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Size estimates for nanoplates

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Abstract

We consider the problem of determining, within an elastic isotropic nanoplate in bending, the possible presence of an inclusion made of different elastic material. Under suitable *a priori* assumptions on the unknown inclusion, we provide quantitative upper and lower estimates for the area of the unknown defect in terms of the works exerted by the boundary data when the inclusion is present and when it is absent.

Keywords: inverse problems, elastic nanoplates, size estimates, unique continuation

1. Introduction

Over the past three decades, micro- and nano-electromechanical systems (MEMS and NEMS) have found wide application as sensors, actuators and for vibration control purposes [15]. Due to their small size and the material properties, they possess superior mechanical, thermal and electrical performance compared to classical devices, allowing extreme miniaturization, high reliability, low costs and reduced energy consumption for their operation. These indisputable advantages have favored their rapid application in strategic areas, such as communications, biological technologies, mechanics and aerospace.

Nanoplates are the core components of MEMS and NEMS, and their proper functionality is an essential requirement for the devices. The demand for higher performances and small sizes (typical size around $1 \div 10 \times 10^{-4}$ m) have led to higher strain/stress states and very challenging operating conditions that can increase the probability of structural failure. Furthermore, defects such as cracks, internal voids, inhomogeneous material properties and abrasions can

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appear during the manufacturing process and can evolve during service, leading to the activation of mechanical device failure [12, 20, 31].

For the reasons stated above, the problem of defect identification is attracting increasing attention from researchers interested in the behavior of MEMS/NEMS devices. In this paper we consider the inverse problem of determining, within an isotropic elastic nanoplate subjected to static bending deformation, the possible presence of an inclusion made by different elastic material from a single boundary data measurement.

Let us formulate the inverse problem. Let us consider a nanoplate in the referential configuration $\Omega \times [-t/2, t/2]$, where Ω is a plane domain with $C^{3,1}$ boundary (see theorems 3.5 and 3.6 and definition 2.1 for more details) representing the middle surface of the nanoplate and t is the uniform thickness, $t << \mathrm{diam}(\Omega)$. Let $\tilde{\Omega}$ be the subset of Ω corresponding to the unknown inclusion. It is well known that classical continuum mechanics, as a length-scale free theory, loses its predictive capacity for nanostructures since it is not able to take into account the presence of size effects in the material response. Here, we shall adopt the simplified strain gradient theory proposed by Lam *et al* [23] to model the mechanical behavior of the material in infinitesimal deformation. Under the kinematic assumptions of Kirchhoff–Love's plate theory, the statical equilibrium problem of the nanoplate loaded at the boundary and under vanishing body forces is described by the following Neumann boundary value problem [21]

$$(M_{\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma,\gamma})_{,\alpha\beta} = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\begin{aligned}\left(M_{\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma,\gamma}\right)_{,\alpha} n_\beta + \left(\left(M_{\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma,\gamma}\right) n_\alpha \tau_\beta\right)_{,s} + \left(\overline{M}^h_{\alpha\beta\gamma}\tau_\alpha\tau_\beta n_\gamma\right)_{,ss} \\ - \left(\overline{M}^h_{\alpha\beta\gamma} n_\gamma (\tau_{\alpha,s}\tau_\beta - n_{\alpha,s} n_\beta)\right)_{,s} = -\widehat{V} \qquad \text{on } \partial\Omega,\end{aligned} \tag{1.2}$$

$$\begin{aligned}&\left(M_{\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma,\gamma}\right) n_\alpha n_\beta + \left(\overline{M}^h_{\alpha\beta\gamma} n_\gamma (\tau_\alpha n_\beta + \tau_\beta n_\alpha)\right)_{,s} \\ &\quad - \overline{M}^h_{\alpha\beta\gamma} n_\gamma (n_{\alpha,s}\tau_\beta) = \widehat{M}_n \qquad \text{on } \partial\Omega,\end{aligned} \tag{1.3}$$

$$\overline{M}^h_{\alpha\beta\gamma} n_\alpha n_\beta n_\gamma = -\widehat{M}^h_n \qquad \text{on } \partial\Omega. \tag{1.4}$$

The functions $M_{\alpha\beta} = M_{\alpha\beta}(u)$, $\overline{M}^h_{\alpha\beta\gamma} = \overline{M}^h_{\alpha\beta\gamma}(u)$, $\alpha, \beta, \gamma = 1, 2$, in the above equations are the Cartesian components of the couple tensor $M = (M_{\alpha\beta})$ and the high-order couple tensor $\overline{M}^h = (\overline{M}^h_{\alpha\beta\gamma})$, respectively, corresponding to the transverse displacement u. The constitutive equations of M and $\overline{M}^h$ are as follows

$$M(u) = -(\chi_{\Omega\setminus\tilde{\Omega}}(\mathbb{P} + \mathbb{P}^h) + \chi_{\tilde{\Omega}}(\widetilde{\mathbb{P}} + \widetilde{\mathbb{P}}^h))D^2 u, \tag{1.5}$$

$$\overline{M}^h(u) = (\chi_{\Omega\setminus\tilde{\Omega}}\mathbb{Q} + \chi_{\tilde{\Omega}}\widetilde{\mathbb{Q}})D^3 u, \tag{1.6}$$

where $\mathbb{P}, \mathbb{P}^h, \mathbb{Q}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{P}}^h, \widetilde{\mathbb{Q}}$ are the tensors expressing the response of the material and are defined in detail in section 3. The vectors τ and n are the unit tangent and the unit outer normal to $\partial\Omega$, and s is an arclength chosen on $\partial\Omega$. The loads acting on $\partial\Omega$ include the shear force $\widehat{V}$, the bending moment $\widehat{M}_n$ and the high-order bending moment $\widehat{M}^h_n$. A full description of the mechanical nanoplate model and the main properties of the direct problem can be found in section 3.1, to which we refer for details.

The first question to be asked in approaching our inverse problem is the question of uniqueness. In particular: *does a single boundary measurement of Neumann data* $\{\widehat{V}, \widehat{M}_n, \widehat{M}_n^h\}$ *and Dirichlet data* $\{u, u_{,n}, u_{,nn}\}$ *uniquely determine the unknown inclusion* $\tilde{\Omega}$? In spite of the simplicity with which it is formulated, this inverse problem is extremely difficult and even in the simpler context of electrical impedance tomography, which involves a second-order elliptic equation, a general uniqueness result is missing. We refer to [2, 3] for an up-to-date overview and an extensive reference list.

In the present note, we discuss another direction of research. In fact, instead of determining the exact shape and location of $\tilde{\Omega}$, we evaluate its size in terms of the data. More precisely we provide quantitative estimate on the area of the unknown inclusion in terms of the quantities

$$W = L(u) = -\int_{\partial\Omega} \widehat{V}u + \widehat{M}_n u_{,n} + \widehat{M}_n^h u_{,nn}\,, \tag{1.7}$$

$$W_0 = L(u_0) = -\int_{\partial\Omega} \widehat{V}u_0 + \widehat{M}_n u_{0,n} + \widehat{M}_n^h u_{0,nn}, \tag{1.8}$$

which represent the works exerted by the boundary data when the inclusion $\tilde{\Omega}$ is present or absent, respectively. Here u_0 is the transverse displacement of the unperturbed nanoplate without inclusion, namely u_0 satisfies (1.1)–(1.4) when $\tilde{\Omega}$ is the empty set.

In order to treat the inverse problem, we first need to analyze the direct one. In section 3.1, we collect some previous results contained in [24], concerning the well posedness of the direct problem (see theorem 3.1) and, for the case in which the inclusion is absent, the H^4 regularity up to the boundary of the solution of the Neumann problem (1.1)–(1.4) (see theorem 3.2) and the H^6 regularity in the interior for solutions of the underlying equation (1.1) (see theorem 3.3), under suitable regularity assumptions on the coefficients and on the boundary of Ω.

In section 3.2 we rigorously formulate the inverse problem and state our main *a-priori* assumptions. In section 3.3 we present our main results that can be summarized as follows.

(i) In theorem 3.4 we consider the case when $\tilde{\Omega}$ is a general measurable set compactly contained in Ω and we provide the following lower bound of its size

$$|\tilde{\Omega}| \geqslant C\left|\frac{W - W_0}{W_0}\right|, \tag{1.9}$$

where the constant $C > 0$ is estimated in terms of the *a priori* data. The main idea underlying this estimate is that the integral

$$\int_{\tilde{\Omega}} |D^2u_0|^2 + |D^3u_0|^2 \ \text{ is comparable to } \ |W_0 - W|\,. \tag{1.10}$$

The above behavior follows from energy estimates for the Neumann problem (1.1)–(1.4) both when the inclusion is present and when it is absent (see lemma 4.1 for a precise statement). By using interior regularity estimate for the sixth order elliptic equation we can control from below the size of $\tilde{\Omega}$ in terms of the integral in (1.10) and achieve the desired bound (1.9).

(ii) In theorem 3.5 we prove an upper bound for the size of $\tilde{\Omega}$ under the following *fatness condition* on $\tilde{\Omega}$ itself. Namely, given $h > 0$ and denoting $\tilde{\Omega}_h = \{x \in \tilde{\Omega} : \operatorname{dist}(x, \partial\tilde{\Omega}) > h\}$, if we assume

$$|\tilde{\Omega}_h| \geqslant \frac{1}{2}|\tilde{\Omega}|, \tag{1.11}$$

then we have

$$|\tilde{\Omega}| \leqslant C\left|\frac{W-W_0}{W_0}\right|, \tag{1.12}$$

where the constant $C>0$ is estimated in terms of the *a priori* data. Although in order to obtain (1.12) we still make use of (1.10), in this step a deeper analysis is required. Indeed, in estimating the integral $\int_{\tilde{\Omega}} |D^2u_0|^2 + |D^3u_0|^2$ from below, one has to face the possible vanishing of D^2u_0 and D^3u_0 at interior points. In this respect, we prove the following unique continuation result known as *Lipschitz propagation of smallness* (see proposition 6.1 for a precise statement).

There exists $\chi > 1$ depending on the *a priori* data such that, for every $\rho > 0$ and every $x \in \Omega$ for which $\mathrm{dist}(x, \partial\Omega) > \chi\rho$, we have

$$\int_{B_\rho(x)} |D^2u_0|^2 \geqslant C\int_\Omega |D^2u_0|^2, \tag{1.13}$$

where the constant $C>0$ is estimated in terms of ρ and the *a priori* data. From such an estimate, inequality (1.12) follows by covering $\tilde{\Omega}_h$ by non overlapping squares of side $\epsilon = O(h)$.

(iii) In theorem 3.6, we remove the fatness condition on $\tilde{\Omega}$ (compactly contained in Ω) and we state an upper bound for the size of $\tilde{\Omega}$ of the following form

$$|\tilde{\Omega}| \leqslant C\left|\frac{W-W_0}{W_0}\right|^{1/p}, \tag{1.14}$$

where $C>0$, $p>1$ are estimated in terms of the *a priori* data.

In this case, in contrast with the *fatness condition*, we need to introduce a further sophisticated argument arising in the theory of Muckenhoupt weight [16] (see proposition 7.2). By combining the A_p-estimates with a covering argument and (1.10) we end up with the desired estimate.

Let us recall that the prototype of the present class of inverse problems is the determination of the size of an inclusion within an electrostatic conductor [8, 9, 11, 22] and that such an issue has been extended to more complicated equations and systems [4, 5, 10, 14, 18, 19, 24, 25–29]. However, although our strategy belongs to the ones adopted in this line of research, the treatment of a higher order partial differential equation has required the development of new mathematical tools.

Indeed, in section 5 we present new estimates of unique continuation in the form of a doubling inequality and a three sphere inequality for the Hessian of the solution of the unperturbed nanoplate. The iterated use of such three sphere inequalities allows us to obtain (1.13). Moreover, we also provide a global doubling inequality expressed in terms of the known boundary data (see proposition 7.1) which establishes a bridge with the theory of Muckenhoupt weight.

