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Calculus of Variations

Quantitative symmetry in a mixed Serrin-type problem for a constrained torsional rigidity

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Abstract

We consider a mixed boundary value problem in a domain Ω contained in a half-ball B_+ and having a portion $\overline{T}$ of its boundary in common with the curved part of ∂B_+. The problem has to do with some sort of constrained torsional rigidity. In this situation, the relevant solution u satisfies a Steklov condition on T and a homogeneous Dirichlet condition on $\Sigma = \partial\Omega \setminus \overline{T} \subset B_+$. We provide an integral identity that relates (a symmetric function of) the second derivatives of the solution in Ω to its normal derivative u_ν on Σ. A first significant consequence of this identity is a rigidity result under a quite weak overdetermining integral condition for u_ν on Σ: in fact, it turns out that Σ must be a spherical cap that meets T orthogonally. This result returns the one obtained by Guo and Xia under the stronger pointwise condition that the values of u_ν be constant on Σ. A second important consequence is a set of stability bounds, which quantitatively measure how Σ is far uniformly from being a spherical cap, if u_ν deviates from a constant in the norm $L^1(\Sigma)$.

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1 Introduction

Let B and $S = \partial B$ be the (open) unit ball and the unit sphere in $\mathbb{R}^N$, centered at the origin, and set $B_+ = \{x = (x_1, \dots, x_N) \in B : x_N > 0\}$. Consider in B_+ a bounded domain Ω (i.e., a bounded open connected set) whose boundary Γ is the union of $\overline{\Sigma}$ and $T = \Gamma \setminus \overline{\Sigma}$, where

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Σ is a smooth hypersurface contained in B_+, T is a subset of S, and $\overline{T}$ meets $\overline{\Sigma}$ at a common $(N-2)$-dimensional submanifold $\Lambda = \overline{\Sigma} \cap \overline{T}$ of S.

In Ω, we consider the following mixed boundary value problem:

$$\Delta u = N \text{ in } \Omega, \quad u = 0 \text{ on } \Sigma, \quad u_\nu = u \text{ on } T. \tag{1.1}$$

Here and in what follows, u_ν denotes the derivative of u in the direction of the outward unit normal ν to Γ. In particular, we have that

$$\nu(x) = x \text{ on } T.$$

For the existence and uniqueness of a solution u of (1.1), which is smooth in $\overline{\Omega} \setminus \Lambda$ and belongs to $C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$, we refer the reader to [8, Proposition 2.2]; more precisely, Lieberman [12] guarantees that $u \in C^{0,\gamma}(\overline{\Omega})$, whereas the classical regularity theory for elliptic equations gives that, for $k \in \mathbb{N}$ with $k \geq 2$ and $\gamma \in (0, 1)$, $u \in C^{k,\gamma}(\overline{\Omega} \setminus \Lambda)$ provided that Σ is of class $C^{k,\gamma}$. In [8], the solution of (1.1) is obtained as a suitably normalized solution of the variational problem:

$$\sup_{0 \neq v \in W_0^{1,2}(\Omega, \Sigma)} \frac{\left(\int_\Omega v \, dx\right)^2}{\int_\Omega |\nabla v|^2 dx - \int_T v^2 dS_x}.$$

Here, $W_0^{1,2}(\Omega, \Sigma)$ denotes the subspace of functions in $W^{1,2}(\Omega)$ vanishing in a Sobolev sense on Σ. The supremum can be interpreted as some sort of *relative* or *constrained torsion* $\mathcal{T}(\Omega, \Sigma)$ for domains contained in B. In fact, if $\overline{\Omega} \subset B$ (and hence $T = \varnothing$), we recover a definition of the standard torsion of Ω (see [13]).

The issue of regularity up to the (whole) boundary for (1.1) is delicate. The regularity of the solution u strongly depends on how $\overline{\Sigma}$ and $\overline{T}$ intersect. As done in [8], we shall further assume that u belongs to $W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ to ensure that we can integrate by parts. As shown in [8, Proposition 3.5], such an assumption is surely satisfied whenever $\overline{\Sigma}$ and $\overline{T}$ intersect orthogonally.

The aim of this paper is to study a Serrin-type overdetermined boundary value problem for (1.1). In fact, similarly to [23, 24] and as done in [8], we add the extra condition

$$u_\nu = R \text{ on } \Sigma, \tag{1.2}$$

where R is some given constant. In [8], under suitable regularity assumptions, it is shown that the problem (1.1)–(1.2) arises naturally in a shape optimization problem. If the relative torsion $\mathcal{T}(\Omega, \Sigma)$ is stationary with respect to volume-preserving transformations at a domain Ω, then the corresponding function u that attains $\mathcal{T}(\Omega, \Sigma)$ satisfies (1.1)–(1.2) (see [8, Proposition 4.2]).

A rigidity result for problem (1.1)–(1.2) has been proved by Guo and Xia [8]. In our slightly different setting, the main result in [8] states that, if a suitably regular overdetermined solution exists, then $R > 0$, Σ must be the spherical cap defined by

$$\{x \in B_+ : |x - z| = R\} \text{ with } |z| = \sqrt{1 + R^2}, \tag{1.3}$$

and u must be equal on $\overline{\Omega}$ to the quadratic polynomial defined for $x \in \mathbb{R}^N$ by

$$\frac{1}{2}\,(|x - z|^2 - R^2).$$

We shall compute in Proposition 2.4 the exact value of R in terms of Ω as

$$R = N \frac{\int_\Omega x_N \, dx}{\int_\Sigma x_N \, dS_x}. \tag{1.4}$$

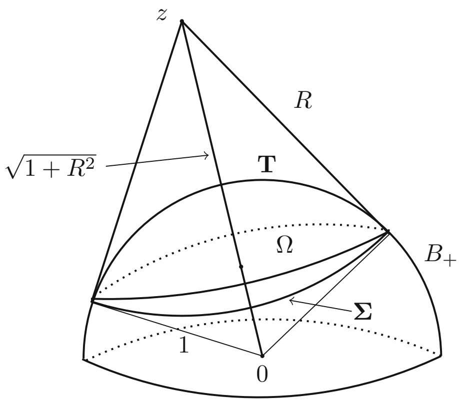

Fig. 1 The construction of a symmetric domain Ω. The R-spherical cap Σ meets orthogonally the unit spherical cap T

Thus, Σ must be a spherical cap and Ω results as a lenticular domain, as shown in Fig. 1. In this paper, we shall study the problem (1.1)–(1.2) from a quantitative point of view. In other words, we will estimate how close Σ is to a spherical cap in terms of the deviation of u_ν from the constant R in some Lebesgue norm on Σ.

In order to do that, we first refine Guo and Xia's rigidity result. In fact, by shadowing the arguments in [8], we obtain the following integral identity for the solution of (1.1):

$$\int_\Omega x_N(-u)\left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\} dx = \frac{1}{2}\int_\Sigma \left(u_\nu^2 - R^2\right)\left[u_\nu x_N - \langle X^q, \nu\rangle\right] dS_x. \tag{1.5}$$

Here, X^q is the conformal *Killing field* defined by

$$X^q = x_N\, x - \frac{1}{2}\left(|x|^2 + 1\right) e_N = x_N\, \nabla q(x) - q(x)\, e_N, \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $e_N = (0, \dots, 1) \in \mathbb{R}^N$, and we set: $q(x) = (1+|x|^2)/2$. Integral identities of this kind have been obtained for the Alexandrov's Soap Bubble Theorem and the classical Serrin's problem by the authors of this note (see [14–16, 20]). In those cases, the role of the field X^q in the identity was played by the identity field $\mathbb{R}^N \ni x \mapsto x$. Note that, on the unit sphere S, X^q is the projection of $-e_N$ on the tangent space to S.

In [8], it is proved that, if u satisfies (1.1)–(1.2), then the left-hand side of (1.5) must be zero. Since $x_N > 0$ in $\Omega \subset B_+$ and $u < 0$ in Ω by [8, Proposition 2.3], the function in the braces at the left-hand side of (1.5) must vanish identically on Ω, since it is always non-negative by the Cauchy–Schwarz inequality. As a by-product, one infers that u must be a spherically symmetric quadratic polynomial, as noted in [14]. Thus, Σ must be a portion of a sphere, since $u = 0$ on Σ. The lenticular shape of Ω then ensues quite easily.

Now, observe that, from (1.5) it is evident that its right-hand side (and hence its left-hand side) is null if (1.2) holds. However, (1.5) gives more information for at least two reasons. One is that Guo and Xia's rigidity result can be merely obtained under the weaker assumption that the right-hand side of (1.2) is non-positive. The second and more important reason is that the identity gives quantitative information. In fact, if we know that u_ν deviates from R by little in some integral norm, then the integral at the left-hand side of (1.5) is small.

Now, notice that, if we consider a quadratic polynomial as defined by

$$Q(x) = \frac{1}{2}|x - z|^2 - q_0 \ \text{ for } x, z \in \mathbb{R}^N,\ q_0 \in \mathbb{R},$$

and we set $h = Q - u$, then it turns out that

$$|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} = |\nabla^2 h|^2.$$

Thus, the square root of the first integral in (1.5) can be seen as the weighted (second order) $W^{2,2}$-seminorm in Ω of h with respect to the positive measure $x_N\,[-u(x)]\,dx$. Also, notice that $h = Q$ on Σ, and hence Q has to do with the distance of the point z to points in Σ. Therefore, we will see that, in order to obtain an estimate of closeness of Σ from the spherical cap defined in (1.3), it is just the matter of proving that the oscillation of $h = Q$ on Σ can be controlled in terms of the aforementioned weighted $W^{2,2}$-seminorm of h.

We are now going to present our quantitative rigidity estimates. We need to recall some notation from the subsequent sections.

As in [8], we assume that $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Under this assumption, since $u \in W^{1,\infty}(\Omega)$ and $u = 0$ on Σ, then $u \in C^{0,1}(\overline{\Omega})$. In fact, we can extend u by 0 outside Ω to the whole B, thus obtaining a function in $W^{1,\infty}(B)$, which coincides with $C^{0,1}(\overline{B}) \supset C^{0,1}(\overline{\Omega})$, since B is convex. Thus, we let L to be an upper bound[1] of the Lipschitz seminorm defined in (3.2), i.e. $L \geq [u]_{C^{0,1}(\overline{\Omega})}$.

Also, we present our stability results under the assumption that $\overline{\Sigma}$ and $\overline{T}$ intersect on Λ in a way that Ω satisfies the (θ, a)-uniform interior cone condition, for given parameters θ and a (see Sect. 3 for the definition). We adopt this condition to avoid an excessively technical presentation. Nevertheless, our arguments could be adapted and the same stability result of Theorem 1.1 below achieved in more general cases (see Remark 4.8).

