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OSKAR BECKER AND THE MODAL TRANSLATION OF INTUITIONISTIC LOGIC*
PREPRINT

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ABSTRACT. We reconsider Oskar Becker's pioneering contributions to modal logic in *On the Logic of Modalities* (1930), in particular Becker's unjustly neglected anticipation of the idea of a modal interpretation of intuitionistic logic, which was realized three years later by Kurt Gödel.

1. INTRODUCTION

"Heyting's intuitionistic propositional calculus **IPC** can be soundly and faithfully translated into the classical modal system **S4**": this is — rephrased in the now current terminology — the well known main result (comprehensive of a conjecture later proved to be true in [McKinsey and Tarski, 1948]) that is contained in the short, deservedly celebrated paper published in 1933 by Kurt Gödel with the title *An interpretation of the intuitionistic propositional calculus.*[1]

Yet, the idea of a modal translation of intuitionistic logic was not new: three years earlier, in the *Appendix to Part I* of his essay *On the Logic of Modalities*[2], Oskar Becker not only had seriously considered that very idea, but he had also actually tried — although unsuccessfully — to realize it at a formal level.

Now, the fact is that Gödel was aware of Becker's aim: indeed, in 1931 he had reviewed[3] *On the Logic of Modalities.* In the *Review*, Gödel is pretty accurate in describing Becker's main intent of extending Lewis's "Survey system" to a modal system with a linearly ordered, finite number of positive modalities (we will say more on this in Section 3) and in drawing attention to some weak points in Becker's formal "experiments" as well. On the other side, he is rather hasty and dismissive in commenting the *Appendix to Part I* and Becker's explicit intent of translating intuitionistic logic into the "Survey system":

> In conclusion the author discusses, from a formal as well as a phenomenological standpoint, the connections that in his opinion obtain between modal logic and the intuitionistic

[1] [Gödel, 1933].

[2] [Becker, 1930] (here quoted according to the original pagination). The forthcoming volume [Centrone and Minari, 2020] contains the first English translation of Becker's *Zur Logik der Modalitäten*, together with an extensive commentary.

[3] [Gödel, 1931].

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> logic of Brouwer and Heyting. It seems doubtful, however, that the steps here taken to deal with this problem on a formal plane will lead to success. ([Gödel, 1931], 217)

All the more surprising is that in his *An interpretation of the intuitionistic propositional calculus* Gödel does indeed mention Becker (once), but only concerning a certain modal axiom (see below) introduced in *On the Logic of Modalities*; and not — *quite unfairly* — for having (at least) anticipated the idea of a modal translation of intuitionistic logic.[4]

Aim of the present note is to reconsider this unjustly neglected contribution by Oskar Becker, and other interesting ones as well.

2. Gödel's result

Of course, it is not our intention to diminish the importance of Gödel's own result. So, to start with, let us briefly summarize Gödel's key accomplishments in [Gödel, 1933]. The target modal system $\mathfrak{S}$ introduced by Gödel features a language containing a modal operator B — *beweisbar*, intended to mean 'provable by any correct means' — in addition to the usual boolean connectives $\neg, \vee, \wedge, \rightarrow$, and is axiomatically presented as an extension of the classical propositional calculus **CPC** by means of four postulates, namely three axiom schemas[5] and a new rule of inference:

($\mathfrak{S}$.1) $\mathsf{B}\alpha \rightarrow \alpha$
($\mathfrak{S}$.2) $\mathsf{B}\alpha \rightarrow (\mathsf{B}(\alpha \rightarrow \beta) \rightarrow \mathsf{B}\beta)$
($\mathfrak{S}$.3) $\mathsf{B}\alpha \rightarrow \mathsf{BB}\alpha$
($\mathfrak{S}$.4) from α to infer $\mathsf{B}\alpha$

$\mathfrak{S}$ coincides, up to the use of 'B' in place of '$\Box$', with Lewis's system **S4** *recasted in the now familiar axiomatic presentation.* It is worth to remark that this is the first time ever that the "user-friendly" format "**CPC** + specific modal rules and axioms" for the axiomatic presentation of a (classical) modal system, which has become standard since the 1950s[6], is adopted. Indeed, **S4** had been *officially* introduced one year earlier in the *Appendix II* of [Lewis and Langford, 1932][7] in the much less perspicuous "Lewis-style" axiomatization not constructed as an extension of **CPC**, which has been in force during the early development of modal logic up until the 1940s.

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[4]Likewise, no mention of Becker is found in A. S. Troelstra's "Introductory note to *An interpretation of the intuitionistic propositional calculus*", in [Feferman et al., 1986], 296–299.

[5]Actually, in Gödel's paper the axioms are not given in schematic form, and the rule of substitution is assumed.

[6]Thanks to [Feys, 1950], [Prior, 1955] and, in particular, [Lemmon, 1957].

