Uniqueness and reconstruction of finite lattice sets from their line sums

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Abstract

If an unknown finite set $C \subset \mathbb{Z}^2$ is cut by lines parallel to given directions, then one may count the number of points of C that are intercepted by each line, that is, the projections of C in the given directions. The inverse problem consists in reconstructing the set C, interpreted as a binary image, from the knowledge of its projections.
In general, this challenging combinatorial problem, also related to the tomographic reconstruction of an unknown homogeneous object by means of X-rays, is ill-posed, meaning that different binary images exist that match the available projections. Therefore, as a preliminary step, one can try to find conditions to be imposed on the considered directions in order to limit the number of allowed solutions.
In this paper we address the above problems for sets C contained in a finite assigned lattice grid, and generalize some results known in literature.
First, we describe special sets of lattice directions, called simple cycles, and focus on some of their properties. Then we prove that uniqueness of reconstruction for binary images is guaranteed if and only if the line sums are computed along suitable simple cycles having even cardinality.
As a second item, we prove that the unique binary solution can be explicitly reconstructed from a real-valued solution having minimal Euclidean norm. This leads to an explicit reconstruction algorithm, tested on four different phantoms and compared with previous results, which points out a significant improvement of the corresponding performance.
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1. Introduction

In the lattice $\mathbb{Z}^2$ of points having integer coordinates, let $\mathcal{A} = \{(\xi, \eta) \in \mathbb{Z}^2 : 0 \leq \xi < M, 0 \leq \eta < N\}$ be a grid of size $M \times N$, with M, N positive integers. Let $S = \{u_1, \ldots, u_d\}$ be a given set of lattice directions. A stimulating combinatorial problem is the reconstruction of an unknown set $C \subseteq \mathcal{A}$

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from the knowledge of the number of points of C intercepted by the straight lines of $\mathbb{Z}^2$ that are parallel to the directions in S.
From a functional point of view, this consists in looking for a function $f : \mathcal{A} \to \{0, 1\}$ having C as its support, namely $C = supp(f) = \{(\xi, \eta) \in \mathcal{A} : f(\xi, \eta) = 1\}$, to be determined from the knowledge of the sums of its values along all lines parallel to the given directions and intersecting $\mathcal{A}$. In this perspective, the set C is a binary image and the combinatorial problem represents the discrete analogous of the tomographic reconstruction of homogeneous (namely, with a single density value) objects by means of projection data, collected by detectors, and corresponding to the absorption of energy along the given directions (see [15, 16] for additional details).
The classical theoretical model of Computerized Tomography goes back to the earlier results by J. Radon, and bases on the Radon Transform and its inversion, which requires the knowledge of the projections along all possible directions (see for instance the English reprint [19] of the original paper). However, in the real world only a finite number of projections can be collected, and this leads to the loss of injectivity of the Radon transform. This results in the existence of *S-ghosts*, namely, non-trivial images that are invisible along the considered set S of directions. An S-ghost can be added to any solution of the tomographic problem without changing the X-ray data, so providing a different solution.
An S-ghost is represented by a non-trivial function $g : \mathcal{A} \to \mathbb{R}$ having zero line sums along all lines parallel to the given directions. In particular, if g has range $\{-1, 0, 1\}$, then we speak of *binary S-ghost*. Adding a binary S-ghost g to a binary solution $f : \mathcal{A} \to \{0, 1\}$ could return a function $f + g$ that is still binary and consequently, having the same line sums as f, could provide a different solution of the same problem. Differently, $f + g$ is never a binary solution if $|g(\xi, \eta)| > 1$ for some $(\xi, \eta) \in \mathcal{A}$. A *weakly bad configuration* $\mathcal{F}_S \subset \mathbb{Z}^2$ is the set of minimal size supporting an S-ghost, namely, $\mathcal{F}_S$ consists of positively and negatively weighted lattice points, summing to zero along all lines in the directions belonging to S. For any $u \in \mathbb{R}^2$ such that $\mathcal{F}_S + u \subset \mathcal{A}$, we also consider the *elementary* S-ghost $g_u : \mathcal{A} \to \mathbb{R}$, whose support is $\mathcal{G}_u = \mathcal{F}_S + u$. Any other S-ghost is a linear combination of elementary ghosts [14].
A fruitful area of research consists in looking for some kind of extra information to be incorporated in the tomographic problem, with the aim of understanding the combinatorial and the geometric structure of ghosts, so reducing the number of allowed solutions and, possibly, to prove uniqueness results (for several results by different perspectives see for instance [1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 18, 20], and also [15, 16] for a general overview and related topics).
For some special classes of objects, uniqueness results are known independently of a grid $\mathcal{A}$, so in the whole lattice $\mathbb{Z}^2$. This is the case, for instance, of convex bodies, when S is any set of seven lattice directions, or any set of four directions having cross-ratio different from $2, 3, 4$, up to rearrangement of the order of the directions [11, 12]. However, working with no restrictions on the object to be reconstructed does not avoid binary ghosts, since the linear combination of a sufficiently large number of elementary ghosts could provide a binary ghost. This suggests confining the tomographic problem to a lattice grid $\mathcal{A}$.
In this paper we deal with the above problem, first by proving a uniqueness theorem, and then by applying the result to get an explicit reconstruction algorithm.
Since we deal with homogeneous sets (namely, with binary images), we must impose that no binary S-ghost exists. In particular, we must guarantee that some points of the weakly bad configuration $\mathcal{F}_S$ have weight greater than 1, and this remains true for any other S-ghost $g : \mathcal{A} \to \mathbb{R}$.
In [4, 5], a solution has been obtained for a set S of four directions in $\mathbb{Z}^2$, and in [6] the result has been generalized by exploiting sets of d independent directions in $\mathbb{Z}^n$, with $3 \leq d \leq n$. As a first step,

we note that the assumption of independence adopted in [6] can be relaxed by using the notion of *simple cycle*, which provides sets of directions whose corresponding weakly bad configuration always has a multiple point, positively or negatively weighted, whenever the weakly bad configuration has to remain inside $\mathcal{A}$. In order to preserve such a multiple point under translations, the set of allowed translations inside the given lattice grid $\mathcal{A}$ is determined. This gives the so-called *enlarging region* of each point of $\mathcal{F}_S$, and leads to Theorem 2. It provides a necessary and sufficient condition for a simple cycle S to be a set of uniqueness in a bounded lattice grid $\mathcal{A}$, meaning that the line sums taken along the directions in S uniquely determine any subset of $\mathcal{A}$.

The second step of our approach consists in applying the obtained uniqueness theorem to get an explicit reconstruction algorithm for binary images. For sets S of four suitable lattice directions, this is done by the Binary Reconstruction Algorithm (BRA) in [10], which is based on the results in [5]. Though perfect reconstructions are always obtained, a drawback of such an approach is that the directions in S are usually *long*, meaning that these have a large norm, and consequently the lines parallel to the given directions intercept a small number of lattice points. This is not convenient from a pure application point of view, since the acquisition system should contain a huge number of detectors, and consequently would be very expensive. In addition, X-rays in very skew directions would be strongly affected by physical and technical constraints, so would easily lead to the presence of artifacts in the reconstructed images.

Consequently, having extended the uniqueness result obtained in [5] for four directions to simple cycles, we are encouraged to adopt the same strategy as in [10], but exploiting simple cycles consisting of $d > 4$ lattice directions. This allows us to reduce the norm of each employed direction, so collecting a larger number of lattice points on each line.

The paper is organized as follows. In Section 2 we introduce the adopted notation and the main definitions. Section 3 is devoted to reach a uniqueness result for simple cycles. In Section 4 we state and prove Theorem 3, that provides a binary rounding theorem for simple cycles of uniqueness. This leads to the algorithm e-BRA (extended BRA), that uniquely and perfectly reconstructs any binary image by exploiting the real-valued solution having minimum Euclidean norm. In Section 5 we test e-BRA on four different binary phantoms, comparing the results with those of BRA, which shows an improvement of the performance. Section 6 resumes the obtained results and gives some details on possible further generalizations.

2. Preliminaries

We work in the lattice $\mathbb{Z}^2$, consisting of points having integer coordinates. A *lattice set* is a finite subset $C \subset \mathbb{Z}^2$. We denote by $|C|$ the cardinality of C, and by $C + u$ the lattice set obtained by translating each point of C along a vector $u \in \mathbb{Z}^2$. We denote by $-C$ the symmetric of C with respect to the origin. A partition of a set C as the disjoint union of two subsets $C_1, C_2 \subset C$ is denoted by $C = C_1 \dot{\cup} C_2$.

A finite set of points $C \subset \mathbb{Z}^2$ can also be considered as a *binary image*, whose minimum bounding grid, up to translation, is $\mathcal{A} = \{(\xi, \eta) \in \mathbb{Z}^2 \ : \ 0 \leq \xi < M,\ 0 \leq \eta < N\}$ for some positive integers M, N. When referring to the elements of a binary image we can say *pixels* instead of points, so $MN =: n$ is the number of pixels. Any image can also be considered as a function $f : \mathcal{A} \to \mathbb{R}$. In particular, a binary image is $f : \mathcal{A} \to \{0, 1\}$. Denote by $\|f\|$ the *norm* of f, namely, $\|f\| = \max_{(\xi,\eta)\in\mathcal{A}}\{|f(\xi, \eta)|\}$.

By *(lattice) direction* we mean a pair (a, b) of coprime integers such that $a \geq 0$, where we assume $b = 1$ if $a = 0$. A *line of direction* $(a, b) \in \mathbb{Z}^2$ has equation $ay = bx + t$, $t \in \mathbb{Z}$.

