Extensions of the Simpson voting rule to the committee selection setting

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Committee selection rules are procedures selecting sets of candidates of a given size on the basis of the preferences of the voters. There are in the literature two natural extensions of the well-known single-winner Simpson voting rule to the multiwinner setting. The first method gives a ranking of candidates according to their minimum number of wins against the other candidates. Then, if a fixed number *k* of candidates are to be elected, the *k* best ranked candidates are chosen as the overall winners. The second method gives a ranking of committees according to the minimum number of wins of committee members against committee nonmembers. Accordingly, the committee of size *k* with the highest score is chosen as the winner. We propose an in-depth analysis of those committee selection rules, assessing and comparing them with respect to several desirable properties among which unanimity, fixed majority, non-imposition, stability, local stability, Condorcet consistency, some kinds of monotonicity, resolvability and consensus committee. We also investigate the probability that the two methods are resolute and suffer the reversal bias, the Condorcet loser paradox and the leaving member paradox. We compare the results obtained with the ones related to further well-known committee selection rules. The probability assumption on which our results are based is the widely used Impartial Anonymous Culture.

Keywords:
Multiwinner Elections, Committee Selection Rule, Simpson Voting Rule, Paradoxes, Probability

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Abstract

Committee selection rules are procedures selecting sets of candidates of a given size on the basis of the preferences of the voters. There are in the literature two natural extensions of the well-known single-winner Simpson voting rule to the multiwinner setting. The first method gives a ranking of candidates according to their minimum number of wins against the other candidates. Then, if a fixed number k of candidates are to be elected, the k best ranked candidates are chosen as the overall winners. The second method gives a ranking of committees according to the minimum number of wins of committee members against committee nonmembers. Accordingly, the committee of size k with the highest score is chosen as the winner. We propose an in-depth analysis of those committee selection rules, assessing and comparing them with respect to several desirable properties among which unanimity, fixed majority, non-imposition, stability, local stability, Condorcet consistency, some kinds of monotonicity, resolvability and consensus committee. We also investigate the probability that the two methods are resolute and suffer the reversal bias, the Condorcet loser paradox and the leaving member paradox. We compare the results obtained with the ones related to further well-known committee selection rules. The probability assumption on which our results are based is the widely used Impartial Anonymous Culture.

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1 Introduction

A voting rule is a procedure for associating with any preference profile, that is a collection of individual preferences expressed as linear orders on a set of candidates, one of the candidates to be interpreted as the winner of the election. There are many situations where instead of choosing a single winner a society is required to select a given number of candidates from the set of available options. This may begin with the choice of representatives in an elementary school class to the choice of national legislators or international representatives of a country (e.g., in the European Parliament). That leads to introduce the concept of committee selection rule (CSR), namely a procedure which associates with any preference profile and any positive integer k a subset of candidates of size k. There are many CSRs studied in the literature based on different ideas and principles. Most of them are extensions of well-known voting rules to the multiwinner setting, that is, they reduce to those voting rules when $k = 1$.

According to this approach, Barberà and Coelho (2008) take into consideration an extension of the Simpson voting rule,[1] called here the *Simpson* CSR and denoted by $\mathfrak{S}$. Recall that, given a preference profile, the Simpson voting rule first associates with any candidate x the number of voters preferring x to y for all other candidate y. The minimum of these quantities is called the Simpson score of x. Next, a candidate is selected if she has the highest Simpson score. If now k is the target size of the committee to be elected, a set W of candidates of size k is selected by $\mathfrak{S}$ if the Simpson score of each candidate in W is at least as great as the one of each nonmember of W. Barberà and Coelho (2008) show that, even though the Simpson voting rule is Condorcet consistent, $\mathfrak{S}$ is not stable, that is, it may fail to select weak k-Condorcet sets when $k \geq 2$. Recall that, following Gehrlein (1985), a subset of candidates of size k is called a weak k-Condorcet set if no candidate in this subset can be defeated by any candidate from outside the subset on the basis of pairwise majority comparisons. Obviously, this concept is a generalization to the committee selection setting of the well-known concept of the weak Condorcet winner. We further notice that $\mathfrak{S}$ presents much more severe distortions. Assume, for instance, that only three candidates x, y and z are considered and that each voter has preferences given by $x \succ y \succ z$. If we focus on committees having size $k = 2$, $\mathfrak{S}$ selects the two sets $\{x, y\}$ and $\{x, z\}$, violating a very natural unanimity principle which should admit $\{x, y\}$ as the unique selected set.

Coelho (2004) proposes another possible extension of the Simpson voting rule to the committee selection setting,[2] called the *Minimal Size of External Opposition* CSR and here denoted by $\mathfrak{M}$. Given a preference profile, the method associates with any nonempty set W a score, called the Maximin score of W, computed as follows: for every x in W and y not in W the number of voters preferring x to y is considered and then the minimum of these numbers is taken. Once the size k of the committee is fixed, $\mathfrak{M}$ selects the sets of candidates which obtain the greatest Maximin score among the ones of size k. The main difference between the two extensions lies in the fact that $\mathfrak{S}$ determines its outcome on the basis of a ranking of candidates while the outcome of $\mathfrak{M}$ is directly obtained by means of a ranking of committees. Notice that $\mathfrak{M}$ selects the subset $\{x, y\}$ as the unique winning committee in the above considered example. This depends on the fact that the Maximin score of $\{x, y\}$ equals the total number of voters, while the Maximin score of the other two possible subsets of size two is zero.

The first contribution of this paper is an in-depth analysis of the properties of the two CSR $\mathfrak{S}$ and $\mathfrak{M}$. We mainly rely on the papers of Kamwa (2017a,b) as well as the paper of Elkind et al. (2017). In Kamwa (2017a) some of the properties considered in our paper are already studied for $\mathfrak{M}$ and other stable CSRs which can be considered as possible extensions of well-known voting rules to the committee selection setting. Such a paper is extended by Kamwa (2017b) who compares the considered CSRs when the candidates are three and the size of the committee to be elected is two on the basis of what the author calls the '*divergence on outcomes*' defined as the situation where a committee selected by a CSR differs from the committee made by the best candidates of the corresponding voting rule from which the CSR is adapted. Elkind et al. (2017) represents a

[1] The Simpson voting rule is also known as the Simpson-Kramer, the Condorcet and the Minimax voting rule. See Kramer (1977) and Simpson (1969).

[2] Another extension of the Simpson voting rule, called Minimax Approval Voting, has been introduced by Brams et al. (2005) in the context of approval-based preferences.

preliminary attempt to develop a formal framework for the study of the characteristics of CSRs when the voters' preferences are linear orders, where several properties are introduced and discussed. We take into account all the properties considered in Kamwa (2017a,b) and Elkind et al. (2017) and, in addition, we formulate further desirable properties against which the two CSRs $\mathfrak{S}$ and $\mathfrak{M}$ are judged. Table 1 in Section 3 provides a summary of our results. It is worth noting that, in Elkind et al. (2017), two families of CSRs were introduced to describe many classic CSRs in a unified framework: the best-k rules and the committee scoring rules (see Section 3.2 in Elkind et al., 2017). It can be easily seen that $\mathfrak{S}$ is a best-k rule, while it is not a committee scoring rule (Proposition 17). Even though those families surely cover a lot of interesting CSRs, there is room for other kinds of rules based on a quite different approach and which do not fall into one of those categories. Indeed, $\mathfrak{M}$ is neither a best-k rule nor a committee scoring rule (Propositions 11 and 17).

In the course of our investigations, we also observe that both $\mathfrak{S}$ and $\mathfrak{M}$ fail to be resolute and immune to the reversal bias and suffer the Condorcet loser and the leaving member paradoxes. We compute then the probability of these particular situations and we compare the results with those obtained for the Borda, the Plurality, the Negative Plurality and the Bloc CSRs. The probability assumption on which our results are based is the widely used Impartial Anonymous Culture (IAC). We are also concerned about the probability that $\mathfrak{S}$ and $\mathfrak{M}$ agree on the same committee as well as the probability of the agreement between each of them and the Borda CSR. This identifies the second contribution of this paper. Our results show distinct performances of the CSRs according to the number of candidates, the numbers of voters and the target size of the committee. In many cases, $\mathfrak{M}$ does not have the highest level of performance, but in general it behaves better than $\mathfrak{S}$.

The paper is structured as follows. Section 2 is devoted to basic notations and definitions. Section 3 presents our results about the properties of $\mathfrak{S}$ and $\mathfrak{M}$. Section 4 presents the computational probabilistic analysis related to $\mathfrak{S}$ and $\mathfrak{M}$ and the other considered CSRs.

2 Definitions and notations

We assume $0 \notin \mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $k \in \mathbb{N}$ and X be a finite set. We denote by $[k]$ the set $\{1, \ldots, k\}$ and by $|X|$ the size of X. Moreover, we call a subset of X of size k a k-subset of X and denote by 2^X_k the set of the k-subsets of X. If $|X| = m \geq 1$, we denote by $\mathcal{L}(X)$ the set of linear orders on X. Every $q \in \mathcal{L}(X)$ is represented by a writing of the type $x_1 \succ_q x_2 \succ_q \ldots \succ_q x_m$ (the subscript q is generally omitted) or by a column vector $[x_1, \ldots, x_m]^T$, where $x_1, \ldots, x_m$ are the distinct elements of X.

Consider a countably infinite set $\mathbf{C}$ whose elements are to be interpreted as potential candidates and a countably infinite set $\mathbf{V}$ whose elements are to be interpreted as potential voters. For simplicity, we identify $\mathbf{C}$ and $\mathbf{V}$ with $\mathbb{N}$. An election is a vector (C, V, p), where C is a finite subset of $\mathbf{C}$ having at least two elements, V is a nonempty and finite subset of $\mathbf{V}$, and p is a function from V to $\mathcal{L}(C)$, the set of linear orders on C. Given an election (C, V, p), $p(v)$ is interpreted as the preference relation of voter v on the set of candidates C. For $x, y \in C$ with $x \neq y$, $x \succ_{p(v)} y$ means that x is strictly preferred to y in $p(v)$. The function p is called the preference profile of the election. When $V = [n]$, p can be represented by a matrix whose v-th column is $p(v)$ for all $v \in V$.

The set of elections is denoted by $\mathcal{E}$. A committee selection rule (CSR) $\mathfrak{R}$ is a function which associates with every vector $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$, a nonempty set of k-subsets of C.

Let $(C, V, p) \in \mathcal{E}$ and $x \in C$. The Simpson score of x is defined by

$$S(C, V, p, x) = \min_{y \neq x} c_p(x, y),$$

where, for every $x, y \in C$ with $x \neq y$,

$$c_p(x, y) = |\{v \in V : x \succ_{p(v)} y\}|.$$

Following Barberà and Coelho (2008), the *Simpson* CSR $\mathfrak{S}$ is defined, for every $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$, by

$$\mathfrak{S}(C, V, p, k) = \Big\{ W \in 2^C_k : \forall x \in W, \forall y \notin W, \; S(C, V, p, x) \geq S(C, V, p, y) \Big\}.$$

The Simpson CSR restricted to the case $k = 1$ agrees, by definition, with the classical Simpson voting rule. Note that, according to Elkind et al. (2017, Definition 1), the Simpson CSR is the best-k rule associated with the social preference function naturally induced by the Simpson score. That observation allows to obtain some properties for $\mathfrak{S}$ as application of general results about best-k rules proved in Elkind et al. (2017).

Note also that the relation on C defined as

$$R_S(C, V, p) = \{(x, y) \in C^2 : S(C, V, p, x) \geq S(C, V, p, y)\}$$

is an order and that $\mathfrak{S}(C, V, p, k) = C_k(R_S(C, V, p))$ within the notation in Bubboloni and Gori (2018, Section 3). That observation also allows to obtain some properties for $\mathfrak{S}$ as an application of results proved in Bubboloni and Gori (2018).

Coelho (2004) proposes another CSR which agrees with the Simpson voting rule when $k = 1$. Here we introduce it via an equivalent definition. Given $(C, V, p) \in \mathcal{E}$ and $W \subseteq C$ with $|W| \in [|C| - 1]$, the Maximin score of W is defined by

$$M(C, V, p, W) = \min_{x \in W, y \notin W} c_p(x, y).$$

The *Minimal Size of External Opposition* CSR $\mathfrak{M}$ is defined, for every $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$, by[3]

$$\mathfrak{M}(C, V, p, k) = \underset{W \in 2^C_k}{\arg\max}\, M(C, V, p, W).$$

We stress that, even though $\mathfrak{S}$ and $\mathfrak{M}$ agree when $k = 1$, that is not true in general for other values of k. Indeed, considering $(C, V, p) \in \mathcal{E}$ with $C = [4]$, $V = [9]$ and

$$p = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 4 & 4 & 4 \\ 2 & 2 & 2 & 4 & 1 & 2 & 1 & 1 & 2 \\ 4 & 4 & 4 & 2 & 2 & 4 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 & 4 & 1 & 3 & 3 & 1 \end{bmatrix},$$

we have that $\mathfrak{M}(C, V, p, 2) = \{\{1, 2\}\}$ and $\mathfrak{S}(C, V, p, 2) = \{\{1, 4\}\}$. As we are going to show in the next section, $\mathfrak{S}$ and $\mathfrak{M}$ present important differences also in terms of satisfied properties.

In what follows, when the reference to the set of candidates C and voters V is clear, we usually omit those symbols. For instance, the writing $S(C, V, p, x)$ simply becomes $S(p, x)$.

3 An analysis of the Simpson and the Maximin CSRs

Anonymity, neutrality and homogeneity are well-known properties a CSR may meet. The proof of the following result is straightforward and thus omitted.