[7][Lewis and Langford, 1932] is not mentioned by Gödel. He says that $\mathfrak{S}$ is equivalent to "Lewis's system of strict implication", that is the *Survey system* ([Lewis, 1918], emended in [Lewis, 1920] and eventually named '**S3**' in the mentioned *Appendix II*), supplemented by "Becker's axiom" $\Box(\Box\alpha \rightarrow \Box\Box\alpha)$.

Gödel's translation $(\ldots)^G$ from the formulas of the language $\mathcal{L}_I$ of **IPC** into the formulas of the language of $\mathfrak{S}$ *alias* **S4**[8] is inductively defined as follows[9]:

- $p^G \ := \ p$, where p is a propositional atom
- $(\neg\beta)^G \ := \ \neg\Box\beta^G$
- $(\beta \wedge \gamma)^G \ := \ \beta^G \wedge \gamma^G$
- $(\beta \vee \gamma)^G \ := \ \Box\beta^G \vee \Box\gamma^G$
- $(\beta \to \gamma)^G \ := \ \Box\beta^G \to \Box\gamma^G$

Next, Gödel claims (without giving the proof[10]) that the translation is *sound*, that is for all $\alpha \in \mathcal{L}_I$:

$$\vdash_{\mathbf{IPC}} \alpha \quad \Rightarrow \quad \vdash_{\mathbf{S4}} \alpha^G$$

and conjectures that the translation is also, as we use to say today, *faithful*, that is for all $\alpha \in \mathcal{L}_I$:

$$\vdash_{\mathbf{S4}} \alpha^G \quad \Rightarrow \quad \vdash_{\mathbf{IPC}} \alpha$$

Fifteen years later the conjecture was indeed solved in the positive by J. C. C. McKinsey and A. Tarski[11] by introducing a suitable algebraic-semantics characterization for **S4** together with the following auxiliary translation $(\ldots)^*$:

- $p^* \ := \ \Box p$, where p is a propositional atom
- $(\neg\beta)^* \ := \ \Box\neg\beta^*$
- $(\beta \wedge \gamma)^* \ := \ \beta^* \wedge \gamma^*$
- $(\beta \vee \gamma)^* \ := \ \beta^* \vee \gamma^*$
- $(\beta \to \gamma)^* \ := \ \Box(\beta^* \to \gamma^*)$

Gödel's accomplishments, beyond their intrinsic conceptual and technical interest, and their more or less immediate consequences as well[12], have played an important role in a number of subsequent developments and investigations. Let us just briefly mention three of the most significative ones.

First of all, Saul Kripke's invention of the relational semantics for intuitionistic logic was actually inspired by his own possible-worlds semantics for modal logics[13] together with "the known mappings of intuitionistic logic

[8] We use the same symbols for the intuitionistic and the classical connectives $(\neg, \wedge, \vee, \to)$, and henceforth replace throughout B with $\Box$ and $\mathfrak{S}$ with **S4**.

[9] Gödel also indicates as variants: $(\neg\beta)^G \ := \ \Box\neg\Box\beta^G$ and/or $(\beta \wedge \gamma)^G \ := \ \Box\beta^G \wedge \Box\gamma^G$.

[10] A syntactical proof is indeed straightforward (and tedious).

[11] [McKinsey and Tarski, 1948], Theorems 5.2 and 5.3 (the latter for Gödel's variant of the translation $(\ldots)^G$), while Theorem 5.1 states the soundness and faithfulness of their own translation $(\ldots)^*$.

[12] E.g. the *disjunction property* for **IPC**, which follows from the translation theorem together with Gödel's conjecture that $\vdash_{\mathbf{S4}} \Box\alpha \vee \Box\beta$ implies $\vdash_{\mathbf{S4}} \Box\alpha$ or $\vdash_{\mathbf{S4}} \Box\beta$ (later proved in [McKinsey and Tarski, 1948]). The first "official" proof of the *disjunction property* for **IPC** was given by Gerhard Gentzen in 1935, via cut-elimination [Gentzen, 1935].

[13] [Kripke, 1963], [Kripke, 1965b].

into the modal system **S4**"[14]. Next, Gödel's remark that the modal operator B of $\mathfrak{S}$/**S4** *cannot* be read as 'provable in a given formal system' like e.g. **PA** (first-order Peano arithmetic), together with his 1931 incompleteness theorems, paved the way in the 1970s to the birth of the *provability logics*, a new family of modal logics, relevant also from a foundational perspective, which feature instead a box-like operator meant to capture in an abstract, modal setting the structural properties of the notion of provability in **PA** (or in another fixed, typically arithmetical, theory).[15] On the other side, Gödel's way of capturing the intended semantics of the intuitionistic logical operators through a modal, **S4**-like notion of 'provability by any correct means' has motivated in the 1990's, starting with the work of S. Artemov, the elaboration of the *logic of proofs* **LP**, a sort of explicit version of **S4** in whose syntax *proof terms*, representing (classical) proofs, become first class citizens, and its subsequent generalization to the so called *justification logic*.[16]

3. Oskar Becker and the search for a "System of closed modalities"

Oskar Becker's 1930 essay *On the Logic of Modalities* sets out explicitly as an attempt to deal with issues pertaining to modal logic by supplementing the method provided by the 'calculus of logic' (in the Russell and Lewis tradition) with the 'phenomenological' method[17] — an enterprise that appears from the outset not to be easy at all.