In a binary tomography perspective, we work in the so-called *grid model*, where pixels correspond to lattice points having integer coordinates and X-rays[1] are discrete lattice lines. The *line sum*, or *projection*, of a function $f : \mathcal{A} \to \{0,1\}$ along the lattice line $ay = bx + t$ is $\sum_{a\eta=b\xi+t} f(\xi,\eta)$, so a projection counts the number of lattice points intercepted by X-rays taken in the assigned direction. The algebraic approach to image reconstruction bases on solving the linear system

$$W\mathbf{x} = \mathbf{p}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ encodes the image to be reconstructed, $\mathbf{p} \in \mathbb{R}^s$ is the vector collecting the measures and $W = [w_{ij}]$ is the $s \times n$ *projection matrix*, whose entries are computed differently according to different models. In the grid model, $w_{ij} = 1$ if the j-th pixel belongs to the i-th X-ray, and $w_{ij} = 0$ otherwise. For a given set $S = \{(a_r, b_r) : r = 1, \ldots, d\}$ of d lattice directions, each one of the s rows of W provides the contribution of all the n pixels along a single line (usually called *bin*), having one of the directions in S.
Referring to (1), we are in the case $\mathbf{p} \in \mathbb{N}^s$ and look for solutions $\mathbf{x} \in \{0,1\}^n$. Let be $h := \sum_{r=1}^{d} a_r$ and $k := \sum_{r=1}^{d} |b_r|$. Uniqueness of reconstruction is guaranteed inside the grid $\mathcal{A}$ by the so-called Katz condition [17]:

$$h \geq M \quad \text{or} \quad k \geq N. \tag{2}$$

Differently, if $h < M$ and $k < N$ we say that S is a *valid* set of directions for $\mathcal{A}$. In this case, uniqueness of reconstruction is not guaranteed without introducing some extra information, since binary S-ghosts $g : \mathcal{A} \to \{-1,0,1\}$ could exist, namely, non-zero solutions $\mathbf{x}_g$ of the homogeneous system $W\mathbf{x} = \mathbf{0}$. A binary ghost, added to a binary solution $f : \mathcal{A} \to \{0,1\}$, could still provide a $\{0,1\}$-ranged function $f + g$, i.e., a different binary solution of the same problem. The set *supp*$(g) = \{(\xi,\eta) \in \mathcal{A} : g(\xi,\eta) \neq 0\}$ is the *support* of g. If *supp*$(g) = \emptyset$, then g is called *trivial ghost*.
For $r = 1, \ldots, d$, let us consider

$$f_{(a_r,b_r)}(x,y) = \begin{cases} x^{a_r} y^{b_r} - 1 & \text{if } a_r \neq 0, b_r > 0, \\ x^{a_r} - y^{-b_r} & \text{if } a_r \neq 0, b_r < 0, \\ x - 1 & \text{if } a_r = 1, b_r = 0, \\ y - 1 & \text{if } a_r = 0, b_r = 1, \end{cases}$$

and define

$$F_S(x,y) = \prod_{r=1}^{d} f_{(a_r,b_r)}(x,y).$$

For a vector $u = (a,b) \in \mathbb{Z}^2$, we often simply write $\mathbf{x}^u$ to denote the monomial $x^a y^b$ [2]. For each $A \subseteq S$, let be $u(A) = \sum_{u \in A} u$, with $u(\emptyset) = 0 \in \mathbb{Z}^2$. It results that $F_S(x,y) = \sum_{A \subseteq S} (-1)^{|S|-|A|} \mathbf{x}^{u(A)}$, where we underline that the sign of the monomial $\mathbf{x}^{u(A)}$ is determined by the parity of (the cardinality of) the set $S \backslash A$.

[1] The term *X-ray* is usually employed in discrete tomography either to mean the measurement (see for instance [12]) or a line intercepting grid points (see [10]). Here we use the second option.
[2] We use the same symbol $\mathbf{x}$ for both a solution of (1) and the monomial $\mathbf{x}^u = x^a y^b$. However, due to the presence of the exponent, no misinterpretation exists.

A monomial $mx^\xi y^\eta \in \mathbb{Z}[x,y]$ can be associated to the lattice point (ξ,η), together with its *weight* m. We say that $|m|$ is the *multiplicity* of (ξ,η). For any function $g : \mathcal{A} \to \mathbb{R}$, the corresponding *generating function* is the polynomial defined by

$$G_g(x,y) = \sum_{(\xi,\eta)\in\mathcal{A}} g(\xi,\eta)x^\xi y^\eta.$$

Remark 1. *The above definition of $f_{(a_r,b_r)}(x,y)$ avoids negative exponents for y, so that $F_S(x,y)$ is in fact a polynomial. This implies that, for each $A \subseteq S$, the exponent $u(A)$ of the associated monomial of $F_S(x,y)$ corresponds to the lattice point $u(A) + (0,q)$, where $q = \sum_{(a,b)\in S, b<0} |b|$.*

Note that if the generating function of $f : \mathcal{A} \to \mathbb{R}$ is the polynomial $F_S(x,y)$, then f has zero sums along the lines taken in the directions in S [14]. Moreover, being S valid for $\mathcal{A}$, $supp(f)$ is contained in $\mathcal{A}$.

For a polynomial $G(x,y)$, we denote by $G^+(x,y)$ (respectively, $G^-(x,y)$) the polynomial as the sum of the monomials of $G(x,y)$ having positive (respectively, negative) coefficients. The sets consisting of the lattice points (counted with their multiplicities) corresponding to $G(x,y), G^+(x,y)$ and $G^-(x,y)$ are here denoted by $\mathcal{G}$, $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively. If $G(x,y) = G_g(x,y)$, g a ghost, then the pair $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-)$ is said to be a *(weakly) bad configuration.* This consists of two sets having the same absolute sums along all lines with directions taken in S, up to count each pixel with its proper multiplicity. In case $\mathcal{G} = (\mathcal{G}^+, \mathcal{G}^-)$ does not contain points having multiplicity greater than 1, then $\mathcal{G}$ is said to be a *bad configuration.*

Let $\mathcal{F}_S = \{\lambda_t : t \in I^- \cup I^+\}$ be the (weakly) bad configuration associated to a set S of valid directions for a lattice grid $\mathcal{A} = \{(\xi,\eta) \in \mathbb{Z}^2 : 0 \le \xi < M, 0 \le \eta < N\}$, where I^+ (respectively, I^-) is the set of indices t such that $\lambda_t \in \mathcal{F}_S$ has positive (respectively, negative) weight. The *enlarging region* associated to $\mathcal{F}_S$ is the rectangle $E = \{(\xi,\eta) : 0 \le \xi \le M-h-1, 0 \le \eta \le N-k-1\}$. Further, for each $(\xi,\eta) \in \mathcal{A}$, define $E^+(\xi,\eta) = \{u \in E : (\xi,\eta) = \lambda_i + u, i \in I^+\}$ and $E^-(\xi,\eta) = \{u \in E : (\xi,\eta) = \lambda_t + u, t \in I^-\}$. The *enlarging region associated to a pixel* $\lambda \in \mathcal{F}_S$ is the set $\lambda + E$.

Indeed, when looking for a uniqueness result in $\mathcal{A}$, the union of the enlarging regions associated to all pixels of $\mathcal{F}_S$ plays a special role, since it provides the region where $\mathcal{F}_S$ can be moved still remaining inside the grid [10]. In what follows we refer to a set S of uniqueness for a bounded lattice grid $\mathcal{A}$, meaning that any binary lattice set contained in $\mathcal{A}$ can be exactly reconstructed from the knowledge of its projections along the directions in S.

3. A uniqueness theorem

We wish to investigate the problem of reconstructing an unknown binary image by exploiting special sets of directions when the Katz's inequalities (2) do not hold. In particular, we are interested in sets consisting of few directions, taken with short Euclidean norm (with respect to the grid size), as well as in related algorithms that, assuming there is no noise, lead to perfect reconstructions.

Definition 1. *A set $S = \{u_1, \ldots, u_d\}$ of lattice directions is a* cycle *if there exists a partition of $S = I \dot\cup J$ such that $u(I) = u(J)$. A cycle S is called* simple *if no proper subset S' of S is a cycle, namely, no other pair A, B of disjoint subsets of S exists such that $A \dot\cup B = S'$ and $u(A) = u(B)$.*

Example 1. *The set* $S_1 = \{(5,-3),(10,11),(15,4),(19,4),(20,13),(15,-29),(7,6),(7,26)\}$ *is a simple cycle. It can be obtained as* $I \dot{\cup} J$, *with* $I = \{(5,-3),(10,11),(15,4),(19,4)\}$ *and* $J = \{(20,13),(15,-29),(7,6),(7,26)\}$, *while no other complementary pair of disjoint subsets* A, B *of* S *such that* $u(A) = u(B)$ *exists, as one can easily (but tediously) check by sorting all the subsets of* S. *On the other hand, the cycle* $S_2 = \{(1,2),(2,1),(4,7)\} \cup \{(0,1),(4,5),(3,4)\}$ *is not a simple cycle, since* $u(A) = u(B)$ *for the subsets* $A = \{(1,2),(3,4)\}$ *and* $B = \{(0,1),(4,5)\}$.

Proposition 1. *Let* $S = I \dot{\cup} J = \{u_1, \dots, u_d\}$ *be a simple cycle. Then* $\mathcal{F}_S$ *has a multiple point if and only if* d *is even. Moreover, there is exactly one coefficient of* $F_S(x,y)$ *not in* $\{-1,0,1\}$, *and its value is either* 2 *or* -2.