Proposition 1. *$\mathfrak{S}$ and $\mathfrak{M}$ are anonymous, neutral and homogeneous.*

Let $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in 2^C_k$. Following Gehrlein (1985), we say that W is a weak k-Condorcet set if, for every $x \in W$ and $y \in C \setminus W$,

$$c_p(x, y) \geq \frac{|V|}{2},$$

and that W is a k-Condorcet set if, for every $x \in W$ and $y \in C \setminus W$,

$$c_p(x, y) > \frac{|V|}{2}.$$

[3] In Coelho (2004) $\mathfrak{M}$ is denoted by SEO and it is defined as

$$\mathfrak{M}(C, V, p, k) = \underset{W \in 2^C_k}{\arg\min} \max_{y \notin W, x \in W} c_p(y, x).$$

Clearly W is a weak k-Condorcet set if and only if $M(C, V, p, W) \geq \frac{|V|}{2}$; W is a k-Condorcet set if and only if $M(C, V, p, W) > \frac{|V|}{2}$; if W is a k-Condorcet set, then it is a weak k-Condorcet set. Moreover, it is easily checked that if a k-Condorcet set exists, then it is unique. On the other hand, it is possible to have more than one weak k-Condorcet set.

A CSR $\mathfrak{R}$ is called *stable* if, for every $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$ such that there exists a weak k-Condorcet set for (C, V, p), we have that every $W \in \mathfrak{R}(C, V, p, k)$ is a weak k-Condorcet set;[4] $\mathfrak{R}$ is *Condorcet consistent* if, for every $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$ such that there exists a k-Condorcet set W for (C, V, p), we have that $\mathfrak{R}(C, V, p, k) = \{W\}$;[5] $\mathfrak{R}$ satisfies *fixed majority* (resp. *strong unanimity*) if, for every $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in 2^C_k$ such that a strict majority of the voters (resp. all the voters) rank all the members of W above all the non-members of W, we have that $\mathfrak{R}(C, V, p, k) = \{W\}$;[6] $\mathfrak{R}$ satisfies *weak unanimity* if, for every $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in 2^C_k$ such that all the voters rank all the members of W above all the non-members of W, we have that $W \in \mathfrak{R}(C, V, p, k)$;[6] $\mathfrak{R}$ satisfies *strong non-imposition* if, for every $C \subseteq \mathbf{C}$ and $V \subseteq \mathbf{V}$ finite sets with $|C| \geq 2$, $|V| \geq 1$, $k \in [|C| - 1]$ and $W \in 2^C_k$, there exists $(C, V, p) \in \mathcal{E}$ such that $\mathfrak{R}(C, V, p, k) = \{W\}$; $\mathfrak{R}$ satisfies *non-imposition* if, for every finite set $C \subseteq \mathbf{C}$ with $|C| \geq 2$, $k \in [|C| - 1]$ and $W \in 2^C_k$, there exists $(C, V, p) \in \mathcal{E}$ such that $\mathfrak{R}(C, V, p, k) = \{W\}$. It is easily proved that if $\mathfrak{R}$ is stable, then it is Condorcet consistent. Indeed, the existence of a k-Condorcet set W for $(C, V, p) \in \mathcal{E}$ implies that the set of the weak k-Condorcet sets for (C, V, p) is $\{W\}$. Then the following chain of implications holds true:

$$\text{Stability} \Rightarrow \text{Condorcet Consistency} \Rightarrow \text{Fixed Majority} \Rightarrow \text{Strong Unanimity} \Rightarrow \text{Weak Unanimity} \quad (1)$$

Note also that strong unanimity implies strong non-imposition.

The next proposition shows not only that $\mathfrak{S}$ fails stability, as shown by Barberà and Coelho (2008, Proposition 3), but that it even fails strong unanimity. On the contrary, $\mathfrak{M}$ is stable. This is a crucial difference between the two CSRs.

Proposition 2. *$\mathfrak{S}$ is not strongly unanimous. In particular, $\mathfrak{S}$ does not satisfy fixed majority, is not Condorcet consistent and is not stable.*

Proof. Consider (C, V, p) with $C = \{1, 2, 3\}$ and $1 \succ_{p(v)} 2 \succ_{p(v)} 3$ for all $v \in V$ and $k = 2$. The strong unanimity principle implies that $\{1, 2\}$ is the only selected committee. However, $S(p, 1) = |V|$, $S(p, 2) = 0$, $S(p, 3) = 0$ so that $\mathfrak{S}(p, 2) = \{\{1, 2\}, \{1, 3\}\}$. □

Proposition 3. *$\mathfrak{S}$ is weakly unanimous, satisfies non-imposition and does not satisfy strong non-imposition.*

Proof. Let $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$. Assume that $W \in 2^C_k$ is such that all the voters rank all the members of W above all the non-members of W. We want to show that $W \in \mathfrak{S}(p, k)$, that is, for every $x \in W$ and every $y \in C \setminus W$, we have $S(p, x) \geq S(p, y)$. Fix then $x \in W$ and $y \in C \setminus W$. Then

$$S(p, y) = \min_{z \neq y} c_p(y, z) \leq c_p(y, x) = 0.$$

Thus, trivially, $S(p, x) \geq S(p, y) = 0$. This shows that $\mathfrak{S}$ is weakly unanimous.

In order to show non-imposition, let C be a finite set with $|C| = m \geq 2$, $k \in [|C| - 1]$ and $W^* \in 2^C_k$. Let $W^* = \{x_1, \ldots, x_k\}$ and $C \setminus W^* = \{y_{k+1}, \ldots, y_m\}$. If $k = 1$, consider just a single voter having x_1 ranked first. Assume next that $k \geq 2$. In this case, consider the election $(C, [2], p)$, where

$$x_1 \succ_{p_1} x_2 \succ_{p_1} \cdots \succ_{p_1} x_k \succ_{p_1} y_{k+1} \succ_{p_1} \cdots \succ_{p_1} y_m$$

and

$$x_k \succ_{p_2} x_{k-1} \succ_{p_2} \cdots \succ_{p_2} x_1 \succ_{p_2} y_{k+1} \succ_{p_2} \cdots \succ_{p_2} y_m.$$

[4] Barberà and Coelho (2008, Definition 3).
[5] Gehrlein (1985).
[6] Elkind et al. (2017).

Then we have $S(p, x_i) = 1$ for all $i \in [k]$, and $S(p, y_i) = 1$ for all $i \in \{k+1, \dots, m\}$. Thus $\mathfrak{S}(p, k) = \{W^*\}$.

Finally note that $\mathfrak{S}$ does not satisfy strong non-imposition because the selection of two alternatives when a single voter is considered never leads to a unique outcome. □

The next proposition shows, among other things, that $\mathfrak{M}$ is stable. We note that Coelho (2004) just introduces $\mathfrak{M}$ as an example of stable CSR.

Proposition 4. *$\mathfrak{M}$ is stable. In particular, $\mathfrak{M}$ is Condorcet consistent, satisfies fixed majority, strong unanimity, weak unanimity and strong non-imposition.*

Proof. Let $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$ be such that there exists a weak k-Condorcet set, say W^*. We know that, $M(p, W^*) \geq \frac{|V|}{2}$. Let now $W \in \mathfrak{M}(p, k)$ and prove that W is a weak k-Condorcet set. Indeed, assume by contradiction that W is not a weak k-Condorcet set. Then $M(p, W) < \frac{|V|}{2}$, so that $M(p, W) < M(p, W^*)$ and $W \notin \mathfrak{M}(p, k)$, a contradiction.

Since $\mathfrak{M}$ is stable, applying the chain of implications (1), all the remaining facts are immediately established. □

Kamwa (2017a) proposes the next property which refers to the behaviour of a CSR when a Condorcet winner exists. A CSR $\mathfrak{R}$ satisfies the *Condorcet winner criterion* if, for every $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$, $W \in \mathfrak{R}(C, V, p, k)$ and $x \in C$ a Condorcet winner, we have that $x \in W$.

Proposition 5. *$\mathfrak{S}$ satisfies the Condorcet winner criterion.*

Proof. Consider $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in \mathfrak{S}(C, V, p, k)$. Assume that $x \in C$ is a Condorcet winner. Then $S(p, x) > \frac{|V|}{2}$. Moreover, for every $y \in C \setminus \{x\}$, we have $S(p, y) \leq c_p(y, x) < \frac{|V|}{2}$ so that $S(p, y) < S(p, x)$. Assume now that $x \notin W$ and pick $y \in W$. Then $S(p, y) \geq S(p, x)$, a contradiction □

Proposition 6. *$\mathfrak{M}$ does not satisfy the Condorcet winner criterion.*

Proof. See Kamwa (2017a, Example 3). □

Following Aziz et al. (2017), given $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in 2^C_k$, we say that W is a *k-locally stable set* with respect to the Droop quota if, for every $y \in C \setminus W$, we have that

$$|\{v \in V : \forall x \in W, y \succ_{p(v)} x\}| \leq \left\lfloor \frac{|V|}{k+1} \right\rfloor.$$

We say then that a CSR $\mathfrak{R}$ is *locally stable* if, for every $(C, V, p) \in \mathcal{E}$ and $k \in [|C| - 1]$ such that there exists a k-locally stable set, we have that every $W \in \mathfrak{R}(C, V, p, k)$ is a k-locally stable set.

Proposition 7. *There exists no CSR $\mathfrak{R}$ which is stable and locally stable.*

Proof. Assume by contradiction that $\mathfrak{R}$ is a stable and locally stable CSR and consider $C = [4]$, $V = [7]$ and

$$p = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 3 & 1 \\ 3 & 3 & 4 & 3 & 3 & 4 & 3 \\ 4 & 4 & 1 & 4 & 4 & 2 & 4 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 \end{bmatrix}$$

Then $\{3, 4\}$ is the unique 2-Condorcet set and $\{1, 2\}$ and $\{1, 3\}$ are the only 2-locally stable sets. Then, by stability, $\mathfrak{R}(C, V, p, 2) = \{\{3, 4\}\}$ while, by local stability, $\mathfrak{R}(C, V, p, 2) \subseteq \{\{1, 2\}, \{1, 3\}\}$, a contradiction. □

Proposition 8. *$\mathfrak{S}$ and $\mathfrak{M}$ are not locally stable.*

Proof. By Proposition 7, since $\mathfrak{M}$ is stable, it cannot be locally stable. For what concerns $\mathfrak{S}$, consider $C = [4]$, $V = [8]$ and

$$p = \begin{bmatrix} 1 & 2 & 2 & 2 & 3 & 4 & 4 & 4 \\ 3 & 1 & 1 & 3 & 1 & 1 & 2 & 3 \\ 4 & 3 & 3 & 4 & 4 & 2 & 3 & 1 \\ 2 & 4 & 4 & 1 & 2 & 3 & 1 & 2 \end{bmatrix} \tag{2}$$

It is immediately checked that $\{2, 4\}$ is the unique 2-locally stable set while $\mathfrak{S}(C, V, p, 2) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. □

Following Elkind et al. (2017), we say that a CSR $\mathfrak{R}$ satisfies *committee monotonicity* if, for every $(C, V, p) \in \mathcal{E}$ with $|C| \geq 3$ and $k \in [|C| - 2]$, we have that:

(1) if $W \in \mathfrak{R}(C, V, p, k)$, then there exists $W' \in \mathfrak{R}(C, V, p, k+1)$ such that $W \subseteq W'$,

(2) if $W \in \mathfrak{R}(C, V, p, k+1)$, then there exists $W' \in \mathfrak{R}(C, V, p, k)$ such that $W' \subseteq W$.

By Theorem 2 in Elkind et al. (2017), a best-k rule is committee monotone. As a consequence, we immediately get the following result.

Proposition 9. $\mathfrak{S}$ *satisfies committee monotonicity.*

As shown by Barberà and Coelho (2008), stability and committee monotonicity cannot co-exist. In particular, from the example proposed in the proof of Proposition 8 in Barberà and Coelho (2008), we get the next result.

Proposition 10. $\mathfrak{M}$ *fails both condition* (1) *and condition* (2) *of committee monotonicity. In particular,* $\mathfrak{M}$ *fails committee monotonicity.*

As pointed out by Elkind et al. (2017), the fact that committee monotonicity fails is not a matter of concern in itself, because the desirability of that property strongly depends on the application. The above negative result allows to shed light onto the nature of $\mathfrak{M}$.

Proposition 11. $\mathfrak{M}$ *is not a best-k rule.*

Proof. By Theorem 2 in Elkind et al. (2017), every best-k rule is committee-monotone. Since, by Proposition 10, $\mathfrak{M}$ fails committee monotonicity, we deduce that $\mathfrak{M}$ is not a best-k rule. □

Let us consider now three further kinds of monotonicity. A CSR $\mathfrak{R}$ is said to satisfy *candidate monotonicity* (resp. *non-crossing monotonicity*; *membership monotonicity*) if, for every $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$, $W^* \in \mathfrak{R}(C, V, p, k)$, $x^* \in W^*$, and p' obtained by p by shifting x^* one position forward in the preference relation of some voter by swapping its position with some $\hat{x} \in C$, we have that there exists $W' \in \mathfrak{R}(C, V, p', k)$ such that $x^* \in W'$ (resp. $W^* \in \mathfrak{R}(C, V, p', k)$ provided that $\hat{x} \notin W^*$; $W^* \in \mathfrak{R}(C, V, p', k)$). Notice that candidate monotonicity and non-crossing monotonicity are proposed in Elkind et al. (2017), while membership monotonicity in Kamwa (2017a). Note also that membership monotonicity implies non-crossing monotonicity.

In order to study these three properties for $\mathfrak{S}$ and $\mathfrak{M}$, we need to observe that, referring to the notation used in the above definition, for every $W \in 2^C_k$, the following facts hold true:

$$c_{p'}(x^*, \hat{x}) = c_p(x^*, \hat{x}) + 1, \tag{3}$$

$$c_{p'}(\hat{x}, x^*) = c_p(\hat{x}, x^*) - 1, \tag{4}$$

$$\text{if } x, y \in C \text{ with } x \neq y \text{ and } \{x, y\} \neq \{x^*, \hat{x}\}, \text{ then } c_{p'}(x, y) = c_p(x, y), \tag{5}$$

$$S(p', \hat{x}) \in \{S(p, \hat{x}), S(p, \hat{x}) - 1\}, \tag{6}$$

$$S(p', x^*) \in \{S(p, x^*), S(p, x^*) + 1\}, \tag{7}$$

$$\text{if } x \in C \setminus \{x^*, \hat{x}\}, \text{ then } S(p', x) = S(p, x), \tag{8}$$

$$\text{if } \{x^*, \hat{x}\} \subseteq W \text{ or } \{x^*, \hat{x}\} \subseteq C \setminus W, \text{ then } M(p', W) = M(p, W), \tag{9}$$

$$\text{if } x^* \in W, \text{ then } M(p', W) \geq M(p, W), \tag{10}$$

$$\text{if } x^* \notin W, \text{ then } M(p', W) \leq M(p, W). \tag{11}$$

The proofs of (3)-(11) are elementary and thus omitted.

