\mathbb{R}_{\ell,m} := \mathrm{Span}(v_\ell, \dots, v_m)\,.$$

Its unit $(m-\ell)$-sphere is denoted by $\mathbb{S}_{\ell,m}$ and is a subset of $\mathbb{S}_A$ if, and only if, $\ell \ge 1$. For instance: $\mathbb{R}_{0,n} = \mathbb{R}^{n+1}$ and $\mathbb{S}_{1,n} = \{\sum_{t=1}^n x_t v_t \in A : \sum_{t=1}^n x_t^2 = 1\} = \mathbb{S}_A \cap \mathbb{R}^{n+1}$.

Definition 3.1. To each *number of steps* $\tau \in \{0,\dots,n\}$ and each *list of steps* $T = (t_0, \dots, t_\tau) \in \mathbb{N}^{\tau+1}$, with $0 \le t_0 < t_1 < \ldots < t_\tau = n$, we associate the *$T$-fan*

$$\mathbb{R}_{0,t_0} \subsetneq \mathbb{R}_{0,t_1} \subsetneq \ldots \subsetneq \mathbb{R}_{0,t_\tau} = \mathbb{R}^{n+1}\,.$$

The first subspace, $\mathbb{R}_{0,t_0}$, is called the *mirror*. We define the *T-torus* as

$$\mathbb{T} := \mathbb{S}_{t_0+1,t_1} \times \ldots \times \mathbb{S}_{t_{\tau-1}+1,t_\tau}$$

when $\tau \ge 1$ and as $\mathbb{T} := \emptyset$ when $\tau = 0$.

We assume henceforth $\tau \in \{0, \dots, n\}$ and $T = (t_0, \dots, t_\tau) \in \mathbb{N}^{\tau+1}$ (with $0 \le t_0 < t_1 < \dots < t_\tau = n$) are fixed. Necessarily, $n \ge t_0 + \tau$. The mirror $\mathbb{R}_{0,t_0}$ of the T-fan is either the real axis $\mathbb{R}$ or a hypercomplex subspace of A, while all other elements of the T-fan are hypercomplex subspaces of A. Moreover, if $\tau \ge 1$ then, for every $h \in \{1, \dots, \tau\}$, the sphere $\mathbb{S}_{t_{h-1}+1,t_h}$ has dimension $t_h - t_{h-1} - 1$, whence the T-torus $\mathbb{T}$ is a subset of $(\mathbb{S}_A)^\tau$ of dimension $n - t_0 - \tau$. On the other hand, $\mathbb{T} = \emptyset$ if $\tau = 0$.

Example 3.2. For paravectors in $C\ell(0,n)$, Example 2.5, the T-fan is

$$\mathbb{R}^{t_0+1} \subsetneq \mathbb{R}^{t_1+1} \subsetneq \dots \subsetneq \mathbb{R}^{t_\tau+1} = \mathbb{R}^{n+1}.$$

Example 3.3. In $\mathbb{H}$, Example 2.8, the 3-fan is $\mathbb{H}$, the $(2,3)$-fan is $\mathbb{R} + i\mathbb{R} + j\mathbb{R} \subsetneq \mathbb{H}$, the $(1,3)$-fan is $\mathbb{C} \subsetneq \mathbb{H}$, the $(0,3)$-fan is $\mathbb{R} \subsetneq \mathbb{H}$, the $(1,2,3)$-fan is $\mathbb{C} \subsetneq \mathbb{R} + i\mathbb{R} + j\mathbb{R} \subsetneq \mathbb{H}$, the $(0,2,3)$-fan is $\mathbb{R} \subsetneq \mathbb{R} + i\mathbb{R} + j\mathbb{R} \subsetneq \mathbb{H}$, the $(0,1,3)$-fan is $\mathbb{R} \subsetneq \mathbb{C} \subsetneq \mathbb{H}$, and the $(0,1,2,3)$-fan is $\mathbb{R} \subsetneq \mathbb{C} \subsetneq \mathbb{R} + i\mathbb{R} + j\mathbb{R} \subsetneq \mathbb{H}$.

Remark 3.4. Each $x = \sum_{\ell=0}^{n} v_\ell x_\ell \in \mathbb{R}^{n+1}$ can be decomposed as $x = x^0 + x^1 + \dots + x^\tau$, where $x^h := \sum_{\ell=t_{h-1}+1}^{t_h} x_\ell v_\ell \in \mathbb{R}_{t_{h-1}+1,t_h}$ (with $t_{-1} := -1$). This decomposition is orthogonal, whence unique. Moreover, if $\tau \ge 1$ then there exist $\beta = (\beta_1, \dots, \beta_\tau) \in \mathbb{R}^\tau$ and $J = (J_1, \dots, J_\tau) \in \mathbb{T}$ such that

$$x = x^0 + \beta_1 J_1 + \dots + \beta_\tau J_\tau . \tag{1}$$

This equality holds true exactly when, for each $h \in \{1, \dots, \tau\}$: either $x^h \neq 0$, $\beta_h = \pm\|x^h\|$ and $J_h = \frac{x^h}{\beta_h}$; or $x^h = 0$, $\beta_h = 0$ and J_h is any element of $\mathbb{S}_{t_{h-1}+1,t_h}$.

We now wish to define J-monogenicity. This requires a preliminary lemma.

Lemma 3.5. *If $\tau \ge 1$, fix $J = (J_1, \dots, J_\tau) \in \mathbb{T}$ and set*

$$\mathbb{R}_J^{t_0+\tau+1} := \mathrm{Span}(\mathcal{B}_J), \quad \mathcal{B}_J := (v_0, v_1, \dots, v_{t_0}, J_1, \dots, J_\tau).$$

If $\tau = 0$ (whence $t_0 = n \ge 1$), set $J := \emptyset, \mathcal{B}_\emptyset := (v_0, v_1, \dots, v_{t_0}), \mathbb{R}_\emptyset^{t_0+1} := \mathrm{Span}(\mathcal{B}_\emptyset) = \mathbb{R}^{n+1}$. In either case, $\mathcal{B}_J$ is a hypercomplex basis of $\mathbb{R}_J^{t_0+\tau+1}$, which is therefore a hypercomplex subspace of A contained in $\mathbb{R}^{n+1}$. Moreover, if $J' \in \mathbb{T}$, then the equality $\mathbb{R}_J^{t_0+\tau+1} = \mathbb{R}_{J'}^{t_0+\tau+1}$ is equivalent to $J' \in \{\pm J_1\} \times \dots \times \{\pm J_\tau\}$.

Proof. If $\tau = 0$, then $\mathcal{B}_\emptyset := (v_0, v_1, \dots, v_{t_0}) = (v_0, v_1, \dots, v_n)$ is a hypercomplex basis of $\mathbb{R}_\emptyset^{t_0+1} = \mathbb{R}^{n+1}$ by construction.

Now assume $\tau \ge 1$. By construction, $v_0 = 1, v_1, \dots, v_{t_0}, J_1, \dots, J_\tau \in \mathbb{S}_{1,n} = \mathbb{S}_A \cap \mathbb{R}^{n+1}$ and $t(v_\ell v_m^c) = 0$ for all distinct $\ell, m \in \{1, \dots, t_0\}$. Moreover, for all $\ell \in \{1, \dots, t_0\}$ and for all distinct $h, h' \in \{1, \dots, \tau\}$, v_ℓ, J_h belong to the mutually orthogonal subspaces $\mathbb{R}_{0,t_0}, \mathbb{R}_{t_{h-1}+1,t_h}$ and J_h, $J_{h'}$ belong to the mutually orthogonal subspaces $\mathbb{R}_{t_{h-1}+1,t_h}, \mathbb{R}_{t_{h'-1}+1,t_{h'}}$. This proves that $\mathcal{B}_J$ is a basis of $\mathbb{R}_J^{t_0+\tau+1}$ and yields, thanks to Theorem 2.4, the remaining desired equalities $t(v_\ell J_h^c) = \langle v_\ell, J_h \rangle = 0$ and $t(J_h J_{h'}^c) = \langle J_h, J_{h'} \rangle = 0$. The last statement follows immediately from Remark 3.4. □