> The concurrent usage of these methods of research, which are so different as to their essence and to their methodological technique, could appear to be questionable and is, actually, not bare of difficulties. Nevertheless, it seems to be unavoidable, if one does not want to end up in two "polar" unilateralities, namely, the mathematical combination of mostly empty concept-constructs, on the one hand, and the

[14][Kripke, 1965a], 92. Indeed, an intuitionistic model is exactly the pre-image of a **S4** model under the McKinsey-Tarski translation.

[15]See [Artemov and Beklemishev, 2004] for a survey.

[16]See [Artemov and Fitting, 2019].

[17]Oskar Becker (Leipzig 1889 – Bonn 1964) is often remembered as one of the most prominent students of Edmund Husserl. He graduated in mathematics in 1914 [Becker, 1914], and in 1922 he wrote under Husserl's supervision his *Habilitationsschrift*, *Contributions Toward a Phenomenological Foundation of Geometry and Its Physical Applications* [Becker, 1923]. In 1927 Becker published what is considered to be his masterpiece, *Mathematical Existence* [Becker, 1927], in the *Jahrbuch für Philosophie und phänomenologische Forschung* (he was, together with Martin Heidegger, Moritz Geiger, Alexander Pfänder, Adolf Reinach and Max Scheler a member of the editorial board of this journal). In 1952 — when the study of modal logic was already well beyond its pioneering era — Becker would come back to this subject with the monograph *Investigations on the Modal Calculus* [Becker, 1952], perhaps too old-fashioned for the time, cp. [Martin, 1969]. For a complete bibliography of Becker's works see [Zimny, 1969].

> quite shortsighted description of obvious, more or less arbitrarily assembled concrete cases, on the other hand, which latter has been called, joking, "empiricism of the apriori". ([Becker, 1930], 1)

Indeed, we might better say that in *On the Logic of Modalities* Becker pursued two loosely related goals. The first one, more technical in character, was to find axiomatic conditions that reduced to the finite the number of logically non-equivalent combinations arising from the iterated application of the operators "not" and "it is impossible that (...)" in Lewis's *Survey system.* The second one, more philosophically oriented and, in a sense, much more ambitious, was to treat the logic of modalities from a phenomenological perspective and to understand, from this perspective, the philosophical and logical-ontological problems underlying, and posed by, the Intuitionism.

In the present paper we focus exclusively on Part I of the essay, entitled *On the rank order and reduction of logical modalities* and dealing with the first of the above mentioned goals, and (in Sect. 4) on the *Appendix to Part I* entitled *The logic of modalities and the Brouwer-Heyting "intuitionistic" logical calculus.*

3.1. **The "Survey system", alias S3, in a nutshell.** The "System of Strict Implication", or "Survey system", the real object of the investigations of Part I, was introduced by Lewis in *A Survey of Symbolic Logic* (1918), and by him emended two years later[18] after Emil L. Post's discovery that the original system proved the (strict) equivalence between the negation of p and the impossibility of p, thus *collapsing* into classical logic. It eventually received the now familiar name '**S3**', which we will use henceforth, in the already mentioned *Appendix II* of [Lewis and Langford, 1932].

As we said, the formal language and the style of axiomatization employed by Lewis in the *Survey* and followed by Becker in his essay are different from the now current ones.

As primitives, they take the unary operators "$-$" and "$\sim$", respectively for *negation* and *impossibility*, and the binary operators "$\times$" and "$=$", respectively for *conjunction* and *strict equivalence.*[19]

In turn, Lewis's (and Becker's) axiomatization of the system is *not* given as an extension of an axiomatic calculus for classical logic by means of additional axioms and inference rules. Actually, it is not at all trivial to prove that all classical tautologies are theorems of this axiomatization of **S3**.[20]

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[18] [Lewis, 1920].

[19] Thus "$-\alpha$", "$\sim \alpha$", "$\alpha \times \beta$", "$\alpha = \beta$" correspond, respectively, to "$\neg\alpha$", "$\neg\Diamond\alpha$", "$\alpha \wedge \beta$", "$\Box(A \leftrightarrow B)$" in the now current notation. The other logical boolean and "strict" operators, in particular "$\supset$" (*material implication*), "$<$" (*strict implication*) and "$+$" (boolean disjunction) are instead *defined* in the expected way.

[20] See [Lewis and Langford, 1932], 136 ff.