In order to measure the deviation of Σ from a spherical cap, for a given point $z \in \mathbb{R}^N$, we define two quantities,

$$\rho_e = \max_{x \in \overline{\Sigma}} |x - z| \quad \text{and} \quad \rho_i = \min_{x \in \overline{\Sigma}} |x - z|,$$

so that we have:

$$\overline{\Sigma} \subseteq \left[\overline{B}_{\rho_e}(z) \setminus B_{\rho_i}(z)\right] \cap \overline{B}_+.$$

The point z must be conveniently chosen. A good choice of z is a somewhat modified center of mass of Ω:

$$z = \frac{1}{|\Omega|}\left\{\int_\Omega x\,dx - \int_T u(x)\,x\,dS_x\right\}. \tag{1.7}$$

With this choice, we have that the mean value of the field ∇h is zero. This will allow the use of certain suitable Hardy-Poincaré-type inequalities.

We now present our stability results

Our most general quantitative estimates are contained in Theorem 4.9. Here, we prefer to present three special instances of that result in three relevant situations, which better depict the dependence of the estimates on certain geometrical assumptions on the surface Σ.

In the next theorem, Σ is not allowed to touch the flat part of B_+.

Theorem 1.1 (Σ does not touch $\partial B_+ \setminus \partial B$) *Set $N \geq 2$. Let Ω be a domain contained in B_+ and satisfying the (θ, a)-uniform interior cone condition. Assume that there exists a positive number m such that*

[1] When $\overline{\Sigma}$ and $\overline{T}$ intersect orthogonally, [8, Proposition 3.5] ensures that $u \in C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$: their argument is based on spherical reflection. The global $C^{1,\gamma}(\overline{\Omega})$ regularity of u is also guaranteed whenever Σ is a capillary surface with contact angle $\theta \in (0, \pi/2)$: see [11, Theorem 3.2].

$$\overline{\Omega} \subset \{x \in \overline{B}_+ : x_N \geq m\}. \tag{1.8}$$

Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ *be the solution of* (1.1) *and assume that* $L \geq [u]_{C^{0,1}(\overline{\Omega})}$. *Moreover, let* R *and* z *be the number and point defined in* (1.4) and (1.7). *Then, it holds that*

$$\rho_e - \rho_i \leq c \begin{cases} \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/2} \max\left\{\log\left(\|u_\nu^2 - R^2\|_{1,\Sigma}^{-1/2}\right), 1\right\} & \text{for } N = 2, \\ \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/N} & \text{for } N \geq 3, \end{cases}$$

for some non-negative constant $c = c(N, \theta, a, L, m)$.

In Sect. 4.3 we show that the assumption 1.8 can be removed at the cost of getting a slightly worse stability exponent, namely $1/(N+1)$ in place of $1/N$ for $N \geq 3$ (see Theorem 4.9). Such a generalization is non-trivial and requires a new and careful analysis, which is provided in Sect. 4.3.

The next result considers the case where Ω satisfies an interior sphere condition relative to B_+. In fact, the same stability rate of Theorem 1.1 can also be obtained if (1.8) is dropped and replaced by the assumption that Ω satisfies the *strong* r_i*-uniform interior sphere condition relative to* B_+. Such a condition, which is introduced in Sect. 4.1 following the spirit of [21, Section 4.1], is surely satisfied whenever $\overline{\Sigma}$ and $\overline{T}$ intersect orthogonally.

Theorem 1.2 (Ω satisfies a strong sphere condition) *Set* $N \geq 2$ *and let* Ω *be a domain contained in* B_+. *Assume that* Ω *satisfies the* (θ, a)*-uniform interior cone condition and the strong* r_i*-uniform interior sphere condition relative to* B_+.

Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ *be the solution of* (1.1) *and assume that* $L \geq [u]_{C^{0,1}(\overline{\Omega})}$. *Moreover, let* R *and* z *be the number and point defined in* (1.4) and (1.7). *Then, it holds that*

$$\rho_e - \rho_i \leq c \begin{cases} \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/2} \max\left\{\log\left(\|u_\nu^2 - R^2\|_{1,\Sigma}^{-1/2}\right), 1\right\} & \text{for } N = 2, \\ \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/N} & \text{for } N \geq 3, \end{cases}$$

for some non-negative constant $c = c(N, \theta, a, L, r_i)$.

The rate of stability further improves if both additional assumptions are in force.

Theorem 1.3 (Ω satisfies a strong sphere condition and Σ does not touch $\partial B_+ \setminus \partial B$) *Set* $N \geq 2$ *and let* Ω *be a domain contained in* B_+. *Assume that* Ω *satisfies the* (θ, a)*-uniform interior cone condition and the strong* r_i*-uniform interior sphere condition relative to* B_+. *In addition, suppose that there exists* $m > 0$ *such that* (1.8) *holds.*

Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ *be the solution of* (1.1) *and assume that* $L \geq [u]_{C^{0,1}(\overline{\Omega})}$. *Moreover, let* R *and* z *be the number and point defined in* (1.4) and (1.7). *Then, it holds that*

$$\rho_e - \rho_i \leq c \begin{cases} \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/2} & \text{for } N = 2, \\ \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/2} \max\left\{\log\left(\|u_\nu^2 - R^2\|_{1,\Sigma}^{-1/2}\right), 1\right\} & \text{for } N = 3, \\ \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/(N-1)} & \text{for } N \geq 4, \end{cases}$$

$c = c(N, \theta, a, L, r_i, m)$.

The paper is organized as follows. In Sect. 2, we derive our fundamental integral identity (1.5). In Sect. 3, we prepare the proofs of Theorems 1.1–1.3 and 4.9, by collecting a pointwise estimate from below for $-u$ in terms of the distance of a point x to the boundary Γ and some Poincaré-type estimates in weighted spaces. These adapt to the constrained case $\Omega \subset B_+$ similar bounds obtained in [14–16] (see also [7]). Finally, in Sect. 4, we carry out the proofs of Theorems 1.1–1.3 and 4.9.

2 A fundamental identity

In this section, we shall prove the identity (1.5).

For later use, we preliminarily recall some easily verified properties of the Killing field X^q defined in (1.6) and the solution u of (1.1). In fact, it holds that

$$\operatorname{div} X^q = N x_N \text{ in } \mathbb{R}^N; \quad X^q = x_N x - e_N, \ \langle X^q, \nu\rangle = \langle X^q, x\rangle = 0 \text{ on } S, \tag{2.1}$$

$$\nabla(\Delta u) = 0 \text{ in } \Omega, \quad \nabla u = u_\nu\, \nu \text{ on } \Sigma, \quad \langle \nabla^2 u\, \nu, \omega\rangle = 0 \text{ on } T, \tag{2.2}$$

for every direction ω which is tangential to T. The last two conditions follow from the fact that Σ and T are level surfaces for u and $u_\nu - u = \langle x, \nabla u\rangle - u$.

The proof of (1.5) is inspired by calculations carried out in [8]. Essentially, those are a combination of repeated integrations by parts and the application of conditions (2.1) and (2.2).

We begin by adapting to our aims and notations an identity in [8, Proposition 3.3]. We introduce the so-called P-*function* by setting:

$$P = \frac{1}{2}\,|\nabla u|^2 - u \text{ in } \Omega. \tag{2.3}$$

Lemma 2.1 (A Pohozaev-type identity) *Let* $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ *be the solution of* (1.1). *Then, the following identity holds:*

$$N \int_\Omega x_N P\, dx = \frac{1}{2} \int_\Sigma u_\nu^2 \langle X^q, \nu\rangle\, dS_x. \tag{2.4}$$

Remark 2.2 Being as $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$, all the integration by parts performed in this section are allowed (see, e.g., the version of the divergence theorem stated in [8, Proposition 3.2]).

Proof of Lemma 2.1 The proof of [8, Proposition 3.3] can be summarized and reorganized as follows. By straightforward computations, we see that the following differential identity holds true:

$$N\, x_N\, P = \operatorname{div}\left\{\langle X^q, \nabla u\rangle\, \nabla u - N\, u\, X^q - \frac{1}{2}\,|\nabla u|^2 X^q\right\}$$
$$+(N-1)\operatorname{div}\left\{x_N\, u\, \nabla u - \frac{1}{2}\, u^2 e_N\right\}.$$

Next, we integrate on Ω and use the divergence theorem. We have that

$$N \int_\Omega x_N\, P\, dx = \int_\Sigma \langle X^q, \nabla u\rangle\, u_\nu\, dS_x + \int_T \langle X^q, \nabla u\rangle\, u_\nu\, dS_x +$$
$$-\frac{1}{2}\int_\Sigma u_\nu^2 \langle X^q, \nu\rangle\, dS_x + (N-1)\int_T x_N\, u\, u_\nu\, dS_x - \frac{1}{2}\,(N-1)\int_T x_N\, u^2 dS_x.$$

Here, we have used that $u = 0$ on Σ and $\langle X^q, \nu\rangle = 0$ on T.

Now, we use that $\nabla u = u_\nu\, \nu$ on Σ and $u_\nu = u$ on T, and hence infer that

$$N \int_\Omega x_N\, P\, dx$$
$$= \frac{1}{2}\int_\Sigma u_\nu^2 \langle X^q, \nu\rangle\, dS_x + \int_T \langle -(e_N)_T, \nabla_T u\rangle\, u\, dS_x + \frac{1}{2}\,(N-1)\int_T x_N\, u^2 dS_x.$$

Here, we have also noticed that

$$\begin{aligned}\langle X^q, \nabla u\rangle\, u_\nu &= \langle X^q, u_\nu\, \nu + \nabla u - u_\nu\, \nu\rangle\, u \\ &= \langle X^q, \nabla u - u_\nu\, \nu\rangle\, u = \langle -(e_N)_T, \nabla_T u\rangle\, u \ \text{ on } \ T.\end{aligned}$$

where with $(e_N)_T$ and $\nabla_T u$ we denote the tangential components of e_N and ∇u on T.

Thus, we are left to prove that the two integrals on T sum up to zero. This ensues by applying the divergence theorem on the surface T:

$$\begin{aligned}0 &= \int_\Lambda u^2 \langle (e_N)_T, \nu_\Lambda\rangle\, d\ell_x = \int_T \operatorname{div}_T(u^2 (e_N)_T)\, dS_x \\ &= \int_T \left\{u^2 \operatorname{div}_T\, ((e_N)_T) + 2\, \langle (e_N)_T, \nabla_T u\rangle\, u\right\} dS_x.\end{aligned}$$

Here, div_T denotes the tangential divergence. The first integral is zero, because $u = 0$ on Λ. The conclusion follows by noting that $\operatorname{div}_T\, ((e_N)_T) = -(N-1)\, x_N$. □

We are now ready to prove the main result of this section.