Proof. If d is odd, then $|I|$ and $|J|$ have different parities, so the monomials $\mathbf{x}^{u(I)}$ and $\mathbf{x}^{u(J)}$ in F_S have opposite coefficients and therefore vanish.
Conversely, if d is even then $|I|$ and $|J|$ have the same parity, and the coefficients of $\mathbf{x}^{u(I)}$ and $\mathbf{x}^{u(J)}$ are both 1 or -1, giving the monomial $\pm 2\mathbf{x}^{u(I)}$. Since S is a simple cycle, then no other pair $\{A,B\} \neq \{I,J\}$ exists such that $u(A) = u(B)$, and consequently no other monomial in F_S having coefficient greater than 1 in absolute value. $\square$

Example 2. *The set* $S = \{(3,-1),(1,2),(1,3),(1,-3),(7,3),(3,-4)\}$ *is a simple cycle of length six partitioned in two sets,* $I = \{(3,-1),(1,2),(1,3),(3,-4)\}$ *and* $J = \{(1,-3),(7,3)\}$, *so that* $u(I) = u(J) = (8,0)$. *Since* $q = 8$, *according to Remark 1 the unique double point of the weakly bad configuration* $\mathcal{F}_S$ *is* $(8,8)$. *The associated polynomial is*

$$F_S(x,y) = x^{16}y^8 - x^{15}y^11 - x^{15}y^6 - x^{15}y^5 + x^{14}y^9 + x^{14}y^8 + x^{14}y^3 - x^{13}y^{12} - x^{13}y^9 - x^{13}y^6 + x^{12}y^{15} + x^{12}y^{12} + x^{12}y^{10} + x^{12}y^9 + x^{12}y^7 + x^{12}y^6 - x^{11}y^{13} - x^{11}y^{12} - x^{11}y^{10} - x^{11}y^9 - x^{11}y^7 - x^{11}y^4 + x^{10}y^{13} + x^{10}y^{10} + x^{10}y^7 - x^9y^{16} - x^9y^{11} - x^9y^{10} - x^9y^5 + x^8y^{14} + x^8y^{13} + \boxed{2x^8y^8} + x^8y^3 + x^8y^2 - x^7y^{11} - x^7y^6 - x^7y^5 - x^7 + x^6y^9 + x^6y^6 + x^6y^3 - x^5y^{12} - x^5y^9 - x^5y^7 - x^5y^6 - x^5y^4 - x^5y^3 + x^4y^{10} + x^4y^9 + x^4y^7 + x^4y^6 + x^4y^4 + x^4y - x^3y^{10} - x^3y^7 - x^3y^4 + x^2y^{13} + x^2y^8 + x^2y^7 - xy^{11} - xy^{10} - xy^5 + y^8.$$

The boxed monomial is the only one in $F_S(x,y)$ *that has a coefficient greater than* 1 *in absolute value. Figure 1 shows the displacement of the weakly bad configuration* $\mathcal{F}_S$ *associated to* $F_S(x,y)$, *contained in an* 18×18 *lattice grid, and consisting of all lattice points having coordinates equal to* $u(A) + (0,8)$ *for all* $A \subseteq S$. *In this case,* $M = N = 18$ *and* $h = k = 16$, *therefore the enlarging region is* $E = \{(\xi,\eta) : 0 \le \xi \le 1,\ 0 \le \eta \le 1\}$. *It consists of four points, so each pixel* $\lambda \in \mathcal{F}_S$ *can be moved in the region* $\lambda + E$ *still preserving the double point and without exceeding the grid size.*

A necessary condition for S to be a set of binary uniqueness for a finite grid $\mathcal{A}$ is that $\mathcal{F}_S$ represents a weakly bad configuration in $\mathcal{A}$, namely, $\mathcal{F}_S$ must have a multiple point. In particular, due to Proposition 1, even simple cycles have a single double point, so these represent the easiest case to be investigated. This leads us to characterize the sets of directions S that provide even simple cycles, so from now on we will assume $d \ge 4$ and even.
Note that for a simple cycle S, once $I \subset S$ has been selected, then there exists only one $J \subset S$ such that $S = I \dot{\cup} J$. If S consists of $d = 2d'$ directions, then there exist d' partitions of $|S|$ as the sum of two integers $|I|, |J|$ having the same parity (two partitions obtained by exchanging the values of $|I|$ and $|J|$ are considered as the same partition).
The case $d = 4$ has been already considered in [5, 11, 13], where, assuming for instance $|I| \ge |J|$, the two possible partitions correspond to the choices $|I| = 3$ and $|J| = 1$, and $|I| = |J| = 2$, respectively.

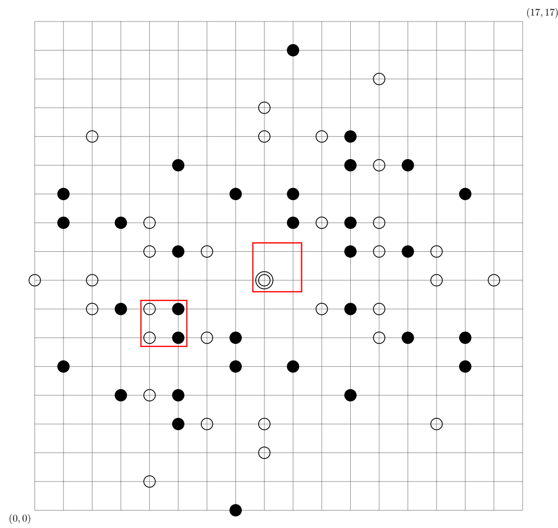


Figure 1: The set $S = \{(3,-1),(1,2),(1,3),(1,-3),(7,3),(3,-4)\}$ and the corresponding weakly bad configuration in a grid of dimension 18×18. Black points correspond to monomials with negative sign, white points to the positive ones. There is one only multiple (positive) point, marked with a double circle and framed enlarging region. The enlarging regions of other pixels could overlap, as shown on the lower-left side of the double pixel.

In order to provide a sufficient condition for uniqueness, we now consider the shift of the S-weakly bad configuration by a vector $w \in \mathbb{Z}^2$, thus defining f_-^w and f_+^w as the maps whose generating functions are $G_{f_-^w}(x,y) = (\mathbf{x}^w - 1)F_S(x,y)$ and $G_{f_+^w}(x,y) = (\mathbf{x}^w + 1)F_S(x,y)$, respectively.
We study in which cases f_-^w and f_+^w correspond to bad configurations, i.e., we seek the translations w such that the double point in F_S vanishes.

Lemma 1. *Let $S = \{u_1, \ldots, u_d\}$ be an even simple cycle, and $I \dot\cup J$ be the partition such that $u(I) = u(J)$ is the double point in the corresponding weakly bad configuration. Then*

1. $\|f_-^w\| \le 1$ *if and only if* $w \in \{\pm(u(I') - u(J')) \,:\, I' \subseteq I,\, J' \subseteq J,\, |I'| \equiv_2 |J'|\}$;
2. $\|f_+^w\| \le 1$ *if and only if* $w \in \{\pm(u(I') - u(J')) \,:\, I' \subseteq I,\, J' \subseteq J,\, |I'| \not\equiv_2 |J'|\}$.

Proof. We focus on f_-^w, since the other case can be treated similarly.
Suppose first that $\|f_-^w\| \le 1$, namely, all the coefficients of $G_{f_-^w}(x,y) = \mathbf{x}^w F_S(x,y) - F_S(x,y)$ belong to the set $\{-1,0,1\}$.
Being $|S|$ even, we can write $F_S(x,y) = \sum_{A \subseteq S} (-1)^{|A|} \mathbf{x}^{u(A)}$, and

$$G_{f_-^w}(x,y) = \sum_{A \subseteq S} (-1)^{|A|} \mathbf{x}^{w+u(A)} - \sum_{B \subseteq S} (-1)^{|B|} \mathbf{x}^{u(B)}. \tag{3}$$

Since S is an even simple cycle, then $|I| \equiv_2 |J|$. Also, by Proposition 1, the only monomials in $\mathbf{x}^w F_S(x,y)$ and $F_S(x,y)$ having coefficients outside the set $\{-1,0,1\}$ are

- $(-1)^{|I|} 2\mathbf{x}^{w+u(I)}$, that is obtained by adding the monomials $(-1)^{|I|}\mathbf{x}^{w+u(I)}$ and $(-1)^{|J|}\mathbf{x}^{w+u(J)}$ in the first sum;

- $(-1)^{|I|}2\mathbf{x}^{u(I)}$, that is obtained by adding the monomials $(-1)^{|I|}\mathbf{x}^{u(I)}$ and $(-1)^{|J|}\mathbf{x}^{u(j)}$ in the second sum.

Since $\|f_-^w\| \leq 1$, the monomial $(-1)^{|I|}2\mathbf{x}^{w+u(I)}$ must have the same exponent and the same sign of a monomial in the second sum. Then, there exists $K \subset S$ such that $|K| \equiv_2 |I|$ and $w + u(I) = u(K)$, that is $w = u(K) - u(I)$. Since $S = I \dot\cup J$, then there exist $T \subseteq I$, $J_1' \subseteq J$ such that $K = T \dot\cup J_1'$. Therefore $u(K) = u(T) + u(J_1')$, and consequently

$$w = u(T) + u(J_1') - u(I) = u(J_1') - (u(I) - u(T)) = u(J_1') - u(I_1'),$$

where I_1' is the complement of T in I. Note that $|I_1'| = |I| - |T| = |I| - |K| + |J_1'| \equiv_2 |J_1'|$.
Similarly, the monomial $(-1)^{|I|}2\mathbf{x}^{u(I)}$ must have the same exponent and the same sign of a monomial in the first sum. Then there exists $H \subset S$ such that $|H| \equiv_2 |I|$ and $u(I) = w + u(H)$, that is $w = u(I) - u(H)$. Therefore, there exist $Q \subseteq I$, $J_2' \subseteq J$ such that $H = Q \dot\cup J_2'$, so that $u(H) = u(Q) + u(J_2')$, and consequently

$$w = u(I) - u(Q) - u(J_2') = u(I_2') - u(J_2'),$$

where I_2' is the complement of Q in I. Note that $|I_2'| = |I| - |Q| = |I| - |H| + |J_2'| \equiv_2 |J_2'|$.
Therefore, in any case, $w \in \{\pm(u(I') - u(J'))\}$ for some I', J' with $I' \subseteq I$, $J' \subseteq J$, $|I'| \equiv_2 |J'|$.

