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Estimates of the topological degree of a class of piecewise linear maps with applications

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Abstract

We provide some new estimates for the topological degree of a class of continuous and piecewise linear maps based on a classical integral computation formula. We provide applications to some nonlinear problems that exhibit a local PC^1 structure.

1 Introduction

In this paper we continue our work [18, 20] on estimates of the topological degree of piecewise linear maps introducing some new results based on a classical integral formula, see, e.g., [9, 15]. Such maps appear naturally in the context of optimal control, see [1, 17, 19, 21], in the area of operational research as the generalization of the differential of PC^1 maps see e.g., [10, 16], and in the applied sciences, compare for instance [12, 14] and [24] where also discontinuous systems are considered. Our interest is motivated by a result of Jong-Shi Pang and Daniel Ralph in [16], Theorem 1 below, that provides a necessary and sufficient condition for the invertibility of continuous piecewise linear maps in terms of their topological degree.

In this paper we follow a substantially different strategy from that of [18, 20]. In fact, in those papers the approach used for the degree was inspired by a Kronecker-like formula, see [3, 6, 23], based on surface integrals. That strategy allowed us to keep track of the topology of the problems (which indeed was required by the optimal control problems studied in [17, 19]) whereas the present approach stresses the importance of the linear selection functions of the map under consideration. By means of a few explicit examples, we highlight the differences between the results obtained via these

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methodologies and explain the relative advantages as, for instance, that the "geometric" approach taken in this paper yields finer results at the price of some extra complication.

For completeness, we also notice that recognizing a map as a composition product of simpler continuous piecewise linear maps can be an effective way of computing its degree (see the comments in Remark 4). However, it can be difficult to recognize such factorizations, as shown by Example 2. For this reason we do not pursue further this idea.

As an illustration of our result we deduce a very natural version of the implicit function theorem for PC^1 functions.

2 Notation and background notions

We denote with $B(x_0, r)$ the ball in $\mathbb{R}^k$ centered at x_0 with radius r. We also write ω_k for the Lebesgue measure of the unit ball of $\mathbb{R}^k$, so that $k\omega_k$ is the $(k-1)$-dimensional measure of the unit sphere of $\mathbb{R}^k$.

The Euclidean norm of vectors $x \in \mathbb{R}^k$ is denoted by $\|x\|$ while, given a $k \times k$ real matrix $A = (a_{ij})_{i,j=1,\dots k}$, by $\|A\|$ we mean the operator norm and by $\|A\|_{\mathcal{F}}$ we denote the Frobenius norm:

$$\|A\| := \max_{\|x\|=1} \|Ax\|\,, \qquad \|A\|_{\mathcal{F}} := \left(\sum_{i,j=1}^{k} a_{ij}^2\right)^{1/2}.$$

We recall that $\|A\| \le \|A\|_{\mathcal{F}}$, see, e.g. [7, Ch. 2§3.2].

We shall also use the notion of *singular value* of a real matrix. Namely, given a matrix A, a singular value of A is the square root of an eigenvalue of the positive semi-definite symmetric matrix A^tA, where A^t denotes the transpose matrix of A.

2.1 Topological degree

A central notion in this paper is that of Brouwer degree of a continuous map. Since many good references for this topic exist, see for instance, [13, 11, 6], we only provide here a very cursory introduction. We say that a map $\psi\colon \Omega \to \mathbb{R}^k$ with $\Omega \subseteq \mathbb{R}^k$ an open set, is *proper* if the preimage of any compact set is compact. A triple (f, U, p), with $p \in \mathbb{R}^k$ and f a continuous and proper map defined in some neighborhood of the closure $\overline{U}$ of the open set $U \subseteq \mathbb{R}^k$, is said to be *admissible* if $f^{-1}(p) \cap U$ is compact. Given an admissible triple (f, U, p), it is defined an integer $\deg(f, U, p)$, called the *degree of f in U respect to p*.

A fruitful viewpoint, assumed also in the above mentioned books, is to regard $\deg(f, U, p)$ as an algebraic count of the elements of $f^{-1}(p) \cap U$. A different approach, following Kronecker's pioneering work, is to use a formula

based on surface integrals (see, e.g., [3, 23] also [6, Ch. 1§6.6] in the two-dimensional case). The latter is the position we assumed in our previous studies [18, 20]. We base this paper on yet another formula ([9, Ch. 2 §1], see also e.g. [15]) based on volume integration that will allow us, in Section 3, to derive some estimates for the degree of strongly piecewise linear maps (see the next subsection for a definition). This formula, for an admissible triple (f, U, p) with $U \subseteq \mathbb{R}^k$ bounded and $f \in C^1(U) \cap C(\overline{U})$, is the following:

$$\deg(f, U, p) = \int_U \varphi\big(\,|f(x) - p|\,\big) \det\big(\mathrm{D}f(x)\big)\,\mathrm{d}x, \tag{1}$$

where $\varphi\colon [0, \infty) \to \mathbb{R}$ is continuous and satisfies:

- Its compact support is contained in the interval

$$\Big[0\,,\ \inf_{x \in \partial U} |f(x) - p|\,\Big);$$

- $\int_{\mathbb{R}^k} \varphi(|x|)\,\mathrm{d}x = 1$.

We will not present here the many properties of the topological degree, as they are easily found in the references above, but only mention a few facts that we use at some point.
• If $f\colon \mathbb{R}^k \to \mathbb{R}^k$ is continuous and proper then $\deg(f, \mathbb{R}^k, p)$ is well-defined for any $p \in \mathbb{R}^k$ and it is actually independent of the choice of p. In this case we shall simply write $\deg(f)$ instead of the bulkier expression $\deg(f, \mathbb{R}^k, p)$.
• Let f be an $\mathbb{R}^k$-valued continuous and proper function defined in some neighborhood of the closure of an open bounded set $U \subseteq \mathbb{R}^k$. If $p \notin f(\partial U)$ then (f, U, p) is admissible. Also, if g is another continuous and proper function defined in a neighborhood of $\overline{U}$ such that $\min_{x \in \partial U} |f(x) - g(x)| \leq \min_{x \in \partial U} |f(x) - p|$ then (g, U, p) is admissible and $\deg(f, U, p) = \deg(g, U, p)$.
• Let (f, U, p) be admissible. If $V \subseteq U$ is such that $f^{-1}(p) \cap U \subseteq V$, then (f, V, p) is admissible and $\deg(f, U, p) = \deg(f, V, p)$. Thus, for $f\colon \mathbb{R}^k \to \mathbb{R}^k$ continuous and proper then $\deg(f) = \deg(f, V, p)$ for any V such that $f^{-1}(p) \subseteq V$. In particular, by (1),

$$\deg(f) = \int_{\mathbb{R}^k} \varphi\big(\,|f(x) - p|\,\big) \det\big(\mathrm{D}f(x)\big)\,\mathrm{d}x \tag{2}$$

where $\varphi\colon [0, \infty) \to \mathbb{R}$ is any continuous map with compact support and such that $\int_{\mathbb{R}^k} \varphi(|x|)\,\mathrm{d}x = 1$. We will show, later, that the continuity assumption on φ in (1) can be relaxed in our case (lemmas 1 and 2).

2.2 Some notions of nonsmooth analysis

In this Section we define the features of the maps we shall deal with and we give some basic definitions from nonsmooth analysis. For the sake of readability we adapt such definitions to our framework.

Following [10], a continuous function $f\colon U \subseteq \mathbb{R}^k \to \mathbb{R}^m$ is a *continuous selection of C^1 functions* if there exists a finite number of C^1 functions $f_1, \dots, f_\ell$, of U into $\mathbb{R}^m$ such that the *active index set* $\mathcal{I} := \{i \in \{1, \dots, \ell\}\colon f(x) = f_i(x)\}$ is nonempty for each $x \in U$. The functions f_i's are called *selection functions* of f. The function f is called a PC^1 *function* if at every point $x \in U$ there exists a neighborhood V such that the restriction of f to V is a continuous selection of C^1 functions.

