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# Phase-field Approximation for Cohesive Models: Variational Analysis and Applications to Fracture Mechanics

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# Chapter 1

# Introduction

**State of the art and motivation** Variational methods provide a unifying language to describe a wide range of problems in applied mathematics, physics, and engineering. Among these, a particularly rich class is formed by *free-discontinuity* problems, introduced by De Giorgi and Ambrosio in the seminal paper [DGA88], where both the state variable and its discontinuity set are unknown and must be determined as part of the minimisation process. Such problems provide a rigorous framework for the mathematical analysis of models featuring a competition between bulk and surface terms, for instance as in the formation of cracks in solids, the detection of edges in images, or the emergence of sharp phase boundaries. A key feature is the intrinsic coupling between the unknown field and its discontinuity set, which makes the analysis mathematically challenging and requires several tools from Geometric Measure Theory and Functional Analysis.

A cornerstone in the *approximation* of free-discontinuity energies, both from numerical and variational perspectives, is the *phase-field* model introduced by Ambrosio and Tortorelli in [AT90, AT92], originally designed to approximate the Mumford–Shah functional in the context of image segmentation. For $\varepsilon > 0$, the phase-field functionals read as

$$AT_\varepsilon(u, v) := \int_O \left(v^2|\nabla u|^2 + \frac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\right) \mathrm{d}x, \tag{1.0.1}$$

with $(u, v) \in H^1(O, \mathbb{R} \times [0, 1])$. This approximation replaces the sharp discontinuity set with a smooth transition layer governed by the auxiliary phase-field variable $v$, that distinguishes between intact regions ($v \approx 1$) and interfaces ($v \approx 0$). From a mathematical standpoint, its relevance is secured by the theory of $\Gamma$-*convergence*, which provides the appropriate notion of variational convergence to capture both the limiting energy and the convergence of minimisers. The theory of $\Gamma$-convergence, introduced by De Giorgi and Franzoni in [DGF75], ensures, under suitable assumptions, that the minimization of the regularized phase-field energies asymptotically yields the same solutions as the original free-discontinuity problem (see [DM93, Bra98]). In (1.0.1) the competition between the penalization term $\frac{(1-v)^2}{\varepsilon}$, which enforces $v \approx 1$ away from discontinuities, and the corresponding gradient term for $v$, which diffuses the transition layer, ensures that as $\varepsilon \to 0$ the phase-field $v$ localizes around sharp interfaces. Consequently, the following $\Gamma$-convergence result (see [AT90, AT92]) has been obtained

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} AT_\varepsilon(u,v) = \int_O |\nabla u|^2 \,\mathrm{d}x + 2\mathcal{H}^{n-1}(J_u) \tag{1.0.2}$$

when $u \in (G)SBV(O)$ and $v = 1$ $\mathcal{L}^n$-a.e., otherwise is $\infty$.

Beyond its initial application in image processing, the Ambrosio–Tortorelli scheme has become a prototype for the regularisation of free-discontinuity problems at large. This rigorous justification bridges abstract functional analysis with concrete approximation schemes, guaranteeing consistency between discrete computations and the underlying variational model. From the computational point of view, the smooth nature of phase-field functionals allows for efficient numerical implementations, often via alternating minimisation or gradient-flow methods (see e.g., [Bou07, Cha04, Wu17]), which would be intractable in the original sharp-interface setting. This dual perspective—rigorous mathematical convergence through Γ-convergence and practical feasibility through numerical realisation—explains the pivotal role of the Ambrosio–Tortorelli approach and its numerous extensions in fields ranging from material science to image processing and phase-transition modelling.

The adaptation of these ideas to the theory of fracture has been particularly influential. Over the last century, Griffith's theory of brittle fracture [Gri21] has established itself as one of the most fundamental and widely adopted frameworks in fracture mechanics. The variational formulation introduced in [FM98, BFM08] has had a profound impact on the field and has also inspired the development of more general damage models (cf. [PM10a, PM10b]).

Within this variational framework, crack initiation and evolution are governed by the competition between elastic and fracture energies, subject to irreversibility and stability conditions, so that fracture arises as the outcome of an energetic minimization process. In the case of a hyperelastic material subjected to antiplane shear, Griffith's energy can be expressed as

$$G(u) := \int_O \mathcal{W}(\nabla u)\,\mathrm{d}x + G_c\,\mathcal{H}^{n-1}(J_u), \tag{1.0.3}$$

where $u \in (G)SBV(O)$ denotes the displacement in the direction of deformation, $\mathcal{W}$ is the elastic energy density with quadratic growth, and $G_c$ represents the material toughness. In this formulation, the fracture energy is assumed to be entirely dissipated upon crack formation, with no residual interactions between the crack faces, irrespective of the displacement discontinuity across them (Figure 1.1).

The free-discontinuity energy minimization problem for brittle fracture admits a regularization, first investigated in [BFM00, Foc01]. In the framework of Γ-convergence, the crack trajectory arises from the localization of a smooth phase-field as the regularization length tends to zero [AT90, AT92]. The phase-field regularization of brittle fracture has then enabled the numerical simulation of complex fracture phenomena that were previously unattainable with classical methods [BMMS14, MBK15]. Today, the phase-field methodology stands as a leading paradigm in fracture mechanics, capable of capturing crack nucleation both in the presence and absence of pre-existing flaws, as well as reproducing highly intricate crack patterns. Its relatively straightforward numerical implementation, often relying on alternate minimization schemes, has further contributed to its widespread adoption. The contribution of [BFM00] represents a pivotal bridge between rigorous mathematical developments and engineering applications in the field of fracture mechanics.

Despite its elegance and historical relevance, Griffith's model exhibits two major shortcomings: first it

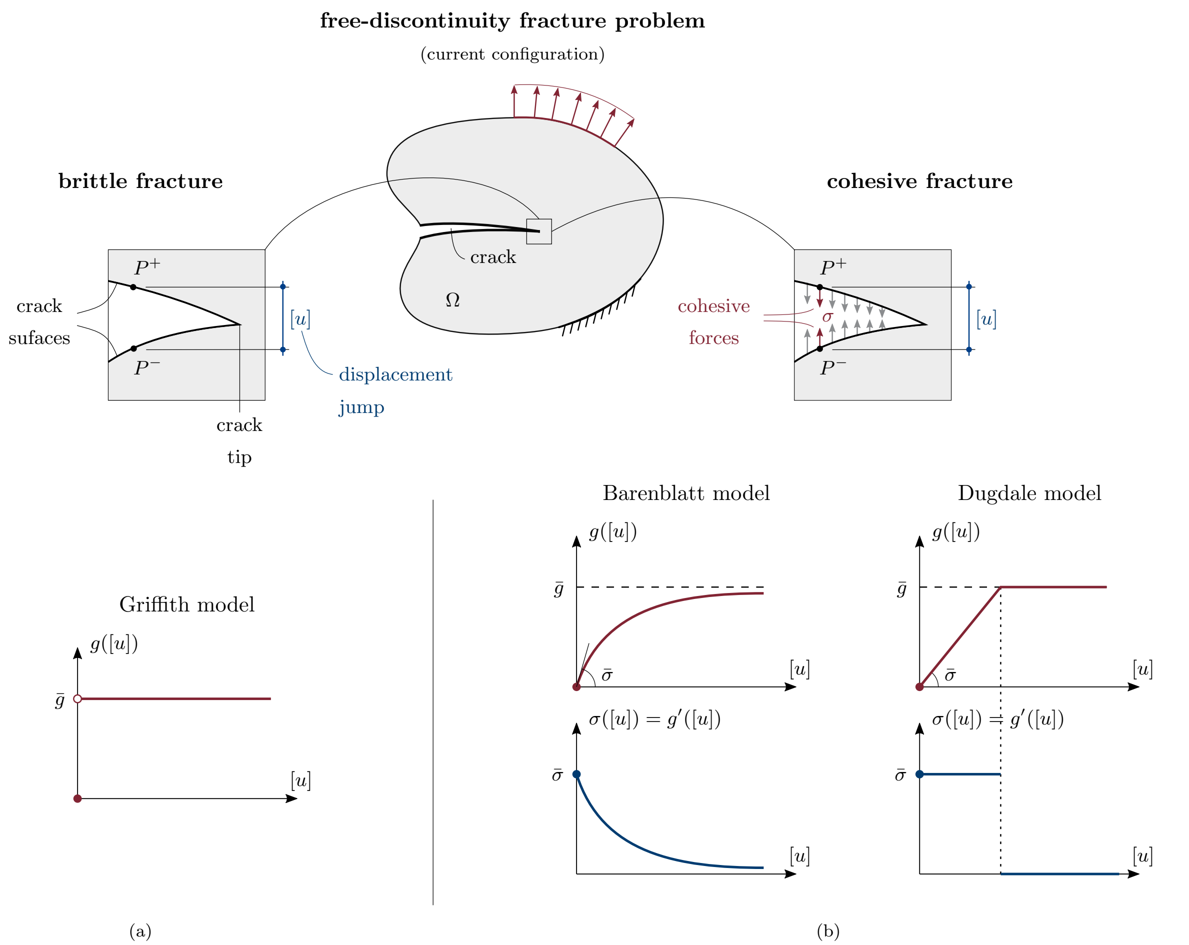



Figure 1.1: Qualitative trends of the surface energy density $g$ and associated cohesive stress $\bar{\sigma}$ with respect to the crack opening (displacement jump) $\delta$ for brittle and cohesive fracture models: (a) Griffith, (b) Dugdale and Barenblatt cohesive models. $\overline{g}$ and $\overline{\sigma}$ denote the fracture toughness and the initial critical stress, respectively.

is unable to predict crack initiation in an initially defect-free elastic body, as it requires the presence of a pre-existing flaw [Mar10, TLB+18, KBFLP20], and second, it produces unrealistic size effects in fracture predictions [Mar23]. Besides, phase-field models for brittle fracture inherit a fundamental limitation from Griffith's formulation: the inability to decouple the critical strength, the fracture toughness (or critical energy release rate), and the regularization length. Consequently, the nucleation stress threshold cannot be directly related to the sharp-interface description of Griffith's model. This drawback restricts the development of a versatile and general framework capable of accurately capturing crack nucleation in both smooth and notched domains [BFM08, TLB+18, LPDFL24]. *Cohesive* fracture models address these limitations by introducing nonzero tractions—referred to as cohesive forces—acting between crack faces. Originally formulated by Dugdale and Barenblatt [Dug60, Bar62], these models prescribe a surface energy density dependent on the displacement discontinuity, thereby offering a more faithful description of fracture processes (Fig. 1.1). Defined through a critical stress and a characteristic length, cohesive laws successfully overcome the deficiencies inherent in Griffith's formulation [Mar23]. Extending the variational reformulation of Griffith's fracture as a free-discontinuity energy minimization problem [FM98, AB95], cohesive fracture models have likewise been reinterpreted within an analogous variational framework [BFM08]. As emphasized in [Mar23], cohesive models provide a natural pathway to overcome the deficiencies of Griffith's brittle fracture theory. Building on the ideas of [PM10b, PM10a], [LCK11] proposed an autonomous gradient-damage (phase-field) formulation that is not derived from a free-discontinuity problem. Initially explored in one dimension—where closed-form solutions were obtained—this model was subsequently extended to higher-dimensional settings and applied to large-scale simulations of failure processes [LG11]. Despite its promise, this cohesive phase-field model initially received limited attention within the fracture mechanics community.

A rigorous attempt to establish a mathematically consistent variational phase-field framework regularizing the free-discontinuity cohesive fracture problem proposed in [BFM08] was undertaken by [CFI16] (see also [CFI24a, CFI25] for vector-valued extensions). For $\varepsilon > 0$, the folllowing phase-field functional has been introduced

$$CFI_\varepsilon(u,v) = \int_O \left(\left(1 \wedge \tfrac{\varepsilon l(v)}{(1-v)^2}\right)|\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon|\nabla v|^2\right) \mathrm{d}x \tag{1.0.4}$$

defined for $(u,v) \in H^1(O, \mathbb{R}\times[0,1])$, where $l \in C^0([0,1])$ is non-negative and vanishes only at the origin. As in the classical Ambrosio-Tortorelli approximation (cf. (1.0.1)) the last two terms act as penalization mechanisms, enforcing the concentration of the damage variable $v$ within narrow transition layers and accounting for the energetic cost of their formation. The distinctive feature of this model lies in the multiplicative weight in front of the elastic energy term, which introduces a damage-dependent degradation of the bulk response. In this framework, the following $\Gamma$-convergence result holds

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} CFI_\varepsilon(u,v) = \int_O h(\nabla u)\,\mathrm{d}x + l^{\frac{1}{2}}(1)|D^c u|(O) + \int_{J_u} g(|[u]|)\mathrm{d}\mathcal{H}^{n-1} \tag{1.0.5}$$

when $u \in (G)BV(O)$ and $v = 1$ $\mathcal{L}^n$-a.e., otherwise $\infty$. In (1.0.5) the limit functional combines the bulk contribution, the Cantor part of the derivative, and a cohesive surface energy depending on the jump amplitude. Here $h^{**}$ is the convex envelope of $h$, defined by $h(t) = t^2 \wedge l(1)t$, while $g$ is obtained by solving a one-dimensional coupled minimization problem given by

$$\begin{aligned} g(s) := \inf\Big\{ \int_0^1 \big(l(\beta)|\gamma'|^2 + (1-\beta)^2|\beta'|^2\big)^{\frac{1}{2}}\mathrm{d}t \ :\ &(\gamma,\beta) \in H^1((0,1),\mathbb{R}\times[0,1]),\\ &\gamma(0)=0,\ \gamma(1)=s,\ \beta(0)=\beta(1)=1\Big\}, \end{aligned} \tag{1.0.6}$$

that remark the key difference with the Ambrosio-Tortorelli approximation, indeed in this case the optimal profiles for the damage variable $v$ and the elastic displacement $u$ cannot be determined separately. They instead arise from a joint vectorial minimization problem which defines the surface energy $g$, specified in (1.0.6). This ensures that $g$ is linear for small crack openings and bounded for large ones, coherently with cohesive fracture laws.

Similar to [BFM00], the results in [CFI16] represent a valuable point of convergence between the mathematical and engineering communities. Its subsequent numerical implementation in [FI17] encountered several difficulties, including the design of a backtracking algorithm and the need to further regularize the degradation function with respect to the internal length. Moreover, the challenge of calibrating the elastic degradation function to reproduce a prescribed cohesive law—combined with the adoption of a fixed quadratic dissipation function—likely reduced the model's flexibility and limited its widespread adoption in fracture mechanics. From a numerical perspective, significant advancements were made by [Wu17, WN18], who, building upon the pioneering contribution of [LCK11], introduced a new generation of phase-field models for cohesive fracture. For $\varepsilon > 0$, the functional essentially takes the form

$$W_\varepsilon(u,v) := \int_O \frac{\varepsilon v^2}{\varepsilon v^2 + Q(1-v)}|\nabla u|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|\nabla v|^2 \,\mathrm{d}x, \tag{1.0.7}$$

defined for $(u,v) \in H^1(O, \mathbb{R}\times[0,1])$, where $Q$ and $\omega$ are polynomial potentials. In contrast to $CFI_\varepsilon$ (cf. (1.0.4)), the distinctive feature of this formulation is the use of a regularized degradation function, which significantly helps the numerical implementation.

Like their predecessors, these models are embedded in the well-established variational framework of gradient-damage formulations [PM10b, PM10a]. By introducing polynomial crack geometric functions together with rational energetic degradation functions, they permit the independent calibration of the critical strength, fracture toughness, and regularization length in accordance with prescribed softening laws. A central feature of these models—adopted by [CFI16, Wu17, WN18]—is the explicit inclusion of the regularization length within the elastic degradation function, which enhances the accuracy of cohesive fracture descriptions.

The primary objective of this thesis is to develop a general and unified framework for the phase-field approximation of free-discontinuity problems governed by cohesive models. The work pursues a twofold aim. The first one is to investigate Γ-convergence and homogenization results for families of phase-field functionals approximating free-discontinuity energies of cohesive type. Second, it is devoted to applying these analytical results to the theory of fractures through the reconstruction and characterization of cohesive laws, thereby establishing a consistent bridge between mathematical analysis and engineering applications, by demonstrating how the proposed models can reproduce physically relevant fracture behaviors. This dual perspective not only advances the theoretical understanding of variational approximations of cohesive models but also enhances their relevance for practical problems in fracture mechanics.

**Main novelties and methods.** In addition to the new statements proved throughout the thesis, it is useful to highlight the methodological content of the three main chapters, which varies in nature.

Chapter 3 provides a new unified Γ-convergence framework that extends and systematizes existing phase-field models (cf. [AT90, CFI16, Wu17, FFL21]) within a single variational formulation; the overall strategy follows a well-established line in the literature on phase-field approximations of free-discontinuity problems (cf. [CFI24b]), but the increased generality of the models requires several nontrivial technical refinements.

Chapter 4 addresses an inverse identification problem of direct interest in mechanics, namely the design of phase-field ingredients leading, in the sharp-interface limit, to a prescribed traction–separation law. At the analytical level, the core object is the vectorial minimum problem defining the cohesive density $g$: we develop a new and alternative analysis (w.r.t [BCI21]) which makes the internal structure of the problem transparent. In particular, this approach brings to light a remarkably rich structure of the reconstruction mechanism from the phase-field potentials to the limiting cohesive law and, among other consequences, clarifies when the effective response selects a Dugdale-type regime as opposed to a genuinely nonlinear Barenblatt-type regime. Furthermore in the latter case, the generality of our unified model allows for a finer characterisation of $g$, leading to an explicit analysis based on an Abel-type integral equation. This framework not only recovers several reconstruction paradigms already proposed in the literature (cf. [LCM25, FFL21]), but also yields new families of phase-field models producing same and new target cohesive laws.

Finally, Chapter 5 is conceived as a preliminary step towards the stochastic homogenisation of the cohesive phase-field models developed in this thesis. It establishes a stochastic homogenisation result for Alicandro–Braides–Shah type phase-field energies (cf. [ABS99, AF02]) in the *linear-growth* regime. While the proof relies on robust homogenisation tools, the main difficulty is precisely the linear growth: bulk and surface effects do not decouple, and the limiting cohesive surface density emerges from an interaction between the volume term and the phase-field localisation; this contrasts with the superlinear-growth setting (cf. [BMZ23]), where a decoupling phenomenon suppresses this interaction.

**Outline of the thesis** The present dissertation is essentially organized as a collection of five research articles. Each work addresses a distinct aspect of the overarching research question, while together they provide a coherent and original contribution to the field. The articles are listed below in the order in which they logically appear in the thesis:

(i) *Phase-field modelling of cohesive fracture. Part I:* $\Gamma$*-convergence results* [ACF25a] (accepted on SIAM Journal on Mathematical Analysis);

(ii) *Vectorial phase-field modelling of cohesive energies: the linear case* [Col25] (in preparation);

(iii) *Phase-field modelling of cohesive fracture. Part II: Reconstruction of the cohesive law* [ACF25b] (submitted for pubblication) ;

(iv) *Phase-field modelling of cohesive fracture. Part III: From mathematical results to engineering applications* [ACF25c] (submitted for pubblication);

(v) *Homogenisation of phase-field functionals with linear growth* [CFZ25] (submitted for pubblication).

In (i) we establish a general, unified and mathematically consistent framework for the phase-field modeling of cohesive fracture, in close analogy with the seminal contribution of [BFM08] in the setting of brittle fracture. The central aim is to bridge the gap highlighted above by extending the $\Gamma$-convergence results of [CFI16], thereby providing a rigorous mathematical foundation for cohesive phase-field models. More precisely, we introduce, in the 1-dimensional scalar case, a general class of phase-field functionals

$$\mathcal{F}_\varepsilon(u,v) := \int_O \left( \varphi\big(\varepsilon \tfrac{l(l)}{Q(1-v)}\big)|u'|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon |v'|^2 \right) \mathrm{d}x \tag{1.0.8}$$

defined for $(u,v) \in H^1(O, \mathbb{R}\times[0,1])$. The precise assumptions on the potentials $\varphi, l, Q$ and $\omega$ will be discussed in details in Section 3.1. It is important to highlight that the introduction of the potential $\varphi$ plays a crucial unifying role within the proposed framework. In the 1-dimensional scalar setting, appropriate choice of $\varphi$ allow us to recover as particular cases both the $CFI_\varepsilon$ and $W_\varepsilon$ models (cf. (1.0.4) and (1.0.7)). Moreover, by choosing a suitable scaling $\gamma_\varepsilon$ ($\gamma_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$) in the argument of $\varphi$, our model provides also a generalization of the classical Ambrosio-Tortorelli model $AT_\varepsilon$ (cf. (1.0.1)) for the approximation of brittle energies. This framework not only brings together, within a single formulation, the most representative brittle and cohesive phase-field models available in the literature, but also offers a versatile setting in which different modeling strategies can be rigorously compared and systematically analyzed. More precisely, we establish in one dimension a Γ-convergence result for a broad class of phase-field energies which, in a unified formulation, include the models proposed in [CFI16], [Wu17, WN18], [FFL21], and [LCM23, LCM25]. The main result is established in Theorem 3.1.1 where, under suitable assumptions on the potentials $\varphi, l, Q$ and $\omega$, we state the following Γ-convergence

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u,v)=\int_O h_{\overline{\varsigma}}^{**}(|u'|)\,\mathrm{d}x+\varphi'(0^+)\overline{\varsigma}^{\frac{1}{2}}|D^c u|(O)+\int_{J_u} g(|[u]|)\mathrm{d}\mathcal{H}^0$$

when $u\in(G)BV(O)$ and $v=1$ $\mathcal{L}^1$-a.e., otherwise $\infty$. Here $h_{\overline{\varsigma}}^{**}$ denotes the convex envelope of the function $h_{\overline{\varsigma}}$, that is defined by the following minimization problem

$$h_{\overline{\varsigma}}(t):=\inf_{\tau\in(0,\infty)}\left\{\varphi\left(\frac{1}{\tau}\right)t^2+\frac{\overline{\varsigma}^2}{4}\tau\right\} \tag{1.0.9}$$

where $\overline{\varsigma}^2:=\lim\limits_{t\to 0}\frac{\omega(t)}{Q(t)}$. Moreover, consistently with (1.0.6), the cohesive surface energy density $g$ is given by

$$\begin{aligned} g(s):=\inf\Big\{\int_0^1\left(\theta(\beta)|\alpha'|^2+\omega(1-\beta)|\beta'|^2\right)^{\frac{1}{2}}\mathrm{d}t\ :\ &(\alpha,\beta)\in H^1((0,1),\mathbb{R}\times[0,1]),\\ &\alpha(0)=0,\ \alpha(1)=s,\ \beta(0)=\beta(1)=1\Big\}, \end{aligned} \tag{1.0.10}$$

where $\theta(t):=\varphi'(0^+)l(t)\frac{\omega(1-t)}{Q(1-t)}$. Furthermore, by appropriately tuning the parameters $\varphi'(0^+),\varphi(\infty)$ and $\overline{\varsigma}$ one can identify distinct asymptotic regimes for $\Gamma\text{-}\lim_{\varepsilon\to 0}\mathcal{F}_\varepsilon$ (cf. Section 3.2.4). Each regime gives rise to a different class of free-discontinuity energies, thereby highlighting the versatility of the proposed formulation in capturing a broad spectrum of fracture behaviors. In particular, this parametric flexibility allows the model to interpolate between brittle and cohesive responses, thus providing a unified variational framework for the derivation and comparison of various fracture theories. Moreover, this first study and the generality of the model play a key role in the reconstruction of the cohesive law and in the subsequent extension to the multidimensional vectorial setting.

In (ii), we extend the Γ-convergence result of (i) to the multidimensional vectorial setting, within the framework of linear cohesive fracture. More precisely, following the approach of [CFI24a], we analyse the vectorial counterpart of the phase-field energy (1.0.8), given by

$$\int_O\left(\varphi\big(\varepsilon\tfrac{l(l)}{Q(1-v)}\big)\Phi(\nabla u)+\frac{\omega(1-v)}{4\varepsilon}+\varepsilon|\nabla v|^2\right)\mathrm{d}x \tag{1.0.11}$$

for $(u,v)\in H^1(O,\mathbb{R}^N\times[0,1])$, where $\Phi\in C^0(\mathbb{R}^{N\times n})$ satisfies quadratic growth conditions for at infinity (cf. (3.1.4)). Our analysis focuses on the *geometrically nonlinear* regime: although a rigorous treatment of the impenetrability and the blow-up of the elastic energy as $\det\nabla u\to 0$ is still missing, due to the absence of the necessary analytical tools, the Γ-convergence can nevertheless be established. In the *linearized* setting, however, even proving Γ-convergence is not possible with the same techniques employed in this thesis because of the lack of an analogue of truncation techniques in the $BD$ framework.

In the main result (cf. Theorem 3.1.1), we prove that the limit functional of the energies in (1.0.11), has the following form

$$\int_O h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u)\,\mathrm{d}x + \int_O h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\mathrm{d}D^c u) + \int_{J_u} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1},$$

where $h^{\mathrm{qc}}_{\overline{\varsigma}}$ denotes the quasi-convex envelope of $h_{\overline{\varsigma}}$ (defined analogously to (1.0.9)), and $h^{\mathrm{qc},\infty}_{\overline{\varsigma}}$ is its recession function. The characterization of the surface energy density $g$ is significantly more intricate than in the 1-dimensional scalar case (cf. (3.1.11)), due to the possible asymptotic anisotropies introduced by the integrand $\Phi$. It is worth emphasizing that, in the particular case $\Phi_\infty(\xi) = |\xi|^2$ (see (3.1.5)), the expression of the surface energy density $g$ simplifies substantially, reducing exactly to the formula in (1.0.10) (cf. Corollary 3.3.6). Moreover, with a suitable choice of the potential $\varphi$ the bulk function $h_{\overline{\varsigma}}$ turns out to be directly convex (cf. Proposition 3.1.5), thereby avoiding the need for quasiconvexification. This special setting not only highlights the consistency of our general framework with previously established scalar models, but also illustrates how appropriate structural assumptions on the potentials can lead to a significant simplification of the limiting functional.

One of the main objectives of (iii) is to provide an analytical validation and a generalization of the model-construction process aimed at reproducing prescribed cohesive fracture laws. Building on the Γ-convergence results established in (i) and (ii), in (iii) we develop a general and versatile methodology for the design of phase-field models capable of reproducing cohesive traction–separation laws $g$. The framework accommodates both linear and superlinear behaviors for small displacement jumps, both relevant for modeling cohesive responses (indeed superlinear laws have been highlighted as especially pertinent to ductile-fracture phenomena in strain-gradient plasticity models [FCO14]). This kind of analysis can be achieved by prescribing the degradation function and deriving the corresponding damage potential, or conversely by fixing the damage potential and identifying the appropriate degradation function (cf. Theorems 4.2.1, 4.2.5, 4.2.2 and 4.2.4). One of the pivotal steps of this work has been the analysis of the auxiliary integral function $\Lambda : (0,1) \to [0,\infty)$ (see Section 4.1.3), depending on the potentials of the phase-field functionals $\mathcal{F}_\varepsilon$, defined as

$$\Lambda(m) = \int_m^1 \sqrt{\frac{\theta(m)\omega(1-t)}{\theta(t)(\theta(t)-\theta(m))}}\mathrm{d}t,$$

whose monotonicity properties are shown to be strictly connected with the qualitative behavior of the surface energy density $g$. In the linear cohesive case, one of the main results of this thesis (cf. Theorems 4.1.16 and 4.1.13) establishes a precise dichotomy:

- if $\Lambda$ is non-decreasing, then the resulting cohesive law $g$ coincides with the Dugdale model, namely

$$g(s) = (\varphi'(0))^{\frac{1}{2}}\overline{\varsigma}s \wedge 2\Psi(1)$$

  where $\Psi(t) = \int_0^t \omega^{\frac{1}{2}}(1-\tau)\mathrm{d}\tau$;

- if $\Lambda$ is strictly decreasing then $g$ turns out to be of class $C^1([0,\infty))$, with the explicit representation

$$g(s) = 2\int_{\Lambda^{-1}(s)}^1 \sqrt{\frac{\theta(t)\omega(1-t)}{\theta(t)-\theta(\Lambda^{-1}(s))}}\mathrm{d}t, \qquad g'(s) = \theta^{\frac{1}{2}}(\Lambda^{-1}(s)).$$

This analysis shows that the monotonicity of $\Lambda$ dictates whether the limiting cohesive law reduces to the classical Dugdale model (cf. Section 4.3.9), or instead gives rise to a smooth nonlinear law, thereby enriching

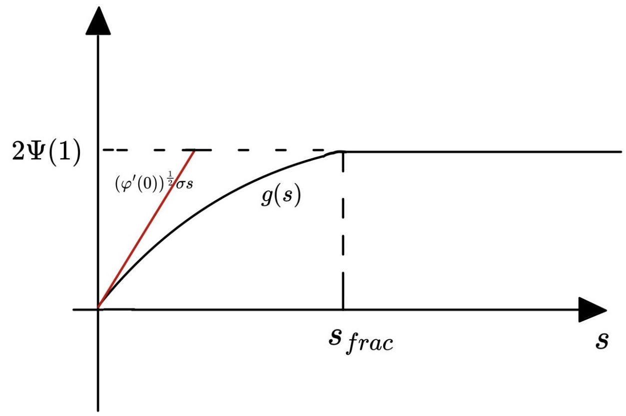



Figure 1.2: Cohesive (Barenblatt-type) surface energy $g(s)$ versus crack opening $s$: initially linear with slope $(\varphi'(0))^{1/2}\sigma$, then saturating to the toughness level $2\Psi(1)$ for $s \geq s_{\text{frac}}$.

the class of admissible traction–separation responses. Similar results are obtained also in the superlinear regime, thus confirming the central role of the auxiliary function $\Lambda$ across different cohesive settings. Our approach rigorously extends and justifies the linear case treated in [FFL21], allowing for the systematic derivation of multiple phase-field models that reproduce the same target cohesive law while exhibiting distinct localized phase-field evolutions, as illustrated through several worked examples (cf. Section 4.2.3).