Theorem 2.3 (Fundamental identity) *Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ be the solution of (1.1). Then, for any given constant c, the following identity holds:*

$$\int_\Omega x_N(-u) \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\} dx = \frac{1}{2} \int_\Sigma \left(u_\nu^2 - c^2\right) \left[x_N\, u_\nu - \langle X^q, \nu\rangle\right] dS_x. \tag{2.5}$$

Proof Taking the vector field $X^u = x_N\, \nabla u - u\, e_N$, we compute that

$$\operatorname{div}(X^u) = N\, x_N \ \text{ in } \ \Omega, \quad \langle X^u, \nu\rangle = 0 \ \text{ on } \ T, \quad \langle X^u, \nu\rangle = x_N\, u_\nu \ \text{ on } \ \Sigma, \tag{2.6}$$

and hence, by the divergence theorem and (2.1),

$$0 = \int_\Omega \operatorname{div}(X^u - X^q)\, dx = \int_\Gamma \langle X^u - X^q, \nu\rangle\, dS_x = \int_\Sigma [x_N\, u_\nu - \langle X^q, \nu\rangle]\, dS_x.$$

Thus, it is sufficient to prove (2.5) for $c = 0$.

Next, observe that

$$\Delta P = |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N},$$

and hence, the Gauss-Green formula gives:

$$\begin{aligned}\int_\Omega x_N u\, \Delta P\, dx &= \int_\Omega \Delta(x_N u)\, P\, dx + \int_\Omega \operatorname{div}\left\{x_N u\, \nabla P - P\, \nabla(x_N u)\right\} dx \\ &= \int_\Omega [2\, u_{x_N} + x_N\, \Delta u]\, P\, dx + \int_\Gamma \left\{x_N u\, P_\nu - [\langle e_N, \nu\rangle\, u + x_N\, u_\nu]\, P\right\} dS_x \\ &= \int_\Omega [2\, u_{x_N} + N\, x_N]\, P\, dx - \int_\Sigma x_N u_\nu P\, dS_x + \int_T \left\{x_N u\, P_\nu - x_N\, (u + u_\nu) P\right\} dS_x.\end{aligned}$$

Here, we used that $\Delta u = N$ in Ω, $u = 0$ on Σ, and $\langle e_N, \nu(x)\rangle = x_N$ for $x \in T$. Consequently, we deduce that

$$\begin{aligned}\int_\Omega x_N u\, \Delta P\, dx = 2 \int_\Omega u_{x_N} P\, dx + \frac{1}{2} \int_\Sigma \langle X^q, \nu\rangle u_\nu^2\, dS_x \, + \\ -\frac{1}{2} \int_\Sigma x_N u_\nu^3\, dS_x + \int_T x_N u\, P_\nu\, dS_x - 2 \int_T x_N u\, P\, dS_x,\end{aligned}$$

since $u_\nu = u$ on T and $u = 0$ and $|\nabla u| = u_\nu$ on Σ. Here, the second summand at the right-hand side is obtained by applying (2.4). All in all, we have that

$$\int_\Omega x_N u\, \Delta P\, dx = -\frac{1}{2}\int_\Sigma u_\nu^2 \left[x_N u_\nu - \langle X^q, \nu\rangle\right] dS_x$$
$$+2\int_\Omega u_{x_N} P\, dx + \int_T x_N u\, P_\nu\, dS_x - 2\int_T x_N u\, P\, dS_x,$$

and hence we are left to prove that the last three integrals sum up to zero.

The integral on Ω can be treated by integrating on Ω the differential identity:

$$\operatorname{div}\left\{\left[(2\,u\,P + u^2)\,I - u^2\nabla^2 u\right] e_N\right\} = 2\,u_{x_N} P.$$

Here, I denotes the $N\times N$ identity matrix. In this calculation, we have used the first identity in (2.2). Thus, by the definition of P and divergence theorem, we get:

$$2\int_\Omega u_{x_N} P\, dx = \int_T \left\{x_N\left[u\,|\nabla u|^2 - u^2\right] - u^2\,\langle\nabla^2 u\, e_N, \nu\rangle\right\} dS_x. \tag{2.7}$$

Again, we used that $u = 0$ on Σ and $\langle e_N, \nu(x)\rangle = x_N$ for $x \in T$.

Next, we directly compute on T that

$$P_\nu = \langle\nabla^2 u\,\nabla u, \nu\rangle - u_\nu = \langle\nabla^2 u\,(u_\nu\nu + \omega), \nu\rangle - u_\nu = u\,\langle\nabla^2 u\,\nu, \nu\rangle - u.$$

In the first equality, we have decomposed ∇u into the sum of its normal and tangential components $u_\nu\,\nu$ and ω. In the second equality, we used the third identity in (2.2) and that $u_\nu = u$ on T. Moreover, we observe that on T it holds that

$$\langle\nabla^2 u\, e_N, \nu\rangle = \langle\nabla^2 u\,\left(x_N\,\nu - X^q\right), \nu\rangle = x_N\,\langle\nabla^2 u\,\nu, \nu\rangle,$$

by the second identity in (2.1) (being as $\nu(x) = x$ on S) and the third identity in (2.2) (being as X^q tangent to T).

Therefore, with this and the identity for P_ν in mind, we finally conclude that

$$2\int_\Omega u_{x_N} P\, dx + \int_T x_N u\, P_\nu\, dS_x - 2\int_T x_N u\, P\, dS_x = 0,$$

thanks to (2.7). This was what we were left to prove. □

A convenient choice of the constat c in (2.5) is suggested by the following proposition.

Proposition 2.4 (The value of R) *Let $u \in W^{1,\infty}(\Omega)\cap W^{2,2}(\Omega)$ be the solution of* (1.1). *If $u_\nu = R$ on Σ, then we have that*

$$R = N\,\frac{\int_\Omega x_N\, dx}{\int_\Sigma x_N\, dS_x} = \frac{N\,|\Omega|}{|\Sigma|}\,\frac{c_N^\Omega}{c_N^\Sigma}, \tag{2.8}$$

where c_N^E denotes the N-th coordinate of the center of mass of a set E.

Proof By using the divergence theorem and (2.6), we compute that

$$N\int_\Omega x_N\, dx = \int_\Omega \operatorname{div}(X^u)\, dx = \int_\Gamma \langle X^u, \nu\rangle\, dS_x = \int_\Sigma x_N\, u_\nu\, dS_x = R\int_\Sigma x_N\, dS_x.$$

Thus, (2.8) follows at once. □

As a consequence of this proposition and Theorem 2.3, we obtain a more general version of Guo and Xia's rigidity result.

Corollary 2.5 *Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ be the solution of* (1.1).

If the right-hand side of (2.5) *is non-positive for some $c \in \mathbb{R}$, then*

$$u(x) = \frac{1}{2}\left(|x-z|^2 - R^2\right) \;\; for \;\; x \in \Omega$$

and Σ must be the spherical cap $\{x \in B_+ : |x-z| = R\}$, where R is given by (2.8), *and $z = (z', z_N)$ is such that $|z| = \sqrt{1+R^2}$ and $|z'| \le 1$. The same conclusion holds true, in particular, if u_ν is constant on Σ.*

Proof By Theorem 2.3, our assumption clearly gives that the volume integral at the left-hand side of (2.5) must be zero. Since $u < 0$ in Ω by [8, Proposition 2.3] and $x_N > 0$ in $\Omega \subset B_+$, we infer that

$$0 \equiv |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} = |\nabla^2 u|^2 - \frac{\langle \nabla^2 u, I\rangle^2}{N} \;\; \text{in} \;\; \Omega.$$

Thus, the Cauchy–Schwarz inequality for the N^2-vectors $\nabla^2 u$ and I holds with the sign of equality. As already observed in [15], we have that u must be a quadratic polynomial of the form:

$$u(x) = \frac{1}{2}\left(|x-z|^2 - q_0\right) \;\; \text{for some} \;\; q_0 \in \mathbb{R}.$$

Since $u = 0$ on Σ, we infer that $q_0 > 0$ and Σ must equal $\{x \in B_+ : |x-z| = \sqrt{q_0}\}$ — a spherical cap.

We now determine q_0 and z. On one hand, observe that

$$\begin{aligned} N \int_\Omega x_N \, dx &= \int_\Omega \operatorname{div}(X^u)\, dx = \int_\Gamma \langle X^u, \nu\rangle \, dS_x \\ &= \int_\Sigma x_N \, u_\nu \, dS_x = \int_\Sigma x_N \, |x-z| \, dS_x = \sqrt{q_0} \int_\Sigma x_N \, dS_x, \end{aligned}$$

i.e. we have that $q_0 = R^2$. In particular, we infer that $u_\nu = R$ on Σ. On the other hand, for $x \in T$, we must have that

$$0 = u_\nu(x) - u(x) = \langle x - z, \nu\rangle - \frac{1}{2}\left(|x-z|^2 - q_0\right) = \frac{1}{2}\left(1 + q_0 - |z|^2\right),$$

being as $\nu(x) = x$ for $x \in T$. Hence, $|z| = \sqrt{1+q_0} = \sqrt{1+R^2}$. Finally, we have that $|z'| \le 1$, since $\overline{T}$ is required to be contained in the upper hemisphere of ∂B_+.

If u_ν is constant on Σ, then Proposition 2.4 tells us that the constant must equal the number R in (2.8). Choosing $c = R$ gives the the right-hand side of (2.5) is zero. $\square$

Remark 2.6 It is just an exercise to check that any spherical cap of the form specified in the corollary meets T orthogonally.

3 Weighted Sobolev-type bounds

In this section, we collect some notations, definitions, and preliminary lemmas. We will provide the proofs only when they are not available in the literature.

Given $\theta \in (0, \pi/2]$ and $a > 0$, we say that a set E satisfies the (θ, a)-*uniform interior cone condition*, if for every $x \in \partial E$ there is a unit vector $\omega = \omega_x$ such that the cone with vertex at the origin, axis ω, opening width θ, and height a defined by

$$\mathcal{C}_\omega = \{y \, : \, \langle y, \omega \rangle > |y| \cos(\theta), \; |y| < a\}$$

is such that

$$w + \mathcal{C}_\omega \subset E \;\text{ for every }\; w \in B_a(x) \cap \overline{E}. \tag{3.1}$$

Such a condition is equivalent to Lipschitz-regularity of the domain; more precisely, it is equivalent to the strong local Lipschitz property of Adams [1, Pag 66] and to the uniform Lipschitz regularity in [5, Section III] and [22, Definition 2.1].

In the sequel, we shall always consider a domain $\Omega \subset B_+$ that satisfies this cone condition. We then denote by $C^{0,1}(\overline{\Omega})$ the class of Lipschitz continuous functions on $\overline{\Omega}$. If $u \in C^{0,1}(\overline{\Omega})$, we set L to be the Lipschitz constant of u in $\overline{\Omega}$, i.e.

$$L = [u]_{C^{0,1}(\overline{\Omega})} = \sup \left\{ \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} \; : \; x_1, x_2 \in \overline{\Omega}, \; x_1 \neq x_2 \right\}. \tag{3.2}$$

The Hardy-Poincaré-type inequalities in the lemma and corollary below are adapted from [19, Section 3.2] and [16, Lemma 2.1] and can be deduced by the works of Bojarski [3] and Hurri-Syrjänen [9, 10]. For a domain $E \subset \mathbb{R}^N$, we denote by d_E its diameter.