A function $f\colon \mathbb{R}^k \to \mathbb{R}^m$ is said to be *piecewise linear* if it is a continuous selection of a finite number of linear functions.

We recall that a cone $C \subseteq \mathbb{R}^k$ with vertex at the origin is a positively homogeneous set, i.e. if $v \in C$ then $\alpha v \in C$ for all $\alpha \geq 0$. In particular

Definition 1. *Let $\pi_1 \neq \pi_2 \subset \mathbb{R}^k$ be two half hyper-planes with common boundary $\partial\pi_1 = \partial\pi_2$ containing the origin. Thus $\mathbb{R}^k \setminus (\pi_1 \cup \pi_2)$ is an open set with two connected components A_1 and A_2. We call each connected component an* open wedge *of $\mathbb{R}^k$. The closure of an open wedge of $\mathbb{R}^k$ is called a* wedge *of $\mathbb{R}^k$. An* admissible cone (with vertex at the origin) *$C \subseteq \mathbb{R}^k$ is a cone with nonempty interior and vertex at the origin which is given by the intersection of a finite number of wedges of $\mathbb{R}^k$, hence it is closed.*

Remark 1. *Notice that an admissible cone may not be convex.*

Given an admissible cone $C \subseteq \mathbb{R}^k$, its *spanning angle (at the origin)*, say α, is the Lebesgue $(k-1)$-dimensional measure of $C \cap S^{k-1}$. Observe that the Lebesgue k-dimensional measure of $C \cap B(0,r)$ is given by

$$\operatorname{meas}(C \cap B(0,r)) = \frac{\alpha r^k}{k\omega_k}\omega_k = \frac{\alpha r^k}{k}. \tag{3}$$

Definition 2. *A* decomposition in admissible cones *of $\mathbb{R}^k$ is a finite collection $C_1, \dots, C_N$ of admissible cones with pairwise disjoint interiors and such that $\mathbb{R}^k = \cup_{i=1}^N C_i$. Notice that $C_i \cap C_j = \partial C_i \cap \partial C_j$ for any $1 \leq i < j \leq N$.*

Definition 3 (SPL map)**.** *A* strongly piecewise linear map (at the origin of $\mathbb{R}^k$)*, SPL map in what follows, is a continuous function $G\colon \mathbb{R}^k \to \mathbb{R}^k$ such that there exist a decomposition $C_1, \dots, C_N$ of $\mathbb{R}^k$ in admissible cones, and linear maps $L_1, \dots, L_N$ with*

$$G(x) = L_i x \quad \text{for } x \in C_i.$$

We say that G is nondegenerate *if* $\operatorname{sign}(\det L_i)$ *is constant and nonzero for all $i = 1, \dots, N$.*

We also recall the notion of Bouligand derivative. Given an open set $U \subseteq \mathbb{R}^s$ and a locally Lipschitz function $f\colon U \to \mathbb{R}^m$, we say that f is

Bouligand differentiable at $x_0 \in U$ if there exists a positively homogeneous function, $f'(x_0, \cdot)\colon \mathbb{R}^s \to \mathbb{R}^m$ with the property that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - f'(x_0, x - x_0)\|}{\|x - x_0\|} = 0.$$

This uniquely determined function $f'(x_0, \cdot)$ is called the *Bouligand derivative* of f at x_0 (see Examples 5.1 and 5.2 [18]). Notice that the Bouligand derivative plays the same role of the differential in the approximation of the function f, the only difference consists in the positive homogeneity of the derivative in place of linearity.

An important fact proved by Kuntz/Scholtes [10] is the following:

Proposition 1 ([10, Prop. 2.1])**.** *Let $U \subseteq \mathbb{R}^s$ be an open set. Any PC^1 function $f\colon U \to \mathbb{R}^m$ is locally Lipschitz and, at every $x_0 \in U$, it has a piecewise linear Bouligand derivative $f'(x_0, \cdot)$ which is a continuous selection of the Fréchet derivatives of the selection functions of f at x_0.*

Following [16] we consider a generalization of the notion of Jacobian matrix $\nabla f(x)$ of a function $f\colon \mathbb{R}^k \to \mathbb{R}^k$ at a Fréchet differentiability point x. Let $f\colon \mathbb{R}^k \to \mathbb{R}^k$ be locally Lipschitz at x_0. We define $\mathrm{Jac}(f, x_0)$ as the set of limit points of sequences $\{\nabla f(x_j)\}$ where $\{x_j\}$ is a sequence converging to x_0 and such that f is Fréchet differentiable at x_j with Jacobian $\nabla f(x_j)$. By Rademacher's Theorem $\mathrm{Jac}(f, x_0)$ is nonempty, see [16]. Moreover it can be shown (see [4, 5]) that the convex hull of $\mathrm{Jac}(f, x_0)$ is equal to the Clarke generalized Jacobian $\partial f(x_0)$ of f at x_0.

For a PC^1 function $f\colon U \subseteq \mathbb{R}^k \to \mathbb{R}^k$, with selection functions f_i, the Bouligand derivative and the above generalized notion of Jacobian are related by the following formula [16, Lemma 2]:

$$\mathrm{Jac}\left(f'(x_0, \cdot), 0\right) \subseteq \mathrm{Jac}(f, x_0) = \left\{\nabla f_i(x_0) : i \in \bar{\mathcal{I}}(x_0)\right\},$$

where $\bar{\mathcal{I}}(x_0) = \left\{i : x_0 \in \mathrm{cl\,int}\{x \in U : i \in \mathcal{I}(x)\}\right\}$, see e.g. [10]. By Proposition 1, one has that $f'(x_0, \cdot)$ is continuous and piecewise linear, hence it is locally Lipschitz. Thus $\mathrm{Jac}\left(f'(x_0, \cdot), 0\right)$ is well defined.

The following result, due to [16], shows how degree theory relates to the above notions of nonsmooth analysis. We quote it here as it will play a fundamental role in what follows and, for the readers convenience, we formulate it with our notation.

Theorem 1. *Let $f\colon U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ be a PC^1 function. Then f is a Lipschitz local homeomorphism at $x_0 \in U$ if and only if $\mathrm{Jac}(f, x_0)$ consists of matrices whose determinants have the same nonzero sign and, for a sufficiently small neighborhood U_0 of x_0, $\deg(f, U_0, y_0)$, $y_0 := f(x_0)$, is well-defined and has value ± 1.*

A second result by [16] shows how the local invertibility of a PC^1 map and of its Bouligand derivative are related by means of their Jac.

Theorem 2. *Let $f\colon U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ be a PC^1 function, and let $x_0 \in U$. Assume that*

$$\operatorname{Jac}(f, x_0) = \operatorname{Jac}\big(f'(x_0, \cdot), 0\big),$$

then the following statements are equivalent:

1. *f is a Lipschitz local homeomorphism at $x_0 \in U$;*
2. *$f'(x_0, \cdot)$ is bijective;*
3. *$f'(x_0, \cdot)$ is a Lipschitz (global) homeomorphism.*

Moreover, if any of (1)–(3) holds, then f is a local PC^1 homeomorphism at x_0.

3 Estimates on the degree of SPL maps

Consider a nondegenerate SPL relative to the decomposition $C_1, \dots, C_N$ of $\mathbb{R}^k$ in admissible cones.

$$G\colon x \in \mathbb{R}^k \mapsto A_i x \in \mathbb{R}^k \quad \text{for } x \in C_i, \tag{4}$$

where $A_1, \dots A_N$ are $k \times k$ invertible matrices satisfying the compatibility condition $A_i x = A_j x \quad \forall x \in C_i \cap C_j$ and whose determinants have the same sign.