The work (iv) represents the natural continuation of the theoretical developments of (i) and (iii), with the explicit aim of embedding their mathematical insights into an engineering-oriented framework. It focuses on the identification procedure that associates a phase-field model with a prescribed traction–separation cohesive law, and on the validation of the resulting models through their mechanical response. In Section 4.3.4, the analysis is enriched by a series of concrete examples based on canonical cohesive laws, which provide a systematic comparison of the predicted tensile behaviors and reveal distinctive features of the different models. This comprehensive study not only substantiates the theoretical findings of the first two parts, but also demonstrates their practical relevance, thereby consolidating the bridge between rigorous variational analysis and the applied perspective of fracture mechanics. A visual summary of the state of the art discussed above is provided in Figure 1.3, which highlights the logical connections among the works along a timeline.

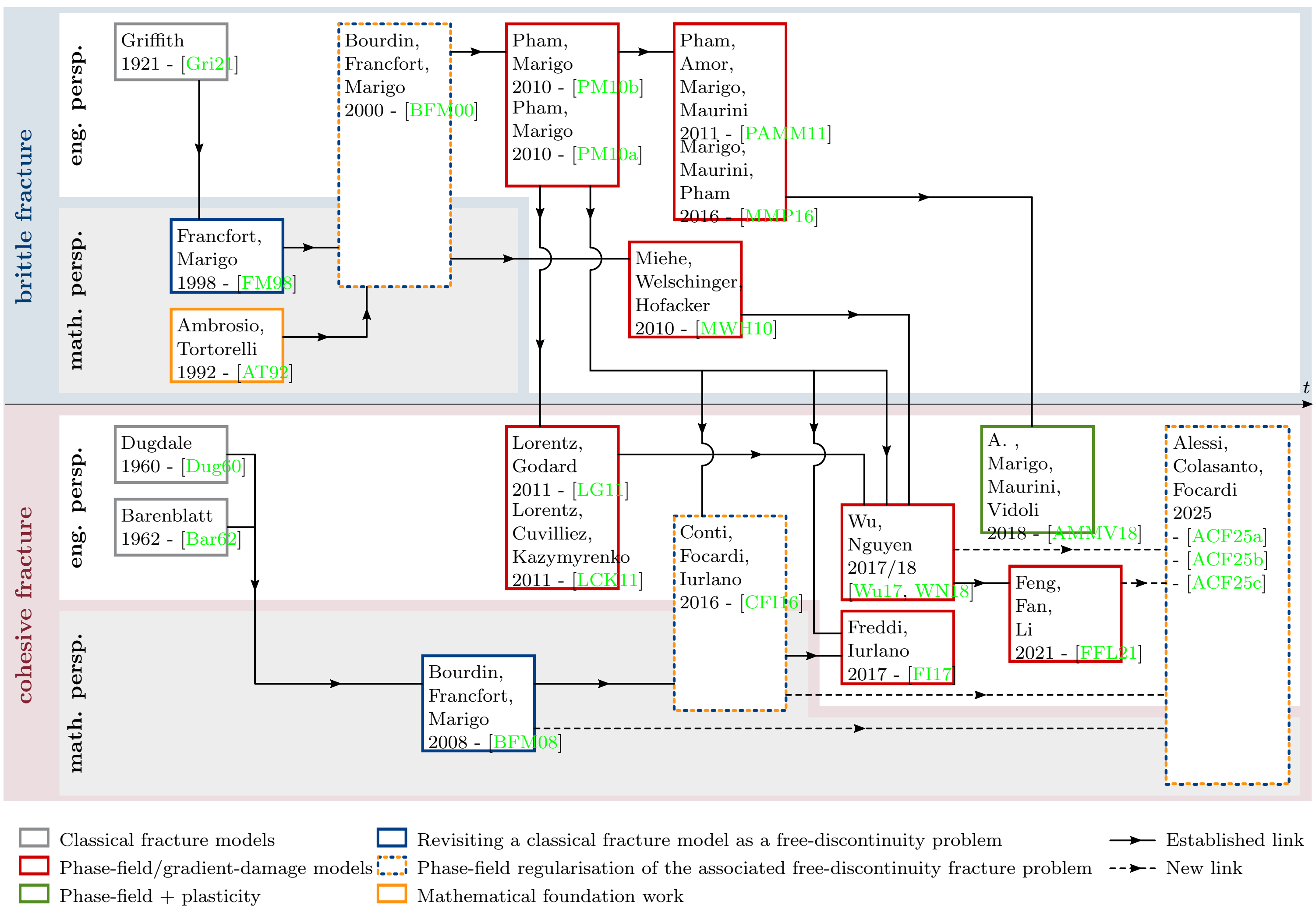



Figure 1.3: Overview of the state of the art with both a mathematical and engineering perspective of the phase-field models describing brittle and cohesive fractures. Specifically, milestones, their logic connection and the collocation of this three part work within the framework of the variational approach to fracture are highlighted.

The final contribution (v) is motivated by the overarching aim of analysing the stochastic homogenisation of phase–field functionals $\mathcal{F}_\varepsilon$ (cf. (1.0.8)), a problem of central importance for fracture mechanics in heterogeneous and anisotropic materials. As a preparatory step towards this challenging objective, we investigate a simpler yet nontrivial family of elliptic functionals in which the diffuse part exhibits only linear growth in the gradient variable. While the specific class of functionals considered here, characterised by linear growth of the bulk integrand, cannot be directly applied to fracture mechanics, it nonetheless fits naturally within the broader framework of phase–field approximations for cohesive models, research theme of the thesis, and finds a well–established application in image segmentation.

More precisely, the topic of the last chapter is devoted to studying the combined effects of homogenisation and elliptic regularisation for random phase–field functionals of the form

$$\mathcal{H}_\varepsilon(\omega)(u,v,A) = \int_A \left(v^2 f(\omega, \tfrac{x}{\varepsilon}, \nabla u) + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\right) \mathrm{d}x, \tag{1.0.12}$$

defined for $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1])$ and $\omega \in \Omega$ (sample space), where $\varepsilon > 0$ characterises both the oscillation and the regularisation scale. We assume that the integrand $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,+\infty)$ satisfies uniform linear growth and coercivity conditions in the gradient variable, namely

$$C^{-1}|\xi| \leq f(\omega,x,\xi) \leq C(|\xi|+1). \tag{1.0.13}$$

Beyond (1.0.13), only mild assumptions on $f$ are imposed, in particular no spatial periodicity is required (cf. Definition 5.1.1). This general setting allows us to establish homogenisation results that also encompass the case of random stationary integrands (cf. Theorem 5.1.7).

The elliptic functionals in (1.0.12) are reminiscent of the Ambrosio–Tortorelli approximation $AT_\varepsilon$ (cf. (1.0.1)). When $f(\omega,x,\xi) = |\xi|$, the functional in (1.0.12) reduces to the model proposed by Shah [Sha96a, Sha96b]

$$\mathcal{S}_\varepsilon(u,v,A) := \int_A \left(v^2|\nabla u| + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\right) \mathrm{d}x,$$

as a possible regularisations of an image-segmentation model, alternative to the Mumford-Shah functional. This formulation provides a unified framework for image segmentation and isotropic curve evolution in Computer Vision, overcoming several limitations of earlier models (cf. [Sha00]). The key distinction between the Ambrosio–Tortorelli functional and the class in (1.0.12) lies in the growth of $f$, superlinear in the former versus linear in the latter. This difference has deep consequences: linear growth admits interactions between the competing terms in (1.0.12), a feature typical of free-discontinuity problems in the linear-growth regime [BFM98, CDMSZ22]. As proven in [ABS99] (scalar isotropic case) and [AF02] (vector-valued anisotropic case), the resulting limit surface energy densities are of cohesive type. By contrast, when $f$ present a superlinear growth, the limit functional is supported on $u \in (G)SBV$, and only brittle surface energies independent of the jump amplitude of $u$ can be obtained [BEMZ22, BEMZ23, BMZ23].

Motivated by applications to anisotropic curve evolution [Sha96a, Sha96b, Sha00], we investigate the homogenisation of (1.0.12) in the presence of highly oscillatory, possibly random metrics. Since image-segmentation problems are sensitive to heterogeneities and noise, homogenisation plays a crucial role in mitigating spurious oscillations, enhancing boundary detection, and improving robustness with respect to illumination and texture variations.

Our main result rigorously characterises the limit of $\mathcal{H}_\varepsilon$ as $\varepsilon \to 0$ without assuming periodicity of $f$. Following the spirit of [CDMSZ19b, CDMSZ22], our assumptions are sufficiently general to include the case of random stationary integrands. Theorem 5.1.7 establishes the following homogenisation formula (almost surely)

$$\big(\Gamma\text{-}\lim_{\varepsilon} \mathcal{H}_\varepsilon(\omega)\big)(u, v; A) = \int_A f_{\mathrm{hom}}(\omega, \nabla u)\,\mathrm{d}x + \int_A f^\infty_{\mathrm{hom}}(\omega, \tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u| + \int_{J_u \cap A} g_{\mathrm{hom}}(\omega, [u], \nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{1.0.14}$$

if $u \in (GBV(A))^N$ and $v = 1$ $\mathcal{L}^n$-a.e., otherwise $\infty$. The precise definitions of the homogenized bulk $f_{\mathrm{hom}}$ and surface densities $g_{\mathrm{hom}}$ are given respectively in (5.3.2) and (5.3.3), and furthermore Proposition 5.2.7 shows that $g_{\mathrm{hom}}$ exhibits the qualitative features of a cohesive law, thereby confirming the cohesive nature of the limiting functional.

Finally, we recall that the homogenisation problem studied here can be actually regarded as a prototype for homogenisation in gradient-damage models of cohesive fracture [CFI16, Wu17, WN18, FFL21, CFI24a, CFI25, LCM25]. In this broader setting, the effective surface densities are also expected to emerge from competing asymptotic mechanisms, though the analysis is more demanding due to the superlinear growth of the bulk energy density and the parameter-dependent degeneracy of the phase-field degradation functions.

# Chapter 2

# Notations and Preliminaries

Throughout the thesis, $n$ and $N$ denote fixed positive integers, while $d$ is used to indicate a generic positive integer. The symbols $\cdot$ and $|\cdot|$ stand, respectively, for the Euclidean scalar product and the associated norm on $\mathbb{R}^d$, and $\{e_1, \dots, e_d\}$ denotes the standard basis of $\mathbb{R}^d$. For any real number $t$, the notation $\lfloor t \rfloor$ denotes its integer part.

We denote by

$$\mathbb{S}^{n-1} := \{y \in \mathbb{R}^n : |y| = 1\}$$

the $(n-1)$–dimensional unit sphere in $\mathbb{R}^n$. For convenience, we also introduce the subsets

$$\widehat{\mathbb{S}}^{n-1}_{\pm} := \{y \in \mathbb{S}^{n-1} : \pm y_{i(y)} > 0\}, \tag{2.0.1}$$

where $i(y)$ denotes the largest index $i \in \{1, \dots, n\}$ such that $y_i \neq 0$. By construction, the sphere decomposes as $\mathbb{S}^{n-1} = \widehat{\mathbb{S}}^{n-1}_{+} \cup \widehat{\mathbb{S}}^{n-1}_{-}$, and both $\widehat{\mathbb{S}}^{n-1}_{\pm}$ are Borel subsets of $\mathbb{S}^{n-1}$.

For every $\nu \in \mathbb{S}^{n-1}$, we denote by $\Pi_\nu \subset \mathbb{R}^n$ the hyperplane orthogonal to $\nu$, namely

$$\Pi_\nu := \{y \in \mathbb{R}^n : y \cdot \nu = 0\}, \tag{2.0.2}$$

and by $\Pi_\nu^x := x + \Pi_\nu$ its translation through the point $x \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^N$, we define the *jump function* $u_{x,\zeta,\nu} \colon \mathbb{R}^n \to \mathbb{R}^N$ by

$$u_{x,\zeta,\nu}(y) := \begin{cases} \zeta, & \text{if } (y-x)\cdot\nu \geq 0, \\ 0, & \text{if } (y-x)\cdot\nu < 0, \end{cases} \tag{2.0.3}$$

and we adopt the shorthand notation $u_{\zeta,\nu} := u_{0,\zeta,\nu}$. This function represents a simple discontinuity with jump $\zeta$ across the plane $\Pi_\nu$ and will be repeatedly used throughout the thesis in the cell formulas defining the surface energy densities of limit functionals.

Before proceeding, we recall a geometric construction, provided in [CDMSZ19a, Example A.1], that will play a crucial role in Chapter 5, for the stochastic homogenisation analysis. In particular, in the next remark we define a matrix-valued mapping $\nu \mapsto R_\nu$ which assigns to each direction $\nu \in \mathbb{S}^{n-1}$ an orthogonal matrix sending the canonical vector $e_n$ onto $\nu$. This mapping will be instrumental in performing rotations and coordinate changes adapted to $\nu$, a technique that will be repeatedly employed in Section 5.6 in combination with Lemma 5.2.6.

**Remark 2.0.1.** *Assume $n > 1$ and let $\phi_\pm\colon \mathbb{S}^{n-1} \setminus \{\pm e_n\} \to \mathbb{R}^{n-1}$ denote the stereographic projections from the poles $\pm e_n$ onto the hyperplane $\Pi_{e_n}$, and let $\psi_\pm\colon \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \setminus \{\pm e_n\}$ be their inverse mappings. For every $\nu \in \widehat{\mathbb{S}}^{n-1}_\pm$, we define the tangent vectors*

$$\xi_i(\nu) := \partial_i \psi_\mp(\phi_\mp(\nu)), \qquad i = 1, \dots, n-1.$$

*Each vector $\xi_i(\nu)$ lies in the tangent space to $\mathbb{S}^{n-1}$ at the point $\nu$, and hence satisfies $\xi_i(\nu) \cdot \nu = 0$. Since the maps $\psi_\mp$ are conformal, the vectors $\xi_i(\nu)$ are mutually orthogonal, that is, $\xi_i(\nu) \cdot \xi_j(\nu) = 0$ whenever $i \neq j$. We normalize them by setting $\nu_i(\nu) := \xi_i(\nu)/|\xi_i(\nu)|$. The family $\{\nu_1(\nu), \dots, \nu_{n-1}(\nu), \nu\}$ thus forms an orthonormal basis of $\mathbb{R}^n$, whose columns define an orthogonal matrix, denoted by $R_\nu$. By construction, $R_\nu e_n = \nu$, and the mapping $\nu \mapsto R_\nu$ is continuous when restricted to either $\widehat{\mathbb{S}}^{n-1}_+$ or $\widehat{\mathbb{S}}^{n-1}_-$. Moreover, since $\phi_+(-\nu) = -\phi_-(\nu)$ for every $\nu \in \mathbb{S}^{n-1} \setminus \{e_n, -e_n\}$, it follows that $\psi_+(-y) = -\psi_-(y)$ for all $y \in \mathbb{R}^{n-1} \setminus \{0\}$. Consequently, $\xi_i(-\nu) = \xi_i(\nu)$ and hence $\nu_i(-\nu) = \nu_i(\nu)$ for every $\nu \in \mathbb{S}^{n-1} \setminus \{e_n, -e_n\}$. This symmetry also holds at the poles, since $\nu_i(\pm e_n) = e_i$.*

*In addition, one readily verifies that $\nu \in (\mathbb{S}^{n-1} \cap \mathbb{Q}^n) \setminus \{e_n, -e_n\}$ if and only if $\phi_\pm(\nu) \in \mathbb{Q}^{n-1} \setminus \{0\}$. Therefore, $\mathbb{S}^{n-1} \cap \mathbb{Q}^n$ is dense in $\mathbb{S}^{n-1}$. Furthermore, the explicit formulas for $\partial_i \psi_\pm$ show that $\nu_i(\nu) \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, and consequently $R_\nu \in \mathbb{Q}^{n \times n}$ for all such $\nu$.*

For $x \in \mathbb{R}^n$ and $\rho > 0$, we denote by

$$B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\} \quad \text{and} \quad Q_\rho(x) := \{y \in \mathbb{R}^n : |(y - x) \cdot e_i| < \tfrac{\rho}{2} \text{ for } i = 1, \dots, n\}.$$

We write $B_\rho := B_\rho(0)$ and $Q_\rho := Q_\rho(0)$, and we set $B := B_1$ and $Q := Q_1$. Moreover, for every $\nu \in \mathbb{S}^{n-1}$ we define

$$Q^\nu_\rho(x) := x + R_\nu(Q_\rho), \tag{2.0.4}$$

where $R_\nu$ is the rotation introduced in Remark 2.0.1, and we adopt the shorthand notations $Q^\nu_\rho := Q^\nu_\rho(0)$ and $Q^\nu := Q^\nu_1$.

For $k \in \mathbb{N}$ we define the elongated rectangle

$$Q^{\nu,k}_\rho(x) := x + Q^{\nu,k}_\rho, \tag{2.0.5}$$

where

$$Q^{\nu,k}_\rho := R_\nu\Big((-\tfrac{k\rho}{2}, \tfrac{k\rho}{2})^{n-1} \times (-\tfrac{\rho}{2}, \tfrac{\rho}{2})\Big),$$

and we distinguish the parts of its boundary according to their orientation:

$$\partial^\perp Q^{\nu,k}_\rho(x) := \partial Q^{\nu,k}_\rho(x) \cap R_\nu\Big((-\tfrac{k\rho}{2}, \tfrac{k\rho}{2})^{n-1} \times \mathbb{R}\Big), \tag{2.0.6}$$

$$\partial^\parallel Q^{\nu,k}_\rho(x) := \partial Q^{\nu,k}_\rho(x) \cap R_\nu\Big(\mathbb{R}^{n-1} \times (-\tfrac{\rho}{2}, \tfrac{\rho}{2})\Big). \tag{2.0.7}$$

If $O$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{A}(O)$, $\mathcal{A}_\infty(O)$, and $\mathcal{B}(O)$ the families of bounded open, Lipschitz bounded open, and Borel subsets of $O$, respectively. For every $B \subseteq \mathbb{R}^d$, we write $\chi_B\colon \mathbb{R}^d \to \{0,1\}$ for the characteristic function of $B$.

If $\mu$ is a Borel measure and $B$ a Borel set, the restriction of $\mu$ to $B$ is denoted by $\mu \llcorner B$ and is defined by $\mu \llcorner B(A) := \mu(A \cap B)$. We denote by $\mathcal{L}^d$ the $d$–dimensional Lebesgue measure in $\mathbb{R}^d$ and by $\mathcal{H}^k$ the $k$–dimensional Hausdorff measure for $k \geq 0$.

Finally, we employ the standard notation for Lebesgue spaces $L^p(O;\mathbb{R}^k)$ ($L^p_{\mathrm{loc}}(O;\mathbb{R}^k)$) and Sobolev spaces $W^{1,p}(O;\mathbb{R}^k)$ ($W^{1,p}_{\mathrm{loc}}(O;\mathbb{R}^k)$) for $p \in [1,\infty]$, with the usual convention $H^1(O;\mathbb{R}^k) := W^{1,2}(O;\mathbb{R}^k)$ for $k \in \mathbb{N}$.

### 2.0.1 $BV$ and $GBV$ functions

Let $u : O \subseteq \mathbb{R}^n \to \mathbb{R}^N$ be a measurable function, let $T := \mathbb{R}^N \cup \{\infty\}$ be the one–point compactification of $\mathbb{R}^N$, and fix $x \in O$. We say that $\zeta \in S$ is the *approximate limit* of $u$ at $x$ with respect to $O$, and we write

$$\zeta = \operatorname*{ap-lim}_{y \to x} u(y),$$

if for every neighborhood $U$ of $\zeta$ in $T$ there holds

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \mathcal{L}^n\big(\{y \in O : |y - x| < \varepsilon,\ u(y) \notin U\}\big) = 0.$$

Denote by $S_u$ the complement in $O$ of the set of points where the approximate limit of $u$ exists; it is well known that $\mathcal{L}^n(S_u) = 0$. Define the function $\tilde{u} : O \setminus S_u \to T$ by

$$\tilde{u}(x) = \operatorname*{ap-lim}_{y \to x} u(y),$$

thus $u$ is equal a.e. to $\tilde{u}$. Notice that $\tilde{u}$ is allowed to take the value $\infty$, but $\mathcal{L}^n(\{\tilde{u} = \infty\}) = 0$. Moreover, we say that $u$ is *approximately differentiable* at a point $x \in O \setminus S_u$ such that $\tilde{u}(x) \neq \infty$, if there exists a matrix $L \in \mathbb{R}^{N \times n}$ such that

$$\operatorname*{ap-lim}_{y \to x} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} = 0. \tag{2.0.8}$$

If $u$ is approximately differentiable at $x$, the matrix $L$ uniquely determined by (2.0.8) is denoted by $\nabla u(x)$ and is called the *approximate gradient* of $u$ at $x$.

#### Functions of bounded variation

We recall some definitions and basic results on functions with bounded variation. Our main reference is the book [AFP00].

**Definition 2.0.2.** *Let $u \in L^1(O;\mathbb{R}^N)$. We say that $u$ is a* function of bounded variation *in $O$, and we write $u \in BV(O,\mathbb{R}^N)$, if the distributional derivative $Du$ of $u$ is representable by a $\mathbb{R}^{N\times n}$–valued Radon measure on $O$ with finite total variation $|Du|(O)$, whose entries are denoted by $D_j u_i$. That is, if $\phi \in C^1_c(O)$, then*

$$\int_O u_i \frac{\partial \phi}{\partial x_j}\, \mathrm{d}x = -\int_O \phi \, \mathrm{d}D_j u_i.$$

If $u \in BV(O;\mathbb{R}^N)$, then $u$ is approximately differentiable a.e. (cf. [AFP00, Theorem 3.83]) and $S_u$ turns out to be countably $(n-1)$–rectifiable (cf. [AFP00, Theorem 3.78]), i.e.

$$S_u = N \cup \bigcup_{i=1}^{\infty} K_i,$$

where $\mathcal{H}^{n-1}(N) = 0$ and each $K_i$ is a compact subset of a $C^1$ manifold. Hence, for $\mathcal{H}^{n-1}$–a.e. $x \in S_u$, one can define an exterior unit normal $\nu_u(x)$ to $S_u$, as well as inner and outer traces of $u$ on $S_u$ by

$$u^{\pm}(x) = \operatorname*{ap-lim}_{y\to x,\, y\in H^{\pm}(x,\nu_u(x))} u(y), \tag{2.0.9}$$

where $H^{\pm}(x,\nu_u(x)) = \{y \in \mathbb{R}^n : \pm\langle y-x, \nu_u(x)\rangle > 0\}$. In such a case we write $x \in J_u$.

Let us consider the Lebesgue decomposition of $Du$ with respect to $\mathcal{L}^n$:

$$Du = D^a u + D^s u,$$

where $D^a u$ is the absolutely continuous part and $D^s u$ the singular one. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides a.e. with the approximate gradient $\nabla u$ (cf. [AFP00, Theorem 3.83]). Define the *jump part* $D^j u$ of $Du$ as the restriction of $D^s u$ to $S_u$, and the *Cantor part* $D^c u$ as the restriction of $D^s u$ to $O \setminus S_u$. Thus,

$$Du = D^a u + D^j u + D^c u.$$

We denote by $C_u$ the support of the measure $D^c u$. The representation

$$D^j u = [u] \otimes \nu_u \, \mathcal{H}^{n-1} \llcorner J_u$$

holds true (cf. [AFP00, Theorem 3.77]), where $[u] := u^+ - u^-$ (with the convention $\infty - \infty = \infty$ in $T$) and, given $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, $a \otimes b$ is the matrix with entries $(a_i b_j)$ for $1 \leq i \leq N$, $1 \leq j \leq n$.

### Generalized functions of bounded variation

**Definition 2.0.3.** *Let $u : O \subseteq \mathbb{R}^n \to \mathbb{R}^N$ be a Borel function. We say that $u$ is a* ***generalized function of bounded variation*** *in $O$, and we write $u \in GBV(O;\mathbb{R}^N)$, if $g(u) \in BV_{\mathrm{loc}}(O)$ for every $g \in C^1(\mathbb{R}^N)$ whose gradient $\nabla g$ has compact support.*

For every $u \in GBV(O;\mathbb{R}^N)$, the function is approximately differentiable a.e. in $O$, and the singular set $S_u$ is $(n-1)$–rectifiable. Furthermore, there exist a subset $J_u \subset S_u$, with $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, and a Borel normal $\nu_u : J_u \to \mathbb{S}^{n-1}$ such that the traces (2.0.9) are well defined on $J_u$ (see [Amb90, Proposition 1.3]). This allows the variation of the absolutely continuous part $|D^a u|$ and of the jump part $|D^j u|$ of the derivative to be naturally defined also in the $GBV$ framework.

For our purposes, it is more convenient to restrict to the space $(GBV(O))^N$. It is well known that a function $u$ belongs to $\in (GBV(O))^N$ if and only if, for every $i \in \{1,\dots,N\}$ and every $M > 0$, the truncation $\tau_M(u_i)$ belongs to $BV_{\mathrm{loc}}(O)$, where $u_i$ is the $i$-component of $u$ and $\tau_M$ denotes the standard $M$–truncation function defined by $\tau_M(s) = (s \wedge M) \vee (-M)$. Within this framework, the Cantor part of the derivative can be described in a straightforward way. Indeed, if $u \in (GBV(O))^N$, one can associate to $u$ a positive measure $|D^c u|$ defined by

$$|D^c u|(B) := \sup_{k\in\mathbb{N}} |D^c \mathcal{T}_k(u)|(B) \qquad \text{for every } B \in \mathcal{B}(O),$$

where the truncation maps $\mathcal{T}_k \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ are given by

$$\mathcal{T}_k(\zeta) := \begin{cases} \zeta, & \text{if } |\zeta| \leq a_k, \\ 0, & \text{if } |\zeta| \geq a_{k+1}, \end{cases} \tag{2.0.10}$$

with $\mathrm{Lip}(\mathcal{T}_k) \leq 1$, and where $(a_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive numbers satisfying $a_k \uparrow \infty$. Actually, the supremum above is independent of the truncation performed on $u$ and it is also the pointwise limit and the least upper bound measure of the family $\big(|D^c\mathcal{T}_k(u)|\big)_{k\in\mathbb{N}}$.

In order to give a rigorous meaning to the functionals considered, when extended to the space $(GBV(O))^N$, it is crucial to recall a structural property ensuring the regularity of the jump set and the consistency of the Cantor part of the derivative. Such a result provides the measure-theoretic foundation on which the definition of energies acting on $GBV$ functions is based. More precisely, if for every $i = 1, \dots, N$ one has

$$\int_O |\nabla u_i|\,\mathrm{d}x + |D^c u_i|(O) + \int_{J_u} (1 \wedge |[u_i]|)\mathrm{d}\mathcal{H}^{n-1} < \infty$$

for every $i \in \{1, \dots, N\}$, then (cf. [AFP00, Theorem 4.40])

$$\mathcal{H}^{n-1}(\{x \in J_u \ : \ u^+(x) = \infty \text{ or } u^-(x) = \infty\}) = 0 \tag{2.0.11}$$

and the sequence $(D^c\mathcal{T}_k(u))_k$ pointwise converges to a vector measure $\lambda$ (not depending on the sequence $(\mathcal{T}_k)_{k\in\mathbb{N}}$) such that

$$|\lambda|(B) = |D^c u|(B),$$

for every $B \in \mathcal{B}(O)$ (cf. [AF02, Lemma 2.10]).

### 2.0.2 Abel's Equation

In this section we collect and discuss some results concerning *Abel's integral equation*, a prototypical example of a weakly singular Volterra equation, which plays a central role in our framework for the reconstruction of cohesive laws (cf. Section 4.2). Our presentation follows the classical theory, and our main reference throughout this section is the monograph [GV91].

The classical form of Abel's equation is given by

$$r(x) = \int_0^x \frac{w(t)}{(x-t)^{1/2}}\,\mathrm{d}t =: (\mathcal{V}w)(x), \qquad x \in (0,1], \tag{2.0.12}$$

where $\mathcal{V}$ denotes the so-called *Abel operator*. In the expression above, the function $r$ is prescribed, while the aim is to determine the unknown function $w$ that satisfies the equation. The Abel operator $\mathcal{V}$ is a classical example of a fractional integral operator of order $\frac{1}{2}$, and it appears in a variety of contexts such as potential theory, inverse problems, and the analysis of fractional differential equations.