Lemma 3.1 *Let $E \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, a)-uniform interior cone condition.*

Consider three numbers r, p, α such that, either

$$1 \le p \le r \le \frac{N\,p}{N - p\,(1 - \alpha)}, \quad p\,(1 - \alpha) < N, \quad 0 \le \alpha \le 1, \tag{3.3}$$

or

$$r = p \in [1, \infty)\,, \quad \alpha = 0. \tag{3.4}$$

Then, there exists a positive constant, $c = c(N, r, p, \alpha, \theta, a, d_E)$ such that

$$\|f - f_E\|_{r,E} \le c\, \|\delta_{\partial E}^{\alpha}\, \nabla f\|_{p,E}, \tag{3.5}$$

for every function $f \in L^1_{loc}(E)$ such that $\delta_{\partial E}^{\alpha}\, \nabla f \in L^p(E)$. Here, f_E denotes the mean value of f on E.

If $E \subset B_+$, the dependence of c on d_E can be removed, being as $d_E \le 2$.

Corollary 3.2 *Let $E \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain satisfying the (θ, a)-uniform interior cone condition and let f be a function such that $\nabla f \in L^1_{loc}(E)$ and $\delta_{\partial E}^{\alpha}\, \nabla^2 f \in L^p(E)$. Consider three numbers r, p, α satisfying either (3.3) or (3.4). If*

$$\int_E \nabla f \, dx = 0,$$

then it holds that

$$\|\nabla f\|_{r,E} \le c\, \|\delta_{\partial E}^{\alpha}\, \nabla^2 f\|_{p,E},$$

where c is the same constant appearing in (3.5).

Remark 3.3 (On the proof of Lemma 3.1 and Corollary 3.2) Lemma 3.1 and Corollary 3.2 hold true in the more general case where E is a John domain: we refer the reader [16, proof of item(i) of Lemma 2.1 and item (i) of Corollary 2.3]) for details. Roughly speaking, a domain is a b-John domain if it is possible to travel from one point of the domain to another without going too close to the boundary (see Section A for the precise definition). The class of John domains contains Lipschitz domains but also very irregular domains with fractal boundaries as, e.g., the Koch snowflake.

For b-John domains, (see [16, items (i),(ii) of Remark 2.4]), the following explicit bounds for the constant c hold true:

$$c \le k_{N,r,p,\alpha}\, b^N |E|^{\frac{1-\alpha}{N}+\frac{1}{r}-\frac{1}{p}}, \quad \text{if } r, p, \alpha \text{ are as in (3.3)},$$

$$c \le k_{N,p}\, b^{3N(1+\frac{N}{p})}\, d_E, \quad \text{if } r, p, \alpha \text{ are as in (3.4)}.$$

Of course, the volume appearing in the first inequality can be easily estimated by means of $|E| \le |B|\, d_E^N$. Moreover, as we show in Lemma A.2, if a domain E satisfies the (θ, a)-uniform interior cone condition, then it is a b-John domain and b can be explicitly estimated in terms of a, θ, d_E only.

We thus obtain that (3.5) holds true with some constant c that depends only on $N, r, p, \alpha, \theta, a, d_E$. If $E \subset B_+$, the dependence on d_E can be removed, being as $d_E \le 2$.

We conclude this section by providing an adaptation of [18, Theorems 2.4 and 2.7] (see also the errata corrige in Sect. A.2). We warn the reader that in [18] we adopted a different normalization in the definition of the L^p-type norms.

Lemma 3.4 *Let $1 \le p < q \le \infty$. Let $E \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, a)-uniform interior cone condition.*

(i) If $p > N$, then there is a non-negative constant $c = c(N, p, \theta, a, d_E)$ such that

$$\max_{\overline{E}} f - \min_{\overline{E}} f \le c\, \|\nabla f\|_{p,E},$$

for any $f \in W^{1,q}(E)$.

(ii) If $1 \le p \le N$ and

$$\alpha_{p,q} = \frac{p\,(q-N)}{N\,(q-p)},$$

then there is a non-negative constant $c = c(N, p, q, \theta, a, d_E)$ such that

$$\max_{\overline{E}} f - \min_{\overline{E}} f \le c \begin{cases} \|\nabla f\|_{p,E}^{\alpha_{p,q}} \|\nabla f\|_{q,E}^{1-\alpha_{p,q}} & \text{if } 1 \le p < N, \\ \|\nabla f\|_{N,E} \log\left(e\, |E|^{\frac{1}{N}-\frac{1}{q}} \frac{\|\nabla f\|_{q,E}}{\|\nabla f\|_{N,E}} \right) & \text{if } p = N, \end{cases}$$

for any $f \in W^{1,q}(E)$.

Explicit bounds for the constants c can be computed.

If $E \subset B_+$, the dependence of the constants c on d_E can be removed, being as $d_E \le 2$.

Remark 3.5 For sub-harmonic functions, a similar estimate in the case where $1 \le p < N$ and $q = \infty$ can also be obtained by putting together [20, Lemma 3.14] with the Hardy-Poincarè-type inequalities mentioned in Lemma 3.1. See also [17, Theorems 3.1 and 3.2] for adaptations to either domains satisfying a weaker cone-type condition or John-type domains.

4 Quantitative stability results

In this section, we shall give the proofs of Theorems 1.1–1.3 and of the more general Theorem 4.9 below. We begin by recalling some notations and other facts from the Introduction.

4.1 Some geometrical facts

Lemma 4.2 below is an adaptation of [15, Lemma 3.1] to the case of the mixed problem (1.1), which takes inspiration from [21, Section 4.1]. We also mention that a fractional version of [15, Lemma 3.1] has been recently used in [6].

We will make use of the following maximum principle for mixed boundary value problems in B_+, which is a reformulation of [8, Proposition 2.3].

Lemma 4.1 *Let $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \Lambda)$ satisfy*

$$\Delta f \geq 0 \text{ in } \Omega, \quad f \leq 0 \text{ on } \overline{\Sigma}, \quad f_\nu \leq f \text{ on } T,$$

and assume that $f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then, we have that $f \leq 0$ on $\overline{\Omega}$.

Proof Set f_+ to be the positive part of f. By using the boundary conditions on f, an integration by parts, and the inequality $\Delta f \geq 0$ in Ω, we find that

$$\int_\Omega |\nabla f_+|^2 \, dx - \int_T f_+^2 \, dS_x \leq \int_\Omega |\nabla f_+|^2 \, dx - \int_{T \cup \Sigma} f_+ f_\nu \, dS_x = - \int_\Omega f_+ \Delta f \, dx \leq 0.$$

On the other hand, [8, (2.5)] informs us that

$$0 \geq \int_\Omega |\nabla f_+|^2 \, dx - \int_T f_+^2 \, dS_x \geq \lambda_1 \int_\Omega f_+^2 \, dx \geq 0,$$

where λ_1 is the first Robin-Dirichlet eigenvalue. Hence, we easily infer that $f_+ \equiv 0$ in $\overline{\Omega}$. □

In the spirit of [21, Section 4.1], we now introduce some appropriate sphere conditions peculiar to B_+. In fact, we say that Ω satisfies the *r_i-uniform interior sphere condition relative to B_+*, if for each $x \in \overline{\Sigma}$ there exists a touching ball $B_{r_i}(x_0)$ of radius r_i such that: (i) its center x_0 satisfies $|x_0|^2 \leq 1 + r_i^2$ and (ii) its closure intersects $\overline{B}_+ \setminus \Omega$ only at x.

Notice that the requirements of this definition are related to how $\overline{\Sigma}$ and $\overline{T}$ intersect. In fact, a necessary condition for the validity of (i) and (ii) is that $\langle \nu_\Sigma(x), \nu_{\partial B}(x) \rangle \geq 0$ for $x \in \partial\Sigma$.

Since in our setting Σ is smooth, we must have that $x_0 = x - r_i \, \nu(x)$ for $x \in \Sigma$. However, notice that this may not be the only possibility for the points on $\partial\Sigma$.

We say that Ω satisfies the *strong r_i-uniform interior sphere condition relative to B_+* if, besides satisfying the r_i-uniform interior sphere condition relative to B_+, Ω has the property that, for any $x \in \overline{\Omega}$ such that its closest point $\underline{x}$ to $\overline{\Sigma}$ belongs to $\partial\Sigma$, the ball with radius r_i and centered at the point $\underline{x} + r_i \frac{x - \underline{x}}{|x - \underline{x}|}$ is a touching ball at $\underline{x}$ relative to B_+ (as in the previous definition). We notice that this condition is surely satisfied if $\overline{\Sigma}$ and $\overline{T}$ intersect orthogonally.

Here and in the sequel, $\delta_A(x)$ will denote the distance of a point $x \in \mathbb{R}^N$ to a set A.

Lemma 4.2 (A geometric bound) *Let u be the solution of (1.1). Then*

$$-u(x) \geq \frac{1}{2} \, \delta_\Sigma(x)^2 \text{ for every } x \in \overline{\Omega}. \tag{4.1}$$

Moreover, if Ω satisfies the strong r_i-uniform interior sphere condition relative to B_+, then it holds that

$$-u(x) \geq \frac{r_i}{2} \, \delta_\Sigma(x) \text{ for every } x \in \overline{\Omega}. \tag{4.2}$$

Proof From Lemma 4.1, we know that $u \leq 0$ in $\overline{\Omega}$.

Fix $x \in \overline{\Omega} \setminus \overline{\Sigma}$, let $r = \delta_\Sigma(x)$, and consider the ball $B_r(x)$ with radius r centered at x. It is easy to check that the function defined by $w(y) = (|y-x|^2 - r^2)/2$ for $y \in \overline{B}_r(x)$ satisfies

$$\Delta w = N \ \text{ in } \ B_r(x), \quad w = 0 \ \text{ on } \ \partial B_r(x),$$

and

$$w_\nu \geq w \ \text{ on } T \cap B_r(x).$$

The last inequality follows from the direct computations

$$w = \frac{1 + |x|^2 - r^2}{2} - \langle x, y \rangle \quad \text{and} \quad w_\nu = 1 - \langle x, y \rangle, \text{ for } y \in T \cap B_r(x) \subset \partial B,$$

and the trivial inequality $r^2 \geq 0 \geq |x|^2 - 1$, which holds true for any $x \in \overline{B} \supset \overline{\Omega}$.

If we choose $f = u - w$ and $\Omega = B_r(x) \cap B_+$ in Lemma 4.1, we then have that $w \geq u$ in $\overline{B_r(x) \cap B_+}$ and hence, in particular, $-r^2/2 = w(x) \geq u(x)$. Thus, (4.1) is proved.