The equivalent, now more familiar axiomatization[21] of **S3** — in a language having as primitives the boolean connectives and the modal operator $\Box$, with $\Diamond\alpha$ defined as $\neg\Box\neg\alpha$ — looks as follows:

Axioms and axiom schemas:

- all classical tautologies
- $\Box\alpha \to \alpha$ (schema T)
- $\Box(\alpha \to \beta) \to \Box(\Box\alpha \to \Box\beta)$ (schema K^+)

Inference rules:

$$\frac{\alpha \to \beta \qquad \alpha}{\beta}\ \text{MP} \qquad (\textit{modus ponens})$$

$$\frac{\alpha}{\Box\alpha}\ \text{RN}^- \qquad \text{provided } \alpha \text{ is an (instance of an) axiom (schema)}$$

Notice that the necessitation rule (RN^-) comes in a *restricted* form. Indeed, it turns out that **S3** is not closed under the unrestricted necessitation rule (RN) and so that *it is not a normal modal system.* For instance, denoting by "$\top$" any classical tautology, say $p \to p$, $\Box\top$ is a theorem of **S3**, while by contrast $\Box\Box\top$ is not a theorem. Indeed, one can prove that **S3** extended with the axiom $\Box\Box\top$ is equivalent to the *normal* system **S4** mentioned in the previous Section.

In the following we will give an idea of Becker's investigations on **S3** by rephrasing them — for the reader's convenience — in terms of the $\Box$-language and the above axiomatization, while making reference to the Lewis-style setting only when it necessary to understand certain motivations underlying Becker's proposals.

3.2. **Becker's "completions" of S3.**

A *nested modality* (in short, *modality*) is a possibly empty string of $\Box$ and $\neg$, which prefixed to a formula yields another formula:

$$\_\ (\text{empty modality: } \_\alpha \equiv \alpha),\ \neg,\ \Box,\ \Box\Box,\ \neg\Box,\ \neg\Box\neg\Box\Box\neg,\ \ldots$$

or, in Lewis's and Becker's formalism, a possibly empty string of $\sim$ and $-$:

$$\_\,,\ -,\ \sim-,\ \sim-\sim-,\ -\sim-,\ -\sim\sim-\sim,\ \ldots$$

In the $\{\Box, \neg\}$ notation, a modality is *positive* (*negative*) iff it has an even (odd) number of occurrences of the negation; the other way around in the $\{\sim, -\}$ notation.

Given two modalities $\mathsf{m}_1, \mathsf{m}_2$, these are said to be **S3**-*equivalent* iff, for all formulas α,

$$\vdash_{\mathbf{S3}} \Box(\mathsf{m}_1\alpha \leftrightarrow \mathsf{m}_1\alpha)$$

while m_1 is said to be *stronger* than m_2 — in symbols: $\mathsf{m}_1 \rightarrowtail \mathsf{m}_2$ — iff

$$\vdash_{\mathbf{S3}} \Box(\mathsf{m}_1\alpha \to \mathsf{m}_2\alpha) \text{ for all } \alpha, \text{ and } \nvdash_{\mathbf{S3}} \Box(\mathsf{m}_2\alpha \to \mathsf{m}_1\alpha) \text{ for some } \alpha$$

[21]Introduced in [Lemmon, 1957].

Trivially, the number of modalities is infinite. But: how many *irreducible*, i.e. pairwise non equivalent, or "logically distinct", *modalities* are there in Lewis's **S3**?

Lewis did not consider this question, and Becker starts his investigation with the (implicit) conjecture that **S3** has indeed an infinite number of irreducible modalities — a conjecture which in 1939 William Parry will prove to be false: **S3** has exactly 42 irreducible modalities[22]. Becker's aim is to (axiomatically) extend **S3** in such a way that the set of irreducible modalities of the resulting system **S** be

(i) *finite* and, possibly
(ii) such that the positive modalities are *linearly ordered* by the above relation $\rightarrowtail$ (and, dually, the negative modalities as well),[23]

so that, in other words, any two positive (negative) modalities of **S** would be comparable with respect to logical strength.[24]

> Before going into the actual meaning of the reduction problem of infinitely many nested modalities, which arise through the iteration and composition of the symbols "$\sim$" and "$-$", we present a purely formal investigation, by which Lewis's system becomes a closed system, thanks to the addition of a further axiom. This can be done in several ways. Here we will consider two of them.
>
> The assumptions introduced by Lewis are (apparently) not sufficient to obtain a closed system of irreducible modalities.
>
> Therefore we add to Lewis's axioms the new axiom 1.9:
>
> $$(1.9) \qquad -(\sim p) < \sim(\sim p)$$
>
> This represents, to some extent, an *ad hoc* choice of the additional axiom. Later on we will make different choices and draw the corresponding conclusions. ([Becker, 1930], 11)