It is important to note that, in general, $\mathcal{V}$ is not well defined for an arbitrary function $w \in L^1((0,1))$, due to the weak singularity of the kernel $(x-t)^{-1/2}$. However, under suitable summability assumptions on $u$, the integral in (2.0.12) is well defined for every $x \in (0,1]$, and the operator $\mathcal{V}$ enjoys regularizing properties. The following result, which we recall from [GV91, Theorem 4.1.4], provides a precise formulation of this fact.

**Theorem 2.0.4.** *Let $p \in (2,\infty)$ and $w \in L^p((0,1))$. Then the map $t \mapsto w(t)(x-t)^{-1/2}$ belongs to $L^1((0,x))$ for every $x \in (0,1]$, and the Abel transform satisfies*

$$\mathcal{V}w \in C^{0,\alpha}([0,1]),$$

*where $\alpha := \frac{1}{2} - \frac{1}{p}$ and $(\mathcal{V}w)(0) := 0$.*

The previous theorem shows that the Abel operator maps $L^p$ functions, with $p > 2$, into Hölder continuous functions, with an explicit gain in regularity of order $\frac{1}{2} - \frac{1}{p}$. This smoothing property will be particularly relevant later, when studying the regularity of auxiliary quantities arising in the reconstruction procedure (see Section 4.2.1).

Concerning the solution of the integral equation (2.0.12), the main result is the *Abel inverse formula*. For completeness, and for convenience within our framework, we recall it here in a slightly simplified form that is sufficient for our purposes (see [GV91, Theorem 1.A.1]).

**Theorem 2.0.5.** *Assume $p \in (2,\infty)$ and $r \in W^{1,p}((0,1))$ with $r(0) = 0$. Define the function $w : [0,1] \to \mathbb{R}$ by*

$$w(x) := \frac{1}{\pi}(\mathcal{V}r')(x) = \frac{1}{\pi}\int_0^x \frac{r'(t)}{(x-t)^{1/2}}\,\mathrm{d}t. \tag{2.0.13}$$

*Then $w$ is the unique function in $L^1((0,1))$ satisfying equation (2.0.12). Moreover, if $r$ is convex and strictly increasing then $w$ is strictly increasing as well.*

Theorems 2.0.4 and 2.0.5 together provide a description of both the direct and inverse aspects of Abel's integral operator. They ensure that the operator $\mathcal{V}$ is well posed on appropriate Lebesgue spaces and that the inversion process is stable within the Sobolev setting $W^{1,p}$, for $p > 2$. These results will play a crucial role in Section 4.2.1, where we employ Abel-type integral representations to reconstruct cohesive laws in the *linear regime*. As will become clear in Section 4.2.2, when extending the analysis to the superlinear setting, a more refined version of these results is required. This is precisely the content of the following theorem, for which we include the proof since we could not find a precise reference in the literature.

**Theorem 2.0.6.** *Let $r \in C^1([0,\infty))$ be a convex, strictly decreasing function such that $r(t) \to 0$ as $t \to \infty$, and the map $t \mapsto r'(t)t^{-1/2}$ belongs to $L^1((0,\infty))$. Let $w : [0,\infty) \to [0,\infty]$ be the function given by*

$$w(x) := -\frac{1}{\pi}\int_x^\infty \frac{r'(t)}{(t-x)^{1/2}}\mathrm{d}t.$$

*Then $w(x) \in (0,\infty)$ for every $x \in [0,\infty)$, $w \in C^0([0,\infty))$ is strictly decreasing and the following formula holds*

$$r(t) = \int_x^\infty \frac{w(\tau)}{(\tau-t)^{1/2}}\mathrm{d}\tau < \infty. \tag{2.0.14}$$

*Proof.* First we observe that $w$ is well defined and strictly positive being $r'(t) < 0$ for every $t \in [0, \infty)$. Moreover, for every $x_1, x_2 \in [0, \infty)$ with $x_1 < x_2$ we have that

$$\begin{aligned} w(x_1) &= -\frac{1}{\pi} \int_{x_1}^{\infty} \frac{r'(t)}{(t - x_1)^{1/2}} \mathrm{d}t = -\frac{1}{\pi} \int_{x_2}^{\infty} \frac{r'(t + x_1 - x_2)}{(t - \tau_2)^{1/2}} \mathrm{d}t \\ &\geq -\frac{1}{\pi} \int_{x_2}^{\infty} \frac{r'(t)}{(t - x_2)^{1/2}} \mathrm{d}t = w(x_2), \end{aligned}$$

being $r'$ non-decreasing. Therefore, $w$ is non increasing and $w(x) \in (0, \infty)$ for every $x \in [0, \infty)$, because

$$w(0) = -\frac{1}{\pi} \int_0^{\infty} \frac{r'(t)}{t^{1/2}} \mathrm{d}t$$

by assumptions.

To prove the continuity of $w$ fix $x_0 \in [0, \infty)$ and $(x_j)_{j \in \mathbb{N}}$ such that $x_j \to x_0$ as $j \to \infty$. For every $\varepsilon \in (0, 1)$ we have that

$$w(x_j) = -\frac{1}{\pi} \int_{x_j}^{x_j + \varepsilon} \frac{r'(t)}{(t - x_j)^{1/2}} \mathrm{d}t - \frac{1}{\pi} \int_{x_j + \varepsilon}^{\infty} \frac{r'(t)}{(t - x_j)^{1/2}} \mathrm{d}t.$$

By Lebesgue Dominated Convergence Theorem we have that

$$\lim_{j \to \infty} \int_{x_j + \varepsilon}^{\infty} \frac{r'(t)}{(t - x_j)^{1/2}} \mathrm{d}t = \int_{x_0 + \varepsilon}^{\infty} \frac{r'(t)}{(t - x_0)^{1/2}} \mathrm{d}t,$$

while

$$\left| \int_{x_j}^{x_j + \varepsilon} \frac{r'(t)}{(t - x_j)^{1/2}} \mathrm{d}t \right| \leq 2 |r'(x_j)| \varepsilon^{1/2}.$$

Thus

$$\limsup_{j \to \infty} |w(x_j) - w(x_0)| \leq 4 |r'(x_0)| \varepsilon^{1/2}.$$

Taking $\varepsilon \to 0$ we infer the continuity of $w$.

To obtain (4.2.9), we use Fubini's Theorem and the elementary identity (cf. [GV91, last Remark of Section 1.1])

$$\int_{t_1}^{t_2} \frac{1}{(\tau - t_1)^{1/2} (t_2 - \tau)^{1/2}} \mathrm{d}\tau = \pi$$

for every $t_1$, $t_2 \in \mathbb{R}$ with $t_1 < t_2$. Indeed, for every $t \in [0, \infty)$ we obtain

$$\begin{aligned} \int_t^{\infty} \frac{w(\tau)}{(\tau - t)^{1/2}} \mathrm{d}\tau &= -\frac{1}{\pi} \int_t^{\infty} \frac{1}{(\tau - t)^{1/2}} \mathrm{d}\tau \int_{\tau}^{\infty} \frac{r'(x)}{(x - \tau)^{1/2}} \mathrm{d}x \\ &= -\frac{1}{\pi} \int_t^{\infty} r'(x) \int_t^x \frac{1}{(\tau - t)^{1/2} (x - \tau)^{1/2}} \mathrm{d}\tau \mathrm{d}x = r(t) \end{aligned}$$

because $R(\tau) \to 0$ as $\tau \to \infty$. $\square$

### 2.0.3 Subadditive Processes

In this section we fix the notation and recall some preliminary results that will be used throughout Chapter 5. In particular, we present a variant of the *Pointwise Subadditive Ergodic Theorem* of Akçoglu and Krengel [AK81, Theorem 2.7], which will play an important role in our analysis (see also [LM02, Theorem 4.1] for a related statement). We recall here the essential definitions and the version of the theorem that is suited to our framework.

Let $d \in \mathbb{N}$. We denote by $(\Omega, \mathcal{M}, P)$ a probability space, and by $\tau := (\tau_z)_{z\in\mathbb{Z}^d}$ a group of *P-preserving transformations* acting on $(\Omega, \mathcal{M}, P)$. That is, $\tau$ is a family of measurable maps $\tau_z : \Omega \to \Omega$ satisfying the following properties:

- $\tau_z\tau_{z'} = \tau_{z+z'}$ and $\tau_z^{-1} = \tau_{-z}$ for every $z, z' \in \mathbb{Z}^d$;
- $\tau$ preserves the probability measure $P$, that is, $P(\tau_z E) = P(E)$ for every $z \in \mathbb{Z}^d$ and every $E \in \mathcal{M}$.

If, in addition, every $\tau$-invariant set $E \in \mathcal{M}$ (that is, every measurable set satisfying $\tau_z E = E$ for all $z \in \mathbb{Z}^d$) has either probability 0 or 1, then $\tau$ is called *ergodic*.

For every $a, b \in \mathbb{R}^d$ with $a_i < b_i$ for $i = 1, \dots, d$, we define the half-open rectangular box

$$[a, b) := \{x \in \mathbb{R}^d : a_i \le x_i < b_i \text{ for } i = 1, \dots, d\},$$

and we denote by

$$\mathcal{I}_d := \{[a, b) : a, b \in \mathbb{R}^d,\ a_i < b_i \text{ for } i = 1, \dots, d\}$$

the collection of all such sets.

The following notion, originally introduced by Akçoglu and Krengel (see [AK81]), formalizes the concept of a stochastic process that is subadditive with respect to disjoint unions of boxes.

**Definition 2.0.7** (Subadditive process)**.** *Let $\tau := (\tau_z)_{z\in\mathbb{Z}^d}$ be a group of $P$-preserving transformations on $(\Omega, \mathcal{M}, P)$. A **d-dimensional subadditive process** is a function $\mu : \Omega \times \mathcal{I}_d \to \mathbb{R}$ satisfying the following properties:*

*(a) for every $A \in \mathcal{I}_d$, the map $\omega \mapsto \mu(\omega, A)$ is $\mathcal{M}$-measurable;*

*(b) for every $\omega \in \Omega$, $A \in \mathcal{I}_d$, and $z \in \mathbb{Z}^d$, one has the stationarity property*

$$\mu(\omega, A + z) = \mu(\tau_z\omega, A);$$

*(c) for every $A \in \mathcal{I}_d$ and every finite family $(A_i)_{i\in I} \subset \mathcal{I}_d$ of pairwise disjoint sets such that $\bigcup_{i\in I} A_i = A$, the subadditivity property holds:*

$$\mu(\omega, A) \le \sum_{i\in I} \mu(\omega, A_i) \quad \textit{for every } \omega \in \Omega;$$

*(d) there exists a constant $c > 0$ such that*

$$0 \le \mu(\omega, A) \le c\,\mathcal{L}^d(A)$$

*for all $\omega \in \Omega$ and all $A \in \mathcal{I}_d$.*

In order to state the ergodic theorem, we introduce the notion of a regular family of sets, which provides a controlled way of exhausting $\mathbb{R}^d$ through nested boxes with uniformly bounded growth.

**Definition 2.0.8** (Regular family of sets)**.** *A family of sets $(A_t)_{t>0} \subset \mathcal{I}_d$ is said to be **regular** with constant $M \in (0, +\infty)$ if there exists another family $(A'_t)_{t>0} \subset \mathcal{I}_d$ such that:*

- *$A_t \subset A'_t$ for every $t > 0$;*
- *$A'_s \subset A'_t$ whenever $0 < s < t$ (monotonicity);*
- *$0 < \mathcal{L}^d(A'_t) \leq M\,\mathcal{L}^d(A_t)$ for every $t > 0$ (uniform volume control);*
- *$\bigcup_{t>0} A'_t = \mathbb{R}^d$ (exhaustion property).*

We can now state the subadditive ergodic theorem in the form that will be used later on (cf. [AK81, LM02]). It ensures the almost sure convergence of normalized subadditive processes along arbitrary regular families of sets, and plays a fundamental role in stochastic homogenization and related variational problems.

**Theorem 2.0.9** (Subadditive Ergodic Theorem)**.** *Let $\tau = (\tau_z)_{z\in\mathbb{Z}^d}$ be a group of $P$-preserving transformations on $(\Omega, \mathcal{M}, P)$. Let $\mu : \Omega \times \mathcal{I}_d \to [0, +\infty)$ be a $d$-dimensional subadditive process. Then there exist a $\mathcal{M}$-measurable function $\vartheta : \Omega \to [0, +\infty)$ and a set $\Omega' \in \mathcal{M}$ with $P(\Omega') = 1$ such that*

$$\lim_{t\to+\infty} \frac{\mu(\omega, A_t)}{\mathcal{L}^d(A_t)} = \vartheta(\omega)$$

*for every regular family of sets $(A_t)_{t>0} \subset \mathcal{I}_d$ and for every $\omega \in \Omega'$. If, in addition, $\tau$ is ergodic, then $\vartheta$ is constant $P$-a.e.*

This result generalizes the classical pointwise ergodic theorem to the context of subadditive processes indexed by families of subsets of $\mathbb{R}^d$, and will later provide the probabilistic foundation for the stochastic homogenisation analysis carried out in Section 5.6.

# Chapter 3

# $\Gamma$-convergence of the phase-field model

This chapter is devoted to the $\Gamma$-convergence analysis for the family of phase-field functionals $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ (cf. (3.1.3)) that constitute one of the central objects of study in the thesis. We begin with the scalar one-dimensional setting (cf. Section 3.2), which offers a clear perspective on the asymptotic behaviour of the model. In this context, we establish the $\Gamma$-limit functional (cf. (3.2.2)) and investigate how its form depends on some parameters depending on the potentials of the functional $\mathcal{F}_\varepsilon$ (see Theorem 3.2.1). By varying these quantities, different asymptotic regimes emerge, ranging from cohesive responses with linear or superlinear surface densities to brittle limits consistent with Griffith-type models (cf. Section 3.2.4). The analysis is then extended to the vectorial setting, for the linear cohesive framework (cf. Section 3.3). Unlike the scalar case, the treatment of vector-valued functions of bounded variation requires the use of nontrivial tools from geometric measure theory, reflecting the richer and more intricate structure of the problem. These challenges, combined with the anisotropic effects and the need for quasi-convexification of the bulk term, make the analysis considerably more demanding. Nevertheless, the $\Gamma$-convergence results confirm that the proposed framework remains consistent, yielding cohesive free-discontinuity energies also in higher dimensions (cf. Theorem 3.1.1).

## 3.1 Presentation of the phase-field model

We begin by introducing the family of phase-field functionals $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$. For every $t \in [0,1)$ we define

$$f(t) := \left(\frac{l(t)}{Q(1-t)}\right)^{1/2}, \tag{3.1.1}$$

where the functions $l, Q \in C^0([0,1],[0,\infty))$ are required to satisfy the following structural conditions:

(Hp 1) $l, Q \in C^0([0,1],[0,\infty))$ with $l(1)=1$, $l^{-1}(\{0\}) = Q^{-1}(\{0\}) = \{0\}$, $Q$ is non-decreasing in a right neighbourhood of the origin, and the map $[0,1) \ni t \mapsto f(t)$ is non-decreasing both in a right neighbourhood of the origin and in a left neighbourhood of 1.

Notice that the assumption $l(1)=1$ is not restrictive, since it can always be enforced by rescaling $Q$ by a multiplicative factor. Next, we introduce a second condition:

(Hp 2) $\omega \in C^0([0,1],[0,\infty))$ is non-decreasing, with $\omega^{-1}(\{0\}) = \{0\}$, and the following limit exists:

$$\lim_{t\to 0^+} \left(\frac{\omega(t)}{Q(t)}\right)^{1/2} =: \overline{\varsigma} \in [0,\infty]. \tag{3.1.2}$$

We also consider a non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ and a Borel function $\Phi : \mathbb{R}^{N\times n} \to [0,\infty]$. With these components, the functionals $\mathcal{F}_\varepsilon : L^1(O,\mathbb{R}^{N+1}) \times \mathcal{A}(O) \to [0,\infty]$ are defined as

$$\mathcal{F}_\varepsilon(u,v,A) := \int_A \Big(\varphi(\varepsilon f^2(v))\Phi(\nabla u) + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x, \tag{3.1.3}$$

if $(u,v) \in H^1(O,\mathbb{R}^N \times [0,1])$, and $\infty$ otherwise. In (3.1.3), the functions $[0,1) \ni t \mapsto \varphi(\varepsilon f^2(t))$ are extended by continuity to $t = 1$ with value $\varphi(\infty)$, for every $\varepsilon > 0$.

In the main results of the section, we further assume:

(Hp 3) $\varphi$ is continuous, increasing, and bounded, with $\varphi(\infty) := \lim_{t\to\infty} \varphi(t) \in (0,\infty)$.

(Hp 4) $\varphi^{-1}(0) = 0$, and $\varphi$ is (right) differentiable at 0, with $\varphi'(0) \in (0,\infty)$.

For the sake of a coherent presentation across both the one-dimensional scalar and the general vectorial frameworks, we adopt the same hypotheses. We remark, however, that in the one-dimensional case the analysis would still hold under milder assumptions, without requiring monotonicity of $\omega$ nor $f$ in the origin.

(Hp 5) $\Phi : \mathbb{R}^{N\times n} \to [0,\infty)$ is a continuous function satisfying, for every $\xi \in \mathbb{R}^{N\times n}$,

$$\left(\frac{1}{C}|\xi|^2 - C\right) \vee 0 \le \Phi(\xi) \le C(|\xi|^2 + 1), \tag{3.1.4}$$

for some constant $C \ge 1$. Moreover $\Phi$ admits a quadratic recession function

$$\Phi_\infty(\xi) = \lim_{t\to\infty} \frac{\Phi(t\xi)}{t^2}, \tag{3.1.5}$$

where the convergence in (3.1.5) is uniform over the unit sphere $\{\xi \in \mathbb{R}^{N\times n} : |\xi| = 1\}$. In the special case $n = N = 1$, we confine our attention to the isotropic setting by taking $\Phi(\xi) = |\xi|^2$, in view of the mechanical applications discussed in Section 4.3.

We point out that (Hp 5) ensures that $\Phi_\infty(t\xi) = t^2\Phi_\infty(\xi)$, and in particular, for every $\delta > 0$ there exists $t_\delta > 0$ such that

$$|\Phi(\xi) - \Phi_\infty(\xi)| \le \delta|\xi|^2 \qquad \text{for all } |\xi| \ge t_\delta. \tag{3.1.6}$$

Furthermore, the uniform convergence implies $\Phi_\infty \in C^0(\mathbb{R}^{N\times n})$, while condition (3.1.4) yields the quadratic growth bounds

$$\frac{1}{C}|\xi|^2 \le \Phi_\infty(\xi) \le C|\xi|^2. \tag{3.1.7}$$

Furthermore, we note that the restriction to the quadratic growth condition in (3.1.4) is made solely for convenience; the arguments extend to any $p > 1$ without substantial modifications.

Let now $\mathscr{F}_{\overline{\varsigma}} : L^1(O) \times \mathcal{A}(O) \to [0,\infty]$ be given by

$$\mathscr{F}_{\overline{\varsigma}}(u,A) := \begin{cases} \displaystyle\int_A h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u)\,\mathrm{d}x + \int_A h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\mathrm{d}D^c u) + \int_{J_u\cap A} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} & \\ & \text{if } u \in (GBV(O))^N \\ \infty & \text{otherwise;} \end{cases} \tag{3.1.8}$$

for $\varsigma \in [0,\infty]$ we define $h_\varsigma : \mathbb{R}^{N\times n} \to [0,\infty)$ by

$$h_\varsigma(\xi) := \inf_{\tau\in[0,\infty)} \left\{ \varphi\left(\frac{1}{\tau}\right)\Phi(\xi) + \frac{\varsigma^2}{4}\tau \right\}, \tag{3.1.9}$$

where $\varphi(1/\tau)$ is extended by continuity as $\varphi(\infty)$ in $\tau = 0$ (in particular, $h_0 \equiv 0$ and $h_\infty(\xi) = \varphi(\infty)\Phi(\xi)$, adopting the convention $0\cdot\infty = 0$), $h^{\mathrm{qc}}_{\overline{\varsigma}}$ denotes the quasi-convex envelope of $h_{\overline{\varsigma}}$, while $h^{\mathrm{qc},\infty}_{\overline{\varsigma}}$ is the recession function of $h^{\mathrm{qc}}_{\overline{\varsigma}}$, namely

$$h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\xi) := \limsup_{t\to\infty} \frac{h^{\mathrm{qc}}_{\overline{\varsigma}}(t\xi)}{t}; \tag{3.1.10}$$

the surface density $g : \mathbb{R}^N \times \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$g(\zeta,\nu) := \inf \left\{ \liminf_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j, Q^\nu) \ :\ \|u_j - u_{\zeta,\nu}\|_{L^1(Q^\nu,\mathbb{R}^N)} \to 0,\ \varepsilon_j \to 0 \right\}, \tag{3.1.11}$$

where $u_{\zeta,\nu}$ is defined in (2.0.3) and

$$\mathcal{F}^\infty_\varepsilon(u,v,A) := \int_A \left( \varphi'(0)\varepsilon f^2(v)\Phi_\infty(\nabla u) + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|\nabla v|^2 \right) \mathrm{d}x, \tag{3.1.12}$$

adopting the usual convention $0\cdot\infty = 0$.

It is important to mention that in the special case where $\Phi_\infty(\xi) = |\xi|^2$, the surface energy density depends only on the modulus of the jump (cf. Corollary 3.3.6), and for every $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$ one has

$$g(\zeta,\nu) = g_{\mathrm{scal}}(|\zeta|) \tag{3.1.13}$$

where the scalar cohesive law $g_{\mathrm{scal}} : [0,\infty) \to [0,\infty)$ is given by the one-dimensional variational problem

$$g_{\mathrm{scal}}(s) := \inf \Big\{ \int_0^1 \left(\theta(\beta)|\alpha'|^2 + \omega(1-\beta)|\beta'|^2\right)^{\frac{1}{2}} \mathrm{d}t \ :\ (\alpha,\beta) \in H^1((0,1),\mathbb{R}\times[0,1]), \\ \alpha(0) = 0,\ \alpha(1) = s,\ \beta(0) = \beta(1) = 1 \Big\}, \tag{3.1.14}$$

with $\theta(t) := \varphi'(0^+) l(t) \frac{\omega(1-t)}{Q(1-t)}$.

The main results of this section are summarized in the $\Gamma$-convergence theorem stated below.

**Theorem 3.1.1.** *Assume (Hp 1)-(Hp 5) with $\overline{\varsigma} \in (0,\infty]$, and either $n = N = 1$ or $\overline{\varsigma} \neq \infty$. Let $\mathcal{F}_\varepsilon$ be the functional defined in (3.1.3), then, for all $(u,v) \in L^1(O,\mathbb{R}^{N+1})$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) = F_{\overline{\varsigma}}(u,v),$$

*where* $F_{\overline{\varsigma}} : L^1(O,\mathbb{R}^{N+1}) \to [0,\infty]$ *is defined by*

$$F_{\overline{\varsigma}}(u,v) := \begin{cases} \mathscr{F}_{\overline{\varsigma}}(u,O) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e. on } O \\ \infty & \text{otherwise} \end{cases}. \tag{3.1.15}$$

The proof of Theorem 3.1.1 builds upon the approach developed in [CFI16, CFI24a, CFI25] for the $\Gamma$-convergence analysis of the phase-field model $CFI_\varepsilon$ (see (1.0.4)), both in the scalar and in the vectorial setting. Nonetheless, several nontrivial difficulties arise from the increased generality of the present framework.

In passing, note that the models $CFI_\varepsilon$ and $W_\varepsilon$ (cf. (1.0.4) and (1.0.7)) correspond, respectively, to the choices $\varphi_1(t) = 1 \wedge t$, $Q_1(t) = \omega_1(t) = t^2$ and $\varphi_2(t) = \frac{t}{1+t}$, $l(t) = t^2$ in the second setting, with $\Phi(\xi) = |\xi|^2$ in both cases. For $i \in \{1,2\}$, $\varphi_i$ satisfies assumptions (Hp 3), (Hp 4) and (Hp 5) with $\varphi_i(\infty) = \varphi_i'(0) = 1$. Moreover, if $h_\varsigma^{(i)}$ denotes the corresponding functions defined in (3.1.9), then for every $\varsigma > 0$ we obtain

$$(h_\varsigma^{(1)})^{\mathrm{qc}}(\xi) \geq (h_\varsigma^{(1)})^{**}(\xi) = h_\varsigma^{(2)}(\xi) = (h_\varsigma^{(2)})^{**}(\xi) = (h_\varsigma^{(2)})^{\mathrm{qc}}(\xi) = \begin{cases} |\xi|^2 & \text{if } |\xi| \leq \frac{\varsigma}{2} \\ \varsigma|\xi| - \frac{\varsigma^2}{4} & \text{if } |\xi| \geq \frac{\varsigma}{2}, \end{cases}$$

where $^{**}$ and $^{\mathrm{qc}}$ stand respectively for the convex and quasi-convex envelope. By [CFI24a, Lemma 2.5, Item (iii)], the first inequality is strict for some values of $\xi$, whereas the choice of $\varphi_2$ does not require any quasiconvexification, since $h_\varsigma^{(2)}$ is already convex. This feature underscores a structural advantage of this formulation, as it avoids the additional convexification steps that are otherwise necessary in these contexts. More generally, as we will show in Propositon 3.1.5, in the isotropic case $\Phi(\xi) = |\xi|^2$, under suitable assumptions on $\varphi$, then $h_\varsigma$ is convex for every $\varsigma > 0$. This provides a criterion to ensure convexity directly from the structural properties of $\varphi$, thereby simplifying the analysis for a family of models.

Before concluding this section, we wish to emphasize that the bulk energy density $h_{\overline{\varsigma}}$ essentially depends only on the function $\varphi$, while the surface energy density $g$ depends primarily on the potentials $l$, $Q$, and $\omega$. This distinction is particularly relevant in view of the reconstruction of the model: the function $\varphi$ encodes the elastic response of the material and can therefore be identified from bulk behaviour (see Propositions 3.1.3 and 3.1.5), while the potentials $l$, $Q$, and $\omega$ control the cohesive properties and can be thus reconstructed from the analysis of the surface energy density $g$ (see Section 4.2).

### 3.1.1 Technical results

We begin by establishing a collection of auxiliary results that will be instrumental in the proof of the $\Gamma$–convergence theorem. These include compactness estimates, structural properties of the bulk density, and truncation arguments, which together provide the technical foundation for the liminf and limsup inequalities.

We start with the following lemma, which will be employed in the vectorial setting, specifically in Section 3.3.3, to establish the subadditivity of the $\Gamma$–$\limsup$ of a perturbation of the functionals $\mathcal{F}_\varepsilon$.

**Lemma 3.1.2.** *Assume (Hp 1) and (Hp 3–4). Then there exists a constant $K \geq 1$ such that, for every $\varepsilon > 0$ and every $t_1, t_2 \in [0,1)$ with $t_1 \leq t_2$, the following inequalities hold:*

$$f^2(t_1) \leq K f^2(t_2) \quad \text{and} \quad \varphi(\varepsilon f^2(t_1)) \leq K\varphi(\varepsilon f^2(t_2)). \tag{3.1.16}$$

*Proof.* Fix $\varepsilon > 0$. By (Hp 1), there exist $0 < r_1 < r_2 < 1$ such that $f$ is non-decreasing on both $[0, r_1]$ and $[r_2, 1)$. Since $f$ is continuous and $f^{-1}(0) = 0$, there exists a constant $c_0 \geq 1$ such that

$$f^2(r_1), \max_{[r_1,r_2]} f^2 \leq c_0 \min_{[r_1,r_2]} f^2 \quad \text{and} \quad \max_{[r_1,r_2]} f^2 \leq c_0 f^2(r_2).$$

Therefore, we can easily deduce that $f^2(t_1) \leq c_0 f^2(t_2)$, for every $t_1, t_2 \in [0, 1)$ with $t_1 \leq t_2$. To address the second inequality, define the auxiliary function $\hat{\varphi} : (0, \infty) \to (0, \infty)$ by

$$\hat{\varphi}(t) := \frac{\varphi(c_0 t)}{\varphi(t)}$$

that is continuous because $\varphi^{-1}(\{0\}) = \{0\}$. From (Hp 3-4) we infer that

$$\lim_{t\to\infty} \hat{\varphi}(t) = 1 \quad \text{and} \quad \lim_{t\to 0} \hat{\varphi}(t) = c_0 \geq 1.$$

Therefore, if $c_1 := \sup_{t\in(0,\infty)} \hat{\varphi}(t) \in (0, \infty)$ we have that $c_1 \in [1, \infty)$ and $\varphi(c_0 t) \leq c_1 \varphi(t)$ for every $t \in [0, \infty)$, and in particular, by the monotonicity of $\varphi$ we obtain that

$$\varphi(\varepsilon f^2(t_1)) \leq \varphi(\varepsilon c_0 f^2(t_2)) \leq c_1 \varphi(\varepsilon f^2(t_2)),$$

for every $t_1, t_2 \in [0, 1)$ with $t_1 \leq t_2$. Setting $K := \max\{c_0, c_1\}$, we conclude the proof. □

We establish in the next statement several useful properties, especially in the 1-dimensional scalar case, of the densities $h_\varsigma$ for functions $\varphi$ satisfying slightly more general assumptions than (Hp 3) and (Hp 4). In particular, in what follows with $\varphi'(0^+) = \infty$ we mean that the limit of the difference quotient of $\varphi$ in $t = 0$ exists and it is not finite.