Definition 3.6. If $\tau \ge 1$, fix $J = (J_1, \dots, J_\tau) \in \mathbb{T}$. If $\tau = 0$, set $J := \emptyset$. For any open subset V of $\mathbb{R}_J^{t_0+\tau+1}$, we define the *$J$-Cauchy-Riemann operator* $\overline{\partial}_J : \mathscr{C}^1(V, A) \to \mathscr{C}^0(V, A)$ as $\overline{\partial}_J := 2\overline{\partial}_{\mathcal{B}_J}$, according to [19, Definition 2]. We also define the operators $\partial_J : \mathscr{C}^1(V, A) \to \mathscr{C}^0(V, A)$ and $\Delta_J : \mathscr{C}^2(V, A) \to \mathscr{C}^0(V, A)$ as $\partial_J := 2\partial_{\mathcal{B}_J}$ and $\Delta_J := \Delta_{\mathcal{B}_J}$, according to [19, page 7]. Explicitly, referring to the decomposition (1) of the variable x, we have

$$\begin{aligned}
\overline{\partial}_J &= \partial_{x_0} + v_1\partial_{x_1} + \dots + v_{t_0}\partial_{x_{t_0}} + J_1\partial_{\beta_1} + \dots + J_\tau\partial_{\beta_\tau}, \\
\partial_J &= \partial_{x_0} - v_1\partial_{x_1} - \dots - v_{t_0}\partial_{x_{t_0}} - J_1\partial_{\beta_1} - \dots - J_\tau\partial_{\beta_\tau}, \\
\Delta_J &= \partial_{x_0}^2 + \partial_{x_1}^2 + \dots + \partial_{x_{t_0}}^2 + \partial_{\beta_1}^2 + \dots + \partial_{\beta_\tau}^2 .
\end{aligned}$$

The kernel of $\overline{\partial}_J$ is denoted by $\mathrm{Mon}_J(V, A)$ and its elements are called *J-monogenic* functions. The elements of the kernel of Δ_J are called *J-harmonic* functions.

Using the formal [19, Definition 2] is necessary to guarantee, for $J, J' \in \mathbb{T}$,

$$\overline{\partial}_J = \overline{\partial}_{J'} \iff \mathbb{R}_J^{t_0+\tau+1} = \mathbb{R}_{J'}^{t_0+\tau+1} . \tag{2}$$

Similar considerations apply to ∂_J, Δ_J. By [19, Proposition 5 (b)], we remark:

Remark 3.7. Fix any open subset V of $\mathbb{R}_J^{t_0+\tau+1}$. The equalities $\overline{\partial}_J \partial_J = \partial_J \overline{\partial}_J = \Delta_J$ hold true on $\mathscr{C}^2(V, A)$. In particular, every $\mathscr{C}^2$ J-monogenic function is J-harmonic, whence real analytic.

We now define the new concept of T-regular function.

Definition 3.8. If either $J \in \mathbb{T}$ or $J = \emptyset$, for every $Y \subseteq \mathbb{R}^{n+1}$, $f : Y \to A$, we call $Y_J := Y \cap \mathbb{R}_J^{t_0+\tau+1}$ the *J-slice* of Y and consider the restriction $f_J := f_{|Y_J}$. We call a nonempty connected open $\Omega \subseteq \mathbb{R}^{n+1}$ a *domain* in $\mathbb{R}^{n+1}$. A function $f : \Omega \to A$ on a domain Ω is termed *T-regular* if the restriction $f_J : \Omega_J \to A$ is J-monogenic for every $J \in \mathbb{T}$, if $\tau \geq 1$ (for $J = \emptyset$, if $\tau = 0$). If, moreover, $f(\mathbb{R}_J^{t_0+\tau+1}) \subseteq \mathbb{R}_J^{t_0+\tau+1}$ for all $J \in \mathbb{T}$, then f is called *T-slice preserving*. The class of T-regular functions $\Omega \to A$ is denoted by $\mathrm{Reg}_T(\Omega, A)$.

Over $C\ell(0, n)$, T-regularity subsumes some of the best-known function theories.

Example 3.9. Within the paravector subspace $\mathbb{R}^{n+1}$ of $C\ell(0, n)$, Example 2.5, fix a domain Ω. For any function $f : \Omega \to C\ell(0, n)$:

- f is n-regular if, and only if, it is in the kernel of the operator $\partial_{x_0} + e_1\partial_{x_1} + \ldots + e_n\partial_{x_n}$; this is the definition of *monogenic* function (see, e.g., [1,2,16]);
- f is $(0, n)$-regular if, and only if, for any $J_1 \in \mathbb{S}_{1,n} = \mathbb{S}_{C\ell(0,n)} \cap \mathbb{R}^{n+1}$, the restriction f_{J_1} to the planar domain $\Omega_{J_1} \subseteq \mathbb{C}_{J_1}$ is a holomorphic map $(\Omega_{J_1}, J_1) \to (C\ell(0, n), J_1)$; this is the same as being *slice-monogenic*, [3] (or *slice-hyperholomorphic*, [4]).

Over the hypercomplex subspace $\mathbb{H}$ of $\mathbb{H}$, we achieve a complete classification of T-regularity. In particular, we show that T-regularity not only subsumes the best-known function theories, but also includes an entirely new function theory.

Example 3.10. Fix a domain Ω in $\mathbb{H}$ and a function $f : \Omega \to \mathbb{H}$. Then:

- f is 3-regular if, and only if, it belongs to the kernel of the left Cauchy-Riemann-Fueter operator $\partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}$; this is the definition of *left Fueter-regular* function (see, [7,8,20]);
- f is $(2, 3)$-regular if, and only if, $(\partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + J_1\partial_{\beta_1}) f(x_0 + ix_1 + jx_2 + \beta_1 J_1) \equiv 0$ for all $J_1 \in \mathbb{S}_{3,3} = \{\pm k\}$; this is the same as being left Fueter-regular by (2);
- f is $(1, 3)$-regular if, and only if,

  $$\overline{\partial}_{J_1} f(x_0 + ix_1 + \beta_1 J_1) := (\partial_{x_0} + i\partial_{x_1} + J_1\partial_{\beta_1}) f(x_0 + ix_1 + \beta_1 J_1) \equiv 0$$

  for all J_1 in the $(1, 3)$-torus $\mathbb{S}_{2,3}$, which is simply the circle $\mathbb{S}^1 := \mathbb{S}_{\mathbb{H}} \cap (j\mathbb{R} + k\mathbb{R})$; this is a brand new theory, studied in the forthcoming Section 4;
- f is $(0, 3)$-regular if, and only if, for any $J_1 \in \mathbb{S}_{1,3} = \mathbb{S}_{\mathbb{H}}$, the restriction f_{J_1} to the planar domain $\Omega_{J_1} \subseteq \mathbb{C}_{J_1}$ is a holomorphic map $(\Omega_{J_1}, J_1) \to (\mathbb{H}, J_1)$; this is the definition of *slice-regular* function, [9] (or *Cullen-regular* in the original articles [10,11]);
- f is $(1, 2, 3)$-regular if, and only if, $(\partial_{x_0} + i\partial_{x_1} + J_1\partial_{\beta_1} + J_2\partial_{\beta_2}) f(x_0 + ix_1 + \beta_1 J_1 + \beta_2 J_2) \equiv 0$ for all $(J_1, J_2) \in \mathbb{S}_{2,2} \times \mathbb{S}_{3,3} = \{\pm j\} \times \{\pm k\}$; this is the same as being left Fueter-regular by (2);
- f is $(0, 1, 3)$-regular if, and only if, $(\partial_{x_0} + J_1\partial_{\beta_1} + J_2\partial_{\beta_2}) f(x_0 + \beta_1 J_1 + \beta_2 J_2) \equiv 0$ for all $(J_1, J_2) \in \mathbb{S}_{1,1} \times \mathbb{S}_{2,3} = \{\pm i\} \times \mathbb{S}^1$; this is the same as $(1, 3)$-regularity by (2);
- f is $(0, 2, 3)$-regular if, and only if, $(\partial_{x_0} + J_1\partial_{\beta_1} + J_2\partial_{\beta_2}) f(x_0 + \beta_1 J_1 + \beta_2 J_2) \equiv 0$ for all $(J_1, J_2) \in \mathbb{S}_{1,2} \times \mathbb{S}_{3,3} = (\mathbb{S}_{\mathbb{H}} \cap (i\mathbb{R} + j\mathbb{R})) \times \{\pm k\}$; this class is the image of the class of $(0, 1, 3)$-regular functions when the standard basis $(1, i, j, k)$ is replaced by $(1, k, -j, i)$;
- f is $(0, 1, 2, 3)$-regular if, and only if, $(\partial_{x_0} + J_1\partial_{\beta_1} + J_2\partial_{\beta_2} + J_3\partial_{\beta_3}) f(x_0 + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3) \equiv 0$ for all $(J_1, J_2, J_3) \in \mathbb{S}_{1,1} \times \mathbb{S}_{2,2} \times \mathbb{S}_{3,3} = \{\pm i\} \times \{\pm j\} \times \{\pm k\}$; this is the same as being left Fueter-regular by (2).