Our next propositions show that $\mathfrak{S}$ satisfies more monotonicity properties than $\mathfrak{M}$. This result is quite interesting. As noticed by Elkind et al. (2017) and Barberà and Coelho (2008), monotonicity should be a desirable requirement in many situations like, for instance, when the finalists of a competition are to be chosen or when some applicants competing for a position at a university are to be short-listed for an interview. We believe that this expresses one of the main values of $\mathfrak{S}$ with regards to $\mathfrak{M}$.

Proposition 12. *$\mathfrak{S}$ satisfies non-crossing monotonicity and candidate monotonicity.*

Proof. Let $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$, $W^* \in \mathfrak{S}(p, k)$, $x^* \in W^*$, and p' obtained by p by shifting x^* one position forward in the preference relation of some voter by swapping its position with some $\hat{x} \in C$.

We first show non-crossing monotonicity. Assume then that $\hat{x} \notin W^*$. We want to show that $W^* \in \mathfrak{S}(p', k)$. Let $x \in W^*$ and $y \notin W^*$. Then, since $x \neq \hat{x}$ and $y \neq x^*$, by (6)-(8), we get

$$S(p', x) \geq S(p, x) \geq S(p, y) \geq S(p', y).$$

We next show candidate monotonicity. This time we may have $\hat{x} \notin W^*$ or $\hat{x} \in W^*$ and we need to exhibit $W' \in \mathfrak{S}(p', k)$ with $x^* \in W'$.

If $\hat{x} \notin W^*$, by non-crossing monotonicity, we know that $W^* \in \mathfrak{S}(p', k)$ and thus it is enough to choose $W' = W^*$.

Let now $\hat{x} \in W^*$. Then, since $W^* \in \mathfrak{S}(p, k)$, we have $S(p, \hat{x}) \geq S(p, y)$ for all $y \in C \setminus W^*$.

Assume first that, for every $y \in C \setminus W^*$, we have

$$S(p, \hat{x}) > S(p, y). \tag{12}$$

We show that, in this case, $W^* \in \mathfrak{S}(p', k)$. Let $x \in W^*$ and $y \in C \setminus W^*$. Then $y \notin \{x^*, \hat{x}\}$ and thus, by (8), we have $S(p, y) = S(p', y)$. If $x \neq \hat{x}$, using (7), (8) and the fact that $W^* \in \mathfrak{S}(p, k)$, we get

$$S(p', x) \geq S(p, x) \geq S(p, y) = S(p', y).$$

Moreover, by (6) and by (12), we also obtain

$$S(p', \hat{x}) \geq S(p, \hat{x}) - 1 \geq S(p, y) = S(p', y).$$

Suppose next that there exists $y' \in C \setminus W^*$ such that

$$S(p, \hat{x}) = S(p, y'). \tag{13}$$

Note that $y' \notin \{x^*, \hat{x}\}$. Define $W' = (W^* \setminus \{\hat{x}\}) \cup \{y'\} \in 2^C_k$. Note that $x^* \in W'$ and that $C \setminus W' = [C \setminus (W^* \cup \{y'\})] \cup \{\hat{x}\}$. We show that $W' \in \mathfrak{S}(p', k)$. Let $x \in W'$ and $y \in C \setminus W'$. Then $y \neq x^*$.

Consider first the case $x \neq y'$. Then $x \in W^* \setminus \{\hat{x}\}$. If $y \neq \hat{x}$, we have that $y \notin \{x^*, \hat{x}\}$ and $y \in C \setminus W^*$.

Thus, using (7), (8) and the fact that $W^* \in \mathfrak{S}(p, k)$, we get $S(p', x) \geq S(p, x) \geq S(p, y) = S(p', y)$. If instead $y = \hat{x}$, using (6)-(8), (13) and the fact that $W^* \in \mathfrak{S}(p, k)$, we have

$$S(p', x) \geq S(p, x) \geq S(p, y') = S(p, \hat{x}) \geq S(p', \hat{x}).$$

It remains to treat the case $x = y'$, showing that $S(p', y') \geq S(p', y)$. If $y \neq \hat{x}$, then $y \in C \setminus W^*$. Thus, recalling that $y' \notin \{x^*, \hat{x}\}$ and using (8), (13) and the fact that $W^* \in \mathfrak{S}(p, k)$, we have

$$S(p', y') = S(p, y') = S(p, \hat{x}) \geq S(p, y) = S(p', y).$$

Finally note that, by (6), (8) and (13), we also have $S(p', y') = S(p, y') = S(p, \hat{x}) \geq S(p', \hat{x})$. □

Proposition 13. $\mathfrak{S}$ *does not satisfy membership monotonicity.*

Proof. Consider $C = [3]$, $V = [3]$ and

$$p = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

Then $\{1,2\} \in \mathfrak{S}(p, 2) = \{\{1,2\}, \{2,3\}\}$. If now $2 \in \{1,2\}$ is swapped with 1 in the preferences of voter 1, then $\{1,2\} \notin \mathfrak{S}(p', 2) = \{\{2,3\}\}$. □

Kamwa (2017a) proved that $\mathfrak{M}$ does not satisfy membership monotonicity and that it satisfies candidate monotonicity. In Proposition 14 we show that also non-crossing monotonicity, a weaker property than membership monotonicity, fails. For the sake of completeness, we prove in Proposition 15 that $\mathfrak{M}$ is candidate monotonic.

Proposition 14. $\mathfrak{M}$ *does not satisfy non-crossing monotonicity. In particular, it does not satisfy membership monotonicity.*

Proof. Consider $C = [4]$, $V = [7]$ and

$$p = \begin{bmatrix} 4 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 4 & 3 & 3 & 3 & 3 & 4 \\ 3 & 3 & 4 & 4 & 4 & 4 & 3 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

Then $\{1,2\} \in \mathfrak{M}(p, 2) = 2^C_2$. If now $1 \in \{1,2\}$ is swapped with $4 \notin \{1,2\}$ in the preferences of voter 1, then $\{1,2\} \notin \mathfrak{M}(p', 2) = \{\{1,3\}\}$. □

Proposition 15. $\mathfrak{M}$ *satisfies candidate monotonicity.*

Proof. Let $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$, $W^* \in \mathfrak{M}(p, k)$, $x^* \in W^*$, and p' obtained by p by shifting x^* one position forward in the preference relation of some voter by swapping its position with some $\hat{x} \in C$. By (9)-(11), we have that for every $W \in 2^C_k$ such that $x^* \in W$, we have $M(p', W) \geq M(p, W)$; for every $W \in 2^C_k$ such that $x^* \notin W$, we have $M(p', W) \leq M(p, W)$.

Thus, using the fact that $W^* \in \mathfrak{M}(p, k)$, we obtain that, for every $W \in 2^C_k$ with $x^* \notin W$,

$$M(p', W^*) \geq M(p, W^*) \geq M(p, W) \geq M(p', W).$$

Assume now, by contradiction, that $x^* \notin W'$ for all $W' \in \mathfrak{M}(p', k)$. Pick $W' \in \mathfrak{M}(p', k)$. By what shown above we have then that $M(p', W^*) \geq M(p', W') \geq M(p', W^*)$. Thus $M(p', W^*) = M(p', W')$, which says that $W^* \in \mathfrak{M}(p', k)$ against the fact that $x^* \in W^*$. □

Following Elkind et al. (2017) and Kamwa (2017a), we say that $\mathfrak{R}$ satisfies *consistency* (resp. *weak consistency*) if, for every pair of elections $E_1 = (C, V_1, p_1)$ and $E_2 = (C, V_2, p_2)$ over a common set C of candidates and with $V_1 \cap V_2 = \varnothing$ and every $k \in [|C| - 1]$, if $\mathfrak{R}(E_1, k) \cap \mathfrak{R}(E_2, k) \neq \varnothing$, then $\mathfrak{R}(E_1, k) \cap \mathfrak{R}(E_2, k) = \mathfrak{R}(E_1 + E_2, k)$ (resp. $\mathfrak{R}(E_1, k) \cap \mathfrak{R}(E_2, k) \subseteq \mathfrak{R}(E_1 + E_2, k)$), where $E_1 + E_2$ denotes the election $(C, V_1 \cup V_2, p_1 + p_2)$ with $p_1 + p_2$ defined as the preference profile given by $(p_1 + p_2)(v) = p_1(v)$ for all $v \in V_1$ and $(p_1 + p_2)(v) = p_2(v)$ for all $v \in V_2$. Of course, consistency implies weak consistency.

Kamwa (2017a, Theorem 6) proves that $\mathfrak{M}$ fails consistency, even though from the proof we deduce that the weak consistency property fails too. Below we show, using a same example, that both $\mathfrak{S}$ and $\mathfrak{M}$ do not satisfy weak consistency.

Proposition 16. $\mathfrak{S}$ *and* $\mathfrak{M}$ *fail weak consistency. In particular, they fail consistency.*

Proof. Consider $C = [4]$, $V_1 = \{1, \ldots, 11\}$ and $V_2 = \{12, \ldots, 22\}$. Consider then

$$p_1 = \begin{bmatrix} 3 & 1 & 2 & 3 & 1 & 2 & 4 & 4 & 4 & 4 & 4 \\ 2 & 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 1 & 2 & 3 & 1 & 2 \end{bmatrix} \tag{14}$$

$$p_2 = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \\ 2 & 3 & 1 & 2 & 3 & 1 & 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 1 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 1 & 2 & 2 & 1 \end{bmatrix} \tag{15}$$

where, for every $v \in \{1, \ldots, 11\}$, $p_1(v)$ is the v-th column of p_1 and, for every $v \in \{12, \ldots, 22\}$, $p_2(v)$ is the $(v - 11)$-th column of p_2. Then $\mathfrak{S}(C, V_1, p_1, 1) = \mathfrak{M}(C, V_1, p_1, 1) = \mathfrak{S}(C, V_2, p_2, 1) = \mathfrak{M}(C, V_2, p_2, 1) = \{\{4\}\}$ while $\mathfrak{S}(C, V_1 \cup V_2, p_1 + p_2, 1) = \mathfrak{M}(C, V_1 \cup V_2, p_1 + p_2, 1) = \{\{1\}, \{2\}, \{3\}\}$. □

The nature of $\mathfrak{S}$ and $\mathfrak{M}$ can be now better enlightened.

Proposition 17. *$\mathfrak{S}$ and $\mathfrak{M}$ are not committee scoring rules.*

Proof. By Theorem 7 in Elkind et al. (2017), every committee scoring rule satisfies consistency. Since, by Proposition 16, $\mathfrak{S}$ and $\mathfrak{M}$ fail consistency, we deduce that $\mathfrak{S}$ and $\mathfrak{M}$ are not committee scoring rules. □

We say that $\mathfrak{R}$ satisfies *resolvability* (resp. *weak resolvability*) if, for every $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$, $W^* \in \mathfrak{R}(C, V, p, k)$ and $q^* \in \mathcal{L}(C)$ such that, for every $x \in W^*$ and $y \in C \setminus W^*$, $x \succ_{q^*} y$, we have that if $v^* \in \mathbf{V} \setminus V$ and $(C, V^*, p^*) \in \mathcal{E}$ is defined by $V^* = V \cup \{v^*\}$, $p^*(v) = p(v)$ for all $v \in V$ and $p^*(v^*) = q^*$, then $\mathfrak{R}(C, V^*, p^*, k) = \{W^*\}$ (resp. $W^* \in \mathfrak{R}(C, V^*, p^*, k)$). Of course, resolvability implies weak resolvability. Note that we are adapting here the standard notion of resolvability for single-winner voting rules (see, for instance, Schulze, 2011; Tideman, 1987, 2006) to the committee selection setting.

Proposition 18. *$\mathfrak{S}$ does not satisfy weak resolvability. In particular, it does not satisfy resolvability.*

Proof. Let p be the preference profile considered in Proposition 14. It is easily checked that $S(C, V, p, x) = 3$ for all $x \in C = [4]$. Thus $\{2, 3\} \in \mathfrak{S}(C, V, p, 2) = 2^C_2$. Let q^* be the linear order $3 \succ 2 \succ 1 \succ 4$. Since $\mathfrak{S}(C, V^*, p^*, 2) = \{\{1, 3\}\}$, we see that $\{2, 3\} \notin \mathfrak{S}(C, V^*, p^*, 2)$. This shows that $\mathfrak{S}$ does not satisfy weak resolvability. □

Proposition 19. *$\mathfrak{M}$ satisfies weak resolvability and does not satisfy resolvability.*

Proof. We show that, for every $W \in 2^C_k$, we have $M(C, V^*, p^*, W) \leq M(C, V^*, p^*, W^*)$. Note that, for every $x, y \in C$ with $x \neq y$, we have $c_{p^*}(x, y) \leq c_p(x, y) + 1$. Thus, we also have $M(C, V^*, p^*, W) \leq M(C, V, p, W) + 1$. On the other hand, for every $x \in W^*$ and $y \in C \setminus W^*$ we have $c_{p^*}(x, y) = c_p(x, y) + 1$ and therefore $M(C, V, p, W^*) + 1 = M(C, V^*, p^*, W^*)$. Since $M(C, V, p, W) \leq M(C, V, p, W^*)$, we then get

$$M(C, V^*, p^*, W) \leq M(C, V, p, W) + 1 \leq M(C, V, p, W^*) + 1 = M(C, V^*, p^*, W^*).$$

We now investigate resolvability. Let p be the preference profile considered in Proposition 14. Let $\{1, 2\} \in \mathfrak{M}(C, V, p, 2) = 2^C_2$ and let q^* be the linear order $2 \succ 1 \succ 3 \succ 4$. Then $\{1, 2\}$ is not the only element in $\mathfrak{M}(C, V^*, p^*, 2)$ because, for instance, we also have $\{1, 3\} \in \mathfrak{M}(C, V^*, p^*, 2)$. Thus $\mathfrak{M}$ does not satisfy resolvability. □

Note that, since $\mathfrak{M}$ satisfies the property of fixed majority, if one keeps adding voters whose preferences are described by a linear order $q^* \in \mathcal{L}(C)$ such that, for every $x \in W^*$ and $y \notin W^*$, $x^* \succ_{q^*} y^*$, then in a finite number of steps we reach the goal of having W^* as the only outcome of

$\mathfrak{M}$. As it can be immediately understood by the proof of Proposition 2, this does not hold true, in general, for $\mathfrak{S}$.