**Proposition 3.1.3.** *Assume (Hp 4) and let $\varphi \in C^0([0, \infty))$, be positive, non-decreasing with $\varphi^{-1}(\{0\}) = 0$, then $h_\varsigma \in C^0(\mathbb{R}^{N\times n})$ (see (3.1.9)) and*

$$h_\sigma(\xi) \leq h_\varsigma(\xi) \leq (\varsigma/\sigma)^2 h_\sigma(\xi) \qquad \text{for every } \xi \in \mathbb{R}^{N\times n}, \tag{3.1.17}$$

*for all $\sigma, \varsigma \in (0, \infty)$ with $\sigma \leq \varsigma$.*

*In addition, if $n = N = 1$, we have*

*(i) Let $\varsigma \in (0, \infty)$, then $h_\varsigma(0) = 0$, $h_\varsigma$ is strictly increasing, locally-Lipschitz on the interval $(0, \infty)$ and $h_0(t) < h_\varsigma(t) \leq h_\infty(t)$ for every $t > 0$. Moreover, for every $t \in \mathbb{R}$*

$$\lim_{\varsigma'\to\varsigma} h^{**}_{\varsigma'}(t) = h^{**}_\varsigma(t)\,. \tag{3.1.18}$$

*(ii) Let $\varsigma \in (0, \infty)$. If $\varphi$ has right derivative $\varphi'(0^+) \in [0, \infty]$ then*

$$\lim_{t\to\pm\infty} \frac{h^{**}_\varsigma(t)}{t} = \lim_{t\to\pm\infty} \frac{h_\varsigma(t)}{t} = (\varphi'(0^+))^{1/2}\varsigma\,. \tag{3.1.19}$$

*If in addition $\varphi'(0^+) \in [0,\infty)$, then there is $\xi_\varsigma > 0$ such that for every $t \in [0,\infty)$*

$$((\varphi'(0^+))^{1/2}\varsigma t - \xi_\varsigma) \vee 0 \le h_\varsigma^{**}(t) \le (\varphi'(0^+))^{1/2}\varsigma t\,. \tag{3.1.20}$$

*Moreover, if $\varphi'(0^+) \in (0,\infty]$*

$$\lim_{t\to 0}\frac{h_\varsigma^{**}(t)}{t^2} = \lim_{t\to 0}\frac{h_\varsigma(t)}{t^2} = \varphi(\infty) \in (0,\infty]\,. \tag{3.1.21}$$

*(iii) Let $\varsigma = \infty$. If $\varphi(\infty) \in (0,\infty)$ then $h_\infty(t) = \varphi(\infty)t^2$ for every $t \in \mathbb{R}$. If, moreover, $\varphi'(0^+) \in (0,\infty]$ then for every $t \in \mathbb{R}$*

$$\lim_{\varsigma\to\infty} h_\varsigma^{**}(t) = \lim_{\varsigma\to\infty} h_\varsigma(t) = h_\infty(t)\,. \tag{3.1.22}$$

*(iv) Let $\varsigma = 0$, $h_0(t) = 0$ for every $t \in \mathbb{R}$ and*

$$\lim_{\varsigma\to 0^+} h_\varsigma^{**}(t) = \lim_{\varsigma\to 0^+} h_\varsigma(t) = h_0(t)\,. \tag{3.1.23}$$

*Proof.* We fix $\sigma \in (0,\infty)$ and begin by proving the continuity of $h_\sigma$. Let $\xi_0 \in \mathbb{R}^{N\times n}$ and consider a sequence $\xi_k \to \xi_0$ as $k\to\infty$. For each $k$, choose $\tau_k \in [0,\infty)$ such that

$$\varphi\left(\tfrac{1}{\tau_k}\right)\Phi(\xi_k) + \tfrac{\sigma^2}{4}\tau_k - \tfrac{1}{k} < h_\sigma(\xi_k). \tag{3.1.24}$$

Since $\sup_{k\in\mathbb{N}}\Phi(\xi_k) < \infty$ and $\Phi$ is continuous, (3.1.9) ensures that $\limsup_{k\to\infty}\tau_k < \infty$. Hence, we may extract a subsequence $(\tau_{k_j})$ such that $\tau_{k_j} \to \tau_0$ for some $\tau_0 \in [0,\infty)$. By continuity of both $\varphi$ and $\Phi$, from (3.1.24) we obtain

$$h_\sigma(\xi_0) \le \varphi\left(\frac{1}{\tau_0}\right)\Phi(\xi_0) + \frac{\sigma^2}{4}\tau_0 \le \liminf_{j\to\infty} h_\sigma(\xi_{k_j}).$$

Since $\xi_0$ and the sequence $(\xi_k)$ are arbitrary, this shows that $h_\sigma$ is lower semicontinuous. The upper semicontinuity follows directly from the definition of $h_\sigma$ and the continuity of $\Phi$.

Let $\varsigma \in (0,\infty)$ with $\sigma \le \varsigma$. The first inequality in (3.1.17) follows directly from the definition (3.1.9). For the second one, fix $\xi \in \mathbb{R}^{N\times n}$. By continuity of $\varphi$, there exists $\tau_{\sigma,\xi} \in [0,\infty)$ such that

$$h_\sigma(\xi) = \varphi\left(\frac{1}{\tau_{\sigma,\xi}}\right)\Phi(\xi) + \frac{\sigma^2}{4}\tau_{\sigma,\xi}.$$

Since $(\varsigma/\sigma)^2 \ge 1$, we then deduce

$$h_\varsigma(\xi) \le \varphi\left(\frac{1}{\tau_{\sigma,\xi}}\right)\Phi(\xi) + \frac{\varsigma^2}{4}\tau_{\sigma,\xi} \le \left(\frac{\varsigma}{\sigma}\right)^2 h_\sigma(\xi),$$

which yields the claimed inequality.

For the remainder of the proof, which will be divided into several steps, we shall assume $n = N = 1$. By (Hp 5) we have $\Phi(t) = t^2$, and hence $h_\varsigma : \mathbb{R} \to [0,\infty)$ is an even function for every $\varsigma \in [0,\infty]$. Accordingly, it suffices to restrict the analysis to the interval $[0,\infty)$.

**Step 1:** Proof of item (i). It is immediate to observe that $h_\varsigma(0) = 0$ for every $\varsigma \in [0, \infty]$, and moreover

$$h_0(t) = 0 \qquad \text{and} \qquad h_\infty(t) = \varphi(\infty)t^2 \qquad \text{for every } t \geq 0$$

since $\varphi(0) = 0$ and we adopt the convention $0 \cdot \infty = 0$. Now fix $\varsigma \in (0, \infty)$. From $\varphi^{-1}(\{0\}) = \{0\}$ and (3.1.9), we obtain

$$h_0(t) = 0 < h_\varsigma(t) \leq \varphi(\infty)t^2 = h_\infty(t) \qquad \text{for every } t \geq 0.$$

The strict monotonicity of $h_\varsigma$ follows directly from (3.1.9), while the limiting behaviour in (3.1.18) is an immediate consequence of (3.1.17). Finally, by the very definition of $h_\varsigma$, we have

$$h_\varsigma(t) \leq \left(\frac{t}{s}\right)^2 h_\varsigma(s)$$

which in turn yields the local Lipschitz continuity of $h_\varsigma$ on $(0, \infty)$.

**Step 2:** Proof of item (ii).

Assume first that $\varphi'(0^+) \in [0, \infty)$. Let $\varsigma > 0$ and $t > 0$ be fixed, and let $\eta_{\varsigma,t} > 0$ be such that $\varphi(\frac{1}{\eta_{\varsigma,t}})t^2 = \frac{\varsigma^2}{4}\eta_{\varsigma,t}$, its existence follows from the continuity and monotonicity of $\varphi$, then

$$h_\varsigma(t) \leq \left(\varphi(1/\eta_{\varsigma,t})\eta_{\varsigma,t}\right)^{1/2} \varsigma t\,.$$

In particular, it easy to check that $\eta_{\varsigma,t} \to \infty$ as $t \to \infty$, so that

$$\limsup_{t\to\infty} \frac{h_\varsigma^{**}(t)}{t} \leq \limsup_{t\to\infty} \frac{h_\varsigma(t)}{t} \leq \left(\varphi'(0^+)\right)^{1/2}\varsigma\,.$$

If $\varphi'(0^+) = 0$ we are done, otherwise to prove the opposite inequality we assume first $\varphi$ to be bounded. Then $h_\varsigma(t)/t$ is upper bounded on $(0, \infty)$, as by testing the definition of $h_\varsigma(t)$ with $\tau = t$ itself yields

$$\frac{h_\varsigma(t)}{t} \leq \varphi(1/t)t + \frac{\varsigma^2}{4},$$

and the claim follows being $\varphi$ continuous, bounded and differentiable in $t = 0$. Thus, $\tau_{\varsigma,t} \to \infty$ as $t \to \infty$, in turn implying

$$\frac{h_\varsigma(t)}{t} = \varphi(1/\tau_{\varsigma,t})t + \frac{\varsigma^2}{4}\frac{\tau_{\varsigma,t}}{t} \geq \left(\varphi(1/\tau_{\varsigma,t})\tau_{\varsigma,t}\right)^{1/2}\varsigma\,,$$

and we may conclude

$$\liminf_{t\to\infty} \frac{h_\varsigma(t)}{t} \geq \left(\varphi'(0^+)\right)^{1/2}\varsigma\,.$$

Hence, for every $\varepsilon \in (0, (\varphi'(0^+))^{1/2}\varsigma)$ there is $\xi_\varepsilon \geq 0$ such that $\zeta(t) := 0 \vee (((\varphi'(0^+))^{1/2}\varsigma - \varepsilon)t - \xi_\varepsilon) \leq h_\varsigma(t)$ for all $t \geq 0$. In particular, $h_\varsigma^{**} \geq \zeta$, and thus

$$\liminf_{t\to\infty} \frac{h_\varsigma^{**}(t)}{t} \geq \left(\varphi'(0^+)\right)^{1/2}\varsigma - \varepsilon\,,$$

and (3.1.19) for $h_\varsigma^{**}$ follows at once by letting $\varepsilon \to 0$. If $\varphi$ is not bounded, consider $\varphi \wedge j$ and $h_{\varsigma,j}$ the corresponding function in (3.1.9). Then using (3.1.19) for $h_{\varsigma,j}$ yields $\liminf\limits_{t\to 0} \frac{h_\varsigma^{**}(t)}{t} \geq \lim\limits_{t\to 0} \frac{h_{\varsigma,j}^{**}(t)}{t} = (\varphi'(0^+))^{1/2}\varsigma$, and thus the conclusion.

If $\varphi'(0^+)=\infty$, there is a diverging sequence of integers $k_j\in\mathbb{N}$ such that the connected component of $\{t\in(0,\infty):\varphi(t)>k_jt\}$ whose closure contains the origin is bounded. Then consider the function $\varphi_j$ equal to $k_jt$ on such a set and to $\varphi$ otherwise, and let $h_{\varsigma,j}$ be the corresponding function in (3.1.9). Using (3.1.19) for $h_{\varsigma,j}$ gives $\liminf\limits_{t\to\infty}\frac{h^{**}_\varsigma(t)}{t}\geq(k_j)^{1/2}\varsigma$, and thus (3.1.19) for $h_\varsigma$ follows by letting $j\to\infty$.

The growth conditions in (3.1.20) are then a conseguence of the convexity of $h^{**}_\varsigma$, and of the equality $h^{**}_\varsigma(0)=h_\varsigma(0)=0$.

We prove next (3.1.21). Assume first $\varphi(\infty)\in(0,\infty)$, and note that being $h_\varsigma\leq h_\infty$ we have

$$\limsup_{t\to0}\frac{h^{**}_\varsigma(t)}{t^2}\leq\limsup_{t\to0}\frac{h_\varsigma(t)}{t^2}\leq\varphi(\infty)\,.$$

In addition, taking $\tau=t^3$ in the definition of $h_\varsigma(t)$, the monotonicity of $\varphi$ in (Hp 3) yields

$$\frac{h_\varsigma(t)}{t^2}=\varphi(1/\tau_{\varsigma,t})+\frac{\varsigma^2}{4}\frac{\tau_{\varsigma,t}}{t^2}\leq\varphi(\infty)+\frac{\varsigma^2}{4}t\,,$$

from which we infer $\tau_{\varsigma,t}\leq\frac{4\varphi(\infty)}{\varsigma^2}t^2+t^3$. As, for every $\varepsilon\in(0,\varphi(\infty))$ there is $\delta_\varepsilon>0$ such that $\varphi(1/\tau)\geq\varphi(\infty)-\varepsilon$ if $\tau\in(0,\delta_\varepsilon)$, there is $t_{\varsigma,\varepsilon}>0$, such that for $t\in(0,t_{\varsigma,\varepsilon}]$

$$h_\varsigma(t)\geq\varphi(1/\tau_{\varsigma,t})t^2\geq(\varphi(\infty)-\varepsilon)t^2\,.$$

In particular, from this it is easy to infer the equality for the limit of $h_\varsigma$ in (3.1.21). Next, set $\eta_{\varsigma,\varepsilon}:=\inf_{(t_{\varsigma,\varepsilon},\infty)}\frac{h_\varsigma(t)-h_\varsigma(t_{\varsigma,\varepsilon})}{t-t_{\varsigma,\varepsilon}}$. Thanks to (3.1.19), to the strict monotonicity and to the local Lipschitz continuity of $h_\varsigma$, we may assume $\eta_{\varsigma,\varepsilon}>0$, up to substituting $t_{\varsigma,\varepsilon}$ with a slightly smaller value $t'_{\varsigma,\varepsilon}$ such that $h_\varsigma$ is differentiable on $t'_{\varsigma,\varepsilon}$ and $h'_\varsigma(t'_{\varsigma,\varepsilon})>0$. Therefore, $h_\varsigma\geq\psi_{\varsigma,\varepsilon}$, where

$$\psi_{\varsigma,\varepsilon}(t):=\begin{cases}(\varphi(\infty)-\varepsilon)t^2 & \text{if } t\in[0,t_{\varsigma,\varepsilon}]\\(\varphi(\infty)-\varepsilon)t^2_{\varsigma,\varepsilon}+\eta_{\varsigma,\varepsilon}(t-t_{\varsigma,\varepsilon}) & \text{if } t>t_{\varsigma,\varepsilon}\end{cases}\,,$$

in turn implying $h^{**}_\varsigma\geq\psi^{**}_{\varsigma,\varepsilon}$, where

$$\psi^{**}_{\varsigma,\varepsilon}(t)=\begin{cases}(\varphi(\infty)-\varepsilon)t^2 & \text{if } t\in[0,\frac{\eta_{\varsigma,\varepsilon}}{2(\varphi(\infty)-\varepsilon)}]\\\eta_{\varsigma,\varepsilon}t-\frac{\eta^2_{\varsigma,\varepsilon}}{4(\varphi(\infty)-\varepsilon)} & \text{if } t\in(\frac{\eta_{\varsigma,\varepsilon}}{2(\varphi(\infty)-\varepsilon)},\infty)\end{cases}$$

if $\eta_{\varsigma,\varepsilon}<2(\varphi(\infty)-\varepsilon)t_{\varsigma,\varepsilon}$, and $\psi^{**}_{\varsigma,\varepsilon}=\psi_{\varsigma,\varepsilon}$ otherwise. In any case, from this it is easy to infer the equality for the limit of $h^{**}_\varsigma$ in (3.1.21). Finally, if $\varphi(\infty)=\infty$, consider $\varphi\wedge j$ and $h_{\varsigma,j}$ the corresponding function in (3.1.9). Then using (3.1.21) for $h_{\varsigma,j}$ yields $\liminf\limits_{t\to0}\frac{h^{**}_\varsigma(t)}{t^2}\geq j$, and the conclusion follows by letting $j\to\infty$.

**Step 3:** Proof of item (iii).

Let $t>0$, recalling that $\tau_{\varsigma,t}\geq0$ is a minimum point in the definition of $h_\varsigma(t)$, from $h_\varsigma(t)\leq h_\infty(t)$ we conclude that $\tau_{\varsigma,t}\to0^+$ as $\varsigma\to\infty$, and thus $h_\varsigma(t)\to h_\infty(t)$ as $\varsigma\to\infty$.

To establish (3.1.22) for $h^{**}_\varsigma$, let $t>0$ (as $h^{**}_\varsigma(0)=h_\infty(0)=0$) and note that by definition $h^*_\varsigma(t)=-\inf\{h_\varsigma(s)-ts:s\in[0,\infty)\}$. Since $(h_\varsigma(\cdot)-t\cdot)_{\varsigma\geq0}$ is equicoercive for $\varsigma\geq\varsigma_t>t(\varphi'(0^+))^{-1/2}$ thanks to (3.1.20), the pointwise convergence of $h_\varsigma$ to $h_\infty$ and the fundamental theorem of $\Gamma$-convergence imply that $h_\varsigma^*(t) \to h_\infty^*(t)$ as $\varsigma \to \infty$. Moreover, the family $(h_\varsigma^*)_{\varsigma\geq 0}$ is non-increasing with $h_\varsigma^*(t) \geq h_\infty^*(t) = \frac{t^2}{4}$, and thus $(h_\varsigma^*(\cdot) - s\cdot)_{\varsigma\geq 0}$ is equicoercive, and again the pointwise convergence of $(h_\varsigma^*)_{\varsigma\geq 0}$ to $h_\infty^*$ implies $h_\varsigma^{**}(s) \to h_\infty^{**}(s) = h_\infty(s)$ for every $s \geq 0$.

**Step 4:** Proof of item (iv).

The proof of (3.1.23) is immediate by choosing $\tau = {}^1\!/\!_\varsigma$. Indeed, we have $h_\varsigma(t) \leq \varphi(\varsigma)t^2 + \frac{\varsigma}{2}$, so that (3.1.23) follows at once by letting $\varsigma \to 0^+$ by (Hp 2). □

In the following corollary we establish a linear-growth bound for the bulk density $h_\varsigma^{\mathrm{qc}}$.

**Corollary 3.1.4.** *Assume (Hp 3) and (Hp 5), and let $\varsigma \in (0,\infty)$. Then there exists a constant $K_\varsigma > 0$ such that*

$$\Big(\frac{1}{K_\varsigma}|\xi| - K_\varsigma\Big) \vee 0 \;\leq\; h_\varsigma^{\mathrm{qc}}(\xi) \;\leq\; h_\varsigma(\xi) \;\leq\; K_\varsigma\big(|\xi|+1\big) \tag{3.1.25}$$

*and*

$$\frac{1}{K_\varsigma}|\xi| \leq h_\varsigma^{\mathrm{qc},\infty}(\xi) \leq K_\varsigma|\xi| \tag{3.1.26}$$

*for all $\xi \in \mathbb{R}^{N\times n}$.*

*Proof.* Define, for $\xi \in \mathbb{R}^{N\times n}$,

$$\tilde{h}(\xi) := \inf_{\tau\in[0,\infty)} \left\{\varphi\left(\frac{1}{\tau}\right)\frac{1}{C}|\xi|^2 + \frac{\varsigma^2}{4}\tau\right\},$$

where $C$ is the constant in (3.1.4). From (3.1.4) it follows that

$$\tilde{h}(\xi) \leq h_\varsigma(\xi) + C\,\varphi(\infty) \qquad \text{for every } \xi \in \mathbb{R}^{N\times n}.$$

Using (3.1.20), the radial symmetry of $\tilde{h}$ and $\tilde{h}^{**} \leq \tilde{h}^{\mathrm{qc}}$, we deduce the first inequality in (3.1.25).

For the third inequality we introduce the auxiliary function

$$\hat{h}(\xi) := \inf_{\tau\in[0,\infty)} \left\{\varphi\left(\frac{1}{\tau}\right)C|\xi|^2 + \frac{\varsigma^2}{4}\tau\right\}, \qquad \xi \in \mathbb{R}^{N\times n}.$$

By (3.1.4) we have $h_\varsigma(\xi) \leq \hat{h}(\xi) + C\varphi(\infty)$ for every $\xi \in \mathbb{R}^{N\times n}$. Furthermore, by (3.1.19) there exists $K_\varsigma > 0$ such that $\hat{h}(\xi) \leq K_\varsigma\big(|\xi|+1\big)$ for every $\xi \in \mathbb{R}^{N\times n}$, which yields the third inequality in (3.1.25). Together with the definition of $h_\varsigma^{\mathrm{qc},\infty}$ (see (3.1.10)), this concludes the proof. □

In the explicit computation of the limit functional $\mathscr{F}_{\overline{\varsigma}}$ (cf. (3.1.8)), the convexity of $h_\varsigma$ plays a crucial role. While the general case requires quasiconvexification (highly implicit construction), for suitable choices of the phase-field potential $\varphi$, the function $h_\varsigma$ turns out to be already convex, thereby allowing a direct identification of the limiting bulk energy. The next proposition provides sufficient conditions ensuring this property, along with a corresponding regularity statement.

**Proposition 3.1.5.** *Assume $n = N = 1$, $\Phi(t) = t^2$ and (Hp 3) and with $\varphi^{-1}(\{0\}) = \{0\}$. Let $\varsigma \in (0,\infty)$ and define $r : [0,\infty) \to [0,\infty)$ by $r(\tau) = \varphi(\frac{1}{\tau})$, with $r(0) = \varphi(\infty)$. Suppose that*

*i) $r \in C^2([0,\infty))$ with $r''(t) > 0$ for every $t \in [0,\infty)$;*

*ii) $t \mapsto 2tr((r')^{-1}(-\frac{1}{4t^2}))$ is non-decreasing.*

*Then $h_\varsigma \in C^1([0,\infty)) \setminus C^2([0,\infty))$ and it is convex, for every $\varsigma \in (0,\infty)$.*

*Proof.* Fix $\varsigma \in (0,\infty)$ and $t \in \mathbb{R}$. Considering the function

$$q_t(\tau) := r(\tau)t^2 + \frac{\varsigma^2}{4}\tau, \qquad \tau \geq 0,$$

from the definition of $h_\varsigma$ (see (3.1.9)), we have that $h_\varsigma(t) = \min_{\tau\in[0,\infty)} q_t(\tau)$. If $|t| \leq t_* := \frac{\varsigma}{2\sqrt{-r'(0)}}$, the function $q_t$ is non-decreasing and hence $h_\varsigma(t) = \varphi(\infty)t^2$. For $|t| > t_*$ the strict convexity of $r$ ensures the existence of a unique critical point $\tau_t := (r')^{-1}(-\frac{\varsigma^2}{4t^2}) \in (0,\infty)$ at which $q_t'(\tau_t) = 0$. Moreover $q_t'(\tau) > 0$ if and only if $\tau > \tau_t$. Consequently,

$$h_\varsigma(t) = r((r')^{-1}(-\tfrac{\varsigma^2}{4t^2}))t^2 + \tfrac{\varsigma^2}{4}(r')^{-1}(-\tfrac{\varsigma^2}{4t^2})$$

for $|t| > t_*$. As $r \in C^2([0,\infty))$ and it is strictly convex then $h_\varsigma \in C^1(\mathbb{R})$, indeed a direct computation shows that

$$h_\varsigma'(t) = \begin{cases} 2t\varphi(\infty) & \text{if } |t| \leq t_* \\ 2tr((r')^{-1}(-\frac{\varsigma^2}{4t^2})) & \text{if } |t| > t_*. \end{cases}$$

By assumption ii) we have that $h_\varsigma$ is convex. Moreover, turning to the second derivative, we obtain

$$h_\varsigma''(t) = \begin{cases} 2\varphi(\infty) & \text{if } |t| < t_* \\ 2r((r')^{-1}(-\frac{\varsigma^2}{4t^2})) - \frac{\varsigma^4}{8t^4 r''((r')^{-1}(-\frac{\varsigma^2}{4t^2}))} & \text{if } |t| > t_*, \end{cases}$$

and thus $\lim_{t\to t_*^-} h_\varsigma''(t) = 2\varphi(\infty)$ and $\lim_{t\to t_*^+} h_\varsigma''(t) = 2\varphi(\infty) - \frac{2(r'(0))^2}{r''(0)}$, showing that $h_\varsigma \notin C^2([0,\infty))$. $\square$

**Remark 3.1.6.** *We point out that in the multidimensional vectorial setting, if we assume $\Phi(\xi) = \phi(|\xi|)$, where $\phi$ has a quadratic growth in the origin, then the function $h_\varsigma$ is radially symmetric. As a consequence, the one-dimensional analysis developed in Proposition 3.1.5 applies without modification, guaranteeing that, under suitable assumption on the potential $\varphi$, the function $h_\varsigma$ is convex and in particular no quasiconvexification procedure is required to obtain the precise bulk density of the limit functional $\mathscr{F}_{\overline{\varsigma}}$ (cf. (3.1.8)).*

Next, we establish a truncation result following [CFI24a, Lemma 4.4]. To this end, we introduce the function $\Psi : [0,1] \to \mathbb{R}$ defined by

$$\Psi(x) := \int_0^x \omega^{1/2}(1-t)\,\mathrm{d}t. \tag{3.1.27}$$

**Lemma 3.1.7.** *Assume (Hp 1), (Hp 2) with $\overline{\varsigma} \in (0,\infty]$. In addition let $\varphi : [0,\infty) \to [0,\infty)$ be non-decreasing and $\varphi^{-1}(\{0\}) = \{0\}$. Then there exist two functions $\zeta_1, \zeta_2 : (0,1) \to (0,\infty)$ with*

$$\lim_{\delta\to 0} \zeta_1(\delta) = 1 \qquad \text{and} \qquad \lim_{\delta\to 0} \zeta_2(\delta) = 0 \tag{3.1.28}$$

*satisfying the following property. For every $\gamma \in (0, 1\wedge\overline{\varsigma})$ there exists $\delta_\gamma \in (0,1)$ such that for every $\delta \in (0,\delta_\gamma)$, $(u,v) \in H^1 \cap L^\infty(O, \mathbb{R}^N \times [0,1])$, and $A \in \mathcal{A}(O)$, there is $s_\delta \in (1-\delta, 1-\delta^2)$ such that $\chi_{\{v>s_\delta\}} \in BV(A)$, $u^\delta := u\,\chi_{\{v>s_\delta\}} \in SBV(A)$ (the previous quantities actually depend also on $\gamma$ and $A$) and*

$$H_\delta(u^\delta, A) \leq \mathcal{F}_\varepsilon(u,v,A) + h_{\overline{\varsigma}_\gamma}(0)\mathcal{L}^n(A \cap \{v \leq \delta\}) \tag{3.1.29}$$

*where* $H_\delta : L^1(O,\mathbb{R}^N)\times\mathcal{A}(O)\to[0,\infty]$ *is defined for* $\overline{\varsigma}\in(0,\infty)$ *by*

$$H_\delta(w,A)=\zeta_1(\delta)\int_A h_{\overline{\varsigma}_\gamma}(\nabla w)\,\mathrm{d}x+\zeta_2(\delta)\mathcal{H}^0(A\cap J_w) \tag{3.1.30}$$

*if* $w\in SBV(A,\mathbb{R}^N)$*, and* $\infty$ *otherwise with* $\overline{\varsigma}_\gamma:=(\overline{\varsigma}-\gamma)(1-\gamma)$*; and for* $\overline{\varsigma}=\infty$ *by*

$$H_\delta(w,A)=\zeta_1(\delta)\int_A h_{\overline{\varsigma}_\gamma}(\nabla w)\,\mathrm{d}x+\zeta_2(\delta)\mathcal{H}^0(A\cap J_w) \tag{3.1.31}$$

*if* $w\in SBV(A,\mathbb{R}^N)$*, and* $\infty$ *otherwise, with* $\overline{\varsigma}_\gamma:=\frac{1}{\gamma}-1$.

*Moreover, if* $u_\varepsilon\to u$ *in* $L^1(O,\mathbb{R}^N)$ *and* $v_\varepsilon\to 1$ *in measure on* $O$*, then* $u_\varepsilon^\delta\to u$ *in* $L^1(A)$ *as* $\varepsilon\to 0$ *for every* $A\subseteq\mathcal{A}(O)$ *and* $\delta\in(0,1)$*.*

*Proof.* With fixed $\gamma$, $\delta\in(0,1)$ and $(u,v)\in H^1\cap L^\infty(O,\mathbb{R}^N\times[0,1])$ setting $A_\delta:=\{x\in A:\ v(x)>1-\delta\}$, we argue as follows:

$$\begin{aligned}
&\mathcal{F}_\varepsilon(u,v,A)\\
&\quad\geq\int_{A_\delta}\left(\varphi(\varepsilon f^2(v))\Phi(\nabla u)+(1-\delta)\frac{\omega(1-v)}{4\varepsilon}\right)\mathrm{d}x+\int_A\left(\delta\,\frac{\omega(1-v)}{4\varepsilon}+\varepsilon|v'|^2\right)\mathrm{d}x\\
&\quad\geq\int_{A_\delta}\left(\varphi(\varepsilon f^2(v))\Phi(\nabla u)+(1-\delta)\frac{\omega(1-v)}{4\varepsilon}\right)\mathrm{d}x+\delta^{1/2}\int_A|\Psi(v)'|\,\mathrm{d}x\,,
\end{aligned}\tag{3.1.32}$$

where $\Psi$ is the function defined in (3.1.27).