Notice that the classic differential of G, $\mathrm{D}G$ is well defined almost everywhere in $\mathbb{R}^k$. Moreover G enjoys nice approximation properties:

Remark 2. *For any $n \in \mathbb{N}$ let $\theta_n\colon \mathbb{R}^k \to [0,1]$ be a C^1 function with support in the ball centered at the origin of $\mathbb{R}^k$ with radius $1/n$ and $\int_{\mathbb{R}^k} \theta_n(x)\,\mathrm{d}x = 1$. If G is as in (4), define $G_n = \theta_n * G$ that is $G_n(x) = \int_{\mathbb{R}^k} \theta_n(y) G(x-y)\,\mathrm{d}y$. The following properties are readily verified:*

- *For all $n \in \mathbb{N}$ the maps $G_n\colon \mathbb{R}^k \to \mathbb{R}^k$ are C^1;*
- *The sequence $\{G_n\}_{n\in\mathbb{N}}$ converges uniformly to G on compact sets;*
- *For any given $x \in \mathbb{R}^k \setminus \cup_{i=1}^N \partial C_i$, $\mathrm{D}G_n(x)$ is equal to $\mathrm{D}G(x)$ for sufficiently large n; in particular $\{\mathrm{D}G_n\}_{n\in\mathbb{N}}$ converges to $\mathrm{D}G$ pointwise in $x \in \mathbb{R}^k \setminus \cup_{i=1}^N \partial C_i$.*

The following technical lemma shows that the degree of G can be computed using the formula (2) even if G is not C^1:

Lemma 1. *Let G be as above. Then,*

$$\deg(G) = \int_{\mathbb{R}^k} \varphi\left(|G(x)|\right) \det\left(\mathrm{D}G(x)\right) \mathrm{d}x, \tag{5}$$

where φ is any $C^0([0,+\infty))$ function with compact support and with the property that $\int_{\mathbb{R}^k} \varphi\left(|x|\right) \mathrm{d}x = 1$.

Sketch of the proof. Let $U \subset \mathbb{R}^k$ be a relatively compact neighborhood of the origin that contains the support of $\varphi\left(|G(x)|\right)$.

By Remark 2, taking n sufficiently large, we have that $(G_n, U, 0)$ is admissible and,

$$\deg(G_n, U, 0) = \deg(G),$$

So that, $\deg(G) = \lim\limits_{n\to\infty} \deg(G_n, U, 0)$.

Also, for sufficiently large n, one can assume that the support of $x \mapsto \varphi\big(|G_n(x)|\big)$ is contained in U so that, by (1), one obtains

$$\deg(G_n, U, 0) = \int_U \varphi\left(|G_n(x)|\right) \det\left(\mathrm{D}G_n(x)\right) \mathrm{d}x.$$

Thus, again by Remark 2, passing to the limit, by Lebesgue's dominated convergence theorem, one has that

$$\begin{aligned}\deg(G) = \lim_{n\to\infty} \deg(G_n, U, 0) = \lim_{n\to\infty} \int_U \varphi\left(|G_n(x)|\right) \det\left(\mathrm{D}G_n(x)\right) \mathrm{d}x = \\ = \int_U \varphi\left(|G(x)|\right) \det\left(\mathrm{D}G(x)\right) \mathrm{d}x = \int_{\mathbb{R}^k} \varphi\left(|G(x)|\right) \det\left(\mathrm{D}G(x)\right) \mathrm{d}x,\end{aligned}$$

the last equality holds as the support of $\varphi\left(|G(x)|\right)$ is contained in U. □

Actually, formula (5) holds also for some discontinuous functions φ as shown by the following lemma:

Lemma 2. *Formula (5) holds also for $\varphi(t) = \dfrac{1}{\omega_k} \mathbb{1}_{[0,1]}(t)$, where ω_k is the Lebesgue measure of the unit ball of $\mathbb{R}^k$.*

Proof. Consider the sequence $\{p_n\}_{n\in\mathbb{N}}$ of continuous functions given by

$$p_n(x) = \begin{cases} 1 & 0 \le x < 1 - \frac{1}{n}, \\ \frac{1}{2}\big(n(1-x)+1\big) & 1 - \frac{1}{n} \le x \le 1 + \frac{1}{n}, \\ 0 & x > 1 + \frac{1}{n}. \end{cases}$$

Let $\varphi_n(x) := \frac{1}{k\omega_{k-1}} p_n(x)$. The claim follows from Lebesgue dominated convergence theorem. □

Choosing φ as in Lemma 2, and recalling that $C_1, \dots, C_N$ are a partition of $\mathbb{R}^k$ into admissible cones, we obtain

$$\begin{aligned}
\deg(G) &= \frac{1}{\omega_k} \int_{\mathbb{R}^k} \mathbb{1}_{B(0,1)}(G(x)) \det\left(\mathrm{D}G(x)\right) \mathrm{d}x \\
&= \frac{1}{\omega_k} \sum_{i=1}^{N} \int_{C_i} \mathbb{1}_{B(0,1)}(A_i x) \det(A_i) \, \mathrm{d}x \\
&= \frac{1}{\omega_k} \sum_{i=1}^{N} \det(A_i) \int_{C_i} \mathbb{1}_{B(0,1)}(A_i x) \, \mathrm{d}x \\
&= \frac{1}{\omega_k} \sum_{i=1}^{N} \det(A_i) \operatorname{meas}\left(C_i \cap A_i^{-1}(B(0,1))\right).
\end{aligned} \tag{6}$$

Taking into account the fact that G is a nondegenerate SPL, we have that the sign of the determinants $\det(A_i)$ agree. Thus, by (6), we have

$$|\deg G| = \frac{1}{\omega_k} \sum_{i=1}^{N} |\det(A_i)| \operatorname{meas}\left(C_i \cap A_i^{-1}(B(0,1))\right). \tag{7}$$

We can use formula (7) to estimate $|\deg(G)|$ in terms of the singular values of the matrices $A_1, \dots, A_N$. Let α_i be the measure of the solid angle spanned by C_i. Let $\sigma_{i,\min}$ and $\sigma_{i,\max}$ be the minimum and the maximum singular value of A_i, respectively. Then

$$\left\|A_i^{-1}x\right\|^2 = \left\langle A_i^{-1}x \, , \, A_i^{-1}x \right\rangle = \left\langle (A_i^{-1})^t A_i^{-1}x \, , \, x \right\rangle = \left\langle (A_i A_i^t)^{-1}x \, , \, x \right\rangle,$$

so that $\left\|A_i^{-1}x\right\| \in \left(\sigma_{i,\max}^{-1} \|x\| \, , \, \sigma_{i,\min}^{-1} \|x\|\right)$. Thus, we have

$$B\left(0, \sigma_{i,\max}^{-1}\right) \subseteq A_i^{-1}\left(B(0,1)\right) \subseteq B\left(0, \sigma_{i,\min}^{-1}\right),$$

whence, by equation (3),

$$\frac{\alpha_i}{k\sigma_{i,\max}^k} \leq \operatorname{meas}\left(C_i \cap A_i^{-1}(B(0,1))\right) \leq \frac{\alpha_i}{k\sigma_{i,\min}^k}.$$

Thus, plugging the above inequality into (6), we obtain

$$\underline{\gamma}_G := \frac{1}{k\omega_k} \sum_{i=1}^{N} \frac{\alpha_i \, |\det(A_i)|}{\sigma_{i,\max}^k} \leq |\deg(G)| \leq \frac{1}{k\omega_k} \sum_{i=1}^{N} \frac{\alpha_i \, |\det(A_i)|}{\sigma_{i,\min}^k} =: \overline{\gamma}_G, \tag{8}$$

which is our main estimate of the degree.