2. Notation

Let $P=(x_1(P),x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes, of length $2\,a$ and $2\,b$, namely

$$R_{a,b}(P)=\{x=(x_1,x_2)\mid |x_1-x_1(P)|<a,\ |x_2-x_2(P)|<b\}. \tag{2.1}$$

Definition 2.1 ($C^{k,\alpha}$ regularity). Let Ω be a bounded domain in $\mathbb{R}^2$. Given k,α, with $k\in\mathbb{N}$, $k\geqslant 1$, $0<\alpha\leqslant 1$, we say that a portion S of $\partial\Omega$ is of *class* $C^{k,\alpha}$ *with constants* r_0, $M_0>0$, if, for any $P\in S$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$\Omega\cap R_{r_0,2M_0r_0}=\{x\in R_{r_0,2M_0r_0}\quad|\quad x_2>g(x_1)\},$$

where g is a $C^{k,\alpha}$ function on $[-r_0,r_0]$ satisfying

$$g(0)=g'(0)=0,$$

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])}\leqslant M_0r_0,$$

with

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])}=\sum_{i=0}^{k}r_0^i\sup_{[-r_0,r_0]}|g^{(i)}|+r_0^{k+\alpha}|g|_{k,\alpha},$$

$$|g|_{k,\alpha}=\sup_{\substack{t,s\in[-r_0,r_0]\\ t\neq s}}\frac{|g^{(k)}(t)-g^{(k)}(s)|}{|t-s|^{\alpha}}.$$

We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance, given a function $u:\Omega\to\mathbb{R}$ we denote

$$\|u\|_{H^k(\Omega)}=r_0^{-1}\left(\sum_{i=0}^{k}r_0^{2i}\int_\Omega|D^iu|^2\right)^{\frac{1}{2}} \tag{2.2}$$

where

$$\int_\Omega|D^ku|^2=\int_\Omega\sum_{|\alpha|=k}|D^\alpha u|^2 \tag{2.3}$$

and so on for boundary and trace norms.

For any $h>0$ we set

$$\Omega_h=\{x\in\Omega\ :\ \mathrm{dist}(x,\partial\Omega)>h\}.$$

Given a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,\alpha}$, with $k\geqslant 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P\in\partial\Omega$, let us denote by $\tau=\tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is

$$\tau=e_3\times n, \tag{2.4}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$ and $\{e_1,e_2,e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Given any connected component $\mathcal{C}$ of $\partial\Omega$ and fixed a point $P_0\in\mathcal{C}$, let us define as positive the orientation of $\mathcal{C}$ associated to an arclength parameterization $\psi(s)=(x_1(s),x_2(s))$, $s\in[0,l(\mathcal{C})]$, such that $\psi(0)=P_0$ and $\psi'(s)=\tau(\psi(s))$. Here $l(\mathcal{C})$ denotes the length of $\mathcal{C}$.

Throughout the paper, we denote by $w_{,\alpha}$, $\alpha = 1,2$, $w_{,s}$, and $w_{,n}$ the derivatives of a function w with respect to the x_α variable, to the arclength s and to the unit outer normal n to Ω, respectively, and similarly for higher order derivatives.

We denote by $\mathbb{M}^2, \mathbb{M}^3$ the Banach spaces of second order and third order tensors and by $\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^3$ the corresponding subspaces of tensors having components invariant with respect to permutations of all the indexes.

Let $\mathcal{L}(X,Y)$ be the space of bounded linear operators between Banach spaces X and Y. Given $\mathbb{K} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ and $A, B \in \mathbb{M}^2$, we use the following notation

$$(\mathbb{K}A)_{ij} = \sum_{l,m=1}^{2} K_{ijlm} A_{lm}, \tag{2.5}$$

$$A \cdot B = \sum_{i,j=1}^{2} A_{ij} B_{ij}. \tag{2.6}$$

Similarly, given $\mathbb{K} \in \mathcal{L}(\mathbb{M}^3, \mathbb{M}^3)$ and $A, B \in \mathbb{M}^3$, we denote

$$(\mathbb{K}A)_{ijk} = \sum_{l,m,n=1}^{2} K_{ijklmn} A_{lmn}, \tag{2.7}$$

$$A \cdot B = \sum_{i,j,k=1}^{2} A_{ijk} B_{ijk}. \tag{2.8}$$

Moreover, for any $A \in \mathbb{M}^n$, with $n = 2,3$, we shall denote

$$|A| = (A \cdot A)^{\frac{1}{2}}. \tag{2.9}$$

The linear space of the infinitesimal rigid displacements is defined as

$$\mathcal{R}_2 = \left\{ r(x) = c + Wx,\ c \in \mathbb{R}^2,\ W \in \mathbb{M}^2,\ W + W^T = 0 \right\}. \tag{2.10}$$

We shall assume summation over repeated indexes and in order to simplify our notation, we will denote by $C, C_1, C_2, \ldots$ positive constants which may vary from step to step.

3. Size estimates results

3.1. The direct problem

Let us consider a nanoplate $\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$ with middle surface Ω represented by a bounded domain of $\mathbb{R}^2$ and having constant thickness t, $t << \mathrm{diam}(\Omega)$. We assume that the boundary $\partial\Omega$ of Ω is of class $C^{2,1}$ with constants r_0, M_0 and that

$$|\Omega| \leqslant M_1 r_0^2, \tag{3.1}$$

where M_1 is a positive constant.

Within the kinematic framework of the Kirchhoff–Love theory in infinitesimal deformation, the statical equilibrium problem of the nanoplate loaded at the boundary and under vanishing body forces is described by the following Neumann boundary value problem [21]:

$$\left(M_{\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma,\gamma} \right)_{,\alpha\beta} = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$\left(M_{\alpha\beta}+\overline{M}^h_{\alpha\beta\gamma,\gamma}\right)_{,\alpha}n_\beta+\left(\left(M_{\alpha\beta}+\overline{M}^h_{\alpha\beta\gamma,\gamma}\right)n_\alpha\tau_\beta\right)_{,s}+\left(\overline{M}^h_{\alpha\beta\gamma}\tau_\alpha\tau_\beta n_\gamma\right)_{,ss}$$
$$-\left(\overline{M}^h_{\alpha\beta\gamma}n_\gamma(\tau_{\alpha,s}\tau_\beta-n_{\alpha,s}n_\beta)\right)_{,s}=-\widehat{V}\qquad\text{on }\partial\Omega, \tag{3.3}$$

$$(M_{\alpha\beta}+\overline{M}^h_{\alpha\beta\gamma,\gamma})n_\alpha n_\beta+\left(\overline{M}^h_{\alpha\beta\gamma}n_\gamma(\tau_\alpha n_\beta+\tau_\beta n_\alpha)\right)_{,s}$$
$$-\overline{M}^h_{\alpha\beta\gamma}n_\gamma(n_{\alpha,s}\tau_\beta)=\widehat{M}_n\qquad\text{on }\partial\Omega, \tag{3.4}$$

$$\overline{M}^h_{\alpha\beta\gamma}n_\alpha n_\beta n_\gamma=-\widehat{M}^h_n\qquad\text{on }\partial\Omega. \tag{3.5}$$

The functions $M_{\alpha\beta}=M_{\alpha\beta}(u)$, $\overline{M}^h_{\alpha\beta\gamma}=\overline{M}^h_{\alpha\beta\gamma}(u)$, $\alpha,\beta,\gamma=1,2$, in the above equations are the Cartesian components of the couple tensor $M=(M_{\alpha\beta})$ and the high-order couple tensor $\overline{M}^h=(\overline{M}^h_{\alpha\beta\gamma})$, respectively, corresponding to the transverse displacement $u(x_1,x_2)$, $u:\Omega\to\mathbb{R}$, of the point $(x_1,x_2)=x$ belonging to the middle surface of the nanoplate. To simplify the presentation, the dependence of these quantities on u is not explicitly indicated in (3.2)–(3.5) and in what follows.

We assume that the functions $M_{\alpha\beta}$ can be expressed as

$$M_{\alpha\beta}=-(P_{\alpha\beta\gamma\delta}+P^h_{\alpha\beta\gamma\delta})u_{,\gamma\delta}\quad(M=-(\mathbb{P}+\mathbb{P}^h)D^2u)), \tag{3.6}$$

where the fourth order tensors $\mathbb{P}=\mathbb{P}(x)\in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$, $\mathbb{P}^h=\mathbb{P}^h(x)\in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ are assumed to satisfy the symmetry conditions

$$\mathbb{P}A\cdot B=\mathbb{P}B\cdot A,\quad\text{a.e. in }\Omega, \tag{3.7}$$

$$\mathbb{P}^hA\cdot B=\mathbb{P}^hB\cdot A,\quad\text{a.e. in }\Omega, \tag{3.8}$$

for every $A,B\in\widehat{\mathbb{M}}^2$, and the strong convexity condition

$$(\mathbb{P}+\mathbb{P}^h)A\cdot A\geqslant t^3\xi_{\mathbb{P}}|A|^2,\quad\text{a.e. in }\Omega, \tag{3.9}$$

for every $A\in\widehat{\mathbb{M}}^2$, where $\xi_{\mathbb{P}}$ is a positive constant.

Concerning the functions $\overline{M}^h_{ijk}$ $(i,j,k=1,2)$, we assume that they can be expressed as

$$\overline{M}^h_{ijk}=Q_{ijklmn}u_{,lmn}\quad(\overline{M}^h=\mathbb{Q}D^3u), \tag{3.10}$$

where Q_{ijklmn} are the Cartesian components of the sixth order tensor $\mathbb{Q}=\mathbb{Q}(x)\in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$, and $\mathbb{Q}$ is assumed to satisfy the symmetry conditions

$$\mathbb{Q}A\cdot B=\mathbb{Q}B\cdot A,\quad\text{a.e. in }\Omega, \tag{3.11}$$

for every $A,B\in\widehat{\mathbb{M}}^3$, and the strong convexity condition

$$\mathbb{Q}A\cdot A\geqslant t^5\xi_{\mathbb{Q}}|A|^2,\quad\text{a.e. in }\Omega, \tag{3.12}$$

for every $A\in\widehat{\mathbb{M}}^3$, where $\xi_{\mathbb{Q}}$ is a positive constant.

On the loading data $\widehat{V}$ (shear force), $\widehat{M}_n$ (bending moment) and $\widehat{M}^h_n$ (high-order bending moment) appearing in the equilibrium boundary equations (3.3)–(3.5), we require the following regularity conditions

$$\widehat{V}\in H^{-5/2}(\partial\Omega),\quad\widehat{M}_n\in H^{-3/2}(\partial\Omega),\quad\widehat{M}^h_n\in H^{-1/2}(\partial\Omega) \tag{3.13}$$

and the compatibility conditions (see [21])

$$\int_{\partial\Omega}\widehat{V}=0,\quad\int_{\partial\Omega}\widehat{V}x_1+\widehat{M}_nn_1=0,\quad\int_{\partial\Omega}\widehat{V}x_2+\widehat{M}_nn_2=0. \tag{3.14}$$

The weak formulation of the Neumann problem (3.2)–(3.5), with loading data satisfying (3.13) and (3.14), consists in determining a function $u \in H^3(\Omega)$ (weak solution) such that

$$a(u,w) = L(w), \quad \text{for every } w \in H^3(\Omega), \tag{3.15}$$

where

$$a(u,w) = \int_\Omega -M_{\alpha\beta}(u)w_{,\alpha\beta} + \overline{M}^h_{\alpha\beta\gamma}(u)w_{,\alpha\beta\gamma}$$
$$= \int_\Omega (\mathbb{P}+\mathbb{P}^h)D^2u\cdot D^2w + \mathbb{Q}D^3u\cdot D^3w, \tag{3.16}$$

$$L(w) = -\int_{\partial\Omega} \widehat{V}w + \widehat{M}_n w_{,n} + \widehat{M}^h_n w_{,nn}\,. \tag{3.17}$$

Finally, in order to identify a unique solution, we assume the following normalization conditions

$$\int_\Omega u = 0, \quad \int_\Omega u_{,\alpha} = 0, \quad \alpha = 1,2. \tag{3.18}$$

We are now in position to state the existence, uniqueness and regularity results useful in our analysis. Details of the proofs can be found in [21, 24].