We distinguish the two cases $\overline{\varsigma}\in(0,\infty)$ and $\overline{\varsigma}=\infty$. We start with the former. Then, as $l(1)=1$ and (3.1.2) holds, for every $\gamma\in(0,\overline{\varsigma}\wedge 1)$ there is $\delta_\gamma\in(0,1)$ such that if $t\in(0,\delta_\gamma)$

$$l(1-t)\geq(1-\gamma)^2\,,\qquad \frac{\omega(t)}{Q(t)}\geq(\overline{\varsigma}-\gamma)^2\,.$$

Therefore, the first summand in (3.1.32) can be estimated as follows for $\delta\in(0,\delta_\gamma)$ by monotonicity of $\varphi$

$$\begin{aligned}
&\int_{A_\delta}\left(\varphi(\varepsilon f^2(v))\Phi(\nabla u)+(1-\delta)(\overline{\varsigma}-\gamma)^2\frac{Q(1-v)}{4\varepsilon}\right)\mathrm{d}x\\
&\qquad\geq(1-\delta)\int_{A_\delta}\left(\varphi\left(\varepsilon\frac{(1-\gamma)^2}{Q(1-v)}\right)\Phi(\nabla u)+(\overline{\varsigma}-\gamma)^2\frac{Q(1-v)}{4\varepsilon}\right)\mathrm{d}x\\
&\qquad\geq(1-\delta)\int_{A_\delta}h_{\overline{\varsigma}_\gamma}(\nabla u)\,\mathrm{d}x\,,
\end{aligned}$$

where $\overline{\varsigma}_\gamma=(\overline{\varsigma}-\gamma)(1-\gamma)$. Hence, (3.1.32) yields that for $\delta\in(0,\delta_\gamma)$

$$\mathcal{F}_\varepsilon(u,v,A)\geq(1-\delta)\int_{A_\delta}h_{\overline{\varsigma}_\gamma}(\nabla u)\,\mathrm{d}x+\delta^{1/2}\int_A|\Psi'(v)|\,\mathrm{d}x\,. \tag{3.1.33}$$

If $\overline{\varsigma}=\infty$, by (3.1.2) for every $\gamma\in(0,1)$ there is $\delta_\gamma\in(0,1)$ such that if $t\in(0,\delta_\gamma)$

$$\frac{Q(t)}{\omega(t)}\leq\gamma^2\,,\quad\text{and}\quad l(1-t)\geq(1-\gamma)^2\,, \tag{3.1.34}$$

from which it follows that for $\delta \in (0, \delta_\gamma)$ we have by monotonicity of $\varphi$

$$\begin{aligned}\int_{A_\delta} &\left(\varphi(\varepsilon f^2(v))\Phi(\nabla u) + (1-\delta)\frac{\omega(1-v)}{4\varepsilon}\right) \mathrm{d}x \\ &\geq \int_{A_\delta} \left(\varphi\left(\frac{\varepsilon(1-\gamma)^2}{Q(1-v)}\right) \Phi(\nabla u) + (1-\delta)\frac{\omega(1-v)}{4\varepsilon}\right) \mathrm{d}x \\ &\geq \int_{A_\delta} \left(\varphi\left(\frac{\varepsilon}{\omega(1-v)}\frac{(1-\gamma)^2}{\gamma^2}\right) \Phi(\nabla u) + (1-\delta)\frac{\omega(1-v)}{4\varepsilon}\right) \mathrm{d}x \\ &\geq (1-\delta)\int_{A_\delta} h_{1/\gamma - 1}(\nabla u)\,\mathrm{d}x = (1-\delta)\int_{A_\delta} h_{\bar{\varsigma}_\gamma}(\nabla u)\,\mathrm{d}x.\end{aligned}$$

Therefore, we deduce that for $\delta \in (0, \delta_\gamma)$

$$\mathcal{F}_\varepsilon(u, v, A) \geq (1-\delta)\int_{A_\delta} h_{\bar{\varsigma}_\gamma}(\nabla u)\,\mathrm{d}x + \delta^{1/2}\int_A |\Psi'(v)|\,\mathrm{d}x\,. \tag{3.1.35}$$

To conclude we follow closely the argument in [CFI24a, Lemma 4.4]. We observe that $\Psi$ is strictly increasing, and in particular $\Psi$ is bijective. By the coarea formula,

$$\int_A |\Psi'(v)|\,\mathrm{d}x = \int_0^{\Psi(1)} \mathcal{H}^0(A \cap \partial^*\{\Psi(v) > t\})\mathrm{d}t.$$

Therefore there is $t_\delta \in (\Psi(1-\delta), \Psi(1-\delta^2))$ such that

$$(\Psi(1-\delta^2) - \Psi(1-\delta))\mathcal{H}^0(A \cap \partial^*\{\Psi(v) > t_\delta\}) \leq \int_A |\Psi'(v)|\,\mathrm{d}x.$$

We define $u^\delta := u\chi_{\{\Psi(v)>t_\delta\}\cap A}$ (dropping the dependence on $A$ from $u^\delta$). As $u \in L^\infty(O, \mathbb{R}^N)$, then $u^\delta \in SBV(A, \mathbb{R}^N)$. In particular, for all $\varsigma$ we obtain either by (3.1.33) or by (3.1.35)

$$\begin{aligned}\mathcal{F}_\varepsilon(u, v, A)+&h_{\bar{\varsigma}_\gamma}(0)\mathcal{L}^n(\{v \leq 1-\delta\}) \\ &\geq (1-\delta)\int_A h_{\bar{\varsigma}_\gamma}(|\nabla u^\delta|)\,\mathrm{d}x + \delta^{1/2}(\Psi(1-\delta^2) - \Psi(1-\delta))\mathcal{H}^0(A \cap J_{\tilde{u}}),\end{aligned}$$

and defining $\zeta_1(\delta) := 1-\delta$, $\zeta_2(\delta) := \delta^{1/2}(\Psi(1-\delta^2) - \Psi(1-\delta))$ and $s_\delta := \Psi^{-1}(t_\delta)$ we deduce both (3.1.29) and (3.1.28).

We also remark that $\|u^\delta - u\|_{L^1(A)} \leq \|u\|_{L^1(\{v \leq \Psi^{-1}(t_\delta)\})}$, hence, if the sequence $u_\varepsilon$ is equi-integrable and $v_\varepsilon \to 1$ in measure on $A$, we obtain that $u_\varepsilon - u_\varepsilon^\delta \to 0$ in $L^1(A, \mathbb{R}^N)$ for every $\delta \in (0, 1)$. $\square$

**Remark 3.1.8.** *We point out that Lemma 3.1.7 also holds under the slightly weaker assumptions (Hp 1') and (Hp 2') introduced in Section 3.2 for the analysis of the one-dimensional scalar case.*

To conclude the section, we recall a partial relaxation result from [CFI24a, Proposition 4.2], adapted to the present framework.

**Proposition 3.1.9.** *Let $\zeta \in (0, \infty)$, and $\vartheta : \mathbb{R}^{N\times n} \to [0, \infty)$ be Borel, quasi-convex map such that for some constant $c > 0$*

$$0 \leq \vartheta(\xi) \leq c(1 + |\xi|) \quad \text{for all } t \in \mathbb{R}^{N\times n}\,.$$

*Define* $F : L^1(O,\mathbb{R}^N) \times \mathcal{A}(O) \to [0,\infty)$ *by*

$$F(w,A) := \int_A \vartheta(\nabla w)\,\mathrm{d}x + \zeta\,\mathcal{H}^{n-1}(A\cap J_w)$$

*if* $w \in SBV(O,\mathbb{R}^N)$, *and* $\infty$ *otherwise. Then, for any* $w_j, w \in BV\cap L^\infty(O,\mathbb{R}^N)$, *with* $w_j \to w$ *in* $L^1(A,\mathbb{R}^N)$, *we have*

$$\liminf_{j\to\infty} F(w_j,A) \geq \int_A \vartheta(\nabla w)\,\mathrm{d}x + \int_A \vartheta^\infty(\mathrm{d}D^c u)\,,$$

*where* $\vartheta^\infty$ *is the recession function of* $\vartheta$.

## 3.2 The 1-dimensional scalar case

We start the analysis with the one-dimensional scalar case, assuming $n = N = 1$ and letting $O \subset \mathbb{R}$ be a bounded open interval. Since, in this specific framework, the anisotropic formulation does not provide any additional structural insight relevant to the applications under consideration, we restrict our attention to the isotropic setting $\Phi(t) = t^2$ throughout this section. In particular, the phase-field functionals $\mathcal{F}_\varepsilon : L^1(O,\mathbb{R}^2)\times\mathcal{A}(O) \to [0,\infty]$ are given by

$$\mathcal{F}_\varepsilon(u,v,A) = \begin{cases} \int_A \left(\varphi(\varepsilon f^2(v))|u'|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|v'|^2\right)\mathrm{d}x, & \text{if } (u,v)\in H^1(O,\mathbb{R}\times[0,1]) \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.1}$$

In this case, thanks to (3.1.13) and (3.1.19), for every $\overline{\varsigma} \in [0,\infty]$ the functionals $\mathscr{F}_{\overline{\varsigma}} : L^1(O)\times\mathcal{A}(O) \to [0,\infty]$ and $F_{\overline{\varsigma}} : L^1(O,\mathbb{R}^2) \to [0,\infty]$, defined respectively in (3.1.8) and (3.1.15), take the following explicit form:

$$\mathscr{F}_{\overline{\varsigma}}(u,A) := \begin{cases} \displaystyle\int_A h^{**}_{\overline{\varsigma}}(|u'|)\,\mathrm{d}x + (\varphi'(0^+))^{1/2}\overline{\varsigma}|D^c u|(A) + \int_{J_u\cap A} g_{\mathrm{scal}}(|[u]|)\mathrm{d}\mathcal{H}^0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in GBV(O) \\ \infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; \text{otherwise} \end{cases} \tag{3.2.2}$$

and

$$F_{\overline{\varsigma}}(u,v) := \begin{cases} \mathscr{F}_{\overline{\varsigma}}(u,O) & \text{if } v = 1 \ \mathcal{L}^n\text{-a.e. on } O \\ \infty & \text{otherwise.} \end{cases} \tag{3.2.3}$$

In (3.2.2) $h^{**}_{\overline{\varsigma}}$ denote the convex envelope of $h_{\overline{\varsigma}}$ (cf. (3.1.9) and Proposition 3.1.3), while $g_{\mathrm{scal}} : [0,\infty) \to \mathbb{R}$ is given by

$$g_{\mathrm{scal}}(s) := \inf_{(\gamma,\beta)\in\mathfrak{U}_s} \underbrace{\int_0^1 \left(\theta(\beta)|\gamma'|^2 + \omega(1-\beta)|\beta'|^2\right)^{1/2}\mathrm{d}x}_{\mathcal{G}(\gamma,\beta)}\,, \tag{3.2.4}$$

where $\mathfrak{U}_s$ is defined in (3.2.8) and $\theta(t) = \varphi'(0)l(t)\frac{\omega(1-t)}{Q(1-t)}$. We recall that throughout the thesis we adopt the convention $0\cdot\infty = 0$. Therefore, if $\mathscr{F}_\infty(u,O) < \infty$ then $|D^c u|(O) = 0$ (cf. (3.1.8)), and thus we conclude that $u \in GSBV(O)$ with $u' \in L^2(O)$.

In this section, following the approach of [CFI16, CFI24a], we prove the one–dimensional scalar case of Theorem 3.1.1. We actually carry out the proof under slightly weaker assumptions than those required in the vectorial framework (Hp 1)–(Hp 5) (cf. Section 3.1). More precisely, we shall use the following assumptions:

(Hp 1′) $l, Q \in C^0([0,1],[0,\infty))$ with $l(1)=1$, $l^{-1}(\{0\}) = Q^{-1}(\{0\}) = \{0\}$; the function $Q$ is nondecreasing in a right neighbourhood of the origin, and the map $[0,1) \ni t \mapsto f(t)$ is nondecreasing in a left neighbourhood of 1 (no assumption is made on the monotonicity of $f$ near 0);

(Hp 2′) $\omega \in C^0([0,1],[0,\infty))$ with $\omega^{-1}(\{0\}) = \{0\}$ (no monotonicity assumption on $\omega$), and the following limit exists:

$$\lim_{t\to 0^+} \left(\frac{\omega(t)}{Q(t)}\right)^{1/2} =: \overline{\varsigma} \in [0,\infty]; \tag{3.2.5}$$

and, in the main results of this section, we further assume (as in Section 3.1):

(Hp 3′) $\varphi$ is continuous, increasing, and bounded, with $\varphi(\infty) := \lim_{t\to\infty} \varphi(t) \in (0,\infty)$;

(Hp 4′) $\varphi^{-1}(\{0\}) = \{0\}$, and $\varphi$ is (right) differentiable at 0, with $\varphi'(0) \in (0,\infty)$.

The main result of the section can then be stated as follows.

**Theorem 3.2.1.** *Assume (Hp 1′)-(Hp 4′), with $\overline{\varsigma} \in (0,\infty]$ (cf. (3.2.5)). Then, for every $(u,v) \in L^1(O;\mathbb{R}^2)$, we have*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) = F_{\overline{\varsigma}}(u,v),$$

*where $\mathcal{F}_\varepsilon$ and $F_{\overline{\varsigma}}$ are defined in (3.2.1) and (3.2.3), respectively.*

For this analysis, we point out that the surface energy densities $g_{\mathrm{scal}}$ of the limit functional $\mathscr{F}_{\overline{\varsigma}}$ can have either a linear or a superlinear behaviour for small jump amplitudes, and the difference between the two cases being encoded by the finiteness or otherwise of the value of the limit $\overline{\varsigma}$ (cf. (3.1.2)). The approximation of superlinear surface energy densities slightly departs from the analysis in [CFI16, Section 7.2], there the numerator $l$ of the degradation function is $\varepsilon$ dependent in contrast to our approach, and is closer to the approach in [CFI25].

We recall that another advantage of the introduction in the model of the function $\varphi$ is that a simple change of the scaling in the functionals $\mathcal{F}_\varepsilon$ provide a family of energies including those originally defined by Ambrosio and Tortorelli in [AT92] to approximate the Mumford-Shah energy. Therefore, we provide a unifying phase-field model for the approximation of Griffith energy in brittle fracture ([BFM08]) and cohesive zone models. Indeed, let $\gamma_\varepsilon > 0$ with ${}^{\gamma_\varepsilon}/_{\varepsilon} \to \infty$ as $\varepsilon \to 0^+$, and consider the functionals $\widetilde{\mathcal{F}}_\varepsilon : L^1(O,\mathbb{R}^2) \times \mathcal{A}(O) \to [0,\infty]$ defined by

$$\widetilde{\mathcal{F}}_\varepsilon(u,v,A) := \int_A \left(\varphi(\gamma_\varepsilon f^2(v))|u'|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|v'|^2\right) \mathrm{d}x \tag{3.2.6}$$

if $(u,v) \in H^1(O,\mathbb{R}\times[0,1])$ and $\infty$ otherwise. As a consequence of Theorem 3.1.1 we obtain the following result.

**Theorem 3.2.2.** *Assume the hypothesis of Theorem 3.2.1 and that ${}^{\gamma_\varepsilon}/_{\varepsilon} \to \infty$ as $\varepsilon \to 0^+$. Let $\widetilde{\mathcal{F}}_\varepsilon$ be the functional defined in (3.2.6). Then, for all $(u,v) \in L^1(O,\mathbb{R}^2)$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \widetilde{\mathcal{F}}_\varepsilon(u,v) = \widetilde{F}(u,v)\,,$$

*where,* $\widetilde{F} : L^1(O,\mathbb{R}^2) \times \mathcal{A}(O) \to [0,\infty]$ *is defined by*

$$\widetilde{F}(u,v,A) := \begin{cases} \varphi(\infty) \displaystyle\int_A |u'|^2 \,\mathrm{d}x + 2\Psi(1)\mathcal{H}^0(J_u \cap A) & \\ & \text{if } u \in SBV(O),\ v = 1\ \mathcal{L}^1\text{-a.e. on } O \\ \infty & \text{otherwise} \end{cases} \cdot \tag{3.2.7}$$

*where* $\Psi$ *is defined in* (3.1.27).

Finally, the role of the assumptions $\{\varphi(\infty), \varphi'(0^+)\} \subset (0,\infty)$ are analysed in Section 3.2.4 by discussing the complementary cases.

### 3.2.1 Properties of the surface energy densities

For the remainder of this section, we shall denote by $g$ the scalar surface density $g_{\text{scal}}$ introduced in (3.2.4). We collect below the main structural properties of $g$, which capture the cohesive nature of the limit functional $\mathscr{F}_{\overline{\varsigma}}$ and will play a key role in Chapter 4, where the cohesive law is reconstructed.

To this purpose, for every $s \in [0,\infty)$ and for every $T_1 < T_2$, we define

$$\begin{aligned} \mathfrak{U}_s(T_1,T_2) := \Big\{(\gamma,\beta) \in H^1((T_1,T_2);\mathbb{R}^2) : \gamma(T_1) = 0,\ \gamma(T_2) = s, \\ 0 \le \beta \le 1,\ \beta(T_1) = \beta(T_2) = 1\Big\}, \end{aligned} \tag{3.2.8}$$

and we set $\mathfrak{U}_s := \mathfrak{U}_s(0,1)$. With this notation, for every $s \in [0,\infty)$ we can write

$$g(s) = \inf_{(\gamma,\beta)\in\mathfrak{U}_s} \int_0^1 \Big(\theta(\beta)|\gamma'|^2 + \omega(1-\beta)|\beta'|^2\Big)^{1/2} \mathrm{d}x, \tag{3.2.9}$$

where

$$\theta(t) := \varphi'(0)\, l(t)\, \frac{\omega(1-t)}{Q(1-t)}, \qquad t \in [0,1). \tag{3.2.10}$$

We point out that the function $\theta$ can be continuously extended to $[0,1]$ whenever $\overline{\varsigma} \in (0,\infty)$, otherwise, if $\overline{\varsigma} = \infty$, then $\lim_{t\to 1^-} \theta(t) = \infty$. We also observe that, if $g_1$ is defined as the function $g$ in (3.2.9), but with $\varphi'(0^+)$ replaced by 1 in the definition of $\theta$, then we have

$$g(s) = g_1\big((\varphi'(0^+))^{1/2} s\big) \quad \text{for every } s \in [0,\infty). \tag{3.2.11}$$

We introduce the auxiliary function $\widehat{g} : [0,\infty) \to \mathbb{R}$

$$\widehat{g}(s) := \inf_{x\in(0,1]} \Big\{2\big(\Psi(1) - \Psi(1-x)\big) + \theta^{1/2}(1-x)\, s\Big\}, \tag{3.2.12}$$

where $\Psi$ is defined in (3.1.27), which will be used later as a convenient tool to describe the behaviour of the surface energy density $g$. The following lemma collects some elementary properties of $\widehat{g}$.

**Lemma 3.2.3.** *Let* $\widehat{g}$ *be the function defined in* (3.2.12)*. Then, under the assumptions of Theorem 3.2.1,* *the function $\widehat{g}$ is continuous, concave and non-decreasing with $\widehat{g}(0) = 0$, and it satisfies*

$$g(s) \le \widehat{g}(s) \le \Big(\lim_{t\to 1^-} \theta^{1/2}(t)\Big) s \wedge 2\Psi(1) \qquad \textit{for all } s \in [0,\infty) \tag{3.2.13}$$

*and*

$$2^{-1/2} \le \liminf_{s\to 0^+} \frac{g(s)}{\widehat{g}(s)} \le \limsup_{s\to 0^+} \frac{g(s)}{\widehat{g}(s)} \le 1\,. \tag{3.2.14}$$

*Proof.* We prove only the first inequality in (3.2.13) and the formula in (3.2.14), as the remaining properties follow directly from the definition of $\widehat{g}$ in (3.2.12).

Fix $s \in (0,\infty)$ and consider the functions $\gamma$ and $\beta$ defined as follows: $\gamma = 0$ on $[0, 1/3]$, $\gamma = s$ on $[2/3, 1]$, and $\gamma$ is linearly interpolated between 0 and $s$ on $[1/3, 2/3]$; $\beta = 1-\delta$ on $[1/3, 2/3]$ and is linearly interpolated between $\delta$ and 1 on $[0, 1/3] \cup [2/3, 1]$, where $\delta \in (0,1]$ is arbitrary. In particular, $(\gamma,\beta) \in \mathfrak{U}_s$, and by the subadditivity of the square root we obtain

$$g(s) \le 2\big(\Psi(1) - \Psi(1-\delta)\big) + \theta^{1/2}(1-\delta)s,$$

which yields the desired inequality.

To prove (3.2.14), we first observe that the second inequality follows directly from (3.2.13). For the second one let $\{s_j\}_{j\in\mathbb{N}}$ be an infinitesimal sequence realizing the inferior limit in (3.2.14), set $\lambda_j = o(\widehat{g}(s_j)) \ge 0$ as $j\to\infty$, and $(\gamma_j, \beta_j) \in \mathfrak{U}_{s_j}$ (cf. (3.2.8)) be competitors such that

$$\int_0^1 \big(\theta(\beta_j)|\gamma_j'|^2 + \omega(1-\beta_j)|\beta_j'|^2\big)^{1/2}\,\mathrm{d}x \le g(s_j) + \lambda_j\,.$$

Then defining $\xi_j(x) := \theta^{1/2}(\beta_j(x))$, and using the concavity of the square root we have

$$\begin{aligned} g(s_j) + \lambda_j &\ge 2^{-1/2} \int_0^1 \big(\xi_j|\gamma_j'| + |(\Psi(\beta_j))'|\big)\,\mathrm{d}x \\ &\ge 2^{-1/2}\Big(\xi_j(x_j)\, s_j + 2(\Psi(1) - \min_{[0,1]} \Psi(\beta_j))\Big) \\ &\ge 2^{-1/2}\Big(\theta^{1/2}(\beta_j(x_j))\, s_j + 2(\Psi(1) - \Psi(\beta_j(x_j))\Big) \ge 2^{-1/2}\widehat{g}(s_j)\,, \end{aligned}$$

where $x_j \in [0,1]$ denotes an absolute minimum point of $\xi_j$. Then the first inequality in (3.2.14) follows at once being $\lambda_j = o(\widehat{g}(s_j))$ as $j\to\infty$. $\square$

We divide the remainder of this section into two parts. First, we study the linear growth regime of $g$ for small jump amplitudes, corresponding to $\varsigma \in (0,\infty)$. Next, we address the superlinear regime, which occurs when $\varsigma = \infty$.

**Case $\bar{\varsigma} \in (0,\infty)$**

In the next proposition, following the approach of [CFI16, BCI21, BI24], we collect the main properties of $g$ in the linear case.

**Proposition 3.2.4.** *Under the assumptions of Theorem 3.2.1 with $\bar{\varsigma} \in (0,\infty)$, the function $g$ defined in (3.2.9) enjoys the following properties:*

*(i)* $g(0)=0$ *and* $g$ *is subadditive;*

*(ii)* $g$ *is non-decreasing,* $g(s)\le(\varphi'(0^+))^{1/2}\,\overline{\varsigma}s\wedge 2\Psi(1)$*,* $g$ *is Lipschitz continuous with Lipschitz constant equal to* $(\varphi'(0^+))^{1/2}\,\overline{\varsigma}$*;*

*(iii) the ensuing limit exists and*

$$\lim_{s\to\infty} g(s)=2\Psi(1); \tag{3.2.15}$$

*(iv) the ensuing limit exists and*

$$\lim_{s\to 0}\frac{g(s)}{s}=(\varphi'(0^+))^{1/2}\,\overline{\varsigma}. \tag{3.2.16}$$

*(v) the following alternative representation for* $g$ *holds*

$$g(s)=\lim_{T\to 0}\inf_{(\gamma,\beta)\in\mathfrak{U}_s(0,T)}\int_0^T\left(\varphi'(0^+)f^2(\beta)|\gamma'|^2+\frac{\omega(1-\beta)}{4}+|\beta'|^2\right)\mathrm{d}x \tag{3.2.17}$$

*(vi) For all* $\eta\in[0,1]$ *and* $s\in[0,\infty)$

$$0\le g(s)-g_\eta(s)\le 2\lambda_0(\eta)\eta \tag{3.2.18}$$

*where* $\lambda_0(s)=\max_{t\in[0,s]}\omega^{1/2}(t)$*, and* $g_\eta:[0,\infty)\to[0,\infty)$ *is defined by*

$$g_\eta(s):=\inf_{(\gamma,\beta)\in\mathfrak{U}_s^\eta}\int_0^1\left(\theta(\beta)|\gamma'|^2+\omega(1-\beta)|\beta'|^2\right)^{1/2}\mathrm{d}x \tag{3.2.19}$$

*where*

$$\mathfrak{U}_s^\eta:=\{(\gamma,\beta)\in H^1((0,1),\mathbb{R}\times[0,1]):\gamma(0)=0\,,\gamma(1)=s\,,\text{ and }\beta(0),\beta(1)\ge 1-\eta\} \tag{3.2.20}$$

*Proof. Item* $(i)$. The equality $g(0)=0$ follows from Lemma 3.2.3.

In order to prove that $g$ is subadditive we fix $s_1,s_2\in(0,\infty)$ and we consider the minimum problems for $g(s_1)$ and $g(s_2)$, respectively. Let $\eta>0$ and let $(\gamma_1,\beta_1),(\gamma_2,\beta_2)$ be admissible couples respectively for $g(s_1)$ and $g(s_2)$ such that for $i=1,2$

$$\int_0^1\left(\theta(\beta_i)|\gamma_i'|^2+\omega(1-\beta_i)|\beta_i'|^2\right)^{1/2}\mathrm{d}t<g(s_i)+\eta. \tag{3.2.21}$$

Next define $\gamma:=\gamma_1$ in $[0,1]$, $\gamma:=\gamma_2(\cdot-1)+s_1$ in $[1,2]$, $\beta:=\beta_1$ in $[0,1]$, and $\beta:=\beta_2(\cdot-1)$ in $[1,2]$). An immediate computation and the invarince under reparametrization entail the subadditivity of $g$ since $\eta$ is arbitrary.

*Item* $(ii)$. The estimate $g(s)\le(\varphi'(0^+))^{1/2}\,\overline{\varsigma}s\wedge 2\Psi(1)$ follows from (3.2.13), the definition of $\theta$ in (3.2.10) and $\overline{\varsigma}\in(0,\infty)$.

In order to prove that $g$ is nondecreasing we fix $s_1,s_2$ with $s_1<s_2$ and $\eta>0$, and we consider $(\alpha,\beta)$ satisfying a condition analogous to (3.2.21) for $g(s_2)$. Then $(\frac{s_1}{s_2}\alpha,\beta)$ is admissible for $g(s_1)$, thus we infer

$$g(s_1)\le\int_0^1\left(\left(\frac{s_1}{s_2}\right)^2\theta(\beta)|\gamma'|^2+\omega(1-\beta)|\beta'|^2\right)^{1/2}\mathrm{d}t<g(s_2)+\eta,$$

since $s_1/s_2 < 1$. As $\eta \to 0$ we find $g(s_1) \leq g(s_2)$. The Lipschitz continuity of $g$ is an obvious consequence of the facts that $g$ is nondecreasing, subadditive and $g(s) \leq (\varphi'(0^+))^{1/2}\,\overline{\varsigma}s \wedge 2\Psi(1)$ for $s \geq 0$.

*Item (iii)*. Let $(s_j)_{j\in\mathbb{N}}$, be a diverging sequence and let $(\gamma_j, \beta_j)$ be an admissible couple for $g(s_k)$ such that

$$\int_0^1 \left(\theta(\beta_j)|\gamma_j'|^2 + \omega(1-\beta_j)|\beta_j'|^2\right)^{1/2} \mathrm{d}t < g(s_j) + \frac{1}{j}. \tag{3.2.22}$$

If $\inf_{(0,1)} \beta_j \geq \delta$ for some $\delta > 0$ and for every $j$, then, by (3.2.10), there exists a constant $c(\delta) > 0$ such that $\theta(\beta_j) > c(\delta)$. Therefore by (3.2.22) one finds

$$c(\delta)s_j \leq g(s_j) + \frac{1}{j},$$

so that $g(s_j) \to \infty$ as $j \to \infty$ and this contradicts the fact that $g \leq 2\Psi(1)$. Therefore there exists a sequence $x_j \in (0,1)$ such that $\beta_j(x_j) \to 0$ up to subsequences. Since we have already shown that $g \leq 2\Psi(1)$, we conclude the proof of (3.2.15) noticing that (3.2.22) and the definitionof $\Psi$ in (3.1.27) yield

$$2(\Psi(1) - \Psi(\beta_j(x_j))) \leq \int_0^{x_j} \omega^{1/2}(1-\beta_j)|\beta_j'|\mathrm{d}t + \int_{x_j}^1 \omega^{1/2}(1-\beta_j)|\beta_j'|\mathrm{d}t \leq g(s_j) + \frac{1}{j}. \tag{3.2.23}$$

*Item (iv)*. Let $(s_j)_{j\in\mathbb{N}}$ be an infinitesimal sequence and let $(\gamma_j, \alpha_j)$ be an admissible couple for $g(s_j)$ satisfying (3.2.22) with $s_j/j$ in place of $1/j$. If there exists $\delta \in (0,1)$, a not relabeled subsequence of $j$, and a sequence $x_j \in [0,1]$ such that $\beta_j(x_j) < 1-\delta$, then the same computation as in (3.2.23) leads to

$$0 < 2(\Psi(1) - \Psi(1-\delta)) \leq g(s_j) + \frac{s_j}{j}.$$

As $j \to \infty$ this contradicts the fact that $g(s) \leq (\varphi'(0^+))^{1/2}\overline{\varsigma}s$. Therefore, $\beta_j$ converges uniformly to 1 and fixing $\delta > 0$

$$(\varphi'(0^+))^{1/2}\overline{\varsigma}s_j \leq \int_0^1 \theta^{1/2}(\beta_j)|\gamma_j'|\mathrm{d}t \leq g(s_j) + \frac{s_j}{j}$$

holds for $j$ large by (3.2.10) and $\overline{\varsigma} \in (0,\infty)$. Formula (3.2.16) immediately follows dividing both sides of the last inequality by $s_j$, taking first $j \to \infty$ and then $\delta \to 0$, and using the fact that $g(s) \leq (\varphi'(0^+))^{1/2}\overline{\varsigma}s$, for $s \geq 0$.