Over the hypercomplex subspace $\mathbb{O}$ of $\mathbb{O}$, the notions of 7-regular and $(0, 7)$-regular function coincide, respectively, with the notion of *octonionic monogenic* function (based on [5], see the recent [17]) and with the notion of *slice-regular* function (see [12]). Additionally, within $\mathbb{H}$, the nonstandard choice of the 2-fan $\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ with $\mathcal{B} = (1, -k, j), \mathcal{B}' = (1, -k, j, i)$, recovers as 2-regular functions the theory of [18], for the reasons explained in [19, page 30].

A function $f \in \mathrm{Reg}_T(\Omega, A)$ needs not be continuous, even when all restrictions f_J are $\mathscr{C}^2$ (whence real analytic by Remark 3.7). An example of this phenomenon can be found in [9, Example 1.1]. To single out subclasses of better-behaved T-regular functions, we need some further definitions.

Definition 3.11. A domain $\Omega \subseteq \mathbb{R}^{n+1}$ is called a *T-slice domain* if it intersects the mirror $\mathbb{R}_{0,t_0}$ and if, for any $J \in \mathbb{T}$, the J-slice Ω_J is connected. For $D \subseteq \mathbb{R}_{0,t_0} \times \mathbb{R}^\tau$, we set

$$\Omega_D := \{x^0 + \beta_1 J_1 + \ldots + \beta_\tau J_\tau \in \mathbb{R}^{n+1} : (x^0, \beta) \in D,\ J \in \mathbb{T}\}$$

if $\tau \geq 1$ (and $\Omega_D := \{x^0 \in \mathbb{R}^{n+1} : (x^0, 0) \in D\}$ if $\tau = 0$, taking into account that $\mathbb{R}^0 = \{0\}$). A subset of $\mathbb{R}^{n+1}$ is termed *T-symmetric* if it equals Ω_D for some $D \subseteq \mathbb{R}_{0,t_0} \times \mathbb{R}^\tau$. The *$T$-symmetric completion* $\widetilde{Y}$ of a set $Y \subseteq \mathbb{R}^{n+1}$ is the smallest T-symmetric subset of $\mathbb{R}^{n+1}$ containing Y. For each point $x \in \mathbb{R}^{n+1}$, we denote by $\mathbb{S}_x$ the T-symmetric completion of the singleton $\{x\}$.

We now define T-stem functions. For any $h \in \{1, \ldots, \tau\}$, consider the reflection

$$\mathbb{R}^\tau \to \mathbb{R}^\tau, \quad \beta = (\beta_1, \ldots, \beta_\tau) \mapsto \overline{\beta}^h := (\beta_1, \ldots, \beta_{h-1}, -\beta_h, \beta_{h+1}, \ldots, \beta_\tau).$$

For the rest of the present section, we assume D to be a subset of $\mathbb{R}_{0,t_0} \times \mathbb{R}^\tau$, invariant under the reflection $(x^0, \beta) \mapsto (x^0, \overline{\beta}^h)$ for every $h \in \{1, \ldots, \tau\}$. Moreover, we let $\{E_K\}_{K \in \mathscr{P}(\tau)}$ denote the canonical real vector basis of $\mathbb{R}^{2^\tau}$: this is to avoid possible confusion with the basis of A in the special case when A is $C\ell(0,n)$.

Definition 3.12. Let $F : D \to A \otimes \mathbb{R}^{2^\tau}$ be a function $F = \sum_{K \in \mathscr{P}(\tau)} E_K F_K$ with components $F_K : D \to A$. We say that F is a *T-stem function* if

$$F_K(x^0, \overline{\beta}^h) = \begin{cases} F_K(x^0, \beta) & \text{if } h \notin K \\ -F_K(x^0, \beta) & \text{if } h \in K \end{cases}$$

for all $K \in \mathscr{P}(\tau)$, for all $(x^0, \beta) \in D$, and for all $h \in \{1, \ldots, \tau\}$.

Now let us define T-functions.

Definition 3.13. To each T-stem function $F = \sum_{K \in \mathscr{P}(\tau)} E_K F_K : D \to A \otimes \mathbb{R}^{2^\tau}$, we associate the *induced* function $f = \mathcal{I}(F) : \Omega_D \to A$, whose value at $x = x^0 + \beta_1 J_1 + \ldots + \beta_\tau J_\tau \in \Omega_D$ is

$$f(x) := F_\emptyset(x^0, \beta) + \sum_{1 \leq p \leq \tau} \sum_{1 \leq k_1 < \ldots < k_p \leq \tau} J_{k_1}(J_{k_2}(\ldots(J_{k_{p-1}}(J_{k_p} F_{k_1 \ldots k_p}(x^0, \beta)))\ldots)).$$

A function induced by a T-stem function is called a *T-function*. We denote the class of T-functions $\Omega_D \to A$ by $\mathcal{S}(\Omega_D, A)$. If Ω_D is a domain in $\mathbb{R}^{n+1}$, then the elements of the intersection $\mathcal{SR}(\Omega_D, A) := \mathcal{S}(\Omega_D, A) \cap \mathrm{Reg}_T(\Omega_D, A)$ are called *strongly T-regular* functions.

Remark 3.14. The map $\mathcal{I}$ from the class of T-stem functions on D to $\mathcal{S}(\Omega_D, A)$ is well-defined, owing to the reflection symmetries of T-stem functions, and bijective.

In the very special case when $\tau = 0$, every subset of $\mathbb{R}^{n+1}$ is T-symmetric, every domain Ω in $\mathbb{R}^{n+1}$ is a T-symmetric T-slice domain and every function $f : \Omega \to A$ is a T-function, induced by a T-stem function $F = F_\emptyset$.

We postpone any further study of the properties of general T-regular functions to a forthcoming paper. In the present work, we focus henceforth on the new quaternionic function theory discovered in Example 3.10.

4. (1, 3)-regular quaternionic functions

4.1. Foundational material

In this section, we work in $\mathbb{H}$ with the $(1,3)$-fan $\mathbb{C} \subsetneq \mathbb{H}$ and the $(1,3)$-torus $\mathbb{S}_{2,3} = \mathbb{S}^1$, see Examples 2.8, 3.3, and 3.10. We decompose the variable $x = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$ as $x = z + \beta J$ with $z \in \mathbb{C}$, $\beta \in \mathbb{R}$, $J \in \mathbb{S}^1$. If $x \in \mathbb{H} \setminus \mathbb{C}$, this is true exactly when

$$z = x_0 + ix_1, \quad \beta = \pm\sqrt{x_2^2 + x_3^2}, \quad J = \pm\frac{jx_2 + kx_3}{\sqrt{x_2^2 + x_3^2}}.$$

On the other hand, every element z of the mirror $\mathbb{C} \subsetneq \mathbb{H}$ may be expressed as $z + 0J$ for all $J \in \mathbb{S}^1$. Over quaternions $x \in \mathbb{H}$, the $*$-involution considered is the *conjugation* $x = x_0 + ix_1 + jx_2 + kx_3 \mapsto \bar{x} = x_0 - ix_1 - jx_2 - kx_3$; the Euclidean norm $\|x\| = \sqrt{x\bar{x}}$ of x is called *modulus* of x and denoted by the symbol $|x|$.

For $J \in \mathbb{S}^1$, the spaces $\mathbb{R}^{2+1}_J := \mathrm{Span}_{\mathbb{R}}(1, i, J)$ are the hyperplanes of $\mathbb{H}$ through $\mathbb{C}$. We have $\mathbb{R}^{2+1}_J \cap \mathbb{R}^{2+1}_K = \mathbb{C}$ when $K \neq \pm J$ and $\mathbb{R}^{2+1}_J = \mathbb{R}^{2+1}_{-J}$. When $J = j$, the hyperplane $\mathbb{R}^{2+1}_j$ is the standard $\mathbb{R}^{2+1}$ used in the theory of *monogenic* functions (also called $C\ell(0,2)$ *left-holomorphic functions* in [16]): namely, quaternion-valued $\mathscr{C}^1$ functions of the variable

$x_0 + ix_1 + jx_2$ belonging to the kernel of the operator $\partial_{x_0} + i\partial_{x_1} + j\partial_{x_2}$. For any $J \in \mathbb{S}^1$, (i, J, iJ) is a Hamiltonian triple (see [6, §8.1]) and $\psi_J : \mathbb{R}^{2+1} \to \mathbb{R}^{2+1}_J$, $\psi_J(x_0 + ix_1 + jx_2) = x_0 + ix_1 + Jx_2$ is the restriction of a $*$-isomorphism $\mathbb{H} \to \mathbb{H}$. Thus, if $V \subseteq \mathbb{R}^{2+1}_J$ is open, then the right $\mathbb{H}$-module $\mathrm{Mon}_J(V, \mathbb{H})$ of J-monogenic functions has the same properties as a class of monogenic functions over $C\ell(0,2)$. In particular, J-monogenicity implies real analyticity, see Remark 4.12, and is preserved under composition with translations in $\mathbb{R}^{2+1}_J$.

