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Nonlinear multivariate sampling Kantorovich operators: a study of their approximation properties in modular spaces

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To my beloved family,
all those close to my heart,
and the esteemed Professors,
I wish to express my deepest gratitude
for your invaluable presence and support
throughout these challenging three years.

Happiness can be found, even in the darkest of times,
if one only remembers to turn on the light.

Contents

Introduction

In the last 50 years, motivated by both theoretical and practical reasons, several approximate versions of the well-known Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem have been studied. The main purposes behind these generalizations were to weaken the assumptions on the function that can be reconstructed by a given family of sampling values. Indeed, in the classical sampling theorem, only band-limited with finite energy signals can be reconstructed. This means that, as a consequence of the Paley-Wiener theorem, the signal must be the restriction to the real axis of an entire function of exponential type, and therefore a very regular function.

One of the most known approximate version of the sampling theorem is due to P. L. Butzer and his school at the RWTH Polytechnic of Aachen. Butzer introduced a family of generalized sampling series, whose multivariate version is given by

$$(G_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} f\left(\frac{\underline{k}}{w}\right)\chi(w\underline{x}-\underline{k}), \quad \underline{x}\in\mathbb{R}^n, \quad w>0, \tag{I}$$

where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with compact support, and $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded function. The function χ is said to be the *kernel* of the multivariate generalized sampling operators. The family $(G_w f)_{w>0}$ is suitable to reconstruct continuous signals; this is due to their definition in (I) which depends on the pointwise values of the function that must be approximated.

In order to introduce a family of approximation operators that allows to reconstruct not necessarily continuous signals, a Kantorovich version of the univariate generalized sampling series has been introduced in [13]. The multivariate version of the sampling Kantorovich operators has been treated in [46], which, for any given kernel χ, is defined as

$$(S_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x}-t_{\underline{k}}\right)\left[\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})\,d\underline{u}\right], \quad \underline{x}\in\mathbb{R},^n \quad w>0. \tag{II}$$

Here, $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$, and

$$R_{\underline{k}}^w := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times \cdots \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right], \quad w > 0,$$

where $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ is a suitable vector, with $(t_{k_i})_{k_i\in\mathbb{Z}}$ a sequence of real numbers, $A_{\underline{k}} := \Delta_{k_1} \cdot \Delta_{k_2} \cdots \Delta_{k_n}$, and $\Delta_{k_i} := t_{k_i+1} - t_{k_i} > 0$. The choice of $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ allows us to sample signals by an irregular sampling scheme; if $t_{\underline{k}} = \underline{k}$, $\underline{k} \in \mathbb{Z}^n$, we proceed to the uniform case as in (I). Moreover, the multidimensional case revealed to be very useful in order to face the problem of image reconstruction. For more details and, as concerns some applications of the above theory to concrete real-world problems, the readers can see [12, 44, 45]. For additional references on Kantorovich type operators, see, e.g., [2–4, 13, 42, 48–52, 57, 72].

Recently, in [47], the above sampling type series have been extended and generalized by the introduction of the so-called *nonlinear multivariate sampling Kantorovich operators*, defined by

$$(K_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u})\,d\underline{u}\right), \quad \underline{x} \in \mathbb{R}^n, \quad w > 0, \tag{III}$$

where $\chi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a *nonlinear kernel function*, which satisfies suitable assumptions (see Chapter 3). Further, if $\chi(\underline{x}, u) = L(\underline{x})u$, $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, the operators $K_w f$ reduce to the linear multivariate sampling Kantorovich operators considered in (II). Their univariate version has been firstly introduced in [78].

The analysis of (III) goes beyond its mathematical significance and finds practical applications in various fields. One important example can be furnished in signal processing, when one has to describe some nonlinear transformations generated by signals that, during their filtering process, produce new frequencies. Power electronics and wireless communications involve amplifiers that introduce nonlinear distortions to their input signals. Similarly, in radiometric photography and CCD image sensors, the relationship between input radiance and intensity exhibits nonlinearity, even though it is monotone increasing. Amplifier saturation also introduces nonlinear distortions to the input signal.

The pioneer works of the theory of nonlinear operators, in connection with approximation problems, and a wide literature, can be found in [10, 11, 15, 16, 22, 24, 59, 63, 65–69, 77, 79]. As these references show, one of the main difficulties in passing from the linear to the nonlinear setting is that one has to introduce a suitable notion of singularity for the family of kernel functions. Such hypothesis was first introduced by Musielak in [65] in modular spaces and then weakened in [66, 67].

Another problem which arises in connection with estimates and convergence results for nonlinear operators is what kind of assumption one must impose on the kernel function and, in this respect, a kind of generalized Lipschitz condition on the kernel function must be assumed. It follows that the approach and the methods obtained in the nonlinear framework are different from those used in the linear case.

Motivated by these developments, the goal of the present PhD thesis is to examine and analyze the convergence properties and the order of approximation of nonlinear multivariate sampling Kantorovich operators in Orlicz spaces, and to extend the convergence properties within the more general framework of modular spaces. The latter spaces have been firstly introduced in [71] by Nakano as a generalization of Orlicz spaces, which, in turn, have been introduced as a natural extension of the classical Lebesgue spaces.

Building upon the insights presented in [47], which establish both pointwise and uniform convergence, as well as modular convergence within Orlicz spaces, we deal with the problem of the order of approximation. In particular, in [36] we estimate the rate of convergence through both quantitative and qualitative analysis in the space of bounded and uniformly continuous functions and in the setting of Orlicz spaces. In this respect, a crucial role is played by the basical properties of the modulus of continuity and the modulus of smoothness, respectively.

Further, in [43] we provide convergence result in the more general setting of modular spaces via a density approach. First we prove a modular convergence theorem, as well as a Luxemburg norm convergence result, for the nonlinear multivariate sampling Kantorovich operators acting on the space of continuous functions with compact support, then we obtain a modular-type inequality, and finally we exploit a well-known density result for the continuous function with compact support in the modular spaces. The choice to work within modular spaces is driven by the fact that they enable us to provide a unifying approach to several settings of approximation problems. In fact, modular spaces include Musielak-Orlicz spaces, which contain, for instance, weighted-Orlicz spaces and Orlicz spaces, as well as spaces of functions equipped by modulars that are not of integral type.

The structure of the thesis is organized as follows.

Chapter 1 collects basic definitions and preliminary results about modular spaces, establishing the framework for the subsequent chapters. Special emphasis is placed on the Musielak-Orlicz spaces and the Orlicz spaces. Moreover, here we also recall the notion of modulus of smoothness, which plays an important role in Approximation Theory and throughout the thesis, and we discuss its main properties.

In Chapter 2, we propose an historical overview of some important results of the classical WKS-sampling theorem (see, e.g., [28]) and of the generalized sampling operators (see, [9, 29, 30, 35, 75]). We explore both the linear and nonlinear forms of these operators, highlighting their properties. This historical perspective serves as a foundation for Chapter 3, clarifying the crucial transition from the linear to the nonlinear setting and emphasizing its significance, also from an application point of view.

Chapter 3 is devoted to a detailed study of the nonlinear sampling Kantorovich type series. In particular, we present pointwise and uniform approximation results, along with a modular convergence theorem in the setting of Orlicz spaces. Here, by using some special kernels, several examples and graphical representations are also provided.

In Chapter 4, the focus shifts to the quantitative analysis, studied in [36], of the aforementioned operators. In this respect, we establish some quantitative estimates in $C(\mathbb{R}^n)$, and in Orlicz spaces $L^\varphi(\mathbb{R}^n)$, using their typical modulus of smoothness. As a consequence, the qualitative order of convergence can be obtained in the case of functions belonging to suitable Lipschitz classes. In the particular instance of L^p-spaces, using a direct approach, we obtain a sharper estimate than the one that can be deduced from the general case of Orlicz spaces.

Finally, Chapter 5 explores some approximation results in the broader context of modular spaces, that we face in [43]. Modular convergence theorems are proved under suitable assumptions, together with a modular inequality. The convergence results in the Musielak-Orlicz spaces, in the weighted Orlicz spaces, and in the Orlicz spaces follow as particular cases. Moreover, we deduce the convergence in spaces of functions equipped by modulars that lack an integral representation.

The thesis ends with the conclusions, which summarize the main goals achieved, discuss open research questions, and outline potential future directions.

Chapter 1

Preliminaries

In this chapter, we denote by $\Omega = (\Omega, \Sigma_\Omega, \mu_\Omega)$ an arbitrary measure space, i.e., Ω is a non empty set, Σ_Ω is a σ-algebra of subsets of Ω and μ_Ω is a non negative, complete measure in Σ_Ω, which does not vanish identically. Let $M(\Omega)$ denote the space of all extended real-valued, Σ_Ω-measurable functions on Ω, finite μ-a.e, with equality μ-a.e.

Definition 1.0.1. *A functional $K : \Omega \times \Omega \times DomK \to \mathbb{R}$, where $DomK \subset M(\Omega)$, is said to be a kernel functional, if for every $f \in DomK$, the functional $K(s, t, f)$ is measurable in $\Omega \times \Omega$, and if $K(s, t, 0) = 0$, for every $s, t \in \Omega$.*

As a result, $K(s, t, f)$ is Σ_Ω-measurable as a function of the variable t for μ-a.e. $s \in \Omega$, so the kernel functional generates an *integral operator* T by the formula

$$(Tf)(s) = \int_\Omega K(s, t, f)d\mu(t). \tag{IV}$$

To investigate any kind of convergence process for a sequence or family of non-linear operators of the form (IV), some function spaces with a suitable notion of convergence should be presented.

1.1 Modular spaces

In this section we introduce some generalization of L^p-spaces using the concept of Orlicz spaces and, more in general, the one of modular spaces.
Modular spaces were firstly introduced by H. Nakano ([71]) in 1950 and the theory was extensively developed since 1959 by the Polish mathematicians J. Musielak and W. Orlicz ([70]).

Let $X(\Omega)$ be the, real or complex, vector space of all Σ_Ω-measurable functions on Ω.

Definition 1.1.1. *A functional* $\rho : X(\Omega) \to [0, +\infty]$ *is called a modular on* $X(\Omega)$*, if the following conditions are satisfied*

(ρ1) $\rho(f) = 0$*, if and only if* $f \equiv 0$ μ_Ω*-a.e. in* Ω*;*

(ρ2) $\rho(-f) = \rho(f)$ *or* $\rho(e^{it} f) = \rho(f)$*, for every* $t \in \mathbb{R}$*, if* $X(\Omega)$ *is a complex vector space;*

(ρ3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$*, for every* $f, g \in X(\Omega)$*, and* $\alpha, \beta \geq 0$*,* $\alpha + \beta = 1$*.*

Definition 1.1.2. *A modular* ρ *is called* p*-convex,* $0 < p \leq 1$*, if*

$$\rho(\alpha f + \beta g) \leq \alpha^p \rho(f) + \beta^p \rho(g), \ \ \forall \alpha, \beta \geq 0, \ \ \alpha^p + \beta^p = 1.$$

In the case of $p = 1$, we have the usual notion of convexity for ρ.

Example 1.1.3. If $(X, \|\cdot\|)$ is a normed linear space, then the functional $\rho(x) = \|x\|$ is a convex modular in X, as follows from the definition of ρ.

Obviously, in general a modular is not a norm.

Example 1.1.4. Let $X = L^p(\Omega)$, then $\rho(f) = \displaystyle\int_\Omega |f(t)|^p d\mu(t)$, $0 < p < 1$ is a p-convex modular for $0 < p < 1$ and a convex modular for $p \geq 1$.

Now we show some useful properties of a modular ρ.

Proposition 1.1.5. *A modular* ρ *satisfies the following conditions*

i) $\rho(\alpha f) \leq \rho(f)$*, for* $|\alpha| \leq 1$*;*

ii) $\rho\left(\displaystyle\sum_{i=1}^{n} \alpha_i f_i\right) \leq \displaystyle\sum_{i=1}^{n} \rho(f_i)$ *with* $\alpha_i \geq 0$ *and* $\displaystyle\sum_{i=1}^{n} \alpha_i = 1$*;*

iii) $\rho(\lambda_1 f) \leq \rho(\lambda_2 f)$*, for every* $f \in X$ *and let* λ_1, λ_2 *be real constants, with* $\lambda_1 < \lambda_2$*.*

Proof. *i)* Without loss of generality, we assume $0 \leq \alpha \leq 1$, since by (ρ2) of Definition 1.1.1 ρ is symmetric. Now, using the property (ρ3) of the modular with $x \in \mathbb{R}$, $y = 0$, α and $\beta = 1 - \alpha$

$$\rho(\alpha f) = \rho(\alpha f + \beta 0) \leq \rho(f) + \rho(0) = \rho(f).$$

ii) It can be easily proved by induction.
iii) Let λ_1, λ_2 be real constants such that $0 < \lambda_1 < \lambda_2$. Since $0 < \frac{\lambda_1}{\lambda 2} < 1$, it follows that $\rho(\lambda_1 f) = \rho\left(\frac{\lambda_1}{\lambda_2}\lambda_2 f\right) \leq \rho(\lambda_2 f)$ by *i*). □

Definition 1.1.6. *Let ρ be a modular on $X(\Omega)$. We define the modular space generated by ρ, as*

$$L_\rho(\Omega) := \left\{ f \in X(\Omega) \;\; : \;\; \lim_{\lambda \to 0} \rho(\lambda f) = 0 \right\}.$$

It can be proved that any modular space $L_\rho(\Omega)$ is a vector subspace of $X(\Omega)$.

Definition 1.1.7. *Let $X(\Omega)$ be a vector space. Then a functional $|\cdot| : X(\Omega) \to [0, +\infty]$ is called an F-norm if it satisfies the following assumptions*

- *(F1) $|f| = 0$ if and only if $f \equiv 0$ μ_Ω-a.e. in Ω;*
- *(F2) if $X(\Omega)$ is a real vector space, then $|-f| = |f|$ and $|e^{it} f| = |f|$ for every $t \in \mathbb{R}$, if $X(\Omega)$ is a complex vector space;*
- *(F3) $|f + g| \leq |f| + |g|$;*
- *(F4) if $\alpha_n \to \alpha$ and $|f_n - f| \to 0$ as $n \to +\infty$, then $|\alpha_n f_n - \alpha f| \to 0$ as $n \to +\infty$.*

Note that condition *(F4)* of the Definition 1.1.7 is weaker than the positive homogeneity; for this reason in general an F-norm is not a norm, while the converse implication is always true.

In general, a modular is not an F-norm (or a norm); however, starting from a modular, we can define an F-norm. Indeed, the following proposition holds ([64]).

Proposition 1.1.8. *If ρ is a modular on $X(\Omega)$, then the functional*

$$|f|_\rho := \inf \{ u > 0 \;\; : \;\; \rho(f/u) \leq u \}$$

is an F-norm on $L_\rho(\Omega)$ and satisfies the following properties

- *i) if $\rho(\lambda f) \leq \rho(\lambda g)$ for every $f, g \in X_\rho$ and $\lambda > 0$, then $|f|_\rho \leq |g|_\rho$;*
- *ii) if $f \in X_\rho$, then $|\alpha f|_\rho$ is non decreasing with respect to $\alpha > 0$;*
- *iii) if $|f|_\rho < 1$, then $\rho(f) \leq |f|_\rho$.*

Proposition 1.1.9. *Let ρ be a convex modular on $X(\Omega)$, then*

$$\|f\|_\rho := \inf \{ u > 0 \;\; : \;\; \rho(f/u) \leq 1 \}$$

is a norm and it is called the Luxemburg norm.

Proof. If $f \in L_\rho(\Omega)$, then $\rho(f/n) \to 0$, as $n \to +\infty$. Hence the set $\{u > 0 : \rho(f/u) \leq 1\}$ is non empty, and so $0 \leq \|f\|_\rho < +\infty$ and $\|0\|_\rho = 0$. If $\|f\|_\rho = 0$, by convexity of ρ, we have for $0 < u \leq 1$

$$\rho(f) = \rho\left(u\frac{f}{u}\right) \leq u\rho\left(\frac{f}{u}\right) \leq u.$$

Taking $u \to 0^+$, we get $\rho(f) = 0$ and consequently $f = 0$.
In order to get the triangle inequality, let us take any $\varepsilon > 0$ and let us put $u = \|f\|_\rho + \varepsilon$ and $v = \|g\|_\rho + \varepsilon$, where $f, g \in L_\rho(\Omega)$. Then $\rho(f/u) \leq 1$ and $\rho(g/v) \leq 1$. By convexity of ρ, we obtain

$$\rho\left(\frac{f+g}{u+v}\right) = \rho\left(\frac{u}{u+v}\frac{f}{u} + \frac{v}{u+v}\frac{g}{v}\right) \leq \frac{u}{u+v}\rho(f/u) + \frac{v}{u+v}\rho(g/v) \leq 1.$$

Thus $\|f+g\|_\rho \leq u + v = \|f\|_\rho + \|g\|_\rho + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain the triangle inequality for $\|\cdot\|_\rho$.
Finally, we have for $f \in L_\rho(\Omega)$ and $c \in \mathbb{R}$

$$\begin{aligned} \|cf\|_\rho &= \inf\left\{u > 0 \,:\, \rho\left(\frac{|c|f}{u}\right) \leq 1\right\} \\ &= |c| \inf\left\{\frac{u}{|c|} > 0 \,:\, \rho\left(\frac{f}{u/|c|}\right) \leq 1\right\} = |c| \, \|f\|_\rho \,. \end{aligned}$$

This completes the proof. □

We point out that the properties *i)*-*iii)* of Proposition 1.1.8 remain valid if we replace $|\cdot|_\rho$ by $\|\cdot\|_\rho$.

The following statement gives a necessary and sufficient condition for norm convergence of a sequence of functions $f_n \in L_\rho(\Omega)$ in the sense of the F-norm $|\cdot|_\rho$ (or the norm $\|\cdot\|_\rho$).

Theorem 1.1.10. *Let $L_\rho(\Omega)$ be the modular space generated by a modular ρ and let $f \in L_\rho(\Omega)$ and $f_n \in L_\rho(\Omega)$, for $n = 1, 2, ...$ There holds $f_n \to f$, in the sense of the F-norm $|\cdot|_\rho$ (or the norm $\|\cdot\|_\rho$), if and only if $\rho(\lambda(f_n - f)) \to 0$ as $n \to +\infty$, for every $\lambda > 0$.*

In connection with Theorem 1.1.10, one may introduce on $L_\rho(\Omega)$ a weaker concept of convergence.

Definition 1.1.11. *A sequence of functions $f_n \in L_\rho(\Omega)$ is said to be modularly convergent (or ρ-convergent) to a function $f \in L_\rho(\Omega)$, if there exists a constant $\lambda > 0$ such that*

$$\lim_{n \to +\infty} \rho(\lambda(f_n - f)) = 0.$$

We denote this convergence by $f_n \xrightarrow{\rho} f$ as $n \to +\infty$.

The notions of modular and Luxemburg norm convergence are equivalent in $L_\rho(\Omega)$, if and only if, the following condition holds

$$\text{if } f_n \in L_\rho(\Omega),\ \rho(f_n) \to 0 \text{ then } \rho(2f_n) \to 0, \text{ for } n \to +\infty. \qquad (\Delta_2)$$

In Section 1.1.2, we will see that there exist modular spaces (Orlicz spaces of exponential type), where (Δ_2) does not hold, i.e., modular convergence does not imply norm convergence. So it make sense to investigate the problems connected with modular convergence, separately.
This is significant in the development of the theory of modular spaces, because if there would be only norm convergence in $L_\rho(\Omega)$, then the entire purpose of a modular ρ would be reduced to that of defining a norm (or an F-norm) in a vector space. However, because modular convergence is in general not reducible to norm convergence, the modular notion leads to problems that cannot be formulated in the language of the metric vector spaces. For further details concerning modular convergence, see e.g. [70].

Definition 1.1.12. *The space of finite elements of $L_\rho(\Omega)$ is defined by*

$$E_\rho(\Omega) := \{f \in X(\Omega)\ :\ \rho(\lambda f) < +\infty \text{ for every } \lambda > 0\}.$$

In general, $E_\rho(\Omega)$ is a proper subspace of $L_\rho(\Omega)$; however $E_\rho(\Omega) = L_\rho(\Omega)$ if and only if the Δ_2-condition is satisfied (see, e.g., [22, 64]).

Further, we need to recall the following properties of the functional ρ.

Definition 1.1.13. *We say that a modular ρ is*

(a) monotone if $\rho(f) \leq \rho(g)$ whenever $|f| \leq |g|$, for every $f, g \in X(\Omega)$;

(b) finite if the characteristic function $\mathbf{1}_A$ of every measurable set A of finite μ_Ω-measure belongs to $L_\rho(\Omega)$;

(c) strongly finite if each $\mathbf{1}_A$ as above belongs to $E_\rho(\Omega)$;

(d) absolutely finite if ρ is finite and if for every $\varepsilon, \lambda_0 > 0$ there exists $\delta > 0$ such that $\rho(\lambda_0 \mathbf{1}_B) < \varepsilon$ for every $B \in \Sigma_\Omega$ with $\mu_\Omega(B) < \delta$;

(e) absolutely continuous if there is $\alpha > 0$ such that for every $f \in X(\Omega)$ with $\rho(f) < +\infty$, the following two conditions hold:

(i) for every $\varepsilon > 0$ there exists a measurable subset $A \subset \Omega$ with $\mu_\Omega(A) < +\infty$ such that $\rho(\alpha f \mathbf{1}_{\Omega \setminus A}) < \varepsilon$;

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(\alpha f \mathbf{1}_B) < \varepsilon$ for all measurable sets $B \subset \Omega$ with $\mu_\Omega(B) < \delta$.

Note that, if ρ is convex then any strongly finite modular is also finite.

Now, we recall the Lebesgue dominated convergence theorem for modular spaces (see [67]), that will be useful in Chapter 5.

Theorem 1.1.14. *Let ρ be a monotone, finite and absolutely continuous modular on $X(\Omega)$. Let (f_n) be a sequence of functions $f_n \in X(\Omega)$ such that $f_n \to 0$ μ_Ω-a.e. in Ω, as $n \to +\infty$. Moreover, let suppose that there exists a function $g \in L_\rho(\Omega)$ such that $\rho(3g) < +\infty$ and $|f_n(x)| \leq g(x)$ μ_Ω-a.e. in Ω, for $n = 1, 2, \ldots$ Then $\rho(f_n) \to 0$, as $n \to +\infty$.*

Finally, in what follows, we will denote by $C^0(\Omega)$ and $C(\Omega)$ the set of all bounded functions $f : \Omega \to \mathbb{R}$ which are respectively continuous and uniformly continuous on Ω, endowed with the sup-norm.
Denoting by $C_c(\Omega)$ the subspace of $C(\Omega)$ consisting of functions with compact support, we can state the following density result, which will be used in order to state the main modular approximation results of this paper.

Theorem 1.1.15 (see Theorem 1 of [62])**.** *Let ρ be a absolutely continuous, monotone and absolutely finite modular on $X(\Omega)$. Then $\overline{C_c(\Omega)}^\rho = L_\rho(\Omega)$, where the bar represents the closure with respect to the modular topology on $L_\rho(\Omega)$.*

1.1.1 Musielak-Orlicz spaces

In this section, we consider some particular cases of modular spaces: the Musielak-Orlicz spaces, which have been again introduced by Nakano in the 50's, and deeply studied by Musielak and Orlicz (see, e.g., [22, 50, 55, 56, 64, 67]).

Definition 1.1.16. *Let $\varphi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfy the following conditions*

(φ1) $\varphi(\cdot, u)$ is Σ_Ω-measurable for every $u \in \mathbb{R}_0^+$;

(φ2) for every $t \in \Omega$, $\varphi(t, \cdot)$ is continuous and non decreasing on $\mathbb{R}_0^+$ such that $\varphi(t, 0) = 0$, $\varphi(t, u) > 0$ for $u > 0$ and $\varphi(t, u) \to +\infty$ as $u \to +\infty$;

(φ3) φ is τ-bounded, i.e., there are a constant $C \geq 1$ and a measurable function $F : \Omega \times \Omega \to \mathbb{R}_0^+$ such that for every $t, s \in \Omega$ and $u \geq 0$

$$\varphi(t - s, u) \leq \varphi(t, Cu) + F(t, s).$$

A function φ, as above, is said a τ-bounded φ-function, and for a sake of simplicity, we will call it simply a φ-function.

We can observe that if $\varphi(t, \cdot)$ is convex on $\mathbb{R}_0^+$, for every $t \in \Omega$, then φ is also continuous and non decreasing with respect to the second variable u.

Remark 1.1.17. Interested readers may find the original definition of τ-boundness in the monograph of Musielak ([64], pag. 37). We point out that one may also consult [23] for a constructive procedure about examples of Musielak-Orlicz spaces satisfying the τ-boundness with $F \neq 0$. However, from now on, we will only consider φ-functions φ which satisfy condition (φ3) with $F \equiv 0$.

Then, it is easily shown that

$$\rho(f) = \rho^{\varphi}(f) := \int_{\Omega} \varphi(t, |f(t)|) d\mu(t) \tag{1.1}$$

is a modular on the space $M(\Omega)$. In fact, the conditions (ρ1) and (ρ2) of Definition 1.1.1 are obviously satisfied by the assumptions on φ and on the absolute value. Moreover, we remark that if $\rho^{\varphi}(f) = 0$, then $\varphi(t, |f(t)|) = 0$ a.e. $t \in \Omega$, so $f(t) = 0$ a.e. $t \in \Omega$, and viceversa.
Finally we prove (ρ3). Let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ be fixed. Since φ is non decreasing with respect to the second variable, it follows that

$$\begin{aligned}
\rho^{\varphi}(\alpha f + \beta g) &= \int_{\Omega} \varphi(t, |\alpha f(t) + \beta g(t)|) d\mu(t) \leq \int_{\Omega} \varphi(t,\, \alpha |f(t)| + \beta |g(t)|) d\mu(t) \\
&\leq \int_{\Omega} \varphi(t,\, \max\{|f(t)|, |g(t)|\}) d\mu(t) \leq \int_{\Omega} \left[\varphi(t, |f(t)|) + \varphi(t, |g(t)|)\right] d\mu(t) \\
&= \rho^{\varphi}(f) + \rho^{\varphi}(g).
\end{aligned}$$

Moreover, if $\varphi(t, u)$ is a convex function of u for all $t \in \Omega$, then ρ^{φ} is a convex modular on $M(\Omega)$.

The modular space generated by ρ^{φ} is called a *Musielak-Orlicz space*, it is briefly denoted by $L^{\varphi}(\Omega)$ and it is defined as follows

$$L^{\varphi}(\Omega) := L_{\rho^{\varphi}}(\Omega) = \{f \in M(\Omega) \;:\; \lim_{\lambda \to 0} \rho^{\varphi}(\lambda f) = 0\}.$$

Remark 1.1.18 ([22])**.** It is easy to check that the modular ρ defined as in (1.1) satisfies the properties (a)-(e) given in Definition 1.1.13. In fact, ρ is always *monotone*, ρ is *finite* if and only if $\varphi(\cdot, u)$ is locally integrable for small u (i.e. for every $A \in \Sigma$ with $\mu(A) < +\infty$ there is a $u > 0$ such that $\int_A \varphi(t, u) d\mu(t) < +\infty$), and ρ is *absolutely finite* if and only if $\varphi(\cdot, u)$ is locally integrable (i.e. for every $A \in \Sigma$ with $\mu(A) < +\infty$ there holds $\int_A \varphi(t, u) d\mu(t) < +\infty$ for all $u > 0$).
The condition $\rho(f) < +\infty$ in (e) means that the function $\varphi(t, |f(t)|)$ is integrable in Ω. Then the *absolute continuity* of the modular ρ with $\alpha = 1$ follows from the well-known properties of the integral.

Proposition 1.1.19. *The following relation holds*

$$L^{\varphi}(\Omega) = \{f \in M(\Omega) \;:\; \rho^{\varphi}(\lambda f) < +\infty, \; \textit{for some } \lambda > 0\}.$$

Proof. ($\subseteq$) It is trivial by the definition of limit.
($\supseteq$) Let $f \in M(\Omega)$ such that $\rho^\varphi(\overline{\lambda} f) < +\infty$ for $\overline{\lambda} > 0$ fixed. Then for all $\lambda \leq \overline{\lambda}$, we get $\varphi(t, \lambda|f(t)|) \leq \varphi(t, \overline{\lambda}|f(t)|)$ for $t \in \Omega$ a.e., since φ is non decreasing with respect to the second variable and $\lim_{\lambda \to 0} \varphi(t, \lambda|f(t)|) = \varphi(t, 0) = 0$ by the continuity of φ.
Since $\varphi(\cdot, \overline{\lambda}|f(\cdot)|) \in L^1(\Omega)$, by the Lebesgue dominated convergence theorem we have

$$\lim_{\lambda \to 0} \int_\Omega \varphi(t, \lambda|f(t)|) d\mu(t) = \int_\Omega \lim_{\lambda \to 0} \varphi(t, \lambda|f(t)|) d\mu(t) = 0,$$

so $f \in L^\varphi(\Omega)$. □

Theorem 1.1.20. *The Musielak-Orlicz spaces $L^\varphi(\Omega)$ are complete with respect the F-norm $|\cdot|_{\rho^\varphi}$ and so are Banach spaces.*

As a particular case of Musielak-Orlicz spaces, one can consider φ-functions of product type, of the form

$$\varphi(t, u) := \theta(t)\ \widetilde{\varphi}(u), \tag{1.2}$$

with $t \in \Omega$, $u \in \mathbb{R}_0^+$, which satisfy the following conditions

($\mathcal{F}1$) $\theta \in M(\Omega)$ and there exist $M \geq m > 0$ such that $m \leq \theta(t) \leq M$, for every $t \in \Omega$;

($\mathcal{F}2$) $\widetilde{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous and non decreasing function such that $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(u) > 0$ for $u > 0$;

($\mathcal{F}3$) for every $\lambda_1 > 0$ there exists $\lambda_2 \geq 1$ such that $\lambda_1 \widetilde{\varphi}(u) \leq \widetilde{\varphi}(\lambda_2 u)$, $u \in \mathbb{R}_0^+$.

It is easy to see that assumptions ($\varphi 1$)-($\varphi 3$) are satisfied, where ($\varphi 3$) holds with $F \equiv 0$ and $C = \lambda_2$. In fact, from ($\mathcal{F}3$) with $\lambda_1 = M/m$, we can write

$$\varphi(t - s, u) = \theta(t - s)\ \widetilde{\varphi}(u) \leq \frac{M}{m} \theta(t)\ \widetilde{\varphi}(u) \leq \theta(t)\ \widetilde{\varphi}(\lambda_2 u) = \varphi(t, \lambda_2 u),$$

for every $u \geq 0$.

Example 1.1.21. If we set $\theta \equiv 1$ in (1.2), i.e., $\varphi = \widetilde{\varphi}$ and it does not depend on the first variable t, we find the case of *Orlicz spaces*, that will be considered in Section 1.1.2.