According to Kamwa (2017a), we say that a CSR $\mathfrak{R}$ satisfies the *Pareto criterion* (resp. *weak Pareto criterion*) if, for every $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$, $W^* \in \mathfrak{R}(C,V,p,k)$ and $x^*, y^* \in C$ with $x^* \neq y^*$ and $c_p(x^*,y^*) = |V|$, having $y^* \in W^*$ implies $x^* \in W^*$ (resp. $x^* \in W'$ for some $W' \in \mathfrak{R}(C,V,p,k)$). Note that if $k=1$ the definition means that if y^* is unanimously beaten by x^* then it cannot be selected (resp. it cannot be the unique winner). Clearly if $\mathfrak{R}$ satisfies the Pareto criterion, then it satisfies the weak Pareto criterion. Here we study those properties for $\mathfrak{S}$ and $\mathfrak{M}$. Notice that the Pareto criterion has already been proved for $\mathfrak{M}$ in Kamwa (2017a). Anyway, for the sake of completeness, we also provide the proof of that fact below.

Proposition 20. *$\mathfrak{S}$ satisfies the weak Pareto criterion but does not satisfy the Pareto criterion.*

Proof. We first show that $\mathfrak{S}$ satisfies the weak Pareto criterion. Let $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$, $W^* \in \mathfrak{S}(C,V,p,k)$, $y^* \in W^*$ and $x^* \in C$, with $x^* \neq y^*$, such that $c_p(x^*,y^*) = |V|$. We need to find $W' \in \mathfrak{S}(C,V,p,k)$ such that $x^* \in W'$. If $x^* \in W^*$ we simply take $W' = W^*$. Assume instead that $x^* \notin W^*$. Note that $c_p(x^*,y^*) = |V|$ implies $c_p(y^*,x^*) = 0$ and thus $S(p,y^*) = 0$. By $y^* \in W^*$ we also have $S(p,y^*) \geq S(p,y)$ for all $y \in C \setminus W^*$. It follows that $S(p,y) = 0$ for all $y \in C \setminus W^*$. Define then $W' = (W^* \setminus \{y^*\}) \cup \{x^*\}$. We show that $W' \in \mathfrak{S}(C,V,p,k)$. Let $z \in W'$ and $t \in C \setminus W'$. Since $C \setminus W' = [C \setminus (W^* \cup \{x^*\})] \cup \{y^*\}$, we have that $t \in C \setminus W^*$ or that $t = y^*$. In both cases, we know that $S(p,t) = 0$ and thus, trivially, $S(p,z) \geq S(p,t)$.

In order to see that $\mathfrak{S}$ does not satisfy the Pareto criterion, consider (C,V,p) with $C = [3]$, $k = 2$ and assume that $1 \succ_{p(v)} 2 \succ_{p(v)} 3$ for all $v \in V$. Then $\{1,3\} \in \mathfrak{S}(p,2)$. Now we have that $c_p(2,3) = |V|$, and $3 \in \{1,3\}$. However $2 \notin \{1,3\}$. □

Proposition 21. *$\mathfrak{M}$ satisfies the Pareto criterion. In particular, $\mathfrak{M}$ satisfies the weak Pareto criterion.*

Proof. Assume, by contradiction, that there exists $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$, $W^* \in \mathfrak{M}(p,k)$ and $x^*, y^* \in C$ with $x^* \neq y^*$ such that $c_p(x^*,y^*) = |V|$, $y^* \in W^*$ and $x^* \notin W^*$. Since $c_p(y^*,x^*) = 0$, we have that $M(p,W^*) = 0$. Consider now $v^* \in V$ and $W \in 2^C_k$ such that, for every $x \in W$ and $y \in C \setminus W$, $x \succ_{p(v^*)} y$. Then, for every $x \in W$ and $y \in C \setminus W$, we have $c_p(x,y) > 0$ and hence $M(p,W) > 0$ so that $W^* \notin \mathfrak{M}(p,k)$ and the contradiction is found. □

Let $E = (C,V,p) \in \mathcal{E}$ and $\mathfrak{R}$ be a CSR. Given $k \in [|C|-1]$, we say that $\mathfrak{R}$ is *k-flat* on E if $\mathfrak{R}(C,V,p,k) = 2^C_k$. We say that $\mathfrak{R}$ is *flat* on E if it is k-flat on E for all $k \in [|C|-1]$. Moreover, $\mathfrak{R}$ is called *symmetric* if it is flat on every $(C,V,p) \in \mathcal{E}$ such that $c_p(x,y) = c_p(y,x)$ for all $x,y \in C$, with $x \neq y$. Note that a preference profile p satisfying the above symmetry request on c_p is given, for instance, by any p such that p and p^r agrees up to a reordering of the names of the voters. The definitions of flatness and symmetry here considered are inspired by the properties by the same name introduced by González-Díaz et al. (2014) in the framework of ranking methods.

We stress that being often flat is considered a defect because if $\mathfrak{R}$ is flat on an election, then it cannot be used to make an effective decision. On the other hand, being flat on the elections with profiles satisfying $c_p(x,y) = c_p(y,x)$ is considered a value. Indeed if the voters collectively do not discriminate between any couple of candidates, it is natural to require that every k-set of candidates must occur as outcome of the voting procedure.

We illustrate now the quite different behaviour of $\mathfrak{S}$ and $\mathfrak{M}$ with respect to flatness and show that they are both symmetric.

Proposition 22. *Let $(C,V,p) \in \mathcal{E}$. The following facts are equivalent:*

(i) *$\mathfrak{S}$ is flat on (C,V,p);*

(ii) *$\mathfrak{S}$ is k-flat on (C,V,p) for some $k \in [|C|-1]$;*

(iii) *the function $S(C,V,p,\cdot) : C \to \mathbb{N}_0$ is constant.*

Proof. $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$ By assumption, we have that $\mathfrak{S}(p,k) = 2^C_k$. Since this can be interpreted as $C_k(R_S(C,V,p)) = 2^C_k$, we can apply Proposition 26 in Bubboloni and Gori (2018) and deduce that $R_S(C,V,p) = C^2$, that is, $S(C,V,p,\cdot)$ is constant.

$(iii) \Rightarrow (i)$ is trivial. □

Proposition 23. *Let $E = (C,V,p) \in \mathcal{E}$. If $\mathfrak{M}$ is flat on E, then $\mathfrak{S}$ is flat on E. The converse does not hold.*

Proof. Let $\mathfrak{M}$ be flat on E. Then $\mathfrak{M}$, in particular, is 1-flat on E. Since $\mathfrak{S}$ and $\mathfrak{M}$ for $k = 1$ coincide, we have that $\mathfrak{S}$ is 1-flat on E. Thus, by Proposition 22, we deduce that $\mathfrak{S}$ is flat.

In order to show that $\mathfrak{S}$ flat on E does not imply $\mathfrak{M}$ flat on E, we consider the election $E = (C,V,p)$ given by $C = [4]$, $V = [5]$ and

$$p = \begin{bmatrix} 2 & 1 & 3 & 4 & 4 \\ 1 & 2 & 1 & 3 & 2 \\ 3 & 3 & 2 & 1 & 3 \\ 4 & 4 & 4 & 2 & 1 \end{bmatrix}.$$

It is immediately checked that $S(p,x) = 2$ for all $x \in C$ and so, by Proposition 22, $\mathfrak{S}$ is flat on E while $\mathfrak{M}(p,3) = \{\{1,2,3\}\}$. Thus $\mathfrak{M}$ is not 3-flat on E. In particular, $\mathfrak{M}$ is not flat on E. □

By the above result, the set of elections in which $\mathfrak{S}$ is flat is larger than the set of elections in which $\mathfrak{M}$ is flat. This represents a critical issue for $\mathfrak{S}$ with respect to $\mathfrak{M}$.

Corollary 24. *The property $\mathfrak{M}$ is flat on $E = (C,V,p)$ is not equivalent to the property $\mathfrak{M}$ is k-flat on E for some $k \in [|C| - 1]$.*

Proof. The example used in Proposition 23 exhibits a case in which $\mathfrak{M}$ is 1-flat but not 3-flat. □

Proposition 25. *The property $\mathfrak{M}$ is k-flat on $E = (C,V,p)$, for some $k \in [|C| - 1]$, does not imply the property $\mathfrak{S}$ is flat on E.*

Proof. Consider the election $E = (C,V,p)$ given by $C = [4]$, $V = [8]$ and p defined in (2). It is immediately checked that $\mathfrak{M}(p,2) = 2^C_2$ so that $\mathfrak{M}$ is 2-flat on E. On the other hand, we have $S(p,1) = 4, S(p,2) = S(p,3) = S(p,4) = 3$. Thus, $\mathfrak{S}$ is not flat on E. □

Proposition 26. *$\mathfrak{S}$ and $\mathfrak{M}$ are symmetric.*

Proof. Let $(C,V,p) \in \mathcal{E}$ such that $c_p(x,y) = c_p(y,x)$ for all $x,y \in C$, with $x \neq y$, and let $k \in [|C| - 1]$. Let $x,y \in C$ with $x \neq y$. Since the preferences in the profile p are linear orders, we have $c_p(x,y) + c_p(y,x) = |V|$. Thus $|V|$ is even and $c_p(x,y) = \frac{|V|}{2}$. In particular, c_p is constant. As a consequence, for every $W \in 2^C_k$ and for every $x \in C$, we have that $M(p,W) = \frac{|V|}{2}$ and $S(p,x) = \frac{|V|}{2}$ so that $\mathfrak{M}(p,k) = \mathfrak{S}(p,k) = 2^C_k$. □

Following Elkind et al. (2017), we say that $\mathfrak{R}$ satisfies *consensus committee* if, for every $(C,V,p) \in \mathcal{E}$, $k \in [|C| - 1]$ and $W \in 2^C_k$ such that each voter ranks some member of W first and each member of W is ranked first by either $\left\lfloor \frac{|V|}{k} \right\rfloor$ or $\left\lceil \frac{|V|}{k} \right\rceil$ voters, we have that $\mathfrak{R}(C,V,p,k) = \{W\}$.

Proposition 27. *$\mathfrak{S}$ and $\mathfrak{M}$ do not satisfy consensus committee.*

Proof. Consider $C = [3]$, $V = [2]$,

$$p = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 2 & 1 \end{bmatrix}$$

and $W = \{1,2\}$. Any CSR satisfying consensus committee should select W as unique winning committee for the election $E = (C,V,p)$. However, both $\mathfrak{S}$ and $\mathfrak{M}$ are 2-flat on E. □

We finally show that $\mathfrak{S}$ and $\mathfrak{M}$ both fail to be resolute and suffer the reversal bias, the Condorcet loser paradox and the leaving member paradox.[7] Recall that a CSR $\mathfrak{R}$ is *resolute* if, for every $(C,V,p) \in \mathcal{E}$ and $k \in [|C|-1]$, $|\mathfrak{R}(C,V,p,k)| = 1$; $\mathfrak{R}$ suffers the *reversal bias* if there exist $(C,V,p) \in \mathcal{E}$ and $k \in [|C|-1]$ such that $|\mathfrak{R}(C,V,p,k)| = 1$ and $\mathfrak{R}(C,V,p,k) = \mathfrak{R}(C,V,p^r,k)$; $\mathfrak{R}$ suffers the *Condorcet loser paradox* if there exist $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$ and $W^* \in 2^C_k$ such that W^* is a k-Condorcet loser set for (C,V,p), that is, for every $x \in W^*$ and $y \in C \setminus W^*$, $c_p(x,y) < \frac{|V|}{2}$, and $\mathfrak{R}(C,V,p,k) = \{W^*\}$; $\mathfrak{R}$ suffers the *leaving member paradox* if there exist $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$ with $k \neq 1$, and $x^* \in C$ such that $\mathfrak{R}(C,V,p,k) = \{W^*\}$, $x^* \in W^*$, $\mathfrak{R}(C',V,p',k) = \{W'\}$ and $W' \cap W^* = \varnothing$, where $C' = C \setminus \{x^*\}$ and p' is the preference profile obtained by p erasing the candidate x^*.[8]

Proposition 28. *$\mathfrak{S}$ and $\mathfrak{M}$ do not satisfy resoluteness.*

Proof. Consider $C = [2]$, $V = [2]$, and

$$p = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Then $\mathfrak{S}(p,1) = \mathfrak{M}(p,1) = \{\{1\},\{2\}\}$. □

Proposition 29. *$\mathfrak{S}$ and $\mathfrak{M}$ suffer the reversal bias.*

Proof. Consider $C = [4]$, $V = [11]$, and p as in (14). Then $\mathfrak{S}(p,1) = \mathfrak{M}(p,1) = \{\{4\}\}$ and $\mathfrak{S}(p^r,1) = \mathfrak{M}(p^r,1) = \{\{4\}\}$. □

Proposition 30. *$\mathfrak{S}$ and $\mathfrak{M}$ suffer the Condorcet loser paradox.*

Proof. Consider $C = [4]$, $V = [11]$, and p as in (14). Then $\{4\}$ is a 1-Condorcet loser set for (C,V,p) and $\mathfrak{S}(p,1) = \mathfrak{M}(p,1) = \{\{4\}\}$. □

Note that, according to Kamwa (2017a), a CSR $\mathfrak{R}$ satisfies the *Condorcet loser criterion* if, for every $(C,V,p) \in \mathcal{E}$, $k \in [|C|-1]$, $W \in \mathfrak{R}(C,V,p,k)$ and $x \in C$ a Condorcet loser, we have that $x \notin W$. The above proposition shows that $\mathfrak{S}$ and $\mathfrak{M}$ fail the Condorcet loser criterion. This fact for $\mathfrak{M}$ was proved in Kamwa (2017a, Theorem 2).