Remark 4.1. Given a domain Ω in $\mathbb{H}$, the class $\mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$ of all $(1,3)$-regular functions $\Omega \to \mathbb{H}$ is a right $\mathbb{H}$-module. Moreover, if $f \in \mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$ and $z_0 \in \Omega \cap \mathbb{C}$, then setting $g(x) := f(x + z_0)$ defines a $g \in \mathrm{Reg}_{(1,3)}(\Omega - z_0, \mathbb{H})$. This g is $(1,3)$-slice preserving if, and only if, f is.

We remark that $(1,3)$-symmetry is circular symmetry with respect to the plane $\mathbb{C}$ within $\mathbb{H}$. We fix henceforth $D \subseteq \mathbb{C} \times \mathbb{R}$, invariant under the reflection $(z, \beta) \mapsto (z, -\beta)$. A function $F = F_\emptyset + E_1 F_1 : D \to \mathbb{H} \otimes \mathbb{R}^2$ is a $(1,3)$-*stem function* if $F_\emptyset, F_1$ are, respectively, even and odd in β. The $(1,3)$-function $f = \mathcal{I}(F)$, defined as $f(z + \beta J) := F_\emptyset(z, \beta) + JF_1(z, \beta)$ for $(z, \beta) \in D$, $J \in \mathbb{S}^1$, is affine along each circle $\mathbb{S}_x$.

4.2. Polynomial functions

For all $k \in \mathbb{N}$, let U_k denote the right $\mathbb{H}$-submodule of $\mathrm{Reg}_{(1,3)}(\mathbb{H}, \mathbb{H})$ consisting of those elements f such that $f(x_0 + ix_1 + jx_2 + kx_3)$ is a k-homogeneous polynomial map in the four real variables x_0, x_1, x_2, x_3. We construct a right $\mathbb{H}$-basis $\mathscr{T}_k$ of U_k, as follows.

Definition 4.2. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, we set $|\mathbf{k}| := k_1 + k_2$. If $k_1 < 0$ or $k_2 < 0$, we define $\mathcal{T}_{\mathbf{k}}(x) :\equiv 0$; we define $\mathcal{T}_{(0,0)} :\equiv 1$; if $\mathbf{k} \in \mathbb{N}^2 \setminus \{(0,0)\}$, we define $\mathcal{T}_{\mathbf{k}}$ by the equality

$$|\mathbf{k}|\mathcal{T}_{\mathbf{k}}(x) := k_1 \mathcal{T}_{(k_1-1,k_2)}(x)(x_1 - (-1)^{k_2} i x_0) + k_2 \mathcal{T}_{(k_1,k_2-1)}(x)(x_0 + jx_2 + kx_3)\,.$$

For all $k \in \mathbb{N}$, we define $\mathscr{T}_k := \{\mathcal{T}_{\mathbf{k}}\}_{|\mathbf{k}|=k}$.

Example 4.3. $\mathscr{T}_0 = \{\mathcal{T}_{(0,0)}\} = \{1\}$, while $\mathscr{T}_1, \mathscr{T}_2$ consist of the functions

$$\mathcal{T}_{(1,0)}(x) = x_1 - ix_0\,, \quad \mathcal{T}_{(0,1)}(x) = x_0 + jx_2 + kx_3\,,$$
$$\mathcal{T}_{(2,0)}(x) = (x_1 - ix_0)^2\,, \quad \mathcal{T}_{(0,2)}(x) = (x_0 + jx_2 + kx_3)^2\,,$$
$$2\mathcal{T}_{(1,1)}(x) = (x_0 + jx_2 + kx_3)(x_1 + ix_0) + (x_1 - ix_0)(x_0 + jx_2 + kx_3)\,.$$

None of the elements of $\mathscr{T}_1, \mathscr{T}_2$ is a slice-regular function $\mathbb{H} \to \mathbb{H}$. Moreover, $\mathcal{T}_{(0,1)}, \mathcal{T}_{(0,2)}, \mathcal{T}_{(1,1)} : \mathbb{H} \to \mathbb{H}$ are not Fueter-regular.

Clearly, each $\mathcal{T}_{\mathbf{k}}$ is a $|\mathbf{k}|$-homogeneous polynomial map in the four real variables x_0, x_1, x_2, x_3. To prove that $\mathscr{T}_k$ is, indeed, a basis of U_k, we rely on some well-known properties of monogenic functions, described in [16, §6.1.2].

Remark 4.4. For any $J \in \mathbb{S}^1, k \in \mathbb{N}$, let U_k^J denote the right submodule of $\mathrm{Mon}_J(\mathbb{R}^{2+1}_J, \mathbb{H})$ consisting of those elements P such that $P(x_0 + ix_1 + \beta J)$ is a k-homogeneous polynomial map in the real variables x_0, x_1, β. Then U_0^J is spanned by $\mathcal{P}^J_{(0,0)} :\equiv 1$, while U_1^J is spanned by $\mathcal{P}^J_{(1,0)} := x_1 - ix_0$, also called the Fueter variable ζ_1, and by $\mathcal{P}^J_{(0,1)} := \beta - Jx_0$, also called the Fueter variable $\zeta_{2,J}$. If we further define $\mathcal{P}^J_{\mathbf{k}} :\equiv 0$ for $\mathbf{k} \in \mathbb{Z}^2$ with $k_1 < 0$ or $k_2 < 0$ and, for all remaining $\mathbf{k} \in \mathbb{N}^2$,

$$|\mathbf{k}|\mathcal{P}^J_{\mathbf{k}} := k_1 \mathcal{P}^J_{(k_1-1,k_2)}\zeta_1 + k_2 \mathcal{P}^J_{(k_1,k_2-1)}\zeta_{2,J} = k_1 \zeta_1 \mathcal{P}^J_{(k_1-1,k_2)} + k_2 \zeta_{2,J}\mathcal{P}^J_{(k_1,k_2-1)}\,,$$

then each $\mathcal{P}^J_{\mathbf{k}}$ preserves $\mathbb{R}^{2+1}_J$ and $\{\mathcal{P}^J_{\mathbf{k}}\}_{|\mathbf{k}|=k}$ is a basis of U_k^J. To be more precise, we need some preparation. For every open $V \subseteq \mathbb{R}^{2+1}_J$ and every $\mathbf{h} = (h_0, h_1, h_2) \in \mathbb{N}^3$, we define $|\mathbf{h}| := h_0 + h_1 + h_2$ and the operator $\nabla_J^{\mathbf{h}} : \mathscr{C}^{|\mathbf{h}|}(V, \mathbb{H}) \to \mathscr{C}^0(V, \mathbb{H})$ as

$$\nabla_J^{\mathbf{h}}\phi(x_0 + ix_1 + \beta J) := \partial_0^{h_0}\partial_1^{h_1}\partial_{2,J}^{h_2}\phi(x_0 + ix_1 + \beta J), \quad \partial_0 := \partial_{x_0}, \partial_1 := \partial_{x_1}, \partial_{2,J} := \partial_\beta\,.$$

More precisely, $\left((\nabla_J^{\mathbf{h}}\phi) \circ \psi_J\right)(x_0 + ix_1 + jx_2) := \left(\left(\partial_{x_0}^{h_0}\partial_{x_1}^{h_1}\partial_{x_2}^{h_2}\right)(\phi \circ \psi_J)\right)(x_0 + ix_1 + jx_2)$. Since the operator $\nabla_J^{\mathbf{h}}$ commutes with $\overline{\partial}_J$ on $\mathscr{C}^{|\mathbf{h}|+1}(V, \mathbb{H})$, it maps J-monogenic functions into J-monogenic functions. With this construction in mind, we see that $\{\mathcal{P}^J_{\mathbf{k}}\}_{|\mathbf{k}|=k}$ is a basis of U_k^J because every $P \in U_k^J$ can be expressed as

$$P(x_0 + ix_1 + \beta J) = \sum_{|\mathbf{k}|=k} \mathcal{P}^J_{\mathbf{k}}(x_0 + ix_1 + \beta J)a_{\mathbf{k}}, \quad a_{\mathbf{k}} := \frac{1}{\mathbf{k}!}\nabla_J^{(0,\mathbf{k})}P(0) \tag{3}$$

for $x_0 + ix_1 + \beta J \in \mathbb{R}^{2+1}_J$. Moreover, P preserves $\mathbb{R}^{2+1}_J$ under the following additional assumptions: $a_{\mathbf{k}} \in \mathbb{R}^{2+1}_J$ when $k_1 = 0 = k_2$; $a_{\mathbf{k}} \in \mathbb{C}_J$ when $k_1 = 0 \neq k_2$, $a_{\mathbf{k}} \in \mathbb{C}$ when $k_1 \neq 0 = k_2$, $a_{\mathbf{k}} \in \mathbb{R}$ when $k_1 \neq 0 \neq k_2$.