Let $\sigma_{i,\min} = \sigma_{i,1} \leq \ldots \leq \sigma_{i,k} = \sigma_{i,\max}$ be the singular values of A_i. Since $\prod_{j=1}^{k} \sigma_{i,j} = \left|\det(A_i)\right|$, we obtain

$$\sigma_{i,\min}^k \leq \left|\det(A_i)\right| \leq \sigma_{i,\max}^k.$$

Notice that for the quantities $\underline{\gamma}_G$ and $\overline{\gamma}_G$ defined in (8), we always have the relation $0 < \underline{\gamma}_G \leq \overline{\gamma}_G$.

We can now deduce some important facts concerning the invertibility of the map G:

Proposition 2. *For any SPL map the following facts hold:*

1. *If* $\overline{\gamma}_G - \underline{\gamma}_G < 1$, *then* $|\deg(G)| = \left\lceil \underline{\gamma}_G \right\rceil = \lfloor \overline{\gamma}_G \rfloor$.
2. *If* $\underline{\gamma}_G > 1$, *then* G *is not invertible.*
3. *If* $\overline{\gamma}_G < 2$, *then* G *is invertible and* $|\det(G)| = 1$.

Proof. Since for the integer $\deg(G)$ we have $\underline{\gamma}_G \leq |\deg(G)| \leq \overline{\gamma}_G$, claim *1* is obvious. Claims *2* and *3* follow from the fact that $\deg(G)$ is an integer and from Theorem 1. □

Remark 3. *Assume* $k = 2$, *so that* $|\det(A_i)| = \sigma_{i,\min}\sigma_{i,\max}$ *for any* $i = 1, \ldots, N$. *Thus*

$$\underline{\gamma}_G = \frac{1}{2\pi} \sum_{i=1}^{N} \alpha_i \, \frac{\sigma_{i,\min}}{\sigma_{i,\max}}, \qquad \overline{\gamma}_G = \frac{1}{2\pi} \sum_{i=1}^{N} \alpha_i \, \frac{\sigma_{i,\max}}{\sigma_{i,\min}}.$$

These formulas emphasize the role played by the ellipticity of the image of the unit circle under each map A_i.

The main difference of Inequality (8), compared to those developed in [18, 20], consists in the fact that, here, the spanning angle α_i on which each index function operates is taken into account. On one hand, one should not get the false impression that the present results are stronger than those of [18, 20]. Indeed, the function G of [18, Example 4.4] is easily seen to have degree 1 whereas, for the corresponding values $\overline{\gamma}_G$ and $\underline{\gamma}_G$, we can get the following numerical estimates:

$$\overline{\gamma}_G \simeq 2407.0433\ldots \qquad \text{and} \qquad \underline{\gamma}_G \simeq 0.2566\ldots,$$

which are inconclusive. Thus, Proposition 2 does not yield the invertibility of G, whereas it can be obtained as an immediate consequence of [18, Thm. 4.2] or [20, Thm. 3.11]. On the other hand, the former Proposition has a greater applicability range than the latter theorems. This fact is illustrated by the following example and by the discussion in the following section that concerns the invertibility of small SPL perturbations of the identity.

Let us also remark the fact that in some cases it is not necessary to have a precise computation of the singular values of the involved matrices. Sometimes sufficient estimates can be obtained (compare [22]). For instance, one might be able to get a lower estimate for $\overline{\gamma}_G$ using as a lower bound the smallest singular values.

Example 1. *Let $k = 2$ and let*

$$A_1 = \begin{pmatrix} 1 & -8/10 \\ 0 & 11/10 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2/10 & 0 \\ 0 & 11/10 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1 & 0 \\ -1/10 & 11/10 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 12/10 \end{pmatrix}.$$

Let C_i, $i = 1, \dots, 4$, be the admissible cones given, in polar coordinates, by the pairs (ρ, θ) with arbitrary nonegative ρ's and θ chosen according to the following table:

C_1	C_2	C_3	C_4
$0 \le \theta \le \frac{\pi}{4}$	$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \frac{3}{4}\pi$	$\frac{3}{4}\pi \le \theta \le 2\pi$

It is not difficult to check that G defined as in (4), i.e. $G(x) = A_i x$ for $x \in C_i$, with the above choice of the A_i's and C_i's, is SPL. By [22, Thm. 2] one immediately sees that

$$\sigma_{1,\min} \ge \frac{2}{10}, \quad \sigma_{3,\min} \ge \frac{9}{10},$$

furthermore, one clearly has $\sigma_{2,\min} = \frac{2}{10}$ and $\sigma_{4,\min} = 1$. Hence

$$\overline{\gamma}_G \le \frac{1}{2\pi}\left(\frac{\pi}{4}\frac{11/10}{(2/10)^2} + \frac{\pi}{4}\frac{22/100}{(2/10)^2} + \frac{\pi}{4}\frac{11/10}{(9/10)^2} + \frac{5\pi}{4}\frac{12/10}{1}\right) = \frac{3269}{2592} < 2.$$

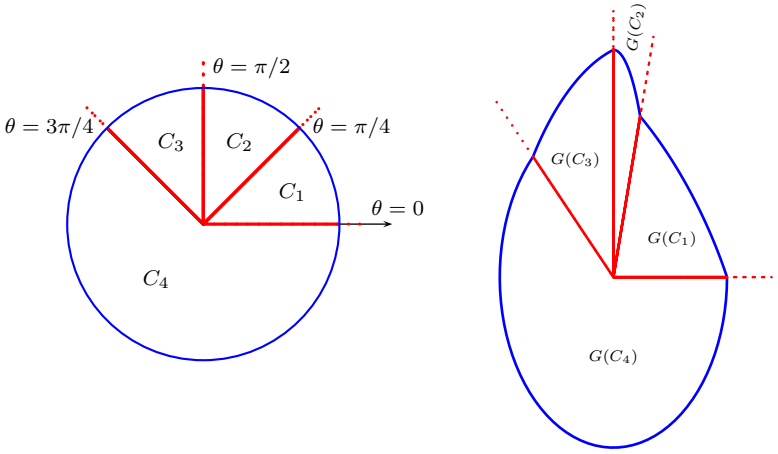
Thus G is invertible by Proposition 2. In Figure 1 we illustrate the invertibility of G by drawing the image of the unit circle S^1.

Observe that the invertibility of the map G in Example 1 does not follow from the results of [18, 20] as the cone C_4 is not convex.

Remark 4. *The degree of an SPL map can sometimes be determined by a "factorization" procedure. To understand how this works, observe that the composition of SPLs is again an SPL (with an appropiate decomposition of $\mathbb{R}^k$, usually with a larger number of decomposing cones). Also notice that, as a consequence of the Multiplication Theorem of the degree (see e.g. [11, Thm. 2.3.1]), if F and G are SPLs then* $\deg(F \circ G) = \deg(F)\deg(G)$. *Thus, given a generic SPL, say H, finding a "factorization" $H = F \circ G$ with simpler F and G could be sufficient. The same clearly applies to the invertibility problem of H. However, finding a factorization may not be simple, as shown by the following example.*

Example 2. *Consider the map*

$$G(x) := \mathbb{1}_{\{x_2 \ge 0\}}(x) A_1 x + \mathbb{1}_{\{x_2 < 0\}} A_2 x,$$



(a) The cones $C_1, \ldots, C_4$ and the unit circle

(b) The images of $C_1, \ldots, C_4$ and of the unit circle

Figure 1: Effects of the map G of Example 1.

with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1/2 \\ 1 & 1/2 \end{pmatrix}.$$