Proposition 31. *$\mathfrak{S}$ and $\mathfrak{M}$ suffer the leaving member paradox.*

Proof. Consider first $\mathfrak{S}$ with $C = [4]$, $V = [4]$, and

$$p = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & 4 & 4 & 1 \\ 4 & 2 & 3 & 3 \\ 2 & 3 & 2 & 4 \end{bmatrix}$$

Then $\mathfrak{S}(p,2) = \{\{1,2\}\}$. Assume now that 1 drops out of the electoral competition for some reason and consider $(C',V,p') \in \mathcal{E}$, where $C' = \{2,3,4\}$ and p' is obtained by p by erasing candidate 1 in each column. Then, we get $\mathfrak{S}(p',2) = \{\{3,4\}\}$.

Consider now $\mathfrak{M}$ with $C = [4]$, $V = [11]$, and

$$p = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 1 & 1 & 4 & 2 & 2 & 4 & 4 & 1 \\ 2 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 2 & 2 & 4 & 4 & 3 & 4 & 4 & 1 & 1 & 3 \end{bmatrix}$$

Then $\mathfrak{M}(p,2) = \{\{1,2\}\}$. Assume now that 1 drops out of the competition for some reason and consider $(C',V,p') \in \mathcal{E}$, where $C' = \{2,3,4\}$ and p' is obtained by p by erasing candidate 1 in each column. Then, we get $\mathfrak{M}(p',2) = \{\{3,4\}\}$. □

[7] Those properties are largely studied in the literature. See for instance, Bubboloni and Gori (2016), Diss and Doghmi (2016), Diss and Gehrlein (2012), Diss and Tlidi (2018), Duggan and Schwartz (2000), Fishburn and Gehrlein (1976), Gehrlein and Lepelley (2010b), Jeong and Ju (2017), Kamwa and Merlin (2015), Saari and Barney (2003), and Staring (1986).

[8] Note that if k were allowed to be 1, then every CSR would suffer the leaving member paradox.

A summary of our results is provided in Table 1.

Table 1: Summary

Properties	$\mathfrak{M}$	$\mathfrak{S}$
Anonymity	✓	✓
Neutrality	✓	✓
Homogeneity	✓	✓
Non-Imposition	✓	✓
Strong Non-Imposition*	✓	✗
Weak Unanimity	✓	✓
Strong Unanimity	✓	✗
Fixed Majority	✓	✗
Consensus Committee	✗	✗
Weak Pareto criterion*	✓	✓
Pareto criterion	✓	✗
Condorcet Consistency	✓	✗
Stability	✓	✗
Local Stability	✗	✗
Condorcet Winner Criterion	✗	✓
Immunity to the Condorcet Loser Paradox	✗	✗
Condorcet Loser Criterion	✗	✗
Committee Monotonicity	✗	✓
Membership Monotonicity	✗	✗
Non-crossing Monotonicity	✗	✓
Candidate Monotonicity	✓	✓
Weak Consistency	✗	✗
Consistency	✗	✗
Weak Resolvability*	✓	✗
Resolvability*	✗	✗
Symmetry*	✓	✓
Resoluteness	✗	✗
Immunity to the Reversal Bias	✗	✗
Immunity to the Leaving Member Paradox	✗	✗

The properties marked with * are new in the committee selection literature.
The other properties have already been introduced and studied for other CSRs.

3.1 Some further propositions

Let us present here other results that will turn out to be useful in the next section.

Proposition 32. *Let* $(C, V, p) \in \mathcal{E}$, $k \in [|C| - 1]$ *and* $W^* \in 2^C_k$. *Then* $\mathfrak{S}(p, k) = \{W^*\}$ *if and only if, for every* $x \in W^*$ *and* $y \in C \setminus W^*$, $S(p, x) > S(p, y)$.

Proof. Let $\mathfrak{S}(p, k) = \{W^*\}$. Assume, by contradiction, that there exists $x^* \in W^*$ and $y^* \in C \setminus W^*$ such that $S(p, x^*) \leq S(p, y^*)$. Since for every $x \in W^*$ and $y \in C \setminus W^*$, we surely have $S(p, x) \geq$

$S(p,y)$, we then have

$$S(p,x^*) = S(p,y^*). \tag{16}$$

Define $W' = (W^* \setminus \{x^*\}) \cup \{y^*\} \in 2^C_k$. We reach the desired contradiction showing that $W' \in \mathfrak{S}(p,k)$. Let $x \in W'$ and $y \in C \setminus W' = C \setminus (W^* \cup \{x^*\}) \cup \{x^*\}$. We see that $s(p,x) \geq S(p,y)$. Let first $x \neq y^*$ and $y \neq x^*$. Then we have $x \in W^*$ and $y \in C \setminus W^*$ and thus, since $W^* \in \mathfrak{S}(p,k)$, we get $S(p,x) \geq S(p,y)$. Let next $x = y^*$ and $y = x^*$. Then by (16), $S(p,x) = S(p,y^*) = S(p,x^*) = S(p,y)$. Let now $x = y^*$ and $y \neq x^*$. Then $y \in C \setminus W^*$ and, by (16) and $W^* \in \mathfrak{S}(p,k)$, we obtain

$$S(p,x) = S(p,y^*) = S(p,x^*) \geq S(p,y).$$

Let finally be $x \neq y^*$ and $y = x^*$. Then $x^* \in W^*$ and $y \in C \setminus W^*$. Hence, by $W^* \in \mathfrak{S}(p,k)$ and (16), we get

$$S(p,x) \geq S(p,y^*) = S(p,x^*) = S(p,y).$$

Assume next that $W^* \in 2^C_k$ is such that for every $x \in W^*$ and $y \in C \setminus W^*$,

$$S(p,x) > S(p,y). \tag{17}$$

Then $W^* \in \mathfrak{S}(p,k)$. We want to show that W^* is the only element in $\mathfrak{S}(p,k)$. Pick $W \in 2^C_k \setminus \{W^*\}$. Since W^* and W have the same size k, there exist $y \in W \setminus W^*$ and $x \in W^* \setminus W$. Thus, by (17), we have $S(p,x) > S(p,y)$, which excludes $W \in \mathfrak{S}(p,k)$. □

Proposition 33. *Let $(C,V,p) \in \mathcal{E}$ be such that $|\mathfrak{S}(p,|C|-1)| = |\mathfrak{S}(p^r,|C|-1)| = 1$. Then $\mathfrak{S}(p,|C|-1) \neq \mathfrak{S}(p^r,|C|-1)$.*

Proof. Assume by contradiction that there exists $W \subseteq C$ with $|W| = |C|-1$ such that

$$\mathfrak{S}(p,|C|-1) = \mathfrak{S}(p^r,|C|-1) = \{W\}.$$

Let x^* be the unique element of the set $C \setminus W$. Then, by Proposition 32, we have that, for every $x \in W$, $S(p,x^*) < S(p,x)$ and $S(p^r,x^*) < S(p^r,x)$. Since, for every $x,y \in C$ with $x \neq y$, $c_p(y,x) = |V| - c_p(x,y)$ and $c_{p^r}(x,y) = c_p(y,x)$, we obtain that, for every $x \in W$,

$$\min_{y \neq x^*} c_p(x^*,y) < \min_{y \neq x} c_p(x,y) \quad \text{and} \quad \max_{y \neq x^*} c_p(x^*,y) > \max_{y \neq x} c_p(x,y).$$

Thus, there exist $y_m, y_M \in W$ such that, for every $x \in W$,

$$c_p(x^*,y_m) < \min_{y \neq x} c_p(x,y) \quad \text{and} \quad c_p(x^*,y_M) > \max_{y \neq x} c_p(x,y).$$

Then, for every $x \in W$ and $y \in C \setminus \{x\}$, $c_p(x^*,y_m) < c_p(x,y) < c_p(x^*,y_M)$. In particular, we have that

$$c_p(x^*,y_m) < c_p(y_m,x^*) < c_p(x^*,y_M) \quad \text{and} \quad c_p(x^*,y_m) < c_p(y_M,x^*) < c_p(x^*,y_M),$$

so that

$$c_p(x^*,y_m) < |V| - c_p(x^*,y_m) < c_p(x^*,y_M) \quad \text{and} \quad c_p(x^*,y_m) < |V| - c_p(x^*,y_M) < c_p(x^*,y_M).$$

Thus, we have that $|V| - c_p(x^*,y_m) < c_p(x^*,y_M)$ and $c_p(x^*,y_m) < |V| - c_p(x^*,y_M)$ from which we deduce the contradiction $|V| < c_p(x^*,y_m) + c_p(x^*,y_M) < |V|$. □

Lemma 34. *Let $(C,V,p) \in \mathcal{E}$ and $W \subseteq C$ with $|W| \in [|C|-1]$. Then $M(p,W) = M(p^r, C \setminus W)$.*

Proof. Simply observe that $C \setminus W \in C$, $|C \setminus W| \in [|C|-1]$ and that

$$M(p,W) = \min_{x \in W, y \in C \setminus W} c_p(x,y) = \min_{x \in W, y \in C \setminus W} c_{p^r}(y,x) = M(p^r, C \setminus W).$$

□

Proposition 35. *Let* $(C,V,p) \in \mathcal{E}$*,* $k \in [|C|-1]$ *and* $W^* \in 2^C_k$*. Then* $W^* \in \mathfrak{M}(p,k)$ *if and only if* $C \setminus W^* \in \mathfrak{M}(p^r,|C|-k)$*. In particular,* $\mathfrak{M}(p,k) = \{W^*\}$ *if and only if* $\mathfrak{M}(p^r,|C|-k) = \{C \setminus W^*\}$*.*

Proof. Consider the following statements:

1. $W^* \in \mathfrak{M}(p,k)$;
2. for every $W \in 2^C_k$, $M(p,W^*) \geq M(p,W)$;
3. for every $W \in 2^C_k$, $M(p^r, C\setminus W^*) \geq M(p^r, C \setminus W)$;
4. for every $W \in 2^C_{|C|-k}$, $M(p^r, C\setminus W^*) \geq M(p^r, W)$;
5. $C \setminus W^* \in \mathfrak{M}(p^r,|C|-k)$.

It is immediate to show that, for every $i \in \{1,2,3,4\}$, statement i is equivalent to statement $i+1$. Then statement 1 is equivalent to statement 5 and the proof is completed. □

Corollary 36. *Let* $(C,V,p) \in \mathcal{E}$ *be such that* $|C|$ *is even and* $|\mathfrak{M}(p,|C|/2)| = |\mathfrak{M}(p^r,|C|/2)| = 1$*. Then* $\mathfrak{M}(p,|C|/2) \neq \mathfrak{M}(p^r,|C|/2)$*.*

Proof. Assume that $\mathfrak{M}(p,|C|/2) = \{W^*\}$. Then by Proposition 35 we have that $\mathfrak{M}(p^r,|C|-|C|/2) = \mathfrak{M}(p^r,|C|/2) = \{C \setminus W^*\}$. Since $W^* \neq C \setminus W^*$, the proof is complete. □

Corollary 37. *Let* $(C,V,p) \in \mathcal{E}$ *and* $k \in [|C|-1]$*. The following conditions are equivalent:*

(i) $\mathfrak{M}(p,k) = \mathfrak{M}(p^r,k)$ *and* $|\mathfrak{M}(p,k)| = 1$*;*

(ii) $\mathfrak{M}(p,|C|-k) = \mathfrak{M}(p^r,|C|-k)$ *and* $|\mathfrak{M}(p,|C|-k)| = 1$*.*

Proof. Assume that (i) holds true. Then there exists $W^* \in 2^C_k$ such that $\mathfrak{M}(p,k) = \mathfrak{M}(p^r,k) = \{W^*\}$. Then, by Proposition 35, we have that $\mathfrak{M}(p,|C|-k) = \mathfrak{M}(p^r,|C|-k) = \{C \setminus W^*\}$ which implies (ii). The proof that (ii) implies (i) is analogous. □

4 Computational results

In this section we better investigate the resoluteness as well as the properties of suffering the reversal bias, the Condorcet loser paradox and the leaving member paradox for $\mathfrak{S}$ and $\mathfrak{M}$. We fix the number $m \in \{3,4\}$ of candidates, the number $n \geq 2$ of the voters and the size $k \in [m-1]$ of the committee to be selected and we focus only on elections (C,V,p) with $|C| = m$ and $|V| = n$.[9] We then study each of the considered properties through a computational approach estimating the probability of its occurrence for $\mathfrak{S}$ and $\mathfrak{M}$ and providing a comparison of the results obtained with those obtained for the Borda, the Plurality, the Negative Plurality and the Bloc CSRs.[10] We emphasize that the Bloc CSR agrees with the Plurality CSR when $k=1$ and with the Negative Plurality CSR when $k=m-1$. Recall also that $\mathfrak{S}$ agrees with $\mathfrak{M}$ when $k=1$. In the last part of the section we also study the probability that $\mathfrak{S}$ and $\mathfrak{M}$ agree on the same unique committee as well as the probability of the agreement of each of them with the Borda CSR. Before giving the results of our analysis, we first describe in Section 4.1 the methodology applied in order to calculate our probabilities.

[9] Note that when, $m \leq 2$ or $n = 1$, the analysis of the considered properties turns out to be straightforward.

[10] See Diss and Doghmi (2016) and Elkind et al. (2017) for the definitions of these CSRs.