We also need the next definition and lemma, where Ω is a fixed domain in $\mathbb{H}$, $J \in \mathbb{S}^1$ is fixed and V is a fixed open subset of $\mathbb{R}^{2+1}_J$.

Definition 4.5. For every $\mathbf{h} = (h_0, h_1, h_2) \in \mathbb{N}^3$, we define an operator $\delta^{\mathbf{h}}$ on functions $f : \Omega \to \mathbb{H}$ such that $f_J \in \mathscr{C}^{|\mathbf{h}|}(\Omega_J, \mathbb{H})$ for all $J \in \mathbb{S}^1$, as follows. For $t \in \mathbb{N}$, we set

$$\delta^{(h_0,h_1,2t)} := \partial_0^{h_0}\partial_1^{h_1}(\partial_0 - i\partial_1)^t(\partial_0 + i\partial_1)^t = \partial_0^{h_0}\partial_1^{h_1}(\partial_0^2 + \partial_1^2)^t\,,$$
$$\delta^{(h_0,h_1,2t+1)} := \partial_0^{h_0}\partial_1^{h_1}(\partial_0 - i\partial_1)^t(\partial_0 + i\partial_1)^{t+1} = \partial_0^{h_0}\partial_1^{h_1}(\partial_0^2 + \partial_1^2)^t(\partial_0 + i\partial_1)\,.$$

The same formulas define operators $\delta_J^{(h_0,h_1,2t)}, \delta_J^{(h_0,h_1,2t+1)}$ on $\mathscr{C}^{|\mathbf{h}|}(V, \mathbb{H})$.

Lemma 4.6. *Fix $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, $\mathbf{h} = (h_0, h_1, h_2) \in \mathbb{N}^3$. The restriction $(\mathcal{T}_{\mathbf{k}})_J$ is the J-monogenic polynomial $\mathcal{P}^J_{\mathbf{k}} J^{k_2} \in U^J_k$. In particular, $(\mathcal{T}_{\mathbf{k}})_J$ maps $\mathbb{R}^{2+1}_J$ into itself when k_2 is even; into $\mathbb{R}^{2+1}_J J = \mathbb{R} + j\mathbb{R} + k\mathbb{R}$ when k_2 is odd. If $\phi \in \mathrm{Mon}_J(V, \mathbb{H})$, then $\delta^{\mathbf{h}}_J \phi = J^{-h_2}\nabla^{\mathbf{h}}_J \phi$. If $f \in \mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$, then $(\delta^{\mathbf{h}} f)_J = J^{-h_2}\nabla^{\mathbf{h}}_J f_J$.*

Proof. We first prove that $(\mathcal{T}_{\mathbf{k}})_J = \mathcal{P}^J_{\mathbf{k}} J^{k_2}$, by induction on $|\mathbf{k}|$. The thesis is obviously true when $k_1 < 0$ or $k_2 < 0$, since in this case $\mathcal{T}_{\mathbf{k}}$ and $\mathcal{P}^J_{\mathbf{k}}$ both vanish identically. Similarly, $(\mathcal{T}_{(0,0)})_J = \mathcal{P}^J_{(0,0)} J^0$ since $\mathcal{T}_{(0,0)} :\equiv 1 \equiv: \mathcal{P}^J_{(0,0)}$ Assume the thesis true for all $\mathbf{k}'$ with $|\mathbf{k}'| = k - 1$. We can prove it for $\mathbf{k}$ with $|\mathbf{k}| = k$ by means of the following computation, where we omit the variable $x_0 + ix_1 + \beta J$:

$$\begin{aligned} k(\mathcal{T}_{\mathbf{k}})_J &= k_1 \mathcal{T}_{(k_1-1,k_2)} \cdot (x_1 - (-1)^{k_2} i x_0) + k_2 \mathcal{T}_{(k_1,k_2-1)} \cdot (x_0 + \beta J) \\ &= k_1 \mathcal{P}^J_{(k_1-1,k_2)} J^{k_2}(x_1 - (-1)^{k_2} i x_0) + k_2 \mathcal{P}^J_{(k_1,k_2-1)} J^{k_2-1}(x_0 + \beta J) \\ &= \left(k_1 \mathcal{P}^J_{(k_1-1,k_2)} \cdot (x_1 - i x_0) + k_2 \mathcal{P}^J_{(k_1,k_2-1)} \cdot (\beta - J x_0)\right) J^{k_2} \\ &= k \mathcal{P}^J_{\mathbf{k}} J^{k_2}\,. \end{aligned}$$

Now we fix $\phi \in \mathrm{Mon}_J(V, \mathbb{H})$ and prove $\nabla^{\mathbf{h}}_J \phi = J^{h_2}\delta^{\mathbf{h}}_J \phi$ by induction on $h_2 \in \mathbb{N}$.

$$\nabla_J^{(h_0,h_1,0)}\phi = \partial_0^{h_0}\partial_1^{h_1}\phi = \delta_J^{(h_0,h_1,0)}\phi = J^0\delta_J^{(h_0,h_1,0)}\phi\,,$$
$$\nabla_J^{(h_0,h_1,1)}\phi = \partial_0^{h_0}\partial_1^{h_1}\partial_{2,J}\phi = \partial_0^{h_0}\partial_1^{h_1} J(\partial_0 + i\partial_1)\phi = J\delta_J^{(h_0,h_1,1)}\phi\,.$$

In the second chain of equalities, we have used the fact that $\partial_{2,J}\phi = J(\partial_0 + i\partial_1)\phi$, which is a consequence of the equality $0 \equiv \overline{\partial}_J \phi = (\partial_0 + i\partial_1 + J\partial_{2,J})\phi$. We remark that, for all $\mathbf{h} \in \mathbb{N}^3$, the function $\nabla^{\mathbf{h}}_J \phi$ is still J-monogenic, whence the equality $\partial_{2,J}\nabla^{\mathbf{h}}_J \phi = J(\partial_0 + i\partial_1)\nabla^{\mathbf{h}}_J \phi$ follows. We are now ready for the induction step. Under the inductive hypothesis $\nabla_J^{(h_0,h_1,2t-1)}\phi = J^{2t-1}\delta_J^{(h_0,h_1,2t-1)}\phi$, we have

$$\begin{aligned} &\nabla_J^{(h_0,h_1,2t)}\phi = \partial_{2,J}\nabla_J^{(h_0,h_1,2t-1)}\phi = J(\partial_0 + i\partial_1)\nabla_J^{(h_0,h_1,2t-1)}\phi \\ &= J(\partial_0 + i\partial_1) J^{2t-1}\delta_J^{(h_0,h_1,2t-1)}\phi = J^{2t}(\partial_0 - i\partial_1)\delta_J^{(h_0,h_1,2t-1)}\phi = J^{2t}\delta_J^{(h_0,h_1,2t)}\phi\,, \\ &\nabla_J^{(h_0,h_1,2t+1)}\phi = \partial_{2,J}\nabla_J^{(h_0,h_1,2t)}\phi = J(\partial_0 + i\partial_1)\nabla_J^{(h_0,h_1,2t)}\phi \\ &= J(\partial_0 + i\partial_1) J^{2t}\delta_J^{(h_0,h_1,2t)}\phi = J^{2t+1}(\partial_0 + i\partial_1)\delta_J^{(h_0,h_1,2t)}\phi = J^{2t+1}\delta_J^{(h_0,h_1,2t+1)}\phi\,. \end{aligned}$$

This completes our induction step, whence our proof of the equality $\nabla^{\mathbf{h}}_J \phi = J^{h_2}\delta^{\mathbf{h}}_J \phi$.

Finally, if $f \in \mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$, then the definitions of $\delta^{\mathbf{h}}, \delta^{\mathbf{h}}_J$ and the equality established for J-monogenic functions yield the desired equality $(\delta^{\mathbf{h}} f)_J = \delta^{\mathbf{h}}_J f_J = J^{-h_2}\nabla^{\mathbf{h}}_J f_J$. □

We are now ready to prove the announced result.