Example 1.1.22. Let us consider the case of functions of several variables by setting $\Omega = \mathbb{R}^n$. Some concrete examples of weighted Orlicz spaces generated by φ-function of the form (1.2) can be obtained by choosing, for instance

$$\theta(\underline{t}) := \frac{5}{\|\underline{t}\|_2^2 + 1} + 1, \quad \underline{t} \in \mathbb{R}^n, \tag{1.3}$$

and as function $\widetilde{\varphi}$ one of the following

$$\widetilde{\varphi}_1(u) := u^p, \qquad \widetilde{\varphi}_2(u) := u^\alpha \log^\beta(u+e),$$

for every $u \geq 0$, with $1 \leq p < +\infty$, $\alpha \geq 1$ and $\beta > 0$, and where $\|\cdot\|_2$ is the usual Euclidean norm defined by $\|\underline{t}\|_2 := (t_1^2 + \cdots + t_n^2)^{1/2}$. Obviously, it is easy to show that the above product-type φ-functions satisfy conditions $(\mathcal{F}1)$, $(\mathcal{F}2)$ and $(\mathcal{F}3)$.
The Musielak-Orlicz spaces generated by $\varphi = \theta\widetilde{\varphi}_1$ are the so-called *weighted L^p-spaces* and the ones generated by $\varphi = \theta\widetilde{\varphi}_2$ are the *weighted Zygmund spaces.*

Example 1.1.23. In $\Omega = \mathbb{R}^n$, we can also consider the following more interesting example of Musielak-Orlicz spaces, which are generated by the φ-functions

$$\varphi_3(\underline{t}, u) := e^{\Psi(\underline{t})u^\gamma} - 1, \tag{1.4}$$

with $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}_0^+$ and $\gamma > 0$, where the function Ψ satisfies the inequality of condition $(\mathcal{F}1)$ for suitable $0 < m \leq M$. By simple computations, it can be shown that also φ-functions of the form as in (1.4) are τ-bounded with $F \equiv 0$ and $C = (M/m)^{1/\gamma}$. As an example of function Ψ one can consider, e.g., the function $\Psi(\underline{t}) = \theta(\underline{t})$ defined in (1.3). Such spaces are the so-called *weighted exponential spaces.*

1.1.2 Orlicz spaces

In this section, we analyze the case presented in Example 1.1.21. So, we now consider the φ-functions of the form $\varphi(t, u) = \widetilde{\varphi}(u)$, $t \in \Omega$, $u \in \mathbb{R}_0^+$. In order to simplify the notation, we will write φ instead of $\widetilde{\varphi}$.

Let φ be a fixed φ-function. Then we can introduce the functional $I^\varphi : X(\Omega) \to [0, +\infty]$, defined by

$$\rho(f) = I^\varphi[f] := \int_\Omega \varphi(|f(t)|) d\mu(t). \tag{1.5}$$

It turns out that I^φ is a modular on the space $M(\Omega)$. The respective modular space is called an *Orlicz space* and it is denoted by $L^\varphi(\Omega)$.
Note that, if φ is a convex φ-function, then $I^\varphi[f]$ is a convex modular functional (see, e.g., [22, 64, 73]).

Remark 1.1.24. If $\Omega = \mathbb{N} = \{1, 2, \cdots\}$ and μ is the counting measure in Ω, the respective Orlicz space is denoted by ℓ^φ and it is called *sequential Orlicz space.*

Remark 1.1.25 ([64])**.** In the case of Orlicz spaces, the modular I^φ and the norms $|\cdot|_{I^\varphi}$ and $\|\cdot\|_{I^\varphi}$ are rearrangement invariant, i.e., if $f, g \in M(\Omega)$ are equimeasurable functions, then $I^\varphi(f) = I^\varphi(g)$, $|f|_{I^\varphi} = |g|_{I^\varphi}$ and $\|f\|_{I^\varphi} = \|g\|_{I^\varphi}$. This property does not hold in general for the Musielak-Orlicz spaces.

If we consider the convex φ-function $\varphi(u) = \widetilde{\varphi}_1(u) = u^p$, $u \geq 0$, $1 \leq p < +\infty$, the Orlicz space generated by φ is the well-known L^p-spaces $L^\varphi(\Omega) = L^p(\Omega)$. In such case, the Luxemburg norm $\|\cdot\|_{I^\varphi}$ in $M(\Omega)$ is equal to

$$\begin{aligned}
\|f\|_{I^\varphi} &= \inf\{u > 0 \ : \ I^\varphi[f/u] \leq 1\} = \inf\left\{u > 0 \ : \ \int_\Omega \left|\frac{f(t)}{u}\right|^p d\mu(t) \leq 1\right\} \\
&= \inf\left\{u > 0 \ : \ \frac{1}{|u|^p}\int_\Omega |f(t)|^p \, d\mu(t) \leq 1\right\} \\
&= \inf\left\{u > 0 \ : \ \int_\Omega |f(t)|^p \, d\mu(t) \leq u^p\right\} \\
&= \inf\left\{u > 0 \ : \ \|f\|_{L^p(\Omega)} \leq u\right\} = \|f\|_{L^p(\Omega)} .
\end{aligned}$$

For a sake of completeness, we recall the Vitali convergence theorem, which provides a characterization of the L^p-convergence and which will be useful in Chapter 3.

Theorem 1.1.26. *Let (f_n) be a sequence in $L^p(\Omega)$, with $1 \leq p < +\infty$. Then $f_n \to f$ in L^p if and only if the following conditions are satisfied*

(i) (f_n) converges in measure to f;

(ii) for every $\varepsilon > 0$ there exists a measurable set E_ε, with $\mu(E_\varepsilon) < +\infty$ (here μ denotes the Lebesgue measure), such that for every $n \in \mathbb{N}$ and for every measurable set F, with $F \cap E_\varepsilon = \varnothing$, we have

$$\int_F |f_n|^p d\mu < \varepsilon^p;$$

(iii) for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for every $n \in \mathbb{N}$ and for any measurable set E with $\mu(E) < \delta(\varepsilon)$, then

$$\int_E |f_n|^p d\mu < \varepsilon^p.$$

Example 1.1.27. Other well-known and useful examples of Orlicz spaces are, e.g., the exponential spaces which can be generated by the convex φ-function $\varphi(u) = \widetilde{\varphi}_3(u) = e^{u^\gamma} - 1$, for $\gamma > 0$ (i.e., the function Ψ defined in (1.4) is identically equal to 1). Moreover, we also have the Zygmund (or interpolation) spaces $L^\alpha \log^\beta L(\Omega)$ which are generated by the φ-functions $\varphi(u) = \widetilde{\varphi}_2(u) = u^\alpha \log^\beta(u + e)$, with $1 \leq \alpha < +\infty$ and $\beta \in \mathbb{R}^+$.

For further results concerning Orlicz spaces, see e.g., [17, 22, 61, 64, 73, 74].

We conclude this section by recalling the Δ_2-condition. In the setting of Orlicz spaces $L^\varphi(\Omega)$, such property can be rewritten in terms of the φ-function φ. We say that φ satisfies (Δ_2) if there exists $M > 0$ such that the inequality

$$\varphi(2u) \leq M\varphi(u) \tag{Δ_2}$$

holds, for every $u \in \mathbb{R}_0^+$.

Example 1.1.28. In $L^p(\Omega)$ and in $L^\alpha \log^\beta L(\Omega)$ the Δ_2-condition is satisfied, then the norm and modular convergence are equivalent and moreover $L^\varphi(\Omega) \equiv E^\varphi(\Omega)$.

Example 1.1.29. In the case of exponential type spaces, e.g. with $\varphi(u) = e^{|u|} - 1$, $u \in \mathbb{R}_0^+$, the Δ_2-condition is not satisfied. Indeed,

$$\frac{\varphi(2u)}{\varphi(u)} = \frac{e^{2|u|} - 1}{e^{|u|} - 1} \to +\infty, \text{ as } u \to +\infty.$$

Therefore $E^\varphi(\Omega)$ is strictly contained in $L^\varphi(\Omega)$ and the modular convergence does not imply the norm convergence.

1.1.3 Another example of modular spaces

In the previous section, we discussed examples of modular spaces characterized by modular functionals in the integral form. In this section, we present an example of modulars that are defined by the supremum, hence that can not be reconducted to integrals.

Let m be a measure on an interval $[a, b[\subset \mathbb{R}$, where b may be equal to $+\infty$, defined on the σ-algebra of all Lebesgue measurable subsets of $[a, b[$. Let W be a nonempty set of indices and let $(a_\ell(\cdot))_{\ell \in W}$ be a family of Lebesgue measurable positive real-valued functions on $[a, b[$. Moreover, let $\Phi : [a, b[\times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function satisfying the following conditions

1) $\Phi(x, u)$ is a non decreasing, continuous function of $u \geq 0$, for every $x \in [a, b[$;

2) $\Phi(x, 0) = 0$, $\Phi(x, u) > 0$ for $u > 0$, and $\Phi(x, u) \to +\infty$ as $u \to +\infty$, for every $x \in [a, b[$;

3) there exists $\lim\limits_{x \to b^-} \Phi(x, u) = \widetilde{\Phi}(u) < +\infty$ for every $u > 0$;

4) $\Phi(x, u)$ is a Lebesgue measurable function on x in $[a, b[$ for every $u \geq 0$.

Then, the functional

$$\mathcal{L}_\Phi(x, f) = \int_\Omega \Phi(x, |f(t)|) d\mu(t)$$

is an Orlicz modular in $M(\Omega)$ for every $x \in [a, b[$. We denote by $M_m(\Omega)$ the subset of $M(\Omega)$ consisting of functions $f \in M(\Omega)$ such that $\mathcal{L}_\Phi(\cdot, f\mathbf{1}_A)$ is Lebesgue measurable in $[a, b[$, for every $A \in \Sigma$. In particular, if $\Phi(x, u)$ is continuous (or a monotone) function of $x \in [a, b[$ for every $u \geq 0$, then $M_m(\Omega) = M(\Omega)$.
We now define an extended functional $\mathcal{A}_\Phi$ on $M_m(\Omega)$ by means of the formula

$$\mathcal{A}_\Phi(f) = \sup_{\ell \in W} \int_a^b a_\ell(x)\mathcal{L}_\Phi(x, f)dm(x),$$

with $f \in M_m(\Omega)$. The functional $\mathcal{A}_\Phi$ is a modular on $M_m(\Omega)$, and in the case when $\Phi(x, u)$ is a convex function of $u \geq 0$, for all $x \in [a, b[$, $\mathcal{A}_\Phi$ is a convex modular.
Under other suitable conditions (see [22], (b) of p. 19 and (b) of p. 23), $\mathcal{A}_\Phi$ is a monotone, strongly finite, absolutely finite and absolutely continuous modular on $M_m(\Omega)$.

1.2 Moduli of smoothness and Lipschitz classes

One of the main task of Approximation Theory, other than the study of convergence properties of sequence or family of operators, is to establish the so-called order of approximation. In order to do this, it is necessary to recall the notion of *modulus of smoothness* of a given function and its main properties.

The introduction of the modulus of smoothness, for bounded and continuous functions, is attributed to Lebesgue (1910) and later to de la Vallée Poussin (1919). Here, it will be defined in the context of modular spaces $L_\rho(\Omega)$ generated by a certain modular ρ. In this case, we assume Ω is provided with an operation $+ : \Omega \times \Omega \to \Omega$ (we do not need to suppose that $(\Omega, +)$ is a group). For a sake of simplicity, we will assume that the operation $+$ is commutative throughout this section. It is not difficult to extend this to the case of a non commutative operation, in such case the notions defined below have right-hand side and left-hand side versions.

In what follows, we will need the notion of a filter $\mathcal{U}$ of subsets Ω and the one of a basis $\mathcal{U}_0$ of a filter.

Definition 1.2.1. *A family $\mathcal{U} \neq \emptyset$ of non empty subsets of Ω is said to be a filter in Ω if it satisfies the following two conditions*

1) if $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$;

2) if $U_1 \in \mathcal{U}$, $U_2 \subset \Omega$ and $U_1 \subset U_2$, then $U_2 \in \mathcal{U}$.

A family $\mathcal{U}_0 \subset \mathcal{U}$ is a basis of the filter $\mathcal{U}$ if for every set $U \in \mathcal{U}$ there exists a set $V \in \mathcal{U}_0$ such that $V \subset U$.

The concept of convergence can be generalized to a general filter $\mathcal{U}$ of subsets of Ω and obviously, we can limit ourself to convergence to zero.

Definition 1.2.2. *A function $f : \Omega \to \mathbb{R}$ is $\mathcal{U}$-convergent to zero if for every $\varepsilon > 0$ there is a set $U_\varepsilon \in \mathcal{U}$ such that $|f(t)| < \varepsilon$, for all $t \in U_\varepsilon$.*

We denote this convergence by $f(t) \xrightarrow{\mathcal{U}} 0$. It is clear that the following equivalence holds: $f : \Omega \to \mathbb{R}$ is $\mathcal{U}$-convergent to zero if and only if for every $\varepsilon > 0$ there is a set $U_\varepsilon \in \mathcal{U}_0$ such that $|f(t)| < \varepsilon$, for all $t \in U_\varepsilon$.
Using the notion of a filter, we will specify a relationship between the operation $+$ in Ω, the σ-algebra Σ, and the measure μ in the measure space (Ω, Σ, μ). For arbitrary $A \in \Sigma$ and $t \in \Omega$, let us denote

$$A_t := \{s \in \Omega \ : \ t + s \in A, \, s \notin A \ \text{ or } \ t + s \notin A, \, s \in A\}.$$

Therefore, we say that $(\Omega, \mathcal{U}, \Sigma, \mu)$ is a *filtered system* with respect to $X(\Omega)$ if

1. the filter $\mathcal{U}$ contains a basis $\mathcal{U}_0$,
2. if $A \in \Sigma$ and $\mu(A) < +\infty$, then $A_t \in \Sigma$ for every $t \in \Omega$ and $\mu(A_t) \xrightarrow{\mathcal{U}} 0$,
3. $X(\Omega)$ invariant with respect to the operation $+$, i.e. if $f \in X(\Omega)$ then $f(t + \cdot) \in X(\Omega)$ for every $t \in \Omega$.

Finally, we can provide the following definition.

Definition 1.2.3. *The ρ-modulus of smoothness is defined as the map $\omega_\rho : X(\Omega) \times \mathcal{U} \to [0, +\infty]$ where*

$$\omega_\rho(f, U) := \sup_{t \in U} \rho\left(f(\cdot + t) - f(\cdot)\right),$$

for every $f \in X(\Omega)$ and $U \in \mathcal{U}$.

This notion has been introduced in [20], and further applied in [21].

The following theorem summarizes the fundamental properties of a modulus of smoothness.

Theorem 1.2.4. *If ρ is a monotone modular on $X(\Omega)$, then*

(a) $\omega_\rho(f, V) \leq \omega_\rho(f, U)$, for $f \in X(\Omega)$ and for $U, V \in \mathcal{U}$, $V \subset U$;

(b) $\omega_\rho(|f|, U) \leq \omega_\rho(f, U)$, for $f \in X(\Omega)$, $U \in \mathcal{U}$;

(c) $\omega_\rho(af, U) \leq \omega_\rho(bf, U)$, for $f \in X(\Omega)$, $U \in \mathcal{U}$, $0 \leq a \leq b$;

(d) $\omega_\rho\left(\sum_{i=1}^{n} f_i, U\right) \leq \sum_{i=1}^{n} \omega_\rho(n f_i, U)$, for $f_1, f_2, \cdots, f_n \in X(\Omega)$, $U \in \mathcal{U}$.

Proof. Properties (a), (c), (d) are trivial. Applying the monotonicity of ρ and the fact that if $f \in X(\Omega)$ then $|f| \in X(\Omega)$, we obtain

$$\rho(|f(\cdot + t)| - |f(\cdot)|) \leq \rho(|f(\cdot + t) - f(\cdot)|) = \rho(f(\cdot + t) - f(\cdot)),$$

which implies (b). □

For further considerations, we need to introduce the notion of boundedness of a modular ρ.

Definition 1.2.5. *A modular ρ is said to be bounded (with respect to the operation $+$ and a filter $\mathcal{U}$ in Ω), if there are a constant $C \geq 1$ and a function $\ell : \Omega \to \mathbb{R}_0^+$ satisfying the conditions $\ell \in L^\infty(\Omega)$, $\ell(t) \xrightarrow{\mathcal{U}} 0$, such that for every function $f \in X(\Omega)$ and every $t \in \Omega$ there holds*

$$\rho(f(\cdot + t)) \leq \rho(Cf) + \ell(t).$$

Hence, we can state the following theorem.

Theorem 1.2.6 (see Theorem 2.4 of [22])**.** *Let $(\Omega, \mathcal{U}, \Sigma, \mu)$ be a filtered system and let ρ be a monotone, absolutely finite, absolutely continuous and bounded modular on $X(\Omega)$. Then for every function $f \in L_\rho(\Omega)$, there exists a constant $\lambda > 0$ such that*

$$\omega_\rho(\lambda f, U) \xrightarrow{\mathcal{U}} 0.$$

Example 1.2.7. Let $\Omega = \mathbb{R}^n$ be provided with the operation of usual addition $+$ component by component and let μ be the Lebesgue measure in the σ-algebra of all Lebesgue measurable subsets on $\mathbb{R}^n$. Let φ be the function from Definition 1.1.16 and let

$$\rho(f) = \rho^\varphi(f) := \int_{\mathbb{R}^n} \varphi(\underline{t}, |f(\underline{t})|) d\underline{t}.$$

The filter $\mathcal{U}$ is the family of all open neighborhoods of the neutral element $\underline{0} \in \mathbb{R}^n$, with the basis consisting of all balls with the center at $\underline{0}$ and radius δ, with $\delta > 0$. Obviously $(\mathbb{R}^n, \mathcal{U}, \Sigma, \mu)$ is a filtered system, where $A_{\underline{t}} = A \triangle (A - \underline{t})$ for every $A \subset \mathbb{R}^n$, $\underline{t} \in \mathbb{R}^n$, denoting by $A \triangle B$ the symmetric difference of sets A and B.
The map $\omega_\rho : M(\mathbb{R}^n) \times \mathcal{U} \to [0, +\infty]$ defined by

$$\omega_\varphi(f, \delta) := \omega_\rho(f, U_\delta) = \sup_{\|\underline{t}\|_2 \leq \delta} \int_{\mathbb{R}^n} \varphi(\underline{s}, |f(\underline{s} + \underline{t}) - f(\underline{s})|) d\underline{s},$$

is the *φ-modulus of smoothness* in the Musielak-Orlicz spaces $L^\varphi(\mathbb{R}^n)$.
If we restrict the modulus ω_φ to the basis $\mathcal{U}_0$, we obtain, by Theorem 1.2.6, that for every $f \in L^\varphi(\mathbb{R}^n)$, there exists $\lambda > 0$ such that $\omega_\varphi(\lambda f, \delta) \to 0$, as $\delta \to 0^+$.

Now, we introduce the Lipschitz classes used in modular spaces in order to study the rate of convergence for a certain family of operators we will deal with in Chapter 4. So, let $\mathcal{T}$ be the class of measurable functions $\tau : \Omega \to [0, +\infty]$ such that $\tau(t) > 0$, $t \in \Omega$, $t \neq \theta$, we

Definition 1.2.8. *For a given $\tau \in \mathcal{T}$, we define the subspace $Lip_\rho(\tau)$ of $L_\rho(\Omega)$ by*

$$Lip_\rho(\tau) := \{f \in L_\rho(\Omega) \,:\, \text{there is } \lambda > 0 \text{ with } \rho(\lambda|f(\cdot+t) - f(\cdot)|) = \mathcal{O}(\tau(t)), \text{ as } t \to \theta\}.$$

The symbol $\mathcal{O}$ means that, for any two functions $f, g \in X(\Omega)$, $f(t) = \mathcal{O}(g(t))$, as $t \to \theta$, if there exist a constant $C > 0$ and $U \in \mathcal{U}_0$ such that $|f(t)| \leq C|g(t)|$ for $t \in U$.

Such classes are called *modular Lipschitz classes*, whose notion was introduced in [25] based on the definition of the classical Zygmund classes of L^p-functions (see example below).

Example 1.2.9. Let $\Omega = \mathbb{R}^n$ with the Lebesgue measure and let $\rho(f) = \|f\|_p$ be the L^p-norm of a function $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$. We recall, for $f \in L^p(\mathbb{R}^n)$, the definition of the first order L^p-modulus of smoothness of f, given by

$$\omega_p(f, \delta) := \sup_{\|\underline{t}\|_2 \leq \delta} \|f(\cdot + \underline{t}) - f(\cdot)\|_p = \sup_{\|\underline{t}\|_2 \leq \delta} \left(\int_{\mathbb{R}^n} |f(\underline{s} + \underline{t}) - f(\underline{s})|^p d\underline{s} \right)^{1/p},$$

with $\delta > 0$. From Theorem 1.2.6, it immediately follows that for every $f \in L^p(\mathbb{R}^n)$, there exists $\lambda > 0$ such that $\omega_p(\lambda f, \delta) \to 0$, as $\delta \to 0^+$. Furthermore, it is interesting to point out that the well-known inequality

$$\omega_p(f, \lambda\delta) \leq (1 + \lambda)\, \omega_p(f, \delta)$$

holds, with $\delta, \lambda > 0$, which is not satisfied in general for the φ-modulus of smoothness instead. This property will be of fundamental importance, as we will see in Chapter 4. In such case, the Lipschitz classes of Zygmund-type in L^p-spaces, with $0 < \nu \leq 1$, are defined as follows

$$Lip_p(\nu) := \{f \in L^p(\mathbb{R}^n) \,:\, \|f(\cdot + \underline{t}) - f(\cdot)\|_p = \mathcal{O}(\|\underline{t}\|_2^\nu), \text{ as } \|\underline{t}\|_2 \to 0\}.$$

Example 1.2.10. As made in the particular context of $L^p(\mathbb{R}^n)$ spaces, we can give the definition of Lipschitz classes in Musielak-Orlicz spaces $L^\varphi(\mathbb{R}^n)$. We define by $Lip_\varphi(\nu)$, $0 < \nu \leq 1$, the set of all functions $f \in L^\varphi(\mathbb{R}^n)$ such that there exists $\lambda > 0$ with

$$\rho^\varphi\left(\lambda\left(f(\cdot + \underline{t}) - f(\cdot)\right)\right) = \int_{\mathbb{R}^n} \varphi\left(\underline{s}, \lambda\left|f(\underline{s} + \underline{t}) - f(\underline{s})\right|\right) d\underline{s} = \mathcal{O}(\|\underline{t}\|_2^\nu),$$

as $\|\underline{t}\|_2 \to 0$.

Example 1.2.11. Let $\Omega = \mathbb{R}^n$ with the Lebesgue measure and let $\rho(f) = \|f\|_\infty = \sup_{\underline{x}\in\mathbb{R}^n} |f(\underline{x})|$ be the usual sup-norm on $\mathbb{R}^n$. For $f \in C(\mathbb{R}^n)$, we define the modulus of continuity as follows

$$\omega(f, \delta) := \sup_{\|\underline{t}\|_2 \le \delta} |f(\cdot + \underline{t}) - f(\cdot)|,$$

with $\delta > 0$. It is easy to prove that the modulus of continuity satisfies the following properties

(a) $\omega(f, \delta)$ is a non decreasing function of δ, i.e., $\omega(f, \delta') \le \omega(f, \delta)$, for $0 < \delta' \le \delta$;

(b) $\omega(f, \lambda\delta) \le (1 + \lambda)\,\omega(f, \delta)$, for every $\lambda > 0$; in particular if $\lambda = n \in \mathbb{N}$, $\omega(f, n\delta) \le n\,\omega(f, \delta)$;

(c) $\omega(f, \delta) \to 0$, as $\delta \to 0^+$;

(d) if $\omega(f, \delta) = o(\delta)$, as $\delta \to 0^+$, then f is constant a.e.

Finally, the Lipschitz classes $Lip(\nu)$, $0 < \nu \le 1$, in the space of bounded and uniformly continuous functions is defined by

$$Lip(\nu) := \{f \in C(\mathbb{R}^n) \ : \ \|f(\cdot + \underline{t}) - f(\cdot)\|_\infty = \mathcal{O}(\|\underline{t}\|_2^\nu), \text{ as } \|\underline{t}\|_2 \to 0\}.$$

Chapter 2

Classical results and approximation by nonlinear generalized sampling operators

In this chapter, we will consider the problem of approximation of a function f, belonging to a certain functional space, by means of the so-called *generalized sampling operators* both in their linear (Section 2.2) and nonlinear (Section 2.3) form.
Several proofs have been omitted, and the interested reader is invited to consult the references.

2.1 The classical WKS sampling theorem

The starting point for us comes directly from the famous sampling theorem, which states that, given a band-limited function $f \in L^2(\mathbb{R})$, it is possible to reconstruct f on the whole real axis by means of an interpolation formula.
This result was firstly proved by Whittaker in 1915; some years later, in 1933 and 1949 respectively, Kotel'nikov and Shannon showed its connection with information theory and its applications. For these reasons, in literature we refer to the classical sampling theorem as the Whittaker-Kotel'nikov-Shannon sampling theorem, or briefly, the WKS sampling theorem.
Here the statement follows.

Theorem 2.1.1. *Let $f \in L^2(\mathbb{R})$ be a continuous and band-limited function, i.e., $supp\widehat{f} \subset [-\pi w, \pi w]$, $w > 0$, where $\widehat{f}$ denotes the Fourier transform of f. Then the following reconstruction formula holds*

$$(S_w f)(x) := \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) sinc[\pi(wx - k)] = f(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where

$$sinc(x) := \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

In other words, the sampling theorem provides the exact reconstruction formula (2.1) taking into account only the behaviour of the function f in its sample values $f(k/w)$ calculated at the nodes k/w, for $k \in \mathbb{Z}$, uniformly spaced on the whole real axis.
Even if the sampling theorem represents a very deep and elegant mathematical theorem, from the point of view of the applications it presents some disadvantages. Indeed, according to (2.1), in order to reconstruct the signal completely, one should know the behaviour of f in an infinite number of sample values, which one usually does not have at disposal. Moreover, if x represents the present time, then formula (2.1) says that one should know the samples of the signal not only in the past of x, but also in the future, that is for $k/w > x$. Still more, the signal should be with finite energy and band-limited, which implies f is the restriction on the real axis of an entire function of exponential type πw, as a consequence of the *Paley-Wiener theorem*. This consists into consider a family of extremely regular signals. Clearly, real world signals are not very regular. As an example, it is sufficient to consider the case of images (both digital and analogue). Indeed, images are multidimensional signals, that in correspondence to the edges of the figures have jumps of gray levels in the gray scale (or in the RGB color scale). These strong luminance variations can be represented, from a mathematical point of view, by discontinuities.
Finally, in view of the *Heisenberg principle*, such a function can not be duration-limited, and in practice most of the signals have the last property.

In the last 60 years, many extensions and generalizations of the classical sampling theorem have been studied, in order to overcome its application problems. Weiss, in 1963 ([80]), and subsequently Brown, in 1967 ([27]), proved the following result.

Theorem 2.1.2. *Let $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ such that $\widehat{f} \in L^1(\mathbb{R})$. Then*

$$|(S_w f)(x) - f(x)| \leq \sqrt{\frac{2}{\pi}} \int_{|v| \geq \pi w} \left| \widehat{f}(v) \right| dv, \tag{2.2}$$

for every $x \in \mathbb{R}$.

By (2.2) we get that $(S_w f)_{w>0}$ is uniformly convergent to f; in fact, since $\widehat{f} \in L^1(\mathbb{R})$, it results that

$$\lim_{w \to +\infty} \int_{|v| \geq \pi w} \left| \widehat{f}(v) \right| dv = 0.$$

In Theorem 2.1.2, the assumption that f is band-limited has been avoided; but it is easy that if we choose a band-limited function, we obtain again Theorem 2.1.1.

Furthermore, it can be shown that in the upper bound (2.2), the constant $\sqrt{\frac{2}{\pi}}$ can not be improved, i.e., it can not be replaced with a lower one.

2.2 Generalized sampling type operators

In the 80's, Butzer and his school (see, e.g., [29,32–35,75]) replaced the sinc function in formula (2.1) by a function χ which is continuous with compact support contained in a real interval, obtaining an approximate sampling formula. Clearly, by using such a function χ, one only needs to know a finite number of sample values. Besides, assuming some additional hypotheses will permit to reconstruct the signal only by using sample values taken from the past, which means to make a prediction of the signal. Namely, they considered a family of discrete operators, called *generalized sampling operators* of the form

$$(G_w f)(x) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \chi(wx - k), \quad x \in \mathbb{R},$$

with $w > 0$. Such operators were firstly introduced in their univariate form, and subsequently extended to the multidimensional setting (see, [30]). The study of the convergence in the multivariate frame is crucial mainly from the applications point of view: in fact, in signal theory, and especially in image processing, we have to work with multivariate signals. Moreover, convergence results that include also the case of not necessarily continuous functions, turn out to be particularly useful in the multivariate setting, since image, for instance, are represented mathematically by bivariate functions with discontinuities in correspondence of the edges of the image itself, where jumps of grey levels occur. Hence, from now on, we shall only consider the multivariate case.

Definition 2.2.1. *Let $\chi \in C_c(\mathbb{R}^n)$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. We define the family of operators $(G_w f)_{w>0}$ as follows*

$$\left(G_w f\right)(\underline{x}) := \sum_{\underline{k} \in \mathbb{Z}^n} f\left(\frac{\underline{k}}{w}\right) \chi(w\underline{x} - \underline{k}), \;\; \underline{x} \in \mathbb{R}^n.$$

The function χ is said to be the kernel of the generalized sampling operators.

Definition 2.2.2. *Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a function and $\beta \geq 0$. We define by*

$$m_\beta(\chi) := \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} \|\underline{u} - \underline{k}\|_2^\beta \left|\chi(\underline{u} - \underline{k})\right|,$$

the discrete absolute moment of order β of the function χ.

Note that in general, it turns out that $0 \leq m_\beta(\chi) \leq +\infty$, $\beta \geq 0$. If $\chi \in C_c(\mathbb{R}^n)$, the moments $m_\beta(\chi)$ are finite for every $\beta \geq 0$.
Let, now, f be a bounded function and $\chi \in C_c(\mathbb{R}^n)$. It is easy to prove the following estimate

$$|(G_w f)(\underline{x})| \leq \sum_{\underline{k} \in \mathbb{Z}^n} \left| f\left(\frac{\underline{k}}{w}\right) \right| |\chi(w\underline{x} - \underline{k})| \leq \|f\|_\infty \sum_{\underline{k} \in \mathbb{Z}^n} |\chi(w\underline{x} - \underline{k})| \leq \|f\|_\infty \, m_0(\chi),$$

for every $\underline{x} \in \mathbb{R}^n$. Since f is bounded and $m_0(\chi) < +\infty$, by the above inequality it follows that the generalized sampling operators are well-defined in $L^\infty(\mathbb{R}^n)$.
The following pointwise and uniform convergence theorem can be stated.

Theorem 2.2.3. *Let $\chi \in C_c(\mathbb{R}^n)$ such that*

$$\sum_{\underline{k} \in \mathbb{Z}^n} \chi(\underline{u} - \underline{k}) = 1, \tag{2.3}$$

for every $\underline{u} \in \mathbb{R}^n$. Then, if $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and continuous in $\underline{x}_0 \in \mathbb{R}^n$, it results that

$$\lim_{w \to +\infty} (G_w f)(\underline{x}_0) = f(\underline{x}_0).$$

In particular, if $f \in C(\mathbb{R}^n)$, we have

$$\lim_{w \to +\infty} \|G_w f - f\|_\infty = 0.$$

In general, checking if a function satisfies the hypothesis (2.3) is not easy. We refer the reader to Remark 3.1.3, where a condition equivalent to (2.3), based on the study of the Fourier transform, is provided.