Conversely, assume that $w \in \{\pm(u(I') - u(J')) : I' \subseteq I, J' \subseteq J, |I'| \equiv_2 |J'|\}$.
Suppose that $A, B \subset S$ correspond to monomials having the same exponent in the first and in the second sum in (3), respectively, namely, $w + u(A) = u(B)$. Then $w = u(B) - u(A)$, so, by definition of w, we get $u(B) - u(A) = u(I') - u(J')$. Let $C = A \cap B$, so we can write $u(B) = u(B \setminus C) + u(C)$ and $u(A) = u(A \setminus C) + u(C)$, so that

$$u(B \setminus C) - u(A \setminus C) = u(I') - u(J'), \tag{4}$$

being $A \setminus C$ and $B \setminus C$ disjoint sets. Let us consider the following sets:

$$\begin{aligned} B_I &= (B \setminus C) \cap I, \qquad & A_I &= (A \setminus C) \cap I, \\ B_J &= (B \setminus C) \cap J, \qquad & A_J &= (A \setminus C) \cap J. \end{aligned}$$

Since $I \cap J = \emptyset$, we can write $u(B \setminus C) = u(B_I \cup B_J) = u(B_I) + u(B_J)$. Analogously, we have $u(A \setminus C) = u(A_I) + u(A_J)$. From (4), we get

$$u(B_I) + u(B_J) - u(A_I) - u(A_J) = u(I') - u(J').$$

Note that $A_I \cap B_I = \emptyset$ and $A_J \cap B_J = \emptyset$, so we can write $u(B_I) - u(A_I) = u(B_I \cup (-A_I))$. Analogously, $u(B_J) - u(A_J) = u(B_J \cup (-A_J))$. Consequently, we have

$$u(B_I \cup (-A_I)) + u(B_J \cup (-A_J)) = u(I') - u(J') = u(I') + u(-J').$$

Since $A_I, B_I, I' \subset I$, $A_J, B_J, J' \subset J$, and $I \cap J = \emptyset$, it must be $B_I \cup (-A_I) = I'$, and $B_J \cup (-A_J) = -J'$. Since $|I'| \equiv_2 |J'|$, then $|B_I \cup (-A_I)| \equiv_2 |B_J \cup (-A_J)|$, and, being $A_I \cap B_I = \emptyset$ and $A_J \cap B_J = \emptyset$, we get

$$|B_I| + |A_I| \equiv_2 |B_J| + |A_J|. \tag{5}$$

If $|B_I| \equiv_2 |B_J|$, then also $|A_I| \equiv_2 |A_J|$, and $|B| = |B_I| + |B_J| \equiv_2 |A_I| + |A_J| = |A|$. If $|B_I| \not\equiv_2 |B_J|$, then also $|A_I| \not\equiv_2 |A_J|$, and we still have $|B| = |B_I| + |B_J| \equiv_2 |A_I| + |A_J| = |A|$. Therefore, in

each case the monomials $(-1)^{|A|}x^{w+u(A)}$ and $(-1)^{|B|}x^{u(B)}$ have the same exponent and the same coefficient. As a consequence, they mutually cancel in $G_{f_-^w}(x,y)$ in case the corresponding lattice points have the same multiplicities (namely, both 1, or both 2), or sum to a monomial having coefficient ± 1 otherwise. Therefore, $\|f_-^w\| \leq 1$ and the statement follows.
When considering f_+^w the proof proceeds similarly, apart changing $\equiv_2$ with $\not\equiv_2$ in (5). $\square$

Example 3. *Let us consider again the simple cycle S as in Example 2 and assume $I' = \{(3,-1)\}$, $J' = \{(7,3)\}$, $w = u(J') - u(I') = (4,4)$. Then we have*
$\mathbf{x}^w F_S(x,y) = x^4y^4F_S(x,y) = x^{20}y^{12} - x^{19}y^{15} - x^{19}y^{10} - x^{19}y^9 + x^{18}y^{13} + x^{18}y^{12} + x^{18}y^7 - x^{17}y^{16} - x^{17}y^{13} - x^{17}y^{10} + x^{16}y^{19} + x^{16}y^{16} + x^{16}y^{14} + x^{16}y^{13} + x^{16}y^{11} + x^{16}y^{10} - x^{15}y^{17} - x^{15}y^{16} - x^{15}y^{14} - x^{15}y^{13} - x^{15}y^{11} - x^{15}y^8 + x^{14}y^{17} + x^{14}y^{14} + x^{14}y^{11} - x^{13}y^{20} - x^{13}y^{15} - x^{13}y^{14} - x^{13}y^9 + x^{12}y^{18} + x^{12}y^{17} + \boxed{2x^{12}y^{12}} + x^{12}y^7 + x^{12}y^6 - x^{11}y^{15} - x^{11}y^{10} - x^{11}y^9 - x^{11}y^4 + x^{10}y^{13} + x^{10}y^{10} + x^{10}y^7 - x^9y^{16} - x^9y^{13} - x^9y^{11} - x^9y^{10} - x^9y^8 - x^9y^7 + x^8y^{14} + x^8y^{13} + x^8y^{11} + x^8y^{10} + x^8y^8 + x^8y^5 - x^7y^{14} - x^7y^{11} - x^7y^8 + x^6y^{17} + x^6y^{12} + x^6y^{11} - x^5y^{15} - x^5y^{14} - x^5y^9 + x^4y^{12}$.

The only monomial of $F_S(x,y)$ having coefficient outside $\{-1,0,1\}$ is $2x^8y^8$, so the only monomial of $\mathbf{x}^w F_S(x,y)$ having coefficient outside $\{-1,0,1\}$ must be the boxed one. Since I' and J' have the same parity, by Lemma 1, the double coefficient must simplify in $G_{f_-^w}(x,y)$. In fact, we get

$(\mathbf{x}^w - 1)F_S(x,y) = (x^4y^4 - 1)F_S(x,y) = x^{20}y^{12} - x^{19}y^{15} - x^{19}y^{10} - x^{19}y^9 + x^{18}y^{13} + x^{18}y^{12} + x^{18}y^7 - x^{17}y^{16} - x^{17}y^{13} - x^{17}y^{10} + x^{16}y^{19} + x^{16}y^{16} + x^{16}y^{14} + x^{16}y^{13} + x^{16}y^{11} + x^{16}y^{10} - x^{16}y^8 - x^{15}y^{17} - x^{15}y^{16} - x^{15}y^{14} - x^{15}y^{13} - x^{15}y^8 + x^{15}y^6 + x^{15}y^5 + x^{14}y^{17} + x^{14}y^{14} + x^{14}y^{11} - x^{14}y^9 - x^{14}y^8 - x^{14}y^3 - x^{13}y^{20} - x^{13}y^{15} - x^{13}y^{14} + x^{13}y^{12} + x^{13}y^6 + x^{12}y^{18} + x^{12}y^{17} - x^{12}y^{15} + x^{12}y^{12} - x^{12}y^{10} - x^{12}y^9 - x^{11}y^{15} + x^{11}y^{13} + x^{11}y^{12} + x^{11}y^7 - x^9y^{13} - x^9y^8 - x^9y^7 + x^9y^5 + x^8y^{11} + x^8y^{10} - x^8y^8 + x^8y^5 - x^8y^3 - x^8y^2 - x^7y^{14} - x^7y^8 + x^7y^6 + x^7y^5 + x^7 + x^6y^{17} + x^6y^{12} + x^6y^{11} - x^6y^9 - x^6y^6 - x^6y^3 - x^5y^{15} - x^5y^{14} + x^5y^{12} + x^5y^7 + x^5y^6 + x^5y^4 + x^5y^3 + x^4y^{12} - x^4y^{10} - x^4y^9 - x^4y^7 - x^4y^6 - x^4y^4 - x^4y + x^3y^{10} + x^3y^7 + x^3y^4 - x^2y^{13} - x^2y^8 - x^2y^7 + xy^{11} + xy^{10} + xy^5 - y^8$.
Let us now assume $I' = \{(1,2),(1,3)\}$, $J' = \{(7,3)\}$, $w = u(J') - u(I') = (5,-2)$. Then we have
$\mathbf{x}^w F_S(x,y) = x^5y^{-2}F_S(x,y) = x^{21}y^6 - x^{20}y^9 - x^{20}y^4 - x^{20}y^3 + x^{19}y^7 + x^{19}y^6 + x^{19}y - x^{18}y^{10} - x^{18}y^7 - x^{18}y^4 + x^{17}y^{13} + x^{17}y^{11} + x^{17}y^9 + x^{17}y^7 + x^{17}y^5 + x^{17}y^4 - x^{16}y^{11} - x^{16}y^{10} - x^{16}y^8 - x^{16}y^7 - x^{16}y^5 - x^{16}y^2 + x^{15}y^{11} + x^{15}y^8 + x^{15}y^5 - x^{14}y^{14} - x^{14}y^9 - x^{14}y^8 - x^{14}y^3 + x^{13}y^{12} + x^{13}y^{11} + \boxed{2x^{13}y^6} + x^{13}y + x^{13} - x^{12}y^9 - x^{12}y^4 - x^{12}y^3 - x^{12} + x^{11}y^7 + x^{11}y^4 + x^{11}y - x^{10}y^{10} - x^{10}y^7 - x^{10}y^5 - x^{10}y^4 - x^{10}y^2 - x^{10}y + x^9y^8 + x^9y^7 + x^9y^5 + x^9y^4 + x^9y^2 + x^9y^{-1} - x^8y^8 - x^8y^5 - x^8y^2 + x^7y^{11} + x^7y^6 + x^7y^5 - x^6y^9 - x^6y^8 - x^6y^3 + x^5y^6$.
Since I' and J' have different parities, according to Lemma 1, the double boxed coefficient must simplify in $G_{f_+^w}(x,y)$. In fact, we get