Theorem 4.7. *The family $\mathscr{F}_k$ is a basis for U_k. Namely, for every $P \in U_k$,*

$$P(x) = \sum_{|\mathbf{k}|=k} \mathcal{T}_{\mathbf{k}}(x) c_{\mathbf{k}}, \quad c_{\mathbf{k}} := \frac{1}{\mathbf{k}!}\delta^{(0,\mathbf{k})}P(0)\,. \tag{4}$$

Moreover, P is $(1,3)$-slice preserving under the following assumptions on $c_{\mathbf{k}}$, valid for all $\mathbf{k}=(k_1,k_2)$ with $|\mathbf{k}|=k$: $c_{\mathbf{k}}\in\mathbb{C}$ when $k_2=0$; $c_{\mathbf{k}}\in\mathbb{R}$ when either $k_1=0\neq k_2$ or $k_1\neq 0\neq k_2\in 2\mathbb{N}$; $c_{\mathbf{k}}=0$ when $k_1\neq 0, k_2\in 2\mathbb{N}+1$.

Proof. Fix $k\in\mathbb{N}$. We take several steps, applying Remark 4.4 and Lemma 4.6 at each step.

We first prove the inclusion $\mathscr{F}_k\subseteq U_k$. We know that each function $\mathcal{T}_{\mathbf{k}}$ is a $|\mathbf{k}|$-homogenous polynomial. Since, for all $J\in\mathbb{S}^1$, the restriction $(\mathcal{T}_{\mathbf{k}})_J=\mathcal{P}^J_{\mathbf{k}}J^{k_2}$ is J-monogenic, we also have $\mathcal{T}_{\mathbf{k}}\in\mathrm{Reg}_{(1,3)}(\mathbb{H},\mathbb{H})$. The desired inclusion follows.

We now prove that the elements of $\mathscr{F}_k$ are linearly independent. For $\{c_{\mathbf{k}}\}_{|\mathbf{k}|=k}\subset\mathbb{H}$, if $P(x):=\sum_{|\mathbf{k}|=k}\mathcal{T}_{\mathbf{k}}(x)c_k$ vanishes identically in $\mathbb{H}$, then $P_J=\sum_{|\mathbf{k}|=k}\mathcal{P}^J_{\mathbf{k}}J^{k_2}c_{\mathbf{k}}$ vanishes identically in $\mathbb{R}^{2+1}_J$. Since $\{\mathcal{P}^J_{\mathbf{k}}\}_{|\mathbf{k}|=k}$ is a basis of U^J_k, it follows that $J^{k_2}c_{\mathbf{k}}=0$ (whence $c_{\mathbf{k}}=0$) for all $\mathbf{k}\in\mathbb{N}^2$ with $|\mathbf{k}|=k$.

We now prove that formula (4) is true for all $P\in U_k$, whence the family $\mathscr{F}_k$ spans U_k. It suffices to prove that the polynomial function $\widetilde{P}:=\sum_{|\mathbf{k}|=k}\mathcal{T}_{\mathbf{k}}c_{\mathbf{k}}$ coincides with P. This is true, because for all $J\in\mathbb{S}^1$

$$\widetilde{P}_J=\sum_{|\mathbf{k}|=k}(\mathcal{T}_{\mathbf{k}})_Jc_{\mathbf{k}}=\sum_{|\mathbf{k}|=k}\mathcal{P}^J_{\mathbf{k}}J^{k_2}c_{\mathbf{k}}=\sum_{|\mathbf{k}|=k}\mathcal{P}^J_{\mathbf{k}}\frac{1}{\mathbf{k}!}\nabla_J^{(0,\mathbf{k})}P_J(0)=P_J\,.$$

The final statement follows from the analogous property of U^J_k, taking into account that $\mathbb{C}=\bigcap_{J\in\mathbb{S}^1}\mathbb{R}^{2+1}_J$, that $\mathbb{R}=\bigcap_{J\in\mathbb{S}^1}\mathbb{C}_J$ and that $\{0\}=\bigcap_{J\in\mathbb{S}^1}J\mathbb{R}$. □

Taking into account Example 4.3, we draw the following conclusion.

Corollary 4.8. *For any domain Ω in $\mathbb{H}$, the function space $\mathrm{Reg}_{(1,3)}(\Omega,\mathbb{H})$ is distinct both from the space $\mathrm{Reg}_3(\Omega,\mathbb{H})$ of left Fueter-regular functions and from the space $\mathrm{Reg}_{(0,3)}(\Omega,\mathbb{H})$ of slice-regular quaternionic functions.*

We point out that we could not have used the J-monogenic polynomials $\mathcal{P}^J_{\mathbf{k}}$ to construct a well-defined $(1,3)$-regular function because $\mathcal{P}^{-J}_{\mathbf{k}}(x_0+ix_1-\beta(-J))=(-1)^{k_2}\mathcal{P}^J_{\mathbf{k}}(x_0+ix_1+\beta J)$. Nor could we have used $\nabla^{\mathbf{h}}_J$ to construct an operator on $(1,3)$-regular functions, because $(\nabla^{\mathbf{h}}_{-J}\phi)(x_0+ix_1-\beta(-J))=(-1)^{h_2}(\nabla^{\mathbf{h}}_J\phi)(x_0+ix_1+\beta J)$. We now prove that the $\mathcal{T}_{\mathbf{k}}$'s are strongly $(1,3)$-regular.

Proposition 4.9. *For all $\mathbf{k}\in\mathbb{Z}^2$, we have $\mathcal{T}_{\mathbf{k}}\in\mathcal{SR}(\mathbb{H},\mathbb{H})$. Indeed, there exist (real) polynomial functions $A_{\mathbf{k}}, B_{\mathbf{k}}:\mathbb{C}\times\mathbb{R}\to\mathbb{C}$ such that*

$$\mathcal{T}_{\mathbf{k}}(x)=A_{\mathbf{k}}\left(x_0+ix_1,x_2^2+x_3^3\right)+(jx_2+kx_3)B_{\mathbf{k}}\left(x_0+ix_1,x_2^2+x_3^3\right)$$

for all $x=x_0+ix_1+jx_2+kx_3\in\mathbb{H}$. Moreover, both $(x_0,x_1,\beta)\mapsto A_{\mathbf{k}}(x_0+ix_1,\beta^2)$ and $(x_0,x_1,\beta)\mapsto\beta B_{\mathbf{k}}(x_0+ix_1,\beta^2))$ are $|\mathbf{k}|$-homogeneous. As a consequence: $\mathcal{T}_{\mathbf{k}}$ preserves $\mathbb{C}$; the modulus $|\mathcal{T}_{\mathbf{k}}(x_0+ix_1+jx_2+kx_3)|$ depends on $\mathbf{k},x_0,x_1,x_2^2+x_3^2$ but is independent of the imaginary unit $\frac{jx_2+kx_3}{\sqrt{x_2^2+x_3^2}}\in\mathbb{S}^1$.

Proof. It suffices to set $A_{\mathbf{k}}:\equiv 0\equiv: B_{\mathbf{k}}$ for all $\mathbf{k}=(k_1,k_2)\in\mathbb{Z}^2$ with $k_1<0$ or $k_2<0$, as well as $A_{(0,0)}:\equiv 1, B_{(0,0)}:\equiv 0$ and, for $\mathbf{k}\in\mathbb{N}^2\setminus\{(0,0)\}$,

$$|\mathbf{k}|A_{\mathbf{k}}(x_0+ix_1,\gamma):=k_1A_{(k_1-1,k_2)}(x_1-(-1)^{k_2}ix_0)+k_2A_{(k_1,k_2-1)}x_0-k_2\overline{B_{(k_1,k_2-1)}}\gamma\,,$$
$$|\mathbf{k}|B_{\mathbf{k}}(x_0+ix_1,\gamma):=k_1B_{(k_1-1,k_2)}(x_1-(-1)^{k_2}ix_0)+k_2\overline{A_{(k_1,k_2-1)}}+k_2B_{(k_1,k_2-1)}x_0\,.$$ □

4.3. Integral and series representation, identity principle

In this subsection, our first aim is providing an integral representation formula for $(1,3)$-regular quaternionic functions. We need the notation $B_J(y_0,R):=B(y_0,R)\cap\mathbb{R}^{2+1}_J$, valid only for $y_0\in\mathbb{R}^{2+1}_J$ and for $R>0$. We begin with Cauchy's integral formula for J-monogenic functions, see [16, Theorem 7.12] (which also specifies the hypersurface integrals considered here). The subsequent proposition follows at once.