Next, assume that Ω satisfies the strong r_i-uniform interior sphere condition relative to B_+. If $\delta_\Sigma(x) \geq r_i$, (4.2) immediately follows from (4.1). If $\delta_\Sigma(x) < r_i$, instead, let $\underline{x}$ be the closest point in $\overline{\Sigma}$ to x and call B_{r_i} the relevant touching ball at $\underline{x} \in \overline{\Gamma}_0$ (with radius r_i and centered at the point $\underline{x} + r_i \frac{x - \underline{x}}{|x - \underline{x}|}$ that we denote by x_0), which contains x. Setting $w(y) = \left(|y - x_0|^2 - r_i^2\right)/2$ and using that the center x_0 of the touching ball satisfies $|x_0|^2 \leq 1 + r_i^2$, we infer that $w_\nu \geq w$ on $T \cap B_{r_i}$. Hence, an application of Lemma 4.1 with $f = u - w$ and $\Omega = B_{r_i} \cap B_+$ gives that $w \geq u$ in $\overline{B_{r_i} \cap B_+}$. As a consequence, being as $x \in B_{r_i} \cap \overline{B_+}$, we obtain that

$$-u(x) \geq \frac{r_i^2 - |x - x_0|^2}{2} = \frac{(r_i + |x - x_0|)(r_i - |x - x_0|)}{2} \geq \frac{r_i}{2}\,(r_i - |x - x_0|).$$

This gives (4.2), since $r_i - |x - x_0| = \delta_\Sigma(x)$. □

Remark 4.3 (i) Notice that the r_i-uniform interior sphere condition relative to B_+ guarantees the validity of the Hopf lemma for u_ν on Σ. The additional "strong" assumption is needed to obtain the Lipschitz growth of u from Σ, i.e., (4.2).

(ii) In the classical setting (where B_+ is replaced by $\mathbb{R}^N$), the improved growth in (4.2) remains true in the more general case in which Ω satisfies an interior pseudoball condition (see [4, Step 2 in the proof of Theorem I] and [2, Theorem 4.4]). In this regard, one may introduce a notion of pseudoball condition relative to B_+.

Let u be the solution of (1.1). We consider the harmonic function

$$h(x) = Q(x) - u(x), \quad x \in \Omega, \tag{4.3}$$

where

$$Q(x) = \frac{1}{2}\,|x - z|^2. \tag{4.4}$$

and $z \in \mathbb{R}^N$ is some point to be chosen. As the next lemma shows, h has to do with the numbers

$$\rho_e = \max_{x \in \overline{\Sigma}} |x - z| \quad \rho_i = \min_{x \in \overline{\Sigma}} |x - z|. \tag{4.5}$$

Note that we have:

$$\overline{\Sigma} \subseteq \left(\overline{B}_{\rho_e}(z) \setminus B_{\rho_i}(z)\right) \cap \overline{B}.$$

Lemma 4.4 *Fix $z \in \mathbb{R}^N$. Then, we have that*

$$|\nabla^2 h|^2 = |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \ \ in \ \ \Omega.$$

Moreover, it holds that

$$\rho_e - \rho_i \le \frac{8}{d_\Omega} \left(\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \right). \tag{4.6}$$

Proof The identity easily follows by direct computation.

Next, let $x_1, x_2 \in \overline{\Omega}$ be such that $d_\Omega = |x_1 - x_2|$. It is clear that $x_1, x_2 \in \overline{\Sigma} \cup T$. If $x_1, x_2 \in \overline{\Sigma}$, then $d_\Omega \le d_\Sigma$. If $x_1, x_2 \in \overline{T}$, then $d_\Omega \le d_T \le d_{\overline{\Sigma} \cap \overline{T}} \le d_\Sigma$. In fact, the second inequality follows from the fact that T is a spherical cap contained in a half sphere. If $x_1 \in \overline{\Sigma}$ and $x_2 \in T$ (or the other way around), we take $y \in \Lambda = \overline{\Sigma} \cap \overline{T}$ and infer that

$$d_\Omega = |x_1 - x_2| \le |x_1 - y| + |y - x_2| \le d_\Sigma + d_T \le 2\, d_\Sigma.$$

Thus, in any case we have that $d_\Omega \le 2\, d_\Sigma$. Now, if $x_1, x_2 \in \overline{\Sigma}$ are such that $d_\Sigma = |x_1 - x_2|$, we easily see that $d_\Sigma = |x_1 - x_2| \le |x_1 - z| + |z - x_2| \le 2\, \rho_e$, so that $d_\Omega \le 4\, \rho_e$. Using the last inequality together with

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h = \frac{1}{2}(\rho_e^2 - \rho_i^2) = \frac{1}{2}\,(\rho_e + \rho_i)(\rho_e - \rho_i) \ge \frac{1}{2}\,\rho_e\,(\rho_e - \rho_i),$$

(4.6) easily follows. □

4.2 Special stability estimates

In this section, we shall give the proof of Theorems 1.1–1.3. To this aim, we must work on the fundamental identity (1.5). We shall see that its right-hand side can be easily estimated in terms of the deviation of $\|u_\nu^2 - R^2\|_{1,\Sigma}$. Thanks to Lemma 4.2, the left-hand side of (1.5), instead, can be bounded from below by the following weighted L^2-norm:

$$\|\delta_\Sigma^\tau \nabla^2 h\|_{2,\Omega}. \tag{4.7}$$

The appropriate exponent τ will be chosen as $\tau = 1$ in Theorems 1.1 and 1.2, $\tau = 1/2$ in Theorem 1.3, and $\tau = 3/2$ in Theorem 4.9 below. The final stability estimates will then result from a bound of $\rho_e - \rho_i$ in terms of those relevant weighted norms. This task will be achieved by means of Lemma 3.4.

Thus, we begin with the following lemma.

Lemma 4.5 (Weighted bounds for the Hessian matrix of h) *Take $N \ge 2$. Let $\Omega \subset \mathbb{R}^N$ be a subdomain of B_+ and define the number*

$$m = \min\{x_N : x \in \overline{\Omega}\}. \tag{4.8}$$

Let u be the solution of (1.1) with Lipschitz constant L be as in (3.2).

For any choice of $z \in \mathbb{R}^N$, let h be the function defined in (4.3). The following statements hold true.

(i) If Ω satisfies the (θ, a)-uniform interior cone condition, then we have that

$$\|\delta_\Sigma^{3/2} \nabla^2 h\|_{2,\Omega}^2 \le (L + 2)\, \|u_\nu^2 - R^2\|_{1,\Sigma}$$

and, if the number m in (4.8) is positive,

$$\|\delta_\Sigma \nabla^2 h\|_{2,\Omega}^2 \le \frac{L+2}{m} \|u_\nu^2 - R^2\|_{1,\Sigma}.$$

(ii) If Ω satisfies the strong r_i-uniform interior sphere condition relative to B_+, then we have that

$$\|\delta_\Sigma \nabla^2 h\|_{2,\Omega}^2 \le \frac{L+2}{r_i} \|u_\nu^2 - R^2\|_{1,\Sigma}$$

and, if the number m in (4.8) is positive,

$$\|\delta_\Sigma^{\frac{1}{2}} \nabla^2 h\|_{2,\Omega}^2 \le \frac{L+2}{m\, r_i} \|u_\nu^2 - R^2\|_{1,\Sigma}.$$

Proof In view of (3.2), we have that $0 < u_\nu \le L$ on Σ. Thus, being as $0 \le x_N \le 1$ for $x \in B_+$, we have that

$$|u_\nu x_N - \langle X^q, \nu\rangle| \le L + 2 \quad \text{on } \Sigma,$$

and hence, from (1.5) and Lemma 4.4, we get:

$$\int_\Omega x_N(-u)|\nabla^2 h|^2 dx \le \frac{L+2}{2} \|u_\nu^2 - R^2\|_{1,\Sigma} \tag{4.9}$$

(i) Notice that

$$x_N \ge \delta_\Sigma(x) \quad \text{for any } x \in \Omega \subset B_+.$$

By this inequality, (4.9), and (4.1), the first desired inequality easily follows. Also, the second desired inequality easily ensues by putting together (4.9), (4.1), and the fact that $m > 0$.

(ii) Since Ω satisfies the strong r_i-uniform interior sphere condition relative to B_+, (4.2) holds true. Thus, we have that

$$-x_N\, u(x) \ge \frac{r_i}{2}\, \delta_\Sigma(x)^2.$$

This bound, together with (4.9) leads to the first desired inequality. Next, by using (4.2) and the fact that $m > 0$, we deduce that

$$-x_N\, u(x) \ge \frac{m\, r_i}{2}\, \delta_\Sigma(x).$$

Inserting this inequality into (4.9) gives the second desired inequality. □

Notice that, as already mentioned, the proofs of Theorems 1.1–1.3 will only entail the cases in this lemma where $0 < \tau \le 1$. The desired conclusions will in fact be obtained by adapting to the present setting the arguments developed by the authors in [16, 18, 20]. The case where $\tau = 3/2$ will instead be used for the proof of the more general result contained in Theorem 4.9, which requires a new and careful analysis.

The proofs of Theorems 1.1–1.3 will result from Theorem 4.6 below. In order to proceed, we recall from the introduction the convenient choice of z:

$$z = \frac{1}{|\Omega|} \left\{ \int_\Omega x\, dx - \int_T u(x)\, x\, dx \right\}.$$

Notice that, with this choice, we have that

$$\int_\Omega \nabla h\, dx = \int_\Omega (x - z)\, dx - \int_\Omega \nabla u\, dx = \int_\Omega x\, dx - z\, |\Omega| - \int_T u(x)\, x\, dx = 0.$$

This ensures that Corollary 3.2 can be applied with $v = h$, $E = \Omega$.

Theorem 4.6 *Let $N \geq 2$ and let $\Omega \subset B_+$ be a domain satisfying the (θ, a)-uniform interior cone condition.*

Let u be solution of (1.1) and consider the function $h = Q - u$, where Q is given by (4.4) with z as in (1.7). Then, the following statements hold true.

(i) *There is some non-negative constant $c = c(N, \tau, \theta, a)$ such that*

$$\rho_e - \rho_i \leq c \begin{cases} \|\delta_\Gamma^\tau \nabla^2 h\|_{2,\Omega} & \text{if } 0 < \tau < 2 - \frac{N}{2}, \\ \|\nabla h\|_{\infty,\Omega}^{1-\kappa_{N,\tau}} \|\delta_\Gamma^\tau \nabla^2 h\|_{2,\Omega}^{\kappa_{N,\tau}} & \text{if } 2 - \frac{N}{2} < \tau \leq 1, \end{cases} \tag{4.10}$$

where

$$\kappa_{N,\tau} = \frac{1}{\tau + N/2 - 1}.$$

(ii) *There is some non-negative constant $c = c(N, \theta, a)$ such that*

$$\rho_e - \rho_i \leq c \, \|\delta_\Gamma^\tau \nabla^2 h\|_{2,\Omega} \max \left\{ \log \left(\frac{\|\nabla h\|_{\infty,\Omega}}{\|\delta_\Gamma^\tau \nabla^2 h\|_{2,\Omega}} \right), 1 \right\} \tag{4.11}$$

with $\tau = 2 - N/2$.