Theorem 3.1 (Existence, uniqueness and H^3-regularity, proposition 3.4 in [24]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ of class $C^{2,1}$ with constant r_0, M_0. Let the tensors $\mathbb{P}$, $\mathbb{P}^h \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ and $\mathbb{Q} \in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ satisfy the symmetry conditions (3.7), (3.8), (3.11) and the strong convexity conditions (3.9), (3.12), respectively. Let the data $\widehat{V}, \widehat{M}_n, \widehat{M}^h_n$ as in (3.13) and satisfying the compatibility conditions (3.14).*

The Neumann problem (3.2)–(3.5) admits a unique weak solution $u \in H^3(\Omega)$ satisfying (3.18) and, moreover,

$$\|u\|_{H^3(\Omega)} \leqslant C\left(\|\widehat{V}\|_{H^{-5/2}(\partial\Omega)} + r_0^{-1}\|\widehat{M}_n\|_{H^{-3/2}(\partial\Omega)} + r_0^{-2}\|\widehat{M}^h_n\|_{H^{-1/2}(\partial\Omega)}\right) \tag{3.19}$$

where the constant $C>0$ only depends on $\frac{t}{r_0}$, M_0, M_1, $\xi_\mathbb{P}$, $\xi_\mathbb{Q}$.

We conclude this section with a global and an improved interior regularity result.

Theorem 3.2 (Global H^4-regularity, theorem 3.5 in [24]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ of class $C^{3,1}$ with constants r_0, M_0, and satisfying (3.1). Let the tensors $\mathbb{P}$, $\mathbb{P}^h \in C^{0,1}(\overline{\Omega},\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ and $\mathbb{Q} \in C^{0,1}(\overline{\Omega},\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ satisfy the symmetry conditions (3.7), (3.8), (3.11) and the strong convexity conditions (3.9), (3.12), respectively. Let $u \in H^3(\Omega)$ be the weak solution of the Neumann problem (3.2)–(3.5) satisfying (3.18), where $\widehat{V} \in H^{-3/2}(\partial\Omega)$, $\widehat{M}_n \in H^{-1/2}(\partial\Omega)$, $\widehat{M}^h_n \in H^{1/2}(\partial\Omega)$ are such that the compatibility conditions (3.14) are satisfied.*

Then $u \in H^4(\Omega)$ and

$$\|u\|_{H^4(\Omega)} \leqslant C\left(\|\widehat{V}\|_{H^{-3/2}(\partial\Omega)} + r_0^{-1}\|\widehat{M}_n\|_{H^{-1/2}(\partial\Omega)} + r_0^{-2}\|\widehat{M}^h_n\|_{H^{1/2}(\partial\Omega)}\right), \tag{3.20}$$

where the constant $C>0$ only depends on $\frac{t}{r_0}$, M_0, M_1, $\xi_\mathbb{P}$, $\xi_\mathbb{Q}$, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$, $\|\mathbb{P}^h\|_{C^{0,1}(\overline{\Omega})}$, $\|\mathbb{Q}\|_{C^{0,1}(\overline{\Omega})}$.

Theorem 3.3 (improved interior regularity, theorem 3.9 in [24]). *Let B_σ be an open ball in $\mathbb{R}^2$ centered at the origin and with radius σ. Let $u \in H^3(B_\sigma)$ be such that*

$$a(u,\varphi) = 0 \qquad \text{for every } \varphi \in H^3_0(B_\sigma), \tag{3.21}$$

with

$$a(u,\varphi)=\int_{B_\sigma}(\mathbb{P}+\mathbb{P}^h)D^2u\cdot D^2\varphi+\mathbb{Q}D^3u\cdot D^3\varphi, \tag{3.22}$$

where the tensors $\mathbb{P},\mathbb{P}^h\in C^{1,1}(\overline{B_\sigma},\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$, $\mathbb{Q}\in C^{2,1}(\overline{B_\sigma},\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ *satisfy the symmetry conditions (3.7), (3.8), (3.11) and the strong convexity conditions (3.9), (3.12), respectively.*

Then $u\in H^6(B_{\frac{\sigma}{8}})$ *and we have*

$$\|u\|_{H^6(B_{\frac{\sigma}{8}})}\leqslant C\|u\|_{H^3(B_\sigma)}, \tag{3.23}$$

where $C>0$ *only depends on* t, $\xi_{\mathbb{P}}$, $\xi_{\mathbb{Q}}$, $\|\mathbb{P}\|_{C^{1,1}(\overline{B_\sigma})}$, $\|\mathbb{P}^h\|_{C^{1,1}(\overline{B_\sigma})}$, $\|\mathbb{Q}\|_{C^{2,1}(\overline{B_\sigma})}$.

3.2. Formulation of the inverse problem

We consider a nanoplate $\Omega\times\left(-\frac{t}{2},\frac{t}{2}\right)$ inside which a possible inclusion $\tilde{\Omega}\times\left(-\frac{t}{2},\frac{t}{2}\right)$ is present, where $\tilde{\Omega}$ is a measurable, possibly disconnected subset of Ω.

Let us consider elasticity tensors $\mathbb{P}$, $\widetilde{\mathbb{P}}$, $\mathbb{P}^h$, $\widetilde{\mathbb{P}}^h\in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ and $\mathbb{Q}$, $\widetilde{\mathbb{Q}}\in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ satisfying the symmetry conditions (3.7), (3.8) and (3.11), respectively.

We shall make the following *a-priori* assumptions on the elasticity tensors.

(i) *Isotropy for* $\mathbb{P}$, $\mathbb{P}^h$, $\mathbb{Q}$.

The Cartesian components of $\mathbb{P}$, $\mathbb{P}^h$, $\mathbb{Q}$ are given by

$$P_{\alpha\beta\gamma\delta}=B((1-\nu)\delta_{\alpha\gamma}\delta_{\beta\delta}+\nu\delta_{\alpha\beta}\delta_{\gamma\delta}), \tag{3.24}$$

$$P^h_{\alpha\beta\gamma\delta}=(2a_2+5a_1)\delta_{\alpha\gamma}\delta_{\beta\delta}+(-a_1-a_2+a_0)\delta_{\alpha\beta}\delta_{\gamma\delta}, \tag{3.25}$$

$$\begin{aligned}Q_{ijklmn}=&\frac{1}{3}(b_0-3b_1)\delta_{ij}\delta_{kn}\delta_{lm}+\frac{1}{6}(b_0-3b_1)(\delta_{ik}(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln})+\delta_{jk}(\delta_{il}\delta_{mn}+\delta_{im}\delta_{ln}))\\&+Q_8(\delta_{kn}(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}))+Q_9(\delta_{jn}(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})+\delta_{in}(\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl})),\end{aligned} \tag{3.26}$$

where $2(Q_8+2Q_9)=5b_1$.

The bending stiffness (per unit length) $B=B(x)$ is given by the function

$$B(x)=\frac{t^3E(x)}{12(1-\nu^2(x))},\quad \text{a.e. in }\Omega, \tag{3.27}$$

where the Young's modulus E and the Poisson's coefficient ν of the material can be written in terms of the Lamé moduli μ and λ as follows

$$E(x)=\frac{\mu(x)(2\mu(x)+3\lambda(x))}{\mu(x)+\lambda(x)},\quad \nu(x)=\frac{\lambda(x)}{2(\mu(x)+\lambda(x))}. \tag{3.28}$$

The coefficients $a_i(x)$, $i=0,1,2$, are given by (see [21])

$$a_0(x)=2\mu(x)tl_0^2,\quad a_1(x)=\frac{2}{15}\mu(x)tl_1^2,\quad a_2(x)=\mu(x)tl_2^2\quad \text{a.e. in }\ \Omega, \tag{3.29}$$

where the material length scale parameters l_i are assumed to be positive constants. We denote

$$l=\min\{l_0,l_1,l_2\}. \tag{3.30}$$

The coefficients $b_i(x)$, $i=0,1$, are given by

$$b_0(x) = 2\mu(x)\frac{t^3}{12}l_0^2, \quad b_1(x) = \frac{2}{5}\mu(x)\frac{t^3}{12}l_1^2 \quad \text{a.e. in } \Omega. \tag{3.31}$$

(ii) *Strong convexity for* $\mathbb{P}+\mathbb{P}^h$, $\mathbb{Q}$.

We assume the following ellipticity conditions on μ and λ:

$$\mu(x) \geqslant \alpha_0 > 0, \quad 2\mu(x)+3\lambda(x) \geqslant \gamma_0 > 0 \quad \text{a.e. in } \Omega, \tag{3.32}$$

where α_0, γ_0 are positive constants. By (3.29), (3.31) and (3.32) we also have

$$a_i(x) \geqslant tl^2\alpha_0^h > 0,\ i=0,1,2, \quad b_j(x) \geqslant t^3l^2\beta_0^h > 0,\ j=0,1, \quad \text{a.e. in } \Omega, \tag{3.33}$$

where $\alpha_0^h = \frac{2}{15}\alpha_0$ and $\beta_0^h = \frac{1}{30}\alpha_0$.

By (3.32) and (3.33) we obtain the following strong convexity conditions on $\mathbb{P}+\mathbb{P}^h$ and $\mathbb{Q}$. For every $A \in \widehat{\mathbb{M}}^2$ we have

$$(\mathbb{P}+\mathbb{P}^h)A\cdot A \geqslant t(t^2+l^2)\xi_{\mathbb{P}}|A|^2 \quad \text{a.e. in } \Omega; \tag{3.34}$$

for every $B \in \widehat{\mathbb{M}}^3$ we have

$$\mathbb{Q}B\cdot B \geqslant t^3l^2\xi_{\mathbb{Q}}|B|^2 \quad \text{a.e. in } \Omega; \tag{3.35}$$

where $\xi_{\mathbb{P}}$, $\xi_{\mathbb{Q}}$ are positive constants only depending on α_0 and γ_0.

(iii) *Bounds on the jumps.*

Either there exists $\eta>0$, $\overline{\eta}>0$ and $\delta>1$, $\overline{\delta}>1$ such that

$$\eta(\mathbb{P}+\mathbb{P}^h) \leqslant (\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h)-(\mathbb{P}+\mathbb{P}^h) \leqslant (\delta-1)(\mathbb{P}+\mathbb{P}^h) \quad \text{a.e. in } \Omega, \tag{3.36}$$

$$\overline{\eta}\mathbb{Q} \leqslant \widetilde{\mathbb{Q}}-\mathbb{Q} \leqslant (\overline{\delta}-1)\mathbb{Q} \quad \text{a.e. in } \Omega, \tag{3.37}$$

or there exists $\eta>0$, $\overline{\eta}>0$ and $0<\delta<1$, $0<\overline{\delta}<1$ such that

$$\eta(\mathbb{P}+\mathbb{P}^h) \leqslant (\mathbb{P}+\mathbb{P}^h)-(\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h) \leqslant (1-\delta)(\mathbb{P}+\mathbb{P}^h) \quad \text{a.e. in } \Omega, \tag{3.38}$$

$$\overline{\eta}\mathbb{Q} \leqslant \mathbb{Q}-\widetilde{\mathbb{Q}} \leqslant (1-\overline{\delta})\mathbb{Q} \quad \text{a.e. in } \Omega. \tag{3.39}$$

Let us note that assumptions (ii) and (iii) ensure that $\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h$ and $\widetilde{\mathbb{Q}}$ are strongly convex a.e. in Ω.

(iv) *Regularity for* $\mathbb{P}$, $\mathbb{P}^h$ and $\mathbb{Q}$.

We assume $\mathbb{P}, \mathbb{P}^h \in C^{1,1}(\overline{\Omega})$ and $\mathbb{Q} \in C^{2,1}(\overline{\Omega})$, with

$$\|\mathbb{P}\|_{C^{1,1}(\overline{\Omega})}+\|\mathbb{P}^h\|_{C^{1,1}(\overline{\Omega})}+r_0^{-2}\|\mathbb{Q}\|_{C^{2,1}(\overline{\Omega})} \leqslant M_2r_0^3, \tag{3.40}$$

with M_2 depending on $\frac{t}{r_0}$, $\frac{l}{r_0}$.