The following corollary shows that, in order to reconstruct a signal f at $\underline{x}_0$, we need of sampling values taken only in the past of $\underline{x}_0$, i.e., by the generalized sampling operators we are able to solve the problem of the linear prediction by samples from the past. This also solves one of the application disadvantages of the WKS-sampling theorem.

Corollary 2.2.4. *Let $\chi \in C_c(\mathbb{R}^n)$, with $\operatorname{supp}\chi \subset (0, +\infty)^n$ and satisfying (2.3). Then for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$, if $\underline{x}_0$ is a point of continuity of f, we get*

$$\lim_{w \to +\infty} (G_w f)(\underline{x}_0) = \lim_{w \to +\infty} \sum_{\underline{k}/w < \underline{x}_0} f\left(\frac{\underline{k}}{w}\right) \chi(w\underline{x}_0 - \underline{k}) = f(\underline{x}_0),$$

where $\underline{k}/w < \underline{x}_0$ means that $k_i/w < x_{0,i}$, for every $i = 1, \cdots, n$.

Remark 2.2.5. Some quantitative estimates of the uniform convergence in the space C^0 in terms of the classical modulus of continuity, together with a Voronovskaja asymptotic formula, have been established in [19]. Here, the authors consider not necessarily compactly supported kernels such that (2.3) holds.

2.3 Nonlinear generalized sampling type operators

In [24], Bardaro and Vinti considered, for the first time, the generalized sampling series in its nonlinear form. From the point of view of the applications, a nonlinear version of the generalized sampling operators may be useful in order to approximate a nonlinear signals, as for example a signal generated by an earthquake, an explosion, and so on.

Definition 2.3.1. *The family of nonlinear generalized operators* $(G_w f)_{w>0}$ *is of the form*

$$(G_w f)(\underline{x}) := \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, f\left(\frac{\underline{k}}{w}\right)\right), \quad \underline{x} \in \mathbb{R}^n,$$

defined for every $f : \mathbb{R}^n \to \mathbb{R}$ *for which the above series converges.*

In recent years, there has been a large increase in interest of the scientific community in nonlinear approximation operators. The pioneer works of the theory of nonlinear integral operators, in connection with approximation problems, can be reconducted to Musielak (see [65–69]). Later, it has been extensively developed in the monograph by Bardaro, Musielak and Vinti ([22]), and studied in various paper by other authors (see, e.g., [17, 63, 78, 79]). This topic, i.e., the possibility to have at disposal a nonlinear constructing procedure for signal reconstruction, is of considerable interest, not only from the mathematical point of view, but also for applications to Signal and Digital Image Processing. In fact, the reconstruction of signals by means of nonlinear sampling-type operators may describe some nonlinear models: for instance, such operators are also suitable in order to describe nonlinear transformations generated by computed signals that, during their filtering process, generate new frequencies.

One of the main problems to be solved in passing from the linear to the nonlinear setting is that of introducing a suitable notion of singularity for the family of kernel functions. Such hypothesis was first introduced by Musielak in [65] in modular spaces and then weakened it in [66, 67]. Another problem which arises in connection with estimates and convergence results for nonlinear operators is what kind of assumption one must impose on the kernel function and, in this respect, a kind of Lipschitz condition on the kernel function must be assumed. This last condition is always used in literature in order to deal with approximation by means of nonlinear operators and here we use a generalized Lipschitz condition.

Let $\chi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a *kernel*, that is, χ satisfies the following assumptions

(χ1) $(\chi(w\underline{x} - \underline{k}, u))_{\underline{k}} \in \ell^1(\mathbb{Z}^n)$, for every $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $w > 0$;

(χ2) $\chi(\underline{x}, 0) = 0$, for every $\underline{x} \in \mathbb{R}^n$;

($\chi3$) χ is an (L,ψ)-Lipschitz kernel, i.e., there exist a measurable function $L : \mathbb{R}^n \to \mathbb{R}_0^+$ and a φ-function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$|\chi(\underline{x}, u) - \chi(\underline{x}, v)| \leq L(\underline{x})\psi(|u - v|),$$

for every $\underline{x} \in \mathbb{R}^n$ and $u, v \in \mathbb{R}$;

($\chi4$) for every $j \in \mathbb{N}$ and $w > 0$, putting

$$\mathcal{T}_w^j(\underline{x}) := \sup_{\frac{1}{j} \leq |u| \leq j} \left| \frac{1}{u} \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, u\right) - 1 \right|,$$

we have $\lim_{w\to+\infty} \mathcal{T}_w^j(\underline{x}) = 0$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

As previously stated, such assumptions are the most reasonable for reproducing classical conditions in the framework of linear operators. Moreover, we need to assume that the function L in the (L,ψ)-condition belongs to $L^1(\mathbb{R}^n)$ and there is a constant $M > 0$ such that

$$\sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - \underline{k}) \leq M, \tag{2.4}$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$. For instance, it is possible to show that (2.4) implies that

$$\lim_{w\to+\infty} \sum_{\|w\underline{x} - \underline{k}\|_2 > \delta w} L(w\underline{x} - \underline{k}) = 0, \tag{2.5}$$

for every $\delta > 0$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$ (see [75]).

In [24], the authors proved the following uniform approximation result, which extends Theorem 2.2.3.

Theorem 2.3.2. *Let $f \in C(\mathbb{R}^n)$, then it results*

$$\lim_{w\to+\infty} \|G_w f - f\|_\infty = 0.$$

Moreover, $G_w : C(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, and for some constant $M > 0$, we have that $\|G_w f\|_\infty \leq M\psi(\|f\|_\infty)$, for every $w > 0$.

Proof. First, we evaluate $\|G_w f\|_\infty$. By conditions ($\chi2$), ($\chi3$) and (2.4), we can write, taking into account that $f \in C(\mathbb{R}^n)$

$$\begin{aligned}|(G_w f)(\underline{x})| &= \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, f\left(\frac{\underline{k}}{w}\right)\right) \right| \leq \sum_{\underline{k} \in \mathbb{Z}^n} L\left(w\underline{x} - \underline{k}\right) \psi\left(\left| f\left(\frac{\underline{k}}{w}\right)\right|\right) \\ &\leq M\psi(\|f\|_\infty),\end{aligned}$$

for every $\underline{x} \in \mathbb{R}^n$; hence, we obtain $\|G_w f\|_\infty \leq M\psi(\|f\|_\infty)$, and so $G_w : C(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$.
We now evaluate $\|G_w f - f\|_\infty$. Let $\underline{x} \in \mathbb{R}^n$ be fixed, by $(\chi 3)$ we get

$$\begin{aligned}
|(G_w f)(\underline{x}) - f(\underline{x})| &= \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, f\left(\frac{\underline{k}}{w}\right)\right) - f(\underline{x}) \right| \\
&\leq \sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - \underline{k}, f\left(\frac{\underline{k}}{w}\right)\right) - \chi\left(w\underline{x} - \underline{k}, f\left(\underline{x}\right)\right) \right| \\
&+ \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, f\left(\underline{x}\right)\right) - f\left(\underline{x}\right) \right| \\
&\leq \sum_{\underline{k}\in\mathbb{Z}^n} L\left(w\underline{x} - \underline{k}\right)\psi\left(\left| f\left(\frac{\underline{k}}{w}\right) - f\left(\underline{x}\right)\right|\right) \\
&+ \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - \underline{k}, f\left(\underline{x}\right)\right) - f\left(\underline{x}\right) \right| \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1. By the uniform continuity of f at $\underline{x}$, for a fixed $\varepsilon > 0$ there exists $\gamma > 0$ such that $|f(\underline{u}) - f(\underline{x})| < \varepsilon$ for every $\|\underline{u} - \underline{x}\|_2 \leq \gamma$. We can use such γ to split I_1 into two additional summands

$$\begin{aligned}
I_1 &= \left\{ \sum_{\|w\underline{x}-\underline{k}\|_2\leq\gamma w} + \sum_{\|w\underline{x}-\underline{k}\|_2>\gamma w} \right\} L\left(w\underline{x} - \underline{k}\right)\psi\left(\left| f\left(\frac{\underline{k}}{w}\right) - f\left(\underline{x}\right)\right|\right) \\
&\leq M\psi(\varepsilon) + \varepsilon\psi(2\|f\|_\infty),
\end{aligned}$$

using both conditions (2.4) and (2.5).
From the boundedness of f, for every $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that $\sup_{\underline{x}\in\mathbb{R}^n} |f(\underline{x})| \leq j$, with $\frac{1}{j} < \varepsilon$. Let $A_j := \{\underline{x} \in \mathbb{R}^n : 0 < f(\underline{x}) < 1/j\}$, taking into account condition $(\chi 2)$, we can rewrite I_2 as follows

$$\begin{aligned}
I_2 &\leq \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}\right) - f(\underline{x})\mathbf{1}_{A_j} \right| \\
&+ \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{\mathbb{R}^n\setminus A_j}\right) - f(\underline{x})\mathbf{1}_{\mathbb{R}^n\setminus A_j} \right|
\end{aligned}$$

$$\leq \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}\right) - f(\underline{x})\mathbf{1}_{A_j} \right| + |f(\underline{x})| \mathcal{T}_w^j(\underline{x})$$
$$=: I_{2,1} + I_{2,2}.$$

It is clear that $I_{2,2} \leq \|f\|_\infty \mathcal{T}_w^j(\underline{x})$, and using $(\chi 4)$ we have that $I_{2,2} \to 0$ as $w \to +\infty$ uniformly with respect to $\underline{x} \in \mathbb{R}^n$. For $I_{2,1}$, we get

$$\begin{aligned} I_{2,1} &\leq \sum_{\underline{k} \in \mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}\right)\right| + \left|f(\underline{x})\mathbf{1}_{A_j}\right| \\ &\leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\psi\left(\left|f(\underline{x})\mathbf{1}_{A_j}\right|\right) + \left|f(\underline{x})\mathbf{1}_{A_j}\right| \\ &\leq M\psi\left(\frac{1}{j}\right) + \frac{1}{j} \leq M\psi(\varepsilon) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily chosen, we have that $I_2 \to 0$ as $w \to +\infty$. □

Remark 2.3.3. In the case of $f(\underline{x}) = u$, for every $\underline{x} \in \mathbb{R}^n$ with $u \neq 0$ a fixed number, it follows that if $\|G_w f - f\|_\infty \to 0$, as $w \to +\infty$, then

$$\left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - \underline{k}, u) - u \right| \to 0,$$

as $w \to +\infty$, which implies that

$$\left| \frac{1}{u} \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - \underline{k}, u) - 1 \right| \to 0,$$

as $w \to +\infty$, i.e., $\mathcal{T}_w^j(\underline{x}) \to 0$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$. This means that the notion of singularity is also necessary in order to have the required approximation theorem.

It would be interesting to formulate such approximation results not only for uniformly continuous functions, but also for functions belonging to L^p-spaces, or more to some general function spaces. So, we are now interested in working in the general setting of Orlicz spaces, and we refer to [77], where the following approximation results are proved.

In order to establish the main convergence theorem for functions belonging to an Orlicz space, we suppose the following growth condition on the composition of the function φ, which generates the Orlicz space, and the function ψ of the (L, ψ)-Lipschitz condition. We assume what follows.

Let φ be a fixed φ-function; we suppose that there is a φ-function η such that, for every $\lambda \in (0,1)$, there exists a constant $C_\lambda \in (0,1)$ satisying

$$\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u), \tag{H$_\varphi$}$$

for every $u \in \mathbb{R}_0^+$, where ψ is the φ-function of the condition $(\chi 3)$.

This condition is quite common working in the approximation theory with a nonlinear setting, for more details, see, e.g., [22].

Assuming L of condition $(\chi 3)$ with compact support on $\mathbb{R}^n$, we establish the following theorem.

Theorem 2.3.4. *Let φ be a convex φ-function. For every $f \in C_c(\mathbb{R}^n)$ and $\lambda > 0$, we have*

$$\lim_{w\to+\infty} I_\varphi[\lambda(G_w f - f)] = 0.$$

Proof. Let $f \in C_c(\mathbb{R}^n)$ and let $\operatorname{supp} f \subset B(\underline{0}, \gamma)$, for a certain constant $\gamma > 0$. Then, by condition $(\chi 2)$, the nonlinear generalized sampling operators reduce to the finite sum

$$(G_w f)(\underline{x}) = \sum_{\|\underline{k}\|_2 \leq w\gamma} \chi\left(w\underline{x} - \underline{k}, f\left(\frac{\underline{k}}{w}\right)\right),$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$. Assuming $\operatorname{supp} L \subset B(\underline{0}, R)$, with $R > 0$, we obtain $L(w\underline{x} - \underline{k}) = 0$, for every $\underline{x} \notin B(\underline{0}, R+\gamma)$, $\underline{k} \in B(\underline{0}, w\gamma)$ and $w \geq 1$. So, from the (L,ψ)-Lipschitz condition, we deduce that $\operatorname{supp} G_w f \subset B(\underline{0}, R+\gamma)$, for every $w \geq 1$.
Now, using similar reasoning to that of Theorem 2.3.2 with $\lim_{w\to+\infty} \mathcal{T}_w^j(\underline{x}) = 0$ for a.e. $\underline{x} \in \mathbb{R}^n$, and taking into account (2.5), we obtain that $\lim_{w\to+\infty}[(G_w f)(\underline{x}) - f(\underline{x})] = 0$, for a.e. $\underline{x} \in \mathbb{R}^n$. Moreover, by the (L,ψ)-Lipschitz condition and (2.4) we estimate $|(G_w f)(\underline{x}) - f(\underline{x})|$ as follows

$$\begin{aligned}
|(G_w f)(\underline{x}) - f(\underline{x})| &= |(G_w f)(\underline{x}) - f(\underline{x})| \mathbf{1}_{B(\underline{0},R+\gamma)} \\
&\leq [|(G_w f)(\underline{x})| + |f(\underline{x})|] \mathbf{1}_{B(\underline{0},R+\gamma)} \\
&\leq \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - \underline{k}) \psi\left(\left|\frac{\underline{k}}{w}\right|\right) + |f(\underline{x})|\right] \mathbf{1}_{B(\underline{0},R+\gamma)} \\
&\leq [M\psi(\|f\|_\infty) + \|f\|_\infty] \mathbf{1}_{B(\underline{0},R+\gamma)},
\end{aligned}$$

for every $\underline{x} \in \mathbb{R}^n$ and $w \geq 1$. Applying a dominated convergence theorem and by continuity of φ, we obtain

$$\lim_{w\to+\infty}\int_{\mathbb{R}^n}\varphi[\lambda|(G_wf)(\underline{x})-f(\underline{x})|]d\underline{x}=$$
$$=\lim_{w\to+\infty}\int_{B(\underline{0},R+\gamma)}\varphi[\lambda|(G_wf)(\underline{x})-f(\underline{x})|]d\underline{x}=0,$$

for every $\lambda>0$. $\square$

Now, given $N>0$, let $\mathcal{L}_N$ be the subset of $L^\eta(\mathbb{R}^n)$ whose elements f satisfy the following assumption

$$\limsup_{w\to+\infty}\frac{1}{w^n}\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(\lambda\left|f\left(\frac{\underline{k}}{w}\right)\right|\right)\le N\int_{\mathbb{R}^n}\eta(\lambda|f(\underline{x})|)d\underline{x}, \tag{2.6}$$

for every $\lambda>0$.

Theorem 2.3.5. *Let φ be a convex φ-function satisfying condition (H_φ) with η convex. Given any two functions f,g in the domain of the operators G_w, $w>0$, such that $f-g\in\mathcal{L}_N$, for $N>0$. Then, there is a constant $P>0$, depending on N, such that for every $\lambda\in(0,1)$ there exists a constant $\mu>0$, for which*

$$\limsup_{w\to+\infty}I_\varphi[\mu(G_wf-G_wg)]\le PI_\eta[\lambda(f-g)].$$

Proof. Let $\lambda\in(0,1)$ be fixed. There exists a constant $C_\lambda\in(0,1)$ such that the condition (H_φ) is satisfied. We choose a constant $\mu>0$ such that $\mu M\le C_\lambda$. Thus, by the (L,ψ)-Lipschitz condition, and applying Jensen inequality and Fubini-Tonelli theorem, we have

$$\begin{aligned}
I_\varphi[\mu(G_wf-G_wg)] &\le \int_{\mathbb{R}^n}\varphi\left(\mu\sum_{\underline{k}\in\mathbb{Z}^n}\left|\chi\left(w\underline{x}-\underline{k},f\left(\frac{\underline{k}}{w}\right)\right)-\chi\left(w\underline{x}-\underline{k},g\left(\frac{\underline{k}}{w}\right)\right)\right|\right)d\underline{x}\\
&\le \int_{\mathbb{R}^n}\varphi\left(\mu\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-\underline{k})\psi\left(\left|f\left(\frac{\underline{k}}{w}\right)-g\left(\frac{\underline{k}}{w}\right)\right|\right)\right)d\underline{x}\\
&\le \frac{1}{M}\int_{\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-\underline{k})\varphi\left(\mu M\psi\left(\left|f\left(\frac{\underline{k}}{w}\right)-g\left(\frac{\underline{k}}{w}\right)\right|\right)\right)d\underline{x}\\
&\le \frac{1}{M}\int_{\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n}L(w\underline{x}-\underline{k})\eta\left(\lambda\left|f\left(\frac{\underline{k}}{w}\right)-g\left(\frac{\underline{k}}{w}\right)\right|\right)d\underline{x}\\
&\le \frac{1}{M}\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(\lambda\left|f\left(\frac{\underline{k}}{w}\right)-g\left(\frac{\underline{k}}{w}\right)\right|\right)\left[\int_{\mathbb{R}^n}L(w\underline{x}-\underline{k})d\underline{x}\right]\\
&= \frac{\|L\|_1}{Mw^n}\sum_{\underline{k}\in\mathbb{Z}^n}\eta\left(\lambda\left|f\left(\frac{\underline{k}}{w}\right)-g\left(\frac{\underline{k}}{w}\right)\right|\right).
\end{aligned}$$

Now, applying (2.6) with $(f-g)$ instead of f, we obtain that

$$\limsup_{w\to+\infty} I_\varphi[\mu(G_w f - G_w g)] \leq \frac{N\,\|L\|_1}{M} I_\eta[\lambda(f-g)].$$

Therefore, the assertion follows with $P = N\,\|L\|_1/M$. □

Thereby, we can state the following modular convergence theorem.

Theorem 2.3.6. *Let φ be a convex φ-function satisfying condition (H_φ) with η convex. Then, for every $f \in L^{\varphi+\eta}(\mathbb{R}^n)$ such that $f - C_c(\mathbb{R}^n) \subset \mathcal{L}_N$, there exists a constant $\mu > 0$ such that*

$$\lim_{w\to+\infty} I_\varphi[\mu(G_w f - f)] = 0.$$

The proof of Theorem 2.3.6 is substantially based on Theorem 2.3.4, Theorem 2.3.5 and on the density of $C_c(\mathbb{R}^n)$ in $L^{\varphi+\eta}(\mathbb{R}^n)$ with respect to the topology induced by the modular convergence.

Remark 2.3.7. We note that the class $\mathcal{L}_N$ includes the set of all functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $h_\lambda(\underline{x}) := \eta(\lambda|f(\underline{x})|)$ is Riemann integrable on $\mathbb{R}^n$ for every $\lambda > 0$ and

$$\lim_{w\to+\infty} \frac{1}{w^n} \sum_{\underline{k}\in\mathbb{Z}^n} \eta\left(\lambda\left|f\left(\frac{\underline{k}}{w}\right)\right|\right) = \int_{\mathbb{R}^n} \eta(\lambda|f(\underline{x})|)d\underline{x}.$$

In [58], it is proved that the Riemann integrable functions of "bounded coarse variation" provide a characterization of the class of functions h_λ satisfying the previous equality, where the concept of "bounded coarse variation" is a generalization of the classical bounded variation in the sense of Jordan. In Chapter 5, we will examine in depth such class in the more general case of modular spaces.

Chapter 3

Nonlinear sampling Kantorovich operators

In this and the succeeding chapters, the focus is on the study of the so-called *nonlinear multivariate sampling Kantorovich operators*, which have been introduced in the one dimensional version in [78], and subsequently, they have been extended in the multidimensional setting in [47].
This kind of operator represents an averaged version, in Kantorovich-sense, of the nonlinear generalized sampling operators, where, instead of the sampling values $f(\underline{k}/w)$, one has an average of f in a n-dimensional interval containing $\underline{k}/w$. This approach allows to reduce the so-called *time-jitter error*, that occurs in signal processing when the sampling values can not be matched exactly at the node, but in a neighborhood of it. In practise, more information is frequently known around a point than at the point itself.
In the present chapter, we show a pointwise and uniform approximation result (Theorem 3.2.1), and a convergence theorem (Theorem 3.3.4) in the setting of Orlicz spaces, in order to cover also the case of not necessarily continuous functions. Convergence results in L^p-spaces, interpolation spaces and exponential spaces follow as particular cases. Several examples of kernels and graphical representations are finally provided in Section 3.5.

3.1 Definitions and preliminary assumptions

Let $\Pi^n = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ be a sequence of vectors defined by $t_{\underline{k}} = (t_{k_1}, \cdots, t_{k_n})$, where each $(t_{k_i})_{k_i\in\mathbb{Z}}$, $i = 1, \cdots, n$, is a sequence of real numbers with $-\infty < t_{k_i} < t_{k_i+1} < +\infty$, $\lim_{k_i\to\pm\infty} t_{k_i} = \pm\infty$, for every $i = 1, \cdots, n$ and such that there exist $\Delta, \delta > 0$ for which $\delta \leq \Delta_{k_i} := t_{k_i+1} - t_{k_i} \leq \Delta$, for every $i = 1, \cdots, n$. Moreover, we denote by

$$R_{\underline{k}}^w := \left[\frac{t_{k_1}}{w}, \frac{t_{k_1+1}}{w}\right] \times \left[\frac{t_{k_2}}{w}, \frac{t_{k_2+1}}{w}\right] \times \cdots \times \left[\frac{t_{k_n}}{w}, \frac{t_{k_n+1}}{w}\right],$$

with $w > 0$,the n-dimensional intervals of $\mathbb{R}^n$ identified by the sequence $\Pi^n = (t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$. We note that the Lebesgue measure of $R_{\underline{k}}^w$ is given by $A_{\underline{k}}/w^n$, where $A_{\underline{k}} := \Delta_{k_1} \cdot \Delta_{k_2} \cdots \Delta_{k_n}$.

A function $\chi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ will be called *kernel* (for the nonlinear multivariate sampling Kantorovich operators) if it satisfies the following conditions

(χ1) $\left(\chi(w\underline{x} - t_{\underline{k}}, u)\right)_{\underline{k}} \in \ell^1(\mathbb{Z}^n)$, for every $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $w > 0$;

(χ2) $\chi(\underline{x}, 0) = 0$, for every $\underline{x} \in \mathbb{R}^n$;

(χ3) χ is an (L, ψ)-Lipschitz kernel, i.e., there exist a measurable function $L : \mathbb{R}^n \to \mathbb{R}_0^+$ and a φ-function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$|\chi(\underline{x}, u) - \chi(\underline{x}, v)| \leq L(\underline{x})\psi(|u - v|),$$

for every $\underline{x} \in \mathbb{R}^n$ and $u, v \in \mathbb{R}$;

(χ4) there exists $\theta_0 > 0$ such that, for every $j \in \mathbb{N}$ and $w > 0$,

1. $$\mathcal{S}_w^j(\underline{x}) := \sup_{0\leq|u|<\frac{1}{j}} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, u\right) - u \right| = \mathcal{O}\left(w^{-\theta_0}\right),$$

2. $$\mathcal{T}_w^j(\underline{x}) := \sup_{\frac{1}{j}\leq|u|} \left| \frac{1}{u} \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, u\right) - 1 \right| = \mathcal{O}\left(w^{-\theta_0}\right),$$

as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

The above assumptions are the same as those established in Section 2.3 for the nonlinear generalized operators, and we have wholly presented them, taking into account that the asset is not equally spaced this time. Concerning (χ4), we stress that such condition, in the present form, is new and can be obtained by the combination of the approximate singularity conditions considered in Section 2.3 and [18]. The choice of $(t_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$ allows us to sample signals by an irregular sampling scheme. If $t_{\underline{k}} = \underline{k}$, $\underline{k} \in \mathbb{Z}^n$, we proceed to the aforementioned uniform case.

Remark 3.1.1. In order to prove the convergence results contained in this chapter, it would be sufficient to weaken condition (χ4), requiring instead that, for every $j \in \mathbb{N}$,

$$\mathcal{T}_w^j(\underline{x}) = \sup_{\frac{1}{j} \leq |u| \leq j} \left| \frac{1}{u} \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, u\right) - 1 \right| \to 0,$$

as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$, as done in Section 2.3, see [47].

Moreover, we assume that the function L of condition (χ3) satisfies the following additional assumptions

(L1) $L \in L^1(\mathbb{R}^n)$ and is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$;

(L2) there exists a number $\beta_0 > 0$ such that

$$m_{\beta_0, \Pi^n}(L) := \sup_{\underline{x} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} L(\underline{x} - t_{\underline{k}}) \left\| \underline{x} - t_{\underline{k}} \right\|_2^{\beta_0} < +\infty,$$

i.e., the discrete absolute moment of order β_0 is finite.

Now, we can introduce the following family of nonlinear operators.

Definition 3.1.2. *The nonlinear multivariate sampling Kantorovich operators for a given kernel χ are defined by*

$$(K_w f)(\underline{x}) := \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u}\right), \quad \underline{x} \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.

Remark 3.1.3. Note that, if $\chi(\underline{x}, u) := L(\underline{x})\, u$, where L satisfies the conditions (L1) and (L2), the operators $K_w f$ reduce to the linear multivariate sampling Kantorovich operators considered in [48]. In such case, conditions 1. and 2. of (χ4) become

$$\begin{aligned}\mathcal{S}_w^j(\underline{x}) &= \sup_{0 \leq |u| < \frac{1}{j}} \left| \sum_{\underline{k} \in \mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) u - u \right| = \sup_{0 \leq |u| < \frac{1}{j}} |u| \cdot \left| \sum_{\underline{k} \in \mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) - 1 \right| \\ &< \frac{1}{j} \left| \sum_{\underline{k} \in \mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) - 1 \right| = \mathcal{O}\left(w^{-\theta_0}\right),\end{aligned}$$

and

$$\mathcal{T}_w^j(\underline{x}) = \sup_{\frac{1}{j} \leq |u|} \left| \frac{1}{u} \sum_{\underline{k} \in \mathbb{Z}} L\left(w\underline{x} - t_{\underline{k}}\right) u - 1 \right| = \left| \sum_{\underline{k} \in \mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) - 1 \right| = \mathcal{O}\left(w^{-\theta_0}\right),$$

as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$, for some $\theta_0 > 0$, that is we deal with the linear case. In the general theory of sampling type operators, a slightly stronger condition is required, that is

$$\sum_{\underline{k} \in \mathbb{Z}^n} L(\underline{u} - t_{\underline{k}}) = 1, \tag{3.1}$$

for every $\underline{u} \in \mathbb{R}^n$. If (3.1) holds, condition $(\chi 4)$ turns out to be satisfied for every $\theta_0 > 0$. When the uniform spaced sequence $t_{\underline{k}} = \underline{k}$ is considered and L is continuous, it is well known that (3.1) is equivalent to

$$\widehat{L}(2\pi\underline{k}) := \begin{cases} 0, & \underline{k} \in \mathbb{Z}^n \setminus \{\underline{0}\}, \\ 1, & \underline{k} = \underline{0}, \end{cases}$$

where $\widehat{L}(\underline{v}) := \int_{\mathbb{R}^n} L(\underline{u}) e^{-i\underline{v}\cdot\underline{u}} d\underline{u}$, $\underline{v} \in \mathbb{R}^n$, denotes the Fourier transform of L (see, e.g., [31]). Such condition is known in literature with the name of Strang-Fix type condition.

Now, we recall the following lemma that will be useful in the next sections and in the next chapter, too. For a proof, the reader can refer to [46].

Lemma 3.1.4. *Let L be a function satisfying conditions (L1) and (L2). We have*

(i) $m_{0,\Pi^n}(L) := \sup_{\underline{x} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} L(\underline{x} - t_{\underline{k}}) < +\infty$*;*

(ii) for every $\gamma > 0$

$$\lim_{w \to +\infty} \sum_{\|w\underline{x} - t_{\underline{k}}\|_2 > \gamma w} L(w\underline{x} - t_{\underline{k}}) = 0,$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$;

(iii) for every $\gamma > 0$ and $\varepsilon > 0$ there exists a constant $M > 0$ such that

$$\int_{\|\underline{x}\|_2 > M} w^n L(w\underline{x} - t_{\underline{k}}) d\underline{x} < \varepsilon,$$

for sufficiently large $w > 0$ and all the elements $t_{\underline{k}}$ such that $\|t_{\underline{k}}\|_2 \leq \gamma w$.

Remark 3.1.5. (a) If we assume that $f \in L^\infty(\mathbb{R}^n)$, by conditions ($\chi 2$), ($\chi 3$) and by (i) of Lemma 3.1.4, we obtain that $(K_w f)_{w>0}$ are well-defined. In fact, it turns out that

$$\begin{aligned} |(K_w f)(\underline{x})| &\leq \sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u} \right) \right| \\ &\leq \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \psi \left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u})| d\underline{u} \right) \\ &\leq m_{0,\Pi^n}(L) \psi(\|f\|_\infty) < +\infty, \end{aligned}$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$.

(b) Instead of assuming that function L is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$ and that $m_{\beta_0,\Pi^n}(L) < +\infty$, one can explicitly assume that for L the properties (i) and (ii) of Lemma 3.1.4 hold.

(c) In the particular case of the equally spaced sequence $t_{\underline{k}} = \underline{k}$, $\underline{k} \in \mathbb{Z}^n$, one can replace the condition $m_{\beta_0,\Pi^n}(L) < +\infty$ and the boundedness assumption upon L by

$$\sup_{\underline{x}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(\underline{x} - \underline{k}) < +\infty,$$

where the convergence of the series is uniform on compact sets.

In the next sections, we show some results of convergence for the nonlinear sampling Kantorovich operators obtained in [47].