Proof In both items, we will use at some point the trivial inequality

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \leq \max_{\overline{\Omega}} h - \min_{\overline{\Omega}} h.$$

(i) Let $2 - N/2 < \tau \leq 1$. By using item (ii) of Lemma 3.4 with $E = \Omega$, $f = h$, $p = N\kappa_{N,\tau} < N$, and $q = \infty$, we find a constant $c = c(N, \tau, \theta, a)$ such that

$$\max_{\overline{\Omega}} h - \min_{\overline{\Omega}} h \leq c \, \|\nabla h\|_{N\kappa_{N,\tau},\Omega}^{\kappa_{N,\tau}} \|\nabla h\|_{\infty,\Omega}^{1-\kappa_{N,\tau}}.$$

By using (4.6) and the trivial inequality, we thus find a constant $c = c(N, \tau, \theta, a)$ such that

$$\rho_e - \rho_i \leq c \, \|\nabla h\|_{N\kappa_{N,\tau},\Omega}^{\kappa_{N,\tau}} \|\nabla h\|_{\infty,\Omega}^{1-\kappa_{N,\tau}}.$$

We point out that in (4.6) $8/d_\Omega$ can be replaced by $8/a$, since Ω contains at least a cone of height a.

By applying Corollary 3.2 with $E = \Omega$, $f = h$, $r = N\kappa_{N,\tau}$, $p = 2$, $\alpha = \tau$, the second inequality in (i) easily follows.

Next, let $\tau < 2 - N/2$. By the Sobolev immersion theorem (here, we are indeed applying item (i) of Lemma 3.4 with $E = \Omega$, $f = h$ and $p = N\kappa_{N,\tau} > N$), we can find a constant $c = c(N, \tau, \theta, a)$ such that

$$\max_{\overline{\Omega}} h - \min_{\overline{\Omega}} h \leq c \, \|\nabla h\|_{N\kappa_{N,\tau},\Omega}.$$

By again using (4.6) and the trivial inequality, we thus infer that

$$\rho_e - \rho_i \leq c \, \|\nabla h\|_{N\kappa_{N,\tau},\Omega},$$

by possibly changing the relevant constant. Hence, the first inequality in (i) follows by using Corollary 3.2 with $E = \Omega$, $f = h$, $r = N\kappa_{N,\tau}$, $p = 2$, $\alpha = \tau$.

(iii) Let $\tau = 2 - N/2$. By using (4.6), the trivial inequality, and item (ii) of Lemma 3.4 with $E = \Omega$, $f = h$, $p = N = 4 - 2\tau$ and $q = \infty$, we find a constant $c = c(N, \theta, a)$ such that

$$\rho_e - \rho_i \le c \, \|\nabla h\|_{N,\Omega} \max \left\{ \log \left(\frac{\|\nabla h\|_{\infty,\Omega}}{\|\nabla h\|_{N,\Omega}} \right), 1 \right\}.$$

The inequality in (ii) then ensues by applying Corollary 3.2 with $E = \Omega$, $f = h$, $r = N$, $p = 2$, and $\alpha = \tau$. □

Remark 4.7 (An explicit bound for $\|\nabla h\|_{\infty,\Omega}$) With the choice (1.7), we can easily obtain the following explicit bound:

$$\|\nabla h\|_{\infty,\Omega} \le 2(L+1).$$

where L is that defined in (3.2).

In fact, we have that

$$|\nabla h(x)| \le |\nabla u(x)| + |x - z| \le L + |x - z| \quad \text{for } x \in \overline{\Omega}. \tag{4.12}$$

Moreover, we see that

$$\begin{aligned} |x - z| &\le \left| x - \frac{1}{|\Omega|} \int_\Omega y \, dy + \int_T u(y) \, y \, dS_y \right| \\ &\le \frac{1}{|\Omega|} \int_\Omega |x - y| \, dy + \frac{1}{|\Omega|} \int_\Omega |\nabla u(y)| \, dy \le d_\Omega + L \le 2 + L, \end{aligned}$$

for $x \in \overline{\Omega}$. In the second inequality, we used that

$$\int_T u(y) \, y \, dS_y = \int_\Omega \nabla u(y) \, dy,$$

by the divergence theorem.

We are now ready for the proofs of Theorems 1.1, 1.2, 1.3.

Proof of Theorem 1.1 The conclusion easily follows by combining the second inequality in item (i) of Lemma 4.5, Theorem 4.6 (with $\tau = 1$), the trivial inequality

$$\delta_\Gamma(x) \le \delta_\Sigma(x) \quad \text{for } x \in \overline{\Omega}, \tag{4.13}$$

and Remark 4.7. □

Remark 4.8 Taking into account Remark 3.5, Theorem 1.1 may be extended to the case where the uniform interior cone condition is dropped and replaced by weaker either cone-type or John-type conditions.

Proof of Theorem 1.2 The desired estimate easily follows by combining the first inequality in item (ii) of Lemma 4.5, Theorem 4.6 (with $\tau = 1$), (4.13), and Remark 4.7. □

Proof of Theorem 1.3 The desired estimate easily follows by using the second inequality in item (ii) of Lemma 4.5, Theorem 4.6 (with $\tau = 1/2$), (4.13), and Remark 4.7. □

4.3 The general stability estimate

In this section, we shall state and prove a stability estimate for general domains satisfying the (θ, a)-uniform interior cone condition. Compared to those proved in Sect. 4.2, in this case the stability rates are slightly poorer, as the following theorem shows.

Theorem 4.9 (General stability) *Set $N \geq 2$ and let Ω be a domain contained in B_+ and satisfying the (θ, a)-uniform interior cone condition.*

Let $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ be the solution of (1.1) and assume that L is a bound for $[u]_{C^{0,1}(\overline{\Omega})}$. Let R be the number defined in (1.4) and set

$$\rho(\Omega) = \inf_{z \in \mathbb{R}^N} (\rho_e - \rho_i), \quad \text{with } \rho_e \text{ and } \rho_i \text{ as in (4.5).}$$

Then, the following estimates hold true.

(i) If $N \geq 3$,

$$\rho(\Omega) \leq c \, \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/(N+1)},$$

for some non-negative constant $c = c(N, \theta, a, L)$.

(ii) If $N = 2$, for any $0 < \eta < 1$,

$$\rho(\Omega) \leq c \, \|u_\nu^2 - R^2\|_{1,\Sigma}^{1/(3+2\eta)},$$

for some non-negative constant $c = c(\theta, a, L, \eta)$.

Notice that, in order to prove the special stability estimate in the previous section, we used Theorem 4.6. Here, we stress that, since its proof is based on Lemma 3.4 and Corollary 3.2, the relevant exponent τ had to be chosen in $[0, 1]$. This fact allowed us to treat the cases of Lemma 4.5 with $\tau = 1/2$ or 1.

However, if we want to treat the general case, we must choose $\tau = 3/2$, as it is clear from Lemma 4.5. Thus, Theorem 4.6 is no longer useful and we must come up with another strategy. The key idea is to obtain inequalities similar to those in Lemma 3.4, but restricting the L^p-norms (appearing on the right-hand sides) to a suitable subset sufficiently far from the boundary.

To this aim, for $\sigma \geq 0$, we define the *parallel set*

$$\Omega_\sigma = \{x \in \Omega : \delta_\Gamma(x) > \sigma\}, \tag{4.14}$$

where Γ denotes the boundary of Ω. Being as Ω a bounded domain (i.e., open and connected) satisfying the (θ, a)-uniform interior cone condition, by Lemma A.1 below, we know that there exists a positive constant $\delta_0 = \delta_0(\theta, a, d_\Omega)$ such that Ω_σ is connected for any $0 \leq \sigma \leq \delta_0$. We now set

$$\sigma_0 = \min\left\{\frac{a}{2}\frac{\sin\theta}{1+\sin\theta}, \delta_0\right\}. \tag{4.15}$$

Notice that for $0 \leq \sigma \leq \sigma_0$, besides being connected, the domain Ω_σ also satisfies the $(\theta, a/2)$-uniform interior cone condition. The second assertion follows from Lemma A.3 noting that $\sigma_0 \leq \frac{a}{2}\frac{\sin\theta}{1+\sin\theta} \leq \frac{a}{4}$. This ensures that Lemma 3.1 and Corollary 3.2 can be applied with $E = \Omega_\sigma$.

The following lemma will be useful in the sequel.

Lemma 4.10 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain satisfying the (θ, a)-uniform interior cone condition. Consider the parallel set Ω_σ for $0 < \sigma \leq \sigma_0$, where σ_0 is that given in (4.15).*

If $1 < p < N$, we have that

$$\max_\Gamma v - \min_\Gamma v \leq c \left\{ \sigma^{1-\frac{N}{p}} \|\nabla v\|_{p,\Omega_\sigma} + [v]_{C^{0,1}(\overline{\Omega})}\, \sigma \right\},$$

for any function $v \in C^{0,1}(\overline{\Omega})$ subharmonic in Ω_σ and some positive constant c depending on $N, p, \theta, a, d_\Omega$.

Proof Let x_1 and x_2 be points on Γ that respectively minimize and maximize v on Γ. For $j = 1, 2$, define the point $y_j = x_j + \frac{2\sigma}{\sin\theta}\, \omega_j$, where ω_j is the axis of a cone $\mathcal{C}_j \subset \Omega$ with vertex at x_j, height a, and opening width θ. Since $\frac{2\sigma}{\sin\theta} \leq \frac{a}{1+\sin\theta}$ (being as $\sigma \leq \sigma_0$), by trigonometry we have that the ball $B_{2\sigma}(y_j)$ is contained in $\mathcal{C}_j \subset \Omega$. Hence, the ball $B_\sigma(y_j)$ is contained in Ω_σ.

Now, the sub-harmonicity of v gives that

$$\begin{aligned}
|v(y_j) - v_{\Omega_\sigma}| &\leq \frac{1}{|B|\,\sigma^N} \int_{B_\sigma(y_j)} |v - v_{\Omega_\sigma}|\, dy \\
&\leq \frac{1}{\left(|B|\,\sigma^N\right)^{1/q}} \left[\int_{B_\sigma(y_j)} |v - v_{\Omega_\sigma}|^q\, dy \right]^{1/q} \\
&\leq \frac{1}{\left(|B|\,\sigma^N\right)^{1/q}} \left[\int_{\Omega_\sigma} |v - v_{\Omega_\sigma}|^q\, dy \right]^{1/q},
\end{aligned}$$

for any $q > 1$, after an application of Hölder's inequality. Thus, by the definition of $[v]_{C^{0,1}(\overline{\Omega})}$, we can infer that

$$\begin{aligned}
|v(x_j) - v_{\Omega_\sigma}| &\leq |v(y_j) - v_{\Omega_\sigma}| + \frac{2\sigma}{\sin\theta}\, [v]_{C^{0,1}(\overline{\Omega})} \\
&\leq c \left\{ \sigma^{-N/q} \left[\int_{\Omega_\sigma} |v - v_{\Omega_\sigma}|^q\, dy \right]^{1/q} + [v]_{C^{0,1}(\overline{\Omega})}\, \sigma \right\},
\end{aligned}$$

for some constant $c = c(N, q, \theta)$. Therefore, by choosing $q = pN/(N-p)$ and applying (3.5) with $E = \Omega_\sigma, r = pN/(N-p), p = p, \alpha = 0$, we conclude that our desired inequality holds with an explicit constant $c = c(N, p, \theta, a, d_\Omega)$. □

Corollary 4.11 *Let Ω, σ, and Ω_σ be as in Lemma 4.10 and take $\tau \geq 1$. For any subharmonic function of class $C^{0,1}(\overline{\Omega})$ in Ω_σ such that*

$$\int_{\Omega_\sigma} \nabla v\, dx = 0,$$

we have the following.