*Item (v)*. Let $s \in (0,\infty)$ and $(\gamma_0, \beta_0) \in \mathfrak{U}_s(0,T_0)$ with $T_0 > 0$. By Cauchy inequality and (3.2.10) we obtain

$$\left(\theta(\beta_0)|\gamma_0'|^2 + \omega(1-\beta_0)|\beta_0'|^2\right)^{1/2} \leq \varphi'(0^+)f^2(\beta_0)|\gamma_0'|^2 + \frac{\omega(1-\beta_0)}{4} + |\beta_0'|^2$$

and integrating

$$\int_0^{T_0} \left(\theta(\beta_0)|\gamma_0'|^2 + \omega(1-\beta_0)|\beta_0'|^2\right)^{1/2} \mathrm{d}t \leq \int_0^{T_0} \left(\varphi'(0^+)f^2(\beta_0)|\gamma_0'|^2 + \frac{\omega(1-\beta_0)}{4} + |\beta_0'|^2\right)\mathrm{d}t\,.$$

Since the first integral is one-homogeneous with respect to the derivatives, we may reparametrize the interval $(0,T_0)$ onto $(0,1)$. Taking the infimum over all admissible $(\gamma,\beta)$ and $T$, we obtain $g(s) \leq \overline{g}(s)$, where

$$\overline{g}(s) := \lim_{T\to 0} \inf_{(\gamma,\beta)\in\mathfrak{U}_s(0,T)} \int_0^T \left(\varphi'(0^+)f^2(\beta)|\gamma'|^2 + \frac{\omega(1-\beta)}{4} + |\beta'|^2\right)\mathrm{d}x. \tag{3.2.24}$$

To prove the converse inequality $\overline{g} \le g$, we first remark that in the minimization problem defining $g$, the functions $\gamma$ and $\beta$ can be assumed to belong to $W^{1,\infty}\big((0,1)\big)$. Fix a small parameter $\eta > 0$ and consider competitors $\gamma, \beta \in W^{1,\infty}\big((0,1)\big)$ admissible for $g(s)$ and satisfying (3.2.9). For every $t \in [0,1]$, we define

$$\beta_\eta(t) := \beta(t) \wedge (1-\eta) \quad \text{and} \quad \psi_\eta(t) := \int_0^t \frac{2}{\omega^{1/2}(1-\beta_\eta)} \big(\eta + \varphi'(0^+) f^2(\beta_\eta)|\gamma'|^2 + |\beta_\eta'|^2\big)^{1/2} \mathrm{d}\tau\,.$$

The function $\psi_\eta : [0,1] \to [0, M_\eta]$ is bilipschitz, where $M_\eta := \psi_\eta(1)$. We define $\tilde\gamma_\eta, \tilde\beta_\eta \in W^{1,\infty}\big((0,M_\eta)\big)$ by

$$\tilde\gamma_\eta := \gamma \circ \psi_\eta^{-1} \text{ and } \tilde\beta_\eta := \beta_\eta \circ \psi_\eta^{-1}\,.$$

We compute, using the definition and the change of variables $x = \psi_\eta(t)$,

$$\begin{aligned}
\int_0^{M_\eta} \frac{\omega(1-\tilde\beta_\eta)}{4}\,\mathrm{d}x &= \int_0^{M_\eta} \frac{\omega(1-\beta_\eta(\psi_\eta^{-1}(x)))}{4}\,\mathrm{d}x = \int_0^1 \frac{\omega(1-\beta_\eta(t))}{4}\psi_\eta'(t)\mathrm{d}t \\
&= \int_0^1 \frac{\omega^{1/2}(1-\beta_\eta)}{2}\big(\eta + \varphi'(0^+) f^2(\beta_\eta)\,|\gamma'|^2 + |\beta_\eta'|^2\big)\mathrm{d}t \\
&\le c\eta^{1/2} + \int_0^1 \frac{\omega^{1/2}(1-\beta_\eta)}{2}\big(\varphi'(0^+) f^2(\beta_\eta)\,|\gamma'|^2 + |\beta_\eta'|^2\big)^{1/2}\mathrm{d}t \\
&= c\eta^{1/2} + \frac{1}{2}\int_0^1 \big(\theta(\beta_\eta)|\gamma'|^2 + \omega(1-\beta_\eta)|\beta_\eta'|^2\big)^{1/2}\mathrm{d}t,
\end{aligned}$$

where we used the subadditivity of the square root. Analogously,

$$\begin{aligned}
\int_0^{M_\eta} \big(\varphi'(0^+) f^2(\tilde\beta_\eta)|\tilde\gamma_\eta'|^2 + |\tilde\beta_\eta'|^2\big)\,\mathrm{d}x &= \int_0^1 \big(\varphi'(0) f^2(\beta_\eta)|\gamma'|^2 + |(\beta_\eta)'|^2\big)\frac{1}{\psi_\eta'}\mathrm{d}t \\
&= \int_0^1 \big(\varphi'(0^+) f^2(\beta_\eta)|\gamma'|^2 + |(\beta_\eta)'|^2\big)\frac{\omega^{1/2}(1-\beta_\eta)}{2\sqrt{\eta + \varphi'(0^+) f^2(\beta_\eta)\,|\gamma'|^2 + |\beta_\eta'|^2}}\mathrm{d}t \\
&\le \int_0^1 \frac{\omega^{1/2}(1-\beta_\eta)}{2}\big(\varphi'(0^+) f^2(\beta_\eta)\,|\gamma'|^2 + |\beta_\eta'|^2\big)^{1/2}\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 \big(\theta(\beta_\eta)|\gamma'|^2 + \omega(1-\beta_\eta)|\beta_\eta'|^2\big)^{1/2}\mathrm{d}t.
\end{aligned}$$

We extend $\tilde\gamma_\eta$ and $\tilde\beta_\eta$ to $(-1, M_\eta+1)$ by setting $\tilde\gamma_\eta = 0$ on $(-1,0)$, $\tilde\gamma_\eta = s$ on $(M_\eta, M_\eta+1)$, and defining $\tilde\beta_\eta$ as the linear interpolation between $1-\eta$ and $1$ on each of these intervals, so that the extended functions satisfy the prescribed boundary conditions for $\overline{g}$ on the enlarged domain. Gathering all the contributions, we obtain

$$\begin{aligned}
\overline{g}(s) &\le \int_{-1}^{M_\eta+1} \Big(\varphi'(0^+) f^2(\tilde\beta_\eta)|\tilde\gamma_\eta'|^2 + \frac{\omega(1-\tilde\beta_\eta)}{4} + |\tilde\beta_\eta'|^2\Big)\,\mathrm{d}x \\
&\le c(\eta^{1/2} + \eta + \eta^2) + \int_0^1 \big(\theta(\beta_\eta)|\gamma'|^2 + \omega(1-\beta_\eta)|\beta_\eta'|^2\big)^{1/2}\mathrm{d}t\,. \qquad (3.2.25)
\end{aligned}$$

It remains to replace $\beta_\eta$ by $\beta$ in the last integral. We observe that $\beta_\eta' = 0$ almost everywhere on the set where $\beta \neq \beta_\eta$ (which coincides with the set $\{\beta > 1-\eta\}$). Therefore

$$\begin{aligned}\int_{\{\beta\neq\beta_\eta\}} \left(\theta(\beta_\eta)|\gamma'|^2 + \omega(1-\beta_\eta)|\beta_\eta'|^2\right)^{1/2}\mathrm{d}t &= \int_{\{\beta\neq\beta^\eta\}} \theta(\beta_\eta)|\gamma'|^2\mathrm{d}t \\ &\leq \int_{\{\beta\neq\beta^\eta\}} \theta(\beta)|\gamma'|\mathrm{d}t + \vartheta(\eta)\int_0^1 |\gamma'|\mathrm{d}t\end{aligned}$$

where $\vartheta$ is the continuity modulus of $\theta(t)$ near $t=1$, and therefore

$$\overline{g}(s) \leq c(\eta^{1/2} + \eta + \eta^2) + \vartheta(\eta)\int_0^1 |\gamma'|\mathrm{d}t + \int_0^1 \left(\theta(\beta)|\gamma'|^2 + \omega(1-\beta)|\beta'|^2\right)^{1/2}\mathrm{d}t\,.$$

Since the last integral is bounded above by $g(s)+\eta$, and $\eta$ can be made arbitrarily small, the proof is complete.

*Item (vi).* We now consider the minimization problems defining $g$ and $g_\eta$ on the intervals $(-1,2)$ and $(0,1)$, respectively. Let $(\gamma_\eta, \beta_\eta)$ be an admissible pair for $g_\eta(s)$, and define $\gamma := 0$ on $(-1,0)$, $\gamma := \gamma_\eta$ on $(0,1)$, and $\gamma := s$ on $(1,2)$. Set $\beta := \beta_\eta$ on $(0,1)$, and extend it linearly to 1 over $(-1,0)$ and $(1,2)$. A straightforward computation then gives

$$g(s) \leq \int_0^1 \left(\theta(\beta_\eta)|\gamma_\eta'|^2 + \omega(1-\beta_\eta)|\beta_\eta'|^2\right)^{1/2}\mathrm{d}t + 2\lambda_0(\eta)\eta,$$

where $\lambda_0(\eta) := \max_{t\in[0,\eta]}\omega^{1/2}(t)$. Taking the infimum over all admissible pairs $(\gamma_\eta, \beta_\eta)$ yields

$$g(s) \leq g_\eta(s) + 2\lambda_0(\eta)\eta.$$

Exchanging the roles of $g$ and $g_\eta$ completes the proof.

□

**Case $\overline{\varsigma} = \infty$**

We turn next to establish the structural properties of the surface energy density $g$ in the superlinear case $\overline{\varsigma} = \infty$. The proof somewhat follows that of the case $\overline{\varsigma} \in (0,\infty)$ to which we refer in case they are an immediate adaptation of the latter. We highlight only the main changes.

**Proposition 3.2.5.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, the function $g$ defined in (3.2.9) enjoys the following properties:*

*(i)* $g(0) = 0$, *$g$ is non-decreasing, and subadditive;*

*(ii)* $g \in C^0([0,\infty))$ *and*

$$\lim_{s\to 0^+} \frac{g(s)}{s} = \infty; \tag{3.2.26}$$

*(iii)*

$$\lim_{s\to\infty} g(s) = 2\Psi(1);$$

*(iv) the alternative representation for $g$ in (3.2.17) holds.*

*Proof.* The arguments for items (i) and (iii) proceed along the same lines as those in Proposition 3.2.4.

To prove (ii), we first observe that, by the subadditivity and monotonicity of $g$, it holds that

$$|g(s_2) - g(s_1)| \leq g(|s_2 - s_1|) \leq \widehat{g}(|s_2 - s_1|) \quad \text{for every } s_1, s_2 \in [0, \infty),$$

by (3.2.13). Hence, the continuity of $g$ follows from the continuity of $\widehat{g}$, established in Lemma 3.2.3.

As for (3.2.26), in view of (3.2.14), it is enough to prove that

$$\lim_{s \to 0^+} \frac{\widehat{g}(s)}{s} = \infty.$$

To this purpose, recall that, by the definition of $\theta$ (see (3.2.10)) and under the assumption $\overline{\varsigma} = \infty$, for every $s > 0$ there exists at least one minimizer of the problem defining $\widehat{g}(s)$ in (3.2.12). Clearly, all minimizers are strictly positive when $s > 0$. Let $\zeta(s)$ denote the smallest minimizer, which exists thanks to the continuity of $\theta$ and $\Psi$. Then $\zeta(s) > 0$ for all $s > 0$, and since $\widehat{g}(s) \geq 2(\Psi(1) - \Psi(1 - \zeta(s)))$, we have $\zeta(s) \to 0^+$ as $s \to 0^+$. Therefore, to establish (3.2.26), it suffices to note that $\lim_{t \to 1^-} \theta(t) = \infty$ (as $\overline{\varsigma} = \infty$) and that

$$\frac{\widehat{g}(s)}{s} \geq \theta^{1/2}(1 - \zeta(s)) \,.$$

The proof of (iv) is essentially the same as that of item (v) in Proposition 3.2.4. The only difference arises in the final part, where we use the fact that $\theta$ admits a modulus of continuity at 1. In the present case, since $\overline{\varsigma} = \infty$, we have $\lim_{t \to 1^-} \theta(t) = \infty$. Nevertheless, we can consider a sequence $\eta_j \to 0$ such that, for every $j$,

$$\theta(1 - \eta_j) \leq \theta(t) \quad \text{for every } t \in [1 - \eta_j, 1),$$

and conclude the argument in the same way.

□

### 3.2.2 Γ-convergence and compactness

In this section we address the Γ-convergence and compactness properties of the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon > 0}$ defined in (3.2.1). We distinguish the cases $\overline{\varsigma} \in (0, \infty)$, $\overline{\varsigma} = \infty$.

**Case $\overline{\varsigma} \in (0, \infty)$**

We begin with establishing compactness and identifying the domain of the eventual Γ-limit.

**Theorem 3.2.6** (Compactness)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} \in (0, \infty)$, let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0} \in L^1(O, \mathbb{R}^2)$ be such that*

$$\sup_{\varepsilon > 0} (\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) + \|u_\varepsilon\|_{L^1(O)}) < \infty \,. \tag{3.2.27}$$

*Then there are $\{\varepsilon_k\}_{k \in \mathbb{N}}$ and $u \in L^1 \cap GBV(O)$ such that $(u_{\varepsilon_k}, v_{\varepsilon_k}) \to (u, 1)$ $\mathcal{L}^1$-a.e. on $O$. If, moreover, $(u_\varepsilon)_\varepsilon$ is equi-integrable then $u_{\varepsilon_k} \to u$ in $L^1(O)$.*

*Proof.* Let $C > 0$ be the supremum on the left hand side of (3.2.27), then

$$\int_O \omega(1 - v_\varepsilon)\,\mathrm{d}x \le C\varepsilon\,,$$

so that $v_\varepsilon \to 1$ in measure on $O$, and moreover $(u_\varepsilon, v_\varepsilon) \in H^1(O, \mathbb{R}\times[0,1])$.

We show next that there is a subsequence of $(u_\varepsilon)_\varepsilon$ converging in measure on $O$. With this aim, fix $\gamma \in (0, \overline{\varsigma}\wedge 1)$, then Lemma 3.1.7 provides $\delta_\gamma > 0$ and functions $u_{\varepsilon,\delta} := u_\varepsilon^\delta \in SBV(O)$ (according to the notation of Lemma 3.1.7) such that $H_\delta(u_{\varepsilon,\delta}) \le \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$ for every $\delta \in (0, \delta_\gamma)$, with bulk density $h_{\overline{\varsigma}_\gamma}$ where $\overline{\varsigma}_\gamma = (\overline{\varsigma} - \gamma)(1-\gamma)$ (cf. (3.1.30)). Therefore, by (3.1.20) we deduce that for some constant $C_0$ depending on $\overline{\varsigma}$, $\gamma$ and $\delta$ we have

$$\sup_\varepsilon \int_O |u'_{\varepsilon,\delta}|\,\mathrm{d}x + \mathcal{H}^0(J_{u_{\varepsilon,\delta}}) \le C_0\,.$$

Considering the truncated functions $u^M_{\varepsilon,\delta} := \tau_M(u_{\varepsilon,\delta}) \in SBV(O)$ for $M \in \mathbb{N}$ (cf. Section 2.0.1), the previous estimate and (3.2.27) yield that $\sup_{\varepsilon>0} \|u^M_{\varepsilon,\delta}\|_{BV(O)} \le C_M < \infty$. The $BV$ compactness Theorem and an elementry diagonal argument imply the existence of a subsequence $\varepsilon_k$ (independent from $M$, but depending on $\delta$) and of $u^M \in BV(O)$ such that $u^M_{\varepsilon_k,\delta} \to u^M$ in $L^1(O)$ for every $M \in \mathbb{N}$.

We recall that in the proof of Lemma 3.1.7 we have set $u_{\varepsilon,\delta} = u_\varepsilon \chi_{\{\Psi(v_\varepsilon) > t_{\varepsilon,\delta}\}}$ for some $t_{\varepsilon,\delta} \in (\Psi(1-\delta), \Psi(1-\delta^2))$, and thus $\mathcal{L}^1(\{\Psi(v_\varepsilon) \le t_{\varepsilon,\delta}\}) \to 0$ as $\varepsilon \to 0$ as $v_\varepsilon \to 1$ in measure on $O$. Thus, we infer the $(u^M_{\varepsilon_k})_k$ convergence in measure on $O$ to $u^M$, i.e.

$$\limsup_k \mathcal{L}^1(\{|u^M_{\varepsilon_k,\delta} - u^M_{\varepsilon_k}| \ge \eta\}) \le \limsup_k \mathcal{L}^1(\{\Psi(v_\varepsilon) < t_{\varepsilon,\delta}\}) = 0\,. \tag{3.2.28}$$

It is easy to check that $u^{M+1} = u^M$ if $|u^{M+1}| \le M$, therefore defining $u := \sup_{M\in\mathbb{N}} u^M$ we conclude that $u \in GBV(O)$. In addition, we have $u \in L^1(O)$ as

$$\begin{aligned} \|u\|_{L^1(O)} &\le \liminf_M \|u^M\|_{L^1(O)} \\ &\le \liminf_M (\liminf_k \|u^M_{\varepsilon_k}\|_{L^1(O)}) \le \liminf_k \|u_{\varepsilon_k}\|_{L^1(O)} \le C\,. \end{aligned} \tag{3.2.29}$$

Finally, for every $\eta > 0$ we have that

$$\begin{aligned} \mathcal{L}^1(|u - u_{\varepsilon_k}| \ge \eta) &\le \mathcal{L}^1(|u| > M) + \mathcal{L}^1(|u^M - u^M_{\varepsilon_k}| \ge \eta/3) \\ &+ \mathcal{L}^1(|u^M_{\varepsilon_k} - u_{\varepsilon_k}| \ge \eta/3) \le \mathcal{L}^1(|u^M - u^M_{\varepsilon_k}| \ge \eta/3) + 2\frac{C}{M}\,, \end{aligned}$$

where in the last inequality we have used (3.2.27) and (3.2.29). From this we conclude that $(u_{\varepsilon_k})_k$ converges in measure on $O$ to $u$ by letting first $k \to \infty$ and then $M \to \infty$.

The claimed $\mathcal{L}^1$-a.e. convergence on $O$ holds up to extracting a further subsequence. Finally, if $(u_\varepsilon)_\varepsilon$ is equi-integrable, by taking into account (3.2.28) we estimate as follows

$$\begin{aligned}\|u-u_{\varepsilon_k}\|_{L^1(O)} &\le \int_{\{|u|>M\}}|u|\,\mathrm{d}x+\|u^M-u^M_{\varepsilon_k}\|_{L^1(O)}+\|u^M_{\varepsilon_k}-u_{\varepsilon_k}\|_{L^1(O)}\\ &\le \int_{\{|u|>M\}}|u|\,\mathrm{d}x+\|u^M-u^M_{\varepsilon_k,\delta}\|_{L^1(O)}\\ &\quad+\int_{\{\Psi(v_{\varepsilon_k})\le t_{\varepsilon_k,\delta}\}}|u_{\varepsilon_k}|\,\mathrm{d}x+\int_{\{|u_{\varepsilon_k}|>M\}}|u_{\varepsilon_k}|\,\mathrm{d}x\,.\end{aligned}$$

The conclusion then follows by letting first $k\to\infty$ and then $M\to\infty$. □

Next, we show the lower bound inequality for the diffuse part, for which we follow [CFI24a, Proposition 4.2].

**Proposition 3.2.7.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma}\in(0,\infty)$, for every $(u_\varepsilon,v_\varepsilon)$, $(u,v)\in L^1(O,\mathbb{R}^2)$ with $(u_\varepsilon,v_\varepsilon)\to(u,v)$ in $L^1(O,\mathbb{R}^2)$, we have*

$$\int_A h^{**}_{\overline{\varsigma}}(|u'|)\,\mathrm{d}x+(\varphi'(0^+))^{1/2}\overline{\varsigma}|D^c u|(A)\le\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A)\,, \tag{3.2.30}$$

*for every $A\in\mathcal{A}(O)$.*

*Proof.* By superadditivity of the inferior limit it suffices to assume that $A\in\mathcal{A}(O)$ is an interval. Moreover, we can suppose the inferior limit on the right hand side of (3.2.30) to be finite, otherwise the claim is obvious. Therefore, $(u_\varepsilon,v_\varepsilon)\in H^1(A,\mathbb{R}\times[0,1])$ for $\varepsilon>0$ sufficiently small, $u\in GBV(A)$ and $v=1$ $\mathcal{L}^1$-a.e. on $A$ by Theorem 3.2.6 applied on $A$ in place of $O$. Let $M\in\mathbb{N}$ and $\gamma\in(0,1)$, using the notation and the results in Lemma 3.1.7 we find $\delta_{\gamma,M}\in(0,1)$ such that for every $\delta\in(0,\delta_{\gamma,M})$ if $u^M_{\varepsilon,\delta}:=\tau_M(u^\delta_\varepsilon)\in SBV(A)$ (where $\tau_M$ is the truncation function defined in Section 2.0.1) such that

$$H_\delta(u^M_{\varepsilon,\delta},A)\le\mathcal{F}_\varepsilon(u^M_\varepsilon,v_\varepsilon,A)\le\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A)\,.$$

Thus, by taking the inferior limit as $\varepsilon\to0$, in view of Proposition 3.1.9 we conclude that for $\overline{\varsigma}_\gamma=(\overline{\varsigma}-\gamma)(1-\gamma)$

$$\zeta_1(\delta)\int_A h^{**}_{\overline{\varsigma}_\gamma}(|(u^M)'|)\,\mathrm{d}x+\zeta_1(\delta)(\varphi'(0^+))^{1/2}\overline{\varsigma}_\gamma|D^c u^M|(A)\le\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A)\,.$$

By letting first $\delta\to0$, $\gamma\to0$ and then $M\to\infty$, we conclude (3.2.30) in view of (3.1.28), (3.1.18) and Beppo Levi's theorem. □

We establish next the lower estimate for the surface part.

**Proposition 3.2.8.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma}\in(0,\infty)$, for every $(u_\varepsilon,v_\varepsilon)$, $(u,v)\in L^1(O,\mathbb{R}^2)$ with $(u_\varepsilon,v_\varepsilon)\to(u,v)$ in $L^1(O,\mathbb{R}^2)$, we have*

$$\int_{J_u\cap A} g(|[u]|)\mathrm{d}\mathcal{H}^0\le\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A)\,, \tag{3.2.31}$$

*for every $A\in\mathcal{A}(O)$, where $g$ is the function defined in (3.1.14).*

*Proof.* By superadditivity of the inferior limit it suffices to assume that $A \in \mathcal{A}(O)$ is an interval. Moreover, we can suppose the inferior limit on the right hand side of (3.2.31) to be finite, otherwise the claim is obvious. Therefore, $(u_\varepsilon, v_\varepsilon) \in H^1(A, \mathbb{R} \times [0,1])$ for $\varepsilon > 0$ sufficiently small, $u \in GBV(A)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $A$ by Theorem 3.2.6.

We claim that it is sufficient to show that if $u \in BV(A)$ we have for every $x_0 \in J_u \cap A$

$$g(|[u](x_0)|) \leq \liminf_{r\to 0} \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, I_r(x)), \tag{3.2.32}$$

where $I_r(x) = (x - {}^r\!/_2, x + {}^r\!/_2)$. Indeed, given (3.2.32) for granted, if $u \in BV(A)$ we can find a nested sequence of finite sets $\{K_m\}_{m\in N}$ such that $\cup_{m\in\mathbb{N}} K_m = J_u \cap A$. Then, for every $m \in \mathbb{N}$, there is $r_m > 0$ such that $\{I_r(x)\}_{x\in K_m}$ are disjoint and contained in $A$ for all $r \in (0, r_m)$. In particular, by the superadditivity of the inferior limit operator and by (3.2.32), for every $m \in \mathbb{N}$ we get

$$\begin{aligned}\int_{K_m} g(|[u](x)|)\mathrm{d}\mathcal{H}^0(x) &\leq \sum_{x\in K_m} \liminf_{r\to 0} \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, I_r(x)) \\ &\leq \liminf_{r\to 0} \sum_{x\in K_m} \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, I_r(x)) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,.\end{aligned}$$

Taking the limit for $m \to \infty$ we conclude (3.2.31) for $u \in BV(A)$.

Moreover, if $u \in GBV \setminus BV(A)$ then $J_u = \cup_{M\in\mathbb{N}} J_{u^M}$ and $[u^M](x_0) = [u](x_0)$ for $M \in \mathbb{N}$ sufficiently large. Noting that $\mathcal{F}_\varepsilon(u^M_\varepsilon, v_\varepsilon, A) \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)$, we conclude (3.2.31) for $u$ thanks to (3.2.31) for $u^M$ and Beppo Levi's theorem.

To establish (3.2.32) for $u \in BV(A)$ we argue by blow up. We can restrict to a subsequence $\mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}, v_{\varepsilon_k})$ such that $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A) = \lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, v_k) < \infty$, where we have set $(u_k, v_k) := (u_{\varepsilon_k}, v_{\varepsilon_k})$. Next, consider an infinitesimal sequence of radii $\{r_j\}_{j\in\mathbb{N}}$ such that

$$\liminf_{r\to 0} \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, v_k, I_r(x_0)) = \lim_{j\to\infty} \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, v_k, I_{r_j}(x_0))\,,$$

and select a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that $\eta_j := {}^{\varepsilon_{k_j}}\!/_{r_j} < {}^1\!/_j$,

$$|\mathcal{F}_{\varepsilon_{k_j}}(u_{k_j}, v_{k_j}, I_{r_j}(x_0)) - \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, v_k, I_{r_j}(x_0))| < {}^1\!/_j$$

and

$$\|v_{k_j} - 1\|_{L^1(O)} + \|u_{k_j} - u\|_{L^1(O)} < {}^{r_j}\!/_j \tag{3.2.33}$$

for all $j \in \mathbb{N}$. Therefore, we have

$$\liminf_{r\to 0} \liminf_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k, v_k, I_r(x_0)) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_{k_j}}(u_{k_j}, v_{k_j}, I_{r_j}(x_0)). \tag{3.2.34}$$

For every $j \in \mathbb{N}$ define the pair $(\tilde{u}_j, \tilde{v}_j) \in H^1(I_1, \mathbb{R}\times[0,1])$ by $\tilde{u}_j(x) := u_{k_j}(x_0 + r_j x)$ and $\tilde{v}_j(x) := v_{k_j}(x_0 + r_j x)$ for all $x \in I_1$. A change of variable yields that

$$\mathcal{F}_{\varepsilon_{k_j}}(u_{k_j}, v_{k_j}, I_{r_j}(x_0)) = \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1) \tag{3.2.35}$$

where $\mathcal{G}_j : L^1(I_1, \mathbb{R}^2) \times \mathcal{A}(I_1) \to [0, \infty]$ is defined for every $(u, v) \in H^1(I_1, \mathbb{R} \times [0, 1])$ and $U \in \mathcal{A}(I_1)$ by

$$\mathcal{G}_j(u, v, U) = \int_U \left( \frac{\eta_j}{\varepsilon_{k_j}} \varphi(\varepsilon_{k_j} f^2(v))|u'|^2 + \frac{\omega(1-v)}{4\eta_j} + \eta_j |v'|^2 \right) \mathrm{d}x \,, \tag{3.2.36}$$

and $\infty$ otherwise. In addition, changing variables it is straightforward to check that inequality (3.2.33) implies that

$$\begin{aligned} &\limsup_{j\to\infty} (\|\tilde{u}_j - u_0\|_{L^1(I_1)} + \|\tilde{v}_j - 1\|_{L^1(I_1)}) \\ &\qquad \le \limsup_{j\to\infty} r_j^{-1} (\|u_{k_j} - u\|_{L^1(O)} + \|v_{k_j} - 1\|_{L^1(O)}) = 0 \,, \end{aligned} \tag{3.2.37}$$

where $u_0 \in BV(I_1)$ is given by $u_0(x) := u(x_0^-)\chi_{(-\frac{1}{2},0)} + u(x_0^+)\chi_{[0,\frac{1}{2})}$, and we have used that $\lim_{r\to0} \|u(x_0 + rx) - u_0(x)\|_{L^1(I_1)} = 0$.