3.2 Pointwise and uniform convergence

Theorem 3.2.1. *Let $f \in C^0(\mathbb{R}^n)$. Then, for every $\underline{x} \in \mathbb{R}^n$,*

$$\lim_{w\to+\infty} (K_w f)(\underline{x}) = f(\underline{x}).$$

Moreover, if $f \in C(\mathbb{R}^n)$, then

$$\lim_{w\to+\infty} \|K_w f - f\|_\infty = 0.$$

Proof. We prove only the first part of the theorem, since the second one can be obtained by similar arguments. Let $\underline{x} \in \mathbb{R}^n$ be fixed. We estimate the error of approximation $|(K_w f)(\underline{x}) - f(\underline{x})|$, obtaining by ($\chi 3$)

$$\begin{aligned}|(K_w f)(\underline{x}) - f(\underline{x})| &\leq \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right)\right| \\ &+ \left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x})\right| \\ &\leq \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})|d\underline{u}\right) \\ &+ \left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x})\right| \\ &=: I_1 + I_2.\end{aligned}$$

We estimate I_1. By the continuity of f at $\underline{x}$, for every fixed $\varepsilon > 0$ there exists $\gamma > 0$ such that $|f(\underline{u}) - f(\underline{x})| < \varepsilon$ whenever $\|\underline{u} - \underline{x}\|_2 \leq \gamma$. Now, we split I_1 into two additional summands, namely $I_1 = I_{1,1} + I_{1,2}$, where

$$I_{1,1} := \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq \frac{\gamma w}{2}} L(w\underline{x} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})|d\underline{u}\right),$$

and

$$I_{1,2} := \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > \frac{\gamma w}{2}} L(w\underline{x} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})|d\underline{u}\right).$$

For every $\underline{u} \in R^w_{\underline{k}} \subset \mathbb{R}^n$, if $\left\|w\underline{x} - t_{\underline{k}}\right\|_2 \leq \frac{\gamma w}{2}$, we have

$$\|\underline{u} - \underline{x}\|_2 \leq \left\|\underline{u} - \frac{t_{\underline{k}}}{w}\right\|_2 + \left\|\frac{t_{\underline{k}}}{w} - \underline{x}\right\|_2 \leq \frac{\Delta\sqrt{n}}{w} + \frac{\gamma}{2} \leq \gamma,$$

since we can choose $w > 0$ sufficiently large to satisfy $\frac{\Delta\sqrt{n}}{w} \leq \frac{\gamma}{2}$. Hence,

$$I_{1,1} \leq \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq \frac{\gamma w}{2}} L(w\underline{x} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \varepsilon\, d\underline{u}\right) \leq m_{0,\Pi^n}(L)\,\psi(\varepsilon),$$

for sufficiently large $w > 0$. For $I_{1,2}$, there holds

$$I_{1,2} \leq \psi(2\,\|f\|_\infty) \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > \frac{\gamma w}{2}} L(w\underline{x} - t_{\underline{k}}).$$

By (ii) of Lemma 3.1.4, it follows that $I_{1,2} \to 0$ as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$. Since $\varepsilon > 0$ is arbitrary and ψ is continuous, then $I_1 \to 0$ as $w \to +\infty$. Now, we estimate I_2. Let $A_j := \{\underline{x} \in \mathbb{R}^n : 0 \leq f(\underline{x}) < 1/j\}$, with $j \in \mathbb{N}$ fixed, taking into account condition $(\chi 2)$, we can rewrite I_2 as follows

$$\begin{aligned} I_2 &\leq \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}\right) - f(\underline{x})\mathbf{1}_{A_j} \right| \\ &+ \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{\mathbb{R}^n \setminus A_j}\right) - f(\underline{x})\mathbf{1}_{\mathbb{R}^n \setminus A_j} \right| \\ &\leq \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}\right) - f(\underline{x})\mathbf{1}_{A_j} \right| + |f(\underline{x})| \mathcal{T}_w^j(\underline{x}) \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

It is clear that $I_{2,1} \leq \mathcal{S}_w^j(\underline{x})$ and $I_{2,2} \leq \|f\|_\infty \mathcal{T}_w^j(\underline{x})$, hence using $(\chi 4)$ we have that $I_2 \to 0$ as $w \to +\infty$ uniformly with respect to $\underline{x} \in \mathbb{R}^n$. This concludes the proof. □

3.3 Modular convergence in Orlicz spaces

Now, in order to obtain approximation results for not necessarily continuous functions, we work in the general setting of Orlicz spaces. From now on, we always consider Orlicz spaces $L^\varphi(\mathbb{R}^n)$ generated by convex φ-functions φ. In order to obtain a modular convergence theorem in Orlicz spaces, we firstly test the modular convergence in $C_c(\mathbb{R}^n)$.

Remark 3.3.1. We can observe that, if $f \in C_c(\mathbb{R}^n)$, there exists a positive constant $\overline{\gamma}$ such that $\operatorname{supp} f \subset B(\underline{0}, \overline{\gamma})$. Let $\gamma > \overline{\gamma} + \Delta$, we have

$$\int_{R_{\underline{k}}^w} f(\underline{u})\, d\underline{u} = 0,$$

for every $t_{\underline{k}} \notin B(\underline{0}, w\gamma)$, being $R_{\underline{k}}^w \cap B(\underline{0}, \overline{\gamma}) = \emptyset$ for sufficiently large $w > 0$.
Then, using condition $(\chi 2)$, the nonlinear multivariate sampling Kantorovich operator of f reduces to the finite sum

$$(K_w f)(\underline{x}) := \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u}\right),$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$.

So, the following Luxemburg-norm convergence theorem can be stated.

Theorem 3.3.2. *Let φ be a convex φ-function. For every $f \in C_c(\mathbb{R}^n)$ and $\lambda > 0$, there holds*

$$\lim_{w\to+\infty} I^{\varphi}\left[\lambda\left(K_w f - f\right)\right] = 0.$$

Proof. We have to prove that

$$\lim_{w\to+\infty} I^{\varphi}\left[\lambda\left(K_w f - f\right)\right] = \lim_{w\to+\infty} \int_{\mathbb{R}^n} \varphi\left(\lambda\left|(K_w f)(\underline{x}) - f(\underline{x})\right|\right) d\underline{x} = 0,$$

for every $\lambda > 0$, which is equivalent to show that the sequence $\left(\varphi\left(\lambda\left|K_w f - f\right|\right)\right)_{w>0}$ converges to zero in $L^1(\mathbb{R}^n)$, for every $\lambda > 0$. In order to do this, it suffices to verify that conditions (i), (ii) and (iii) of the Vitali convergence theorem (Theorem 1.1.26), for $p = 1$ hold.
Let now $\lambda > 0$ be fixed.
(i) Let $f \in C_c(\mathbb{R}^n)$. By Theorem 3.2.1 and the continuity of φ, it is easy to see that $\lim_{w\to+\infty} \varphi(\lambda \left\|K_w f - f\right\|_{\infty}) = 0$, for every $\lambda > 0$. Then, for every fixed $\varepsilon > 0$ there exists $\overline{w} > 0$ such that for every $w \geq \overline{w}$, we have

$$\varphi\left(\lambda\left|(K_w f)(\underline{x}) - f(\underline{x})\right|\right) \leq \varphi\left(\lambda\left\|K_w f - f\right\|_{\infty}\right) < \varepsilon,$$

for every $\underline{x} \in \mathbb{R}^n$. So,

$$\mu\left(\left\{\underline{x} \in \mathbb{R}^n \,:\, \varphi\left(\lambda\left|(K_w f)(\underline{x}) - f(\underline{x})\right|\right) > \varepsilon\right\}\right) = \mu\left(\emptyset\right) = 0,$$

for every $w \geq \overline{w}$. It follows that $\left(\varphi\left(\lambda\left|K_w f - f\right|\right)\right)_{w>0}$ converges in measure to zero.
(ii) Let now $\varepsilon > 0$ be fixed and let $\gamma, \overline{\gamma} > 0$ be as in Remark 3.3.1, i.e., such that $\text{supp}\, f \subset B(\underline{0}, \overline{\gamma})$ and $\gamma > \overline{\gamma} + \Delta$. By Lemma 3.1.4 (iii), there exists a constant $M > 0$ (we can assume $M > \overline{\gamma}$ without any loss of generality), such that

$$\int_{\|\underline{x}\|_2 > M} w^n L(w\underline{x} - t_{\underline{k}}) d\underline{x} < \varepsilon,$$

for sufficiently large $w > 0$ and $\left\|t_{\underline{k}}\right\|_2 \leq w\gamma$. Then, by using Jensen inequality and Fubini-Tonelli theorem, it follows

$$\int_{\|\underline{x}\|_2 > M} \varphi(\lambda|(K_w f)(\underline{x})|) d\underline{x} \leq \int_{\|\underline{x}\|_2 > M} \varphi\left(\lambda \sum_{\left\|t_{\underline{k}}\right\|_2 \leq w\gamma} \left|\chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u}\right)\right|\right) d\underline{x}$$

$$\leq \int_{\|\underline{x}\|_2 > M} \varphi\left(\lambda \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\underline{x} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} |f(\underline{u})|d\underline{u}\right)\right) d\underline{x}$$

$$\leq \int_{\|\underline{x}\|_2 > M} \varphi\left(\lambda \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\underline{x} - t_{\underline{k}})\psi\left(\|f\|_\infty\right)\right) d\underline{x}$$

$$\leq \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \frac{\varphi(\lambda m_{0,\Pi^n}(L)\psi\left(\|f\|_\infty\right))}{w^n m_{0,\Pi^n}(L)} \int_{\|\underline{x}\|_2 > M} w^n L(w\underline{x} - t_{\underline{k}}) d\underline{x}$$

$$< \varepsilon \cdot \frac{\varphi(\lambda m_{0,\Pi^n}(L)\psi\left(\|f\|_\infty\right))}{w^n m_{0,\Pi^n}(L)} \cdot G,$$

where $G > 0$ represents the number of terms of the above sum in fact corresponding to the number of $t_{\underline{k}}/w$ belonging to $B(\underline{0}, \gamma)$. For every $w \geq 1$, we can estimate G as follows

$$G \leq \left(2\left(\left[\frac{\gamma w}{\delta}\right] + 1\right)\right)^n = 2^n \sum_{i=0}^{n} \binom{n}{i} \left[\frac{\gamma w}{\delta}\right]^{n-i} = 2^n w^n \left(\left[\frac{\gamma}{\delta}\right]^n + n\left[\frac{\gamma}{\delta}\right]^{n-1} \frac{1}{w} + \cdots \frac{1}{w^n}\right)$$

$$\leq w^n \left\{2^n \left(\left[\frac{\gamma}{\delta}\right]^n + n\left[\frac{\gamma}{\delta}\right]^{n-1} + \cdots + 1\right)\right\} =: w^n \cdot P,$$

where $[\cdot]$ denotes the integer part. Thus,

$$\int_{\|\underline{x}\|_2 > M} \varphi(\lambda |(K_w f)(\underline{x})|) d\underline{x} < \varepsilon \cdot \frac{\varphi(\lambda m_{0,\Pi^n}(L)\psi\left(\|f\|_\infty\right))}{m_{0,\Pi^n}(L)} \cdot P =: \varepsilon \cdot C,$$

for every $w \geq 1$. Therefore, for $\varepsilon > 0$ there exists a set $E_\varepsilon = B(\underline{0}, M)$ such that for every measurable set F, with $F \cap E_\varepsilon = \emptyset$, we have

$$\int_F \varphi(\lambda |(K_w f)(\underline{x}) - f(\underline{x})|) d\underline{x} = \int_F \varphi(\lambda |(K_w f)(\underline{x})|) d\underline{x}$$

$$\leq \int_{\|\underline{x}\|_2 > M} \varphi(\lambda |(K_w f)(\underline{x})|) d\underline{x} < \varepsilon \cdot C.$$

(iii) Finally, let $B \subset \mathbb{R}^n$ be a measurable set with $\mu(B) < \varepsilon/\tau$, where

$$\tau := \max\{\varphi(2\lambda m_{0,\Pi^n}(L)\psi(\|f\|_\infty)), \varphi(2\lambda \|f\|_\infty)\},$$

$\|f\|_\infty \neq 0$. Using Remark 3.1.5 (a), in correspondence to $\varepsilon > 0$ and for every $w > 0$,

$$\int_B \varphi(\lambda |(K_w f)(\underline{x}) - f(\underline{x})|) d\underline{x} \leq \frac{1}{2}\int_B \varphi(2\lambda |(K_w f)(\underline{x})|) d\underline{x} + \frac{1}{2}\int_B \varphi(2\lambda |f(\underline{x})|) d\underline{x}$$

$$\leq \frac{1}{2}\int_B \varphi(2\lambda m_{0,\Pi^n}(L)\psi(\|f\|_\infty)) d\underline{x} + \frac{1}{2}\int_B \varphi(2\lambda \|f\|_\infty) d\underline{x}$$

$$\leq \int_B \tau \, d\underline{x} = \mu(B)\, \tau < \varepsilon.$$

It follows that the integrals $\int_{(\cdot)} \varphi(\lambda|(K_w f)(\underline{x}) - f(\underline{x})|)d\underline{x}$ are equi-absolutely continuous, and the proof is complete. □

Now, we want to state a modular continuity property for the nonlinear sampling Kantorovich operators in the setting of Orlicz spaces.
We pointed out in Remark 3.1.5 (a) that K_w maps $L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. In [46] it is shown that for linear Kantorovich sampling operators an analogous property holds for the space $L^\varphi(\mathbb{R}^n)$, i.e., they map $L^\varphi(\mathbb{R}^n)$ into itself. However, this property does not hold in the nonlinear case. So, for our operators to be well-defined in $L^\varphi(\mathbb{R}^n)$, we must once again require the growth condition (H_φ), recalled in Section 2.3. In particular, if φ satisfies condition (H_φ), then K_w maps $L^\eta(\mathbb{R}^n)$ into $L^\varphi(\mathbb{R}^n)$.

Theorem 3.3.3. *Let φ be a convex φ-function satisfying condition (H_φ) with η convex. Then, for any $f, g \in L^\eta(\mathbb{R}^n)$, there exist $\lambda \in (0,1)$ and a constant $c > 0$ such that*
$$I^\varphi[c(K_w f - K_w g)] \leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)} I^\eta[\lambda(f-g)].$$

Proof. Let $f, g \in L^\eta(\mathbb{R}^n)$. For $\underline{x} \in \mathbb{R}^n$, by applying condition ($\chi 3$), we have

$$\begin{aligned}
&|(K_w f)(\underline{x}) - (K_w g)(\underline{x})| \\
&\qquad \leq \sum_{\underline{k} \in \mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} g(\underline{u})d\underline{u}\right)\right| \\
&\qquad \leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - g(\underline{u})|d\underline{u}\right).
\end{aligned}$$

Since $f - g \in L^\eta(\mathbb{R}^n)$, there exists $\lambda > 0$ (that can be considered $\lambda \in (0,1)$ without any loss of generality) such that $I^\eta[\lambda(f-g)] < +\infty$. Then, we can choose $c > 0$ such that $c\, m_{0,\Pi^n}(L) \leq C_\lambda$, where $C_\lambda \in (0,1)$ is the parameter arising from condition (H_φ).
Therefore, applying Jensen inequality twice and Fubini-Tonelli theorem, together with the change of variable $w\underline{x} - t_{\underline{k}} = \underline{u}$, we get

$$\begin{aligned}
&I^\varphi[c(K_w f - K_w g)] = \int_{\mathbb{R}^n} \varphi(c|(K_w f)(\underline{x}) - (K_w g)(\underline{x})|)d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \varphi\left(c \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - g(\underline{u})|d\underline{u}\right)\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k} \in \mathbb{Z}^n} \varphi\left(c\, m_{0,\Pi^n}(L) \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - g(\underline{u})|d\underline{u}\right)\right) \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}})d\underline{x}
\end{aligned}$$

$$\leq \frac{\|L\|_1}{w^n\, m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \varphi\left(C_\lambda \psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u})-g(\underline{u})|d\underline{u}\right)\right)$$
$$\leq \frac{\|L\|_1}{w^n\, m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \eta\left(\lambda\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u})-g(\underline{u})|d\underline{u}\right)$$
$$\leq \frac{\|L\|_1}{w^n\, m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \eta\left(\lambda|f(\underline{u})-g(\underline{u})|\right)d\underline{u}$$
$$\leq \frac{\|L\|_1}{\delta^n\, m_{0,\Pi^n}(L)} I^\eta[\lambda(f-g)].$$

The proof is now complete. □

As a consequence of Theorem 3.3.3 (for $g \equiv 0$), we have that K_w maps $L^\eta(\mathbb{R}^n)$ into $L^\varphi(\mathbb{R}^n)$, for every $w > 0$. Finally, we may state the main theorem of this section.

Theorem 3.3.4. *Let φ be a convex φ-function satisfying condition (H_φ) with η convex. If $f \in L^{\varphi+\eta}(\mathbb{R}^n)$, then there exists $c > 0$ such that*

$$\lim_{w\to+\infty} I^\varphi[c(K_w f - f)] = 0.$$

Proof. Let $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. By the density theorem (Theorem 1.1.15), there exists a $\lambda \in (0,1)$ such that, for every $\varepsilon > 0$ there exists a function $g \in C_c(\mathbb{R}^n)$ such that $I^{\varphi+\eta}[\lambda(f-g)] < \varepsilon$. Now, we can fix a constant $c > 0$ such that

$$c \leq \min\left\{\frac{C_\lambda}{3m_{0,\Pi^n}(L)}, \frac{\lambda}{3}\right\},$$

where C_λ is the constant of condition (H_φ). By Theorem 3.3.3 and the properties of the modular I^φ, we can write

$$I^\varphi[c(K_w f - f)] \leq I^\varphi[3c(K_w f - K_w g)] + I^\varphi[3c(K_w g - g)] + I^\varphi[3c(f-g)]$$
$$\leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)} I^\eta[\lambda(f-g)] + I^\varphi[\lambda(K_w g - g)] + I^\varphi[\lambda(f-g)].$$

Let $\kappa := \max\left\{\frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)}, 1\right\}$, we have

$$I^\varphi[c(K_w f - f)] \leq \kappa I^{\varphi+\eta}[\lambda(f-g)] + I^\varphi[\lambda(K_w g - g)]$$
$$\leq \kappa\varepsilon + I^\varphi[\lambda(K_w g - g)].$$

The assertions follows from Theorem 3.3.2. □

3.4 Applications to particular cases of Orlicz spaces

We will now apply the convergence results obtained in Section 3.3 to some special case of Orlicz spaces.
Let $\varphi(u) = u^p$, $1 \leq p < +\infty$, $u \in \mathbb{R}_0^+$, and we consider the Orlicz space generated by such φ. As shown in Section 1.1.2, the modular I^φ coincides with $\|\cdot\|_p^p$ and $L^\varphi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If we choose the function ψ of condition $(\chi 3)$ equal to $\psi(u) = u$, $u \in \mathbb{R}_0^+$, (i.e., χ satisfies a strong Lipschitz condition), we have that φ satisfies condition (H_φ) with $\eta(u) = u^p$ and $C_\lambda = \lambda$. It means that the operators K_w map the whole space $L^p(\mathbb{R}^n)$ into itself and it follows the proposition below.

Proposition 3.4.1. *For every $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, we have*

$$\lim_{w \to +\infty} \|K_w f - f\|_p = 0.$$

Moreover, there holds

$$\|K_w f\|_p \leq \delta^{-n/p} m_{0,\Pi^n}(L)^{(p-1)/p} \|L\|_1^{1/p} \|f\|_p .$$

If the function ψ of condition $(\chi 3)$ is $\psi(u) = u^{q/p}$, $1 \leq q \leq p < +\infty$, instead, condition (H_φ) turns out to be satisfied with $\eta(u) = u^q$ and $C_\lambda = \lambda^{q/p}$. We obtain the following proposition.

Proposition 3.4.2. *Let $1 \leq q \leq p < +\infty$ and φ, ψ as above. Then*

$$\|K_w f\|_p \leq \delta^{-n/p} m_{0,\Pi^n}(L)^{(p-1)/p} \|L\|_1^{1/p} \|f\|_q^{q/p} ,$$

for every $f \in L^q(\mathbb{R}^n)$ and $K_w : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ are well-defined. Moreover, for every $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, we have

$$\lim_{w \to +\infty} \|K_w f - f\|_p = 0.$$

Other important spaces in the applications are the so-called interpolation spaces (see Example 1.1.27), generated by the convex φ-functions $\varphi_{\alpha,\beta}(u) := u^\alpha \log^\beta(e+u)$, $u \geq 0$, $\alpha \geq 1$, $\beta > 0$. The corresponding modular functional is given by

$$I^{\varphi_{\alpha,\beta}}[f] = \int_{\mathbb{R}^n} |f(\underline{x})|^\alpha \log^\beta(e + |f(\underline{x})|) d\underline{x},$$

and $L^{\varphi_{\alpha,\beta}}(\mathbb{R}^n) = L^\alpha \log^\beta L(\mathbb{R}^n)$. Now, choosing $\psi(u) = u$, we obtain for the interpolation spaces that condition (H_φ) is again satisfied for $\eta(u) = \varphi_{\alpha,\beta}(u)$ and $C_\lambda = \lambda$, i.e., $K_w : L^\alpha \log^\beta L(\mathbb{R}^n) \to L^\alpha \log^\beta L(\mathbb{R}^n)$.

Proposition 3.4.3. *For every $f \in L^\alpha \log^\beta L(\mathbb{R}^n)$, with $\alpha \geq 1$ and $\beta > 0$, we have*

$$\lim_{w \to +\infty} \|K_w f - f\|_{L^\alpha \log^\beta L} = 0.$$

Moreover, for every $\lambda > 0$

$$\int_{\mathbb{R}^n} |(K_w f)(\underline{x})|^\alpha \log^\beta (e + \lambda |(K_w f)(\underline{x})|) d\underline{x}$$
$$\leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)^{1-\alpha}} \int_{\mathbb{R}^n} |f(\underline{x})|^\alpha \log^\beta (e + \lambda m_{0,\Pi^n}(L) |f(\underline{x})|) d\underline{x},$$

Finally, we consider the case of the exponential spaces (see, again, Example 1.1.27), generated by the convex φ-function $\varphi_\gamma(u) := e^{u^\gamma} - 1$, $u \in \mathbb{R}_0^+$ for $\gamma > 0$. The modular functional generated by φ_γ is of the following form

$$I^{\varphi_\gamma}[f] = \int_{\mathbb{R}^n} (\exp(|f(\underline{x})|^\gamma) - 1) d\underline{x}.$$

If we set $\psi(u) = u$, condition (H_φ) is fulfilled for $\eta(u) = \varphi_\gamma(u)$ and $C_\lambda = \lambda$, i.e., $K_w : L^{\varphi_\gamma}(\mathbb{R}^n) \to L^{\varphi_\gamma}(\mathbb{R}^n)$.

Proposition 3.4.4. *For every $f \in L^{\varphi_\gamma}(\mathbb{R}^n)$, with $\gamma > 0$. Then*

$$\int_{\mathbb{R}^n} (\exp(\lambda |(K_w f)(\underline{x})|^\gamma) - 1) d\underline{x} \leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} (\exp(\lambda m_{0,\Pi^n}(L) |f(\underline{x})|^\gamma) - 1)) d\underline{x},$$

for every $\lambda > 0$. Moreover, there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} \int_{\mathbb{R}^n} (\exp(\lambda \, |(K_w f)(\underline{x}) - f(\underline{x})|)^\gamma - 1) d\underline{x} = 0.$$

It is cleat that, taking $\psi(u) \neq 0$, one can furnish some estimates and convergence results analogous to Proposition 3.4.2 for the operators $K_w f$ in the interpolation spaces and in the exponential ones.

3.5 Examples of kernels

In this section, we discuss about a suitable procedure in order to construct examples of kernels for the nonlinear multivariate sampling Kantorovich operators. In general, we consider kernel functions of the form

$$\chi(w\underline{x} - t_{\underline{k}}, u) = L(w\underline{x} - t_{\underline{k}}) \, g_w(u),$$

where $(g_w)_{w>0}$, $g_w : \mathbb{R} \to \mathbb{R}$ is a family of functions satisfying $g_w(u) \to u$ uniformly as $w \to +\infty$ and such that there exists a φ-function ψ with

$$|g_w(u) - g_w(v)| \leq \psi(|u - v|), \tag{3.2}$$

for every $u, v \in \mathbb{R}$ and $w > 0$.
For a sake of clarity, all the assumptions made in Section 3.1 on χ and L can be summarized as follows

($\mathcal{L}1$) $\left(L(w\underline{x} - t_{\underline{k}})\right)_{\underline{k}} \in \ell^1(\mathbb{Z}^n)$, for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$, $L \in L^1(\mathbb{R}^n)$ is bounded in a neighborhood of $\underline{0} \in \mathbb{R}^n$ and there exists a number $\beta_0 > 0$ such that

$$m_{\beta_0,\Pi^n}(L) := \sup_{\underline{x}\in\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(\underline{x} - t_{\underline{k}}) \left\|\underline{x} - t_{\underline{k}}\right\|_2^{\beta_0} < +\infty;$$

($\mathcal{L}2$) $g_w(0) = 0$, for every $w > 0$;

($\mathcal{L}3$) there exists $\theta_0 > 0$ such that, for every $j \in \mathbb{N}$ and $w > 0$

1. $$\mathcal{S}_w^j(\underline{x}) := \sup_{0\leq|u|<\frac{1}{j}} \left| g_w(u) \sum_{\underline{k}\in\mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) - u \right| = \mathcal{O}\left(w^{-\theta_0}\right),$$

2. $$\mathcal{T}_w^j(\underline{x}) := \sup_{\frac{1}{j}\leq|u|} \left| \frac{g_w(u)}{u} \sum_{\underline{k}\in\mathbb{Z}^n} L\left(w\underline{x} - t_{\underline{k}}\right) - 1 \right| = \mathcal{O}\left(w^{-\theta_0}\right),$$

as $w \to +\infty$, uniformly with respect to $\underline{x} \in \mathbb{R}^n$.

Example 3.5.1 (see, e.g., [36,78])**.** An example of family $(g_w)_{w>0}$ satisfying all the above assumptions is defined by

$$g_w(u) = \begin{cases} u^{1-1/w}, & \text{if } a < u < 1, \\ u, & \text{otherwise,} \end{cases}$$

with $0 < a < 1/e$ (see, Figure 3.1). It is easy to see that $g_w(u) \to u$ uniformly on $\mathbb{R}$, as $w \to +\infty$. Note that if the function L satisfies condition (3.1), assumption ($\mathcal{L}3$) holds for $\theta_0 = 1$. In fact, the function $g_w(u) - u$ on $(a, 1)$ achieves the maximum at $u_0 := \left(\frac{w-1}{w}\right)^w$ for sufficiently large $w > 0$, $g_w(u) - u = 0$ otherwise, then for every $u \in \mathbb{R}$ we have

$$|g_w(u) - u| \leq |g_w(u_0) - u_0| = \left(\frac{w-1}{w}\right)^w \left(\frac{1}{w-1}\right) \leq \frac{C}{w-1},$$

for sufficiently large $w > 0$, and for a suitable positive constant C. Then

$$\mathcal{S}_w^j(\underline{x}) = \sup_{0\leq|u|<\frac{1}{j}} |g_w(u) - u| \leq \frac{C}{w-1} = \mathcal{O}(w^{-1}),$$

and

$$\mathcal{T}_w^j(\underline{x}) = \sup_{\frac{1}{j}\leq|u|} \left|\frac{g_w(u)}{u} - 1\right| = \sup_{u\in(a,1)} \frac{1}{|u|} \cdot |g_w(u) - u| \leq a^{-1} \cdot \frac{C}{w-1} = \mathcal{O}\left(w^{-1}\right),$$

as $w \to +\infty$.

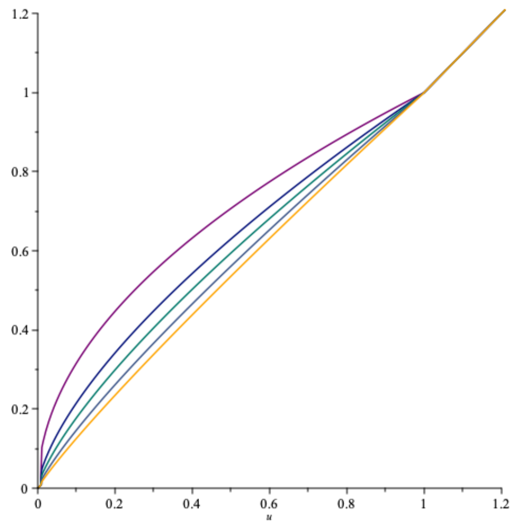

Figure 3.1: Graph of $g_w(u)$ for different values of w and $a = 1/100$.

Moreover, the (L, ψ)-Lipschitz condition turns out to be satisfied with a piecewise concave function $\psi : \mathbb{R}_0^+ \to \mathbb{R}$ defined as follows

$$\psi(u) := g_2|_{\mathbb{R}_0^+}(u) = \begin{cases} \sqrt{u}, & \text{if } a < u < 1, \\ u, & \text{otherwise,} \end{cases}$$

then condition (3.2) holds for sufficiently large $w > 0$. In details, for every $w \geq 2$, if we consider $u, v \geq 1$, or $u, v \leq a$, with $|u - v| \geq 1$, we have

$$|g_w(u) - g_w(v)| = |u - v| = \psi(|u - v|);$$

if we consider $u, v \geq 1$, or $u, v \leq a$, with $|u - v| \in (a, 1)$, we obtain

$$|g_w(u) - g_w(v)| = |u - v| \leq \sqrt{|u - v|} = \psi(|u - v|);$$

if we consider $u, v \in (a, 1)$ with $|u - v| \in (a, 1)$, and since g_w is concave on $(a, 1)$, we can write

$$|g_w(u) - g_w(v)| = |u^{1-1/w} - v^{1-1/w}| \leq |u - v|^{1-1/w} \leq \sqrt{|u - v|} = \psi(|u - v|);$$

if we take $u \in (a,1)$, $v \geq 1$ (or conversely) with $|u-v| \geq 1$, we get

$$|g_w(v) - g_w(u)| = v - u^{1-1/w} \leq v - u = \psi(|u-v|);$$

finally, if we assume $u \in (a,1)$, $v \geq 1$ (or conversely) with $|u-v| \in (a,1)$, we obtain

$$|g_w(v) - g_w(u)| = v - u^{1-1/w} \leq v - u \leq (v-u)^{1-1/w} \leq \sqrt{|u-v|} = \psi(|u-v|).$$

If instead $g_w(u) \equiv u$, $u \in \mathbb{R}$, for every $w > 0$, the function ψ corresponding to $\chi(\underline{x}, u) = L(\underline{x})\, u$ is $\psi(u) = u$, therefore we reduce again to the linear case already studied in [8]. For a sake of simplicity, in what follows, we will consider only the case of the uniform sequence $t_{\underline{k}} = \underline{k}$, $\underline{k} \in \mathbb{Z}^n$.