$(\mathbf{x}^w + 1)F_S(x,y) = (x^5y^{-2} + 1)F_S(x,y) = x^{21}y^6 - x^{20}y^9 - x^{20}y^4 - x^{20}y^3 + x^{19}y^7 + x^{19}y^6 + x^{19}y - x^{18}y^{10} - x^{18}y^7 - x^{18}y^4 + x^{17}y^{13} + x^{17}y^{10} + x^{17}y^8 + x^{17}y^7 + x^{17}y^5 + x^{17}y^4 - x^{16}y^{11}x^{16}y^{10} - x^{16}y^7 - x^{16}y^5 - x^{16}y^2 + x^{15}y^8 - x^{15}y^6 - x^{14}y^{14} + x^{13}y^{11} - x^{13}y^9 + x^{13}y^6 + x^{13}y + x^{13} + x^{12}y^{15} + x^{12}y^{12} + x^{12}y^{10} + x^{12}y^7 + x^{12}y^6 - x^{12}y^4 - x^{12}y^3 - x^{12}y^{-2} - x^{11}y^{13} - x^{11}y^{12} - x^{11}y^{10} - x^{11}y^9 + x^{11}y + x^{10}y^{13} - x^{10}y^5 - x^{10}y^4 - x^{10}y^2 - x^{10}y - x^9y^{16} - x^9y^{11} - x^9y^{10} + x^9y^8 + x^9y^7 + x^9y^4 + x^9y^2 + x^9y^{-1} + x^8y^{14} + x^8y^{13} + x^8y^8 - x^8y^5 + x^8y^3 - x^7 - x^6y^8 + x^6y^6 - x^5y^{12} - x^5y^9 - x^5y^7 - x^5y^4 - x^5y^3 + x^4y^{10} + x^4y^9 + x^4y^7 + x^4y^6 + x^4y^4 + x^4y - x^3y^{10} - x^3y^7 - x^3y^4 + x^2y^{13} + x^2y^8 + x^2y^7 - xy^{11} - xy^{10} - xy^5 + y^8$.

By Lemma 1, for an even simple cycle $S = I \dot\cup J$ the set

$$D = \{\pm(u(I') - u(J')) \,:\, I' \subseteq I,\, J' \subseteq J\} \tag{6}$$

provides the set of the single switchings of F_S that give rise to a bad configuration, so points of multiplicity greater than 1 are not allowed.

In the following theorem we describe the generating polynomial of any $\{-1,0,+1\}$-valued function.

Theorem 1. *Let $S=\{u_1,\ldots,u_d\}$ be an even simple cycle, and $I\dot{\cup}J$ be the partition such that $u(I)=u(J)$ is the double point in the corresponding weakly bad configuration. Let $g:\mathbb{Z}^2\to\mathbb{Z}$ be a non-trivial function having zero line sums along the directions in S. If $\|g\|\leq 1$, then there exists $r\in\mathbb{N}$ such that*

$$G_g(x,y)=\sum_{t=1}^{r}\left(\delta_t\mathbf{x}^{u_t}+\mu_t\mathbf{x}^{v_t}\right)F_S(x,y), \tag{7}$$

where $\delta_t,\mu_t\in\{\pm1\}$ and $u_t-v_t\in D$.

Proof. By Lemma 1, the statement follows by the same arguments as in [4, Theorem 3] and in [6, Theorem 11], up to use the set D as in (6) and replace the independence of the directions with the notion of simple cycle. □

Remark 2. *The two cases when $g=f_-^w$ and $g=f_+^w$ can be obtained for $r=1$, by choosing $u_1=w, v_1=(0,0)$, with $\delta_1=1,\mu_1=-1$, and $\delta_1=\mu_1=1$, respectively.*

Starting from the previous results, we now characterize the simple cycles that are sets of uniqueness for a grid $\mathcal{A}$ of fixed size $M\times N$. Recall that $h=\sum_{r=1}^{d}a_r$ and $k=\sum_{r=1}^{d}|b_r|$.

Theorem 2. *Let $S=\{u_1,\ldots,u_d\}$ be an even simple cycle, valid for a lattice grid $\mathcal{A}=\{(\xi,\eta)\in\mathbb{Z}^2\ :\ 0\leq\xi<M,\ 0\leq\eta<N\}$. Then S is a set of uniqueness for $\mathcal{A}$ if and only if for each $w=(w_1,w_2)\in D$ it holds $|w_1|\geq M-h$ or $|w_2|\geq N-k$.*

Proof. Let be $u_r=(a_r,b_r)$, $r=1,\ldots,d$, and let $g:\mathcal{A}\to\mathbb{Z}$ be a function such that $\|g\|\leq 1$, having zero line sums along the directions in S. We have to show that g is identically zero if and only if for each $w\in D$ it holds $|w_1|\geq M-h$ or $|w_2|\geq N-k$.

First, let us show that a non-trivial function $g:\mathcal{A}\to\mathbb{Z}$ with $\|g\|\leq 1$ exists, if we assume that, for some $w=(w_1,w_2)\in D$, both $|w_1|<M-h$ and $|w_2|<N-k$.

Since $w\in D$, we know that either $\|f_-^w\|\leq 1$ or $\|f_+^w\|\leq 1$ holds (see Lemma 1). By the assumption on w, we also know that $G_{f_-^w}(x,y)$ and $G_{f_+^w}(x,y)$ both have maximum degree equal to $|w_1|+h$ and $|w_2|+k$ w.r.t. x and y, respectively. Therefore, both functions are non-trivial and, since $|w_1|+h<M$ and $|w_2|+k<N$, their support is inside $\mathcal{A}$. Consequently, we can choose $g=f_+^w$ or $g=f_-^w$, depending on which one has norm less than one.

Conversely, assume that for each $w=(w_1,w_2)\in D$ it holds $|w_1|\geq M-h$ or $|w_2|\geq N-k$, and let us show that a function $g:\mathcal{A}\to\mathbb{Z}$ having zero line sums along the directions in S and such that $\|g\|\leq 1$ must be identically zero.

Suppose that some non-trivial function $g:\mathcal{A}\to\mathbb{Z}$ exists having zero line sums along the directions in S and such that $\|g\|\leq 1$. Then $G_g(x,y)$ is a polynomial as in (7) and, being g defined on the grid $\mathcal{A}$, then $G_g(x,y)$ must have degree less than M in x and less than N in y. However, for each $t\in\{1,\ldots,r\}$, $G_g(x,y)$ contains the expression

$$\left(\delta_t\mathbf{x}^{u_t}+\mu_t\mathbf{x}^{v_t}\right)F_S(x,y)=\mathbf{x}^{v_t}\left(\delta_t\mathbf{x}^{u_t-v_t}+\mu_t\right)F_S(x,y).$$

Since $u_t - v_t \in D$, say $u_t - v_t = (w_1(t), w_2(t))$, then, by the assumption on the components of elements of D, we have $|w_1(t)| \geq M - h$ or $|w_2(t)| \geq N - k$. Since $deg_x F_S(x, y) = h$ and $deg_y F_S(x, y) = k$, then we get $deg_x G_g(x, y) \geq M$ or $deg_y G_g(x, y) \geq N$, a contradiction.
It follows that g is identically zero, which completes the proof. □

In case S is an even simple cycle that satisfies the assumptions of Theorem 2, we say that S is a *simple cycle of uniqueness* in the given lattice grid.

Remark 3. *The notion of simple cycle is independent of being in the planar case, so it can be preserved even for sets of directions in $\mathbb{Z}^n$, for any $n \geq 2$. Indeed, the proof of Theorem 2 exploits the same arguments as in [6, Theorem 12], so the uniqueness result in [6] holds true even under the weaker condition that S is a simple cycle of uniqueness in $\mathbb{Z}^n$, $n \geq 3$, instead of a set of independent directions.*

4. Extension of BRA to simple cycles of uniqueness

In [5], a uniqueness theorem for binary tomography has been obtained in a finite lattice grid, by means of X-rays taken along four suitable directions. In [10], a Binary Reconstruction Algorithm (BRA) has been considered, that, based on the uniqueness result in [5], provides a perfect noise-free reconstruction of a binary image in polynomial time, and running with sets of four suitably selected directions. As discussed in the Introduction, a drawback of the considered approach is that the exploited sets of uniqueness consist of long directions, so that each line meets only a small number of lattice points.
In order to overcome the problem, a usual strategy is to highly increase the number of employed directions, so to reduce their size and collect several lattice points on each line. However, without a uniqueness result, this reflects in the introduction of ghosts, and consequently in ambiguous reconstructions (see for instance [2]).
In this paper, thanks to Theorem 2, we have extended to simple cycles of uniqueness the result in [5]. This encourages us to look for a generalization of BRA to such simple cycles of uniqueness consisting of $d > 4$ directions.
To this, we investigate possible links between the unique binary solution (say $\overline{\mathbf{x}}$) of the tomographic problem $W\mathbf{x} = \mathbf{p}_S$ and the solution having minimum Euclidean norm (called *central solution*, see [2]). Let $\mathcal{F}_S = \{\lambda_i : i \in I^- \cup I^+\}$ be the weakly bad configuration associated to a simple cycle of uniqueness S. For $u = (p, q) \in E$ (where E denotes the enlarging region, see Section 2 for its definition), let $\mathcal{G}_u = \mathcal{F}_S + u$, and let $g_u : \mathcal{A} \to \mathbb{R}$ be the S-ghost generated by $x^p y^q F_S(x, y)$, namely,

$$g_u(\xi, \eta) = \begin{cases} 0 & \text{if } (\xi, \eta) \notin \mathcal{G}_u, \\ 1 & \text{if } (\xi, \eta) = \lambda_i + u,\ i \in I^+, \\ -1 & \text{if } (\xi, \eta) = \lambda_i + u,\ i \in I^-. \end{cases}$$