The first extension of **S3** proposed by Becker is thus

$$\mathbf{S3'} \ := \ \mathbf{S3} + \text{ the schema: } \Box(\Diamond\alpha \to \Box\Diamond\alpha)$$

which he calls the "Six modalities system". Indeed, he gives a correct and detailed proof of the fact that $\mathbf{S3'}$ has exactly 6 irreducible modalities:

- positive modalities: $\Box$, $\Diamond$, _ ("factual" truth),
- negative modalities: $\neg\Box$, $\neg\Diamond$, $\neg$ ("factual" falsity),

and that they are linearly ordered, as to logical strength, as follows

$$\Box \rightarrowtail \_ \rightarrowtail \Diamond \text{ (positive)}, \quad \neg\Diamond \rightarrowtail \neg \rightarrowtail \neg\Box \text{ (negative)}$$

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[22][Parry, 1939].

[23]The 21 positive (resp. negative) irreducible modalities — after Parry's result — are indeed *not* linearly ordered w.r.t. $\rightarrowtail$.

[24]It looks like Becker was supposing that (i), possibly together with (ii), would also imply the existence of a decision procedure for the extended calculus.

It is not difficult to prove that **S3′** is equivalent to the normal modal system **S5**, officially introduced for the first time, and thus named, in the *Appendix II* of [Lewis and Langford, 1932][25].

While the characteristic axiom schema of **S3′**, alias **S5**, represented according to Becker more or less the fruit of a formally convenient choice, intuitionistic logic is behind the finding of one of the two characteristic schemas of the second "completion" proposed by him, the "Ten modalities system"(here **S3″**).

As we said, the exploration of the connection between intuitionistic and modal logic is one of Becker's aims. In this context he is thus naturally led, in particular, to interpret the intuitionistic negation ("$\neg$") — which is *stronger* than classical negation — in modal terms, as *impossibility* ("$\sim$") or, as he uses to say, *absurdity* (*Absurdität*).[26] By replacing, in the intuitionistic law

$$\text{(WDN)} \quad \alpha \to \neg\neg\alpha$$

"$\neg$" with "$\sim$" and "$\to$" with "$<$" (strict implication), one gets

$$\alpha < \sim\sim \alpha$$

"Truth — as he puts it[27] — implies the absurdity of the absurdity (but not conversely!)". The schema, in our $\Box$-notation, reads

$$\Box(\alpha \to \Box\Diamond\alpha)$$

which is the "boxed-version" of the modal schema B ("Brouwer's *Axiom*", as Becker called it), a name still current in the literature.

> One can now add (this is the weakest additional postulation we propose) "Brouwer's *Axiom*" to this setting
>
> $$(1.91) \qquad p = --p < \sim\sim p$$
>
> [...] As an [additional] axiom we choose [...]:
>
> $$(1.92) \qquad \sim -p < \sim - \sim -p$$
>
> [...] If one postulates (1.91) $\times$ (1.92) one can thus set up a ten modalities calculus. ([Becker, 1930], 17–18)

Becker rightly realized that the addition of the schema $B^{\Box}$ alone was not sufficient to reduce to the finite the number of irreducible modalities, hence the addition of a second axiom schema, (1.92), corresponding in our notation

[25]Here Becker is acknowledged for having introduced the characteristic schema of the system. In fn. 1, p. 492, the Authors also mention a letter sent by M. Wajsberg to Lewis in 1927, containing "the outline of a system of Strict Implication with the addition of the postulate later suggested in Becker's paper".

[26]Becker refers explicitly to [Heyting, 1930], which contains the first (complete) presentation of intuitionistic logic as a formalized calculus. The paper was published in the same year of *On the Logic of Modalities*, but was circulating since 1928.

[27][Becker, 1930], 17.

to $\Box(\Box\alpha \to \Box\Box\alpha)$, that is the "boxed-version" of the modal schema 4, as it is currently named.[28] Thus

$$\mathbf{S3}'' := \mathbf{S3} + (B^{\Box}): \Box(\alpha \to \Box\Diamond\alpha) + (4^{\Box}): \Box(\Box\alpha \to \Box\Box\alpha)$$

Becker's claim, supported by an elaborate and detailed (putative) proof, is that this system has 10 irreducible modalities:

– positive modalities: $\Box$, $\Diamond\Box$, $\Diamond$, $\Box\Diamond$, _ ("factual" truth),
– negative modalities: $\neg\Box$, $\neg\Diamond\Box$, $\neg\Diamond$, $\neg\Box\Diamond$, $\neg$ ("factual" falsity),

and that they are linearly ordered, as to logical strength, as follows

$$\Box \rightarrowtail \Diamond\Box \rightarrowtail \_ \rightarrowtail \Box\Diamond \rightarrowtail \Diamond \ (+) \quad \neg\Diamond \rightarrowtail \neg\Box\Diamond \rightarrowtail \neg \rightarrowtail \neg\Diamond\Box \rightarrowtail \neg\Box \ (-)$$