On the boundary data appearing in (3.3)–(3.5) we assume

$$\widehat{V} \in H^{-3/2}(\partial\Omega), \quad \widehat{M}_n \in H^{-1/2}(\partial\Omega), \quad \widehat{M}_n^h \in H^{1/2}(\partial\Omega) \tag{3.41}$$

and we obviously assume the compatibility conditions (3.14).

In what follows we denote by u, u_0 the solutions of the equilibrium problem for the nanoplate (3.2)–(3.5) with and without inclusion, namely $u \in H^3(\Omega)$ is the solution to (3.2)–(3.5) when $M(u) = -(\chi_{\Omega\setminus\tilde{\Omega}}(\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}}(\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h))D^2u$, $\overline{M}^h(u) = (\chi_{\Omega\setminus\tilde{\Omega}}\mathbb{Q}+\chi_{\tilde{\Omega}}\widetilde{\mathbb{Q}})D^3u$ and $u_0 \in H^3(\Omega)$ is the solution to (3.2)–(3.5) when $M(u_0) = -(\mathbb{P}+\mathbb{P}^h)D^2u_0$, $\overline{M}^h(u_0) = \mathbb{Q}D^3u_0$. Let us recall that u and u_0 are uniquely determined by the normalization conditions (3.18).

Note that the boundary data $\widehat{V}$, $\widehat{M}_n$, $\widehat{M}_n^h$ associated to the problem for u and u_0 are the *same*. Finally, let us introduce the quantities

$$W = L(u) = -\int_{\partial\Omega} \widehat{V}u + \widehat{M}_n u_{,n} + \widehat{M}_n^h u_{,nn}\,, \tag{3.42}$$

$$W_0 = L(u_0) = -\int_{\partial\Omega} \widehat{V}u_0 + \widehat{M}_n u_{0,n} + \widehat{M}_n^h u_{0,nn}, \tag{3.43}$$

which represent the work exerted by the boundary data when the inclusion $\tilde{\Omega}$ is present or absent, respectively. By the weak formulation of the corresponding problems, the works W and W_0 coincide with the strain energy stored in the deformed microplate, namely

$$W = \int_{\Omega} (\chi_{\Omega\setminus\tilde{\Omega}}(\mathbb{P}+\mathbb{P}^h) + \chi_{\tilde{\Omega}}(\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h))D^2u\cdot D^2u + (\chi_{\Omega\setminus\tilde{\Omega}}\mathbb{Q} + \chi_{\tilde{\Omega}}\widetilde{\mathbb{Q}})D^3u\cdot D^3u, \tag{3.44}$$

$$W_0 = \int_{\Omega} (\mathbb{P}+\mathbb{P}^h)D^2u_0\cdot D^2u_0 + \mathbb{Q}D^3u_0\cdot D^3u_0. \tag{3.45}$$

3.3. *Main results*

We are now in position to state our size estimates results for nanoplates.

Theorem 3.4 (lower bound of $|\tilde{\Omega}|$). *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is of $C^{2,1}$-class with constants r_0, M_0 and satisfying (3.1). Let $\tilde{\Omega}$, $\tilde{\Omega}\subset\subset\Omega$, be a measurable subset of Ω satisfying*

$$dist(\tilde{\Omega},\partial\Omega) \geqslant d_0 r_0, \tag{3.46}$$

where d_0 is a positive constant. Let the tensors $\mathbb{P}$, $\mathbb{P}^h$, $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{P}}^h \in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ and $\mathbb{Q}$, $\widetilde{\mathbb{Q}} \in L^\infty(\Omega,\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ satisfy the symmetry conditions (3.7), (3.8) and (3.11), the strong convexity conditions (3.9) and (3.12), and either the jump conditions (3.36)–(3.37) or (3.38)–(3.39). Moreover, let the tensors $\mathbb{P}$, $\mathbb{P}^h$, $\mathbb{Q}$ satisfy the regularity conditions iv).

If (3.36)–(3.37) hold, then we have

$$|\tilde{\Omega}| \geqslant C_1^+ r_0^2 \frac{W_0 - W}{W}. \tag{3.47}$$

If, conversely, (3.38)–(3.39) hold, then we have

$$|\tilde{\Omega}| \geqslant C_1^- r_0^2 \frac{W - W_0}{W_0}. \tag{3.48}$$

Here the constants C_1^+, C_1^- depend only on $\frac{t}{r_0}$, M_0, M_1, d_0, $\xi_{\mathbb{P}}$, $\xi_{\mathbb{Q}}$, M_2, δ, $\overline{\delta}$.

Theorem 3.5 (upper bound of $|\tilde{\Omega}|$ for fat inclusions). *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is of $C^{3,1}$-class with constants r_0, M_0 and satisfying (3.1). Let $\tilde{\Omega}$ be a measurable subset of Ω satisfying*

$$|\tilde{\Omega}_{h_1 r_0}| \geqslant \frac{1}{2}|\tilde{\Omega}|, \tag{3.49}$$

for a given positive constant h_1. Let the tensors $\mathbb{P}$, $\mathbb{P}^h \in C^{1,1}(\overline{\Omega},\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2))$ and $\mathbb{Q} \in C^{2,1}(\overline{\Omega},\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ satisfy the isotropy conditions (3.24), (3.25) and (3.26), respectively, and let the Lamé moduli μ and λ satisfy the strong convexity conditions (3.32). Let the *tensors* $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{P}}^h \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2))$ *and* $\widetilde{\mathbb{Q}} \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$ *and let us assume the jump conditions (iii).*

If (3.36)–(3.37) hold, then we have

$$|\tilde{\Omega}| \leqslant C_2^+ r_0^2 \frac{W_0 - W}{W_0}. \tag{3.50}$$

If, conversely, (3.38)–(3.39) hold, then we have

$$|\tilde{\Omega}| \leqslant C_2^- r_0^2 \frac{W - W_0}{W_0}. \tag{3.51}$$

Here the constants C_2^+, C_2^- *depend only on* $\frac{t}{r_0}$, $\frac{l}{r_0}$, M_0, M_1, h_1, α_0, γ_0, M_2, η, $\overline{\eta}$, δ, $\overline{\delta}$ *and on the ratio*

$$F = \frac{\|\widehat{V}\|_{H^{-3/2}(\partial\Omega)} + r_0^{-1}\|\widehat{M}_n\|_{H^{-1/2}(\partial\Omega)} + r_0^{-2}\|\widehat{M}_n^h\|_{H^{1/2}(\partial\Omega)}}{\|\widehat{V}\|_{H^{-5/2}(\partial\Omega)} + r_0^{-1}\|\widehat{M}_n\|_{H^{-3/2}(\partial\Omega)} + r_0^{-2}\|\widehat{M}_n^h\|_{H^{-1/2}(\partial\Omega)}}. \tag{3.52}$$

Theorem 3.6 (upper bound of $|\tilde{\Omega}|$ for general inclusions). *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is of $C^{3,1}$-class with constants r_0, M_0 and satisfying (3.1). Let $\tilde{\Omega}$, $\tilde{\Omega} \subset\subset \Omega$, be a measurable subset of Ω satisfying*

$$dist(\tilde{\Omega}, \partial\Omega) \geqslant d_0 r_0, \tag{3.53}$$

where d_0 is a positive constant. Let the tensors $\mathbb{P}$, $\mathbb{P}^h \in C^{1,1}(\overline{\Omega}, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2))$ and $\mathbb{Q} \in C^{2,1}(\overline{\Omega}, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$ satisfy the isotropy conditions (3.24), (3.25) and (3.26), respectively, and let the Lamé moduli μ and λ satisfy the strong convexity conditions (3.32). Let the tensors $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{P}}^h \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2))$ and $\widetilde{\mathbb{Q}} \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$ and let us assume the jump conditions (iii).

If (3.36)–(3.37) hold, then we have

$$|\tilde{\Omega}| \leqslant C_2^+ r_0^2 \left(\frac{W_0 - W}{W_0}\right)^{1/p}. \tag{3.54}$$

If, conversely, (3.38)–(3.39) hold, then we have

$$|\tilde{\Omega}| \leqslant C_2^- r_0^2 \left(\frac{W - W_0}{W_0}\right)^{1/p}. \tag{3.55}$$

Here the constants C_2^+, C_2^- *and* $p > 1$ *depend only on* $\frac{t}{r_0}$, $\frac{l}{r_0}$, M_0, M_1, d_0, α_0, γ_0, M_2, η, $\overline{\eta}$, δ, $\overline{\delta}$ *and on the ratio F given in (3.52).*

4. Proof of theorem 3.4

Let us premise the following energy lemma, which states that the work gap $|W - W_0|$ is estimated from above and from below by the strain energy of the unperturbed solution u_0 stored in the inclusion $\tilde{\Omega}$.

Lemma 4.1 (energy lemma). *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of $C^{2,1}$-class. Let $\tilde{\Omega}$ be a measurable subset of Ω. Let the tensors $\mathbb{P}$, $\mathbb{P}^h$, $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{P}}^h \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2))$ and $\mathbb{Q}$, $\widetilde{\mathbb{Q}} \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$ satisfy the symmetry conditions (3.7), (3.8), (3.11), respectively. Let ξ_0, ξ_1, $\overline{\xi_0}$, $\overline{\xi_1}$, $0 < \xi_0 < \xi_1$, $0 < \overline{\xi_0} < \overline{\xi_1}$, be such that*

$$t^3\xi_0|A|^2 \leqslant (\mathbb{P}(x)+\mathbb{P}^h(x))A\cdot A \leqslant t^3\xi_1|A|^2 \quad \textit{for a.e. } x\in\Omega, \tag{4.1}$$

$$t^5\overline{\xi_0}|B|^2 \leqslant \mathbb{Q}(x)B\cdot B \leqslant t^5\overline{\xi_1}|B|^2 \quad \textit{for a.e. } x\in\Omega, \tag{4.2}$$

for every matrix $A\in\widehat{\mathbb{M}}^2$ and $B\in\widehat{\mathbb{M}}^3$. Let the jumps $(\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h)-(\mathbb{P}+\mathbb{P}^h)$, $\widetilde{\mathbb{Q}}-\mathbb{Q}$ satisfy either (3.36)–(3.37) or (3.38)–(3.39). Let $u,u_0\in H^3(\Omega)$ be the weak solution to the problem (3.2)–(3.5), normalized by (3.18), when the inclusion D is present or absent, respectively, for the Neumann data $\widehat{V}\in H^{-5/2}(\partial\Omega)$, $\widehat{M}_n\in H^{-3/2}(\partial\Omega)$, $\widehat{M}_n^h\in H^{-1/2}(\partial\Omega)$ that fulfill the compatibility conditions (3.14).

If (3.36)–(3.37) hold, then

$$\frac{\eta_*\xi_{0*}t^3}{\delta^*}\int_{\tilde{\Omega}}|D^2u_0|^2+t^2|D^3u_0|^2 \leqslant W_0-W \leqslant (\delta^*-1)\xi_1^*t^3\int_{\tilde{\Omega}}|D^2u_0|^2+t^2|D^3u_0|^2; \tag{4.3}$$

if, conversely, (3.38)–(3.39) hold, then

$$\eta_*\xi_{0*}t^3\int_{\tilde{\Omega}}|D^2u_0|^2+t^2|D^3u_0|^2 \leqslant W-W_0 \leqslant \frac{(1-\delta_*)\xi_1^*t^3}{\delta_*}\int_{\tilde{\Omega}}|D^2u_0|^2+t^2|D^3u_0|^2. \tag{4.4}$$

Here $\eta_*=\min\{\eta,\overline{\eta}\}$, $\delta^*=\max\{\delta,\overline{\delta}\}$, $\xi_{0*}=\min\{\xi_0,\overline{\xi}_0\}$, $\xi_1^*=\max\{\xi_1,\overline{\xi}_1\}$, $\delta_*=\min\{\delta,\overline{\delta}\}$.