(i) If $N \geq 3$, then

$$\max_\Gamma v - \min_\Gamma v \leq c \left\{ \sigma^{2-\frac{N}{2}-\tau} \|\delta_\Gamma^\tau \nabla^2 v\|_{2,\Omega} + [v]_{C^{0,1}(\overline{\Omega})}\, \sigma \right\},$$

for some positive constant $c = c(N, \tau, \theta, a, d_\Omega)$.

(ii) If $N = 2$, then for any $0 < \eta < 1$ we have that

$$\max_{\Gamma} v - \min_{\Gamma} v \le c \left\{ \sigma^{1-\eta-\tau} \| \delta_{\Gamma}^{\tau} \nabla^2 v \|_{2,\Omega} + [v]_{C^{0,1}(\overline{\Omega})} \, \sigma \right\},$$

for some positive constant $c = c(N, \tau, \theta, a, \eta, d_{\Omega})$.

Proof For convenience, we set $\Gamma_\sigma = \partial\Omega_\sigma$.

(i) Let $N \ge 3$. By putting together Lemma 4.10 with $p = 2$ and Corollary 3.2 with $E = \Omega_\sigma$, $f = v$, $r = 2$, $p = 2$, $\alpha = 1$, we find that

$$\begin{aligned} \max_{\Gamma} v - \min_{\Gamma} v &\le c \left\{ \sigma^{1-\frac{N}{2}} \| \delta_{\Gamma_\sigma} \nabla^2 v \|_{2,\Omega_\sigma} + [v]_{C^{0,1}(\overline{\Omega})} \, \sigma \right\} \\ &\le c \left\{ \sigma^{1-\frac{N}{2}} \| \delta_{\Gamma} \nabla^2 v \|_{2,\Omega_\sigma} + [v]_{C^{0,1}(\overline{\Omega})} \, \sigma \right\} \end{aligned}$$

for some constant $c = c(N, \theta, a, d_\Omega)$, being as $\delta_{\Gamma_\sigma}(x) \le \delta_\Gamma(x)$ for any $x \in \Omega_\sigma$. We can now exploit our construction to further increase the exponent of the distance in the first summand at the right-hand side of the last inequality.

In fact, the definition (4.14) of Ω_σ gives that

$$\delta_\Gamma \le \sigma^{1-\tau} \delta_\Gamma^\tau \ \text{ in } \ \Omega_\sigma,$$

and hence

$$\| \delta_\Gamma \nabla^2 v \|_{2,\Omega_\sigma} \le \sigma^{1-\tau} \| \delta_\Gamma^\tau \nabla^2 v \|_{2,\Omega_\sigma} \le \sigma^{1-\tau} \| \delta_\Gamma^\tau \nabla^2 v \|_{2,\Omega}.$$

This is just what was left to prove.

(ii) Let $N = 2$. By combining Lemma 4.10 with $p = 2/(1+\eta)$, the Hölder inequality

$$\| \nabla v \|_{2/(1+\eta),\Omega_\sigma} \le |\Omega_\sigma|^{\eta/2} \, \| \nabla v \|_{2,\Omega_\sigma},$$

and Corollary 3.2 with $E = \Omega_\sigma$, $f = v$, $r = 2$, $p = 2$, $\alpha = 1$, we find that

$$\begin{aligned} \max_{\Gamma} v - \min_{\Gamma} v &\le c \left\{ \sigma^{-\eta} \| \delta_{\Gamma_\sigma} \nabla^2 v \|_{2,\Omega_\sigma} + [v]_{C^{0,1}(\overline{\Omega})} \, \sigma \right\} \\ &\le c \left\{ \sigma^{-\eta} \| \delta_{\Gamma} \nabla^2 v \|_{2,\Omega_\sigma} + [v]_{C^{0,1}(\overline{\Omega})} \, \sigma \right\} \end{aligned}$$

for some positive constant $c = c(\theta, a, \eta, d_\Omega)$. Here, we also estimated the term $|\Omega_\sigma|$ appearing in the Hölder inequality above by means of $|\Omega_\sigma| \le |\Omega| \le |B| \, d_\Omega^N$. The first summand at the right-hand side of the last inequality can be estimated as in the proof of (i), and hence the desired result follows at once. □

Remark 4.12 If $\Omega \subset B_+$, then the dependence on d_Ω in the constants c in Lemma 4.10 and Corollary 4.11 can be removed, being as $d_\Omega \le 2$.

We are now ready to prove our general stability result. We are going to prove the stability result for $\rho_e - \rho_i$ with the choice

$$z = \frac{1}{|\Omega_\sigma|} \left\{ \int_{\Omega_\sigma} x \, dx - \int_{\Omega_\sigma} \nabla u \, dx \right\}, \tag{4.16}$$

for a given value of σ, as specified below in the proof. The result in the statement of Theorem 4.9 will follow noting that $\rho(\Omega) \le \rho_e - \rho_i$. With the choice of z in (4.16), the function h defined in (4.3)–(4.4) satisfies

$$\int_{\Omega_\sigma} \nabla h \, dx = 0,$$

and hence, Corollary 4.11 can be applied with $v = h$.

Proof of Theorem 4.9 Let $\sigma_0 = \sigma_0(\theta, a)$ be that defined in (4.15), where the dependence on d_Ω has been removed in light of Remark 4.12.

(i) Combining item (i) of Corollary 4.11 with $v = h$, $\tau = 3/2$ and the trivial inequality

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le \max_{\Gamma} h - \min_{\Gamma} h \tag{4.17}$$

gives that

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le c \left\{ \sigma^{-\frac{N-1}{2}} \|\delta_\Gamma^{3/2}\, \nabla^2 h\|_{2,\Omega} + [h]_{C^{0,1}(\overline{\Omega})}\, \sigma \right\}, \tag{4.18}$$

for any $0 < \sigma \le \sigma_0$. By Remark 4.12, here $c = c(N, \theta, a)$.

Now, the term $[h]_{C^{0,1}(\overline{\Omega})}$ can be bounded by recalling (4.12) and using that, by (4.16),

$$|x - z| \le \frac{1}{|\Omega_\sigma|} \int_{\Omega_\sigma} |x - y|\, dy + \frac{1}{|\Omega_\sigma|} \int_{\Omega_\sigma} |\nabla u(y)|\, dy \le d_\Omega + L \le 2 + L,$$

being as $\Omega_\sigma \subset \Omega \subset B_+$. As a consequence, we get the bound:

$$[h]_{C^{0,1}(\overline{\Omega})} \le 2\,(L + 1). \tag{4.19}$$

Putting together (4.19), (4.18), (4.13), and the first inequality in item (i) of Lemma 4.5 gives that

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le 2\,c\,(L + 1) \left\{ \sigma^{-\frac{N-1}{2}} \|u_\nu^2 - R^2\|_{2,\Omega}^{1/2} + \sigma \right\}. \tag{4.20}$$

We now fix

$$\sigma = \min \left\{ \|u_\nu^2 - R^2\|_{2,\Omega}^{1/(N+1)}, \sigma_0 \right\},$$

so as to minimize in $\sigma \in (0, \sigma_0]$ the right-hand-side of (4.20). We then distinguish two cases.

If $\|u_\nu^2 - R^2\|_{2,\Omega}^{1/(N+1)} < \sigma_0$, we have that $\sigma = \|u_\nu^2 - R^2\|_{2,\Omega}^{1/(N+1)}$, and hence (4.20) becomes

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le 4c\,(L + 1)\, \|u_\nu^2 - R^2\|_{2,\Omega}^{1/(N+1)}. \tag{4.21}$$

Otherwise, we easily obtain that

$$\begin{aligned} \max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h &\le [h]_{C^{0,1}(\overline{\Omega})}\, d_\Sigma \le [h]_{C^{0,1}(\overline{\Omega})}\, d_\Omega \\ &\le 4(L + 1) \le 4\,\sigma_0^{-1}(L + 1)\, \|u_\nu^2 - R^2\|_{2,\Omega}^{1/(N+1)}, \end{aligned}$$

where, in the third inequality, we used (4.19) and that $d_\Omega \le 2$. Thus, (4.21) always holds for some constant $c = c(N, \theta, a)$. The desired conclusion, then easily follows by recalling (4.6).

(ii) Fix $0 < \eta < 1$. Combining item (ii) of Corollary 4.11 with $v = h$, $\tau = 3/2$, and (4.17) gives that

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le c \left\{ \sigma^{-\eta - 1/2} \|\delta_\Gamma^{3/2}\, \nabla^2 h\|_{2,\Omega} + [h]_{C^{0,1}(\overline{\Omega})}\, \sigma \right\},$$

for any $0 < \sigma \le \sigma_0$. Putting together the last inequality, (4.19), (4.13), and the first inequality in item (i) of Lemma 4.5, we infer:

$$\max_{\overline{\Sigma}} h - \min_{\overline{\Sigma}} h \le 2c\,(L + 1) \left\{ \sigma^{-\eta - 1/2} \|u_\nu^2 - R^2\|_{2,\Omega}^{1/2} + \sigma \right\}.$$

We now fix

$$\sigma = \min\left\{\|u_\nu^2 - R^2\|_{2,\Omega}^{1/(3+2\eta)}, \sigma_0\right\},$$

so as to minimize in $\sigma \in (0, \sigma_0]$ the right-hand-side, and conclude by the same analysis performed in item (i). $\square$

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A: Remarks on the uniform cone condition

In this appendix, we detail some geometrical facts and amend an inaccuracy contained in [18].

A.1 Some geometrical facts

As already mentioned, the uniform (θ, a)-interior cone condition adopted in the present paper is equivalent to the strong local Lipschitz property of Adams [1, p. 66] and to the uniform Lipschitz regularity in [5, Section III] and [22, Definition 2.1]. By putting together [22, Proposition 4.1 in the Appendix] and [5, Proposition III.1], we easily infer the following result.

Lemma A.1 *Let Ω be a bounded domain satisfying the uniform (θ, a)-interior cone condition. There exists a positive constant δ_0 depending on a, θ, and d_Ω such that, for any $\sigma \le \delta_0$, the parallel set $\Omega_\sigma = \{x \in \Omega : \delta_\Gamma(x) > \sigma\}$ is connected.*

A domain Ω in $\mathbb{R}^N$ is a *b-John domain*, with $b \ge 1$, if each pair of distinct points x_1 and x_2 in Ω can be joined by a curve $\psi : [0, 1] \to \Omega$ such that $\psi(0) = x_1$, $\psi(1) = x_2$, and

$$b\, \delta_\Gamma(\psi(t)) \ge \min\{|\psi(t) - x_1|, |\psi(t) - x_2|\}.$$

A curve satisfying the previous inequality is called a *John curve*. By using the previous lemma, we now prove that domains satisfying the uniform (θ, a)-interior cone condition are b-John domains and provide an explicit estimate for b in terms of θ, a, d_Ω.