While Becker was right in the claim that in $\mathbf{S3}''$ any modality is equivalent to one of the above ten modalities, he unfortunately missed to realize that, by the combined effect of the two schemas, the two modalities $\Diamond\Box$ and $\Box\Diamond$ turn out to be equivalent, respectively, with $\Box$ and $\Diamond$. In other words, the "Ten modalities system" $\mathbf{S3}''$ is equivalent to the "Six modalities system" $\mathbf{S3}'$, thus boiling down to an alternative axiomatization of $\mathbf{S5}$.[29]

In conclusion, we would like just to touch upon Becker's very interesting and elaborate attempt[30] to develop a more abstract approach to the problem of "completing" Lewis's calculus in such a way that in the resulting system, independently from the number (finite or infinite) of irreducible modalities, any two positive (or negative) modalities be comparable with respect to logical strength.[31]

Becker conveniently uses here as basic, elementary positive modality the operator "$\Box$" ("N" in his symbolism) and fixes a finite set $\mathcal{R}$ of "rules" which are intended to impose conditions concerning the *preservation* of relations of logical strength between $\{\Box, \neg\}$-modalities under juxtaposition.[32] Now, the point is that Lewis's calculus **S3**, while being closed under these rules, contains however *incomparable* positive modalities. So, this is Becker's very

[28] This is "Becker's additional axiom" mentioned in [Gödel, 1933].

[29] In his *Review*, Gödel indeed remarked that "it is nowhere shown that the [...] systems set up really differ from one another and from Lewis's system (in other words, that the additional axioms are not in fact equivalent and do not follow from Lewis's); nor, furthermore, that the six, or ten, basic modalities obtained cannot be still further reduced."([Gödel, 1931], 201).

[30] In §5 of Part I, [Becker, 1930], 25–30, entitled "On the Calculus of Modalities with least Requirements, which still yields a Linear Rank Order".

[31] See [Centrone and Minari, 2020] for a detailed analysis and discussion.

[32] These rules have not been correctly interpreted and formalized in [Churchman, 1938], the first (and unique, as far as we know) paper where this experiment by Becker is detailedly analyzed. Incidentally, notice that the inference rules

$$\frac{\Box(\alpha \to \beta)}{\Box(\Box\alpha \to \Box\beta)} \quad \text{and} \quad \frac{\Box(\alpha \to \beta)}{\Box(\Diamond\alpha \to \Diamond\beta)}$$

known also in the current literature as *Becker's rules*, were given this name in [Churchman, 1938] because (uncorrectly) regarded as specific instances of one of the rules in $\mathcal{R}$.

interesting idea, one should try to devise the "weakest possible axiomatic conditions" to be added to the $\mathcal{R}$-rules, in order to obtain a calculus in which all the positive modalities are pairwise comparable w.r. to logical strength. At the end of a rather complex argument, he arrives at the claim that a stepwise generalization of Brouwer's schema (in the form $B^{\Box}$, see above) provides an infinite number of axiomatic conditions which, added to **S3** together with the $\mathcal{R}$-rules, produce a calculus — let us call it **SM** — with the requested properties although with a possibly infinite number of irreducible modalities.

Unfortunately again, **SM** turns out to be equivalent to the system **S3′**, that is **S5**, as proved by Churchman in 1938.[33] Notwithstanding, whether **SM** collapses to **S5** also when based on a system *weaker* than **S3** is an interesting question, open as far as we know, and worth to be investigated.

4. Becker's idea of a modal interpretation of intuitionistic logic

The idea is exposed and elaborated in the mentioned *Appendix to Part I*[34], after an introductory short presentation of Heyting's intuitionistic calculus **IPC** and its relationship with the classical ("Russellian") calculus[35]:

> How is now Heyting's calculus related to the uncompleted and the completed Lewis's calculus?
>
> Firstly, the question of an appropriate "translation" of the symbols emerges. ([Becker, 1930], 31)

More precisely, Becker's idea is to define a vocabulary translation associating to each intuitionistic logical operator ($\to$, $\vee$, $\wedge$, $\neg$) a corresponding logical operator of Lewis's calculus **S3**, in such a way that every theorem of the intuitionistic calculus **IPC** becomes, once transformed according to the translation, a theorem of **S3**. He tentatively considers three candidate translations[36]:

(T1) H: $\to, \vee, \wedge, \neg \;\Rightarrow\;$ L: $\to, \vee, \wedge, \neg$

(T2) H: $\to, \vee, \wedge, \neg \;\Rightarrow\;$ L: $<, \vee_s, \wedge, \sim$ (where $\alpha \vee_s \beta =_{df} \Box(\alpha \vee \beta)$)

(T3) H: $\to, \vee, \wedge, \neg \;\Rightarrow\;$ L: $\to, \vee, \wedge, \sim$

As to (T1), he observes that the T1-translation of every intuitionistic theorem is *obviously* a **S3**-theorem, because intuitionistic logic is included in classical non modal logic, and the latter in turn is included in the Lewis's system. On the other side, he stresses that

> [...] this is a worthless triviality. Indeed, the purpose of a comparison between intuitionistic and modal logic can only

[33][Churchman, 1938], 78 ff. The claim is indeed correct, although Churchman's proof thereof is not, because he did not formalize the system **SM** as Becker really intended it.