A direct computation shows that

$$G^2(x) = \begin{cases} A_1^2 x & x_2 \geq 0,\ x_1 + x_2 \geq 0, \\ A_2 A_1 x & x_2 \geq 0,\ x_1 + x_2 \leq 0, \\ A_1 A_2 x & x_2 \leq 0,\ x_1 + \frac{x_2}{2} \geq 0, \\ A_2^2 x & x_2 \leq 0,\ x_1 + \frac{x_2}{2} \leq 0, \end{cases}$$

and, similarly,

$$G^3(x) = \begin{cases} A_1^3 x & x_2 \geq 0,\ x_1 + x_2 \geq 0,\ 2x_1 + x_2 \geq 0, \\ A_2 A_1^2 x & x_2 \geq 0,\ x_1 + x_2 \geq 0,\ 2x_1 + x_2 \leq 0, \\ A_2^2 A_1 x & x_2 \geq 0,\ x_1 + x_2 \leq 0,\ 3x_1 + x_2 \leq 0, \\ A_1^2 A_2 x & x_2 \leq 0,\ x_1 + \frac{x_2}{2} \geq 0,\ x_1 \geq 0, \\ A_1 A_2^2 x & x_2 \leq 0,\ x_1 + \frac{x_2}{2} \leq 0,\ \frac{3}{2}x_1 - \frac{x_2}{4} \geq 0, \\ A_2^3 x & x_2 \leq 0,\ x_1 + \frac{x_2}{2} \leq 0,\ \frac{3}{2}x_1 - \frac{x_2}{4} \leq 0. \end{cases}$$

One should observe that the products $A_1 A_2 A_1$ and $A_2 A_1^2$ do not appear in the expression of G^3. As we see, the powers of G become rapidly unwieldy.

Moreover, suppose one is given an explicit SPL map in $\mathbb{R}^2$ *as the following:*

$$H(x,y) = \begin{cases} (x, 3x+y) & y \geq 0,\ x+y \geq 0,\ 2x+y \geq 0, \\ (-\frac{y}{2}, 2x+\frac{y}{2}) & y \geq 0,\ x+y \geq 0,\ 2x+y \leq 0, \\ (-\frac{x}{4}-\frac{3y}{4}, \frac{5x}{4}-\frac{y}{4}) & y \geq 0,\ x+y \leq 0,\ 3x+y \leq 0, \\ (x-\frac{y}{2}, 3x-\frac{y}{2} & y \leq 0,\ x+\frac{y}{2} \geq 0,\ x \geq 0, \\ (\frac{x}{2}-\frac{2y}{4}, 2x-y) & y \leq 0,\ x+\frac{y}{2} \leq 0,\ \frac{3}{2}x-\frac{y}{4} \geq 0, \\ (-\frac{x}{4}-\frac{5y}{8}, \frac{5x}{4}-\frac{7y}{8}) & y \leq 0,\ x+\frac{y}{2} \leq 0,\ \frac{3}{2}x-\frac{y}{4} \leq 0. \end{cases}$$

It is not easy to recognize at first sight that H *is invertible being, in fact,* $H = G^3$*. Figure 2 shows the image of the unit circle and the relative cones, under the maps,* G*,* G^2 *and* G^3*.*

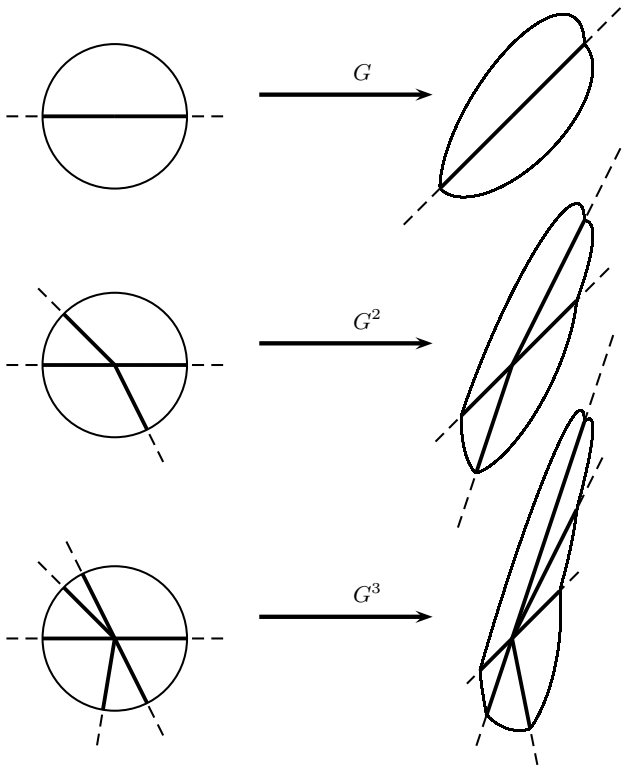


Figure 2: The map G of Example 2. Decomposition in admissible cones for G, G^2 and G^3, and the respective transformations of the unit circle.

We conclude this section with a simple application to local invertibility of nonlinear PC^1 functions. Let us first recall the following result:

Theorem 3 ([18, Thm. 5.1])**.** *Let* f *be an* $\mathbb{R}^k$*-valued* PC^1 *function in a neighborhood of* $x_0 \in \mathbb{R}^k$*. Assume that*

1. *The determinants of all the elements of* $\mathrm{Jac}(f, x_0)$ *have the same nonzero sign;*

2. *The Bouligand differential of* f *at* x_0 *is an invertible piecewise linear map.*

Then f *is locally invertible at* x_0*.*

Let f be an $\mathbb{R}^k$-valued PC^1 function in a sufficiently small ball $B(x_0, \rho) \subseteq \mathbb{R}^k$, and let $\mathcal{I}_0 = \{1, \dots, N\}$ be the active index set in $B(x_0, \rho)$. For each $i \in \mathcal{I}_0$ define

$$S_i := \big\{x \in B(x_0, \rho) : f(x) = f_i(x)\big\}.$$

Let $C_1, \dots, C_N$ be the tangent cones (in the sense of Bouligand) at x_0 to the sectors $S_1, \dots, S_N$. Assume the following:

- The C_i's are admissible cones;
- $\mathrm{d}f_i(x_0)x = \mathrm{d}f_j(x_0)x \quad$ for any $x \in C_i \cap C_j, \quad 1 \le i < j \le N$;
- f satisfies assumption 1 of Theorem 3.

Define

$$G(x) := \mathrm{d}f_i(x_0)x \quad \text{for } x \in C_i.$$

Then G is a nondegenerate SPL. If $\overline{\gamma}_G < 2$, then also Assumption 2 of Theorem 3 is satisfied. We thus obtain the following:

Corollary 1. *Let* f *and* G *be as above, with* $\overline{\gamma}_G < 2$*. Then* f *is a Lipschitz homeomorphism in a sufficiently small neighborhood of* x_0*.*

4 SPL perturbations of the identity

In this section we illustrate the meaning of the results obtained in the previous one by underlining the importance of taking into account the piecewise linear nature of the perturbation. In order to do this, we compare Proposition 2 with a more traditional-style invertibility proposition that does not take this aspect into account.

Consider the special case when G is as in (4) with the matrices A_i given by perturbations of the $k \times k$ identity matrix $\mathbb{I}$ as follows:

$$A_i = \mathbb{I} + M_i, \qquad i = 1, \dots, N.$$

Broadly speaking, as a consequence of degree theory combined with the continuity of the determinant and of [16, Thm. 4], one can prove that "small" continuous piecewise linear perturbations of the identity are invertible. Namely, one can prove the following:

Proposition 3. *Let G and $M_1, \dots, M_N$ be as above. Assume there exists δ such that $\|M_i\|_{\mathcal{F}} \leq \delta$ for any $i = 1, \dots, N$ and*

$$\sqrt{k}\delta(1+\delta)^{k-1} < 1. \tag{9}$$

Then G is invertible.

The proof of Proposition 3, being somewhat out of the context of this paper has been moved to the appendix.