In general, it is not easy to verify if a function L satisfies conditions $(\mathcal{L}1)$ and $(\mathcal{L}3)$. A possible approach to define suitable examples of functions L is to consider functions which are n-fold products of univariate functions satisfying suitable properties. For instance, let $L_1, L_2, \cdots, L_n \in L^1(\mathbb{R})$ such that

$$m_{0,\Pi}(L_i) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} L_i(u-k) < +\infty,$$

where $\Pi = (k)_{k \in \mathbb{Z}}$ and the convergence of the series is uniform on compact series of $\mathbb{R}$. Moreover, we assume that $\sum_{k \in \mathbb{Z}} L_i(u-k) = 1$, for every $u \in \mathbb{R}$ and $i = 1, \cdots, n$. Setting $L(\underline{u}) := \prod_{i=1}^n L_i(u_i)$, we obtain that $L \in L^1(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} L(\underline{u}) d\underline{u} = \int_{\mathbb{R}^n} \prod_{i=1}^n L_i(u_i) du_1 \cdots du_n = \prod_{i=1}^n \int_{\mathbb{R}} L_i(u_i) du_i < +\infty,$$

and

$$m_{0,\Pi^n}(L) = \sup_{\underline{u} \in \mathbb{R}^n} \sum_{\underline{k} \in \mathbb{Z}^n} L(\underline{u} - \underline{k}) = \prod_{i=1}^n m_{0,\Pi}(L_i) < +\infty,$$

and the convergence is uniform on compact subsets of $\mathbb{R}^n$; hence condition $(\mathcal{L}1)$ holds. Furthermore,

$$\sum_{\underline{k} \in \mathbb{Z}^n} L(\underline{u} - \underline{k}) = \prod_{i=1}^n \sum_{k_i \in \mathbb{Z}} L_i(u_i - k_i) = 1,$$

for every $\underline{u} \in \mathbb{R}^n$, then condition $(\mathcal{L}3)$ is satisfied, taking into account that $g_w(u) \to u$, uniformly as $w \to +\infty$.
A first typical example of nonlinear multivariate sampling Kantorovich operators of the above type is based on the multivariate Fejér kernel $\mathcal{F}_n(\underline{x}) := \prod_{i=1}^n F(x_i)$ (see, Figure 3.2), where F is the well-known Fejér kernel of one variable

$$F(x) := \frac{1}{2} \text{sinc}^2 \left(\frac{x}{2} \right), \quad x \in \mathbb{R}.$$

Furthermore, the Fourier transform of F is given by

$$\widehat{F}(v) := \begin{cases} 1 - \left|\frac{v}{\pi}\right|, & |v| \leq \pi, \\ 0, & |v| > \pi. \end{cases}$$

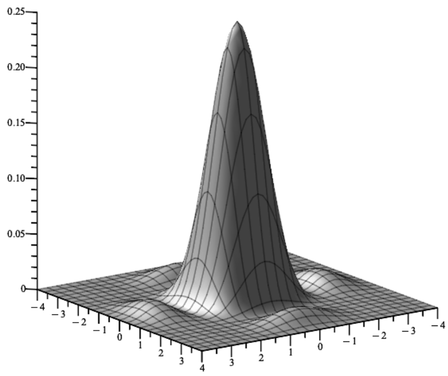

Figure 3.2: Bivariate Fejér kernel $\mathcal{F}_2$.

The multivariate Féjer kernel $\mathcal{F}_n$ plays now the role of the function L. Clearly, we have that $\mathcal{F}_n$ is continuous, non-negative and bounded, belongs to $L^1(\mathbb{R}^n)$ and satisfies all the other required conditions. In particular, it is possible to see that (3.1) holds in view of the Strang-Fix condition recalled in Remark 3.1.3. In this case, we can assume

$$\chi(w\underline{x} - \underline{k}, u) := \mathcal{F}_n(w\underline{x} - \underline{k})\, g_w(u),$$

and therefore, condition ($\mathcal{L}3$) is obviously satisfied as $w \to +\infty$, for some $\theta_0 > 0$. The corresponding nonlinear multivariate sampling Kantorovich operators take now the following form

$$(K_w^{\mathcal{F}_n} f)(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n} \mathcal{F}_n(w\underline{x} - \underline{k})\, g_w \left(\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u} \right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.

Now, in order to support the theory through graphical examples, let's consider several operators based on specific kernels that we will apply to a particular discontinuous function. For instance, we take $(g_w)_{w>0}$ defined as in Example 3.5.1 and we apply the nonlinear sampling operators $K_w^{\mathcal{F}_2}$ to a function $f \in L^p(\mathbb{R}^2)$, $1 \leq p < +\infty$, defined by (Figure 3.3)

$$f(x,y) = \begin{cases} 3, & -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1, \\ \dfrac{6}{x^2+y^2}, & \text{otherwise.} \end{cases} \tag{3.3}$$

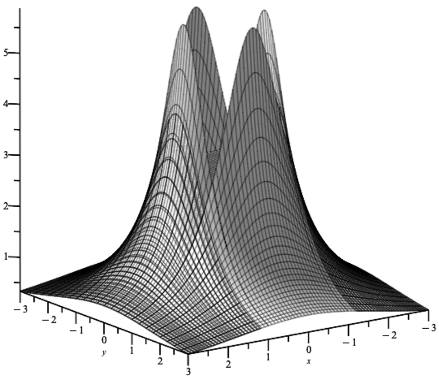

Figure 3.3: Graph of the function f.

The two-dimensional nonlinear sampling Kantorovich operators for the function f defined in (3.3) in case of $w = 5$ and $w = 10$ are given in Figure 3.4 (in an octant of the plane).

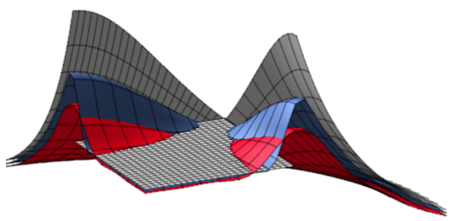

Figure 3.4: The function f (gray) with the bivariate nonlinear sampling Kantorovich operators $K_5^{\mathcal{F}_2} f$ (purple) and $K_{10}^{\mathcal{F}_2} f$ (azure).

Another useful class of kernels is given by the so-called Jackson type kernels of order $s \in \mathbb{N}$, defined in the univariate case by

$$J_s(x) := c_s \,\text{sinc}^{2s}\left(\frac{x}{2s\pi\alpha}\right), \quad x \in \mathbb{R},$$

with $\alpha \geq 1$ and c_s is a non-zero normalization coefficient, given by

$$c_s := \left[\int_{\mathbb{R}} \operatorname{sinc}^{2s} \left(\frac{u}{2s\pi\alpha} \right) du \right]^{-1}.$$

The multivariate Jackson type kernel is given by the n-fold product of the corresponding univariate function, $\mathcal{J}_s^n(\underline{x}) = \prod_{i=1}^n J_s(x_i)$, $\underline{x} \in \mathbb{R}^n$ (see, Figure 3.5). It is easy to prove that all the required assumptions are satisfied and the corresponding multivariate nonlinear sampling Kantorovich operators are given by

$$(K_w^{\mathcal{J}_s^n} f)(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n} \mathcal{J}_s^n(w\underline{x} - \underline{k})\, g_w \left(\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u} \right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.

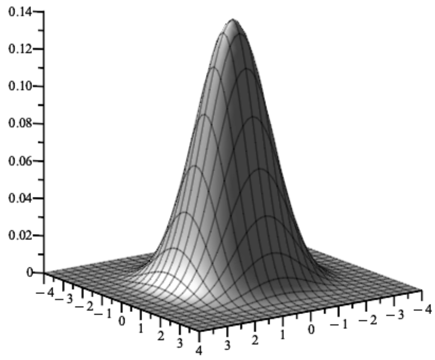

Figure 3.5: Bivariate Jackson kernel $\mathcal{J}_2^3$, with $\alpha = 1$.

As before, in order to give a graphical representation also in this case, we plot the function f, the operators $K_5^{\mathcal{J}_2} f$ and $K_{10}^{\mathcal{J}_2} f$ all together in a same octant of the plane (Figure 3.6).

For what concerns examples of function L with compact support, we can consider the well-known central B-spline (univariate) of order $s \in \mathbb{N}$, defined by

$$M_s(x) := \frac{1}{(s-1)!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} \left(\frac{s}{2} + x - j \right)_+^{s-1}$$

where $x_+ := \max\{x, 0\}$ is the positive part of x. The Fourier transform of M_s is given by

$$\widehat{M_s}(v) = \operatorname{sinc}^s \left(\frac{v}{2\pi} \right), \quad v \in \mathbb{R},$$

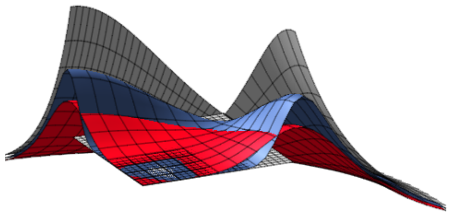

Figure 3.6: The function f (gray) with the bivariate nonlinear sampling Kantorovich operators $K_5^{\mathcal{J}_2} f$ (purple) and $K_{10}^{\mathcal{J}_2} f$ (azure).

and then, we have $\sum_{k \in \mathbb{Z}} M_s(u-k) = 1$, for every $u \in \mathbb{R}$, by Remark 3.1.3, and therefore, condition ($\mathcal{L}3$) is again satisfied. Obviously, each M_n is bounded on $\mathbb{R}$, with compact support on $[-s/2, s/2]$, and hence $M_s \in L^1(\mathbb{R})$, for all $s \in \mathbb{N}$, with $\|M_s\|_1 = 1$. Further, condition (1) is fulfilled for every $\beta_0 > 0$. Thus we can define the multivariate central B-spline of order s, as follows

$$\mathcal{M}_s^n(\underline{x}) := \prod_{i=1}^{n} M_s(x_i), \qquad \underline{x} \in \mathbb{R}^n.$$

So, setting

$$\chi(w\underline{x} - \underline{k}, u) := \mathcal{M}_s^n(w\underline{x} - \underline{k})\, g_w(u),$$

the corresponding multivariate nonlinear sampling Kantorovich operators are given by

$$(K_w^{\mathcal{M}_s^n} f)(\underline{x}) = \sum_{\underline{k} \in \mathbb{Z}^n} \mathcal{M}_s^n(w\underline{x} - \underline{k})\, g_w \left(\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u}) d\underline{u} \right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.
Let's now consider the particular case $s = 3$. First, we recall that the B-spline M_3 (see, Figure 3.7) is given by

$$M_3(x) := \begin{cases} \frac{3}{4} - x^2, & |x| \leq \frac{1}{2}, \\ \frac{1}{2}\left(\frac{3}{2} - |x|\right)^2, & \frac{1}{2} < |x| \leq \frac{3}{2}, \\ 0, & |x| > \frac{3}{2}, \end{cases}$$

for $x \in \mathbb{R}$.

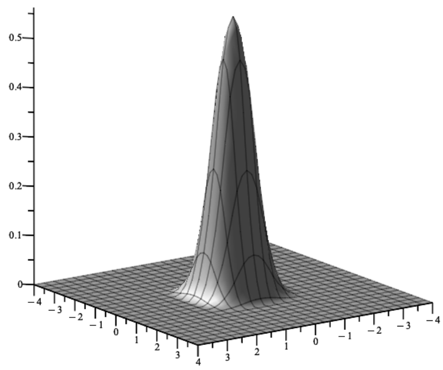

Figure 3.7: Bivariate B-spline kernel $\mathcal{M}_3^2$.

The two-dimensional nonlinear sampling Kantorovich operators generated by $\mathcal{M}_3^2$ applied to the function f defined in (3.3), in case of $w = 5$ and $w = 10$, are displayed together in Figure 3.8.

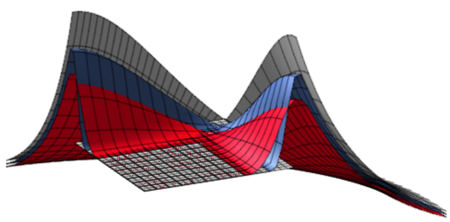

Figure 3.8: The function f (gray) with the bivariate nonlinear sampling Kantorovich operators $K_5^{\mathcal{M}_3^2} f$ (purple) and $K_{10}^{\mathcal{M}_3^2} f$ (azure).

When the graphs of the nonlinear bivariate sampling Kantorovich operators generated by the spline kernels and the Féjer kernels are compared, it is clear that the approximation by the series based on the first is significantly better than the series based on the second (see, e.g, Figure 3.9). This means that, using the series $K_w^{\mathcal{M}_3^2} f$, a reasonable approximation can be obtained by taking into account fewer mean values of f than using the series $K_w^{\mathcal{F}_2} f$.

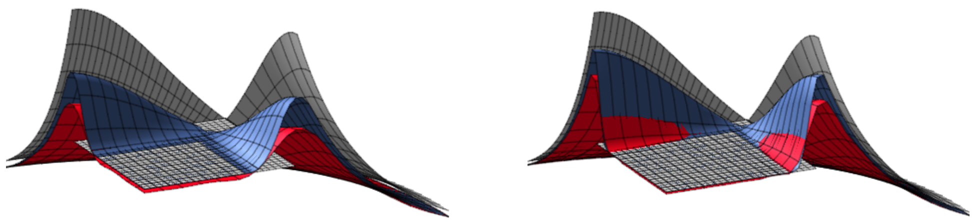

Figure 3.9: The function f (gray) with respectively $K_5^{\mathcal{F}_2} f$ (purple), $K_5^{\mathcal{M}_3^2} f$ (azure) and $K_{10}^{\mathcal{F}_2} f$ (purple), $K_{10}^{\mathcal{M}_3^2} f$ (azure).

One can also use, instead of M_3, linear combination of univariate B-spline of different degree, such as

$$L_1(x) := 4M_3(x) - 3M_4(x), \quad L_2(x) := 5M_4(x) - 4M_5(x), \quad x \in \mathbb{R},$$

$x \in \mathbb{R}$, or linear combinations of translates of B-splines, e.g.,

$$L_3(x) := \frac{5}{4} M_3(x) - \frac{1}{8}\Bigg(M_3(x+1) + M_3(x-1) \Bigg), \quad x \in \mathbb{R},$$

in order to construct examples of multivariate kernels improving the rate of approximation.

Lastly, we mention an example of non-product kernels, which can be of radial type, e.g., represented by the so-called Bochner-Riesz kernel of order $s > 0$, defined as follows

$$b_s^n(\underline{x}) := \frac{2^s}{\sqrt{(2\pi)^n}} \Gamma(s+1) \, \|\underline{x}\|_2^{-s-n/2} \, J_{s+n/2}(\|\underline{x}\|_2), \quad \underline{x} \in \mathbb{R}^n,$$

where J_λ is the Bessel function of order λ, with $\lambda > \frac{n-1}{2}$, and Γ is the usual Euler gamma function.
Since it is well-known that $J_\lambda(\|\underline{x}\|_2) = \mathcal{O}(\|\underline{x}\|_2^{-n/2})$, as $\|\underline{x}\|_2 \to +\infty$, hence $b_s^n(\underline{x}) = \mathcal{O}(\|\underline{x}\|_2^{-s-n})$, as $\|\underline{x}\|_2 \to +\infty$, then $b_s^n \in L^1(\mathbb{R}^n)$. Its Fourier transform is given by

$$\widehat{b_s^n}(\underline{v}) = \begin{cases} (1 - \|\underline{v}\|_2^2)^s, & \|\underline{v}\|_2 \leq 1, \\ 0, & \|\underline{v}\|_2 > 1, \end{cases} \quad \underline{v} \in \mathbb{R}^n,$$

namely, b_s^n is bandlimited (i.e., it belongs to the Bernstein class $B_1^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$). The corresponding nonlinear multivariate sampling Kantorovich operators take the

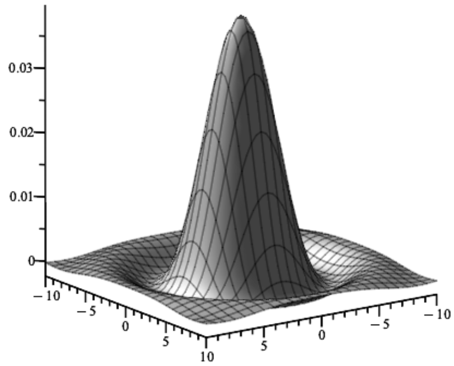

Figure 3.10: Bivariate Bochner-Riesz kernel of order $s = 1$.

following form

$$(K_w^{b_s^n} f)(\underline{x}) = \sum_{\underline{k}\in\mathbb{Z}^n} b_s^n(w\underline{x} - \underline{k})\, g_w \left(\frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} f(\underline{u})d\underline{u} \right),$$

for every $w > 0$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $\underline{x} \in \mathbb{R}^n$.

Now, considering again the two-dimensional framework ($n = 2$), we take as L the bivariate Bochner-Riesz kernel of order $s = 1$ (see, Figure 3.10). Thus, we apply the corresponding bivariate operator $K_5^{b_1^2} f$ and $K_{10}^{b_1^2} f$ to the function f defined in (3.3), in the same octant of the plane (Figure 3.11).

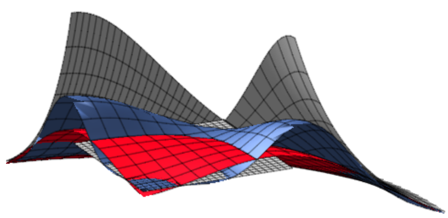

Figure 3.11: The function f (gray) with the bivariate nonlinear sampling Kantorovich operators $K_5^{b_1^2} f$ (purple) and $K_{10}^{b_1^2} f$ (azure).

Finally, for other examples of kernel functions we can refer to the wide existing literature, see, e.g. [6, 7, 37–40, 52–54].

Chapter 4

Quantitative and qualitative estimates

In this chapter, we deal with the study of the order of approximation for the operators $K_w f$. The results that appear in the following sections are partially contained in [36].
Therefore, we prove some quantitative estimates for the nonlinear sampling Kantorovich operators in the multivariate setting using the modulus of smoothness of Orlicz spaces $L^\varphi(\mathbb{R}^n)$. The general frame of Orlicz spaces allows us to deduce the corresponding estimates in L^p-spaces, $1 \leq p < +\infty$, interpolation spaces, exponential spaces and many others instances of Orlicz spaces. In the particular case of L^p-approximation, we can also proceed with the estimation of the aliasing error, by a direct proof.
However, quantitative estimates for $f \in C(\mathbb{R}^n)$ have not been investigated before; hence, Theorem 4.1.1 and Theorem 4.1.2 consist in new results still not published. The qualitative order of approximation is established for functions belonging to suitable Lipschitz classes.

4.1 Quantitative estimates in $C(\mathbb{R}^n)$

In order to establish quantitative estimates for the order of approximation of a family of nonlinear multivariate operators, we briefly recall some facts from Section 1.2.
For $f \in C(\mathbb{R}^n)$, the modulus of continuity is given by

$$\omega(f, \delta) := \sup_{\|\underline{t}\|_2 \leq \delta} |f(\cdot + \underline{t}) - f(\cdot)|,$$

with $\delta > 0$ (see Example 1.2.11).

Theorem 4.1.1. *Let $f \in C(\mathbb{R}^n)$ and let L be a function satisfying condition (L2) with $\beta_0 \geq 1$. Then, we have*

$$\|K_w f - f\|_\infty \leq M_1 \psi\left(\omega\left(f, \frac{1}{w}\right)\right) + M_2 w^{-\theta_0} + M_3 \|f\|_\infty w^{-\theta_0},$$

for sufficiently large $w > 0$, where $M_1 := m_{0,\Pi^n}(L) + \sqrt{n}\,\Delta\, m_{0,\Pi^n}(L) + m_{1,\Pi^n}(L)$, $m_{0,\Pi^n}(L) < +\infty$ and $M_2, M_3, \theta_0 > 0$ are the constants of condition (χ4).

Proof. Let $\underline{x} \in \mathbb{R}^n$ be fixed. We have

$$\begin{aligned}
|(K_w f)(\underline{x}) - f(\underline{x})| &= \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - f(\underline{x}) \right| \\
&\leq \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) \right| \\
&+ \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right| \\
&= I_1 + I_2.
\end{aligned}$$

We estimate I_1. Applying condition (χ3) and taking into account that ψ is non decreasing, we get

$$\begin{aligned}
I_1 &\leq \sum_{\underline{k} \in \mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) \right| \\
&\leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| d\underline{u}\right) \\
&\leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \omega(f, \|\underline{u} - \underline{x}\|_2) d\underline{u}\right) \\
&\leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \omega\left(f, \frac{1}{w}\right) [1 + w\|\underline{u} - \underline{x}\|_2]\, d\underline{u}\right) \\
&= \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\omega\left(f, \frac{1}{w}\right) \left[1 + \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} w\|\underline{u} - \underline{x}\|_2\, d\underline{u}\right]\right),
\end{aligned}$$

for every $w > 0$, where the previous estimate is a consequence of the well-known inequality $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$, with $\lambda = w\|\underline{u} - \underline{x}\|_2$ and $\delta = \frac{1}{w}$. Now, for every $\underline{x}, \underline{u} \in \mathbb{R}^n$, we may write

$$\|\underline{u} - \underline{x}\|_2 \leq \left\| \underline{u} - \frac{t_{\underline{k}}}{w} \right\|_2 + \left\| \frac{t_{\underline{k}}}{w} - \underline{x} \right\|_2 \leq \sqrt{n}\frac{\Delta}{w} + \frac{\left\| w\underline{x} - t_{\underline{k}} \right\|_2}{w}, \tag{4.1}$$

for every $w > 0$; therefore

$$I_1 \leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi \left(\omega \left(f, \frac{1}{w} \right) \left[1 + \sqrt{n}\Delta + \left\| w\underline{x} - t_{\underline{k}} \right\|_2 \right] \right).$$

Since ψ is concave, we have for $u \geq 1$

$$u\psi(v) = u\psi\left(\frac{1}{u} \cdot vu \right) \geq u\frac{1}{u}\psi(vu) = \psi(vu), \tag{4.2}$$

for every $v \geq 0$; consequently, we finally get

$$\begin{aligned}
I_1 &\leq \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \, \left[1 + \sqrt{n}\Delta + \left\| w\underline{x} - t_{\underline{k}} \right\|_2 \right] \psi \left(\omega \left(f, \frac{1}{w} \right) \right) \\
&\leq m_{0,\Pi^n}(L) \, (1 + \sqrt{n}\,\Delta) \, \psi \left(\omega \left(f, \frac{1}{w} \right) \right) + \sum_{\underline{k} \in \mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \, \left\| w\underline{x} - t_{\underline{k}} \right\|_2 \psi \left(\omega \left(f, \frac{1}{w} \right) \right) \\
&\leq m_{0,\Pi^n}(L) \, (1 + \sqrt{n}\,\Delta) \, \psi \left(\omega \left(f, \frac{1}{w} \right) \right) + m_{1,\Pi^n}(L) \, \psi \left(\omega \left(f, \frac{1}{w} \right) \right).
\end{aligned}$$

Now, we estimate I_2. Setting $A_j := \{\underline{x} \in \mathbb{R}^n \, : \, 0 \leq |f(\underline{x})| < 1/j\}$, we can rewrite I_2 as follows

$$\begin{aligned}
I_2 &= \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right| \\
&\leq \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{A_j}(\underline{x})) - f(\underline{x})\mathbf{1}_{A_j}(\underline{x}) \right| \\
&+ \left| \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{\mathbb{R} \setminus A_j}(\underline{x})) - f(\underline{x})\mathbf{1}_{\mathbb{R} \setminus A_j}(\underline{x}) \right|.
\end{aligned}$$

Therefore, by condition $(\chi 4)$, there exist constants $M_2, M_3, \theta_0 > 0$ such that

$$\begin{aligned}
I_2 &\leq \mathcal{S}_w^j(\underline{x}) + |f(\underline{x})| \left| \frac{1}{|f(\underline{x})|} \sum_{\underline{k} \in \mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})\mathbf{1}_{\mathbb{R} \setminus A_j}(\underline{x})) - \mathbf{1}_{\mathbb{R} \setminus A_j}(\underline{x}) \right| \\
&\leq M_2 w^{-\theta_0} + |f(\underline{x})| \mathcal{T}_w^j(x) \\
&\leq M_2 w^{-\theta_0} + |f(\underline{x})| M_3 w^{-\theta_0} \\
&\leq M_2 w^{-\theta_0} + M_3 \, \|f\|_\infty \, w^{-\theta_0},
\end{aligned}$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$, for sufficiently large $w > 0$. This completes the proof. □

It is important to underline that the estimate presented in Theorem 4.1.1 is valid only when the condition $(L2)$ holds with β_0 being greater than or equal to one. However, there exists kernels for which the discrete absolute moments of order $\beta_0 \geq 1$ are not finite, but at the same time, condition $(L2)$ is satisfied for some values $0 < \beta_0 < 1$. In such case, Theorem 4.1.1 cannot be applied. For this reason, we prove the following.

Theorem 4.1.2. *Let $f \in C(\mathbb{R}^n)$ and let L be a function satisfying condition $(L2)$ with $0 < \beta_0 < 1$. Then, we have*

$$\begin{aligned}\|K_w f - f\|_\infty \leq M_4\, \psi\left(\omega\left(f, w^{-\beta_0}\right)\right) + 2^{\beta_0+1}\, \psi(\|f\|_\infty)\, w^{-\beta_0}\, m_{\beta_0,\Pi^n}(L) \\ + M_2 w^{-\theta_0} + M_3 \|f\|_\infty\, w^{-\theta_0},\end{aligned}$$

for sufficiently large $w > 0$, where $M_4 := m_{0,\Pi^n}(L) + m_{\beta_0,\Pi^n}(L) + n^{\beta_0/2}\Delta^{\beta_0} m_{0,\Pi^n}(L)$, $m_{0,\Pi^n}(L) < +\infty$ and $M_2, M_3, \theta_0 > 0$ are the constants of condition $(\chi 4)$.

Proof. Let $\underline{x} \in \mathbb{R}^n$ be fixed. Proceeding as in the proof of Theorem 4.1.1, we can write

$$\begin{aligned}|(K_w f)(\underline{x}) - f(\underline{x})| &\leq \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x}))\right| \\ &+ \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x})\right| \\ &= I_1 + I_2.\end{aligned}$$

It is clear that $I_2 \leq M_2 w^{-\theta_0} + M_3 \|f\|_\infty\, w^{-\theta_0}$, where $M_2, M_3, \theta_0 > 0$ are the constants of condition $(\chi 4)$. On the other hand, we split the series in I_1 as follows

$$\begin{aligned}I_1 &\leq \sum_{\underline{k}\in\mathbb{Z}^n} \left|\chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x}))\right| \\ &\leq \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| d\underline{u}\right) \\ &\leq \sum_{\left\|w\underline{x} - t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}})\, \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| d\underline{u}\right)\end{aligned}$$

$$+ \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > w/2} L(w\underline{x} - t_{\underline{k}})\, \psi \left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - f(\underline{x})| d\underline{u} \right)$$
$$=: I_{1,1} + I_{1,2}.$$

Before estimating $I_{1,1}$, we observe that, for every $\underline{u} \in R^w_{\underline{k}}$, and if $\left\|w\underline{x} - t_{\underline{k}}\right\|_2 \leq w/2$ we have

$$\|\underline{u} - \underline{x}\|_2 \leq \left\| \underline{u} - \frac{t_{\underline{k}}}{w} \right\|_2 + \left\| \frac{t_{\underline{k}}}{w} - \underline{x} \right\|_2 \leq \sqrt{n}\, \frac{\Delta}{w} + \frac{1}{2} \leq 1,$$

for $w > 0$ sufficiently large, and moreover, since $0 < \beta_0 < 1$, it is also easy to see that

$$\omega\left(f, \|\underline{u} - \underline{x}\|_2\right) \leq \omega\left(f, \|\underline{u} - \underline{x}\|_2^{\beta_0}\right).$$

Hence, by using the property for which $\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta)$, with $\lambda = (w\, \|\underline{u} - \underline{x}\|_2)^{\beta_0}$ and $\delta = w^{-\beta_0}$, we get

$$I_{1,1} \leq \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}})\, \psi \left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \omega\left(f, \|\underline{u} - \underline{x}\|_2^{\beta_0}\right) d\underline{u} \right)$$
$$\leq \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}})\, \psi \left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} [w^{\beta_0}\, \|\underline{u} - \underline{x}\|_2^{\beta_0} + 1]\, \omega\left(f, w^{-\beta_0}\right) d\underline{u} \right)$$
$$= \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}})\, \psi \left(\left[1 + \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} w^{\beta_0}\, \|\underline{u} - \underline{x}\|_2^{\beta_0}\, d\underline{u} \right] \omega\left(f, w^{-\beta_0}\right) \right).$$

Since ψ is concave and then subadditive, by (4.2) we have

$$I_{1,1} \leq \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}}) \left[1 + \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} w^{\beta_0}\, \|\underline{u} - \underline{x}\|_2^{\beta_0}\, d\underline{u} \right] \psi\left(\omega\left(f, w^{-\beta_0}\right)\right)$$
$$\leq \psi\left(\omega\left(f, w^{-\beta_0}\right)\right) \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} w^{\beta_0}\, \|\underline{u} - \underline{x}\|_2^{\beta_0}\, d\underline{u}$$
$$+ m_{0,\Pi^n}(L)\, \psi\left(\omega\left(f, w^{-\beta_0}\right)\right),$$

for $w > 0$ sufficiently large. By using (4.1) and by exploiting the subadditivity of the function $\|\cdot\|_2^{\beta_0}$, with $0 < \beta_0 < 1$, we can write

$$I_{1,1} \leq \psi\left(\omega\left(f, w^{-\beta_0}\right)\right)\left[m_{0,\Pi^n}(L) + \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x}-t_{\underline{k}})\left(\left\|w\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0} + n^{\beta_0/2}\Delta^{\beta_0}\right)\right]$$

$$\leq \psi\left(\omega\left(f, w^{-\beta_0}\right)\right)\left[m_{0,\Pi^n}(L) + \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x}-t_{\underline{k}})\left\|w\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0}\right.$$

$$\left. + \, n^{\beta_0/2}\Delta^{\beta_0} \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 \leq w/2} L(w\underline{x}-t_{\underline{k}})\right]$$

$$\leq \psi\left(\omega\left(f, w^{-\beta_0}\right)\right)\left[m_{0,\Pi^n}(L) + m_{\beta_0,\Pi^n}(L) + \, n^{\beta_0/2}\Delta^{\beta_0} m_{0,\Pi^n}(L)\right].$$

Finally, for what concerns $I_{1,2}$ we have

$$I_2 \leq \psi(2\|f\|_\infty) \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > w/2} L(w\underline{x}-t_{\underline{k}})$$

$$\leq \psi(2\|f\|_\infty) \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > w/2} \frac{\left\|w\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0}}{\left\|w\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0}} L(w\underline{x}-t_{\underline{k}})$$

$$\leq \left(\frac{2}{w}\right)^{\beta_0} \psi(2\|f\|_\infty) \sum_{\left\|w\underline{x}-t_{\underline{k}}\right\|_2 > w/2} \left\|w\underline{x}-t_{\underline{k}}\right\|_2^{\beta_0} L(w\underline{x}-t_{\underline{k}})$$

$$\leq 2^{\beta_0+1}\, \psi(\|f\|_\infty)\, w^{-\beta_0}\, m_{\beta_0,\Pi^n}(L).$$

Thus, the theorem is proved. □

If we consider the multivariate Fejér kernel, defined as in Section 3.5, condition $(L2)$ is satisfied only for every $\beta_0 < 1$ (then $m_{\beta_0,\Pi^n}(L) = +\infty$, for $\beta_0 \geq 1$); therefore, Theorem 4.1.1 cannot be applied, while Theorem 4.1.2 holds.

Remark 4.1.3. In general, it is possible to give a condition on the kernels which ensures that $(L2)$ holds for $0 \leq \beta_0 < \nu$, for some $\nu < 1$, and $m_{\beta_0,\Pi^n}(L) = +\infty$, for $\nu < \beta_0 \leq 1$. In this regard, we refer the readers to [42].