[34][Becker, 1930], 30-35.

[35]Becker is aware of Glivenko's *double negation translation* of **CPC** into **IPC**, [Glivenko, 1929], and mentions this paper.

[36]Here we use a convenient mix of Lewis's symbolism and the current symbolism.

> be to make the deficits of the former with respect to the latter comprehensible by interpreting the intuitionistic notions by the specific modal-logical notions, that is Lewis's "strict" notions (strict implication, strict logical sum, impossibility). (*loc. cit.*)

Concerning the second translation (T2), which implements exactly the above proposal, he correctly observes that the T2-translation $\Box(\Box(p \vee p) \leftrightarrow p)$ of $p \vee p \leftrightarrow p$, which is one of the axioms of Heyting's calculus, *is not a theorem* of **S3** — otherwise, as he cleverly shows, the latter would collapse.

Finally, as far as the third of the proposed translations is concerned, Becker claims, without giving a detailed proof, that the T3-translation of the **IPC** axiom $(\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha$, that is the modal formula

$$(*) \qquad (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\Diamond\beta) \rightarrow \neg\Diamond\alpha$$

is not a theorem of **S3**. Indeed, it is not difficult to prove that Becker was right in claiming that $(*)$ is underivable in **S3**. Actually, it is possible to prove even more: every normal modal system containing the schemas T and $(*)$ collapses. This fact therefore implies that also with respect to the extended system(s) **S3′**/**S3″** the translation (T3) would boil down to the *trivial* translation (T1).

Becker thereby concludes his "translation-experiments" as follows:

> At this point a further investigation must begin, with the aim to assess *whether and which additions must be made to the extended Lewis's System (Calculus of 10 Modalities, Calculus of 6 Modalities) so that Heyting's Axiom (11)* [i.e. $(*)$ above] *holds*. Further problems can nevertheless arise because of the difference of the undefined notions in the Heyting's and the Lewis's System. The solution of these tasks and the overcome of these difficulties shall be left to future work. (*ibid*, 33.)

5. Conclusions

As we know, only three years later someone else, namely Kurt Gödel, did the "future work" and "found the solution of these tasks". *Almost* found ..., to be precise: Becker indeed was asking for a translation of **IPC** into **S3**, and Gödel provided a translation into the stronger system **S4**.

So, the very end of the story comes with Ian Hacking's 1963 paper[37], containing (no mention of Becker ..., but) the proof that the McKinsey-Tarski translation provides a sound and faithful translation of **IPC** *already into Lewis's* **S3**.[38] By the way, the McKinsey-Tarski translation is very close

[37][Hacking, 1963]. Hacking's proof-theoretical demonstration, which makes use of a normalization theorem for a suitable natural deduction presentation of **S3**, is rather convoluted. A much simpler semantical proof, obtained by exploiting a conjecture in [Oakes, 1999], can be found in [Centrone and Minari, 2020].

[38]Gödel's translation $(\ldots)^G$ is instead not sound with respect to **S3**.

to Becker's (unsuccessful) translation (T2): apart from the essential fact that atomic formulas become boxed under the McKinsey-Tarski translation, the translation of the intuitionistic connectives is the same, except for the disjunction.

One might be tempted to underrate Becker's *formal* contributions in *On the Logic of Modalities*. On the contrary, and notwithstanding the many shortcomings, Becker's *pioneering* work, containing sophisticated insights and interesting technical solutions, has played an extremely important role in the *early* development of modal logic during the decade 1930–1940, as witnessed by the scientific contributions of other scholars who, at that time, referred to Becker's investigations and to the problems raised by him, and took them as a basis for further developments and researches.[39]

References

[Artemov and Beklemishev, 2004] Artemov, S. and Beklemishev, L. (2004). Provability logic. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic, 2nd Edition*, volume 13, pages 229–403. Kluwer.

[Artemov and Fitting, 2019] Artemov, S. and Fitting, M. (2019). *Justification Logic: Reasoning with Reasons*. Cambridge University Press, New York.

[Becker, 1914] Becker, O. (1914). *Über die Zerlegung eines Polygons in exclusive Dreiecke auf Grund der ebenen Axiome der Verknüpfung und Anordnung*. F. A. Brockhaus, Leipzig.

[Becker, 1923] Becker, O. (1923). *Beiträge zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendung*. Jahrbuch für Philosophie und phänomenologische Forschung VI. Max Niemeyer Verlag, Halle.