4.1 Evaluating the probability of voting situations

Let us first present our main computational assumption, that is, the Impartial Anonymous Culture (IAC). With m candidates, there are $m!$ possible individual preferences (linear orders) which we order in some fashion. A voting situation is a vector $(n_1, \ldots, n_i, \ldots, n_{m!})$ such that $\sum_{i=1}^{m!} n_i = n$, where the integer n_i is the number of individuals voting the i-th linear order. The IAC condition stipulates that all voting situations are equiprobable. This assumption, introduced by Gehrlein and Fishburn (1976), is one of the most used assumption in social choice theory when computing the theoretical probability of electoral events. For more details on the IAC condition and other well-known assumptions, we refer the reader to the recent books by Gehrlein and Lepelley (2011, 2017). Under IAC, obtaining the probability of an electoral event is accomplished by the computation of two elements. The first one is the total number of voting situations, that is, $\binom{n+m!-1}{m!-1}$. The second one is the number of voting situations associated with the property under examination. This can be in general reduced to a finite system of linear constraints with rational coefficients. For instance, if $C = \{x, y, z\}$, there exist 3!=6 linear orders on C: $x \succ y \succ z$ (n_1), $x \succ z \succ y$ (n_2), $y \succ x \succ z$ (n_3), $y \succ z \succ x$ (n_4), $z \succ x \succ y$ (n_5), and $z \succ y \succ x$ (n_6). In this setting, evaluating the situations for which the Plurality CSR selects $\{x, y\}$ as unique outcome for a given number of voters n is equivalent to finding the number of voting situations $(n_1, \ldots, n_6)$ subject to the following conditions: $n_1 + n_2 - n_5 - n_6 > 0$ (the Plurality score of candidate x is greater than the one of z), $n_3 + n_4 - n_5 - n_6 > 0$ (the Plurality score of y is greater than the one of z), $n_i \geq 0$ for each $i \in [6]$, and $\sum_{i=1}^{6} n_i = n$.

As recently pointed out in the literature of social choice theory, Ehrhart polynomials are the appropriate mathematical tool to study such problems (Gehrlein and Lepelley, 2011, 2017; Lepelley et al., 2008; Wilson and Pritchard, 2007). In fact, they have been widely used in numerous studies analyzing the probability of electoral events in the case of three-candidate elections under IAC assumption (Courtin et al., 2015; Diss, 2015; Diss et al., 2012; Gehrlein and Lepelley, 2011, 2017; Gehrlein et al., 2015, 2016, 2018; Kamwa and Valognes, 2017; Lepelley et al., 2017; Smaoui et al., 2016). There exist strong algorithms that enable to specify the Ehrhart polynomials for many problems in the case of three-candidate elections. In this paper, all our results for the case of three candidates are obtained by using the parameterized Barvinok's algorithm.[11] After obtaining the Ehrhart polynomials corresponding to each of the considered properties, we evaluate these polynomials for any needed number of voters and we get the desired probabilities. As noticed in Lepelley et al. (2008), the different algorithms that can be used in the three-candidate framework do not allow to deal with four-candidate elections, where the total number of variables, i.e., possible linear orderings, is $4! = 24$. However, recent developments within the polytope theory may allow to obtain exact results for the case of $m = 4$ and a small number of voters. These results are obtained using the algorithms of Normaliz (Bruns et al., 2017a) which is, to the best of our knowledge, the only program able to compute the number of voting situations in polytopes corresponding to elections with up to four candidates. The reader interested in a deeper understanding of the algorithms of Normaliz is refereed to Bruns et al. (2017b) who describe several results obtained in four-candidate elections. This software can do all computations in dimension 24, but the limitation of the method is that all voting situations $(n_1, \ldots, n_{m!})$ that verify the required conditions of the considered voting event are enumerated and stored. Thus, this method needs relatively high memory when the number of voters increases. Moreover, the computation time rises accordingly. Consequently, exact results are obtained with $n \in \{2, \ldots, 9\}$ and $m = 4$ and we use computer simulations in order to evaluate the probabilities for $n > 9$.[12] We describe now the Monte-Carlo simulation methodology applied in order to estimate our probabilities in the same spirit as the Impartial Anonymous Culture condition. Let us consider as an example the probability of resoluteness.

1. At the beginning of the evaluation, we randomly generate a voting situation of length $m!$.
2. In the second step, we check whether the conditions of the resoluteness are fulfilled or not.

[11] For a detailed description of algorithms computing Ehrhart polynomials, we recommend the report by Verdoolaege et al. (2005).

[12] This is really possible for all the scoring CSRs we are considering here but, unfortunately, this is not true when we consider $\mathfrak{S}$ and $\mathfrak{M}$. For those CSRs exact probabilities can be obtained only for $n \in \{2, 3, 4, 5\}$.

3. These two steps are iterated $1,000,000$ times to obtain the number of voting situations for which resoluteness holds.

4. Finally, the probability of resoluteness is calculated as the quotient of the number obtained in step 3 over the total number of simulated voting situations, i.e., $1,000,000$.[13]

Another technique is used in this paper in order to obtain exact results when the number of candidates is $m = 4$ and the number of voters tends to infinity. In this case, the calculations of the limiting probability under IAC condition are reduced to computation of volumes of convex polytopes. For this, our volumes are found with the use of the algorithm Convex which is a Maple package for convex geometry by Franz (2017). This package works with the same general procedure that was implemented in Cervone et al. (2005) and recently used in other studies, e.g., Diss and Doghmi (2016), Diss and Gehrlein (2012, 2015), Gehrlein et al. (2015), and Moyouwou and Tchantcho (2017).[14]

Finally note that in what follows the probability values 0 and 0.0000 have different meanings. In fact, the first one corresponds to an exact value while the second one is obtained using our simulation method.

4.2 The probability of resoluteness

Our first concern is the probability of resoluteness. The results of our computations are provided in Tables 2 to 6. Recall that a CSR is resolute if it always selects a single committee. We stress that the analysis of the probability of resoluteness of the classical scoring CSRs considered here is new in the literature.

It is intuitive that ties are highly unlikely if the number of voters is large so that the probability of resoluteness is expected to approach 1 as n increases. Moreover, it is natural to understand that the probability of ties highly depends on whether the number of voters is odd or even, with ties more probable for an even number of voters. Those facts are confirmed by our results.

We observe very different behaviours of the considered CSRs for odd and even numbers of voters. For instance, with $m = 3$ and $k = 1$, both $\mathfrak{S}$ and $\mathfrak{M}$ (which agree in this case) perform better than the other CSRs when the number of voters is odd, but the scoring CSRs have a superior performance when the number of voters is even. In this case, the Borda CSR performs very well in comparison with the other scoring CSRs. When $m = 3$, $k = 2$ and n is odd, our results show that $\mathfrak{M}$ is the best CSR, while $\mathfrak{S}$ performs the worst. With n even, $\mathfrak{S}$ still is the worst scenario, but it is found that some scoring CSRs can perform better than $\mathfrak{M}$. Finally, when the number of candidates increases to $m = 4$, it appears that usually the Borda CSR has a greater probability of resoluteness than $\mathfrak{M}$ which in turn leads to a better performance than some scoring CSRs. Note that $\mathfrak{S}$ performs the worst in most cases. This remains in general true independently of the target size of the committee.

It is worth mentioning that some theoretical facts about the resoluteness of the considered CSRs help in building the tables. In particular, via very simple arguments involving the concept of reversal of a preference profile, it can be proved that the probability of resoluteness of the Borda CSR is the same for $k = 1$ and $k = m - 1$; the probability of resoluteness of the Plurality CSR with $k = 1$ is the same as the one of the Negative Plurality CSR with $k = m - 1$; the probability of resoluteness of Negative Plurality CSR with $k = 1$ is the same as the one of Plurality CSR with $k = m - 1$; when m is even and $k = \frac{m}{2}$, the probability of resoluteness of the Plurality CSR and the one for Negative Plurality CSR are the same. Moreover, from Proposition 35, we also get that the probability of resoluteness of $\mathfrak{M}$ is the same for $k = 1$ and $k = m - 1$.

[13]The MATLAB code of our simulations is available upon request.

[14]Once again this technique can not be used for $\mathfrak{S}$ and $\mathfrak{M}$ for which obtaining the probabilities in the limit case is only possible with computer simulations. We consider for them a number of voters $n = 100,000$.

Table 2: The probability of resoluteness: $m = 3$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$	Plurality = Bloc	Borda	Negative Plurality
2	0.4286	0.4286	0.7143	0.5714
3	0.9643	0.8571	0.8571	0.6429
4	0.6190	0.7857	0.8571	0.5952
5	0.9524	0.7857	0.8810	0.7381
6	0.7143	0.8377	0.8831	0.7662
7	0.9545	0.8788	0.9091	0.7424
8	0.7692	0.8298	0.9044	0.8089
9	0.9590	0.8931	0.9201	0.8242
50	0.9530	0.9703	0.9806	0.9588
51	0.9888	0.9725	0.9810	0.9595
100	0.9758	0.9854	0.9900	0.9782
101	0.9941	0.9851	0.9901	0.9788
1000	0.9975	0.9985	0.9990	0.9978
1001	0.9994	0.9985	0.9990	0.9978
∞	1	1	1	1

Table 3: The probability of resoluteness: $m = 3$ and $k = 2$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Borda	Negative Plurality = Bloc
2	0.4286	0.4286	0.5714	0.7143	0.4286
3	0.4286	0.9643	0.6429	0.8571	0.8571
4	0.5714	0.6190	0.5952	0.8571	0.7857
5	0.5952	0.9524	0.7381	0.8810	0.7857
6	0.6623	0.7143	0.7662	0.8831	0.8377
7	0.6818	0.9545	0.7424	0.9091	0.8788
8	0.7226	0.7692	0.8089	0.9044	0.8298
9	0.7373	0.9590	0.8242	0.9201	0.8931
50	0.9413	0.9530	0.9588	0.9806	0.9703
51	0.9423	0.9888	0.9595	0.9810	0.9725
100	0.9697	0.9758	0.9782	0.9900	0.9854
101	0.9700	0.9941	0.9788	0.9901	0.9851
1000	0.9969	0.9975	0.9978	0.9990	0.9985
1001	0.9969	0.9994	0.9978	0.9990	0.9985
∞	1	1	1	1	1

Table 4: The probability of resoluteness: $m = 4$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$	Plurality = Bloc	Borda	Negative Plurality
2	0.2800	0.2800	0.7200	0
3	0.9015	0.6677	0.8185	0.3323
4	0.4807	0.7754	0.8274	0.5169
5	0.8733	0.6769	0.8481	0.5692
6	0.5155	0.6819	0.8618	0.5491
7	0.8259	0.7799	0.8730	0.6063
8	0.5787	0.7772	0.8829	0.6608
9	0.8346	0.7652	0.8901	0.6863
50	0.8841	0.8899	0.9615	0.9245
51	0.9468	0.8886	0.9619	0.9244
100	0.9411	0.9441	0.9804	0.9591
101	0.9716	0.9443	0.9805	0.9601
1000	0.9942	0.9930	0.9993	0.9949
1001	0.9955	0.9933	0.9996	0.9966
∞	1	1	1	1

Table 5: The probability of resoluteness: $m = 4$ and $k = 2$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Borda	Negative Plurality	Bloc
2	0.4400	0.2000	0.7200	0.6400	0.7200	0.2000
3	0.2769	0.8492	0.5815	0.8031	0.5815	0.8031
4	0.5115	0.4048	0.3805	0.7774	0.3805	0.4769
5	0.4508	0.8083	0.5590	0.8227	0.5590	0.7460
6	0.5197	0.4219	0.7017	0.8245	0.7017	0.6103
7	0.4825	0.7292	0.6367	0.8477	0.6367	0.7454
8	0.5495	0.4857	0.6119	0.8517	0.6119	0.6808
9	0.5422	0.7416	0.6841	0.8665	0.6841	0.7603
50	0.8563	0.8448	0.8812	0.9500	0.8825	0.8939
51	0.8593	0.9065	0.8817	0.9477	0.8819	0.8983
100	0.9219	0.9106	0.9340	0.9719	0.9341	0.9459
101	0.9262	0.9455	0.9380	0.9779	0.9383	0.9483
1000	0.9908	0.9739	0.9916	0.9966	0.9989	0.9946
1001	0.9964	0.9825	0.9920	0.9973	0.9949	0.9962
∞	1	1	1	1	1	1

Table 6: The probability of resoluteness: $m = 4$ and $k = 3$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Borda	Negative Plurality = Bloc
2	0.2400	0.2800	0	0.7200	0.2800
3	0.4000	0.9015	0.3323	0.8185	0.6677
4	0.4356	0.4807	0.5169	0.8274	0.7754
5	0.5245	0.8733	0.5692	0.8481	0.6769
6	0.5414	0.5155	0.5491	0.8618	0.6819
7	0.5871	0.8259	0.6063	0.8730	0.7799
8	0.6051	0.5787	0.6608	0.8829	0.7772
9	0.6297	0.8346	0.6863	0.8901	0.7652
50	0.8767	0.8841	0.9245	0.9615	0.8899
51	0.8800	0.9468	0.9244	0.9619	0.8886
100	0.9341	0.9411	0.9591	0.9804	0.9441
101	0.9374	0.9716	0.9601	0.9805	0.9443
1000	0.9920	0.9942	0.9949	0.9993	0.9930
1001	0.9945	0.9955	0.9966	0.9996	0.9933
∞	1	1	1	1	1

4.3 The probability of suffering the reversal bias

In Tables 9 to 11 we collect our results about the probability of suffering the reversal bias. Recall that a CSR suffers the reversal bias when for a given preference profile a unique committee is selected and reversing the preferences of all the voters the same committee is still selected as unique outcome. Saari and Barney (2003) and Bubboloni and Gori (2016) prove several results about the reversal bias when $k = 1$. On the basis of our results and those in Bubboloni and Gori (2016) we can deduce interesting facts concerning the computations we are going to perform. In particular, from Theorem A in Bubboloni and Gori (2016), we get that $\mathfrak{S}$ and $\mathfrak{M}$ are immune to the reversal bias for $m = 3$ and $k = 1$; from Proposition 33, $\mathfrak{S}$ is immune to the reversal bias for $m = 3$ and $k = 2$ as well as for $m = 4$ and $k = 3$; when $m = 4$, from Theorem A in Bubboloni and Gori (2016), we have that $\mathfrak{S}$ and $\mathfrak{M}$ are immune to the reversal bias if and only if $n \in \{2, 3, 4, 5, 7\}$; from Theorem A in Bubboloni and Gori (2016) and Corollary 37, $\mathfrak{M}$ is immune to the reversal bias for $m = 3$ and $k = 2$; from Corollary 36, $\mathfrak{M}$ is immune to the reversal bias for $m = 4$ and $k = 2$; from Corollary 37, the probability for $\mathfrak{M}$ to be immune to the reversal bias for $m = 4$ and $k = 1$ is the same as for $m = 4$ and $k = 3$.