Example 4.1.4. If we consider, for instance, the family $(g_w)_{w>0}$ defined in Example 3.5.1, the (L,ψ)-Lipschitz condition turns out to be satisfied with a function ψ that is only piecewise concave and not globally concave; hence the estimates of Theorem 4.1.1 and Theorem 4.1.2 can not be applied. On the contrary, an example of family $(g_w)_{w>0}$, for which Theorem 4.1.1 and Theorem 4.1.2 hold, is defined in [41] as follows

$$g_w(u) = \begin{cases} u^{1+1/w}, & \text{if } 0 < u < 1, \\ u, & \text{otherwise,} \end{cases}$$

for $w > 0$ (see, Figure 4.1). It is easy to see that $g_w(u) \to u$ uniformly on $\mathbb{R}$, as $w \to +\infty$. In fact, the function $u - g_w(u)$ on $(0,1)$ achieves its maximum at $u_0 := \left(\frac{w}{w+1}\right)^w$ for sufficiently large $w > 0$ ($u - g_w(u) = 0$ otherwise), then for every $u \in \mathbb{R}$ we have

$$|g_w(u) - u| \leq u_0 - g_w(u_0) = \left(\frac{w}{w+1}\right)^w \left(\frac{1}{w+1}\right) \leq \frac{1}{w+1},$$

for sufficiently large $w > 0$. Note that, we are currently in the scenario described in Remark 3.1.3, and if the function L satisfies condition (3.1), assumption ($\mathcal{L}3$) holds for $\theta_0 = 1$. In fact,

$$\mathcal{S}_w^j(\underline{x}) = \sup_{0 \leq |u| < \frac{1}{j}} |g_w(u) - u| \leq \frac{1}{w-1} = \mathcal{O}(w^{-1}),$$

and

$$\mathcal{T}_w^j(\underline{x}) = \sup_{\frac{1}{j} \leq |u|} \left| \frac{g_w(u)}{u} - 1 \right| = \sup_{\frac{1}{j} \leq |u|} \frac{1}{|u|} \cdot |g_w(u) - u| \leq j \cdot \frac{1}{w-1} = \mathcal{O}\left(w^{-1}\right),$$

as $w \to +\infty$. Moreover, if we consider, for instance, the function $\psi : \mathbb{R}_0^+ \to \mathbb{R}$, such that $\psi(u) := \frac{3}{2}u$, the functions $g_w(u)$, $w > 0$ satisfy (3.2) for sufficiently large $w > 0$. In details, if we consider $|u|, |v| \geq 1$, we obtain

$$|g_w(u) - g_w(v)| = |u - v| \leq \psi(|u - v|);$$

if we consider $u, v \in (0,1)$, using the Langrange theorem, we get

$$|g_w(u) - g_w(v)| = |u^{1+1/w} - v^{1+1/w}| \leq \frac{3}{2}|u - v| = \psi(|u - v|),$$

for $w \geq 2$; finally, for $u \in (0,1)$ and $|v| \geq 1$ (or conversely), using again the Lagrange theorem, we obtain

$$\begin{aligned} |g_w(u) - g_w(v)| &= v - u^{1+1/w} = (v - 1) + (1 - u^{1+1/w}) \\ &\leq (v - 1) + \frac{3}{2}(1 - u) \leq \frac{3}{2}(v - 1 + 1 - u) = \psi(|v - u|), \end{aligned}$$

if $v \geq 1$, while if $v \leq -1$

$$\begin{aligned} |g_w(u) - g_w(v)| &= u^{1+1/w} - v \leq \frac{3}{2}u - v = \frac{3}{2}u - v - \frac{1}{2}v + \frac{1}{2}v \\ &= \frac{3}{2}u - \frac{3}{2}v + \frac{1}{2}v \leq \frac{3}{2}u - \frac{3}{2}v = \psi(|u - v|). \end{aligned}$$

Note that, since the above function ψ is linear, hence it is concave on $\mathbb{R}_0^+$, and in this case, we have that the functions g_w, $w > 0$, satisfy a strongly-Lipschitz condition.

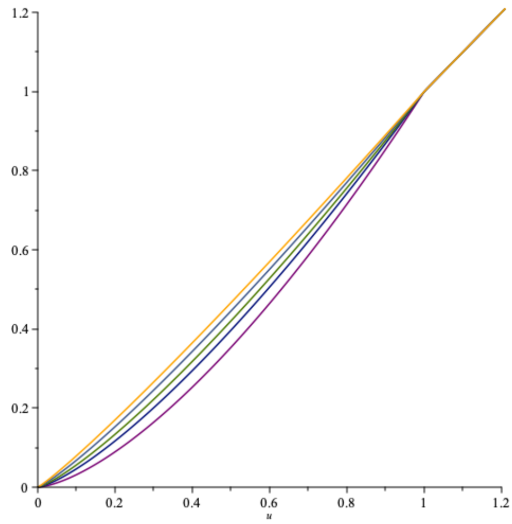

Figure 4.1: Graph of $g_w(u)$ for different values of w.

4.2 Quantitative estimates in Orlicz spaces

For any fixed $f \in L^{\varphi}(\mathbb{R}^n)$, we define the modulus of smoothness in Orlicz spaces $L^{\varphi}(\mathbb{R}^n)$, with respect to the modular I^{φ}, as follows

$$\omega_{\varphi}(f, \delta) := \sup_{\|\underline{t}\|_2 \leq \delta} I^{\varphi}\left[f(\cdot + \underline{t}) - f(\cdot)\right] = \sup_{\|\underline{t}\|_2 \leq \delta} \int_{\mathbb{R}^n} \varphi(|f(\underline{s} + \underline{t}) - f(\underline{s})|) d\underline{s},$$

with $\delta > 0$. It is well-known that, by Theorem 1.2.6, for every $f \in L^{\varphi}(\mathbb{R}^n)$ there exists $\lambda > 0$ such that $\omega_{\varphi}(\lambda f, \delta) \to 0$, as $\delta \to 0^+$.

Hence, denoting by τ the characteristic function of the set $[0, 1]^n$, i.e., $\tau(\underline{u}) = 1$, if $\underline{u} \in [0, 1]^n$, and $\tau(\underline{0}) = 0$ otherwise, we can state the main result of this section.

Theorem 4.2.1. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H_{φ}) with η convex, $f \in L^{\varphi+\eta}(\mathbb{R}^n)$ and also for any fixed $0 < \alpha < 1$, we have*

$$w^n \int_{\|\underline{y}\|_2 > \frac{1}{w^{\alpha}}} L(w\underline{y}) d\underline{y} \leq M_5 w^{-\alpha_0}, \tag{4.3}$$

as $w \to +\infty$, for suitable positive constants M_5, α_0 depending on α and L. Then, there exist $\mu > 0$, $\lambda > 0$ and further parameters $K, \lambda_0 > 0$ such that

$$I^{\varphi}[\mu(K_w f - f)] \leq \frac{\|L\|_1\, m_{0,\Pi^n}(\tau)}{3\delta^n m_{0,\Pi^n}(L)} \omega_\eta \left(\lambda f, \frac{1}{w^\alpha}\right) + \frac{M_5 m_{0,\Pi^n}(\tau) I^\eta[\lambda_0 f]}{3\delta^n m_{0,\Pi^n}(L)} w^{-\alpha_0}$$
$$+ \frac{\Delta^n}{3\delta^n}\omega_\eta\left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right) + \frac{K}{3} w^{-\theta_0} + \frac{I^\varphi[\lambda_0 f]}{3} w^{-\theta_0},$$

for every sufficiently large $w > 0$, where $m_{0,\Pi^n}(L) < +\infty$ by (i) of Lemma 3.1.4, $m_{0,\Pi^n}(\tau) < +\infty$, since τ is bounded and with compact support, and $\theta_0 > 0$ is the constant of condition ($\chi 4$). In particular, if $\mu > 0$ and $\lambda > 0$ are sufficiently small, the above inequality implies the modular convergence of nonlinear multivariate sampling Kantorovich operators $K_w f$ to f.

Proof. Let $\lambda_0, K > 0$ such that $I^\varphi[\lambda_0 f] < +\infty$ and $\int_A \varphi(\lambda_0) d\underline{x} < K$ for every measurable set A of finite measure, by the absolutely finiteness of I^φ. Further, we also fix $\lambda > 0$ such that

$$\lambda < \min\left\{1, \frac{\lambda_0}{2}\right\}.$$

In correspondence to λ, by condition (H_φ), we know that there exists $C_\lambda \in (0,1)$ such that $\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u)$, $u \in \mathbb{R}_0^+$, while by ($\chi 4$), there exist constants θ_0, $M_2, M_3 > 0$ such that

$$\mathcal{S}_w^j(\underline{x}) \leq M_2 w^{-\theta_0}, \quad \mathcal{T}_w(\underline{x}) \leq M_3 w^{-\theta_0},$$

uniformly with respect to $\underline{x} \in \mathbb{R}^n$, for sufficiently large $w > 0$. Now, we choose $\mu > 0$ such that

$$\mu \leq \min\left\{\frac{C_\lambda}{3m_{0,\Pi^n}(L)}, \frac{\lambda_0}{3M_2}, \frac{\lambda_0}{3M_3}\right\}.$$

Taking into account that φ is convex and non-decreasing, for $\mu > 0$, we can write

$$I^\varphi\left[\mu\left(K_w f - f\right)\right] = \int_{\mathbb{R}^n} \varphi\left(\mu \left|(K_w f)(\underline{x}) - f(\underline{x})\right|\right) d\underline{x}$$
$$\leq \frac{1}{3}\Bigg\{\int_{\mathbb{R}^n} \varphi\left(3\mu\left|(K_w f)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\right|\right) d\underline{x}$$
$$+ \int_{\mathbb{R}^n} \varphi\left(3\mu\left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R_{\underline{k}}^w} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right)\right|\right) d\underline{x}$$
$$+ \int_{\mathbb{R}^n} \varphi\left(3\mu\left|\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x})\right|\right) d\underline{x}\Bigg\} =: I_1 + I_2 + I_3,$$

where $\frac{t_{\underline{k}}}{w} = \left(\frac{t_{k_1}}{w}, \frac{t_{k_2}}{w}, \cdots, \frac{t_{k_n}}{w}\right)$.
Now, we estimate I_1. Applying condition $(\chi 3)$, we have

$$
\begin{aligned}
3I_1 &= \int_{\mathbb{R}^n} \varphi \left(3\mu \left|(K_w f)(\underline{x}) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\right|\right) d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \varphi \Bigg(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} \Bigg|\chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})\, d\underline{u}\right) \\
&\qquad\qquad - \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right)\Bigg|\Bigg) d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \varphi\left(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \psi\left(\left|\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right|\right)\right) d\underline{x}.
\end{aligned}
$$

Using Jensen inequality twice, the change of variable $\underline{y} = \underline{x} - \frac{t_{\underline{k}}}{w}$, condition ($\mathrm{H}_\varphi$) and Fubini-Tonelli theorem, we obtain

$$
\begin{aligned}
3I_1 &\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \cdot \\
&\qquad \cdot \varphi\left(3\mu m_{0,\Pi^n}(L) \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left|f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \cdot \\
&\qquad \cdot \varphi\left(3\mu m_{0,\Pi^n}(L) \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left|f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \cdot \\
&\qquad \cdot \varphi\left(C_\lambda \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left|f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right)\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}})\, \eta\left(\lambda \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left|f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u}\right) d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \eta\left(\lambda \left|f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|\right) d\underline{u}\, d\underline{x}
\end{aligned}
$$

$$= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \cdot$$

$$\cdot \frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \eta\left(\lambda\left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|\right) \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} d\underline{x}$$

$$= \frac{1}{m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{y}) \frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \eta\left(\lambda\left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} d\underline{y}$$

$$\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n L(w\underline{y}) \int_{\mathbb{R}^n} \eta\left(\lambda\left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) \sum_{\underline{k}\in\mathbb{Z}^n} \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} d\underline{y}$$

$$\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \int_{\mathbb{R}^n} \eta\left(\lambda\left| f\left(\underline{u}\right) - f\left(\underline{u} + \underline{y}\right)\right|\right) d\underline{u} d\underline{y}$$

$$= \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y})\ I^\eta\left[\lambda\left(f(\cdot) - f(\cdot + \underline{y})\right)\right] d\underline{y},$$

where the constant $m_{0,\Pi^n}(\tau) < +\infty$, since τ is bounded and with compact support (see, e.g., [49]). Now, let $0 < \alpha < 1$ be fixed. We now split the above integral as follows

$$\frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)} m_{0,\Pi^n}(\tau) \left\{ \int_{\|\underline{y}\|_2 \leq \frac{1}{w^\alpha}} + \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} \right\} L(w\underline{y})\ I^\eta\left[\lambda\left(f(\cdot) - f(\cdot + \underline{y})\right)\right] d\underline{y}$$
$$=: I_{1,1} + I_{1,2}.$$

For $I_{1,1}$, one has

$$I_{1,1} \leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) \int_{\|\underline{y}\|_2 \leq \frac{1}{w^\alpha}} L(w\underline{y}) \omega_\eta\left(\lambda f, \|\underline{y}\|_2\right) d\underline{y}$$
$$\leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau)\, \omega_\eta\left(\lambda f, \frac{1}{w^\alpha}\right) \int_{\|\underline{y}\|_2 \leq \frac{1}{w^\alpha}} L(w\underline{y}) d\underline{y}$$
$$\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau)\, \omega_\eta\left(\lambda f, \frac{1}{w^\alpha}\right) \|L\|_1 .$$

On the other hand, taking into account that η is convex, for $I_{1,2}$ we can write

$$I_{1,2} \leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y}) \frac{1}{2} \left(I^\eta[2\lambda f(\cdot)] + I^\eta[2\lambda f(\cdot + \underline{y})]\right) d\underline{y}.$$

Now, observing that

$$I^\eta[2\lambda f(\cdot)] = I^\eta[2\lambda f(\cdot + \underline{y})],$$

for every $\underline{y}$, using (4.3), we finally get

$$\begin{aligned} I_{1,2} &\leq \frac{w^n \delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau) \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y})\; I^\eta[2\lambda f] d\underline{y} \\ &\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)}\, m_{0,\Pi^n}(\tau)\; I^\eta[\lambda_0 f]\; M_5 w^{-\alpha_0}, \end{aligned}$$

for $w > 0$ sufficiently large and for $M_5 > 0$. Now we can proceed estimating I_2. Using the assumption $(\chi 3)$ we immediately have

$$\begin{aligned} 3I_2 &= \int_{\mathbb{R}^n} \varphi\left(3\mu \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) \right| \right) d\underline{x} \\ &\leq \int_{\mathbb{R}^n} \varphi\left(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\; \psi\left(\left| \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right) d\underline{u} - f(\underline{x}) \right|\right)\right) d\underline{x}. \end{aligned}$$

Now, by the change of variable $\underline{y} = \underline{u} - \frac{t_{\underline{k}}}{w}$, we have

$$3I_2 \leq \int_{\mathbb{R}^n} \varphi\left(3\mu \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\; \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right)\right) d\underline{x},$$

where the symbol $\widetilde{R}^w_{\underline{k}} := \left[0, \frac{\Delta_{k_1}}{w}\right] \times \cdots \left[0, \frac{\Delta_{k_n}}{w}\right]$ for every $\underline{k} \in \mathbb{Z}^n$ and $w > 0$. Hence, applying Jensen inequality twice as above, recalling that $3\mu m_{0,\Pi^n}(L) \leq C_\lambda$ and condition (H_φ), we get

$$\begin{aligned} 3I_2 &\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\; \varphi\left(3\mu m_{0,\Pi^n}(L)\; \psi\left(\frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right)\right) d\underline{x} \\ &\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}})\; \eta\left(\lambda \frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}^w_{\underline{k}}} \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right| d\underline{y}\right) d\underline{x} \\ &\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}^w_{\underline{k}}} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y} d\underline{x} \\ &\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \int_{\Delta_w} \eta\left(\lambda \left|f\left(\underline{x} + \underline{y}\right) - f(\underline{x})\right|\right) d\underline{y} d\underline{x}, \end{aligned}$$

where $\Delta_w := \left[0, \frac{\Delta}{w}\right]^n$. Then, by the Fubini-Tonelli theorem, we get

$$\begin{aligned}
3I_2 &\leq \frac{\delta^{-n}}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} w^n m_{0,\Pi^n}(L) \int_{\Delta_w} \eta\left(\lambda \left| f\left(\underline{x}+\underline{y}\right) - f(\underline{x})\right|\right) d\underline{y} d\underline{x} \\
&= \delta^{-n} w^n \int_{\Delta_w} \int_{\mathbb{R}^n} \eta\left(\lambda \left| f\left(\underline{x}+\underline{y}\right) - f(\underline{x})\right|\right) d\underline{x} d\underline{y} \\
&= \delta^{-n} w^n \int_{\Delta_w} I^\eta \left[\lambda\left(f\left(\cdot + \underline{y}\right) - f(\cdot)\right)\right] d\underline{y} \\
&\leq \delta^{-n} w^n \omega_\eta \left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right) \int_{\Delta_w} d\underline{y} \\
&= \delta^{-n} \Delta^n \omega_\eta \left(\lambda f, \sqrt{n}\frac{\Delta}{w}\right).
\end{aligned}$$

For I_3, denoted by $A_j \subseteq \mathbb{R}^n$ the set of all points of $\mathbb{R}^n$ for which $0 \leq |f(\underline{x})| < 1/j$, with $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
3I_3 &= \int_{A_j} \varphi\left(3\mu \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x}) \right|\right) d\underline{x} \\
&+ \int_{\mathbb{R}^n \setminus A_j} \varphi\left(3\mu \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - f(\underline{x}) \right|\right) d\underline{x} \\
&\leq \int_{A_j} \varphi(3\mu \mathcal{S}_w^j(\underline{x})) d\underline{x} + \int_{\mathbb{R}^n \setminus A_j} \varphi\left(3\mu |f(\underline{x})| \left| \frac{1}{f(\underline{x})} \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x})\right) - 1 \right|\right) d\underline{x} \\
&\leq \int_{A_j} \varphi(3\mu \mathcal{S}_w^j(\underline{x})) d\underline{x} + \int_{\mathbb{R}^n \setminus A_j} \varphi\left(3\mu |f(\underline{x})| \mathcal{T}_w^j(x)\right) d\underline{x}.
\end{aligned}$$

By the convexity of φ and condition $(\chi 4)$, we have

$$\begin{aligned}
3I_3 &\leq \int_{A_j} \varphi\left(3\mu M_2 w^{-\theta_0}\right) d\underline{x} + \int_{\mathbb{R}^n \setminus A_j} \varphi\left(3\mu M_3 w^{-\theta_0} |f(\underline{x})|\right) d\underline{x} \\
&\leq w^{-\theta_0} \int_{A_j} \varphi\left(3\mu M_2\right) d\underline{x} + w^{-\theta_0} \int_{\mathbb{R}^n} \varphi\left(3\mu M_3 |f(\underline{x})|\right) d\underline{x} \\
&\leq w^{-\theta_0} \int_{A_j} \varphi\left(\lambda_0\right) d\underline{x} + w^{-\theta_0} \int_{\mathbb{R}^n} \varphi\left(\lambda_0 |f(\underline{x})|\right) d\underline{x} \\
&\leq K w^{-\theta_0} + w^{-\theta_0} I^\varphi\left[\lambda_0 f\right],
\end{aligned}$$

for positive constants M_2, M_3 and θ_0. This completes the proof. □

Remark 4.2.2. Note that, condition (4.3) is obviously fullfilled when the kernel χ satisfies condition ($\chi 3$) with L having compact support. Indeed, if $supp\, L \subset B(\underline{0}, R) \subset \mathbb{R}^n$, $R > 0$, we have

$$w^n \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y}) d\underline{y} = \int_{\|\underline{u}\|_2 > w^{1-\alpha}} L(\underline{u}) d\underline{u} = 0,$$

for every $w > R^{1/(1-\alpha)}$. The above consideration implies that the term $I_{1,2}$ in the proof of Theorem 4.2.1 is null, for sufficiently large $w > 0$. Moreover, in this case, we also have that condition ($L2$) is satisfied for every $\beta_0 > 0$.

Corollary 4.2.3. *Let χ be a kernel satisfying condition ($\chi 3$) with L having compact support. Let φ be a convex φ-function satisfying condition (H_φ) with η convex and $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0 < \alpha < 1$, there exist constants $\mu > 0$, $\lambda > 0$ and further parameters $\lambda_0, K > 0$ such that*

$$\begin{aligned} I^\varphi[\mu(K_w f - f)] \leq &\frac{\|L\|_1 \, m_{0,\Pi^n}(\tau)}{3\delta^n m_{0,\Pi^n}(L)} \omega_\eta \left(\lambda f, \frac{1}{w^\alpha} \right) + \frac{\Delta^n}{3\delta^n} \omega_\eta \left(\lambda f, \sqrt{n} \frac{\Delta}{w} \right) \\ &+ \frac{K}{3} w^{-\theta_0} + \frac{I^\varphi[\lambda_0 f]}{3} w^{-\theta_0}, \end{aligned}$$

for sufficiently large $w > 0$, where $m_{0,\Pi^n}(L) < +\infty$, $m_{0,\Pi^n}(\tau) < +\infty$, and $\theta_0 > 0$ is the constant of condition ($\chi 4$).

Remark 4.2.4. Note that, if L has not compact support, we may require the following condition

$$M^\nu(L) := \int_{\mathbb{R}^n} L(\underline{u}) \, \|\underline{u}\|_2^\nu \, d\underline{u} < +\infty, \tag{4.4}$$

for $\nu > 0$, which results a sufficient condition for (4.3). Indeed, for every $0 < \alpha < 1$, we can write what follows

$$\begin{aligned} w^n \int_{\|\underline{y}\|_2 > \frac{1}{w^\alpha}} L(w\underline{y}) d\underline{y} = \int_{\|\underline{u}\|_2 > w^{1-\alpha}} L(\underline{u}) d\underline{u} \leq \frac{1}{w^{\nu(1-\alpha)}} \int_{\|\underline{u}\|_2 > w^{1-\alpha}} \|\underline{u}\|_2^\nu \, L(\underline{u}) d\underline{u} \\ \leq \frac{M^\nu(L)}{w^{\nu(1-\alpha)}} = \mathcal{O}(w^{\nu(\alpha-1)}), \end{aligned}$$

as $w \to +\infty$. Hence, (4.3) is satisfied with $\alpha_0 = (1-\alpha)\nu$ and $M_5 = M^\nu(L)$.

Remark 4.2.5. Quantitative estimates for the multivariate sampling Kantorovich operators in the linear case have been considered in details in [8]. For more references concerning linear operators, see, e.g., [5, 13, 14, 60, 72, 76].

4.2.1 Application to special kernels

In this section, we give the following corollaries, as particular results of the previous ones, for some special kernels. For a more extensive exposition of kernel functions, we refer to Section 3.5.

As shown in Remark 4.2.4, it is easy to see that the multivariate Féjer kernel $\mathcal{F}_n$ satisfies (4.4) for every $0 < \nu < 1$. Hence, for the corresponding nonlinear operators $K_w^{\mathcal{F}_n}$, from Theorem 4.2.1 we can state the following.

Corollary 4.2.6. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H_φ) with η convex, $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0 < \nu < 1$, $0 < \alpha < 1$, there exist constants $\mu > 0$, $\lambda > 0$ and further parameters $\lambda_0, K > 0$, such that*

$$I^\varphi[\mu(K_w^{\mathcal{F}_n} f - f)]$$
$$\leq \frac{1}{3}\left\{\omega_\eta\left(\lambda f, \frac{1}{w^\alpha}\right) + M_5 I^\eta[\lambda_0 f] w^{-\alpha_0} + \omega_\eta\left(\lambda f, \frac{\sqrt{n}}{w}\right) + K w^{-\theta_0} + I^\varphi[\lambda_0 f] w^{-\theta_0}\right\},$$

for sufficiently large $w > 0$, $\alpha_0 = (1-\alpha)\nu$, $M_5 > 0$ and $\theta_0 > 0$ is the constant of condition ($\mathcal{L}3$).

For the nonlinear sampling Kantorovich operators $K_w^{\mathcal{J}_n}$ based on the multivariate Jackson kernel, we can obtain an analogous result to that one achieved for $K_w^{\mathcal{F}_n}$.

Both the Féjer kernel and the Jackson kernel have unbounded support. Thus, to reconstruct a given signal of f by means of $K_w^{\mathcal{F}_n}$ or $K_w^{\mathcal{J}_n}$, we need to compute an infinite number of mean values $w^n \int_{R_{\underline{k}}^w} f(\underline{u})d\underline{u}$ in order to evaluate the above operators at any fixed $\underline{x} \in \mathbb{R}^n$. Therefore, for a practical application of the above sampling series with L having unbounded support, the sampling series must be truncated and this leads to truncation errors which worsen the quality of reconstruction.

However, considering kernels with L having compact support, the truncation error can be avoided. In this case, the infinite sampling series computed at any fixed $\underline{x} \in \mathbb{R}$ reduce to a finite one. Important examples of such kernels can be generated by using the well-known B-splines. For the corresponding operators $K_w^{\mathcal{M}_s^n}$, from Corollary 4.2.3 we obtain the following corollary.

Corollary 4.2.7. *Let φ be a convex φ-function. Suppose that φ satisfies condition (H_φ) with η convex, $f \in L^{\varphi+\eta}(\mathbb{R}^n)$. Then, for every $0 < \alpha < 1$, there exist constants $\mu > 0$, $\lambda > 0$ and further parameters $\lambda_0, K > 0$, such that*

$$I^\varphi[\mu(K_w^{\mathcal{M}_s^n} f - f)] \leq \frac{1}{3}\left\{\omega_\eta\left(\lambda f, \frac{1}{w^\alpha}\right) + \omega_\eta\left(\lambda f, \frac{\sqrt{n}}{w}\right) + K w^{-\theta_0} + I^\varphi[\lambda_0 f] w^{-\theta_0}\right\},$$

for sufficiently large $w > 0$, where $\theta_0 > 0$ is the constant of condition ($\mathcal{L}3$).

4.3 Quantitative estimates in Lebesgue spaces

Now, we consider some particular cases of Orlicz spaces. Let $\varphi(u) = u^p$, $u \in \mathbb{R}_0^+$, $1 \leq p < +\infty$, the Orlicz space $L^\varphi(\mathbb{R}^n)$ coincides with the space $L^p(\mathbb{R}^n)$. If $\psi(u) = u^{q/p}$, $1 \leq q \leq p$, condition (H_φ) turns out to be satisfied with $\eta(u) = u^q$ and $C_\lambda = \lambda^{p/q}$. In such case, we have $L^{\varphi+\eta}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, which is a proper subspace of $L^p(\mathbb{R}^n)$, and obviously Theorem 4.2.1 and its corollary hold.
From the theory developed in [78], we know that if the function ψ of condition (χ3) is of the form $\psi(u) = u$, $u \in \mathbb{R}$, the operators K_w map the whole space $L^p(\mathbb{R}^n)$ into itself, i.e., K_w are well-defined in $L^p(\mathbb{R}^n)$, and therefore, we can obtain, as particular case, a quantitative estimate in $L^p(\mathbb{R}^n)$.

But thanks to the well-known properties of the first order modulus of smoothness in L^p (see Example 1.2.9), we can establish a direct quantitative estimate, which turns out to be sharper than that one established in the general case considered in Theorem 4.2.1.
In order to obtain the above mentioned result for the nonlinear multivariate sampling Kantorovich operators, we recall, for $f \in L^p(\mathbb{R}^n)$, the definition of the L^p-first order modulus of smoothness of f, given by

$$\omega_p(f, \delta) := \sup_{\|\underline{t}\|_2 \leq \delta} \|f(\cdot + \underline{t}) - f(\cdot)\|_p = \sup_{\|\underline{t}\|_2 \leq \delta} \left(\int_{\mathbb{R}^n} |f(\underline{s} + \underline{t}) - f(\underline{s})|^p d\underline{s} \right)^{1/p},$$

with $\delta > 0$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$.
Therefore, we can prove the following estimate.

Theorem 4.3.1. *Suppose that (χ3) is satisfied with $\psi(u) = u$, $u \in \mathbb{R}$, and*

$$M^p(L) := \int_{\mathbb{R}^n} L(\underline{u}) \, \|\underline{u}\|_2^p \, d\underline{u} < +\infty, \tag{4.5}$$

for some $1 \leq p < +\infty$. Then, for every $f \in L^p(\mathbb{R}^n)$, the following quantitative estimate holds

$$\|K_w f - f\|_p \leq T \omega_p \left(f, \frac{1}{w} \right) + +M_2 w^{-\theta_0} + M_3 \|f\|_p w^{-\theta_0},$$

where

$$T := \delta^{-\frac{n}{p}} \left(m_{0,\Pi^n}(L) \right)^{\frac{p-1}{p}} \cdot \\ \cdot \left\{ 2^{\frac{p-1}{p}} m_{0,\Pi^n}(\tau)^{1/p} \left[\|L\|_1 + M^p(L) \right]^{\frac{1}{p}} + \left(m_{0,\Pi^n}(L) \right)^{\frac{1}{p}} \Delta^{\frac{n}{p}} (1 + \sqrt{n}\Delta) \right\},$$

for sufficiently large $w > 0$, where $m_{0,\Pi^n}(L) < +\infty$ by (i) of Lemma 3.1.4, $m_{0,\Pi^n}(\tau) < +\infty$, τ being the characteristic function of $[0,1]^n$, and M_2, M_3, $\theta_0 > 0$ are the constants of condition ($\chi 4$).