Then we have

$$\mathbf{y}(\xi, \eta) = \overline{\mathbf{x}}(\xi, \eta) + \sum_{u \in E} \alpha_u g_u(\xi, \eta)$$

for any solution $\mathbf{y}$ of $W\mathbf{x} = \mathbf{p}_S$, for all $(\xi, \eta) \in \mathcal{A}$ and for suitable coefficients $\alpha_u \in \mathbb{R}$, where

$$\sum_{u \in E} \alpha_u g_u(\xi, \eta) = \begin{cases} 0 & \text{if } (\xi, \eta) \notin H, \\ \displaystyle\sum_{u \in E(\xi, \eta)} m(\xi, \eta) \alpha_u & \text{otherwise,} \end{cases}$$

being $m(\xi,\eta)$ the multiplicity of (ξ,η) and $H=\bigcup_{u\in E}\mathcal{G}_u=\mathcal{F}_S+E$. Let $\{\alpha_u^*\in\mathbb{R}:\ u\in E\}$ be the set of real-valued coefficients corresponding to the central solution $\mathbf{x}^*$, namely,

$$\mathbf{x}^*(\xi,\eta)=\overline{\mathbf{x}}(\xi,\eta)+\sum_{u\in E}\alpha_u^* g_u(\xi,\eta).$$

Setting $w^*(\xi,\eta)=\sum_{u\in E}\alpha_u^* g_u(\xi,\eta)$, it results $\mathbf{x}^*(\xi,\eta)=\overline{\mathbf{x}}(\xi,\eta)+w^*(\xi,\eta)$, and $\overline{\mathbf{x}}(\xi,\eta)$ can be reconstructed from $\mathbf{x}^*(\xi,\eta)$ once we can explicitly compute $w^*(\xi,\eta)$ for all $(\xi,\eta)\in\mathcal{A}$. Denote by $\mathrm{round}(x)$ the closest integer to x. The following theorem is a generalization to simple cycles of uniqueness of [10, Theorem 13] and [9]. The arguments are the same, however some differences occur, so, for the readers' convenience, we provide the proof explicitly.

Theorem 3. *Let S be a simple cycle of uniqueness for a lattice grid $\mathcal{A}=\{(\xi,\eta)\in\mathbb{Z}^2:\ 0\le\xi<M,\ 0\le\eta<N\}$, and let $\mathbf{x}^*$ be the central solution of $W\mathbf{x}=\mathbf{p}_S$. Then, for all $u\in E$ it results*

$$\alpha_u^*=\mathbf{x}^*(\lambda_\delta+u)-\mathrm{round}(\mathbf{x}^*(\lambda_\delta+u)).$$

Proof. Since S is a simple cycle, the multiplicity of each point of $\mathcal{F}_S$ belongs to the set $\{\pm1,\pm2\}$, with a single point $\lambda_\delta\in\mathcal{F}_S$ such that $|m(\lambda_\delta)|=2$.
For each $u\in E$, let $f:\mathbb{R}\to\mathbb{R}$ be the following function:

$$f(\alpha_u)=\sum_{i\in I^+\cup I^-}\left(\overline{\mathbf{x}}(\lambda_i+u)+\alpha_u m(\lambda_i)\right)^2.$$

For a real-valued solution $\mathbf{y}$ of $W\mathbf{x}=\mathbf{p}_S$, it results

$$\begin{aligned}
\|\mathbf{y}\|_2^2 &= \sum_{(\xi,\eta)\in\mathcal{A}}\mathbf{y}^2(\xi,\eta)=\sum_{(\xi,\eta)\in\mathcal{A}}\left(\overline{\mathbf{x}}(\xi,\eta)+\sum_{u\in E^+(\xi,\eta)\cup E^-(\xi,\eta)}\alpha_u g_u(\xi,\eta)\right)^2\\
&= \sum_{(\xi,\eta)\notin H}\overline{\mathbf{x}}^2(\xi,\eta)+\sum_{(\xi,\eta)\in H}\left(\overline{\mathbf{x}}(\xi,\eta)+\sum_{u\in E^+(\xi,\eta)\cup E^-(\xi,\eta)}\alpha_u g_u(\xi,\eta)\right)^2\\
&= \sum_{(\xi,\eta)\notin H}\overline{\mathbf{x}}^2(\xi,\eta)+\sum_{u\in E}\left[\sum_{i\in I^+\cup I^-}\left(\overline{\mathbf{x}}(\lambda_i+u)+\alpha_u g_u(\lambda_i+u)\right)^2\right]\\
&= \sum_{(\xi,\eta)\notin H}\overline{\mathbf{x}}^2(\xi,\eta)+\sum_{u\in E}\left[\sum_{i\in I^+\cup I^-}\left(\overline{\mathbf{x}}(\lambda_i+u)+m(\lambda_i)\alpha_u\right)^2\right]\\
&= \sum_{(\xi,\eta)\notin H}\overline{\mathbf{x}}^2(\xi,\eta)+\sum_{u\in E}f(\alpha_u).
\end{aligned}$$

The central solution $\mathbf{x}^*$ is obtained when $\|\mathbf{y}\|_2^2$ attains its minimum value. Note that $f(\alpha_u)\ge0$ for all $u\in E$. Therefore, $\|\mathbf{y}\|_2^2$ is the sum of the constant term $\sum_{(\xi,\eta)\notin H}\overline{\mathbf{x}}^2(\xi,\eta)$ and of $|E|$ copies of the non-negative number $f(\alpha_u)$, for all $u\in E$. Consequently, the minimum of $\|\mathbf{y}\|_2^2$ is obtained by minimizing f, separately with respect to each variable α_u. Computing the derivative we get

$$f'(\alpha_u) = 2\sum_{i\in I^+\cup I^-}m(\lambda_i)\left(\overline{\mathbf{x}}(\lambda_i+u)+\alpha_u m(\lambda_i)\right).$$

Therefore, it results

$$\alpha_{u,\min} = \alpha_u^* = -\frac{\sum_{i\in I^+\cup I^-} m(\lambda_i)\overline{\mathbf{x}}(\lambda_i+u)}{\sum_{i\in I^-\cup I^+} m^2(\lambda_i)}.$$

Let $\delta \in I$ be the index such that λ_δ is the unique double point of F_S. Then $m(\lambda_\delta) = -2$ implies $|I^-| = 2^{|S|-1} - 1$ and $|I^+| = 2^{|S|-1}$, while $m(\lambda_\delta) = 2$ implies $|I^-| = 2^{|S|-1}$ and $|I^+| = 2^{|S|-1} - 1$. Therefore, the minimum of α_u^* is attained when $\overline{\mathbf{x}}(\lambda_i) = 0$ for all $i \in I^-$ and $\overline{\mathbf{x}}(\lambda_i) = 1$ for all $i \in I^+$, while the maximum is attained when $\overline{\mathbf{x}}(\lambda_i) = 0$ for all $i \in I^+$ and $\overline{\mathbf{x}}(\lambda_i) = 1$ for all $i \in I^-$. Consequently, we have

$$\alpha_u^* \in \left[-\frac{\sum_{i\in I^+} m(\lambda_i)}{2^{|S|}+2}, \frac{\sum_{i\in I^-} m(\lambda_i)}{2^{|S|}+2}\right] = \left[-\frac{2^{|S|-1}}{2^{|S|}+2}, \frac{2^{|S|-1}}{2^{|S|}+2}\right] \subset \left(-\frac{1}{2}, \frac{1}{2}\right). \quad (8)$$

By [10, Lemma 10], the enlarging region of λ_δ does not intersect the enlarging region of any other $\lambda_i \in F_S$, meaning that, for all $u \in E$, there exists a unique α_u to be computed for each pixel in $\lambda_\delta + E$. Therefore, by (8) it results

$$\mathrm{round}(\mathbf{x}^*(\lambda_\delta+u)) = \mathrm{round}\left(\overline{\mathbf{x}}(\lambda_\delta+u)+\alpha_u^*\right) = \overline{\mathbf{x}}(\lambda_\delta+u).$$

Hence, the unique binary solution $\overline{\mathbf{x}}$ is exactly reconstructed in $\lambda_\delta + E$. This also allows to explicitly compute the value of each α_u^*. In fact, since $\mathbf{x}^*(\lambda_\delta+u) = \overline{\mathbf{x}}(\lambda_\delta+u)+\alpha_u^*$, we have

$$\alpha_u^* = \mathbf{x}^*(\lambda_\delta+u) - \overline{\mathbf{x}}(\lambda_\delta+u) = \mathbf{x}^*(\lambda_\delta+u) - \mathrm{round}(\mathbf{x}^*(\lambda_\delta+u)),$$

which proves the theorem. □

Corollary 1. *Let S be a simple cycle of uniqueness for a lattice grid $\mathcal{A}$. Then the unique binary solution $\overline{\mathbf{x}}$ is uniquely and explicitly reconstructible from $\mathbf{x}^*$.*

Proof. We have

$$\overline{\mathbf{x}}(\xi,\eta) = \mathbf{x}^*(\xi,\eta) - w^*(\xi,\eta) = \begin{cases} \mathbf{x}^*(\xi,\eta) & \text{if } (\xi,\eta)\notin H, \\ \mathbf{x}^*(\xi,\eta) - \sum\limits_{u\in E(\xi,\eta)} m(\xi,\eta)\alpha_u^* & \text{otherwise,} \end{cases}$$

and, by Theorem 3, the value of each α_u^* is known. □

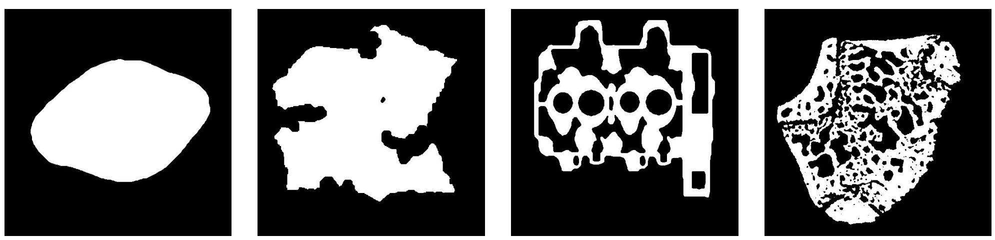
Remark 4. *In case the enlarging regions of the pixels of $\mathcal{F}_S$ are pairwise disjoint, then, for each $(\xi,\eta) \in H$, $\overline{\mathbf{x}}(\xi,\eta)$ differs from $\mathbf{x}^*(\xi,\eta)$ for a single $\alpha_u^* \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, so $\overline{\mathbf{x}}(\xi,\eta) = \mathrm{round}(\mathbf{x}^*(\xi,\eta))$.*