Remark 5. *Let $k = 2$, then Assumption (9) in Proposition 3 becomes $\sqrt{2}\delta(\delta+1) < 1$ that is satisfied for any*

$$0 \leq \delta < \delta^* := \frac{\sqrt{\sqrt{8}+1}-1}{2} \simeq 0.4873\dots.$$

Therefore G is invertible whatever the M_i's, provided $\|M_i\|_{\mathcal{F}} < \delta^*$ for any $i = 1, \dots N$.

Remark 6. *Observe that the assumptions of Proposition 3 are independent of the chosen basis. In fact, if Q is an orthogonal matrix representing a change of coordinates, one has for $i = 1, \dots, N$*

$$Q^T A_i Q = Q^T(\mathbb{I} + M_i)Q = \mathbb{I} + Q^T M_i Q,$$

and

$$\left\|Q^T M_i Q\right\|_{\mathcal{F}} = \sqrt{\mathrm{Tr}\left(Q^T M_i Q \left(Q^T M_i Q\right)^T\right)} = \sqrt{\mathrm{Tr}\left(Q^T M_i M_i^T Q\right)} = \|M_i\|_{\mathcal{F}},$$

the trace being invariant under change of coordinates, see [2, Section 9.2].

The whole point of this section is that Proposition 3, being unaware of the precise structure of the perturbation can be ineffective in some cases. On the contrary, Proposition 2 might still yield invertibility.

For example, for $k = 2$, we may consider G as the SPL map given by

$$G(x,y) = \begin{cases} (x,y) & \text{if } y \geq 0, \\ (x, \frac{3}{2}y) & \text{if } y \leq 0. \end{cases}$$

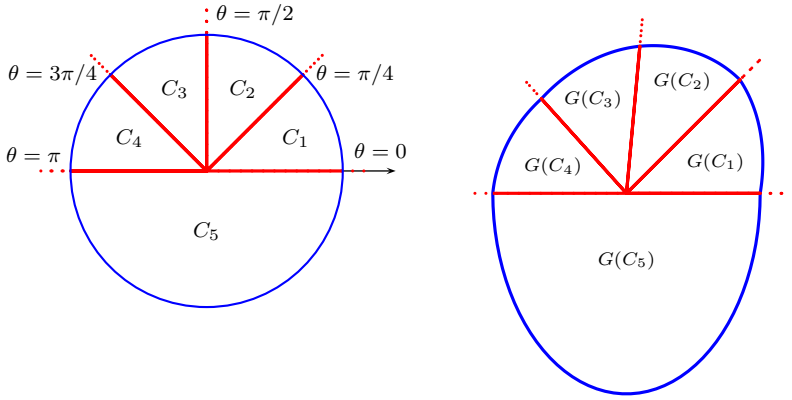
That is clearly continuous. Call C_1 and C_2, respectively, the half-planes $\{y \geq 0\}$ and $\{y \leq 0\}$ and let

$$A_1 = \mathbb{I}, \quad \text{and} \quad A_2 = \mathbb{I} + M_2 = \mathbb{I} + \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

One immediately verifies that M_2 does not verify assumption (9) –see Remark 5– hence Proposition 3 does not apply. On the contrary, we immediately get $\overline{\gamma}_G = 5/4$, so that Proposition 2 yields the invertibility of G. Notice

that the invertibility of this particular G could also be obtained from [18, Thm. 4.2] or [20, Thm. 3.11].

A more interesting example of piecewise linear perturbation of the identity where the invertibility cannot be deduced neither from Proposition 3 nor from the results of [18, 20] but does indeed follow from Proposition 2 could easily be obtained from Example 1. Below we provide another example where there are more than four pieces involved.



(a) The cones $C_1, \ldots, C_5$ and the unit circle

(b) Images $G(C_1), \ldots, G(C_5)$ and of the unit circle

Figure 3: Effects of the map G of Example 3.

Example 3. *Let $k = 2$ and consider the matrices*

$$M_1 = \begin{pmatrix} 0 & 1/5 \\ 0 & 1/5 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1/10 & 1/10 \\ 1/10 & 1/10 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1/10 \\ 1/10 & 1/10 \end{pmatrix},$$
$$M_4 = \begin{pmatrix} 0 & 1/10 \\ 0 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix},$$

and $A_i = \mathbb{I} + M_i$ for $i = 1, \ldots, 5$. Let C_i be the cones given, in polar coordinates, by the pairs (ρ, θ) with arbitrary nonegative ρ's and θ chosen according to the following table:

C_1	C_2	C_3	C_4	C_5
$0 \le \theta \le \frac{\pi}{4}$	$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \frac{3}{4}\pi$	$\frac{3}{4}\pi \le \theta \le \pi$	$\pi \le \theta \le 2\pi$

It is not difficult to check that $G(x) = A_i x$ for $x \in C_i$, with the above choice of the A_i's and C_i's, is SPL.

One sees immediately –see Remark 5– that M_5 violates assumption (9)*, also there are more than 4 cones, thus neither Proposition 3 nor the results*

of [18, 20] apply. With some computation[1]*, though, we find*

$$\overline{\gamma}_G = -\frac{3\sqrt{401}}{1600} + \frac{\sqrt{61}}{240} + \frac{21\sqrt{5}}{1744} + \frac{108319}{104640} \simeq 1.057\ldots < 2$$

Hence, the invertibility of G follows from Proposition 2. We illustrate the invertibility of G by drawing the image of the unit circle S^1, see Figure 3

5 An implicit function theorem for PC^1 functions

In this Section we apply the results on the invertibility of SPL maps to the problem of local invertibility of Bouligand differentiable maps. We begin by stating and proving two technical lemmas:

Lemma 3. *Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^m$ and $\psi\colon \mathbb{R}^k \to \mathbb{R}^k$ be invertible mappings and let $\gamma\colon \mathbb{R}^m \to \mathbb{R}^k$ be given. Then, the map $F\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$ given by*

$$F(x,y) := \big(\varphi(x), \gamma(x) + \psi(y)\big), \quad \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^k,$$

is invertible with inverse given by

$$(\xi,\eta) \mapsto \big(\varphi^{-1}(\xi)\,,\, \psi^{-1}\big(\eta - \gamma\big(\varphi^{-1}(\xi)\big)\big)\big)\,. \tag{10}$$

Proof. Given $(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^k$, we can find a pair $(x,y) \in \mathbb{R}^m \times \mathbb{R}^k$ such that $F(x,y) = (\xi,\eta)$ if and only if

$$\begin{cases} \xi = \varphi(x), \\ \eta = \gamma(x) + \psi(y), \end{cases} \iff \begin{cases} x = \varphi^{-1}(\xi), \\ y = \psi^{-1}\big(\eta - \gamma(x)\big). \end{cases}$$

Or, equivalently,

$$\begin{cases} x = \varphi^{-1}(\xi), \\ y = \psi^{-1}\Big(\eta - \gamma\big(\varphi^{-1}(\xi)\big)\Big), \end{cases}$$

that proves (10). □

Lemma 4. *Let φ, ψ, γ and F be as in Lemma 3 and assume, in addition, that φ, ψ and γ are SPL, with φ and ψ nondegenerate. Then, F and F^{-1} are nondegenerate SPL as well.*

Proof. By a refinement, we can assume that φ and γ have the same underlying decomposition in admissible cones $C_1, \ldots, C_{N_1}$ of $\mathbb{R}^m$. Let $\hat{C}_1, \ldots, \hat{C}_{N_2}$ be the decomposition in admissible cones of $\mathbb{R}^k$ relative to the map ψ, i.e.

$$\begin{aligned} \varphi(x) = A_i x, \;\; \gamma(x) = L_i x, \quad \text{for } x \in C_i, \\ \psi(y) = B_j y, \quad \text{for } y \in \hat{C}_j, \end{aligned}$$