Remark 4.10. Let $J\in\mathbb{S}^1$, $y_0\in\mathbb{R}^{2+1}_J$, $R>0$ and $B_J:=B_J(y_0,R)$. If $\phi\in\mathrm{Mon}_J(V,\mathbb{H})$ for some open neighborhood V of the closure of B_J in $\mathbb{R}^{2+1}_J$, then for all $x_0+ix_1+\beta J\in B_J$

$$\phi(x_0+ix_1+\beta J)=\frac{1}{4\pi}\int_{\partial B_J}\frac{\overline{y-x_0-ix_1-\beta J}}{|y-x_0-ix_1-\beta J|^{2+1}}dy^*\phi(y)\,.\tag{5}$$

Proposition 4.11. *Let Ω be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a $(1,3)$-regular function. If $J \in \mathbb{S}^1$, $y_0 \in \Omega_J$, $R > 0$ are such that Ω contains the closure of $B_J := B_J(y_0, R)$, then for all $x \in B_J$*

$$f(x) = \frac{1}{4\pi} \int_{\partial B_J} \frac{\overline{y-x}}{|y-x|^{2+1}} dy^* f_J(y)\,. \tag{6}$$

Our next aim is expanding $(1,3)$-regular quaternionic functions into polynomial series at every point of the mirror $\mathbb{C}$. To do so, we rely on well-known properties of monogenic functions, described in [16, Theorem 9.24 and Theorem 9.27]:

Remark 4.12. Let $J \in \mathbb{S}^1$, $R > 0$, $B_J := B_J(0,R)$, $\phi \in \mathrm{Mon}_J(B_J, \mathbb{H})$. Then

$$\phi = \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{P}^J_{\mathbf{k}} \frac{1}{\mathbf{k}!} \nabla_J^{(0,\mathbf{k})} \phi(0)\,. \tag{7}$$

Moreover, the series in formula (7) converges normally in B_J. Now, fix a connected open subset V of $\mathbb{R}^{2+1}_J$ and $\phi \in \mathrm{Mon}_J(V, \mathbb{H})$. Then ϕ is real analytic because for every $B_J(y_0, R) \subseteq V$ formula (7) applies to $y \mapsto \phi_{|_{B_J(y_0,R)}}(y + y_0)$. As a consequence: if $\phi' \in \mathrm{Mon}_J(V, \mathbb{H})$ coincides with ϕ on a smooth 2-surface contained in V, then $\phi = \phi'$ throughout V.

Theorem 4.13. *Let Ω be a domain in $\mathbb{H}$, including the open ball $B(z_0, R)$ for some $z_0 \in \mathbb{C}$ and some $R > 0$. If $f : \Omega \to \mathbb{H}$ is a $(1,3)$-regular function, then*

$$f(x) = \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{T}_{\mathbf{k}}(x - z_0) c_{\mathbf{k}}, \quad c_{\mathbf{k}} := \frac{1}{\mathbf{k}!} \delta^{(0,\mathbf{k})} f(z_0) \tag{8}$$

for all $x \in B(z_0, R)$. Moreover, the series on the right-hand side of formula (8) converges normally in $B(z_0, R)$. Finally, f is $(1,3)$-slice preserving if we make the following assumptions: $c_{\mathbf{k}} \in \mathbb{C}$ when $k_2 = 0$; $c_{\mathbf{k}} \in \mathbb{R}$ when either $k_1 = 0 \neq k_2$ or $k_1 \neq 0, k_2 \in 2\mathbb{N}$; $c_{\mathbf{k}} = 0$ when $k_1 \neq 0, k_2 \in 2\mathbb{N} + 1$.

Proof. We assume $z_0 = 0$, without loss of generality because Remark 4.1 allows us to precompose our $(1,3)$-regular function f with the translation $x \mapsto x + z_0$. Let us consider on $B := B(0,R)$ the series

$$\widetilde{f}(x) := \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{T}_{\mathbf{k}}(x) c_{\mathbf{k}}, \quad c_{\mathbf{k}} := \frac{1}{\mathbf{k}!} \delta^{(0,\mathbf{k})} f(0)\,.$$

Fix $J_0 \in \mathbb{S}^1$. For every compact subset C of B, its symmetric completion $\widetilde{C}$ is also a compact subset of B. Moreover, the J_0-slice $(\widetilde{C})_{J_0}$ of $\widetilde{C}$ is a compact subset of B_{J_0}. For $k \in \mathbb{N}$ and $x = x_0 + ix_1 + \beta J \in \widetilde{C}$,

$$\sum_{|\mathbf{k}|=k} \mathcal{T}_{\mathbf{k}}(x) c_{\mathbf{k}} = \sum_{|\mathbf{k}|=k} A_{\mathbf{k}}\left(x_0 + ix_1, \beta^2\right) c_{\mathbf{k}} + \beta J \sum_{|\mathbf{k}|=k} B_{\mathbf{k}}\left(x_0 + ix_1, \beta^2\right) c_{\mathbf{k}}$$

is affine in J and equals $\sum_{|\mathbf{k}|=k} \mathcal{P}^{J_0}_{\mathbf{k}}(x_0 + ix_1 + \beta J_0) \frac{1}{\mathbf{k}!} \nabla_{J_0}^{(0,\mathbf{k})} f_{J_0}(0)$ when $J = J_0$. Here, we have used both Proposition 4.9 and Lemma 4.6. The series $\sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{P}^{J_0}_{\mathbf{k}}(x_0 + ix_1 + \beta J_0) \frac{1}{\mathbf{k}!} \nabla_{J_0}^{(0,\mathbf{k})} f_{J_0}(0)$ is the expansion (7) of f_{J_0}: this expansion converges normally to f_{J_0} in B_{J_0}. It follows that the series $\widetilde{f}(x)$ converges normally in B. Its sum defines a function $\widetilde{f} : B \to \mathbb{H}$. For every $J \in \mathbb{S}^1$,

$$\widetilde{f}_J = \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} (\mathcal{T}_{\mathbf{k}})_J c_{\mathbf{k}} = \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{P}^J_{\mathbf{k}} \frac{1}{\mathbf{k}!} \nabla_J^{(0,\mathbf{k})} f_J(0) = (f_J)_{|_{B_J}}\,,$$

thanks to Lemma 4.6 and formula (7). Thus, $\widetilde{f}$ coincides with $f_{|_B}$ and formula (8) is proven. The final statement follows from its analog in Theorem 4.7. □

Besides its independent interest, formula (8) serves to prove the following principle, valid on $(1,3)$-slice domains.

Theorem 4.14 *(Identity Principle). Let Ω be a $(1,3)$-slice domain in $\mathbb{H}$ and let $f, g : \Omega \to \mathbb{H}$ be $(1,3)$-regular functions. If there exist $J_0 \in \mathbb{S}^1$ and a smooth 2-surface contained in Ω_{J_0} where f and g coincide, then $f = g$ throughout Ω.*

Proof. Since f, g are $(1,3)$-regular functions on Ω, for all $J \in \mathbb{S}^1$ we have $f_J, g_J \in \mathrm{Mon}_J(\Omega_J, \mathbb{H})$ and we can apply Remark 4.12. Our hypotheses yield that f_{J_0} and g_{J_0} coincide in their connected domain Ω_{J_0}. Since Ω is a slice domain, there exist $z_0 \in \mathbb{C}$, $R > 0$ such that the open ball $B := B(z_0, R)$ is contained in Ω. We have

$$\delta^{(0,\mathbf{k})} f(z_0) = J_0^{-k_2} \nabla_{J_0}^{(0,\mathbf{k})} f_{J_0}(z_0) = J_0^{-k_2} \nabla_{J_0}^{(0,\mathbf{k})} g_{J_0}(z_0) = \delta^{(0,\mathbf{k})} g(z_0)\,.$$

This information, when performing expansion (8) at z_0 for both f and g, yields $f_{|_B} = g_{|_B}$. In particular, for every $J \in \mathbb{S}^1$ the real analytic functions f_J and g_J coincide in B_J, whence in their connected domain Ω_J. We conclude that $f = g$ throughout Ω, as announced. □

4.4. Representation formula

We now aim at proving that if a $(1,3)$-symmetric set $\Omega := \Omega_D$ is a $(1,3)$-slice domain then $\mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H}) = \mathcal{SR}(\Omega, \mathbb{H})$, i.e., all $(1,3)$-regular functions on Ω are strongly $(1,3)$-regular. To this end, we establish the next property.