With fixed $\delta \in (0, 1)$, for all $\varepsilon > 0$ set $\delta_\varepsilon := \sup\{t \in [0, 1) : \varepsilon f^2(t) \le \delta\}$. Recalling that $f(0) = 0$ and $f(t) \to \infty$ as $t \to 1^-$, $\delta_\varepsilon \in (0, 1)$ is actually a maximum by the continuity of $f$ on $[0, 1)$. Moreover, we have for all $t \in (\delta_\varepsilon, 1)$

$$\delta < \varepsilon f^2(t) \tag{3.2.38}$$

and actually

$$\varepsilon f^2(\delta_\varepsilon) = \delta \,. \tag{3.2.39}$$

In particular, the latter equation implies $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$.

Moreover, $f$ is increasing in $[\gamma, 1)$ for some $\gamma \in (0, 1)$ by (Hp 1). Define $M := \max_{[0,\gamma]} f$. Having fixed $\delta$, for all $\varepsilon$ sufficiently small we have $\delta/\varepsilon > M^2$, and $\delta_\varepsilon \in (\gamma, 1)$ for $\varepsilon$ small enough, thus for all $t \in [0, \delta_\varepsilon]$ we conclude that

$$\varepsilon f^2(t) \le \delta \,. \tag{3.2.40}$$

Then, for every $j \in \mathbb{N}$ define $\hat{v}_j = \tilde{v}_j \wedge \delta_j$, where $\delta_j := \delta_{\varepsilon_{k_j}}$. We have

$$\begin{aligned} \mathcal{G}_j(\tilde{u}_j, \hat{v}_j, I_1) &= \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, \{\tilde{v}_j < \delta_j\}) \\ &\quad + \int_{\{\tilde{v}_j \ge \delta_j\}} \frac{\eta_j}{\varepsilon_{k_j}} \varphi(\delta)|\tilde{u}_j'|^2 \,\mathrm{d}x + \frac{\omega(1-\delta_j)}{\eta_j} \mathcal{L}^1(\{\tilde{v}_j \ge \delta_j\}) \,. \end{aligned} \tag{3.2.41}$$

We analyze the first term in the last line of (3.2.41): we employ (3.2.38) to infer

$$\int_{\{\tilde{v}_j \ge \delta_j\}} \frac{\eta_j}{\varepsilon_{k_j}} \varphi(\delta)|\tilde{u}_j'|^2 \,\mathrm{d}x \le \int_{\{\tilde{v}_j \ge \delta_j\}} \frac{\eta_j}{\varepsilon_{k_j}} \varphi(\varepsilon_{k_j} f^2(\tilde{v}_j))|\tilde{u}_j'|^2 \,\mathrm{d}x \le \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, \{\tilde{v}_j \ge \delta_j\}). \tag{3.2.42}$$

Instead, for what the second term in the last line of (3.2.41) is concerned, being $\overline{\varsigma}$ finite, the definition of $\eta_j$ and the identity in (3.2.39) imply that

$$\limsup_{j\to\infty} \frac{\omega(1-\delta_j)}{\eta_j} \mathcal{L}^1(\{\tilde{v}_j \ge \delta_j\}) \le \overline{\varsigma}^2 \cdot \lim_{j\to\infty} \frac{Q(1-\delta_j)}{\varepsilon_{k_j}} r_j = \overline{\varsigma}^2 \cdot \lim_{j\to\infty} \frac{l(\delta_j)}{\delta} r_j = 0 \,. \tag{3.2.43}$$

By taking into account (3.2.38) and that $\hat{v}_j \leq \delta_j$ $\mathcal{L}^1$-a.e. on $I_1$, we have by (3.2.40)

$$\frac{1}{\varepsilon_{k_j}}\varphi(\varepsilon_{k_j} f^2(\hat{v}_j)) \geq (1-\vartheta(\delta))\varphi'(0^+) f^2(\hat{v}_j)\,, \tag{3.2.44}$$

where $\vartheta(\delta) \to 0^+$ as $\delta \to 0^+$. In particular, by (3.2.41)-(3.2.44) we conclude that

$$\liminf_{j\to\infty} \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1) \geq \liminf_{j\to\infty} \mathcal{G}_j(\tilde{u}_j, \hat{v}_j, I_1) \geq (1-\theta(\delta)) \liminf_{j\to\infty} \widehat{\mathcal{F}}_{\eta_j}(\tilde{u}_j, \hat{v}_j, I_1)\,, \tag{3.2.45}$$

where $\widehat{\mathcal{F}}_{\eta_j} : L^1(I_1, \mathbb{R}^2) \times \mathcal{A}(I_1) \to [0,\infty]$ is the functional defined for every $(u, v) \in H^1(I_1, \mathbb{R} \times [0, 1])$ and $U \in \mathcal{A}(I_1)$ by

$$\widehat{\mathcal{F}}_{\eta_j}(u, v, U) = \int_U \left(\eta_j \varphi'(0^+) f^2(v)|u'|^2 + \frac{\omega(1-v)}{4\eta_j} + \eta_j |v'|^2\right) \mathrm{d}x\,, \tag{3.2.46}$$

and $\infty$ otherwise.

Up to extracting a subsequence not relabeled, we may use (3.2.37) to find points $-\frac{1}{2} < x_1 < 0 < x_2 < \frac{1}{2}$ such that $\tilde{v}_j(x_i) \to 1$ (so that $\hat{v}_j(x_i) \to 1$), $\tilde{u}_j(x_i) \to u_0(x_i)$, $i \in \{1, 2\}$, and moreover that

$$\liminf_{j\to\infty} \widehat{\mathcal{F}}_{\eta_j}(\tilde{u}_j, \hat{v}_j, I_1) = \lim_{j\to\infty} \widehat{\mathcal{F}}_{\eta_j}(\tilde{u}_j, \hat{v}_j, I_1)\,.$$

In particular, with fixed any $\eta > 0$, for all $j$ sufficiently large by Cauchy-Schwartz inequality we get for $g_\eta$ defined in (3.2.19)

$$\begin{aligned}\widehat{\mathcal{F}}_{\eta_j}(\tilde{u}_j, \hat{v}_j, I_1) &\geq \int_{x_1}^{x_2} \Big(\omega(1-\hat{v}_j)\big(\varphi'(0^+) f^2(\hat{v}_j)|\tilde{u}_j'|^2 + |\hat{v}_j|^2\big)\Big)^{1/2} \mathrm{d}x \\ &\geq g_\eta(|\tilde{u}_j(x_2) - \tilde{u}_j(x_1)|)\,.\end{aligned} \tag{3.2.47}$$

In deriving the last inequality, we have used that the functional to be minimized in the definition of $g_\eta$ is invariant under reparametrization.

The conclusion in (3.2.32) then follows by (3.2.35), (3.2.45), (3.2.47) and item (vi) in Proposition 3.2.4. □

We gather the lower estimates on the diffuse and surface parts obtained in Propositions 3.2.7 and 3.2.8, respectively, via a standard measure theoretic argument.

**Proposition 3.2.9** (Lower Bound inequality)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} \in (0,\infty)$, for every $(u_\varepsilon, v_\varepsilon)$, $(u, v) \in L^1(O, \mathbb{R}^2)$ with $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(O, \mathbb{R}^2)$ we have*

$$F_{\overline{\varsigma}}(u, v) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon), \tag{3.2.48}$$

*where $\mathcal{F}_\varepsilon$ and $F_{\overline{\varsigma}}$ are defined in (3.2.1) and (3.2.3), respectively.*

*Proof.* Without loss of generality, we assume that the inferior limit in (3.2.48) to be finite. Thus, $(u_\varepsilon, v_\varepsilon) \in H^1(O, \mathbb{R} \times [0, 1])$ for every $\varepsilon > 0$ sufficiently small, and moreover $u \in GBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$ in view of Theorem 3.2.6. It is sufficient to establish (3.2.48) if $u \in BV(O)$ by employing a standard truncation argument and the fact that the functionals $\mathcal{F}_\varepsilon$ are decreasing by truncation.

Thus, for $u \in BV(O)$, we may consider the superadditive set function defined on $\mathcal{A}(O)$ by

$$\mu(A) := \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A),$$

if $A \in \mathcal{A}(O)$, and the Radon measure

$$\nu := \mathcal{L}^1 \llcorner O + \mathcal{H}^0 \llcorner J_u + |D^c u| \,.$$

In particular, $\nu$ is the sum of three Radon measures concentrated on mutually disjoint Borel sets $U_1$, $U_2$, $U_3$ partitioning $O$. Then, we may define two Borel functions $\psi_1, \psi_2 : O \to [0, \infty]$ by

$$\psi_1(x) = \begin{cases} g(|[u](x)|) & \text{on } U_2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \psi_2(x) = \begin{cases} h^{**}_{\overline{\varsigma}}(|u'(x)|) & \text{on } U_1 \\ (\varphi'(0^+))^{1/2}\overline{\varsigma} & \text{on } U_3 \\ 0 & \text{otherwise} \end{cases}$$

Propositions 3.2.7 and 3.2.8 then imply for $i = 1, 2$ and $A \in \mathcal{A}(O)$

$$\mu(A) \geq \int_A \psi_i \mathrm{d}\nu \,,$$

thus [Bra98, Proposition 1.16] yields that

$$\mu(O) \geq \int_O (\psi_1 \vee \psi_2) \mathrm{d}\nu = F_{\overline{\varsigma}}(u, 1) \,.$$

□

To show the upper bound inequality we follow in part the strategy in [CFI16, CFI24a, Proposition 5.2] and take advantage of the one-dimensional setting. More generally, we establish it for the perturbed family of functionals $\mathcal{F}^\kappa_\varepsilon : L^1(O, \mathbb{R}^2) \times \mathcal{A}(O) \to [0, \infty]$

$$\mathcal{F}^\kappa_\varepsilon(u, v, A) := \mathcal{F}_\varepsilon(u, v, A) + \kappa_\varepsilon \int_A |u'|^2 \, \mathrm{d}x \,, \tag{3.2.49}$$

if $(u, v) \in H^1(O, \mathbb{R} \times [0, 1])$ and $\infty$ otherwise, where $\kappa_\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0$ and $\kappa_\varepsilon \geq 0$, in order to gain coercivity for applications to Dirichlet boundary value problems (see Section 3.2.3). We denote by $F'' := \Gamma(L^1)\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}^\kappa_\varepsilon$.

We divide the argument into several steps, by providing first a rough bound for the diffuse part in the case of Sobolev functions and then obtaining the sharp bound optimizing upon the former rough one through a relaxation argument. The extension of the upper bound to piecewise Sobolev functions is done thanks to an explicit construction matching the surface energy density at jump points, finally the sharp bound for any $BV$ function is obtained again through relaxation.

In what follows it is convenient to consider the functional $H : L^1(O) \times \mathcal{A}(O) \to [0, \infty]$ defined by

$$H(u, A) := \int_A h_{\overline{\varsigma}}(|u'|) \, \mathrm{d}x \,, \tag{3.2.50}$$

if $u \in W^{1,1}(O)$, and $\infty$ otherwise. Given the continuity of $h_{\overline{\varsigma}}$ (cf. Proposition 3.1.3) and the growth conditions in (3.1.20), the functional $H$ in (3.2.50) is continuous with respect to the strong convergence in $W^{1,1}(O)$. The same remark applies to the functional $\mathscr{F}_{\overline{\varsigma}}$. We will take advantage of this fact by proving in several instances the upper bound inequality (cf. (3.2.51) below) on classes of functions which are dense in $L^1(O)$ or in a stronger topology, and along which $H$ and $\mathscr{F}_{\overline{\varsigma}}$ are continuous, respectively. The $L^1$ lower semicontinuity of $F''$ will allow to extend the validity of (3.2.51) to functions in the $L^1$ closure of such dense classes.

**Proposition 3.2.10** (Upper Bound inequality)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} \in (0, \infty)$, for every $(u, v) \in L^1(O; \mathbb{R}^2)$*

$$F''(u, v) \leq F_{\overline{\varsigma}}(u, v)\,. \tag{3.2.51}$$

*Proof.* It is clearly sufficent to assume $v = 1$ $\mathcal{L}^1$-a.e. on $O$. We split the proof in different

**Step 1.** If $u(x) = \xi x + \eta$ is affine on $O$ then for every interval $I \subseteq O$

$$F''(u, 1, I) \leq H(u, I) = h_{\overline{\varsigma}}(|\xi|)\mathcal{L}^1(I)\,. \tag{3.2.52}$$

Let $\tau_\xi \in [0, \infty)$ be such that $h_{\overline{\varsigma}}(|\xi|) = \varphi(1/\tau_\xi)\xi^2 + \frac{\overline{\varsigma}^2}{4}\tau_\xi$ (with the convention that $\varphi(1/\tau)$ is extended by continuity as $\varphi(\infty) \in (0, \infty)$ in $\tau = 0$). Then set $u_\varepsilon = u$ and $v_\varepsilon = \lambda_\varepsilon$ where $\varepsilon f^2(\lambda_\varepsilon) = \varepsilon \frac{l(\lambda_\varepsilon)}{Q(1-\lambda_\varepsilon)} = 1/\tau_\xi$ (note that $\lambda_\varepsilon = 1$ if $\tau_\xi = 0$). By the very definition $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$. Therefore

$$\begin{aligned} \mathcal{F}^\kappa_\varepsilon(u_\varepsilon, v_\varepsilon, I) &= \Big(\varphi(1/\tau_\xi)\xi^2 + \kappa_\varepsilon \xi^2 + \frac{\omega(1-\lambda_\varepsilon)}{4\varepsilon}\Big)\mathcal{L}^1(I) \\ &= \Big(\varphi(1/\tau_\xi)\xi^2 + \kappa_\varepsilon \xi^2 + \frac{\omega(1-\lambda_\varepsilon)}{4Q(1-\lambda_\varepsilon)} l(\lambda_\varepsilon)\tau_\xi\Big)\mathcal{L}^1(I)\,, \end{aligned}$$

the conclusion then follows by taking into account (3.1.2), and the fact that $l(\lambda_\varepsilon) \to l(1) = 1$.

**Step 2.** The inequality in (3.2.52) holds if $u$ is piecewise affine on $O = (a, b)$, that is $u \in C^0(O)$ and there are $\{a_i\}_{i=1}^N$ where $a_1 = a$, $a_N = b$ and $a_i < a_{i+1}$ for $i \in \{1, \dots N-1\}$ such that

$$u(x) = \sum_{i=1}^N (\xi_i x + \eta_i)\chi_{[a_i, a_{i+1})}(x)\,.$$

First, we show explicitly the claim for the (dense) subclass of piecewise affine functions $u$ such that $u' = 0$ on $(a_i, a_{i+1})$ for $i \in \{2, \dots, N-1\}$ even. Let $\delta \in (0, \min_i(a_{i+1} - a_i))$, set $O_{i,\delta} := (a_i - \delta/2, a_{i+1} + \delta/2)$, and let $\{\phi_i\}_{i=1}^N$ be a partition of unity subordinated to the covering $\{O_{i,\delta}\}_{i=1}^N$ of $O$, i.e. $\phi_i \in C^\infty_c(O_{i,\delta}, [0,1])$, $\phi_i|_{(a_i+\delta/2, a_{i+1}-\delta/2)} = 1$, $\max_{1 \leq i \leq N} \|\phi_i'\|_{C^0(\mathbb{R})} \leq C/\delta$, for some $C > 0$, and $\sum_{i=1}^N \phi_i(x) = 1$ for every $x \in O$.

Set $u_\varepsilon := u$ and $v_\varepsilon := \sum_{i=1}^N \lambda_{\varepsilon,i}\phi_i$, where $\varepsilon f^2(\lambda_{\varepsilon,i}) = 1/\tau_{\xi,i}$ for every $i \in \{1, \dots, N\}$ with $\lambda_{\varepsilon,i} \in [0,1]$ (using the notation introduced in the previous step). In particular, $\lambda_{\varepsilon,i} = 1$ for $i \in \{2, \dots, N-1\}$ even because $\xi_i = 0$ and $h_{\overline{\varsigma}}(0) = 0$. Therefore, from Step 1 we get for $\delta$ sufficiently small

$$\begin{aligned} \mathcal{F}_\varepsilon^\kappa(u_\varepsilon, v_\varepsilon, I) \le H(u, I) + o_\varepsilon + (\delta\varphi(\infty) + \kappa_\varepsilon \mathcal{L}^1(O)) \sum_{i=1}^N |\xi_i|^2 \\ + CN\frac{\varepsilon}{\delta} + C \sum_{i=2}^{N-1} \int_{a_i - \delta/2}^{a_i + \delta/2} \frac{Q(1 - v_\varepsilon)}{\varepsilon} \, \mathrm{d}x \,, \end{aligned} \tag{3.2.53}$$

where $C > 0$ is a universal constant, $o_\varepsilon \to 0$ as $\varepsilon \to 0$, and we have used (3.1.2) and the convergence $v_\varepsilon \to 1$ in $L^\infty(O)$. Next, note that $\phi_{i-1}(x) + \phi_i(x) = 1$ on $(a_i - \delta/2, a_i + \delta/2)$ for $\delta$ sufficiently small, so that on such a set $1 - v_\varepsilon = (1 - \lambda_{\varepsilon,i-1})\phi_{i-1} + (1 - \lambda_{\varepsilon,i})\phi_i$. In addition, using that $u' = 0$ on $(a_i, a_{i+1})$ for even $i \in \{2, \ldots, N-1\}$, then $1 - v_\varepsilon = (1 - \lambda_{\varepsilon,i})\phi_i$ if $i \ge 3$ is odd, and $1 - v_\varepsilon = (1 - \lambda_{\varepsilon,i-1})\phi_{i-1}$ if $i \ge 2$ is even. Hence, if $i$ is odd on $(a_i - \delta/2, a_i + \delta/2)$ we have

$$Q(1 - v_\varepsilon) = Q((1 - \lambda_{\varepsilon,i})\phi_i) \le Q(1 - \lambda_{\varepsilon,i}) = \varepsilon \tau_{\xi_i} l(\lambda_{\varepsilon,i}) \,,$$

where we have used that $Q$ is increasing in a neighbourhood of the origin (cf. (Hp 1)), and $\phi_i \in [0, 1]$. Clearly, an analogous statement holds if $i$ is even. Thanks to this estimate and to (3.2.53) we conclude that

$$\begin{aligned} \mathcal{F}_\varepsilon^\kappa(u_\varepsilon, v_\varepsilon, I) \le H(u, I) + o_\varepsilon + (\delta\varphi(\infty) + \kappa_\varepsilon \mathcal{L}^1(O)) \sum_{i=1}^N |\xi_i|^2 \\ + CN\frac{\varepsilon}{\delta} + C\delta \sum_{i=1}^N \tau_{\xi_i} l(\lambda_{\varepsilon,i}) \,, \end{aligned}$$

from which we conclude (3.2.52) by taking first the superior limit as $\varepsilon \to 0$ and then as $\delta \to 0$.

Finally, given any piecewise affine function $u$ with an underlying partition $\{a_1, .., a_N\}$, consider

$$u_j(x) = \begin{cases} u(x) & \text{if } x \in [a_1, a_2] \\ u(a_i) & \text{if } x \in [a_i, a_i + 1/j] \\ u(a_{i+1}) + \frac{u(a_{i+1}) - u(a_i)}{a_{i+1} - a_i - 1/j}(x - a_{i+1}) & \text{if } x \in [a_i + 1/j, a_{i+1}] \end{cases}$$

for $i \in \{2, .., N-1\}$, where $0 < 1/j < \min_i(a_{i+1} - a_i)$. It is easy to show that $u_j \to u$ in $L^\infty(O)$ and $u_j' \to u'$ in $L^p(O)$ for every $p \in [1, \infty)$ as $j \to \infty$. We conclude using the density argument explained at the beginning of the proof.

**Step 3.** The inequality in (3.2.51) holds on $W^{1,1}(O)$.
We use first the continuity of $H$ with respect to the strong $W^{1,1}(O)$ convergence together with the density of piecewise affine functions in $W^{1,1}(O)$ (this is easily established for the dense class of smooth functions, see also [ET99, Proposition 2.1 in Chapter X]) to extend the validity of inequality (3.2.52) to every $u \in W^{1,1}(O)$ via the density argument explained at the beginning of the proof.

We obtain the bound $F''(u, 1) \le F_{\overline{\varsigma}}(u, 1)$ for $u \in W^{1,1}(\Omega)$ using a classical relaxation result (cf. for instance [But89, Theorem 4.4.1 and Remark 4.4.5], [AF84, Statement III.7]).

**Step 4.** The inequality in (3.2.51) holds if $u$ is $SBV(O)$ with $\mathcal{H}^0(J_u) < \infty$.

We explicit the construction first for $u \in SBV(O)$ with $J_u = \{x_0\}$ such that $u = u(x_0^-)$ on $(x_0 - 2\lambda, x_0)$ and $u = u(x_0^+)$ on $[x_0, x_0+2\lambda)$ for some $\lambda > 0$ with $(x_0-2\lambda, x_0+2\lambda) \subseteq O$, where $u(x_0^-), u(x_0^+)$ are respectively the left and the right limit of $u$ in $x_0$ (without loss of generality we suppose that $u(x_0^-) < u(x_0^+)$).

A simple contradiction argument yields that it is sufficient to show that for every infinitesimal sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ there are a subsequence $\{\varepsilon_{j_k}\}_{k\in\mathbb{N}}$ and $(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) \to (u, 1)$ in $L^1(O, \mathbb{R}^2)$ for which

$$\limsup_{k\to\infty} \mathcal{F}^{\kappa}_{\varepsilon_{j_k}}(u_{\varepsilon_{j_k}}, v_{\varepsilon_{j_k}}) \leq F_{\bar{\varsigma}}(u, 1)\,.$$

Fix, $\{\varepsilon_j\}_{j\in\mathbb{N}}$ infinitesimal, and consider $I_1 := (a, x_0)$ and $I_2 := (x_0, b)$. Being $u \in W^{1,1}(I_1 \cup I_2)$, there are sequences $(u^{(i)}_{\varepsilon_j}, v^{(i)}_{\varepsilon_j}) \to (u, 1)$ in $L^1(I_i, \mathbb{R}^2)$ such that $F''(u, I_i) = \lim_j \mathscr{F}^{\kappa}_{\varepsilon_j}(u^{(i)}_{\varepsilon_j}, v^{(i)}_{\varepsilon_j}, I_i)$. We may then extract a subsequence such that $(u^{(i)}_{\varepsilon_{j_k}}, v^{(i)}_{\varepsilon_{j_k}}) \to (u, 1)$ $\mathcal{L}^1$-a.e. on $I_i$ for $i \in \{1, 2\}$, as well. For the sake of notational simplicity in what follows we denote $(u^{(i)}_{\varepsilon_{j_k}}, v^{(i)}_{\varepsilon_{j_k}})$ simply by $(u^{(i)}_k, v^{(i)}_k)$. By a.e. convergence we can find points $x_1 \in I_1 \cap (x_0 - 2\lambda, x_0 - \lambda)$ and $x_2 \in I_2 \cap (x_0 + \lambda, x_0 + 2\lambda)$, such that $(u^{(i)}_k(x_i), v^{(i)}_k(x_i)) \to (u(x_i), 1)$, with $u(x_1) = u(x_0^-)$ and $u(x_2) = u(x_0^+)$.

By item (v) in Proposition 3.2.4 for every $\eta > 0$ there exist $T_\eta > 0$ and $(\gamma_\eta, \beta_\eta) \in \mathfrak{U}_{|[u](x_0)|}(0, T_\eta)$ such that

$$\int_0^{T_\eta} \left(\varphi'(0^+) f^2(\beta_\eta)|\gamma_\eta'|^2 + \frac{\omega(1-\beta_\eta)}{4} + |\beta_\eta'|^2\right) \mathrm{d}x \leq g(|[u](x_0)|) + \eta\,. \tag{3.2.54}$$

Set $A^\eta_k := \left(x_0 - \frac{\varepsilon_{j_k} T_\eta}{2}, x_0 + \frac{\varepsilon_{j_k} T_\eta}{2}\right)$, then $A^\eta_k \subseteq (x_1+\lambda, x_2-\lambda)$ for $k$ sufficiently big. Then define $\{(u^\eta_k, v^\eta_k)\}_{k\in\mathbb{N}}$ by

$$u^\eta_k(x) = \begin{cases} u^{(1)}_k & x \in (a, x_1) \\ u^{(1)}_k(x_1) + (u(x_0^-) - u^{(1)}_k(x_1))\lambda^{-1}(x - x_1) & x \in [x_1, x_1 + \lambda] \\ u(x_0^-) & x \in (x_1 + \lambda, x_0 - \frac{\varepsilon_{j_k}}{2} T_\eta] \\ u(x_0^-) + \gamma_\eta\left(\frac{x - x_0}{\varepsilon_{j_k}} + \frac{T_\eta}{2}\right) & x \in A^\eta_k \\ u(x_0^+) & x \in [x_0 + \frac{\varepsilon_{j_k}}{2} T_\eta, x_2 - \lambda), \\ u^{(2)}_k(x_2) + (u^{(2)}_k(x_2) - u(x_0^+))\lambda^{-1}(x - x_2) & x \in [x_2 - \lambda, x_2] \\ u^{(2)}_k & x \in (x_2, b)\,, \end{cases}$$

and

$$v^\eta_k(x) = \begin{cases} v^{(1)}_k & x \in (a, x_1) \\ \zeta^{(1)}_k & x \in [x_1, x_0 - \frac{\varepsilon_{j_k} T_\eta}{2}] \\ \beta_\eta\left(\frac{x - x_0}{\varepsilon_{j_k}} + \frac{T_\eta}{2}\right) & x \in A^\eta_k \\ \zeta^{(2)}_k & x \in [x_0 + \frac{\varepsilon_{j_k} T_\eta}{2}, x_2] \\ v^{(2)}_k & x \in (x_2, b)\,, \end{cases}$$

where $\zeta^{(1)}_k(x) = (v^{(1)}_k(x_1) + \frac{1}{\varepsilon_{j_k}}(x - x_1)) \wedge 1$ and $\zeta^{(2)}_k(x) = (v^{(2)}_k(x_2) + \frac{1}{\varepsilon_{j_k}}(x_2 - x)) \wedge 1$. Therefore, if $\zeta^{(i)}_k(x) \leq 1$ then $x \in [x_1, x_1 + \varepsilon_{j_k}(1 - v^{(1)}_k(x_1))]$ if $i = 1$, and $x \in [x_2 - \varepsilon_{j_k}(1 - v^{(2)}_{\varepsilon_{j_k}}(x_2)), x_2]$ if $i = 2$. Note that $(u^\eta_k, v^\eta_k) \in H^1(O, \mathbb{R} \times [0, 1])$ thanks to the assumptions on $u$. Moreover, $(u^\eta_k, v^\eta_k) \to (u, 1)$ in $L^1(O, \mathbb{R}^2)$ as $\gamma_\eta$,

$\beta_\eta \in L^\infty(0,T_\eta)$. To estimate the energy of $\{(u_k^\eta, v_k^\eta)\}_{k\in\mathbb{N}}$ we start with the contribution on $I_1 \cup I_2 \setminus A_k^\eta$:

$$\begin{aligned}
&\limsup_{k\to\infty} \mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^\eta, v_k^\eta, (a,x_1)\cup(x_2,b)) \\
&\qquad\qquad = \limsup_{k\to\infty} \big(\mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^{(1)}, v_k^{(1)}, (a,x_1)) + \mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^{(2)}, v_k^{(2)}, (x_2,b))\big) \\
&\qquad\qquad \le F''(u,1,I_1) + F''(u,1,I_2) \le \int_O h_\varsigma^{**}(|u'|)\,\mathrm{d}x\,, 
\end{aligned} \tag{3.2.55}$$

by the previous step as the intervals are disjoint. Next, we use a change of variable to estimate the contribution on $A_k^\eta$ as follows

$$\begin{aligned}
&\mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^\eta, v_k^\eta, A_k^\eta) \\
&\qquad \le \int_0^{T_\eta} \left(\frac{1}{\varepsilon_{j_k}}(\varphi(\varepsilon_{j_k} f^2(\beta_\eta)) + \kappa_{\varepsilon_{j_k}})|\gamma_\eta'|^2 + \frac{\omega(1-\beta_\eta)}{4} + |\beta_\eta'|^2\right)\mathrm{d}x\,.
\end{aligned}$$

Being $\varphi$ (right) differentiable in 0 and bounded, there is $C>0$ such that $\varphi(t)\le Ct$ for all $t\ge 0$. Hence by Lebesgue dominated convergence theorem, by $\kappa_{\varepsilon_{j_k}} = o(\varepsilon_{j_k})$ as $k\to\infty$, and by (3.2.54) we conclude that

$$\limsup_{k\to\infty} \mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^\eta, v_k^\eta, A_k^\eta) \le g(|[u](x_0)|) + \eta\,, \tag{3.2.56}$$

Finally, we are left with estimating the energy on the set $O_k^\eta := ((a,x_1)\cup(x_2,b))\setminus A_k^\eta$. By construction both $u_k^\eta$ and $v_k^\eta$ are piecewise affine on such a set, then a direct computation yields

$$\begin{aligned}
\mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^\eta, v_k^\eta, O_k^\eta) &\le \frac{1}{\lambda}(\varphi(\infty) + \kappa_{\varepsilon_{j_k}})\big((u(x_0^+) - u_k^{(1)}(x_1))^2 + (u(x_0^+) - u_k^{(2)}(x_2))^2\big) \\
&+ \frac{\kappa_{\varepsilon_{j_k}}}{\varepsilon_{j_k}} \int_0^{T_\eta} |\gamma_\eta'|^2\,\mathrm{d}x + (2 - v_k^{(1)}(x_1) - v_k^{(2)}(x_2))(\sup_{[0,1]}\omega + 1)\,.
\end{aligned}$$

Therefore, recalling that $u$ is constant on $(x_0-2\lambda, x_0)$ and on $(x_0, x_0+2\lambda)$, by the choice of $x_1$ and $x_2$, and as $\kappa_\varepsilon = o(\varepsilon)$ as $\varepsilon\to 0$, we conclude that

$$\limsup_k \mathcal{F}^\kappa_{\varepsilon_{j_k}}(u_k^\eta, v_k^\eta, O_k^\eta) = 0 \tag{3.2.57}$$

By collecting (3.2.55)- (3.2.57) and letting $\eta\to 0$, (3.2.51) follows at once.