It is also immediate to show that the Borda CSR is always immune to the reversal bias; the probability of suffering the reversal bias of the Plurality CSR (Bloc CSR) with $k = 1$ is the same as the one of the Negative Plurality CSR (Bloc CSR) with $k = m - 1$; the probability of suffering the reversal bias of the Negative Plurality CSR with $k = 1$ is the same as the one of the Plurality CSR with $k = m - 1$; when m is even and $k = \frac{m}{2}$, the probability of suffering the reversal bias of the Plurality CSR and the one of the Negative Plurality CSR are the same; the Bloc CSR is immune to the reversal bias.

It turns out that the most interesting cases to study for comparing $\mathfrak{S}$ and $\mathfrak{M}$ to the other CSRs correspond to $m = 4$ with $k = 1$ and $k = 3$. Our computations reveal that, yet again, $\mathfrak{S}$ and $\mathfrak{M}$ perform better than the other CSRs (except Borda). However, $\mathfrak{M}$ appears to be more vulnerable to the reversal bias than $\mathfrak{S}$ for $m = 4$ and $k = 3$.

Finally, the computations made suggest to conjecture that $\mathfrak{S}$ is immune to the reversal bias if $m = 4$ and $k = 2$. This is, at the moment, an open problem.

Table 7: The probability of suffering the reversal bias: $m = 3$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$ & Borda	Plurality = Bloc	Negative Plurality
2	0	0	0.1429
3	0	0	0.1071
4	0	0.0238	0.0714
5	0	0	0.1190
6	0	0.0260	0.1234
7	0	0.0379	0.1061
8	0	0.0210	0.1352
9	0	0.0420	0.1379
50	0	0.0776	0.1856
51	0	0.0789	0.1859
100	0	0.0852	0.1938
101	0	0.0850	0.1942
1000	0	0.0918	0.2027
1001	0	0.0918	0.2027
∞	0	0.0926	0.2037

Table 8: The probability of suffering the reversal bias: $m = 3$ and $k = 2$

n	$\mathfrak{S}$ & $\mathfrak{M}$ & Borda	Plurality	Negative Plurality = Bloc
2	0	0.1429	0
3	0	0.1071	0
4	0	0.0714	0.0238
5	0	0.1190	0
6	0	0.1234	0.0260
7	0	0.1061	0.0379
8	0	0.1352	0.0210
9	0	0.1379	0.0420
50	0	0.1856	0.0776
51	0	0.1859	0.0789
100	0	0.1938	0.0852
101	0	0.1942	0.0850
1000	0	0.2027	0.0918
1001	0	0.2027	0.0918
∞	0	0.2037	0.0926

Table 9: The probability of suffering the reversal bias: $m = 4$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$	Plurality = Bloc	Negative Plurality	Borda
2	0	0	0	0
3	0	0	0.0246	0
4	0	0.0328	0.0492	0
5	0	0.0215	0.0513	0
6	0.0009	0.0163	0.0426	0
7	0	0.0380	0.0548	0
8	0.0011	0.0417	0.0670	0
9	0.0003	0.0381	0.0714	0
50	0.0025	0.0790	0.1326	0
51	0.0022	0.0792	0.1333	0
100	0.0030	0.0924	0.1458	0
101	0.0031	0.0924	0.1471	0
1000	0.0042	0.1086	0.1599	0
1001	0.0044	0.1078	0.1605	0
∞	0.0044	0.1105	0.1608	0

Table 10: The probability of suffering the reversal bias: $m = 4$ and $k = 2$

n	$\mathfrak{S}$	$\mathfrak{M}$ & Borda & Bloc	Plurality & Negative Plurality
2	0	0	0.0800
3	0	0	0.0277
4	0	0	0.0085
5	0	0	0.0254
6	0.0000	0	0.0466
7	0.0000	0	0.0325
8	0.0000	0	0.0271
9	0.0000	0	0.0374
50	0.0000	0	0.0656
51	0.0000	0	0.0653
100	0.0000	0	0.0759
101	0.0000	0	0.0750
1000	0.0000	0	0.0879
1001	0.0000	0	0.0872
∞	0.0000	0	0.0889

Table 11: The probability of suffering the reversal bias: $m = 4$ and $k = 3$

n	$\mathfrak{S}$ & Borda	$\mathfrak{M}$	Plurality	Negative Plurality = Bloc
2	0	0	0	0
3	0	0	0.0246	0
4	0	0	0.0492	0.0328
5	0	0	0.0513	0.0215
6	0	0.0009	0.0426	0.0163
7	0	0	0.0548	0.0380
8	0	0.0011	0.0670	0.0417
9	0	0.0003	0.0714	0.0381
50	0	0.0025	0.1326	0.0790
51	0	0.0022	0.1333	0.0792
100	0	0.0030	0.1458	0.0924
101	0	0.0031	0.1471	0.0924
1000	0	0.0042	0.1599	0.1086
1001	0	0.0044	0.1605	0.1078
∞	0	0.0044	0.1608	0.1105

4.4 The probability of suffering the Condorcet loser paradox

Tables 12 to 16 describe our results about the probability of suffering the Condorcet loser paradox provided a Condorcet loser set exists. Recall that such a paradox occurs when, for a given preference profile, the unique winning committee is a Condorcet loser set, that is, a set having the property that each candidate which does not belong to it is preferred by the majority of voters to every of its elements. It is important to notice that this paradox is also known in the literature as the strong Borda paradox. This type of problem for $k = 1$ is studied by Diss and Gehrlein (2012); Diss and Tlidi (2018); Fishburn and Gehrlein (1976); Gehrlein and Lepelley (2010b), among others.

It is important to notice that, since a k-Condorcet loser set for (C, V, p) is a k-Condorcet winner for (C, V, p^r), if a CSR is Condorcet consistent and immune to the reversal bias then it is immune to the Condorcet loser paradox. Since $\mathfrak{M}$ is Condorcet consistent, the analysis made in the previous section can be used to get some information about the Condorcet loser paradox for $\mathfrak{M}$. For instance, with $m = 4$ and $k = 1$, Table 9 shows that $\mathfrak{M}$ is immune to the reversal bias when $n \in \{2, 3, 4, 5, 7\}$. This remains true for the immunity to the Condorcet loser paradox as shown in Table 14. It is well known that the Borda CSR is immune to the Condorcet loser paradox when $k = 1$ and it is immediate to show that the probability of suffering the Condorcet loser paradox of the Plurality CSR (Bloc CSR) with $k = 1$ is the same as the one of the Negative Plurality CSR (Bloc CSR) with $k = m - 1$; the probability of suffering the Condorcet loser paradox of the Negative Plurality CSR with $k = 1$ is the same as the one of the Plurality CSR with $k = m - 1$; when m is even and $k = \frac{m}{2}$, the probability of suffering the Condorcet loser paradox of the Plurality CSR and the one of the Negative Plurality CSR are the same. Notice that the results in Table 12 for Plurality and Negative Plurality when the number of voters is large are in agreement with results in Gehrlein and Lepelley (2010b).

From the tables we understand that the probability of observing the Condorcet loser paradox can never exceed 0.0315 with $m = 3$ and 0.0240 with $m = 4$ under all the CSRs that we consider in this paper. In other words, the probability of observing this paradox should be rare, but not impossible to observe particularly when the number of voters increases. Our results also show that the use of CSRs like Borda, $\mathfrak{M}$ and $\mathfrak{S}$ will clearly tend to minimize the probability of observing the Condorcet loser paradox, with a certain advantage for $\mathfrak{S}$ and Borda when $m = 4$ and $k = 3$.

Our probability calculations suggest to conjecture that the Borda CSR is immune to the Condorcet loser paradox whatever the values of m, n, and k. In addition, except for the situation with $m = 4$ and $k = 1$, $\mathfrak{S}$ seems to be immune to the Condorcet loser paradox for all the other values of m and k that we consider in this paper. Again these conjectures are, at the moment, open problems.

Table 12: The probability of suffering the Condorcet loser paradox: $m = 3$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$ & Borda	Plurality = Bloc	Negative Plurality
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0.0250
6	0	0	0
7	0	0.0160	0.0080
8	0	0	0.0100
9	0	0.0159	0.0222
50	0	0.0209	0.0245
51	0	0.0258	0.0287
100	0	0.0250	0.0276
101	0	0.0275	0.0300
1000	0	0.0291	0.0311
1001	0	0.0294	0.0313
∞	0	0.0296	0.0315

Table 13: The probability of suffering the Condorcet loser paradox: $m = 3$ and $k = 2$

n	$\mathfrak{S}$ & $\mathfrak{M}$ & Borda	Negative Plurality = Bloc	Plurality
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0.0250
6	0	0	0
7	0	0.0160	0.0080
8	0	0	0.0100
9	0	0.0159	0.0222
50	0	0.0209	0.0245
51	0	0.0258	0.0287
100	0	0.0250	0.0276
101	0	0.0275	0.0300
1000	0	0.0291	0.0311
1001	0	0.0294	0.0313
∞	0	0.0296	0.0315

Table 14: The probability of suffering the Condorcet loser paradox: $m = 4$ and $k = 1$

n	$\mathfrak{S} = \mathfrak{M}$	Plurality = Bloc	Negative Plurality	Borda
2	0	0	0	0
3	0	0	0.0137	0
4	0	0	0	0
5	0	0.0078	0.0059	0
6	0.0000	0	0.0005	0
7	0	0.0081	0.0105	0
8	0.0000	0.0031	0.0052	0
9	0.0000	0.0090	0.0116	0
50	0.0002	0.0115	0.0154	0
51	0.0004	0.0165	0.0208	0
100	0.0004	0.0166	0.0194	0
101	0.0005	0.0194	0.0222	0
1000	0.0006	0.0221	0.0237	0
1001	0.0007	0.0224	0.0236	0
∞	0.0007	0.0228	0.0240	0

Table 15: The probability of suffering the Condorcet loser paradox: $m = 4$ and $k = 2$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Negative Plurality	Bloc	Borda
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0.0027	0.0027	0.0003	0
6	0.0000	0	0.0013	0.0013	0	0
7	0.0000	0	0.0026	0.0026	0.0005	0
8	0.0000	0	0.0003	0.0003	0	0
9	0.0000	0	0.0038	0.0038	0.0006	0
50	0.0000	0	0.0055	0.0051	0.0005	0.0000
51	0.0000	0	0.0068	0.0069	0.0012	0.0000
100	0.0000	0	0.0071	0.0070	0.0009	0.0000
101	0.0000	0	0.0078	0.0076	0.0013	0.0000
1000	0.0000	0	0.0089	0.0089	0.0014	0.0000
1001	0.0000	0	0.0090	0.0089	0.0015	0.0000
∞	0.0000	0	0.0090	0.0090	0.0016	0.0000

Table 16: The probability of suffering the Condorcet loser paradox: $m = 4$ and $k = 3$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Negative Plurality = Bloc	Borda
2	0	0	0	0	0
3	0	0	0.0137	0	0
4	0	0	0	0	0
5	0	0	0.0059	0.0078	0
6	0.0000	0.0000	0.0005	0	0
7	0.0000	0	0.0105	0.0081	0
8	0.0000	0.0000	0.0052	0.0031	0
9	0.0000	0.0000	0.0116	0.0090	0
50	0.0000	0.0002	0.0160	0.0116	0.0000
51	0.0000	0.0004	0.0202	0.0162	0.0000
100	0.0000	0.0003	0.0196	0.0160	0.0000
101	0.0000	0.0005	0.0221	0.0189	0.0000
1000	0.0000	0.0007	0.0233	0.0222	0.0000
1001	0.0000	0.0007	0.0235	0.0225	0.0000
∞	0.0000	0.0007	0.0240	0.0228	0.0000

4.5 The probability of suffering the leaving member paradox

The probability of suffering the leaving member paradox is illustrated in Table 17. Recall that the leaving member paradox occurs when for a certain election a unique committee of size k out of m candidates is elected and if an elected candidate leaves the office for some reason and a new election on the $m-1$ remaining candidates is hold, then a unique committee is elected and such a committee and the original one are disjoint. This paradox has been introduced by Staring (1986). Kamwa and Merlin (2015) considered the probability of this paradox under the four scoring CSRs presented in this paper with the Impartial Culture (IC)[15] assumption for large electorates. Considering the IAC condition, Diss and Doghmi (2016) extended the results of Kamwa and Merlin (2015) in many directions. In particular, the results we propose for the limit case of the scoring CSRs are taken from Diss and Doghmi (2016). Since we are focusing only on $m \in \{3, 4\}$, the leaving member paradox only occurs in the case $m = 4$ and $k = 2$. We deduce then from our results that $\mathfrak{M}$ has a better performance than the other CSRs except for some small number of voters; in most cases $\mathfrak{S}$ performs better than the Plurality CSR and the Bloc CSR. The Bloc CSR appears to be the worst CSR according to this criterion.

[15]This is another assumption widely used for analyzing the probability of electoral events. Under this assumption, the preference relation of each voter is drawn uniformly at random from the set of all possible linear orders.