Proof. Recalling that $I^{\varphi}[f] = \|f\|_p^p$, when $\varphi(u) = u^p$, using the Minkowsky inequality, the concavity (hence the subadditivity) of the function $|\cdot|^{\frac{1}{p}}$, and applying condition ($\chi 3$), we have

$$\begin{aligned}
\|K_w f - f\|_p &= \left(\int_{\mathbb{R}^n} \left| K_w f(\underline{x}) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u} \right) \right.\right.\right. \\
&\qquad \left.\left.\left. - \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) d\underline{u} \right) \right| \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) d\underline{u} \right) \right.\right.\right. \\
&\qquad \left.\left.\left. - \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x}) \right) \right| d\underline{u} \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right| d\underline{u} \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) - f(\underline{x}) \right| d\underline{u} \right]^p d\underline{x} \right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi(w\underline{x} - t_{\underline{k}}, f(\underline{x})) - f(\underline{x}) \right|^p d\underline{x} \right)^{1/p} =: I_1^p + I_2^p + I_3^p.
\end{aligned}$$

Now, proceeding as in the proof of Theorem 4.2.1, i.e., applying Jensen inequality twice and Fubini-Tonelli theorem, we obtain

$$I_1^p = \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) \right| d\underline{u} \right]^p d\underline{x}$$

$$\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} m_{0,\Pi^n}(L) \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right| d\underline{u} \right]^p d\underline{x}$$

$$\leq m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|^p d\underline{u} \right] d\underline{x}.$$

Now, applying the change of variable $\underline{y} = \underline{x} - t_{\underline{k}}/w$ and Fubini-Tonelli theorem, we get

$$I_1^p \leq m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w}\right)\right|^p \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{y}$$

$$= m_{0,\Pi^n}(L)^{p-1} \sum_{\underline{k}\in\mathbb{Z}^n} \int_{\mathbb{R}^n} L(w\underline{y}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y}\right)\right|^p \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{y}$$

$$\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left[\int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y}\right)\right|^p \sum_{\underline{k}\in\mathbb{Z}^n} \tau(w\underline{u} - t_{\underline{k}}) d\underline{u} \right] d\underline{y}$$

$$= \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left[\int_{\mathbb{R}^n} \left| f(\underline{u}) - f\left(\underline{u} + \underline{y}\right)\right|^p d\underline{u} \right] d\underline{y}$$

$$\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y})\, \omega_p\left(f, \|\underline{y}\|_2\right)^p d\underline{y},$$

where the constant $m_{0,\Pi^n}(\tau) < +\infty$ since τ is bounded and with compact support. Exploiting the well-known inequality $\omega_p(f, \lambda\delta) \leq (1+\lambda)\omega_p(f,\delta)$, with $\delta, \lambda > 0$, we finally get

$$I_1^p \leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} m_{0,\Pi^n}(\tau) \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(1 + w\|\underline{y}\|_2\right)^p \omega_p\left(f, \frac{1}{w}\right)^p d\underline{y}$$

$$\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1}\, m_{0,\Pi^n}(\tau)\, 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(1 + w^p \|\underline{y}\|_2^p\right) d\underline{y}$$

$$= \delta^{-n} m_{0,\Pi^n}(L)^{p-1}\, m_{0,\Pi^n}(\tau)\, 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p \cdot$$

$$\cdot \left\{ \int_{\mathbb{R}^n} w^n L(w\underline{y}) d\underline{y} + \int_{\mathbb{R}^n} w^n L(w\underline{y}) \left(w\|\underline{y}\|_2\right)^p d\underline{y} \right\}$$

$$= \delta^{-n} m_{0,\Pi^n}(L)^{p-1}\, m_{0,\Pi^n}(\tau)\, 2^{p-1} \omega_p\left(f, \frac{1}{w}\right)^p \left\{ \|L\|_1 + M^p(L) \right\},$$

for every $w > 0$, where $\|L\|_1$ and $M^p(L)$ are both finite, in view of $(L1)$ and (4.5). Now we estimate I_2^p. Using Jensen inequality twice, the change of variable $\underline{y} =$

$\underline{u} - t_{\underline{k}}/w$ and Fubini-Tonelli theorem, we have

$$\begin{aligned}
I_2^p &= \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{R_{\underline{k}}^w} \left| f\left(\underline{u} + \underline{x} - \frac{t_{\underline{k}}}{w} \right) - f(\underline{x}) \right| d\underline{u} \right]^p d\underline{x} \\
&\leq \int_{\mathbb{R}^n} \left[\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}_{\underline{k}}^w} \left| f\left(\underline{x} + \underline{y} \right) - f(\underline{x}) \right| d\underline{y} \right]^p d\underline{x} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[\frac{w^n}{A_{\underline{k}}} \int_{\widetilde{R}_{\underline{k}}^w} m_{0,\Pi^n}(L) \left| f\left(\underline{x} + \underline{y} \right) - f(\underline{x}) \right| d\underline{y} \right]^p d\underline{x} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^{p-1} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{x} - t_{\underline{k}}) \left[w^n \int_{\Delta_w} \left| f\left(\underline{x} + \underline{y} \right) - f(\underline{x}) \right|^p d\underline{y} \right] d\underline{x} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\mathbb{R}^n} w^n \int_{\Delta_w} \left| f\left(\underline{x} + \underline{y} \right) - f(\underline{x}) \right|^p d\underline{y} d\underline{x} \\
&= \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\Delta_w} w^n \left[\int_{\mathbb{R}^n} \left| f\left(\underline{x} + \underline{y} \right) - f(\underline{x}) \right|^p d\underline{x} \right] d\underline{y} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \int_{\Delta_w} w^n \left[\omega_p \left(f, \sqrt{n} \frac{\Delta}{w} \right) \right]^p d\underline{y} \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \Delta^n \omega_p \left(f, \sqrt{n} \frac{\Delta}{w} \right)^p \\
&\leq \delta^{-n} m_{0,\Pi^n}(L)^p \Delta^n (1 + \sqrt{n}\Delta)^p \omega_p \left(f, \frac{1}{w} \right)^p,
\end{aligned}$$

where $\Delta_w := \left[0, \frac{\Delta}{w}\right]^n$.
Finally, denoted again by $A_j \subseteq \mathbb{R}^n$ the set of all points of $\mathbb{R}^n$ for which $0 \leq |f(\underline{x})| < 1/j$, with $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
I_3^p &= \int_{A_j} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x}) \right) - f(\underline{x}) \right|^p d\underline{x} \\
&\quad + \int_{\mathbb{R}^n\setminus A_j} \left| \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x}) \right) - f(\underline{x}) \right|^p d\underline{x} \\
&\leq \int_{A_j} [\mathcal{S}_w^j(\underline{x})]^p d\underline{x} + \int_{\mathbb{R}^n\setminus A_j} |f(\underline{x})|^p \left| \frac{1}{f(\underline{x})} \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\underline{x} - t_{\underline{k}}, f(\underline{x}) \right) - 1 \right|^p d\underline{x} \\
&\leq \int_{A_j} M_2^p w^{-p\theta_0} d\underline{x} + \int_{\mathbb{R}^n\setminus A_j} |f(\underline{x})|^p [\mathcal{T}_w^j(x)]^p d\underline{x}
\end{aligned}$$

$$\leq \int_{A_j} M_2^p w^{-p\theta_0} d\underline{x} + \int_{\mathbb{R}^n \setminus A_j} |f(\underline{x})|^p M_3^p w^{-p\theta_0} d\underline{x}$$
$$\leq M_2^p w^{-p\theta_0} \int_{A_j} d\underline{x} + M_3^p w^{-p\theta_0} \int_{\mathbb{R}^n} |f(\underline{x})|^p d\underline{x}$$
$$\leq M_2^p w^{-p\theta_0} + M_3^p w^{-p\theta_0} \|f\|_p^p$$

for positive constants M_2, M_3 and θ_0. This proves the theorem. □

4.4 Qualitative order of convergence in Lipschitz classes

Recalling the definition of Lipschitz classes $Lip(\nu)$, $0 < \nu \leq 1$, namely

$$Lip(\nu) := \{f \in C(\mathbb{R}^n) \ : \ \|f(\cdot + \underline{t}) - f(\cdot)\|_\infty = \mathcal{O}(\|\underline{t}\|_2^\nu), \text{ as } \|\underline{t}\|_2 \to 0\},$$

from Theorem 4.1.1, we immediately obtain the following corollary.

Corollary 4.4.1. *Let $f \in Lip(\nu)$, $0 < \nu \leq 1$, and let L be a function satisfying condition (L2) with $\beta_0 \geq 1$. In addition, we suppose that ψ of condition (χ3) satisfies the following assumption*

$$\psi(u) = \mathcal{O}(u^q), \tag{4.6}$$

as $u \to 0^+$ and for some $0 < q \leq 1$. Then, there exists a constant $C > 0$ such that

$$\|K_w f - f\|_\infty \leq C w^{-l},$$

for sufficiently large $w > 0$, with $l := \min\{\nu q, \theta_0\}$, where $\theta_0 > 0$ is the constant of condition (χ4).

Whereas, from Theorem 4.1.2 we deduce the following result.

Corollary 4.4.2. *Let $f \in Lip(\nu)$, $0 < \nu \leq 1$, L be a function satisfying condition (L2) with $0 < \beta_0 < 1$, and ψ of condition (χ3) satisfying assumption (4.6) as $u \to 0^+$ and for some $0 < q \leq 1$. Then, there exists a constant $C > 0$ such that*

$$\|K_w f - f\|_\infty \leq C w^{-l},$$

for sufficiently large $w > 0$, with $l := \min\{\nu \beta_0 q, \beta_0, \theta_0\}$, where $\theta_0 > 0$ is the constant of condition (χ4).

Now we, recall the definition of Lipschitz classes in Orlicz spaces $L^\varphi(\mathbb{R}^n)$. We define by $Lip_\varphi(\nu)$, $0 < \nu \leq 1$, as follows

$$Lip_\varphi(\nu) := \{f \in L^\varphi(\mathbb{R}^n) \ : \ \exists \lambda > 0, \ I^\varphi\left[\lambda\left(f(\cdot + \underline{t}) - f(\cdot)\right)\right] = \mathcal{O}(\|\underline{t}\|_2^\nu), \text{ as } \|\underline{t}\|_2 \to 0\}$$

From Theorem 4.2.1, we obtain the following corollary.

Corollary 4.4.3. *Under the assumptions of Theorem 4.2.1 with* $0 < \alpha < 1$ *and for any* $f \in Lip_\eta(\nu)$, $0 < \nu \leq 1$, *there exist* $C > 0$ *and* $\mu > 0$ *such that*

$$I^\varphi \left[\mu \left(K_w f - f\right)\right] \leq C w^{-l},$$

for sufficiently large $w > 0$, *with* $l := \min\{\alpha\nu, \alpha_0, \theta_0\}$, *where* $\theta_0 > 0$ *is the constant of condition* $(\chi 4)$.

As made in the general context of Orlicz spaces, from Theorem 4.3.1 we can directly deduce the qualitative order of approximation, assuming f in suitable Lipschitz spaces.
We recall that the Lipschitz class of Zygmund-type in L^p-spaces, with $0 < \nu \leq 1$, are defined as follows

$$Lip_p(\nu) := \{f \in L^p(\mathbb{R}^n) \ : \ \|f(\cdot + \underline{t}) - f(\cdot)\|_p = \mathcal{O}(\|\underline{t}\|_2^\nu), \text{ as } \|\underline{t}\|_2 \to 0\}.$$

Now, we can state the following result.

Corollary 4.4.4. *Under the assumptions of Theorem 4.3.1, for every* $f \in Lip_p(\nu)$, *with* $0 < \nu \leq 1$, $1 \leq p < +\infty$, *the following qualitative estimate holds*

$$\|K_w f - f\|_p \leq TC \frac{1}{w^\nu} + M_2 w^{-\theta_0} + M_3 \|f\|_p w^{-\theta_0},$$

for sufficiently large $w > 0$, *where* $m_{0,\Pi^n}(L) < +\infty$, T *is the constant of Theorem 4.3.1,* M_2, M_3, $\theta_0 > 0$ *are the constants of condition* $(\chi 4)$ *and* $C > 0$ *is the constant arising from the class* $Lip_p(\nu)$.

Chapter 5

Extensions to modular spaces

This chapter is based on the original results contained in [43], and herein we replace the setting of Orlicz spaces by the more general one of modular spaces ([1,26,54,61,64,70]). This extends the field of applications and enables us to give a unifying approach to several kinds of approximation problems. The theory of modular spaces contains, as we have already seen in Section 1.1, the Musielak-Orlicz and the Orlicz spaces, which are, for instance, generalizations of the weighted L^p-spaces and the classical L^p-spaces, respectively.
Since, the framework we are now interested in is very general and abstract, it requires the use of some technical conditions on the modulars taken into consideration which generate the involved spaces and the kernel function χ. However, we will show that these technical conditions are satisfied in several concrete cases.
In this context, our main result is a modular convergence theorem (Theorem 5.1.5), that has been proved via a density approach. More in detail, first we prove a modular convergence result for the nonlinear multivariate sampling Kantorovich operators acting on the space of continuous functions with compact support (Theorem 5.1.2). Then we obtain a modular-type inequality (Theorem 5.1.4) for our operators, and finally we exploit the well-known density result (Theorem 1.1.15) for the continuous function with compact support in the modular spaces.
In the last two sections of this chapter, we investigate the case of Musielak-Orlicz spaces and the spaces of functions equipped by modulars that are not of integral type.

5.1 Convergence results in modular spaces

Here, we consider $\mathbb{R}^n = (\mathbb{R}^n, \Sigma_{\mathbb{R}^n}, \mu_{\mathbb{R}^n})$ and $\mathbb{Z}^n = (\mathbb{Z}^n, \Sigma_{\mathbb{Z}^n}, \mu_{\mathbb{Z}^n})$, where $\mu_{\mathbb{R}^n}$ and $\mu_{\mathbb{Z}^n}$ are the Lebesgue and the counting measures respectively, while $\Sigma_{\mathbb{R}^n}$ and $\Sigma_{\mathbb{Z}^n}$ are their corresponding σ-algebras, i.e., the families of all measurable sets (with respect to the above measures) which are closed with respect to the operations of

countable union and complementary.
Let $\rho_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}$ be two modular functionals on $X(\mathbb{R}^n)$ and $X(\mathbb{Z}^n)$, respectively. In order to study convergence results in the general setting of modular spaces, we need to introduce a condition of compatibility between a given kernel χ and the modulars $\rho_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}$.
We will say that the kernel χ is *L-compatible* with $\rho_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}$ if there exist two constants $D_1, D_2 > 0$ and a net $(b_w)_{w>0}$ of positive numbers with $b_w \to 0$ as $w \to +\infty$, such that

$$\rho_{\mathbb{R}^n}\left(\sum_{\underline{k}\in\mathbb{Z}^n} g_{\underline{k}} L(w\cdot - t_{\underline{k}})\right) \leq \frac{1}{w^n} D_1 \rho_{\mathbb{Z}^n}(D_2 g) + b_w \tag{5.1}$$

for any non negative $g = (g_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n}$, $g \in X(\mathbb{Z}^n)$, and for sufficiently large $w > 0$, where L is the function of condition $(\chi 3)$.

Furthermore, we also need to introduce an additional assumption which relates L of condition $(\chi 3)$ with the modular $\rho_{\mathbb{R}^n}$.
We assume that for any fixed $\gamma > 0$ and $a > 0$, there exist a constant $T > 0$ and a measurable set $\mathcal{I} \subset \mathbb{R}^n$, with $\mu_{\mathbb{R}^n}(\mathcal{I}) < +\infty$ such that

$$\rho_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\cdot - t_{\underline{k}})\right) \leq T, \tag{5.2}$$

for sufficiently large $w > 0$.

Remark 5.1.1. a) The compatibility condition (5.1) has been firstly introduced in [63] in a more general form, and then, it has been recalled in [51] for the linear version of sampling Kantorovich operators.

b) If χ is a kernel satisfying condition $(\chi 3)$ with L having compact support, the assumption (5.2) is obviously satisfied. In fact, let $\gamma > 0$ and $a > 0$ be fixed, and $\operatorname{supp} L \subset B(\underline{0}, R) \subset \mathbb{R}^n$, with $R > 0$. It is easy to see that, for every $t_{\underline{k}} \in B(\underline{0}, w\gamma)$, it turns out that $L(w\underline{x} - t_{\underline{k}}) = 0$ for every $\underline{x} \notin B(\underline{0}, \gamma + R/w)$, $w > 0$.
So, taking $\mathcal{I} := B(\underline{0}, \gamma + R)$ and using condition $(\rho 1)$, we have

$$\rho_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\cdot - t_{\underline{k}})\right) = 0,$$

for every $w \geq 1$.

Now, we can prove the following modular convergence theorem for the family of the nonlinear sampling Kantorovich operators acting on functions which belong to $C_c(\mathbb{R}^n)$.

Theorem 5.1.2. *Let $\rho_{\mathbb{R}^n}$ be a convex, monotone, strongly finite and absolutely continuous modular on $X(\mathbb{R}^n)$. Moreover let χ be a kernel which satisfies assumption (5.2) together with $\rho_{\mathbb{R}^n}$. Then for every $f \in C_c(\mathbb{R}^n)$ and for every $0 < \lambda \leq \alpha/2$, we have*

$$\lim_{w \to +\infty} \rho_{\mathbb{R}^n}(\lambda(K_w f - f)) = 0,$$

where α is the parameter of the absolutely continuity of $\rho_{\mathbb{R}^n}$.

Proof. Let $f \in C_c(\mathbb{R}^n)$ and let $\overline{\gamma}$ be a positive constant such that $\operatorname{supp} f \subset B(\underline{0}, \overline{\gamma})$. Let $\gamma > \overline{\gamma} + \Delta$, we have

$$\int_{R^w_{\underline{k}}} f(\underline{u})\, d\underline{u} = 0,$$

for every $t_{\underline{k}} \notin B(\underline{0}, w\gamma)$, being $R^w_{\underline{k}} \cap B(\underline{0}, \overline{\gamma}) \neq \emptyset$ for sufficiently large $w > 0$.
Then, as made in Remark 3.3.1, the nonlinear multivariate sampling Kantorovich operator of f reduces to the finite sum

$$(K_w f)(\underline{x}) := \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u}\right),$$

for every $\underline{x} \in \mathbb{R}^n$ and $w > 0$.
Note that, it turns out that $K_w f \in X(\mathbb{R}^n)$, since χ is measurable, and $f \in X(\mathbb{R}^n)$ being continuous; moreover, also $K_w f - f \in X(\mathbb{R}^n)$. Now, by the (L, ψ)-Lipschitz condition, we can write what follows

$$\begin{aligned} |(K_w f)(\underline{x}) - f(\underline{x})| &\leq |(K_w f)(\underline{x})| + |f(\underline{x})| \\ &= \left| \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u}) d\underline{u}\right) \right| + |f(\underline{x})| \\ &\leq \psi(\|f\|_\infty) \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\underline{x} - t_{\underline{k}}) + \|f\|_\infty, \end{aligned} \tag{5.3}$$

for $\underline{x} \in \mathbb{R}^n$ and $w > 0$. Furthermore, if $\underline{x} \notin B(\underline{0}, \overline{\gamma})$ the above upper-bound can be sharpened, namely

$$|(K_w f)(\underline{x}) - f(\underline{x})| = |(K_w f)(\underline{x})| \leq \psi(\|f\|_\infty) \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\underline{x} - t_{\underline{k}}).$$

By the above inequalities, using the monotonicity of $\rho_{\mathbb{R}^n}$ together with the property ($\rho 3$), we get

$$\rho_{\mathbb{R}^n}(K_w f - f) \le \rho_{\mathbb{R}^n}\left(\frac{1}{2}\cdot 2\cdot \psi(\|f\|_\infty) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}}) + \frac{1}{2}\cdot 2\cdot \|f\|_\infty \cdot \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right)$$
$$\le \rho_{\mathbb{R}^n}\left(2\psi(\|f\|_\infty) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right) + \rho_{\mathbb{R}^n}\left(2\|f\|_\infty \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right),$$

where $\mathbf{1}_{B(\underline{0},\overline{\gamma})}$ is the characteristic function on $B(\underline{0},\overline{\gamma})$, with $\mu_{\mathbb{R}^n}(B(\underline{0},\overline{\gamma})) \le 2^n\overline{\gamma}^n < +\infty$. Now, applying condition (5.2), with γ above fixed and $a := 4\psi(\|f\|_\infty)$, there exist $T > 0$ and a measurable set $\mathcal{I} \subset \mathbb{R}^n$, with $\mu_{\mathbb{R}^n}(\mathcal{I}) < +\infty$, such that

$$\rho_{\mathbb{R}^n}\left(4\psi(\|f\|_\infty)\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right) \le T, \tag{5.4}$$

for $w > 0$ sufficiently large. Recalling that the modular $\rho_{\mathbb{R}^n}$ is strongly finite, we have that $\mathbf{1}_{\mathcal{I}}, \mathbf{1}_{B(\underline{0},\overline{\gamma})} \in E_{\rho_{\mathbb{R}^n}}(\mathbb{R}^n)$ and therefore, using ($\rho 3$), the monotonicity of $\rho_{\mathbb{R}^n}$, and (5.4), we have

$$\rho_{\mathbb{R}^n}(K_w f - f) \le \rho_{\mathbb{R}^n}\left(2\psi(\|f\|_\infty)(\mathbf{1}_{\mathcal{I}}(\cdot) + \mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot)) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right)$$
$$+ \rho_{\mathbb{R}^n}\left(2\|f\|_\infty \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right)$$
$$= \rho_{\mathbb{R}^n}\left(\frac{1}{2}\cdot 4\psi(\|f\|_\infty)\mathbf{1}_{\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}}) + \frac{1}{2}\cdot 4\psi(\|f\|_\infty)\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right)$$
$$+ \rho_{\mathbb{R}^n}\left(2\|f\|_\infty \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right)$$
$$\le \rho_{\mathbb{R}^n}\left(4\psi(\|f\|_\infty)\mathbf{1}_{\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right)$$
$$+ \rho_{\mathbb{R}^n}\left(4\psi(\|f\|_\infty)\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \le w\gamma} L(w\cdot - t_{\underline{k}})\right) + \rho_{\mathbb{R}^n}\left(2\|f\|_\infty \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right)$$
$$\le \rho_{\mathbb{R}^n}\left(4\psi(\|f\|_\infty) m_{0,\Pi^n}(L)\mathbf{1}_{\mathcal{I}}(\cdot)\right) + T + \rho_{\mathbb{R}^n}\left(2\|f\|_\infty \mathbf{1}_{B(\underline{0},\overline{\gamma})}\right) < +\infty,$$

for sufficiently large $w > 0$.
Now, we denote by $\alpha > 0$ the constant of the absolute continuity of $\rho_{\mathbb{R}^n}$, and let

$\varepsilon > 0$ be fixed. In correspondence to $\varepsilon/2$, we have that there exists a measurable subset $A \subset \mathbb{R}^n$, with $\mu_{\mathbb{R}^n}(A) < +\infty$ such that

$$\rho_{\mathbb{R}^n}\left(\alpha(K_w f - f)\mathbf{1}_{\mathbb{R}^n \setminus A}\right) < \frac{\varepsilon}{2}. \tag{5.5}$$

In particular, since $\mu_{\mathbb{R}^n}(A) < +\infty$ and $\rho_{\mathbb{R}^n}$ is a convex, strongly finite modular, which implies that it is also finite, one has that $\mathbf{1}_A \in E_{\rho_{\mathbb{R}^n}}(\mathbb{R}^n) \subset L_{\rho_{\mathbb{R}^n}}(\mathbb{R}^n)$, i.e., $\lim_{\lambda \to 0} \rho_{\mathbb{R}^n}(\lambda \mathbf{1}_A) = 0$. Then in correspondence to $\varepsilon/2$, there exists a sufficient small $\lambda_\varepsilon > 0$ such that

$$\rho_{\mathbb{R}^n}\left(\lambda_\varepsilon \mathbf{1}_A\right) < \frac{\varepsilon}{2}. \tag{5.6}$$

Moreover, since $f \in C_c(\mathbb{R}^n)$ and by Theorem 3.2.1, we also have

$$\alpha \left\| K_w f - f \right\|_\infty < \lambda_\varepsilon, \tag{5.7}$$

for sufficiently large $w > 0$.
Now, let $0 < \lambda \leq \alpha/2$ arbitrary fixed. We can write what follows

$$\begin{aligned} \lambda \left|(K_w f)(\underline{x}) - f(\underline{x})\right| &\leq \frac{\alpha}{2}\left|(K_w f)(\underline{x}) - f(\underline{x})\right| \\ &\leq \frac{1}{2}\left[\alpha \left|(K_w f)(\underline{x}) - f(\underline{x})\right| \mathbf{1}_{\mathbb{R}^n \setminus A}(\underline{x}) + \alpha \left|(K_w f)(\underline{x}) - f(\underline{x})\right| \mathbf{1}_A(\underline{x})\right], \end{aligned}$$

with $\underline{x} \in \mathbb{R}^n$. Now, using the monotonicity of $\rho_{\mathbb{R}^n}$, condition ($\rho 3$) and the conditions (5.5)-(5.7), we finally obtain

$$\begin{aligned} \rho_{\mathbb{R}^n}\left(\lambda |K_w f - f|\right) &\leq \rho_{\mathbb{R}^n}\left(\alpha |K_w f - f| \mathbf{1}_{\mathbb{R}^n \setminus A}\right) + \rho_{\mathbb{R}^n}\left(\alpha |K_w f - f| \mathbf{1}_A\right) \\ &< \frac{\varepsilon}{2} + \rho_{\mathbb{R}^n}(\lambda_\varepsilon \mathbf{1}_A) < \varepsilon, \end{aligned}$$

for $w > 0$ sufficiently large. This completes the proof. □

If the function L of condition ($\chi 3$) has compact support, we can state the following theorem, that is a Luxemburg-norm convergence result for $K_w f$, with $f \in C_c(\mathbb{R}^n)$.

Theorem 5.1.3. *Let $\rho_{\mathbb{R}^n}$ be a convex, monotone, strongly finite and absolutely continuous modular on $X(\mathbb{R}^n)$. Moreover, let χ be a kernel satisfying condition ($\chi 3$) with L having compact support. Then, for any $f \in C_c(\mathbb{R}^n)$ and for every $\lambda > 0$, we have*

$$\lim_{w \to +\infty} \rho_{\mathbb{R}^n}(\lambda(K_w f - f)) = 0.$$

Proof. Assuming $\operatorname{supp} L \subset B(\underline{0}, R) \subset \mathbb{R}^n$, with $R > 0$, we obtain that $L(w\underline{x} - t_{\underline{k}}) = 0$ for every $\underline{x} \notin B(\underline{0}, R + \gamma)$, $t_{\underline{k}} \in B(\underline{0}, w\gamma)$ and $w \geq 1$, where $\gamma > 0$ is defined as in the proof of Theorem 5.1.2. Hence, using (χ3), we get

$$\begin{aligned} 0 \leq |(K_w f)(\underline{x})| &\leq \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} \left| \chi\left(w\underline{x} - t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u} \right) \right| \\ &\leq \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\underline{x} - t_{\underline{k}}) \psi \left(\frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u} \right) = 0, \end{aligned}$$

if $\underline{x} \notin B(\underline{0}, R+\gamma)$, i.e., $\operatorname{supp} K_w f \subset B(\underline{0}, R+\gamma)$ for every $w \geq 1$. Now, for every fixed $\lambda > 0$, using (5.3), we have

$$\begin{aligned} \lambda|(K_w f)(\underline{x}) - f(\underline{x})| &= \lambda|(K_w f)(\underline{x}) - f(\underline{x})| \mathbf{1}_{B(\underline{0},R+\gamma)} \\ &\leq \lambda \left[\psi(\|f\|_\infty) m_{0,\Pi^n}(L) + \|f\|_\infty\right] \mathbf{1}_{B(\underline{0},R+\gamma)}, \end{aligned}$$

for every $\underline{x} \in \mathbb{R}^n$ and $w \geq 1$ sufficiently large.
Since the modular $\rho_{\mathbb{R}^n}$ is strongly finite, we have that $\mathbf{1}_{B(\underline{0},R+\gamma)} \in E_{\rho_{\mathbb{R}^n}}(\mathbb{R}^n)$ and therefore, by the monotonicity of the modular

$$\rho_{\mathbb{R}^n} \left(\lambda|(K_w f)(\underline{x}) - f(\underline{x})|\right) \leq \rho_{\mathbb{R}^n} \left(\lambda \left[\psi(\|f\|_\infty) m_{0,\Pi^n}(L) + \|f\|_\infty\right] \mathbf{1}_{B(\underline{0},R+\gamma)}\right) < +\infty.$$

By Theorem 3.2.1 we can observe that, for every $\underline{x} \in \mathbb{R}^n$

$$\lim_{w \to +\infty} \lambda|(K_w f)(\underline{x}) - f(\underline{x})| = 0\,,$$

and so, with $g(\underline{x}) := \lambda \left[\psi(\|f\|_\infty) m_{0,\Pi^n}(L) + \|f\|_\infty\right] \mathbf{1}_{B(\underline{0},R+\gamma)} \in E_{\rho_{\mathbb{R}^n}}(\mathbb{R}^n)$, it is possible to apply Theorem 1.1.14,, and so the thesis immediately follows. □

Now, in order to study the convergence of the operators $K_w f$ in the general case of functions belonging to modular spaces, we need to require the following growth condition, that provides a connection between pairs of modulars on $X(\mathbb{Z}^n)$ and the function ψ of the condition (χ3).
Let $\rho_{\mathbb{Z}^n}, \eta_{\mathbb{Z}^n}$ be two modulars on $X(\mathbb{Z}^n)$. We suppose that, for every $\lambda \in (0,1)$, there exists a constant $C_\lambda \in (0,1)$ satisfying

$$\rho_{\mathbb{Z}^n}(C_\lambda \psi(g)) \leq \eta_{\mathbb{Z}^n}(\lambda g) \tag{H}$$

for any $g = (g_{\underline{k}})_{\underline{k} \in \mathbb{Z}^n} \in X(\mathbb{Z}^n)$. We point out that this assumption is similar to condition (H$_\varphi$), recalled in Section 3.3.

Moreover, in order to prove that the operators $K_w f$ are well-defined in the setting of modular spaces, we will exploit the following non trivial subset of $X(\mathbb{R}^n)$.
Given $E, K > 0$ and $\eta_{\mathbb{R}^n}$, $\eta_{\mathbb{Z}^n}$ two modulars on $X(\mathbb{R}^n)$ and $X(\mathbb{Z}^n)$ respectively, we

define the subset $\mathcal{L}_{E,K}(\mathbb{R}^n)$ of $L_{\eta_{\mathbb{R}^n}}(\mathbb{R}^n)$ whose elements f are locally absolutely integrable and satisfy the following assumption (see [51, 63])

$$\limsup_{w\to+\infty} \frac{1}{w^n}\, \eta_{\mathbb{Z}^n}(\lambda F_w) \leq E\eta_{\mathbb{R}^n}(\lambda K f), \tag{5.8}$$

for every $\lambda > 0$, where $F_w = (f_{w,\underline{k}})_{\underline{k}\in\mathbb{Z}^n} \in X(\mathbb{Z}^n)$, $w > 0$, with

$$f_{w,\underline{k}} := \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u})| d\underline{u}\,. \tag{5.9}$$

Therefore, we can establish the following theorem.

Theorem 5.1.4. *Let $\rho_{\mathbb{R}^n}, \eta_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}, \eta_{\mathbb{Z}^n}$ be two pairs of modulars on $X(\mathbb{R}^n)$ and $X(\mathbb{Z}^n)$, respectively. Assume in addition that $\rho_{\mathbb{R}^n}$ is monotone, $\rho_{\mathbb{Z}^n}, \eta_{\mathbb{Z}^n}$ satisfy condition (H), and χ is an L-compatible kernel with $\rho_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}$. Then, given any two functions $f, g \in X(\mathbb{R}^n)$, such that $f - g \in \mathcal{L}_{E,K}(\mathbb{R}^n)$, for some $E, K > 0$, there exists a constant $0 < c \leq C_\lambda/D_2$, for which*

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq E D_1 \eta_{\mathbb{R}^n}\left(\lambda K(f-g)\right),$$

holds, for every $\lambda \in (0, 1)$, for a suitable $C_\lambda \in (0, 1)$, and where D_1, D_2 are the constants of the compatibility condition.

Proof. Let $\lambda \in (0, 1)$ be fixed and let $C_\lambda \in (0, 1)$ be the corresponding parameter arising from condition (H). Now, we can choose a positive constant c such that $c \leq C_\lambda/D_2$, where D_2 is the constant of the compatibility condition (5.1). By the monotonicity of $\rho_{\mathbb{R}^n}$ combined with the (L, ψ)-Lipschitz condition, we have

$$\begin{aligned}
&\rho_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) = \\
&= \rho_{\mathbb{R}^n}\left(c\left(\sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\cdot -t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \sum_{\underline{k}\in\mathbb{Z}^n} \chi\left(w\cdot -t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} g(\underline{u})d\underline{u}\right)\right)\right) \\
&\leq \rho_{\mathbb{R}^n}\left(c\sum_{\underline{k}\in\mathbb{Z}^n}\left|\chi\left(w\cdot -t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} f(\underline{u})d\underline{u}\right) - \chi\left(w\cdot -t_{\underline{k}}, \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} g(\underline{u})d\underline{u}\right)\right|\right) \\
&\leq \rho_{\mathbb{R}^n}\left(c\sum_{\underline{k}\in\mathbb{Z}^n} L(w\cdot -t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \left|(f-g)(\underline{u})\right| d\underline{u}\right)\right).
\end{aligned}$$

Applying the compatibility condition (5.1), for sufficiently large $w > 0$, we can write

$$\begin{aligned}
\rho_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) &\leq \frac{1}{w^n} D_1 \rho_{\mathbb{Z}^n}\left(c D_2 \psi((F-G)_w)\right) + b_w \\
&\leq \frac{1}{w^n} D_1 \rho_{\mathbb{Z}^n}\left(C_\lambda \psi((F-G)_w)\right) + b_w,
\end{aligned}$$

where $(F-G)_w = ((f-g)_{w,\underline{k}})$ denotes a net defined as in (5.9), i.e.

$$(f-g)_{w,\underline{k}} = \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u}) - g(\underline{u})| d\underline{u}\,.$$

and $b_w \to 0$, as $w \to +\infty$. Then, using (H) we get

$$\rho_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq \frac{1}{w^n} D_1 \eta_{\mathbb{Z}^n}\left(\lambda(F-G)_w\right) + b_w.$$

Now, since $f - g \in \mathcal{L}_{E,K}(\mathbb{R}^n)$, we obtain

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq E D_1 \eta_{\mathbb{R}^n}\left(\lambda K(f-g)\right),$$

and this completes the proof. □

Now, we are ready to prove the main theorem of this section.