[1] We performed these computations with the help of the computer algebra system Maxima 5.41.0 running on Linux

for $A_i \in \mathbb{R}^{m\times m}$, $L_i \in \mathbb{R}^{k\times m}$, $B_j \in \mathbb{R}^{k\times k}$. Then, one can write

$$F(x,y) = (\, A_i x \,,\, L_i x + B_j y) \quad \text{for } (x,y) \in C_i \times \hat{C}_j,$$

and

$$F^{-1}(\xi,\eta) = \Big(A_i^{-1}\xi \,,\, B_j^{-1}\big(\eta - \gamma(A_i^{-1}\xi)\big)\Big) \quad \text{for } (\xi,\eta) \in A_i^{-1}C_i \times B_j^{-1}\hat{C}_j.$$

Hence both F and F^{-1} are SPL. Their nondegeneracy follows from Proposition 4.1 in [18]. □

Let $f\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ be a PC^1 map in a neighborhood of (x_0, y_0). Consider the partial functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ that, clearly, admit Bouligand derivatives $\partial_1 f(x_0, y_0)(\cdot)$ at $x_0 \in \mathbb{R}^m$ and $\partial_2 f(x_0, y_0)(\cdot)$ at $y_0 \in \mathbb{R}^k$, respectively. We refer to the latter differential as to the *partial differentials* of f at (x_0, y_0).

Proposition 4. *Let $f\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ be PC^1. Assume that, for each $(v, w) \in \mathbb{R}^m \times \mathbb{R}^k$, the following assumption holds:*

$$f'(x_0, y_0)(v, w) = \partial_1 f(x_0, y_0)(v) + \partial_2 f(x_0, y_0)(w). \tag{11}$$

Suppose also that $\partial_2 f(x_0, y_0)$ is SPL and nondegenerate. Then, there exists a neighborhood U of $0 \in \mathbb{R}^k$ and an unique (for a given U) Lipschitz function $h\colon U \to \mathbb{R}^m$ such that

$$f\big(x, h(x)\big) = 0, \quad \forall x \in U. \tag{12}$$

Proof. Define the map $\Phi\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$ by

$$\Phi(x, y) = \big(x, f(x, y)\big), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^k,$$

so that $\Phi(x_0, y_0) = (x_0, 0)$.

Let us denote by $\partial_1 f(x_0, y_0)(\cdot)$ and $\partial_2 f(x_0, y_0)(\cdot)$ the Bouligand derivatives of the partial functions $x \mapsto f(x, y_0)$ at $x_0 \in \mathbb{R}^m$ and of $y \mapsto f(x_0, y)$ at $y_0 \in \mathbb{R}^k$, respectively. Then, by (11), the Bouligand derivative of Φ at (x_0, y_0) can be written as

$$\Phi'(x_0, y_0)(v, w) = \big(v, \partial_1 f(x_0, y_0)(v) + \partial_2 f(x_0, y_0)(w)\big).$$

By Lemma 4 we have that $\Phi'(x_0, y_0)$ is an invertible and nondegenerate SPL map. Theorem 5.1 of [18] ensures that Φ is locally invertible with Lipschitz inverse at (x_0, y_0). Let us denote by Φ^{-1} its local inverse. Clearly, we can write

$$\Phi^{-1}(x, y) = \big(x, H(x, y)\big),$$

for some $\mathbb{R}^k$-valued Lipschitz function H defined on a neighborhood of $(x_0, 0) \in \mathbb{R}^m \times \mathbb{R}^k$. Define $h(x) = H(x, 0)$. Thus one has

$$\Big(x, f\big(x, h(x)\big)\Big) = \Phi\big(x, h(x)\big) = \Phi\big(x, H(x,0)\big) = \Phi\big(\Phi^{-1}(x,0)\big) = (x,0),$$

whence (12).

Let us now prove uniqueness. Suppose that h_0 is some $\mathbb{R}^k$-valued continuous map defined in a neighborhood of $x_0 \in \mathbb{R}^m$ with the property that $f\big(x, h_0(x)\big) = 0$ in its domain. Let U be some small neighborhood of x_0 contained in the intersection of the domains of h and h_0. Hence $\Phi\big(x, h(x)\big) = \Phi\big(x, h_0(x)\big)$ for all $x \in U$; the invertibiliy of Φ implies $h(x) = h_0(x)$. □

The following lemma shows that the assumption (11) in Proposition 4 holds when some continuity of the partial differentials is assumed.

Lemma 5. *Let $f\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ be a PC^1 map in a neighborhood of (x_0, y_0) and assume that, for each $(v, w) \in \mathbb{R}^m \times \mathbb{R}^k$, at least one of the following assumptions hold:*

1. *$y \mapsto \partial_1 f(x_0, y)(v)$ is continuous in a neighborhood of y_0,*
2. *$x \mapsto \partial_2 f(x, y_0)(w)$ is continuous in a neighborhood of x_0.*

Then, for each $(v, w) \in \mathbb{R}^m \times \mathbb{R}^k$,

$$f'(x_0, y_0)(v, w) = \partial_1 f(x_0, y_0)(v) + \partial_2 f(x_0, y_0)(w).$$

Proof. We prove the assertion when (2) holds, the other case being analogous. Given $(v, w) \in \mathbb{R}^m \times \mathbb{R}^k$ with $\|(v, w)\| = 1$, let us define, for $\alpha > 0$ sufficiently small, the function $\rho\colon (0, \alpha) \to \mathbb{R}$ by

$$\rho(t) = \frac{\|f(x_0 + tv, y_0 + tw) - f(x_0, y_0) - \partial_1 f(x_0, y_0)(tv) - \partial_2 f(x_0, y_0)(tw)\|}{t}$$

We only need to show that $\rho(t) \to 0$ as $t \to 0^+$. Then the assertion will follow from the arbitrary choice of (v, w).

In order to prove the above limit property for ρ it is convenient to let

$$h(t, s) := \|f(x_0 + sv, y_0 + tw) - f(x_0 + sv, y_0) - \partial_2 f(x_0 + sv, y_0)(tw)\|.$$

Since the partial differentials in the sense of Bouligand exist in a neighborhood of (x_0, y_0) and by assumption (2), the function h is well-defined and continuous in a sufficiently small neighborhood of $(0, 0) \in \mathbb{R}^2$. By triangle inequality we have

$$\begin{aligned} \rho(t) \leq \frac{1}{t}\Big(h(t,t) &+ \|f(x_0 + tv, y_0) - f(x_0, y_0) - \partial_1 f(x_0, y_0)(tv)\| + \\ &+ \|\partial_2 f(x_0 + tv, y_0)(tw) - \partial_2 f(x_0, y_0)(tw)\|\Big). \end{aligned} \tag{13}$$

Observe that by the definition of Bouligand derivative of the partial function $y \mapsto f(x_0, y)$ at y_0 we can write

$$h(t,s) = tg(t,s),$$

with g a continuous function in a neighborhood of $(0,0)$ such that $g(0,s) = 0$ for all s close to 0. Thus, by inequality (13), taking also into account the positive homegeneity of the Bouligand derivative, we get

$$\begin{aligned}\rho(t) \leq g(t,t) + \frac{1}{t} \|f(x_0 + tv, y_0) - f(x_0, y_0) - \partial_1 f(x_0, y_0)(tv)\| + \\ + \|\partial_2 f(x_0 + tv, y_0)(w) - \partial_2 f(x_0, y_0)(w)\|\end{aligned} \tag{14}$$

for sufficiently small $t > 0$.