Remark 4.2. Let us note that if the materials constituting the inclusion $\tilde{\Omega}$ and the surrounding material in $\Omega\setminus\tilde{\Omega}$ are isotropic with Lamé moduli $\widetilde{\mu}$, $\widetilde{\lambda}$ and μ, λ, respectively, then the jump conditions (3.36)–(3.39) can be written in terms of the difference $\widetilde{\mu}-\mu$ and $\widetilde{\kappa}-\kappa$, where $\widetilde{\kappa}=\frac{2\widetilde{\mu}(2\widetilde{\mu}+3\widetilde{\lambda})}{2\widetilde{\mu}+\widetilde{\lambda}}$, $\kappa=\frac{2\mu(2\mu+3\lambda)}{2\mu+\lambda}$.

Proof. Let us assume conditions (3.36)–(3.37) (i.e. the material of the inclusion $\tilde{\Omega}$ is stiffer than the surrounding material in $\Omega\setminus\tilde{\Omega}$) and prove inequalities (4.3). The proof of the estimates (4.4) is similar.

We start by determining some basic identities. Let us denote by $u_1\in H^3(\Omega)$ and $u_2\in H^3(\Omega)$ the weak solution to (3.2)–(3.5) when the inclusion $\tilde{\Omega}_1$ or $\tilde{\Omega}_2$ is present, respectively. Since the boundary data are the same, for every $w\in H^3(\Omega)$ we have

$$\begin{aligned}&\int_\Omega((\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{P})D^2u_1\cdot D^2w+(\mathbb{Q}+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{Q})D^3u_1\cdot D^3w\\&\quad=\int_\Omega((\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}_2}\mathbb{H}_\mathbb{P})D^2u_2\cdot D^2w+(\mathbb{Q}+\chi_{\tilde{\Omega}_2}\mathbb{H}_\mathbb{Q})D^3u_2\cdot D^3w,\end{aligned} \tag{4.5}$$

where we have defined

$$\mathbb{H}_\mathbb{P}=(\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h)-(\mathbb{P}+\mathbb{P}^h),\quad \mathbb{H}_\mathbb{Q}=\widetilde{\mathbb{Q}}-\mathbb{Q}. \tag{4.6}$$

Note that the tensors $\mathbb{H}_\mathbb{P}$, $\mathbb{H}_\mathbb{Q}$ satisfy the symmetry conditions (3.7)–(3.8) and (3.11), respectively.

By subtracting $\int_\Omega((\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{P})D^2u_2\cdot D^2w+(\mathbb{Q}+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{Q})D^3u_2\cdot D^3w$ to both sides of (4.5) we obtain

$$\begin{aligned}&\int_\Omega((\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{P})D^2(u_1-u_2)\cdot D^2w+(\mathbb{Q}+\chi_{\tilde{\Omega}_1}\mathbb{H}_\mathbb{Q})D^3(u_1-u_2)\cdot D^3w\\&\quad=\int_\Omega(\chi_{\tilde{\Omega}_2}-\chi_{\tilde{\Omega}_1})(\mathbb{H}_\mathbb{P}D^2u_2\cdot D^2w+\mathbb{H}_\mathbb{Q}D^3u_2\cdot D^3w),\end{aligned} \tag{4.7}$$

for every $w\in H^3(\Omega)$.

Let us choose $w = u_1$ in (4.7). By using the symmetry conditions on $\mathbb{P}$, $\mathbb{P}^h$, $\mathbb{Q}$, $\mathbb{H}_{\mathbb{P}}$ and $\mathbb{H}_{\mathbb{Q}}$, and considering $u_1 - u_2$ as test function in the weak formulation for u_1, we have

$$
\begin{aligned}
&-\int_{\partial\Omega} \widehat{V}(u_1-u_2) + \widehat{M}_n(u_{1,n}-u_{2,n}) + \widehat{M}_n^h(u_{1,nn}-u_{2,nn}) \\
&\quad = \int_{\Omega} (\chi_{\tilde{\Omega}_2} - \chi_{\tilde{\Omega}_1})(\mathbb{H}_{\mathbb{P}} D^2u_2 \cdot D^2u_1 + \mathbb{H}_{\mathbb{Q}} D^3u_2 \cdot D^3u_1).
\end{aligned} \tag{4.8}
$$

Next, we choose $w = u_1 - u_2$ in (4.7). By using (4.8), after simple algebra we obtain the following identity

$$
\begin{aligned}
&\int_{\Omega} ((\mathbb{P}+\mathbb{P}^h) + \chi_{\tilde{\Omega}_1}\mathbb{H}_{\mathbb{P}})D^2(u_1-u_2)\cdot D^2(u_1-u_2) + (\mathbb{Q}+\chi_{\tilde{\Omega}_1}\mathbb{H}_{\mathbb{Q}})D^3(u_1-u_2)\cdot D^3(u_1-u_2) \\
&+ \int_{\tilde{\Omega}_2\setminus\tilde{\Omega}_1} \mathbb{H}_{\mathbb{P}} D^2u_2\cdot D^2u_2 + \mathbb{H}_{\mathbb{Q}} D^3u_2\cdot D^3u_2 \\
&\quad = -\int_{\partial\Omega} \widehat{V}(u_1-u_2) + \widehat{M}_n(u_{1,n}-u_{2,n}) + \widehat{M}_n^h(u_{1,nn}-u_{2,nn}) \\
&\qquad + \int_{\tilde{\Omega}_1\setminus\tilde{\Omega}_2} \mathbb{H}_{\mathbb{P}} D^2u_2\cdot D^2u_2 + \mathbb{H}_{\mathbb{Q}} D^3u_2\cdot D^3u_2.
\end{aligned} \tag{4.9}
$$

By choosing $\tilde{\Omega}_1 = \tilde{\Omega}$ (i.e. $u_1 = u$) and $\tilde{\Omega}_2 = \emptyset$ (i.e. $u_2 = u_0$) in (4.9), we obtain the first fundamental identity

$$
\begin{aligned}
&\int_{\Omega} ((\mathbb{P}+\mathbb{P}^h)\chi_{\Omega\setminus\tilde{\Omega}} + (\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h)\chi_{\tilde{\Omega}})D^2(u-u_0)\cdot D^2(u-u_0) \\
&+ \int_{\Omega} (\mathbb{Q}\chi_{\Omega\setminus\tilde{\Omega}} + \widetilde{\mathbb{Q}}\chi_{\tilde{\Omega}})D^3(u-u_0)\cdot D^3(u-u_0) - \int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{P}} D^2u_0\cdot D^2u_0 + \mathbb{H}_{\mathbb{Q}} D^3u_0\cdot D^3u_0 \\
&\quad = -\int_{\partial\Omega} \widehat{V}(u-u_0) + \widehat{M}_n(u_{,n}-u_{0,n}) + \widehat{M}_n^h(u_{,nn}-u_{0,nn}) = W - W_0.
\end{aligned} \tag{4.10}
$$

A second fundamental identity is obtained by choosing $\tilde{\Omega}_1 = \emptyset$ ($u_1 = u_0$) and $\tilde{\Omega}_2 = \tilde{\Omega}$ ($u_2 = u$) in (4.9):

$$
\begin{aligned}
&\int_{\Omega} (\mathbb{P}+\mathbb{P}^h)D^2(u_0-u)\cdot D^2(u_0-u) + \mathbb{Q}D^3(u_0-u)\cdot D^3(u_0-u) + \int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{P}} D^2u\cdot D^2u + \mathbb{H}_{\mathbb{Q}} D^3u\cdot D^3u \\
&\quad = -\int_{\partial\Omega} \widehat{V}(u_0-u) + \widehat{M}_n(u_{0,n}-u_{,n}) + \widehat{M}_n^h(u_{0,nn}-u_{,nn}) = W_0 - W.
\end{aligned} \tag{4.11}
$$

Next, let us choose $w = u_0$ as test function in the weak formulation (3.15) of the Neumann problem (3.2)–(3.5) when the inclusion $\tilde{\Omega}$ is present, obtaining

$$\int_{\Omega}((\mathbb{P}+\mathbb{P}^h)+\chi_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}})D^2u\cdot D^2u_0+(\mathbb{Q}+\chi_{\tilde{\Omega}}\mathbb{H}_{\mathbb{Q}})D^3u\cdot D^3u_0=-\int_{\partial\Omega}\widehat{V}u_0+\widehat{M}_nu_{0,n}+\widehat{M}_n^hu_{0,nn}. \tag{4.12}$$

Conversely, choosing the solution u of (3.2)–(3.5) when the inclusion $\tilde{\Omega}$ is present as test function in the weak formulation (3.15) when the inclusion is absent, we have

$$\int_{\Omega}(\mathbb{P}+\mathbb{P}^h)D^2u_0\cdot D^2u+\mathbb{Q}D^3u_0\cdot D^3u=-\int_{\partial\Omega}\widehat{V}u+\widehat{M}_nu_{,n}+\widehat{M}_n^hu_{,nn}. \tag{4.13}$$

By subtracting (4.13) from (4.12), we obtain a third fundamental identity

$$\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}}D^2u\cdot D^2u_0+\mathbb{H}_{\mathbb{Q}}D^3u\cdot D^3u_0=\int_{\partial\Omega}\widehat{V}(u_0-u)+\widehat{M}_n(u_{0,n}-u_{,n})+\widehat{M}_n^h(u_{0,nn}-u_{,nn}). \tag{4.14}$$

We are now in position to derive the estimates (4.3).

Using the positivity of $\mathbb{P}+\mathbb{P}^h$, $\widetilde{\mathbb{P}}+\widetilde{\mathbb{P}}^h$, $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, from the first identity (4.10) we obtain

$$\begin{aligned}W_0-W&=-\int_{\partial\Omega}\widehat{V}(u_0-u)+\widehat{M}_n(u_{0,n}-u_{,n})+\widehat{M}_n^h(u_{0,nn}-u_{,nn})\\&\leqslant\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}}D^2u_0\cdot D^2u_0+\mathbb{H}_{\mathbb{Q}}D^3u_0\cdot D^3u_0\end{aligned} \tag{4.15}$$

and the estimate from above of the work gap W_0-W in (4.3) easily follows from (3.36)–(3.37) and from (4.1)–(4.2).

To get the estimate from below of W_0-W, we use the following inequality

$$\begin{aligned}\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}}D^2u_0\cdot D^2u_0+\mathbb{H}_{\mathbb{Q}}D^3u_0\cdot D^3u_0\leqslant{}&(1+\epsilon)\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}}D^2(u-u_0)\cdot D^2(u-u_0)\\&+\left(1+\frac{1}{\epsilon}\right)\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{P}}D^2u\cdot D^2u\\&+(1+\overline{\epsilon})\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{Q}}D^3(u-u_0)\cdot D^3(u-u_0)\\&+\left(1+\frac{1}{\overline{\epsilon}}\right)\int_{\tilde{\Omega}}\mathbb{H}_{\mathbb{Q}}D^3u\cdot D^3u,\end{aligned} \tag{4.16}$$

for every $\epsilon>0$, $\overline{\epsilon}>0$. The above inequality stems from the following arguments. We notice that

$$\begin{aligned}\mathbb{H}_{\mathbb{P}}D^2u_0\cdot D^2u_0={}&\mathbb{H}_{\mathbb{P}}D^2(u_0-u)\cdot D^2(u_0-u)+\mathbb{H}_{\mathbb{P}}D^2u\cdot D^2u\\&+\ 2\mathbb{H}_{\mathbb{P}}D^2(u_0-u)\cdot D^2u.\end{aligned} \tag{4.17}$$

By the positivity condition (3.36), we have that

$$\mathbb{H}_{\mathbb{P}}\left(\sqrt{\epsilon}D^2(u_0-u)-\frac{1}{\sqrt{\epsilon}}D^2u\right)\cdot\left(\sqrt{\epsilon}D^2(u_0-u)-\frac{1}{\sqrt{\epsilon}}D^2u\right)\geqslant0. \tag{4.18}$$

By the symmetry properties (3.7), (3.8) we have that

$$\mathbb{H}_{\mathbb{P}}D^2(u_0-u)\cdot D^2u=\mathbb{H}_{\mathbb{P}}D^2u\cdot D^2(u_0-u)$$

which combined with (4.18) leads

$$2\mathbb{H}_{\mathbb{P}}D^2(u_0-u)\cdot D^2u\leqslant\epsilon\,\mathbb{H}_{\mathbb{P}}D^2(u_0-u)\cdot D^2(u_0-u)+\frac{1}{\epsilon}\,\mathbb{H}_{\mathbb{P}}D^2u\cdot D^2u. \tag{4.19}$$

By (4.17) and (4.19) we have that

$$\mathbb{H}_{\mathbb{P}} D^2 u_0 \cdot D^2 u_0 \leqslant (1+\epsilon)\mathbb{H}_{\mathbb{P}} D^2(u_0-u)\cdot D^2(u_0-u) + \left(1+\frac{1}{\epsilon}\right)\mathbb{H}_{\mathbb{P}} D^2 u \cdot D^2 u.$$

With similar arguments we may handle the terms involving the sixth order tensor $\mathbb{H}_{\mathbb{Q}}$ obtaining

$$\mathbb{H}_{\mathbb{Q}} D^3 u_0 \cdot D^3 u_0 \leqslant (1+\overline{\epsilon})\mathbb{H}_{\mathbb{Q}} D^3(u_0-u)\cdot D^3(u_0-u) + \left(1+\frac{1}{\overline{\epsilon}}\right)\mathbb{H}_{\mathbb{Q}} D^3 u \cdot D^3 u.$$

The last two inequalities lead to (4.16).