Lemma A.2 *Let Ω be a bounded domain satisfying the uniform (θ, a)-interior cone condition. Then, Ω is a b-John domain with*

$$b \le \max\left\{\frac{1}{\sin(\theta)}, \frac{d_\Omega}{\min\left\{\frac{a}{2}\frac{\sin\theta}{1+\sin\theta}, \delta_0\right\}}\right\},$$

where δ_0 is the constant appearing in Lemma A.1.

Proof Set $\sigma = \min\left\{\frac{a}{2}\frac{\sin\theta}{1+\sin\theta},\ \delta_0\right\}$. Lemma A.1 guarantees that any two points $x_1, x_2 \in \Omega_\sigma$ can be joined by a curve $\psi : [0, 1] \to \Omega_\sigma$. Also, we easily compute that

$$\frac{\min\{|\psi(t) - x_1|, |\psi(t) - x_2|\}}{\delta_\Gamma(\psi(t))} \leq \frac{d_\Omega}{\delta_\Gamma(\psi(t))} \leq \frac{d_\Omega}{\min\left\{\frac{a}{2}\frac{\sin\theta}{1+\sin\theta},\ \delta_0\right\}}.$$

On the other hand, if x_j (for $j = 1$ and/or 2) is a point in $\Omega \setminus \Omega_\sigma$, then we can find a point $y_j \in \Omega_\sigma$ and another curve ϕ_j, joining x_j to y_j, such that

$$\frac{\min\left\{|\phi_j(t) - x_1|, |\phi_j(t) - x_2|\right\}}{\delta_\Gamma(\phi_j(t))} \leq \frac{1}{\sin\theta}.$$

In fact, we have that $\delta_\Gamma(x_j) \leq \sigma \leq \frac{a}{2}\frac{\sin\theta}{1+\sin\theta} \leq a/4$. Hence, if x^j is the projection of x_j on Γ, (3.1) gives that $x_j + \mathcal{C}_\omega \subset \Omega$ with $\omega = \omega_{x^j}$. If we set $y_j = x_j + \frac{a}{1+\sin\theta}\omega$ (which is a point on the axis of the cone $x_j + \mathcal{C}_\omega$), by some trigonometry we have that $\delta_\Gamma(y_j) \geq \delta_{\partial(x_j+\mathcal{C}_\omega)}(y_j) = a\frac{\sin\theta}{1+\sin\theta} > \frac{a}{2}\frac{\sin\theta}{1+\sin\theta}$. In particular, $y_j \in \Omega_\sigma$.

For $\ell > 0$, the choice

$$\phi_j(t) = \begin{cases} x_1 + \frac{t}{\ell}(y_1 - x_1) & \text{if } j = 1, \\ y_2 + \frac{t}{\ell}(x_2 - y_2) & \text{if } j = 2, \end{cases} \qquad t \in [0, \ell],$$

is clearly admissible. Moreover, for any $x_1, x_2 \in \Omega$, it allows to create a suitable curve from x_1 to x_2 by joining together ϕ_1 (if $x_1 \in \Omega \setminus \Omega_\sigma$), a curve contained in Ω_σ, and ϕ_2 (if $x_2 \in \Omega\setminus\Omega_\sigma$).

In any case, for any $x_1, x_2 \in \Omega$ we can always find a John curve ψ from x_1 to x_2 such that

$$\frac{\min\{|\psi(t) - x_1|, |\psi(t) - x_2|\}}{\delta_\Gamma(\psi(t))} \leq \max\left\{\frac{1}{\sin(\theta)}, \frac{d_\Omega}{\min\left\{\frac{a}{2}\frac{\sin\theta}{1+\sin\theta},\ \delta_0\right\}}\right\},$$

and the conclusion follows. □

We now prove the following useful result.

Lemma A.3 *Let Ω satisfy the (θ, a)-uniform interior cone condition. Then, the parallel set $\Omega_\sigma = \{x \in \Omega : \delta_\Gamma(x) > \sigma\}$ satisfies the $(\theta, a/2)$-uniform interior cone condition, for any $\sigma \leq a/4$.*

Proof Let x be any point on $\partial\Omega_\sigma$ and let y be a point in Γ (not necessarily unique) such that $\delta_\Gamma(A) = |x - y| = \sigma$. Since Ω satisfies the (θ, a)-uniform interior cone condition, we set $\mathcal{C}_\omega$ to be a cone satisfying (3.1) (with $x = y$). Since $B_\sigma(x) \subset \Omega$, by using (3.1) we can easily verify that $x + \mathcal{C}_\omega \cap B_{a/2} \subset \Omega_\sigma$ (Fig. 2).

Moreover, we can also check that

$$w + \mathcal{C}_\omega \cap B_{a/2} \subset \Omega_\sigma \quad \text{for every} \quad w \in B_{a/2}(x) \cap \overline{\Omega}_\sigma.$$

Since x is chosen arbitrarily in $\partial\Omega_\sigma$, the last inclusion gives that Ω_σ satisfies the $(\theta, a/2)$-uniform interior cone condition. The last inclusion holds by noting that, for any $w \in B_{a/2}(x) \cap \overline{\Omega}_\sigma$, we have that $B_\sigma(w) \subset \Omega$ (by definition of Ω_σ) and $B_\sigma(w) \subset B_a(y)$ (being as $\sigma \leq a/4$). Hence, we can argue as above to get that $w + \mathcal{C}_\omega \cap B_{a/2} \subset \Omega_\sigma$. □

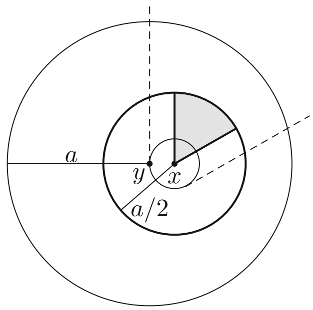

Fig. 2 The construction of Lemma A.3. Here, $x \in \partial\Omega_\sigma$ and $y \in \Gamma = \partial\Omega$ is such that $|x - y| = \delta_\Gamma(x) = \sigma \le a/4$. The shaded region is the cone $x + \mathcal{C}_\omega \cap B_{a/2}$. By (3.1), the region bounded by the dashed lines and containing the smallest disk is contained Ω

A.2 Errata corrige of [18, Corollary 2.3 and Theorems 2.4 and 2.7]

In [18], we assumed the following notion of cone condition, which is strictly weaker than the one adopted in the present paper. A bounded domain $\Omega \subset \mathbb{R}^N$ with boundary Γ satisfies the (θ, a)-uniform interior cone condition if, for every $x \in \overline{\Omega}$, there is a cone $\mathcal{C}_x$ with vertex at x, opening width θ, and height a, such that $\mathcal{C}_x \subset \Omega$ and $\overline{\mathcal{C}}_x \cap \Gamma = \{x\}$, whenever $x \in \Gamma$. We will refer to this definition as the *old cone condition*. It is easy to check that this condition is verified (with same θ and a), if Ω satisfies the (new) (θ, a)-uniform interior cone condition adopted in Sect. 3.

It is a classical result [1, 22] that if Ω is a bounded domain satisfying the old cone condition, then there exists a positive constant $C_p(\Omega)$ — the (p, p)-Poincaré constant — such that

$$\|f - f_\Omega\|_{p,\Omega} \le C_p(\Omega)\|\nabla f\|_{p,\Omega} \ \text{ for any } \ f \in W^{1,p}(\Omega).$$

We realized that the proof of [18, Corollary 2.3] contains a mistake. Here, we correct that proof. The amended proof below shows that the constant c in [18, Corollary 2.3] depends not only on N, p, θ, a, *but also* on $C_p(\Omega)$. As a consequence, the dependence on $C_p(\Omega)$ should be added also in the constants c of [18, Theorems 2.4 and 2.7]. Since, when Ω is of class C^2, $C_p(\Omega)$ can be estimated in terms of the radius r_i of the uniform interior sphere condition and the diameter d_Ω (see [16, item (iii) of Remark 2.4]), [18, Lemma 3.2] remains true with a constant $c = c(N, p, r_i, d_\Omega)$ and the rest of the paper remains unchanged.

Amended proof of [18, Corollary 2.3]

By using [18, (2.3)], we have that

$$|f(x) - f_{\mathcal{C}_x}| \le c_{N,p}\, a \left(\frac{1}{|\mathcal{C}_x|} \int_{\mathcal{C}_x} |\nabla f|^p \, dx \right)^{1/p} \le c_{N,p} \, \frac{a}{|\mathcal{C}_x|^{1/p}} \|\nabla f\|_{p,\Omega}.$$

(Note that in [18], differently from the present paper, the L^p norms were normalized by the Lebesgue measure of the domain.)

Next, we easily infer that

$$\begin{aligned} |f_{\mathcal{C}_x} - f_\Omega| &\le \frac{1}{|\mathcal{C}_x|} \int_{\mathcal{C}_x} |f - f_\Omega| \, dx \le \frac{1}{|\mathcal{C}_x|^{1/p}} \left(\int_{\mathcal{C}_x} |f - f_\Omega|^p \, dx \right)^{1/p} \\ &\le \frac{1}{|\mathcal{C}_x|^{1/p}} \|f - f_\Omega\|_{p,\Omega} \le \frac{C_p(\Omega)}{|\mathcal{C}_x|^{1/p}} \, \|\nabla f\|_{p,\Omega}. \end{aligned}$$

All in all, we conclude that

$$|f(x) - f_\Omega| \le |f(x) - f_{\mathcal{C}_x}| + |f_{\mathcal{C}_x} - f_\Omega| \le c \, \|\nabla f\|_{p,\Omega},$$

for some constant c that depends on N, p, θ, a, and $C_p(\Omega)$. □

Remark A.4 As pointed out in Remark 3.3, if Ω is a bounded b-John domain, $C_p(\Omega)$ can be estimated in terms of b and d_Ω. In turn, if Ω satisfies the new cone condition of the present paper, the John parameter b, and hence $C_p(\Omega)$, can be estimated in terms of the parameters θ, a of the relevant definition, and d_Ω. From this observation, the statement of Lemma 3.4 easily follows.

On the contrary, the old cone condition adopted in [18] is not sufficient to give an estimate of the (p, p)-Poincaré constant (see, e.g., [22]), and hence neither of the John parameter. In fact, reasoning as in [22, Example 2.6], one can construct a family of (uniformly) bounded domains Ω^ε sharing the same (fixed) parameters of the old cone condition and a sequence $u_\varepsilon \in W^{1,2}(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} u_\varepsilon \, dx = 0, \quad \int_{\Omega^\varepsilon} |\nabla u_\varepsilon|^2 \, dx \to 0,$$

while $\int_{\Omega^\varepsilon} u_\varepsilon^2 \, dx$ remains bounded away from zero.

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