It is enough to prove that each summand on the right hand side of (14) tends to zero as $t \to 0^+$. This assertion follows by continuity of g for the first summand, from Bouligand differentiability of the partial function $x \mapsto f(x, y_0)$ for the second, and from assumption (2) for the third one. The assertion follows. □

Theorem 4. *Let $f\colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^k$ be PC^1 with SPL Bouligand derivative at (x_0, y_0). Assume that, for each $(v, w) \in \mathbb{R}^m \times \mathbb{R}^k$, at least one of the following assumptions hold:*

1. *$y \mapsto \partial_1 f(x_0, y)(v)$ is continuous in a neighborhood of y_0,*
2. *$x \mapsto \partial_2 f(x, y_0)(w)$ is continuous in a neighborhood of x_0.*

Suppose that

$$\partial_2 f(x_0, y_0)(v) = A_i v \quad \text{for } v \in C_i,$$

where $C_1, \ldots, C_N$ is a partition of $\mathbb{R}^k$ in admissible cones and $A_1, \ldots A_N$ are invertible $k \times k$ matrices having determinants of the same sign and satisfying the compatibility condition $A_i x = A_j x \quad \forall x \in C_i \cap C_j$. Let $\sigma_{i,\min}$ and $\sigma_{i,\max}$ be the minimum and the maximum singular value of A_i, respectively. If

$$\frac{1}{k\omega_k} \sum_{i=1}^{N} \frac{\alpha_i \, |\det(A_i)|}{\sigma_{i,\min}^k} < 2, \tag{15}$$

then there exists a neighborhood U of $0 \in \mathbb{R}^k$ and a unique (for a given U) Lipschitz function $h\colon U \to \mathbb{R}^m$ such that $f\big(x, h(x)\big) = 0, \quad \forall x \in U$.

Proof. The claim easily follows from Proposition 2, Lemma 5 and Proposition 4. □

Remark 7. *Condition (15) in Theorem 4 can be dropped if either $N = 1, 2, 3$ or if $N = 4$ with C_i convex for $i = 1, \ldots, 4$. In fact, in these cases, Proposition 2 can be replaced by Theorem 3.11 in [20].*

In order to illustrate Theorem 4 we consider the following elementary example:

Example 4. *Consider the set*

$$C := \Big\{(x,y) \in \mathbb{R}^2 : |x-1| + \sin\big(\max\{y-x, 2y\}\big) = 0\Big\}.$$

Clearly, the point $p_0 = (1,0)$ *belongs to* C*. By Remark 7 (with* $N = 1$*) we have that* C *is a Lipschitz curve in a neighborhood of* p_0*. Figure 4 shows the set* C*.*

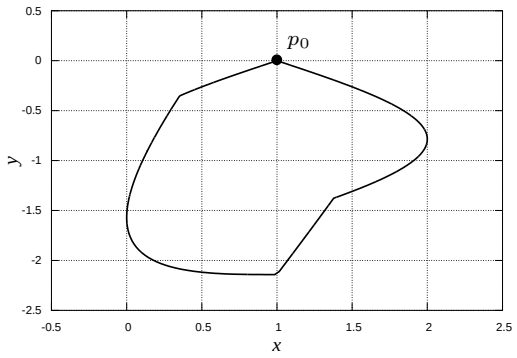


Figure 4: The curve C of Example 4.

6 Appendix

This section is devoted to the proof of Proposition 3. We need two technical lemmas whose proof we provide for the sake of completeness.

Lemma 6. *Let* A*,* B *be* $k \times k$ *real matrices. Denote by* $A_{(1)}, \dots, A_{(k)}$ *and by* $B_{(1)}, \dots, B_{(k)}$ *the columns of* A *and* B*, respectively. Then*

$$|\det(A) - \det(B)| \le \sum_{j=1}^{k} \big\|A_{(1)}\big\| \dots \big\|A_{(j-1)}\big\| \, \big\|A_{(j)} - B_{(j)}\big\| \, \big\|B_{(j+1)}\big\| \dots \big\|B_{(k)}\big\|$$

Proof. Since the determinant of a matrix is a linear function of each of the columns, we can write

$$\det(A) - \det(B) = \sum_{j=1}^{k} \det\left[A_{(1)}\big| \cdots \big|A_{(j-1)}\big|A_{(j)} - B_{(j)}\big|B_{(j+1)}\big| \cdots \big|B_{(k)}\right].$$

By Hadamard's inequality: $|\det(A)| \leq \prod_{j=1}^{k} \left\|A_{(j)}\right\|$ (see, e.g., [8, Cor. 7.8.2]), we have

$$\begin{aligned}|\det(A) - \det(B)| \leq & \sum_{j=1}^{k} \left|\det\left[A_{(1)}\middle|\cdots\middle|A_{(j)} - B_{(j)}\middle|\cdots\middle|B_{(k)}\right]\right| \\ \leq & \sum_{j=1}^{k} \left\|A_{(1)}\right\| \cdots \left\|A_{(j)} - B_{(j)}\right\| \cdots \left\|B_{(k)}\right\|.\end{aligned}$$

□

Lemma 7. *Let M be a $k \times k$ matrix such that $\|M\|_{\mathcal{F}} \leq \delta$. Then*

$$|1 - \det(\mathbb{I} + M)| \leq \sqrt{k}\delta(1+\delta)^{k-1}.$$

Proof. Let us first recall that, given positive real numbers $a_1, \ldots, a_k$, as a consequence of the convexity of the function $x \mapsto x^2$, one has

$$\left(\sum_{j=1}^{k} a_j\right)^2 \leq k \sum_{j=1}^{k} a_j^2. \tag{16}$$

Since, for any $j = 1, \ldots, k$ $\left\|(\mathbb{I} + M)_{(j)}\right\| \leq 1 + \delta$, applying Lemma 6 with $A = \mathbb{I}$ and $B = \mathbb{I} + M$, and inequality (16), we obtain

$$\begin{aligned}|1 - \det(\mathbb{I} + M)| \leq & \sum_{j=1}^{k} (1+\delta)^{k-1} \left\|M_{(j)}\right\| \\ \leq & \sqrt{k}(1+\delta)^{k-1} \left(\sum_{j=1}^{k} \left\|M_{(j)}\right\|^2\right)^{\frac{1}{2}} = \sqrt{k}\delta(1+\delta)^{k-1}.\end{aligned}$$

That proves the assertion. □

of Proposition 3. Recall that $\|M_i\| \leq \|M_i\|_{\mathcal{F}}$. Clearly $\sqrt{k}\delta(1+\delta)^{k-1} < 1$ implies $\delta < 1$, hence one has

$$\begin{aligned}\sup_{x \in S^{k-1}} \left\|G(x) - x\right\| = & \max_{i=1,\ldots,N} \sup_{x \in S^{k-1} \cap C_i} \left\|A_i x - x\right\| \\ \leq & \max_{i=1,\ldots,N} \sup_{x \in S^{k-1}} \left\|A_i x - x\right\| \leq \max_{i=1,\ldots,N} \left\|M_i\right\| \\ \leq & \max_{i=1,\ldots,N} \|M_i\|_{\mathcal{F}} < 1.\end{aligned} \tag{17}$$

Consider the map $\mathcal{H}\colon [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$ given by,

$$\mathcal{H}(\lambda, x) := \lambda x + (1-\lambda)G(x).$$

Since, by (17), $\|G(x) - x\| < \|x\|$ for all $x \in S^{k-1}$, we find

$$\|\mathcal{H}(x)\| = \|\lambda x + (1-\lambda)G(x)\| \geq \big|\lambda \|G(x) - x\| - \|x\|\big| > 0.$$

That is, $\mathcal{H}$ is an admissible homotopy between G and the identity on the unit disk. Hence the degree of G is one. To be able to apply [16, Thm. 4] and so obtain the invertibility of G one only needs to observe that by Lemma 7 the determinant of all the matrices A_i, $i = 1, \ldots, N$, are positive. □

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