By using the jump conditions (3.36)–(3.37), by choosing $\epsilon = (\delta-1)^{-1}$, $\overline{\epsilon} = \left(\overline{\delta}-1\right)^{-1}$ in (4.16), and employing identity (4.11) we get

$$\begin{aligned}
&\int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{P}} D^2 u_0 \cdot D^2 u_0 + \mathbb{H}_{\mathbb{Q}} D^3 u_0 \cdot D^3 u_0 \\
&\quad \leqslant (1+\epsilon)\int_{\tilde{\Omega}} (\delta-1)(\mathbb{P}+\mathbb{P}^h)D^2(u-u_0)\cdot D^2(u-u_0) + \left(1+\frac{1}{\epsilon}\right)\int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{P}} D^2 u\cdot D^2 u \\
&\qquad + (1+\overline{\epsilon})\int_{\tilde{\Omega}} (\overline{\delta}-1)\mathbb{Q}D^3(u-u_0)\cdot D^3(u-u_0) + \left(1+\frac{1}{\overline{\epsilon}}\right)\int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{Q}} D^3 u\cdot D^3 u \\
&\quad \leqslant \max\{\delta,\overline{\delta}\}\left\{\int_{\Omega} (\mathbb{P}+\mathbb{P}^h)D^2(u_0-u)\cdot D^2(u_0-u) + \mathbb{Q}D^3(u_0-u)\cdot D^3(u_0-u)\right. \\
&\qquad \left. + \int_{\tilde{\Omega}} \mathbb{H}_{\mathbb{P}} D^2 u\cdot D^2 u + \mathbb{H}_{\mathbb{Q}} D^3 u \cdot D^3 u\right\} = \max\{\delta,\overline{\delta}\}(W_0-W).
\end{aligned} \tag{4.20}$$

Finally, the estimate from below of $W_0 - W$ in (4.3) follows from (4.20) and (3.36)–(3.37). □

Proof of theorem 3.4. To fix the ideas, let us assume that the jump conditions (3.36)–(3.37) hold. Let us estimate the right hand side of (4.3). Let us notice that there exists $d^*, 0 < d^* < d_0$, only depending on M_0, such that $\Omega_{d^* r_0}$ is of Lipschitz class with constants γr_0, $\gamma' M_0$, where $0<\gamma<1$ and $\gamma'>1$ only depend on M_0, and $\tilde{\Omega} \subset \Omega_{d^* r_0}$(see [17, lemma 14.16] for details). By Sobolev Imbedding theorem [1, chapter 5, theorem 5.4]

$$\begin{aligned}
W_0 - W &\leqslant C r_0^3 \int_{\tilde{\Omega}} |D^2 u_0|^2 + r_0^2 |D^3 u_0|^2 \leqslant C r_0^3 |\tilde{\Omega}| \left(\|D^2 u\|^2_{L^\infty(\tilde{\Omega})} + r_0^2 \|D^3 u\|^2_{L^\infty(\tilde{\Omega})}\right) \\
&\leqslant C\frac{1}{r_0}|\tilde{\Omega}|\|u_0\|^2_{H^6(\Omega_{d^* r_0})}.
\end{aligned} \tag{4.21}$$

We observe that by a covering argument and interior regularity estimates (3.23) we have

$$\|u_0\|_{H^6(\Omega_{d^* r_0})} \leqslant C\|u_0\|_{H^3(\Omega)}\ . \tag{4.22}$$

Hence, by (4.21) and (4.22), standard Poincarè inequality (see [25, proposition 3.3]), by (3.34) and (3.35), we obtain

$$\begin{aligned}
W_0 - W &\leqslant C\frac{1}{r_0}|\tilde{\Omega}|\|u_0\|^2_{H^3(\Omega)} \leqslant C r_0 |\tilde{\Omega}| \int_{\Omega} |D^2 u_0|^2 + r_0^2 |D^3 u_0|^2 \\
&\leqslant C\frac{|\tilde{\Omega}|}{r_0^2}\int_{\Omega} (\mathbb{P}+\mathbb{P}^h)D^2 u_0\cdot D^2 u_0 + \mathbb{Q}D^3 u_0\cdot D^3 u_0 = \frac{C}{r_0^2}|\tilde{\Omega}|W_0,
\end{aligned}$$

where $C>0$ depends on $d_0, \delta, \overline{\delta}, \frac{t}{r_0}, M_0, M_1, \xi_{\mathbb{P}}, \xi_{\mathbb{Q}}, M_2$. Hence, estimate (3.47) follows. □

5. Doubling and three spheres inequality for the Hessian

This section is devoted to strong unique continuation estimates for solutions to equation (3.2) for the isotropic case only. Such estimates are given in the form of doubling inequality and three spheres inequality for the Hessian of the solutions. The latter are crucial tools of unique continuation needed in the proof of upper bound estimates for the size of both the considered inclusions. We shall premise the proof of the main result of this section with some auxiliary results which are contained in [24].

For simplicity of notation in this section we denote by u a weak solution to the partial differential equation (3.2).

Proposition 5.1 (doubling inequality and three sphere inequality for solutions to (3.2)). *Let* $\mathbb{P},\mathbb{P}^h \in C^{1,1}(\overline{B_1},\mathcal{L}(\widehat{\mathbb{M}}^2,\widehat{\mathbb{M}}^2)),\mathbb{Q} \in C^{2,1}(\overline{B_1},\mathcal{L}(\widehat{\mathbb{M}}^3,\widehat{\mathbb{M}}^3))$ *be given by (3.24), (3.25), (3.26) and satisfying the regularity condition (3.40), the strong convexity conditions (3.34), (3.35), respectively. Let* $u \in H^6(B_1)$ *be a weak solution to (3.2).*

Then there exists an absolute constant $R_1 \in (0,1]$ *such that for every* $r \leqslant s \leqslant \frac{R_1}{2^8}$*, we have*

$$\int_{B_s} u^2 \leqslant CN^{\bar{k}} \left(\frac{s}{r}\right)^{\log_2\left(CN^{\bar{k}}\right)} \int_{B_r} u^2, \tag{5.1}$$

where N *is given by*

$$N = \frac{\int_{B_{R_1}} u^2}{\int_{B_{R_1/2^7}} u^2}$$

and $\bar{k} = 8$.

In addition, if $2r \leqslant s \leqslant \frac{R_1}{2^8}$ *then we have*

$$\int_{B_s} u^2 \leqslant \left(C\int_{B_{R_1}} u^2\right)^{1-\widetilde{\theta}(s,r)} \left(\int_{B_r} u^2\right)^{\widetilde{\theta}(s,r)}, \tag{5.2}$$

where

$$\widetilde{\theta}(s,r) = \frac{1}{1+2\bar{k}\log_2\frac{s}{r}}$$

and the constant $C>0$ *only depends on* $M_2,\alpha_0,\gamma_0,t,l$.

Proof. See corollary 4.10 in [24]. □

Lemma 5.2 (Caccioppoli-type inequality). *Let us assume that the hypothesis of proposition 5.1 are satisfied. Then, for every* r, $0<r<1$, *we have*

$$\|D^h u\|_{L^2(B_{\frac{r}{2}})} \leqslant \frac{C}{r^h}\|u\|_{L^2(B_r)}, \quad \forall h = 1,\ldots,6, \tag{5.3}$$

where $C>0$ *is a constant only depending on* $M_2,\alpha_0,\gamma_0,t,l$ *only.*

Proof. For the proof we refer to [24, lemma 4.7]. □

Let us now recall a Poincaré-type inequality. Let R,r positive numbers such that $r \leqslant R$. For a given function $u \in H^2(B_R)$ denote

$$(u)_r = \frac{1}{|B_r|}\int_{B_r} u, \qquad (Du)_r = \frac{1}{|B_r|}\int_{B_r} Du \tag{5.4}$$

and

$$\widetilde{u}_r = u(x) - (u)_r - (Du)_r \cdot x \tag{5.5}$$

Proposition 5.3 (Poincaré inequality). *There exists a positive absolute constant C such that*

$$\int_{B_R} |\widetilde{u}_r|^2 + R^2 \int_{B_R} |D\widetilde{u}_r|^2 \leqslant C\frac{R^6}{r^2} \int_{B_R} \left|D^2\widetilde{u}_r\right|^2, \tag{5.6}$$

for every $u \in H^2(B_R)$ and for every $r \in (0, R]$.

Proof. See [6] and [25, proposition 6.1]. □

Proposition 5.4 (doubling inequality and three sphere inequality for the Hessian). *Let us assume that the hypothesis of proposition 5.1 are satisfied. Then there exists $C > 1$, only depending on $M_2, \alpha_0, \gamma_0, t, l$, such that, for every $0 < r < \frac{R_1}{2^{11}}$ we have*

$$\int_{B_{2r}} \left|D^2u\right|^2 \leqslant C\overline{N}^{3\bar{k}} \int_{B_r} \left|D^2u\right|^2, \tag{5.7}$$

where

$$\overline{N} = \frac{\left\|D^2u\right\|^2_{L^2(B_{R_1})}}{\|D^2u\|^2_{L^2\left(B_{R_1/2^9}\right)}} \tag{5.8}$$

and $\bar{k} = 8$.

In addition, if $2r \leqslant s \leqslant \frac{R_1}{2^{11}}$ then we have

$$\int_{B_s} \left|D^2u\right|^2 \leqslant \left(C\int_{B_{R_1/2}} \left|D^2u\right|^2\right)^{1-\theta(s,r)} \left(\int_{B_r} \left|D^2u\right|^2\right)^{\theta(s,r)}, \tag{5.9}$$

where

$$\theta(s,r) = \frac{1}{1 + 6\bar{k}\log_2 \frac{s}{r}} \tag{5.10}$$

with $\bar{k} = 8$.

Proof. Let

$$0 < 4r < \frac{R_2}{2^8}, \tag{5.11}$$

with $R_2 = \frac{R_1}{2}$. We define $v = \widetilde{u}_{R_2}$ and we observe that since $|D^2u| = |D^2v|$ we may as well prove (5.7) and (5.9) for v instead.

Let us note that v is still a solution to (3.2).