[Becker, 1927] Becker, O. (1927). *Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene*. Jahrbuch für Philosophie und phänomenologische Forschung VIII. Max Niemeyer Verlag, Halle.

[Becker, 1930] Becker, O. (1930). *Zur Logik der Modalitäten*. Jahrbuch für Philosophie und phänomenologische Forschung XI. Max Niemeyer Verlag, Halle.

[Becker, 1952] Becker, O. (1952). *Untersuchungen über den Modalkalkül*. Westkulturverlag Anton Hain, Meisenheim/Glan.

[Centrone and Minari, 2020] Centrone, S. and Minari, P. (2020). *Oskar Becker, On the Logic of Modalities (1930): Translation, Commentary and Analysis*. Springer, Berlin. (To appear).

[Churchman, 1938] Churchman, C. W. (1938). On finite and infinite modal systems. *The Journal of Symbolic Logic*, 3:77–82.

[Feferman et al., 1986] Feferman, S., Kleene, S., Moore, G., Solovay, R., and van Heijenoort, J., editors (1986). *Kurt Gödel Collected Works. I: Publications 1929–1936*. Oxford University Press, Oxford.

[Feys, 1937] Feys, R. (1937). Les logiques nouvelles des modalités. *Revue néo-scolastique de philosophie*, 56:517–553.

[Feys, 1938] Feys, R. (1938). Les logiques nouvelles des modalités (suite et fin). *Revue néo-scolastique de philosophie*, 58:217–252.

[Feys, 1950] Feys, R. (1950). Les systèmes formalisés des modalités aristotéliciennes. *Revue philosophique de Louvain*, 4:478–509.

[39]In this regard, [Feys, 1937] and [Feys, 1938] should at least be added to the already mentioned works by Gödel, Lewis and Langford, Churchman and Parry.

[Gentzen, 1935] Gentzen, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431.

[Glivenko, 1929] Glivenko, V. (1929). Sur quelques points de la logique de M. Brouwer. *Acad. Roy. Belg. Bull. Cl. Sci., Sér. 5*, 15:183–188.

[Gödel, 1931] Gödel, K. (1931). Besprechung von *Becker 1930*: Zur Logik der Modalitäten. *Monatshefte für Mathematik und Physik (Literaturberichte)*, 38:5–6. (Reprinted in Feferman et. al. 1986, 216–217).

[Gödel, 1933] Gödel, K. (1933). Eine Interpretation des intuitionistischen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, 4:39–40. (Reprinted in Feferman et. al. 1986, 300–301).

[Hacking, 1963] Hacking, I. (1963). What is strict implication? *The Journal of Symbolic Logic*, 28:51–71.

[Heyting, 1930] Heyting, A. (1930). Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse*, pages 42–56, 57–71, 158–169.

[Kripke, 1963] Kripke, S. (1963). Semantical analysis of modal logic, I. Normal modal propositional calculi. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 9:67–96.

[Kripke, 1965a] Kripke, S. (1965a). Semantical analysis of intuitionistic logic. In Crossley, J. and Dummett, M., editors, *Formal Systems and Recursive Functions*, pages 92–130. North Holland.

[Kripke, 1965b] Kripke, S. (1965b). Semantical analysis of modal logic, II. Non-normal modal propositional calculi. In Addison, J. W., Henkin, L., and Tarski, A., editors, *The Theory of Models. Proceedings of the 1963 international Symposium at Berkeley*, pages 206–220. North Holland.

[Lemmon, 1957] Lemmon, E. J. (1957). New foundations for Lewis modal systems. *The Journal of Symbolic Logic*, 22:176–186.

[Lewis, 1918] Lewis, C. I. (1918). *A Survey of Symbolic Logic*. University of California Press, Berkeley.

[Lewis, 1920] Lewis, C. I. (1920). Strict implication – an emendation. *Journal of Philosophy, Psychology and Scientific Methods*, 17:300–302.

[Lewis and Langford, 1932] Lewis, C. I. and Langford, C. H. (1932). *Symbolic Logic*. The Century Co., New York. (Dover Publications, New York ^{2}1959).

[Martin, 1969] Martin, G. (1969). Oskar Beckers Untersuchungen über den Modalkalkül. *Kant-Studien*, 60:312–318.

[McKinsey and Tarski, 1948] McKinsey, J. C. C. and Tarski, A. (1948). Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic*, 13:1–15.

[Oakes, 1999] Oakes, C. (1999). Interpretations of intuitionistic logic in non-normal modal logics. *Journal of Philosophical Logic*, 28:47–60.

[Parry, 1939] Parry, W. T. (1939). Modalities in the Survey System of strict implication. *The Journal of Symbolic Logic*, 4:137–154.

[Prior, 1955] Prior, A. (1955). *Formal Logic*. Oxford University Press, Oxford.

[Zimny, 1969] Zimny, L. (1969). Oskar Becker – Bibliographie. *Kant-Studien*, 60:319–330.

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