Theorem 4.15 *(General representation formula). Assume $\Omega := \Omega_D$ to be a $(1,3)$-slice domain and let $f \in \mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$. For all $I, J, K \in \mathbb{S}^1$ with $J \neq K$ and for all $z + \beta I \in \Omega$,*

$$\begin{aligned} f(z+\beta I) &= (J-K)^{-1}(Jf(z+\beta J) - Kf(z+\beta K)) + I(J-K)^{-1}(f(z+\beta J) - f(z+\beta K)) \\ &= \big((J-K)^{-1}J + I(J-K)^{-1}\big) f(z+\beta J) - \big((J-K)^{-1}K + I(J-K)^{-1}\big) f(z+\beta K)\,. \end{aligned} \tag{9}$$

In particular,

$$\begin{aligned} f(z+\beta I) &= \frac{1}{2}(f(z+\beta J) + f(z-\beta J)) + I\frac{J}{2}(f(z-\beta J) - f(z+\beta J)) \\ &= \frac{1-IJ}{2} f(z+\beta J) + \frac{1+IJ}{2} f(z-\beta J)\,. \end{aligned} \tag{10}$$

Proof. To prove formula (10), we show that the function $g : \Omega \to \mathbb{H}$ defined as

$$g(z+\beta I) := \frac{1-IJ}{2} f_J(z+\beta J) + \frac{1+IJ}{2} f_{-J}(z-\beta J)$$

(for $I \in \mathbb{S}^1$, $z + \beta I \in \Omega$) coincides with f. A long computation proves that

$$(\overline{\partial}_I g_I)(z+\beta I) = \frac{1-IJ}{2}(\overline{\partial}_J f_J)(z+\beta J) + \frac{1+IJ}{2}(\overline{\partial}_{-J} f_{-J})(z-\beta J)\,.$$

This expression vanishes identically because $f_J = f_{-J}$ is both J-monogenic and $(-J)$-monogenic. A direct computation for $I = J$ shows that $g_J = f_J$. By the Identity Principle 4.14, f and g coincide throughout Ω.

With formula (10) available, we can prove formula (9) by direct computation. We start with the last expression in formula (9) and substitute $\frac{1-KJ}{2} f(z+\beta J) + \frac{1+KJ}{2} f(z-\beta J)$ for $f(z+\beta K)$. A long but straightforward computation shows that the last expression in formula (9) equals $\frac{1-IJ}{2} f(z+\beta J) + \frac{1+IJ}{2} f(z-\beta J)$, which in turn equals $f(z+\beta I)$ by formula (10). □

We are now ready for the announced result.

Theorem 4.16. *If $\Omega := \Omega_D$ is a $(1,3)$-slice domain, $\mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H}) = \mathcal{SR}(\Omega, \mathbb{H})$. Namely, every $f \in \mathrm{Reg}_{(1,3)}(\Omega, \mathbb{H})$ is a $(1,3)$-function, i.e., there exist unique real analytic functions $F_\emptyset, F_1 : D \to \mathbb{H}$, respectively even and odd in β, such that*

$$f(z+\beta J) = F_\emptyset(z,\beta) + J F_1(z,\beta) \tag{11}$$

for all $J \in \mathbb{S}^1$ and all $z + \beta J \in \Omega$. In particular, f is real analytic. Moreover,

$$\begin{aligned} &(\partial_{x_0} + i\partial_{x_1}) F_\emptyset - \partial_\beta F_1 = 0\,, \\ &\partial_\beta F_\emptyset + (\partial_{x_0} - i\partial_{x_1}) F_1 = 0\,, \end{aligned} \tag{12}$$

whence $F_\emptyset(x_0 + ix_1, \beta)$, $F_1(x_0 + ix_1, \beta)$ are harmonic in x_0, x_1, β. Finally, f is $(1,3)$-slice preserving if, and only if, $F_\emptyset(D) \subseteq \mathbb{C}$ and $F_1(D) \subseteq \mathbb{R} + j\mathbb{R} + k\mathbb{R}$.

Proof. If a stem function $F := F_\emptyset + E_1 F_1 : D \otimes \mathbb{R}^2 \to \mathbb{H}$ such that $f = \mathcal{I}(F)$ exists, then it is unique because $\mathcal{I}$ is bijective. Let us prove existence. Fix $J \in \mathbb{S}^1$: since f_J is real analytic, setting

$$F_\emptyset(z,\beta) := \frac{1}{2}(f_J(z+\beta J) + f_J(z-\beta J)) \tag{13}$$
$$F_1(z,\beta) := \frac{J}{2}(f_J(z-\beta J) - f_J(z+\beta J))$$

defines two real analytic functions $F_\emptyset, F_1 : D \to \mathbb{H}$, with $F_\emptyset(z,\beta)$ even in β and $F_1(z,\beta)$ odd in β. By the Representation Formula (10), equality (11) holds. Now,

$$\begin{aligned} 0 &\equiv \overline{\partial}_J f_J(z+\beta J) \\ &= (\partial_{x_0} + i\partial_{x_1} + J\partial_\beta)F_\emptyset + (\partial_{x_0} + i\partial_{x_1} + J\partial_\beta)(JF_1) \\ &= (\partial_{x_0} + i\partial_{x_1})F_\emptyset - \partial_\beta F_1 + J(\partial_\beta F_\emptyset + (\partial_{x_0} - i\partial_{x_1})F_1)\,. \end{aligned}$$

(with $J \in \mathbb{S}^1$ arbitrary) implies equalities (12). For $\Delta = \partial_{x_0}^2 + \partial_{x_1}^2 + \partial_\beta^2$,

$$\begin{aligned} \Delta F_\emptyset &= (\partial_{x_0} - i\partial_{x_1})(\partial_{x_0} + i\partial_{x_1})F_\emptyset + \partial_\beta^2 F_\emptyset = (\partial_{x_0} - i\partial_{x_1})\partial_\beta F_1 - \partial_\beta(\partial_{x_0} - i\partial_{x_1})F_1\,, \\ \Delta F_1 &= (\partial_{x_0} + i\partial_{x_1})(\partial_{x_0} - i\partial_{x_1})F_1 + \partial_\beta^2 F_1 = -(\partial_{x_0} + i\partial_{x_1})\partial_\beta F_\emptyset + \partial_\beta(\partial_{x_0} + i\partial_{x_1})F_\emptyset \end{aligned}$$

both vanish identically. Finally, taking into account formula (13), we see that f is $(1,3)$-slice preserving if, and only if, $F_\emptyset$ takes values in $\bigcap_{J\in\mathbb{S}^1} \mathbb{R}_J^{2+1} = \mathbb{C}$ and F_1 takes values in $\bigcap_{J\in\mathbb{S}^1} J\mathbb{R}_J^{2+1} = \mathbb{R} + j\mathbb{R} + k\mathbb{R}$. □

We conclude with a relevant example and with a more general remark.

Example 4.17. Let $z_0 \in \mathbb{C}$, $R > 0$, $B := B(z_0, R)$, $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^2} \subset \mathbb{H}$ be such that

$$f(x) = \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} \mathcal{T}_{\mathbf{k}}(x - z_0)c_{\mathbf{k}}$$

converges normally in B, defining a $(1,3)$-regular function $f : B \to \mathbb{H}$. Referring to the polynomial functions $A_{\mathbf{k}}, B_{\mathbf{k}} : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ of Proposition 4.9, we have $f(z+\beta J) = A(z,\beta^2) + J\beta B(z,\beta^2)$ with

$$\begin{aligned} A(z,\gamma) &= \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} A_{\mathbf{k}}(z - z_0, \gamma)c_{\mathbf{k}}\,, \\ B(z,\gamma) &= \sum_{k\in\mathbb{N}} \sum_{|\mathbf{k}|=k} B_{\mathbf{k}}(z - z_0, \gamma)c_{\mathbf{k}}\,. \end{aligned}$$

Here, absolute and uniform convergence is guaranteed by the inequalities $|A_{\mathbf{k}}(x_0 + ix_1 - z_0, x_2^2 + x_3^2)|$, $|B_{\mathbf{k}}(x_0 + ix_1 - z_0, x_2^2 + x_3^2)| \le |\mathcal{T}_{\mathbf{k}}(x - z_0)|$, valid for all $x = x_0 + ix_1 + jx_2 + kx_3 \in B$.

Remark 4.18. With the notations set in Theorem 4.16, let $D' := \{(z,\beta^2) \in \mathbb{C} \times \mathbb{R} : (z,\beta) \in D\}$. By Whitney's Theorem, see [21], there exist an open neighborhood W of D' in $\mathbb{C} \times \mathbb{R}$, with $W \cap (\mathbb{C} \times \mathbb{R}^{\geqslant}) = D'$, and real analytic functions $A, B : W \to \mathbb{H}$ such that $F_\emptyset(z,\beta) = A(z,\beta^2)$ and $F_1(z,\beta) = \beta B(z,\beta^2)$, for all $(z,\beta) \in D$.

Data availability

No data was used for the research described in the article.

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