Hence by lemma 5.2 we have that for every $r \in (0, R_1]$ the following holds

$$r^4 \int_{B_{2r}} \left|D^2v\right|^2 = r^4 \int_{B_{2r}} \left|D^2\widetilde{v}_r\right|^2 \leqslant C \int_{B_{4r}} |\widetilde{v}_r|^2 \,. \tag{5.12}$$

By (5.6) we have

$$\int_{B_r} |\widetilde{v}_r|^2 \leqslant Cr^4 \int_{B_r} \left|D^2\widetilde{v}_r\right|^2 = Cr^4 \int_{B_r} \left|D^2v\right|^2. \tag{5.13}$$

Now, denote by

$$\widetilde{N}_r = \frac{\int_{B_{R_2}} |\widetilde{v}_r|^2}{\int_{B_{R_2/2^7}} |\widetilde{v}_r|^2}. \tag{5.14}$$

By (5.1) and (5.12)–(5.14) we have

$$r^4 \int_{B_{2r}} \left|D^2 v\right|^2 \leqslant C \int_{B_{4r}} |\widetilde{v}_r|^2 \leqslant C\widetilde{N}_r^{3\bar{k}} \int_{B_r} |\widetilde{v}_r|^2 \leqslant Cr^4 \widetilde{N}_r^{3\bar{k}} \int_{B_r} \left|D^2 v\right|^2.$$

Hence, for every r that satisfies $0 < 4r < \frac{R_2}{2^8}$, we have

$$\int_{B_{2r}} \left|D^2 v\right|^2 \leqslant C\widetilde{N}_r^{3\bar{k}} \int_{B_r} \left|D^2 v\right|^2. \tag{5.15}$$

Now we estimate $\widetilde{N}_r$ from above. By lemma 5.2 we have

$$\int_{B_{R_2/2^7}} |\widetilde{v}_r|^2 \geqslant \frac{1}{C}\left(\frac{R_2}{2^7}\right)^4 \int_{B_{R_2/2^8}} \left|D^2 \widetilde{v}_r\right|^2 = \frac{1}{C}\left(\frac{R_2}{2^7}\right)^4 \int_{B_{R_2/2^8}} \left|D^2 v\right|^2. \tag{5.16}$$

Moreover, by triangle inequality and the Sobolev embedding theorem, [17, chapter 7], we get

$$\begin{aligned} \|\widetilde{v}_r\|_{L^2(B_{R_2})} &\leqslant \|v\|_{L^2(B_{R_2})} + CR_2 \left|(v)_r\right| + CR_2^2 \left|(Dv)_r\right| \\ &\leqslant CR_2 \|v\|_{L^\infty(B_{R_2})} + CR_2^2 \|Dv\|_{L^\infty(B_{R_2})} \\ &\leqslant C \|v\|_{H^3(B_{R_2})}. \end{aligned} \tag{5.17}$$

Therefore, by (5.17) and (5.6) we have

$$\int_{B_{R_2}} |\widetilde{v}_r|^2 \leqslant CR_2^4 \left\|D^2 v\right\|^2_{L^2(B_{R_2})} + CR_2^6 \left\|D^3 v\right\|^2_{L^2(B_{R_2})} \tag{5.18}$$

and, by (5.3), we have that

$$\int_{B_{R_2}} |\widetilde{v}_r|^2 \leqslant CR_2^4 \left\|D^2 v\right\|^2_{L^2(B_{R_2})} + C \|v\|^2_{L^2(B_{2R_2})}. \tag{5.19}$$

Using again (5.6) we have that

$$\int_{B_{R_2}} |\widetilde{v}_r|^2 \leqslant CR_2^4 \left\|D^2 v\right\|^2_{L^2(B_{2R_2})}. \tag{5.20}$$

Hence combining (5.20) and (5.16) we have that

$$\widetilde{N}_r \leqslant \frac{C \left\|D^2 v\right\|^2_{L^2(B_{2R_2})}}{\|D^2 v\|^2_{L^2\left(B_{R_2/2^8}\right)}}. \tag{5.21}$$

Now, recalling that $R_2 = \frac{R_1}{2}$ and that $|D^2 v| = |D^2 u|$, by (5.15) we get (5.7).

Finally, by using the same argument used in corollary 4.10 in [24], by (5.7) we obtain (5.9) easily. □

6. Proof of theorem 3.5

Proposition 6.1 (Lipschitz propagation of smallness). *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of class $C^{3,1}$, with constants r_0, M_0 and satisfying (3.1). Let the tensor $\mathbb{P}, \mathbb{P}^h \in C^{1,1}(\overline{\Omega}, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2)), \mathbb{Q} \in C^{2,1}(\overline{\Omega}, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$, be given by (3.24), (3.25) and (3.26), respectively, and satisfying the ellipticity condition (3.32). Let $u_0 \in H^3(\Omega)$ be the unique solution to the problem (3.2)–(3.5) normalized by (3.18), with Neumann data $\widehat{V} \in H^{-3/2}(\partial\Omega), \widehat{M}_n \in H^{-1/2}(\partial\Omega), \widehat{M}_n^h \in H^{1/2}(\partial\Omega)$ satisfying the compatibility condition (3.14). There exists $\chi > 1$ only depending on α_0, γ_0, M_2, $\frac{t}{r_0}$ and $\frac{l}{r_0}$ such that for every $s > 0$ and for every $x \in \Omega_{\chi s r_0}$ we have that*

$$\int_{B_{sr_0}(x)} |D^2 u_0|^2 \geqslant C_s \int_\Omega |D^2 u_0|^2, \tag{6.1}$$

with $C_s > 0$ only depending on $M_0, M_1, \frac{t}{r_0}, \frac{l}{r_0}, \alpha_0, \gamma_0, M_2, s$ and on the ratio F given in (3.52).

We premise the following Lemma.

Lemma 6.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{3,1}$ with constants r_0, M_0 satisfying (3.1). Let the tensors $\mathbb{P}, \mathbb{P}^h \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^2, \widehat{\mathbb{M}}^2)), \mathbb{Q} \in L^\infty(\Omega, \mathcal{L}(\widehat{\mathbb{M}}^3, \widehat{\mathbb{M}}^3))$ satisfy the symmetry conditions (3.7), (3.8) and (3.11) and the strong convexity assumptions (3.9) and (3.12). Let $u_0 \in H^3(\Omega)$ be the unique weak solution to problem (3.2)–(3.5), satisfying the normalization condition (3.18) with the boundary data satisfying (3.13) and (3.14). We have*

$$\|\widehat{V}\|_{H^{-5/2}(\partial\Omega)} + r_0^{-1}\|\widehat{M}_n\|_{H^{-3/2}(\partial\Omega)} + r_0^{-2}\|\widehat{M}_n^h\|_{H^{-1/2}(\partial\Omega)} \leqslant C\|u_0\|_{H^3(\Omega)}, \tag{6.2}$$

where $C > 0$ depends on $M_0, M_1, \|\mathbb{P}\|_{L^\infty(\Omega)}, \|\mathbb{P}^h\|_{L^\infty(\Omega)}, \|\mathbb{Q}\|_{L^\infty(\Omega)}$.

Proof. Let us estimate the first term on the left hand side of (6.2). Similar arguments allow to estimate the other two terms. Given $g \in H^{5/2}(\partial\Omega)$, by extension results there exists $w \in H^3(\Omega)$ be such that $w = g, w_{,n} = 0, w_{,nn} = 0$ on $\partial\Omega$, and

$$\|w\|_{H^3(\Omega)} \leqslant C\|g\|_{H^{5/2}(\partial\Omega)}, \tag{6.3}$$

where $C > 0$ depends on M_0, M_1 (see for example [30]).

We have

$$\int_{\partial\Omega} \widehat{V}g = \int_{\partial\Omega} \widehat{V}w + \widehat{M}_n w_{,n} + \widehat{M}_n^h w_{,nn} = -\int_\Omega (\mathbb{P} + \mathbb{P}^h) D^2 u_0 \cdot D^2 w + \mathbb{Q} D^3 u_0 \cdot D^3 w. \tag{6.4}$$

By (6.4), Cauchy–Schwartz inequality and the *a priori* regularity bound (3.40) we deduce

$$\begin{aligned}\int_{\partial\Omega} \widehat{V}g &\leqslant C r_0^5 \left(\|D^2 u_0\|_{L^2(\Omega)} \cdot \|D^2 w\|_{L^2(\Omega)} + r_0^2 \|D^3 u_0\|_{L^2(\Omega)} \cdot \|D^3 w\|_{L^2(\Omega)}\right) \\ &\leqslant C r_0 \|u_0\|_{H^3(\Omega)} \cdot \|w\|_{H^3(\Omega)} \leqslant C r_0 \|u_0\|_{H^3(\Omega)} \cdot \|g\|_{H^{5/2}(\partial\Omega)}.\end{aligned} \tag{6.5}$$

Therefore, by (6.5),

$$\|\widehat{V}\|_{H^{-5/2}(\partial\Omega)} = \sup_{\|g\|_{H^{5/2}(\partial\Omega)}=1} \frac{1}{r_0} \int_{\partial\Omega} \widehat{V}g \leqslant C\|u_0\|_{H^3(\Omega)}. \tag{6.6}$$

□

Proof of proposition 6.1. Let us assume for this proof $r_0 = 1$. By following the lines of the proof of proposition 5.2 in [25], a process of iteration of the three spheres inequality (5.9) leads to

$$\frac{\|D^2u_0\|_{L^2(\Omega_{(\chi+1)s)})}}{\|D^2u_0\|_{L^2(\Omega)}} \leqslant \frac{C}{s}\left(\frac{\|D^2u_0\|_{L^2(B_s(x))}}{\|D^2u_0\|_{L^2(\Omega)}}\right)^{\theta_0{}^{L-1}} \tag{6.7}$$

for every $s \leqslant \frac{s_0}{\chi}$. Here $\theta_0 \in (0,1)$, $C > 0$ and $\chi > 1$ only depend on $\alpha_0, \gamma_0, M_0, M_1, M_2, t, l$; s_0 only depends on M_0 and is such that $\Omega_{\chi s}$ is connected for $s \leqslant \frac{s_0}{\chi}$ (see for instance Proposition 5.5 in [7]); moreover, the parameter L is such that $0 < L < \frac{M_1}{\pi s^2}$.

Let us rewrite the left hand side of (6.7) as follows

$$\frac{\|D^2u_0\|^2_{L^2(\Omega_{(\chi+1)s)})}}{\|D^2u_0\|^2_{L^2(\Omega)}} = 1 - \frac{\int_{\Omega\setminus\Omega_{(\chi+1)s}}|D^2u_0|^2}{\int_\Omega |D^2u_0|^2}\ . \tag{6.8}$$

By Hölder and Sobolev inequalities we have that

$$\begin{aligned}\|D^2u_0\|^2_{L^2(\Omega\setminus\Omega_{(\chi+1)s})} &\leqslant |\Omega\setminus\Omega_{(\chi+1)s}|^{\frac{1}{2}}\|D^2u_0\|^2_{L^4(\Omega\setminus\Omega_{(\chi+1)s})}\\ &\leqslant Cs^{\frac{1}{2}}\|D^2u_0\|^2_{H^{1/2}(\Omega)} \leqslant Cs^{\frac{1}{2}}\|u_0\|^2_{H^3(\Omega)},\end{aligned} \tag{6.9}$$

where $C > 0$ only depends on M_0, M_1, α_0, γ_0, M_2, t, l.

Let us notice that in the last step we have used the bound

$$|\Omega\setminus\Omega_{(\chi+1)s}| \leqslant Cs, \tag{6.10}$$

where $C > 0$ only depends on M_0 (see [8]).

Let us recall the following interpolation inequality: for any $u \in H^4(\Omega)$, we have (see [17, theorem 7.25])

$$\|u\|_{H^3(\Omega)} \leqslant C\|u\|^{\frac{1}{2}}_{H^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^4(\Omega)} \tag{6.11}$$

where $C > 0$ depends on M_0, M_1 only.