Table 17: The probability of suffering the leaving member paradox: $m = 4$ and $k = 2$

n	$\mathfrak{S}$	$\mathfrak{M}$	Plurality	Borda	Negative Plurality	Bloc
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0.0021	0	0	0	0	0.0055
5	0.0011	0	0.0033	0	0	0
6	0.0038	0.0000	0.0018	0.0002	0	0.0085
7	0.0020	0.0002	0.0008	0.0006	0	0.0252
8	0.0049	0.0000	0.0082	0.0008	0	0.0081
9	0.0035	0.0006	0.0060	0.0014	0	0.0253
50	0.0201	0.0032	0.0321	0.0058	0.0137	0.0485
51	0.0205	0.0041	0.0324	0.0059	0.0154	0.0512
100	0.0262	0.0051	0.0386	0.0073	0.0199	0.0612
101	0.0267	0.0060	0.0399	0.0076	0.0193	0.0605
1000	0.0338	0.0076	0.0480	0.0090	0.0278	0.0738
1001	0.0344	0.0082	0.0485	0.0090	0.0286	0.0756
∞	0.0350	0.0084	0.0485	0.0094	0.0286	0.0757

4.6 The probability of agreement

We know that for certain voting situations the set of committees selected by $\mathfrak{S}$ differs from the set of committees selected by $\mathfrak{M}$. However, in some cases, they both select the same unique committee. Here we study the probability of such a situation, which we shortly refer to as an agreement between $\mathfrak{S}$ and $\mathfrak{M}$. Recall that $\mathfrak{S}$ equals $\mathfrak{M}$ when $k = 1$. Moreover, when $m = 3$, $k = 2$ and under the IAC hypothesis, Kamwa (2017b) computes the probability of having voting situations where $\mathfrak{S}$ and $\mathfrak{M}$ coincide.

The results of our calculations are described in Table 18. We can make some observations based upon this table. First of all, the probability of agreement is smaller when there is an even number of voters than when there is an odd number of voters. One of the possible reasons for that is that having ties for $\mathfrak{S}$ or $\mathfrak{M}$, which are more likely in the even case, implies no agreement between the two CSRs. We also observe that, our results indicate that increasing an odd (even) number of voters implies an increase in the probability of having the agreement. Moreover, when the number of voters is large the probability of the agreement reaches 70% for $m = 3$ and $k = 2$ and 60% with $m = 4$ and $k = 2$ or $k = 3$. Finally, it seems from the tables that, for fixed m and n, with n large, the probability of the agreement decreases as the size of the committee to be elected increases.

We study also the probability of agreement between $\mathfrak{S}$ and the Borda CSR (Table 19), and between $\mathfrak{M}$ and the Borda CSR (Table 20). Our results show that there exists, at least in three-candidate and four-candidate elections, an important set of voting situations where the Borda CSR and one of the two extensions of the Simpson voting rule agree. Surprisingly, this probability reaches more than 80% for many cases with only 101 voters. Moreover, our results indicate that the Borda CSR is more likely to agree with $\mathfrak{M}$ than with $\mathfrak{S}$ except for a small even number of voters ($n \leq 6$). Since the probability of resoluteness tends to rapidly increase as the number of voters increases, we know that each of the studied CSRs usually selects a unique committee as the number of voters is large. Hence, for large electorates, the probability of the agreement here considered should be close to the probability of equality. Note that, using a simple argument based on the concept of reversal of a preference profile, it can be proved that the probability of agreement between $\mathfrak{M}$ and the Borda when $k = 1$ is the same as the one when $k = m - 1$.

Table 18: The probability of agreement between $\mathfrak{S}$ and $\mathfrak{M}$

n	$m=3$		$m=4$		
	$k=1$	$k=2$	$k=1$	$k=2$	$k=3$
2	1	0.1429	1	0.0800	0.0400
3	1	0.3214	1	0.1846	0.2615
4	1	0.2857	1	0.1593	0.1730
5	1	0.4286	1	0.2794	0.3206
6	1	0.3766	1	0.1806	0.2305
7	1	0.4848	1	0.2840	0.3502
8	1	0.4336	1	0.2197	0.2766
9	1	0.5215	1	0.3146	0.3749
50	1	0.6346	1	0.4908	0.4900
51	1	0.6524	1	0.5168	0.5193
100	1	0.6603	1	0.5442	0.5355
101	1	0.6694	1	0.5603	0.5517
1000	1	0.6847	1	0.5983	0.5807
1001	1	0.6856	1	0.6045	0.5832
∞	1	0.6875	1	0.6074	0.5866

Table 19: The probability of agreement between $\mathfrak{S}$ and the Borda CSR

n	$m=3$		$m=4$		
	$k=1$	$k=2$	$k=1$	$k=2$	$k=3$
2	0.4286	0.4286	0.2800	0.3600	0.2400
3	0.8571	0.3214	0.7631	0.2123	0.3077
4	0.6190	0.5238	0.4561	0.4075	0.3651
5	0.8333	0.4762	0.7287	0.3509	0.4120
6	0.6883	0.5762	0.4793	0.4147	0.4362
7	0.8409	0.5530	0.6944	0.3749	0.4607
8	0.7273	0.6107	0.5286	0.4323	0.4789
9	0.8452	0.5964	0.6999	0.4198	0.4937
50	0.8624	0.7528	0.7523	0.6344	0.6526
51	0.8812	0.7522	0.7861	0.6348	0.6554
100	0.8787	0.7716	0.7908	0.6706	0.6866
101	0.8880	0.7714	0.8055	0.6753	0.6887
1000	0.8941	0.7896	0.8253	0.7118	0.7199
1001	0.8950	0.7896	0.8245	0.7146	0.7228
∞	0.8958	0.7917	0.8272	0.7168	0.7262

Table 20: The probability of agreement between $\mathfrak{M}$ and the Borda CSR

n	$m=3$	$m=4$		
	$k=1$ & $k=2$	$k=1$	$k=2$	$k=3$
2	0.4286	0.2800	0.1333	0.2800
3	0.8571	0.7631	0.4800	0.7631
4	0.6190	0.4561	0.2525	0.4561
5	0.8333	0.7287	0.4453	0.7287
6	0.6883	0.4793	0.3835	0.4756
7	0.8409	0.6944	0.6009	0.6944
8	0.7273	0.5286	0.4333	0.5266
9	0.8452	0.6999	0.6073	0.7004
50	0.8624	0.7523	0.7031	0.7500
51	0.8812	0.7861	0.7337	0.7858
100	0.8787	0.7908	0.7459	0.7886
101	0.8880	0.8055	0.7663	0.8057
1000	0.8941	0.8253	0.7865	0.8237
1001	0.8950	0.8245	0.7909	0.8256
∞	0.8958	0.8272	0.7945	0.8288

References

Aziz, H., Elkind, E., Faliszewski, P., Lackner, M., and Skowron, P. (2017) The Condorcet principle for multiwinner election: from short listing to proportionality. Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence (IJCAI-17), 84-90.

Barberà, S., and Coelho, D. (2008) How to choose a non-controversial list with k names. Social Choice and Welfare, 31: 79-96.

Bubboloni, D., and Gori, M. (2016) On the reversal bias of the Minimax social choice correspondence. Mathematical Social Sciences, 81: 53-61.

Bubboloni, D., and Gori, M. (2018) The flow network method. Social Choice and Welfare, https://doi.org/10.1007/s00355-018-1131-7.

Brams, S.J., Kilgour, D.M., and Sanver, M.R. (2005) A minimax procedure for electing committees. Public Choice, 132(34): 401-420.

Bruns, W., Ichim, B., Römer, T., Sieg, R., and Söger, C. (2017) Normaliz: Algorithms for rational cones and affine monoids. Available at http://normaliz.uos.de.

Bruns, W., Ichim, B., and Söger, C. (2017) Computations of volumes and Ehrhart series in four candidates elections. Working paper, Cornell University Library.

Cervone, D., Gehrlein, W.V., and Zwicker, W. (2005) Which scoring rule maximizes Condorcet efficiency under IAC? *Theory and Decision*, 58: 145-185.

Coelho, D. (2004) *Understanding, evaluating and selecting voting rules through games and axioms* (Ph.d. thesis), Universitat Autònoma de Barcelona, Departament d'Economia i d'Història Econòmica. Available at http://hdl.handle.net/10803/4056

Courtin, S., Martin, M., and Moyouwou, I. (2015) The q-majority efficiency of positional rules. Theory and Decision, 79: 31-49.

Diss, M. (2015) Strategic manipulability of self-selective social choice rules. Annals of Operations Research, 229: 347-376.

Diss, M., and Doghmi, A. (2016) Multi-winner scoring election methods: Condorcet consistency and paradoxes. Public Choice, 169: 97-116

Diss, M., and Gehrlein, W.V. (2015) The true impact of voting rule selection on Condorcet efficiency. Economics Bulletin, 35: 2418-2426.

Diss, M., and Gehrlein, W.V. (2012) Borda's paradox with weighted scoring rules. Social Choice and Welfare, 38: 121-136.

Diss, M., Louichi, A., Merlin, V., and Smaoui, H. (2012) An example of probability computations under the IAC assumption: The stability of scoring rules. Mathematical Social Sciences, 64: 57-66.

Diss, M., and Tlidi, A. (2018) Another perspective on Borda's paradox. Theory and Decision, 84(1): 99-121.

Duggan, J. and Schwartz, T. (2000) Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. Social Choice and Welfare, 17(1): 85-93.

Elkind, E., Faliszewski P., Skowron, P., and Slinko, A. (2017) Properties of multiwinner voting rules. Social Choice and Welfare, 48(3): 599-632.

Fishburn, P.C., and Gehrlein, W.V. (1976) Borda's rule, positional voting, and Condorcet's simple majority principle. *Public Choice*, 28:79-88.

Franz, M. (2017) Convex - a Maple package for convex geometry, version 1.2 available at http://www-home.math.uwo.ca/ mfranz/convex/.

Gehrlein, W.V. (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences, 10: 199-209.

Gehrlein, W.V., and Fishburn, P.C. (1976) The probability of the paradox of voting: A computable solution. Journal of Economic Theory, 13: 14-25.

Gehrlein, W.V, and Lepelley, D. (2010) On the probability of observing Borda's paradox. *Social Choice and Welfare*, 35:1-23.

Gehrlein, W.V., and Lepelley, D. (2011) Voting paradoxes and group coherence. Springer, Berlin/Heidelberg.

Gehrlein, W.V., and Lepelley, D. (2017) Elections, voting rules and paradoxical outcomes. Springer-Verlag.

Gehrlein, W.V., Lepelley, D., and Plassmann, F. (2018) An Evaluation of the Benefit of Using Two-Stage Election Procedures. Homo Oeconomicus, 35(1-2): 53-79.

Gehrlein, W.V., Lepelley, D., and Moyouwou, I. (2016) A note on Approval Voting and electing the Condorcet loser. Mathematical Social Sciences, 80: 115-122.

Gehrlein, W.V., Lepelley, D., and Moyouwou, I. (2015) Voters' preference diversity, concepts of agreement and Condorcet's paradox. Quality & Quantity, 49(6): 2345-2368.

González-Díaz, J., Hendrickx, R., and Lohmann, E. (2014) Paired comparisons analysis: an axiomatic approach of ranking methods. Social Choice and Welfare, 42: 139-169.

Jeong, H., and Ju, B.G. (2017) Resolute majority rules. Theory and Decision, 82: 31-39.

Kamwa, E. (2017a) On stable rules for selecting committees. Journal of Mathematical Economics, 70: 36-44.

Kamwa, E. (2017b) Stable rules for electing committees and divergence on outcomes. Group Decision and Negotiation, 26: 547-564.

Kamwa, E., and Merlin, V. (2015) Scoring rules over subsets of alternatives: Consistency and paradoxes. Journal of Mathematical Economics, 61: 130-138.

Kamwa, E., and Merlin, V. (2018) Coincidence of Condorcet committees. Social Choice and Welfare, 50(1): 171-189.

Kamwa, E., and Valognes, F. (2017) Scoring rules and preference restrictions: The strong Borda paradox revisited. Revue d'Economie Politique, 127(3): 375-395.

Kramer, G.H. (1977) A dynamical model of political equilibrium. Journal of Economic Theory, 16, 310-333.

Lepelley, D., Moyouwou, I., and Smaoui, H. (2018) Monotonicity paradoxes in three-candidate elections using scoring elimination rules. Social Choice and Welfare, 50(1): 1-33.

Lepelley, D., Louichi, A., and Smaoui, H. (2008) On Ehrhart polynomials and probability calculations in voting theory. Social Choice and Welfare, 30: 363-383.

Moyouwou, I., and Tchantcho, H. (2017) A note on Approval Voting and electing the Condorcet loser. Mathematical Social Sciences, 89: 70-82.

Saari, D.G., and Barney, S. (2003) Consequences of reversing preferences. The Mathematical Intelligencer, 25: 17-31.

Schulze, M. (2011) A new monotonic, clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method, Social Choice and Welfare, 36: 267-303.

Simpson, P.B. (1969) On defining areas of voter choice: Professor Tullock on stable voting. Quarterly Journal of Economics, 83, 478-490.

Smaoui, H., Lepelley, D., and Moyouwou, I. (2016) Borda elimination rule and monotonicity paradoxes in three-candidate elections. Economics Bulletin, 36(3): 1722-1728.

Staring, M. (1986) Two paradoxes of committee elections. Mathematics Magazine, 59: 158-159.

Tideman, N. (1987) Independence of clones as a criterion for voting rules. Social Choice and Welfare, 4: 185-206.

Tideman, N. (2006) Collective decisions and voting: The potential for public choice, Routledge.

Verdoolaege, S., Woods, K.M., Bruynooghe, M., and Cools, R. (2005) Computation and manipulation of enumerators of integer projections of parametric polytopes. Technical Report CW 392, Katholieke Universiteit Leuven.

Wilson, M.C., and Pritchard, G. (2007) Probability calculations under the IAC hypothesis. Mathematical Social Sciences, 54: 244-256.