Our results allow to define a new algorithm, *e-BRA*, for the reconstruction of binary images under sets of uniqueness having an arbitrary even cardinality, that is a natural extension of the algorithm BRA introduced in [10]. The implementation is precisely the same, with the only difference that projections are collected along a set of directions S that form a simple cycle of any even size $d \geq 4$. The solution $\mathbf{x}^*$ having minimum Euclidean norm is computed by means of the conjugate gradient least square subroutine (CGLS).
The computational cost of e-BRA is the same of BRA, mainly related to CGLS, and depending on the number of iterations and on the sparsity of the projection matrix W. It is estimated as $O(max\{s\sqrt{MN}, (M-h)(N-k)(MN)\})$ [10], where s is the number of rows of W. In particular, we underline that the cost also depends on the size of the enlarging region, $(M-h)(N-k)$, that

consequently plays an important role in selecting a suitable simple cycle of uniqueness S as an input of the algorithm.
Actually, as highlighted in the experimental results reported in the next section, a careful choice of S allows to reach a perfect reconstruction even avoiding the process of update of weights, that was the key point of the algorithm BRA [10]. As a matter of fact, it is sufficient to select the set S such that, in the corresponding weakly bad configuration, no overlapping occurs among the enlarging regions of the points of $\mathcal{F}_S$. In this way, the explicit computation of the weights α_v^* becomes superfluous (see Remark 4), and the unique existing binary solution $\overline{\mathbf{x}}$ can be computed by the simple integer rounding of the central solution $\mathbf{x}^*$, thus noticeably improving the computational cost, and so the whole performance of the reconstruction strategy.

5. Experimental results

In this section we provide some experimental results deriving from the application of e-BRA to the reconstruction of four binary images of size 512×512. We test our algorithm on the same phantoms used in [10] (see Figure 2), and compare its performance to that of BRA.



(a) (b) (c) (d)

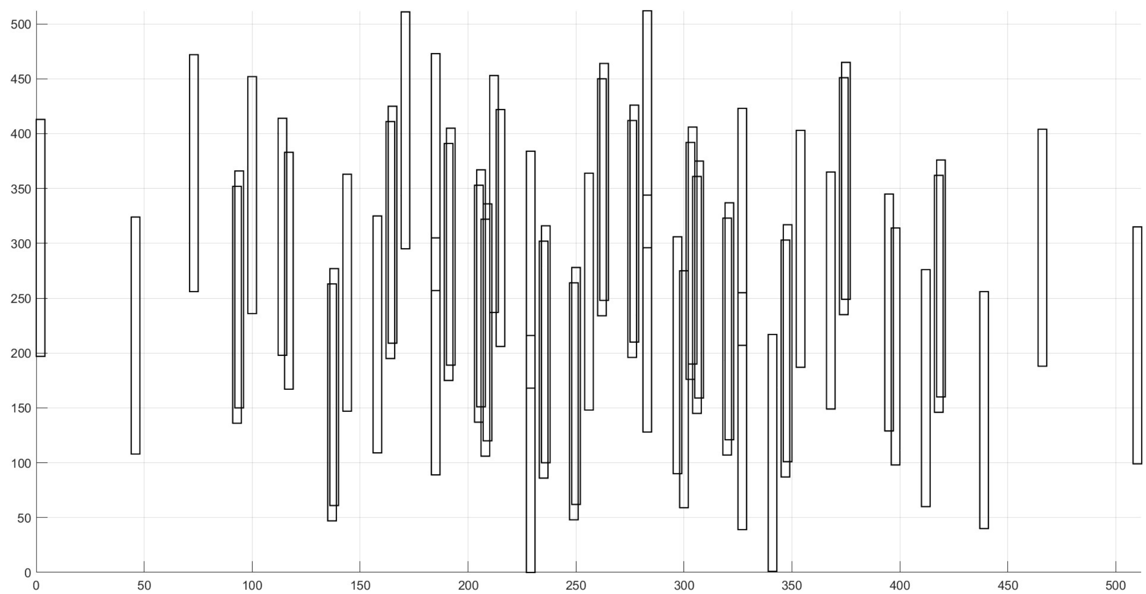
Figure 2: The four binary images which are used to test the sets of directions.

We employed simple cycles of six directions, chosen among 10^6 randomly generated sets of size six, and then imposing the conditions of Theorem 2 for uniqueness. For the reasons outlined in the Introduction, we prefer directions that are not too long, and whose norms do not differ too much. This reflects in generating simple cycles $S = I \dot{\cup} J$ in which I and J have the same cardinality, namely $|I| = |J| = 3$.
After the random generation, we have selected two simple cycles of uniqueness, according to two different parameters.
For the first test, we considered the cycle in which the norm of the longest direction is minimum among all the randomly generated sets. For the second test, we used (and minimized all over the random generated sets of directions) the parameter $R = \frac{n_{max}^2 \cdot |E|}{MN}$, where n_{max} is the length of the longest direction of the cycle, $|E|$ is the area of the associated enlarging region and MN is the size of the binary image we wish to reconstruct (namely, the size of the lattice grid $\mathcal{A}$). Different motivations lead to the choice of such parameters. On one side, directions of short length allow increasing the number of lattice points collected along each line, and consequently are preferable in view of real applications. On the other side, a small parameter R reflects in having enlarging regions of small size for the different pixels of $\mathcal{F}_S$, which increases the probability of empty intersection among them, so providing faster reconstructions and lower computational costs. Let us also remark

that, since the computation of $\mathbf{x}^*$ is returned after a finite number of iterations, of course a final integer rounding step is required to remove the numerical errors and get the unique binary solution.
In Tables 1 and 3 we report the results of the reconstruction test obtained with the simple cycle of minimum norm, that is $S_1 = \{(92, -47), (91, -61), (71, 59), (44, -89), (98, 39), (112, 1)\}$, with its partition $I = \{(92, -47), (91, -61), (71, 59)\} \dot\cup J = \{(44, -89), (98, 39), (112, 1)\}$.
By increasing progressively the number of iterations in the CGLS subroutine, returning the corresponding approximations of the central solution $\mathbf{x}^*$, we compared the results with the output obtained in [10] w.r.t. the choice of the simple cycle of size four $S = \{(80, 77), (81, 91), (80, 83), (241, 251)\}$.
It turns out that, using S_1, drastically reduces the number of CGLS iterations that are required to get the exact reconstruction of each phantom. In particular, for the first phantom only 10 iterations are enough, against the 350 required when using S. Similarly for the other phantoms, with only 45, 40 and 85 iterations instead of 500, 650 and 550, respectively (see [10]).
Tables 2 and 4 show the performance related to the second simple cycle of directions, namely $S_2 = \{(98, -81), (99, 19), (58, -55), (65, 68), (1, 51), (189, -236)\}$, obtained for the minimum value of the parameter R over all the randomly generated cycles. Here the results are even more surprising, since the exact reconstruction of any phantom can be reached with at most 14 iterations of the CGLS. The choice of this set also reveals to be more efficient in terms of computational cost. In fact, the size of the enlarging regions is very small, namely $|E| = 4$, so, differently from what happens in the case of the simple cycle S_1, no overlapping exists among them (see also Figure 3), which considerably reduces the running time of e-BRA, that actually coincides with the rounding of the central solution computed by the CGLS (see Remark 4).
In the case of S_1, due to the non-empty intersections between various enlarging regions, perfect reconstructions are obtained with an average running time of $\sim$ 49 minutes. However, the only integer rounding of the central solution computed by the CGLS subroutine performs in $\sim$ 10 seconds only (see the caption of Table 1), still providing perfect reconstructions using less iterations, which is better than BRA [10] even from this point of view.
As highlighted in [10], the performance of the reconstruction is strongly affected by the shape of the binary image we want to detect, such as its boundary or the presence of holes, even if the perfect reconstruction is always achieved in a limited number of iterations.

6. Conclusions and perspectives

We have proven that any set C contained in a finite lattice grid $\mathcal{A}$ can be uniquely and perfectly reconstructed from the knowledge of the number of points intercepted in C by lines parallel to special sets of lattice directions. Such sets, called simple cycles of uniqueness, have even cardinality and generalize the sets of four directions considered in [5].
The obtained uniqueness theorem has been later matched with Theorem 3, a useful rounding result that leads to an explicit reconstruction algorithm. This has been tested on four phantoms by using two simple cycles of uniqueness, selected among 10^6 randomly generated sets of six directions. The selection has been carried out by imposing the required conditions of uniqueness determined in Theorem 2. Then, we have extracted two simple cycles of uniqueness among the resulting ones, by optimizing two different parameters related to the underlying geometric structure of the problem. In both cases the algorithm e-BRA showed a significant improvement of the performance if compared with the results in [10], so providing perfect reconstruction with a largely decreased number of iterations. Furthermore, we showed that a careful choice of the set S allows to replace