Theorem 5.1.5. *Let $\rho_{\mathbb{R}^n}$, $\eta_{\mathbb{R}^n}$ be convex, monotone, strongly finite, absolutely finite and absolutely continuous modulars on $X(\mathbb{R}^n)$, and let $\rho_{\mathbb{Z}^n}$, $\eta_{\mathbb{Z}^n}$ be two modulars on $X(\mathbb{Z}^n)$ satisfying condition (H). Further, let χ be a kernel L-compatible with $\rho_{\mathbb{R}^n}$ and $\rho_{\mathbb{Z}^n}$, and satisfying assumption (5.2) together with $\rho_{\mathbb{R}^n}$. Then, for every $f \in L_{\rho_{\mathbb{R}^n}+\eta_{\mathbb{R}^n}}(\mathbb{R}^n)$, such that $f - C_c(\mathbb{R}^n) \subset \mathcal{L}_{E,K}(\mathbb{R}^n)$, for some $E, K > 0$, there exists a constant $c > 0$ such that*

$$\lim_{w\to+\infty} \rho_{\mathbb{R}^n}(c(K_w f - f)) = 0.$$

Proof. Let f be as in the statement. By the density theorem (Theorem 1.1.15), there is a positive constant $\overline{\lambda}$ (we may take $\overline{\lambda} < 1$) such that, for every $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R}^n)$ with

$$(\rho_{\mathbb{R}^n} + \eta_{\mathbb{R}^n})\left(\overline{\lambda}(f-g)\right) < \varepsilon. \tag{5.10}$$

Fix $\varepsilon > 0$ and $g \in C_c(\mathbb{R}^n)$ as above. Since $g \in C_c(\mathbb{R}^n)$, by Theorem 5.1.2, for every $0 < \widetilde{\lambda} \leq \alpha/2$, where α is the parameter of the absolute continuity of $\rho_{\mathbb{R}^n}$, we can write

$$\lim_{w\to+\infty} \rho_{\mathbb{R}^n}(\widetilde{\lambda}(K_w g - g)) = 0. \tag{5.11}$$

In correspondence to the above $\overline{\lambda} > 0$, choose now $\lambda > 0$ so small that $\lambda K < \overline{\lambda}$, where $K \geq 1$ is the constant of the definition (5.8) and let

$$c \leq \min\left\{\frac{C_\lambda}{3D_2}, \frac{\alpha}{6}, \frac{\lambda}{3}\right\},$$

where D_2 is the constant of the compatibility condition and C_λ is the constant of condition (H) corresponding to λ. Then, by the monotonicity of $\rho_{\mathbb{R}^n}$ and condition $(\rho 3)$ of $\rho_{\mathbb{R}^n}$

$$\begin{aligned}\rho_{\mathbb{R}^n}(c(K_w f - f)) &\leq \rho_{\mathbb{R}^n}(3c(K_w f - K_w g)) + \rho_{\mathbb{R}^n}(3c(K_w g - g)) + \rho_{\mathbb{R}^n}(3c(f-g)) \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Now, applying Theorem 5.1.4 to I_1, we obtain

$$\limsup_{w\to+\infty} \rho_{\mathbb{R}^n}\left(3c(K_w f - K_w g)\right) \leq ED_1 \eta_{\mathbb{R}^n}\left(\lambda K(f-g)\right) \leq ED_1 \eta_{\mathbb{R}^n}\left(\overline{\lambda}(f-g)\right),$$

where, without loss of generality, we can suppose $ED_1 > 1$. For what concerns I_2, applying (5.11), we can write

$$\rho_{\mathbb{R}^n}(3c(K_w g - g)) \leq \rho_{\mathbb{R}^n}\left(\frac{\alpha}{2}(K_w g - g)\right) < \varepsilon,$$

for $w > 0$ sufficiently large.
As concerns I_3, we have $\rho_{\mathbb{R}^n}(3c(f-g)) \leq \rho_{\mathbb{R}^n}(\lambda(f-g)) \leq \rho_{\mathbb{R}^n}(\overline{\lambda}(f-g))$, since $\lambda K < \overline{\lambda}$, $K \geq 1$.
In conclusion, by (5.10), we have

$$\rho_{\mathbb{R}^n}(c(K_w f - f)) < ED_1(\rho_{\mathbb{R}^n} + \eta_{\mathbb{R}^n})(\overline{\lambda}(f-g)) + \varepsilon < (ED_1 + 1)\varepsilon,$$

for $w > 0$ sufficiently large. Since ε is chosen at will, the above estimation implies that $\rho_{\mathbb{R}^n}(c(K_w f - f)) \to 0$, as $w \to +\infty$, as desired. $\square$

5.2 Particular case: convergence results in Musielak-Orlicz spaces

In this section, we consider some particular cases of modular spaces in which the theory developed in the previous section holds: the *Musielak-Orlicz spaces*.
In order to see that the previous convergence theorem can be applied to these spaces, we need to recall the expression of the modular functionals characterizing these spaces.
Let φ and ξ be two fixed φ-functions, and set

$$\rho^{\varphi}_{\mathbb{R}^n}(f) := \int_{\mathbb{R}^n} \varphi(\underline{t}, |f(\underline{t})|)d\underline{t}, \qquad \eta^{\xi}_{\mathbb{R}^n}(f) := \int_{\mathbb{R}^n} \xi(\underline{t}, |f(\underline{t})|)d\underline{t},$$

where $f \in X(\mathbb{R}^n)$. Both $\rho^{\varphi}_{\mathbb{R}^n}$ and $\eta^{\xi}_{\mathbb{Z}^n}$ are modulars on $X(\mathbb{R}^n)$, which satisfy the properties (a)-(e) given in Section 1.1.
Now, let $\rho^{\varphi}_{\mathbb{Z}^n}$ and $\eta^{\xi}_{\mathbb{Z}^n}$ be two modulars on $X(\mathbb{Z}^n)$, defined by

$$\rho^{\varphi}_{\mathbb{Z}^n}(g) := \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(t_{\underline{k}}, |g_{\underline{k}}|) \qquad \eta^{\xi}_{\mathbb{Z}^n}(g) := \sum_{\underline{k}\in\mathbb{Z}^n} \xi(t_{\underline{k}}, |g_{\underline{k}}|),$$

with $g = (g_{\underline{k}})_{\underline{k}\in\mathbb{Z}^n} \in X(\mathbb{Z}^n)$.

In this frame, the growth condition (H) can be deduced requiring the following inequality involving the φ-functions φ and ξ: for every $\lambda \in (0,1)$ there exists $C_\lambda \in (0,1)$ satisfying

$$\varphi(\underline{t}, C_\lambda \psi(u)) \leq \xi(\underline{t}, \lambda u), \tag{H$_\varphi$}$$

for every $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}^+_0$.

Furthermore, the compatibility condition (5.1) is easily seen in this setting. In fact, by the Jensen inequality, the Fubini-Tonelli theorem and the change of variable $w\underline{t} - t_{\underline{k}} = \underline{y}$, we have

$$\begin{aligned}
\rho^{\varphi}_{\mathbb{R}^n}\left(\sum_{\underline{k}\in\mathbb{Z}^n} g_{\underline{k}} L(w\cdot - t_{\underline{k}})\right) &= \int_{\mathbb{R}^n} \varphi\left(\underline{t}, \left|\sum_{\underline{k}\in\mathbb{Z}^n} g_{\underline{k}} L(w\underline{t} - t_{\underline{k}})\right|\right) d\underline{t} \\
&\leq \int_{\mathbb{R}^n} \varphi\left(\underline{t}, \sum_{\underline{k}\in\mathbb{Z}^n} |g_{\underline{k}}| L(w\underline{t} - t_{\underline{k}})\right) d\underline{t} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{t} - t_{\underline{k}}) \varphi(\underline{t}, m_{0,\Pi^n}(L)|g_{\underline{k}}|) d\underline{t} \\
&\leq \frac{\|L\|_1}{w^n\, m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \varphi(\underline{t}, m_{0,\Pi^n}(L)|g_{\underline{k}}|) \\
&= \frac{\|L\|_1}{w^n\, m_{0,\Pi^n}(L)} \rho^{\varphi}_{\mathbb{Z}^n}(m_{0,\Pi^n}(L)\, g),
\end{aligned}$$

from which (5.1) follows with $D_1 = \|L\|_1 / m_{0,\Pi^n}(L)$, $D_2 = m_{0,\Pi^n}(L)$ and $b_w = 0$.

The following lemmas are related to assumption (5.2).

Lemma 5.2.1. *Let φ be a fixed φ-function which satisfies the following additional assumption*

(φ4) for sufficiently large $N > 0$

$$\sup_{\|\underline{t}\|_2 > N} \varphi(\underline{t}, u) =: K_u < +\infty,$$

for every $u \in \mathbb{R}^+_0$.

Then assumption (5.2) holds.

Proof. Let $\gamma > 0$ and $a > 0$ be fixed. Moreover, we consider the ball centered at the origin and with the radius N, $\mathcal{I} := B(\underline{0}, N) \subset \mathbb{R}^n$, where $N > 0$ is the parameter of assumption ($\varphi 4$). Using Jensen inequality and applying conditions ($\varphi 2$) and ($\varphi 4$), we have

$$\begin{aligned}
\rho^{\varphi}_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot)\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\cdot - t_{\underline{k}})\right) &= \int_{\mathbb{R}^n}\varphi\left(\underline{t}, a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\underline{t})\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\underline{t}-t_{\underline{k}})\right)d\underline{t}\\
&= \int_{\|\underline{t}\|_2>N}\varphi\left(\underline{t}, a\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\underline{t}-t_{\underline{k}})\right)d\underline{t}\\
&\le \frac{1}{m_{0,\Pi^n}(L)}\int_{\|\underline{t}\|_2>N}\varphi\left(\underline{t}, am_{0,\Pi^n}(L)\right)\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\underline{t}-t_{\underline{k}})d\underline{t}\\
&\le \frac{K_{a\,m_{0,\Pi^n}(L)}}{m_{0,\Pi^n}(L)}\sum_{\|t_{\underline{k}}\|_2\le w\gamma}\int_{\|\underline{t}\|_2>N} L(w\underline{t}-t_{\underline{k}})d\underline{t}\ ,
\end{aligned}$$

for $w > 0$, where $K_{a\,m_{0,\Pi^n}(L)}$ is the constant of assumption ($\varphi 4$) with $u = a\,m_{0,\Pi^n}(L)$. By the change of variable $\underline{y} = w\underline{t} - t_{\underline{k}}$, we obtain

$$\begin{aligned}
\rho^{\varphi}_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot)\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\cdot - t_{\underline{k}})\right) &\le \frac{K_{a\,m_{0,\Pi^n}(L)}}{w^n\ m_{0,\Pi^n}(L)}\sum_{\|t_{\underline{k}}\|_2\le w\gamma}\int_{\mathbb{R}^n} L(\underline{y})d\underline{y}\\
&\le \frac{K_{a\,m_{0,\Pi^n}(L)}}{w^n\ m_{0,\Pi^n}(L)}\|L\|_1\cdot G,
\end{aligned}$$

where $G > 0$ represents the number of terms of the above sum in fact corresponding to the number of $t_{\underline{k}}/w$ belonging to $B(\underline{0},\gamma)$. For every $w \ge 1$, we can estimate G as follows

$$\begin{aligned}
G &\le \left(2\left(\left[\frac{\gamma w}{\delta}\right]+1\right)\right)^n = 2^n\sum_{i=0}^{n}\binom{n}{i}\left[\frac{\gamma w}{\delta}\right]^{n-i} = 2^n w^n\left(\left[\frac{\gamma}{\delta}\right]^n + n\left[\frac{\gamma}{\delta}\right]^{n-1}\frac{1}{w} + \cdots \frac{1}{w^n}\right)\\
&\le w^n\left\{2^n\left(\left[\frac{\gamma}{\delta}\right]^n + n\left[\frac{\gamma}{\delta}\right]^{n-1} + \cdots + 1\right)\right\} =: w^n\cdot P,
\end{aligned}$$

where $[\cdot]$ denotes the integer part. Thus

$$\rho^{\varphi}_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot)\sum_{\|t_{\underline{k}}\|_2\le w\gamma} L(w\cdot - t_{\underline{k}})\right) \le \frac{K_{a\,m_{0,\Pi^n}(L)}}{m_{0,\Pi^n}(L)}\|L\|_1\,P,$$

for every $w \ge 1$. This completes the proof. □

Lemma 5.2.2. *Let φ be a fixed φ-function which satisfies the following additional assumption*

(φ5) $\varphi(\cdot, u) \in L^1(\mathbb{R}^n)$ for every $u \in \mathbb{R}_0^+$.

Then assumption (5.2) holds.

Proof. Let $\gamma > 0$, $a > 0$ and $\mathcal{I} := B(\underline{0}, N)$, $N > 0$, be fixed. We easily get

$$\rho^{\varphi}_{\mathbb{R}^n}\left(a\mathbf{1}_{\mathbb{R}^n\setminus\mathcal{I}}(\cdot) \sum_{\|t_{\underline{k}}\|_2 \leq w\gamma} L(w\cdot - t_{\underline{k}}) \right) \leq \int_{\|\underline{t}\|_2 > N} \varphi\left(\underline{t}, a\, m_{0,\Pi^n}(L)\right) d\underline{t}$$
$$\leq \left\|\varphi\left(\cdot, a\, m_{0,\Pi^n}(L)\right)\right\|_1 < +\infty.$$

□

In the previous lemmas we have provided two different sufficient conditions for the assumption (5.2) based on properties of φ.

Now, we can state the following lemma concerning the space $\mathcal{L}_{E,K}(\mathbb{R}^n)$, generated by $\eta^{\xi}_{\mathbb{R}^n}$ and $\eta^{\xi}_{\mathbb{Z}^n}$. In particular, we show that $\mathcal{L}_{E,K}(\mathbb{R}^n)$ is not trivial and not empty.

Lemma 5.2.3. *Let $f \in X(\mathbb{R}^n)$ be a locally integrable function. Then $f \in \mathcal{L}_{E,K}(\mathbb{R}^n)$, with $E := \delta^{-n}$ and $K := C$, where δ is one of the parameters of the sequence Π^n and C is the constant of condition (φ3).*

Proof. For every $\lambda > 0$, applying the Jensen inequality and conditions (φ2) and (φ3), we can write

$$\begin{aligned}
\frac{1}{w^n}\eta^{\xi}_{\mathbb{Z}^n}(\lambda F_w) &= \frac{1}{w^n}\sum_{\underline{k}\in\mathbb{Z}^n} \xi\left(t_{\underline{k}}, \lambda\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} |f(\underline{u})|d\underline{u}\right)\\
&\leq \frac{1}{w^n}\sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \xi\left(t_{\underline{k}}, \lambda|f(\underline{u})|\right) d\underline{u}\\
&= \frac{1}{w^n}\sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}} \xi\left(\underline{u} - \underline{u} + t_{\underline{k}}, \lambda|f(\underline{u})|\right) d\underline{u}\\
&\leq \delta^{-n}\int_{\mathbb{R}^n} \xi(\underline{u}, \lambda C|f(\underline{u})|)d\underline{u}.
\end{aligned}$$

Now, passing to the limsup as w approaches $+\infty$, we obtain the thesis

$$\limsup_{w\to+\infty} \frac{1}{w^n}\eta^{\xi}_{\mathbb{Z}^n}(\lambda F_w) \leq \delta^{-n}\eta^{\xi}_{\mathbb{R}^n}(\lambda C f),$$

where $F_w = (f_{w,\underline{k}})_{\underline{k}\in\mathbb{Z}^n} \in X(\mathbb{Z}^n)$, $w > 0$, with $f_{w,\underline{k}}$ defined as in (5.9), with $E = \delta^{-n}$ and $K = C$. □

The following theorem summarize all the results obtained in Section 5.1 in the case of Musielak-Orlicz spaces.

Theorem 5.2.4. *Let φ be a fixed φ-function which satisfies at least one between conditions (φ4) and (φ5). Hence, the following statements hold*

1. *for $f \in C_c(\mathbb{R}^n)$*
$$\lim_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}(\lambda(K_w f - f)) = 0,$$
for every $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolutely continuity of $\rho^{\varphi}_{\mathbb{R}^n}$. In particular, if the function L of condition (χ3) has compact support such convergence result holds for every $\lambda > 0$.

2. *For every locally integrable functions f, $g \in X(\mathbb{R}^n)$ and assuming that φ satisfies condition (H_φ), there exists a constant $0 < c \leq C_\lambda/m_{0,\Pi^n}(L)$, for which*
$$\limsup_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq \frac{\delta^{-n}\, \|L\|_1}{m_{0,\Pi^n}(L)} \eta^{\xi}_{\mathbb{R}^n}\left(\lambda C^2 (f-g)\right),$$
holds, for every $\lambda \in (0,1)$, where $C_\lambda \in (0,1)$ is the constant of condition (H_φ), δ is one of the parameters of sequence Π^n and C is the constant of condition (φ3).

3. *For every $f \in L^{\varphi+\xi}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that*
$$\lim_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}(c(K_w f - f)) = 0.$$

Proof. 1. As we said, $\rho^{\varphi}_{\mathbb{R}^n}$ is a modular satisfying the properties (a)-(e) of Section 1.1, and from Lemmas 5.2.1 and 5.2.2, φ is a φ-function which satisfy condition (5.2). Hence, the first statement follows applying Theorem 5.1.2.
If the kernel function χ satisfies condition (χ3) with L having compact support, we can exploit Theorem 5.1.3 to get the second part of the thesis.

2. Proceeding as in the proof of Theorem 5.1.4, for every $f, g \in X(\mathbb{R}^n)$, we can choose a positive constant c such that $c \leq C_\lambda/m_{0,\Pi^n}(L)$, where C_λ is the constant of condition (H_φ).
Using Jensen inequality twice, condition (H_φ), assumptions (φ2) and (φ3),

and Fubini-Tonelli theorem, we have

$$\rho^{\varphi}_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq \rho^{\varphi}_{\mathbb{R}^n}\left(c\sum_{\underline{k}\in\mathbb{Z}^n} L(w\cdot - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)\right)$$

$$= \int_{\mathbb{R}^n}\varphi\left(\underline{t}, c\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{t} - t_{\underline{k}})\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)\right)d\underline{t}$$

$$\leq \frac{1}{m_{0,\Pi^n}(L)}\int_{\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{t} - t_{\underline{k}})\varphi\left(\underline{t}, c\, m_{0,\Pi^n}(L)\psi\left(\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)\right)d\underline{t}$$

$$\leq \frac{1}{m_{0,\Pi^n}(L)}\int_{\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{t} - t_{\underline{k}})\xi\left(\underline{t}, \lambda\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)d\underline{t}$$

$$\leq \frac{1}{m_{0,\Pi^n}(L)}\int_{\mathbb{R}^n}\sum_{\underline{k}\in\mathbb{Z}^n} L(w\underline{t} - t_{\underline{k}})\xi\left(t_{\underline{k}}, \lambda C\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)d\underline{t}$$

$$= \frac{\|L\|_1}{w^n m_{0,\Pi^n}(L)}\sum_{\underline{k}\in\mathbb{Z}^n}\xi\left(t_{\underline{k}}, \lambda C\frac{w^n}{A_{\underline{k}}}\int_{R^w_{\underline{k}}}|(f-g)(\underline{u})|d\underline{u}\right)$$

$$\leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)}\sum_{\underline{k}\in\mathbb{Z}^n}\int_{R^w_{\underline{k}}}\xi\left(t_{\underline{k}}, \lambda C|(f-g)(\underline{u})|\right)d\underline{u}$$

$$\leq \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)}\sum_{\underline{k}\in\mathbb{Z}^n}\int_{R^w_{\underline{k}}}\xi\left(\underline{u}, \lambda C^2|(f-g)(\underline{u})|\right)d\underline{u}$$

$$= \frac{\|L\|_1}{\delta^n m_{0,\Pi^n}(L)}\eta^{\xi}_{\mathbb{R}^n}\left(\lambda C^2(f-g)\right).$$

3. This statement can be proved by the previous ones following the same steps of the proof of Theorem 5.1.5.

□

5.2.1 Convergence results in weighted Orlicz spaces

As a particular example of Musielak-Orlicz spaces, one can consider φ-functions of product type, of the form

$$\varphi(\underline{t}, u) := \theta(\underline{t})\,\widetilde{\varphi}(u),$$

with $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}^+_0$, which satisfy conditions ($\mathcal{F}1$)-($\mathcal{F}3$) (see p. 8 of Section 1.1.1). It is easy to check that the assumption ($\varphi4$) is also satisfied, in addition to assumptions ($\varphi1$)-($\varphi3$). Therefore, from Theorem 5.2.4, we immediately obtain the following corollary.

Corollary 5.2.5. *Let φ be a fixed φ-function of the form $\varphi(\underline{t}, u) := \theta(\underline{t})\ \widetilde{\varphi}(u)$, $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}_0^+$. Hence, the following statements hold*

1. *for $f \in C_c(\mathbb{R}^n)$*
$$\lim_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}(\lambda(K_w f - f)) = 0,$$
for every $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolutely continuity of $\rho^{\varphi}_{\mathbb{R}^n}$. In particular, if the function L of condition ($\chi 3$) has compact support such convergence result holds for every $\lambda > 0$.

2. *For every locally integrable functions f, $g \in X(\mathbb{R}^n)$ and assuming that φ satisfies condition (H_φ), there exists a constant $0 < c \leq C_\lambda / m_{0,\Pi^n}(L)$, for which*
$$\limsup_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}\left(c(K_w f - K_w g)\right) \leq \frac{\delta^{-n}\left\|L\right\|_1}{m_{0,\Pi^n}(L)} \eta^{\xi}_{\mathbb{R}^n}\left(\lambda C^2 (f-g)\right),$$
holds, for every $\lambda \in (0,1)$, where $C_\lambda \in (0,1)$ is the constant of condition (H_φ), δ is one of the parameters of sequence Π^n and C is the constant of condition ($\varphi 3$).

3. *For every $f \in L^{\varphi+\xi}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that*
$$\lim_{w\to+\infty} \rho^{\varphi}_{\mathbb{R}^n}(c(K_w f - f)) = 0.$$

Remark 5.2.6. The convergence results established in the present section extend for the nonlinear sampling Kantorovich operators (and also for the linear ones, see Remark 3.1.3) the convergence results proved in [2]. More precisely, in [2] the convergence has been proved for the usual weighted sup-norm, while here we can deduce the convergence with respect to the weighted L^p-norm, $1 \leq p < +\infty$.

5.2.2 Convergence results in Orlicz spaces

In this section, we consider the φ-functions of the form $\varphi(\underline{t}, u) = \widetilde{\varphi}(u)$, $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}_0^+$, presented in Example 1.1.21 for $\Omega = \mathbb{R}^n$. In order to simplify the notation, we write φ instead of $\widetilde{\varphi}$, as in Section 1.1.2. So, we take the following modulars on $X(\mathbb{R}^n)$

$$I^{\varphi}[f] := \rho^{\varphi}_{\mathbb{R}^n}(f) = \int_{\mathbb{R}^n} \varphi(|f(\underline{t})|) d\underline{t}, \qquad I^{\xi}[f] := \eta^{\xi}_{\mathbb{R}^n}(f) = \int_{\mathbb{R}^n} \xi(|f(\underline{t})|) d\underline{t}.$$

Hence, we can state the following corollary, whose thesis follows as a consequence of Theorem 5.2.4.

Corollary 5.2.7. *Let φ be a fixed convex φ-function of the form $\varphi(\underline{t}, u) = \varphi(u)$, $\underline{t} \in \mathbb{R}^n$, $u \in \mathbb{R}_0^+$. Hence, the following statements hold*

1. *for $f \in C_c(\mathbb{R}^n)$*
$$\lim_{w \to +\infty} I^{\varphi}[\lambda(K_w f - f)] = 0,$$
for every $0 < \lambda \leq \alpha/2$, where α is the parameter of the absolutely continuity of I^{φ}. In particular, if L has compact support such convergence result holds for every $\lambda > 0$.

2. *For every locally integrable functions f, $g \in X(\mathbb{R}^n)$ and assuming that φ satisfies condition (H_{φ}), there exists a constant $0 < c \leq C_{\lambda}/m_{0,\Pi^n}(L)$, for which*
$$\limsup_{w \to +\infty} I^{\varphi}\left[c(K_w f - K_w g)\right] \leq \frac{\delta^{-n} \|L\|_1}{m_{0,\Pi^n}(L)} I^{\xi}\left(\lambda(f - g)\right),$$
holds, for every $\lambda \in (0,1)$, where $C_{\lambda} \in (0,1)$ and δ is one of the parameters of sequence Π^n.

3. *For every $f \in L^{\varphi+\xi}(\mathbb{R}^n)$, there exists a constant $c > 0$ such that*
$$\lim_{w \to +\infty} I^{\varphi}[c(K_w f - f)] = 0.$$

5.3 Particular case: modulars without an integral representation

In the previous section, we discussed examples of modular spaces characterized by modular functionals in the integral form. In this section, we discuss examples of modulars that are defined by the supremum, as in Section 1.1.3.
We recall the definition of the modulars characterizing these spaces

$$\rho_{\mathbb{R}^n}^{\Phi}(f) := \sup_{\ell \in W} \int_a^b a_{\ell}(x) \left[\int_{\mathbb{R}^n} \Phi(x, |f(\underline{t})|) d\underline{t}\right] dm(x),$$

$$\rho_{\mathbb{Z}^n}^{\Phi}(g) := \sup_{\ell \in W} \int_a^b a_{\ell}(x) \left[\sum_{\underline{k} \in \mathbb{Z}^n} \Phi(x, |g_{\underline{k}}|)\right] dm(x),$$

with $f \in X(\mathbb{R}^n)$ and $g = (g_{\underline{k}})_{\underline{k} \in \mathbb{Z}^n} \in X(\mathbb{Z}^n)$. Considering a given kernel χ with L of condition $(\chi 3)$ having compact support, we have that the compatibility condition (5.1) holds. In fact, by the Jensen inequality and the Fubini-Tonelli theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi\left(x, \sum_{\underline{k}\in\mathbb{Z}^n} |g_{\underline{k}}| L(w\underline{t} - t_{\underline{k}})\right) d\underline{t} \\
&\leq \frac{1}{m_{0,\Pi^n}(L)} \int_{\mathbb{R}^n} \sum_{\underline{k}\in\mathbb{Z}^n} \Phi\left(x, m_{0,\Pi^n}(L)|g_{\underline{k}}|\right) L(w\underline{t} - t_{\underline{k}}) d\underline{t} \\
&\leq \frac{\|L\|_1}{w^n m_{0,\Pi^n}(L)} \sum_{\underline{k}\in\mathbb{Z}^n} \Phi\left(x, m_{0,\Pi^n}(L)|g_{\underline{k}}|\right),
\end{aligned}$$

for which (5.1) follows with $D_1 = \|L\|_1 / m_{0,\Pi^n}(L)$, $D_2 = m_{0,\Pi^n}(L)$ and $b_w = 0$. Since $L \in C_c(\mathbb{R}^n)$, in view of Remark 5.1.1, it follows that also (5.2) is satisfied. Finally, by defining the modulars $\eta^{\Psi}_{\mathbb{R}^n}$ and $\eta^{\Psi}_{\mathbb{Z}^n}$ analogously as above (with the function $\Psi : [a, b[\times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ in place of Φ), it turns out that any $f \in X(\mathbb{R}^n)$ belongs to $\mathcal{L}_{E,K}(\mathbb{R}^n)$, with $E = 1/\delta^n$ and $K = 1$. In fact, using Jensen inequality again

$$\begin{aligned}
\frac{1}{w^n} \eta^{\Psi}_{\mathbb{Z}^n}(\lambda F_w) &= \frac{1}{w^n} \sup_{\ell\in W} \int_a^b a_\ell(x) \left[\sum_{\underline{k}\in\mathbb{Z}^n} \Psi\left(x, \lambda \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} |f(\underline{u})| d\underline{u}\right)\right] dm(x) \\
&\leq \frac{1}{w^n} \sup_{\ell\in W} \int_a^b a_\ell(x) \left[\sum_{\underline{k}\in\mathbb{Z}^n} \frac{w^n}{A_{\underline{k}}} \int_{R^w_{\underline{k}}} \Psi\left(x, \lambda |f(\underline{u})|\right) d\underline{u}\right] dm(x) \\
&\leq \frac{1}{\delta^n} \sup_{\ell\in W} \int_a^b a_\ell(x) \left[\int_{\mathbb{R}^n} \Psi\left(x, \lambda |f(\underline{u})|\right) d\underline{u}\right] dm(x) = \frac{1}{\delta^n} \eta^{\Psi}_{\mathbb{R}^n}(\lambda f),
\end{aligned}$$

for $\lambda > 0$ sufficiently small. Thus, all the assumptions of Theorem 5.1.5 are satisfied.

Conclusions and future developments

This thesis provides a contribution to the study of *nonlinear multivariate sampling Kantorovich operators* by combining the theoretical framework of modular spaces with the investigation of convergence results and quantitative and qualitative estimates.

This kind of operators represents an averaged version, in Kantorovich-sense, of the generalized sampling operators introduced by P. L. Butzer in the years 80s and includes, as special case, the linear ones, that can be obtained by taking the kernel χ of the form $\chi(\underline{x}, u) = L(\underline{x})u$, $\underline{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$. Therefore, it is natural to introduce some suitable assumptions due to the nonlinear setting which furnishes a generalization, not only from the mathematical point of view, but also in terms of practical applications.

Another important point of the thesis is the multivariate setting that allows to reconstruct, from the point of view of the applications, also signals that are mathematically modeled by functions of several variables such as, for example, digital images, paving the way for their possible application in the Image Processing field. Furthermore, the general setting of modular spaces in which we work allows us to deduce the convergence results in several concrete cases, such as Musielak-Orlicz spaces, Orlicz spaces, and many others presented throughout the thesis, by a unifying approach.

The results obtained in this thesis, in addition to providing contributions and tools for the study of the approximation properties of families of nonlinear operators, lay the foundations for future developments of the theory, both in general settings, such as modular spaces, and in terms of new approximation properties to investigate. In this respect, quantitative estimates in the general setting of modular spaces is an open problem which constitutes the natural development of this thesis and which we have already begun to examine.

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