By (6.9), a standard Poincarè inequality, (6.11), (3.20) and lemma 6.2, we have

$$\frac{\int_{\Omega\setminus\Omega_{(\chi+1)s}}|D^2u_0|^2}{\int_\Omega |D^2u_0|^2} \leqslant Cs^{\frac{1}{2}}\frac{\|u_0\|^2_{H^3(\Omega)}}{\|u_0\|^2_{H^2(\Omega)}} \le Cs^{\frac{1}{2}}\left(\frac{\|u_0\|_{H^4(\Omega)}}{\|u_0\|_{H^3(\Omega)}}\right)^2 \leqslant Cs^{\frac{1}{2}}F^2 \leqslant \frac{1}{2} \tag{6.12}$$

for $s \leqslant \bar{s}$, where $\bar{s}$ only depends on $M_0, M_1, t, l, \alpha_0, \gamma_0, M_2$ and F. Finally, the thesis follows from (6.7), (6.8) and (6.12).

□

Proof of theorem 3.5. We can cover $\tilde{\Omega}_{h_1 r_0}$ with internally non-overlapping closed squares Q_k of side ϵr_0, where $k = 1, \ldots, L$ and $\epsilon = \min\left\{\frac{2h_1}{\chi+\sqrt{2}}, \frac{h_1}{\sqrt{2}}\right\}$, where $\chi > 1$ has been defined in proposition 6.1. By construction, all the squares are contained in $\tilde{\Omega}$ and $|\tilde{\Omega}_{h_1 r_0}| \leqslant L\epsilon^2 r_0$. Let $\bar{k}$ be such that $\int_{Q_{\bar{k}}}|D^2u_0|^2 = \min_{k=1,\ldots,L}\int_{Q_k}|D^2u_0|^2$. By the fatness assumption (3.49), we have

$$\int_{\tilde{\Omega}}|D^2u_0|^2 \geqslant \frac{|\tilde{\Omega}|}{2r_0^2\epsilon^2}\int_{Q_{\bar{k}}}|D^2u_0|^2\ . \tag{6.13}$$

Let $\bar{x}$ be the center of the square $Q_{\bar{k}}$. By applying the Lipschitz propagation of smallness estimate (6.1) with $x=\bar{x}$ and $s=\frac{\epsilon}{2}$, we have

$$\int_{\tilde{\Omega}} |D^2u_0|^2 \geqslant \frac{C|\tilde{\Omega}|}{r_0^2}\int_{\Omega}|D^2u_0|^2, \tag{6.14}$$

with $C>0$ only depending on $h_1,\alpha_0,\gamma_0,M_0,M_1,M_2,\frac{t}{r_0},\frac{l}{r_0},F$. By (6.14), by applying Poincarè inequality, interpolation inequality (6.11), the regularity estimate (3.20), lemma 6.2, (4.1)–(4.2) and the weak formulation of the problem (3.2)–(3.5), we have

$$\begin{aligned}\int_{\tilde{\Omega}} |D^2u_0|^2 &\geqslant C\frac{|\tilde{\Omega}|}{r_0^4}\|u_0\|^2_{H^2(\Omega)} \geqslant C\frac{|\tilde{\Omega}|}{r_0^4}\frac{\|u_0\|^2_{H^3(\Omega)}}{\|u_0\|^2_{H^4(\Omega)}}\|u_0\|^2_{H^3(\Omega)}\\ &\geqslant C|\tilde{\Omega}|F^{-2}(\|D^2u_0\|^2_{L^2(\Omega)}+r_0^2\|D^3u_0\|^2_{L^2(\Omega)}) \geqslant C|\tilde{\Omega}|F^{-2}r_0^{-5}W_0,\end{aligned} \tag{6.15}$$

where $C>0$ depends on $M_0,M_1,\frac{t}{r_0},\frac{l}{r_0},\alpha_0,\gamma_0,M_2,h_1$. Estimates (3.50) and (3.51) follow from (6.15) and from the left hand side of (4.3) and (4.4) respectively. □

7. Proof of theorem 3.6

Proposition 7.1 (doubling inequality for the Hessian in terms of the boundary data). *Under the hypothesis of theorem 3.6, let $u_0\in H^3(\Omega)$ be the unique solution to (3.2)–(3.5) satisfying (3.18), with $\widehat{V},\widehat{M}_n,\widehat{M}_n^h$ satisfying (3.13) and (3.14). There exists a constant $\theta, 0<\theta<1$, only depending on $\alpha_0,\gamma_0,M_2,\frac{t}{r_0},\frac{l}{r_0}$, such that for every $\bar{r}>0$ and for every $x_0\in\Omega_{\bar{r}r_0}$, we have*

$$\int_{B_{2r}(x_0)}|D^2u_0|^2 \leqslant K\int_{B_r(x_0)}|D^2u_0|^2 \tag{7.1}$$

for every $r, 0<r<\frac{\theta}{2}\bar{r}r_0$, where $K>0$ only depends on $\alpha_0,\gamma_0,M_2,M_0,M_1,\bar{r},\frac{t}{r_0},\frac{l}{r_0}$ and the ratio F given by (3.52).

Proof. By applying a scaling argument to (5.7) and (5.8), there exists an absolute constant $\theta, 0<\theta<1$ such that for every $\bar{r}>0$ and for every $x_0\in\Omega_{\bar{r}r_0}$ we have

$$\int_{B_{2r}(x_0)}|D^2u_0|^2 \leqslant K\int_{B_r(x_0)}|D^2u_0|^2 \tag{7.2}$$

for every $r, 0<r<\frac{\theta}{2}\bar{r}r_0$, where $K>0$ only depends on $\alpha_0,\gamma_0,M_2,\frac{t}{r_0},\frac{l}{r_0}$ and $\bar{r}$ and the increasingly on the ratio

$$N=\frac{\int_{B_{\bar{r}r_0}(x_0)}|D^2u_0|^2}{\int_{B_{\frac{\bar{r}r_0}{2^9}}(x_0)}|D^2u_0|^2}. \tag{7.3}$$

By applying (6.1) to bound from below the denominator, we trivially obtain the desired bound. □

Proposition 7.2 (A_p property). *Let the assumptions of proposition 7.1 be satisfied. For every $\bar{r}>0$ there exist $B>0$ and $p>1$ such that for every $x_0\in\Omega_{\bar{r}r_0}$ we have*

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|D^2u_0|^2\right)\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|D^2u_0|^{-2/(p-1)}\right)^{p-1}\leqslant B,$$

$$\text{for every } r,\ 0<r\leqslant\frac{\vartheta}{2}\bar{r}r_0, \tag{7.4}$$

where ϑ is as in proposition 7.1 and where B, p only depend on $\alpha_0,\gamma_0,M_2,M_0,M_1,\bar{r},\frac{t}{r_0},\frac{l}{r_0}$ and the ratio F given by (3.52).

Proof. In view of the results in [13], it is sufficient to prove a reverse Hölder's inequality for $|D^2u_0|^2$. Let us introduce

$$v_0=u_0+ax_1+bx_2+c, \tag{7.5}$$

such that

$$\int_{B_{2r}(x_0)}v_0\mathrm{d}x=0,\quad \int_{B_{2r}(x_0)}v_{0,\alpha}\,\mathrm{d}x=0,\quad \alpha=1,2. \tag{7.6}$$

By interior regularity estimates (see, for instance, [25, theorem 8.3]), by Poincaré inequality (see, for instance, [25, proposition 3.3]) and by proposition 7.1 we have

$$\begin{aligned}\|D^2u_0\|_{L^\infty(B_r(x_0))}&=\|D^2v_0\|_{L^\infty(B_r(x_0))}\leqslant\frac{C}{r^2}\|v_0\|_{H^2(B_{2r}(x_0))}\\&\leqslant C\|D^2v_0\|_{L^2(B_{2r}(x_0))}=C\|D^2u_0\|_{L^2(B_{2r}(x_0))}\leqslant C\|D^2u_0\|_{L^2(B_r(x_0))},\end{aligned} \tag{7.7}$$

where $C>0$ only depends on $\alpha_0,\gamma_0,M_2,M_0,M_1,\bar{r},\frac{t}{r_0},\frac{l}{r_0}$ and the ratio F given by (3.52). □

Proof of theorem 3.6. Let us cover $\tilde{\Omega}$ with internally non overlapping closed cubes Q_j, $j=1,\ldots,J$, with side $\epsilon=\frac{\theta d_0}{4\sqrt{2}}r_0$, where $\theta<1$ has been introduced in proposition 7.1. Let $p>1$ be the exponent introduced in proposition 7.2. By Hölder's inequality we have

$$|\tilde{\Omega}|\leqslant\left(\int_{\bigcup_{j=1}^J Q_j}|D^2u_0|^{-\frac{2}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_{\tilde{\Omega}}|D^2u_0|^2\right)^{\frac{1}{p}}. \tag{7.8}$$

By applying proposition 7.2, with $\bar{r}=\frac{d_0}{2}$ to the balls B_j circumscribing each Q_j, $j=1,\ldots,J$, we have

$$\begin{aligned}\left(\int_{\bigcup_{j=1}^J Q_j}|D^2u_0|^{-\frac{2}{p-1}}\right)^{\frac{p-1}{p}}&\leqslant\left(\frac{\pi}{2}\epsilon^2\sum_{j=1}^J\frac{1}{|B_j|}\int_{B_j}|D^2u_0|^{-\frac{2}{p-1}}\right)^{\frac{p-1}{p}}\\&\leqslant\left(\frac{\pi}{2}\epsilon^2\sum_{j=1}^J\left(\frac{B}{\frac{1}{|B_j|}\int_{B_j}|D^2u_0|^2}\right)^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}\leqslant\frac{\frac{\pi}{2}\left(J\epsilon^2\right)^{\frac{p-1}{p}}B^{\frac{1}{p}}\epsilon^{2/p}}{\min_j\left(\int_{B_j}|D^2u_0|^2\right)^{\frac{1}{p}}},\end{aligned} \tag{7.9}$$

where the constants p and B only depend on α_0, γ_0, M_2, M_1, M_0, $\frac{t}{r_0}$, $\frac{l}{r_0}$, d_0 and the ratio F given by (3.52). By (3.1) we have

$$J\epsilon^2=\sum_{j=1}^J|Q_j|\leqslant|\Omega|\leqslant M_1r_0^2. \tag{7.10}$$

Consequently, from (7.8)–(7.10) and recalling the definition of ϵ, we have

$$|\tilde{\Omega}| \leqslant C r_0^2 \left(\frac{\int_{\tilde{\Omega}} |D^2 u_0|^2}{\int_{B_{\bar{j}}} |D^2 u_0|^2} \right)^{\frac{1}{p}}, \tag{7.11}$$

with $\bar{j}$ such that $\int_{B_{\bar{j}}} |D^2 u_0|^2 = \min_j \int_{B_j} |D^2 u_0|^2$.

By proposition 6.1, by standard Poincarè inequality, by the interpolation inequality (6.11), by (6.2), (3.20), (3.40) and (3.45), we have

$$\int_{B_{\bar{j}}} |D^2 u_0|^2 \geqslant C \int_\Omega |D^2 u_0|^2 \geqslant \frac{C}{r_0^2} \|u_0\|^2_{H^2(\Omega)} \geqslant \frac{C}{r_0^2} \left(\frac{\|u_0\|_{H^3(\Omega)}}{\|u_0\|_{H^4(\Omega)}} \right)^2 \cdot \|u_0\|^2_{H^3(\Omega)}$$
$$\geqslant C F^{-2} \left(\int_\Omega |D^2 u_0|^2 + r_0^2 |D^3 u_0|^2 \right) \geqslant \frac{C}{r_0^3} W_0. \tag{7.12}$$

By (7.11) and (7.12) we have

$$|\tilde{\Omega}| \leqslant C r_0^2 \left(\frac{r_0^3 \int_{\tilde{\Omega}} |D^2 u_0|^2}{W_0} \right)^{\frac{1}{p}}, \tag{7.13}$$

with the constant $C > 0$ only depending on $\alpha_0, \gamma_0, M_2, M_1, M_0, \frac{t}{r_0}, \frac{l}{r_0}, d_0$ and the ratio F given by (3.52).

Finally, from the left hand side of (4.3) and (4.4) and from (7.13) we end up with the upper bounds for $|\tilde{\Omega}|$ in (3.54) and (3.55). □

Data availability statement

No new data were created or analyzed in this study.

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