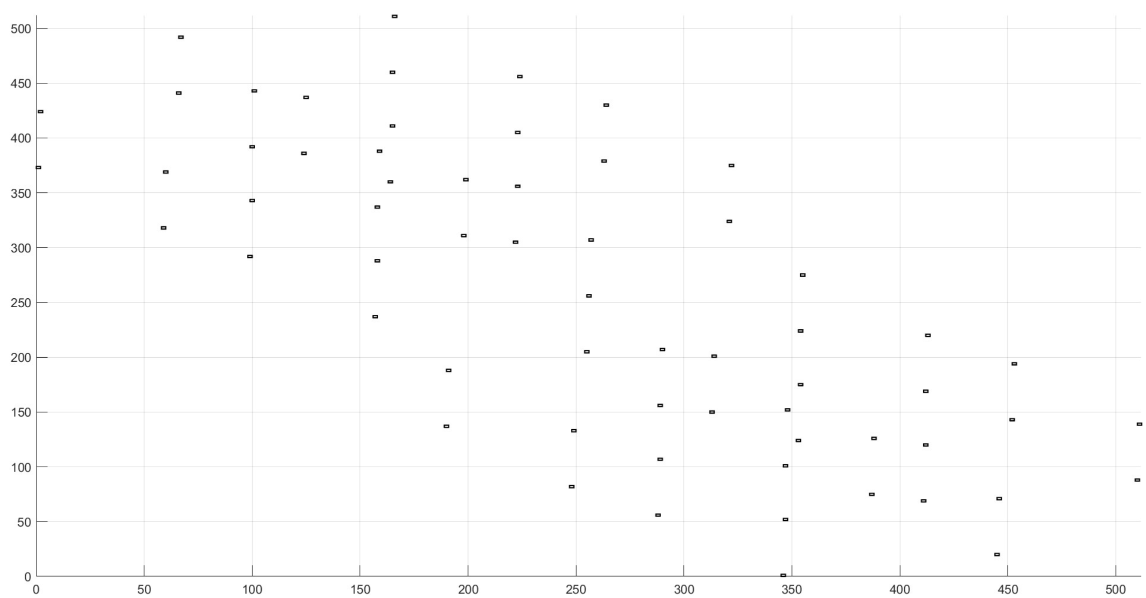


Figure 3: Top: The enlarging regions corresponding to each pixel of the weakly bad configuration $\mathcal{F}_{S_1}$ associated to the simple cycle S_1. Note the various intersections among them. Bottom: The enlarging regions corresponding to each pixel of the weakly bad configuration $\mathcal{F}_{S_2}$ associated to the simple cycle S_2. In this case, the enlarging regions are pairwise disjoint.

	Phantom (a)		Phantom (b)		Phantom (c)		Phantom (d)	
	% reconstruction		% reconstruction		% reconstruction		% reconstruction	
♯ iterations	**BRA**	**r(CGLS)**	**BRA**	**r(CGLS)**	**BRA**	**r(CGLS)**	**BRA**	**r(CGLS)**
5	87, 4092	99, 7715	91, 2003	98, 8316	96, 5935	99, 6399	95, 3346	99, 2786
10	89, 4630	100	93, 1129	99, 6948	98, 0015	99, 8970	96, 9589	99, 7692
15	90, 9096	100	93, 9625	99, 7490	98, 8770	99, 9287	97, 7921	99, 8798
20	92, 9287	100	94, 5007	99, 8356	99, 1268	99, 9619	98, 2510	99, 9359
25	93, 1965	100	94, 8883	99, 9325	99, 1817	99, 9771	98, 5619	99, 9699
30	93, 1419	100	94, 9764	99, 9805	99, 2161	99, 9889	98, 7358	99, 9866
35	93, 1881	100	95, 0813	99, 9958	99, 2851	99, 9981	98, 8308	99, 9931
40	93, 2270	100	95, 1820	99, 9985	99, 3397	100	98, 8640	99, 9962
45	93, 2526	100	95, 2492	100	99, 4194	100	98, 9540	99, 9985
50	93, 3460	100	95, 4098	100	99, 4907	100	99, 0292	99, 9992
70	94, 1795	100	96, 4542	100	99, 6288	100	99, 1974	99, 9996
85	95, 1431	100	97, 2759	100	99, 6510	100	99, 2123	100

Table 1: The table shows the percentage of pixels that were correctly reconstructed by round(CGLS), w.r.t. the number of iterations of CGLS, by choosing the cycle of uniqueness S_1. For each phantom the performance of the new algorithm is compared with the results reached with BRA. The average running time of round(CGLS) is 9.6828s (Phantom (a)), 9.6798s (Phantom (b)), 9.6894s (Phantom (c)) and 9.7351s (Phantom (d)), against 35.9991s, 37.1793s, 36.2200s, and 36.7735s performed by BRA, respectively. If we include the computation of the values α_u^*, namely, we move from round(CGLS) to e-BRA, the running time of the algorithm increases to $\sim$ 49 minutes, since the enlarging regions in $\mathcal{F}_{S_1}$ are not pairwise disjoint. In any case, the number of required iterations to reach perfect reconstructions does not significantly differ from those ones required by round(CGLS), since the overlappings are not so substantial (see Fig. 3). The improvement of the performance using six directions instead of four is evident.

	Phantom (a)		Phantom (b)		Phantom (c)		Phantom (d)	
	% reconstruction		% reconstruction		% reconstruction		% reconstruction	
♯ iterations	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)
1	72, 0032	99, 9474	67, 5571	97, 3408	80, 6828	87, 7895	77, 6054	83, 6227
2	78, 6480	100	79, 6726	98, 7370	88, 6410	97, 8024	85, 7597	96, 5656
3	85, 1456	100	87, 1723	99, 7883	93, 3338	99, 5544	91, 3975	98, 7869
4	87, 4401	100	91, 1255	99, 9462	95, 8172	99, 9268	94, 3275	99, 4404
5	87, 4092	100	91, 2003	99, 9905	96, 5935	99, 9928	95, 3346	99, 7250
6	87, 9543	100	91, 7496	100	96, 9959	100	95, 9316	99, 8768
7	88, 5426	100	92, 1745	100	97, 2382	100	96, 2200	99, 9245
8	89, 0572	100	92, 6483	100	97, 5616	100	96, 5168	99, 9439
9	89, 3127	100	92, 9394	100	97, 8333	100	96, 7274	99, 9638
10	89, 4630	100	93, 1129	100	98, 0015	100	96, 9589	99, 9840
11	89, 5687	100	93, 2533	100	98, 1678	100	97, 1081	99, 9924
12	89, 8548	100	93, 4521	100	98, 3734	100	97, 2919	99, 9962
13	90, 1440	100	93, 6581	100	98, 5573	100	97, 4491	99, 9985
14	90, 5163	100	93, 8484	100	98, 7583	100	97, 6395	100

Table 2: The table shows the percentage of pixels that were correctly reconstructed by round(CGLS), w.r.t. the number of iterations of CGLS, by choosing the cycle of uniqueness S_2. For each phantom the performance of the new algorithm is compared with the results reached with BRA. The average running time of round(CGLS) is 16.4376s (Phantom (a)), 16.5854s (Phantom (b)), 16.6367s (Phantom (c)) and 16.8537s (Phantom (d)), against 37.2501s, 36.4100s, 35.8227s, and 36.3038s performed by BRA, respectively. In this case, e-BRA coincides with round(CGLS), since all the enlarging regions in $\mathcal{F}_{S_2}$ are pairwise disjoint (see Remark 4). The improvement of the performance w.r.t. BRA is evident.

	Phantom (a)		Phantom (b)		Phantom (c)		Phantom (d)	
	♯ wrong pixel		♯ wrong pixel		♯ wrong pixel		♯ wrong pixel	
♯ iterations	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)
5	33006	599	23068	3063	8930	944	12230	1891
10	27622	0	18054	800	5239	270	7972	605
15	23830	0	15827	658	2944	187	5788	315
20	18537	0	14416	431	2289	100	4585	168
25	17835	0	13400	177	2145	60	3770	79
30	17978	0	13169	51	2055	29	3314	35
35	17857	0	12894	11	1874	5	3065	18
40	17755	0	12630	4	1731	0	2978	10
45	17688	0	12454	0	1522	0	2742	4
50	17443	0	12033	0	1335	0	2545	2
70	15258	0	9295	0	973	0	2104	1
85	12732	0	7141	0	915	0	2065	0

Table 3: The table shows the number of wrong pixels in the reconstruction of the binary image when choosing the cycle of uniqueness S_1, w.r.t. the number of iterations selected for CGLS. The performance is also compared with the algorithm BRA.

	Phantom (a)		Phantom (b)		Phantom (c)		Phantom (d)	
	♯ wrong pixel		♯ wrong pixel		♯ wrong pixel		♯ wrong pixel	
♯ iterations	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)	BRA	r(CGLS)
1	73392	138	85047	6971	50639	32009	58706	42932
2	55973	0	53287	3311	29777	5761	37330	9003
3	38940	0	33627	555	17475	1168	22551	3180
4	32925	0	23264	141	10965	192	14870	1467
5	33006	0	23068	25	8930	19	12230	721
6	31577	0	21628	0	7875	0	10665	323
7	30035	0	20514	0	7240	0	9909	198
8	28686	0	19272	0	6392	0	9131	147
9	28016	0	18509	0	5680	0	8579	95
10	27622	0	18054	0	5239	0	7972	42
11	27345	0	17686	0	4803	0	7581	20
12	26595	0	17165	0	4264	0	7099	10
13	25837	0	16625	0	3782	0	6687	4
14	24861	0	16126	0	3255	0	6188	0

Table 4: The table shows the number of wrong pixels in the reconstruction of the binary image when choosing the cycle of uniqueness S_2, w.r.t. the number of iterations selected for CGLS. The performance is also compared with the algorithm BRA.

the algorithm e-BRA with round(CGLS), drastically reducing the computational cost and running time of the performance.
Even if not explicitly included in the present paper, we have also considered simple cycles of uniqueness consisting of eight directions, and tested the algorithm on the same previous four phantoms. The number of required iterations has been further reduced, even if the improvement is not so evident as when moving from four to six directions. Also, in view of real applications to tomographic reconstructions, we emphasize that working with as few directions as possible is highly desirable, in order to reduce the employed radiation dose. Therefore, it seems that simple cycles having cardinality six are an optimal choice for the considered tomographic problem. As a further step, it could be of interest to explore the robustness to noise of the proposed approach, as well as a possible extension to non homogeneous objects.
We also remark that simple cycles can be defined independently of the lattice dimension, so that the same notion could be considered even in $\mathbb{Z}^n, n > 2$, and consequently adopted in a possible extension of the reconstruction algorithm to higher dimensions.

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