To remove the assumption that $u$ is piecewise constant close to the jump point $x_0$, we define the sequence

$$u_j(x) = u(x)\chi_{O\setminus[x_0-\frac{1}{j}, x_0+\frac{1}{j}]} + u(x_0 - \tfrac{1}{j})\chi_{(x_0-\frac{1}{j}, x_0]} + u(x_0+\tfrac{1}{j})\chi_{(x_0, x_0+\frac{1}{j})}\,. \tag{3.2.58}$$

We have $u_j\to u$ in $L^1(O)$ and $u_j' = u'\chi_{O\setminus[x_0-\frac{1}{j}, x_0+\frac{1}{j}]}$ $\mathcal{L}^1$-a.e. on $O$, and thus we conclude using the density argument explained before the statement.

Finally, since our construction modifies recovery sequences of $F''$ on each sub-interval on which $u$ is in $W^{1,1}$ only in a neighbourhood of the endpoints, we can extend the validity of (3.2.51) to every $u\in SBV(O)$ with $\mathcal{H}^0(J_u)<\infty$ arguing locally.

**Step 5.** The inequality in (3.2.51) holds if $u\in SBV(O)$.

We claim that each function $u \in SBV(O)$ can be approximated with a sequence of functions $u_j \in SBV(O)$ and $\mathcal{H}^0(J_{u_j}) < \infty$ for every $j \in \mathbb{N}$, converging to $u$ in $L^1(O)$ and with energies $\mathscr{F}_{\overline{\varsigma}}(u_j) \to \mathscr{F}_{\overline{\varsigma}}(u)$. Indeed, consider the decomposition $u = u^{(a)} + u^{(s)}$, where $u_a \in W^{1,1}(O)$ and $u_s(x) = \sum_{y \in J_u \cap (a,x]} [u](y)$ is piecewise constant, and define $u_j := u^{(a)} + \sum_{y \in I_j \cap (a,x]} [u](y)$, where $I_j := \{y \in J_u : |[u](y)| \geq 1/j\}$.

**Step 6.** The inequality in (3.2.51) holds if $u \in GBV(O)$.
Note that if (3.2.51) holds on $BV(O)$, we may conclude it on $GBV(O)$ by means of the sequence of truncations $\tau_M$ (cf. Section 2.0.1), together with the standard density argument.

To show inequality (3.2.51) on $BV(O)$ we use a relaxation argument. Indeed, thanks to Step 5 we may apply [BBB95, Theorem 3.1] to get that the relaxed functional with respect to the weak$^*$ $BV$ topology of $\widetilde{\mathscr{F}_{\overline{\varsigma}}} : BV(O) \to [0, \infty]$ defined as $\widetilde{\mathscr{F}_{\overline{\varsigma}}} = \mathscr{F}_{\overline{\varsigma}}$ on $SBV(O)$ and $\infty$ otherwise is given by

$$\widetilde{\mathscr{F}_{\overline{\varsigma}}}(u) = \int_O (h^{**}_{\overline{\varsigma}} \nabla g^0)(|u'|)\,\mathrm{d}x + \int_{J_u} ((h^{**}_{\overline{\varsigma}})^\infty \nabla g)(|[u]|)\mathrm{d}\mathcal{H}^0 + \int_O (h^{**}_{\overline{\varsigma}} \nabla g^0)^\infty \mathrm{d}D^c u\,, \tag{3.2.59}$$

for every $u \in BV(O)$, where $\psi_1 \nabla \psi_2(t) := \inf\{\psi_1(x) + \psi_2(t-x) : x \in [0,t]\}$ is the usual infimal convolution of two functions, and for every $s \geq 0$

$$g^0(s) := \limsup_{t \to 0} \frac{g(st)}{t}, \quad (h^{**}_{\overline{\varsigma}})^\infty(s) := \lim_{t \to \infty} \frac{h^{**}_{\overline{\varsigma}}(st)}{t}\,.$$

In particular, $g^0(s) = (h^{**}_{\overline{\varsigma}})^\infty(s) = \overline{\varsigma}(\varphi'(0^+))^{1/2} s$ for every $s \geq 0$ by Lemma 3.1.3 and (3.2.16), respectively. We have $\psi_1 \nabla \psi_2 \leq \psi_1 \wedge \psi_2$ by the very definition. On the one hand, item (ii) in Proposition 3.2.4 implies $g \leq (h^{**}_{\overline{\varsigma}})^\infty$ so that $(h^{**}_{\overline{\varsigma}})^\infty \nabla g \leq g$ and actually the equality holds by sub-additivity of $g$. On the other hand, (3.1.20) in Proposition 3.1.3 implies $h^{**}_{\overline{\varsigma}} \leq g^0$ so that $h^{**}_{\overline{\varsigma}} \nabla g^0 \leq h^{**}_{\overline{\varsigma}}$ and actually the equality holds thanks to the convexity of $h^{**}_{\overline{\varsigma}}$, and the fact that if $\zeta$ belongs to the subdifferential of $h^{**}_{\overline{\varsigma}}$ in $t$, then $\zeta \leq g^0(1)$ by $g^0(s) = (h^{**}_{\overline{\varsigma}})^\infty(s) = \overline{\varsigma}(\varphi'(0^+))^{1/2} s$.

The conclusion then follows at once from (3.2.59) by taking into account that $F'' \leq \widetilde{\mathscr{F}_{\overline{\varsigma}}}$ on $SBV(O) \times \{1\}$ by Step 5, and that $F''$ is $L^1$ lower semicontinuous. □

**Case $\overline{\varsigma} = \infty$**

In this section we consider the case $\overline{\varsigma} = \infty$.

**Theorem 3.2.11** (Compactness)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0} \in L^1(O, \mathbb{R}^2)$ be such that*

$$\sup_{\varepsilon > 0} (\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) + \|u_\varepsilon\|_{L^1(O)}) < \infty\,, \tag{3.2.60}$$

*then there are $\{\varepsilon_k\}_{k \in \mathbb{N}}$ and $u \in L^1 \cap GSBV(O)$ with $u' \in L^2(O)$ such that $(u_{\varepsilon_k}, v_{\varepsilon_k}) \to (u, 1)$ $\mathcal{L}^1$-a.e. on $O$. If, moreover, $(u_\varepsilon)_\varepsilon$ is equi-integrable then $u_{\varepsilon_k} \to u$ in $L^1(O)$.*

*Proof.* The proof follows the same strategy of Theorem 3.2.6. We adopt the same notation used there and we highlight only the main changes. The convergences of $(v_\varepsilon)_\varepsilon$ to 1 in measure on $O$, and of a subsequence $(u_{\varepsilon_k})_k$ to some $u \in L^1 \cap GBV(O)$ follow from the same argument. The $\mathcal{L}^1$-a.e. convergence on $O$, and the $L^1(O)$ convergence if $(u_\varepsilon)_\varepsilon$ is equi-integrable, are analogous.

We need only to show that $u \in GSBV(O)$ and that $u' \in L^2(O)$. With this aim we note that in case $\overline{\varsigma} = \infty$, the estimate $H_\delta(\tilde{u}_{\varepsilon,\delta}) \le \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon)$ established in Lemma 3.1.7 holds with bulk density $h_{1/\gamma - 1}$, for every $\gamma \in (0,1)$ (cf. (3.1.31)). Thus, in view of Proposition 3.1.9 we conclude that for every $M > 0$, $\gamma \in (0,1)$ and $\delta \in (0, \delta_\gamma)$

$$\zeta_1(\delta) \int_O h^{**}_{1/\gamma - 1}(|(u^M)'|)\,\mathrm{d}x + \zeta_1(\delta)\left(\frac{1}{\gamma} - 1\right)|D^c u^M|(O) \le C\,,$$

where $u^M := \tau_M(u)$ and $\tau_M$ is the truncation function defined in Section 2.0.1. By letting first $\delta \to 0$ and then $\gamma \to 0$ we conclude $|D^c u^M|(O) = 0$ for every $M \in \mathbb{N}$, that is equivalently $u^M \in SBV(O)$, from which we conclude that $u \in GSBV(O)$. In addition, the latter estimate and Proposition 3.1.3 imply

$$\varphi(\infty) \int_O |(u^M)'|^2\,\mathrm{d}x \le C\,.$$

The conclusion then follows at once by letting $M \to \infty$. □

The lower bound inequality in the case $\overline{\varsigma} = \infty$ follows as the analogous result in Proposition 3.2.7.

**Proposition 3.2.12.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, for every $(u_\varepsilon, v_\varepsilon)$, $(u, v) \in L^1(O, \mathbb{R}^2)$ with $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(O, \mathbb{R}^2)$, we have*

$$\int_A h_\infty(|u'|)\,\mathrm{d}x \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,, \tag{3.2.61}$$

*for every $A \in \mathcal{A}(O)$.*

*Proof.* First, let us assume the inferior limit on the right hand side of (3.2.61) is finite, otherwise the claim is obvious. Therefore, $(u_\varepsilon, v_\varepsilon) \in H^1(O, \mathbb{R} \times [0,1])$ for $\varepsilon > 0$ sufficiently small, $u \in GSBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. by Theorem 3.2.11. We show the result in case $u \in SBV(O)$, the general case follows as in Proposition 3.2.7.

Fix $\gamma \in (0,1)$ and $A \in \mathcal{A}(O)$, using the notation and the results in Lemma 3.1.7 we find $\delta_\gamma \in (0,1)$ such that for every $\delta \in (0, \delta_\gamma)$ there is $\tilde{u}_{\varepsilon,\delta} := (\widetilde{u_\varepsilon})_\delta \in SBV(A)$ (for every $\varepsilon$ small enough) we have

$$H_\delta(\tilde{u}_{\varepsilon,\delta}, A) \le \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,,$$

with density $h_{1/\gamma}$ (cf. (3.1.31)). Thus, in view of Proposition 3.1.9 we conclude that

$$\zeta_1(\delta) \int_A h^{**}_{1/\gamma - 1}(|u'|)\,\mathrm{d}x \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,,$$

recalling that $|D^c u|(O) = 0$. By letting first $\delta \to 0$ and then $\gamma \to 0$ we conclude (3.2.61) in view of (3.1.28), (3.1.22) in Proposition 3.1.3, and Beppo Levi's theorem. □

The proof of the lower bound inequality for the surface part takes advantage of Proposition 3.2.8 and of [CFI25, Step 2 in Corollary 4.8].

**Proposition 3.2.13.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, for every $(u_\varepsilon, v_\varepsilon)$, $(u, v) \in L^1(O, \mathbb{R}^2)$ with $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(O, \mathbb{R}^2)$, we have*

$$\int_{J_u \cap A} g(|[u]|)\mathrm{d}\mathcal{H}^0 \le \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A) \tag{3.2.62}$$

*for every $A \in \mathcal{A}(O)$, where $g$ is defined in (3.2.9).*

*Proof.* The proof is similar to that of Proposition 3.2.7, therefore we highlight only the necessary changes adopting the notation introduced there. We proceed up to formula (3.2.37), noticing that Theorem 3.2.11 implies that $u \in GSBV(O)$ with $u' \in L^2(O)$. Next, we change the argument as the truncation procedure in (3.2.38)-(3.2.40) does not work in this setting. As an outcome of the blow up procedure we are given a sequence $(\tilde{u}_j, \tilde{v}_j) \to (u_0, 1)$ in $L^1(I_1)$, where $I_1 = (-1/2, 1/2)$ and $u_0(x) = u(x_0^-)\chi_{(-\frac{1}{2},0)} + u(x_0^+)\chi_{[0,\frac{1}{2})}$, such that $\liminf_{j\to\infty} \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1) < \infty$, $\mathcal{G}_j$ defined in (3.2.36).

Let $\gamma > 0$ and $\delta > 0$ such that $\frac{\varphi(t)}{t} \geq \varphi'(0^+) - \gamma$ for all $t \in (0, \delta)$. Because of the continuity of $f$ on $[0, 1)$, the assumption that $f$ is non-decreasing in a left neighbourhood of $t = 1$, and $f(t) \to \infty$ as $t \to 1^-$, then $\{t \in [0, 1) : \varepsilon_j f^2(t) > \delta\} = (\delta_j, 1)$ for $j$ large, where $\delta_j \to 1^-$ as $j \to \infty$. Hence, $\{t \in I_1 : \varepsilon_j f^2(\tilde{v}_j) > \delta\} = \{t \in I_1 : \tilde{v}_j > \delta_j\} = \cup_i (a_j^i, b_j^i)$, being $\tilde{v}_j \in H^1(I_1)$. Consider the function $\hat{u}_j := \tilde{u}_j\chi_{\{\tilde{v}_j\leq\delta_j\}} + \sum_i \tilde{u}_j(a_j^i)\chi_{(a_j^i,b_j^i)}$, then $\hat{u}_j \in SBV(I_1)$ with $D\hat{u}_j = \tilde{u}_j'\chi_{\{\tilde{v}_j\leq\delta_j\}}\mathcal{L}^1 \llcorner I_1 + \sum_i(\tilde{u}_j(b_j^i) - \tilde{u}_j(a_j^i))\delta_{t=b_j^i}$. Take its absolutely continuous part $w_j$ in the standard decomposition of $BV$ functions, namely

$$w_j(x) := \tilde{u}_j(-1/2) + \int_{-1/2}^{x} \hat{u}_j'(t)\mathrm{d}t\,.$$

We claim that $w_j \to u_0$ in $L^1(I_1)$ and that

$$\begin{aligned}\hat{\mathcal{F}}_j(w_j, \tilde{v}_j, I_1) &:= \int_{I_1} \left(\eta_j(\varphi'(0^+) - \gamma)f^2(\tilde{v}_j)|w_j'|^2 + \frac{\omega(1 - v_j)}{4\eta_j} + \eta_j|v_j'|^2\right)\mathrm{d}x \\ &\leq \mathcal{G}_j(w_j, \tilde{v}_j, I_1) \leq \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j.I_1). \end{aligned} \tag{3.2.63}$$

Indeed, first note that

$$\frac{\eta_j}{\varepsilon_j}\varphi(\delta)\int_{\{\tilde{v}_j>\delta_j\}} |\tilde{u}_j'|^2\,\mathrm{d}x \leq \frac{\eta_j}{\varepsilon_j}\int_{\{\tilde{v}_j>\delta_j\}} \varphi(\varepsilon_j f^2(\tilde{v}_j))|\tilde{u}_j'|^2\,\mathrm{d}x \leq \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1)\,,$$

and then that $w_j' = \hat{u}_j'\chi_{\{\tilde{v}_j\leq\delta_j\}} = \tilde{u}_j'\chi_{\{\tilde{v}_j\leq\delta_j\}}$ $\mathcal{L}^1$-a.e. on $I_1$. Hence, $w_j \to u_0$ in $L^1(I_1)$ as

$$\|w_j - \tilde{u}_j\|_{L^\infty(I_1)} \leq \int_{\{\tilde{v}_j>\delta_j\}} |\tilde{u}_j'|\,\mathrm{d}x \leq \left(\frac{\varepsilon_j}{\varphi(\delta)\eta_j}\mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1)\right)^{1/2}\,,$$

and furthermore we have

$$\begin{aligned}\hat{\mathcal{F}}_j(w_j, \tilde{v}_j) \leq& \mathcal{G}_j(w_j, \tilde{v}_j, I_1) = \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, \{\tilde{v}_j \leq \delta_j\}) \\ &+ \int_{\{\tilde{v}_j>\delta_j\}} \left(\frac{\omega(1 - \tilde{v}_j)}{\varepsilon_j} + \varepsilon_j|\tilde{v}_j{}'|^2\right)\mathrm{d}x \leq \mathcal{G}_j(\tilde{u}_j, \tilde{v}_j, I_1)\,,\end{aligned}$$

and thus (3.2.63) follows.

The final argument is similar to that employed in Proposition 3.2.8 using Proposition 3.2.5 rather than Proposition 3.2.4, and finally letting $\gamma \to 0^+$. □

The lower bound inequality follows arguing analogously as in Proposition 3.2.9.

**Proposition 3.2.14** (Lower Bound inequality)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, for every $(u_\varepsilon, v_\varepsilon)$, $(u, v) \in L^1(O, \mathbb{R}^2)$ with $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(O, \mathbb{R}^2)$ we have*

$$F_\infty(u, v) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon), \tag{3.2.64}$$

*where $\mathcal{F}_\varepsilon$ and $F_\infty$ are defined in (3.2.1) and (3.2.3), respectively.*

To conclude we show the upper bound inequality in this setting for the perturbed family $\{\mathcal{F}_\varepsilon^\kappa\}_{\varepsilon>0}$ defined in (3.2.49). We recall the notation $F'' = \Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon^\kappa$.

**Proposition 3.2.15** (Upper Bound inequality)**.** *Under the assumptions of Theorem 3.2.1 with $\overline{\varsigma} = \infty$, let $(u, v) \in L^1(O; \mathbb{R}^2)$ then*

$$F''(u, v) \leq F_\infty(u, v)\,. \tag{3.2.65}$$

*Proof.* Without loss of generality we assume $u \in GSBV(O)$ with $u' \in L^2(O)$, and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, the inequality being trivial otherwise. Moreover, we can reduce to $u \in SBV(O)$ with $u' \in L^2(O)$ by the density argument explained before Proposition 3.2.10 by using the sequence of truncations $\tau_M$ (cf. Section 2.0.1). Furthermore, we may even assume that $\mathcal{H}^0(J_u) < \infty$ by using the construction in Step 5 of Proposition 3.2.10.

First, consider $u \in SBV(O)$ with $u' \in L^2(O)$, $J_u = \{x_0\}$ and $u = u(x_0^-)$ on $(x_0 - \lambda, x_0)$ and $u = u(x_0^+)$ on $[x_0, x_0 + \lambda)$ for some $\lambda > 0$ with $(x_0 - \lambda, x_0 + \lambda) \subseteq O$, where $u(x_0^-), u(x_0^+)$ are respectively the left and the right limit of $u$ in $x_0$ (without loss of generality we can suppose that $u(x_0^-) < u(x_0^+)$), and argue as in Step 4 of Proposition 3.2.10. In particular, the assumption that $u$ is constant near the jump point $x_0$ is not restrictive up to a density argument and the construction in (3.2.58).

By item (iv) in Proposition 3.2.5 for every $\eta > 0$ there exist $T_\eta > 0$ and $(\gamma_\eta, \beta_\eta) \in \mathfrak{U}_{|[u](x_0)|}(0, T_\eta)$ such that

$$\int_0^{T_\eta} \left( \varphi'(0^+) f^2(\beta_\eta) |\gamma_\eta'|^2 + \frac{\omega(1 - \beta_\eta)}{4} + |\beta_\eta'|^2 \right) \mathrm{d}x \leq g(|[u](x_0)|) + \eta\,. \tag{3.2.66}$$

Set $A_\varepsilon^\eta := \left( x_0 - \frac{\varepsilon T_\eta}{2}, x_0 + \frac{\varepsilon T_\eta}{2} \right)$, then $A_\varepsilon^\eta \subseteq (x_0 - \lambda, x_0 + \lambda)$ for $\varepsilon$ sufficiently small. Define $\{(u_\varepsilon^\eta, v_\varepsilon^\eta)\}_\varepsilon$ by

$$u_\varepsilon^\eta(x) = \begin{cases} u(x_0^-) + \gamma_\eta \left( \frac{x - x_0}{\varepsilon} + \frac{T_\eta}{2} \right) & x \in A_\varepsilon^\eta \\ u & x \in O \setminus A_\varepsilon^\eta \end{cases}$$

and

$$v_\varepsilon^\eta(x) = \begin{cases} \beta_\eta \left( \frac{x - x_0}{\varepsilon} + \frac{T_\eta}{2} \right) & x \in A_\varepsilon^\eta \\ 1 & x \in O \setminus A_\varepsilon^\eta\,. \end{cases}$$

Note that $(u_\varepsilon^\eta, v_\varepsilon^\eta) \in H^1(O, \mathbb{R} \times [0, 1])$ thanks to the assumptions on $u$. Moreover, $(u_\varepsilon^\eta, v_\varepsilon^\eta) \to (u, 1)$ in $L^1(O, \mathbb{R}^2)$ as $\gamma_\eta$, $\beta_\eta \in L^\infty((0, T_\eta))$.

Next, we estimate the energy of the family $\{(u_\varepsilon^\eta, v_\varepsilon^\eta)\}_\varepsilon$. We start with the contribution on $O \setminus \overline{A_\varepsilon^\eta}$ by taking into account that $v_\varepsilon^\eta = 1$ and $u_\varepsilon^\eta = u$ on such a set to get (recall that $\varphi(\varepsilon f^2)$ is extended by continuity with value $\varphi(\infty)$ to $t = 1$)

$$\mathcal{F}_\varepsilon^\kappa(u_\varepsilon^\eta, v_\varepsilon^\eta, O \setminus \overline{A_\varepsilon^\eta}) \leq (\varphi(\infty) + \kappa_\varepsilon) \int_{O \setminus \overline{A_\varepsilon^\eta}} |u'|^2 \,\mathrm{d}x\,. \tag{3.2.67}$$

For the contribution on $A^\eta_\varepsilon$ we change variable to get

$$\mathcal{F}^\kappa_\varepsilon(u^\eta_\varepsilon, v^\eta_\varepsilon, A^\eta_\varepsilon) \le \int_0^{T_\eta} \left(\frac{1}{\varepsilon}(\varphi(\varepsilon f^2(\beta_\eta)) + \kappa_\varepsilon)|\gamma'_\eta|^2 + \frac{\omega(1-\beta_\eta)}{4} + |\beta'_\eta|^2\right) \mathrm{d}x\,.$$

Being $\varphi$ (right) differentiable in 0 and bounded (cf. (Hp 3-4)), there is $c > 0$ such that $\varphi(t) \le ct$ for all $t \ge 0$. Hence by Lebesgue dominated convergence theorem, $\kappa_\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0$, and by (3.2.66) we conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{F}^\kappa_\varepsilon(u^\eta_\varepsilon, v^\eta_\varepsilon, A^\eta_\varepsilon) \le g(|[u](x_0)|) + \eta\,, \tag{3.2.68}$$

By collecting (3.2.67) and (3.2.68) and letting $\eta \to 0$, (3.2.65) follows at once.

Finally, arguing locally we can extend the validity of (3.2.65) to every $u \in SBV^2(O)$ with $\mathcal{H}^0(J_u) < \infty$ (cf. Step 4 of Proposition 3.2.10). $\square$

### 3.2.3 Dirichlet boundary values problem

In this section we impose Dirichlet boundary conditions on the approximating energies, determine the related $\Gamma$-limit and discuss the convergence of the related minimum problems. With this aim define $\mathcal{D}_\varepsilon : L^1(O,\mathbb{R}^2) \times \mathcal{A}(O) \to [0,\infty]$ by

$$\mathcal{D}_\varepsilon(u, v, A) := \mathcal{F}^\kappa_\varepsilon(u, v, A) \tag{3.2.69}$$

if $(u, v) \in H^1(O, \mathbb{R}\times[0, 1])$ and $u(a^+) = 0$, $u(b^-) = L$, $v(a^+) = v(b^-) = 1$ and $\infty$ otherwise, where $O = (a, b)$ and $\mathcal{F}^\kappa_\varepsilon$ is defined in (3.2.49). Here, $\kappa_\varepsilon = o(\varepsilon)$ and it is strictly positive.

**Theorem 3.2.16.** *Assume the hypothesis of Theorem 3.2.1, and let $\mathcal{D}_\varepsilon$ be the functional defined in (3.2.69). Then, for all $(u, v) \in L^1(O, \mathbb{R}^2)$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{D}_\varepsilon(u, v) = D_{\overline{\varsigma}}(u, v)\,, \tag{3.2.70}$$

*where*

$$D_{\overline{\varsigma}}(u, v) := \begin{cases} \mathscr{F}_{\overline{\varsigma}}(u) + g(|u(a^+)|) + g(|u(b^-) - L|) & \text{if } v = 1\ \mathcal{L}^1\text{-a.e. on } O \\ \infty & \text{otherwise}\,. \end{cases} \tag{3.2.71}$$

*Moreover, if $(u_\varepsilon, v_\varepsilon) \in \mathrm{argmin}_{L^1(O,\mathbb{R}^2)}\mathcal{D}_\varepsilon$, there are a subsequence (not relabeled) and a function $u \in GBV(O)$ such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(O, \mathbb{R}^2)$ and*

$$\lim_{\varepsilon\to 0} \mathcal{D}_\varepsilon(u_\varepsilon, v_\varepsilon) = D_{\overline{\varsigma}}(u, 1)\,. \tag{3.2.72}$$

*Proof.* We first show how to deduce (3.2.70) thanks to the results in Theorem 3.2.1. With this aim, for every $(u, v) \in H^1(O, \mathbb{R}^2)$ denote by $(U, V)$ the extensions given by

$$U(x) = \begin{cases} u(x) & \text{if } x \in O \\ 0 & \text{if } x \in (a-1, a] \\ L & \text{if } x \in (b, b+1] \end{cases} \qquad V(x) = \begin{cases} v(x) & \text{if } x \in O \\ 1 & \text{if } x \in (a-1, a] \\ 1 & \text{if } x \in (b, b+1] \end{cases}$$

Note that for every $(u,v) \in L^1(O,\mathbb{R}^2)$ and $\eta \in (0,1)$ we have

$$\mathcal{D}_\varepsilon(u,v) = \mathcal{F}_\varepsilon(U,V,(a-\eta,b+\eta)) .$$

Therefore, given $(u_\varepsilon, v_\varepsilon) \to (u,v)$ in $L^1(O,\mathbb{R}^2)$ then $(U_\varepsilon, V_\varepsilon) \to (U,V)$ in $L^1((a-\eta,b+\eta),\mathbb{R}^2)$ and either by Proposition 3.2.9 if $\overline{\varsigma} \in (0,\infty)$ or by Proposition 3.2.14 if $\overline{\varsigma} = \infty$, we deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{D}_\varepsilon(u_\varepsilon,v_\varepsilon) &= \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(U_\varepsilon,V_\varepsilon,(a-\eta,b+\eta)) \\ &\geq F_{\overline{\varsigma}}(U,V,(a-\eta,b+\eta)) = D_{\overline{\varsigma}}(u,v) .\end{aligned}$$

It is sufficient to show the upper bound inequality for $u \in GBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$. In this case, by using the standard density argument we can reduce to functions which are constant on a neighborhood of the boundary of $O$ by considering $\lambda \in (0,1)$ and $u_\lambda(x) := U(\frac{a+b}{2} + \lambda(x - \frac{a+b}{2}))$, for $x \in (a-1,b+1)$, and $v_\lambda = 1$ $\mathcal{L}^1$-a.e. on $O$. Indeed, we have

$$\lim_{\lambda\to 1^-} F_{\overline{\varsigma}}(u_\lambda, v_\lambda) = D_{\overline{\varsigma}}(u,1) .$$

For this class of functions the upper bound inequality follows from Propositions 3.2.10 if $\overline{\varsigma} \in (0,\infty)$ or from Proposition 3.2.12 if $\overline{\varsigma} = \infty$.

Finally, for every $\varepsilon > 0$ the functional $\mathcal{D}_\varepsilon$ is coercive. Denote by $(u_\varepsilon, v_\varepsilon)$ a minimizing sequence. By truncation we can assume that $u_\varepsilon \in [0,L]$ $\mathcal{L}^1$-a.e. on $O$. Therefore, Theorem 3.2.6 if $\overline{\varsigma} \in (0,\infty)$ or Theorem 3.2.11 if $\overline{\varsigma} = \infty$ provide the $L^1(O,\mathbb{R}^2)$ convergence of $(u_\varepsilon, v_\varepsilon)$ up to a subsequence not relabeled. The fundamental theorem of $\Gamma$-convergence yields (3.2.72). □

### 3.2.4 Corollaries of Theorem 3.1.1

In this section we collect several consequences of Theorem 3.2.1 essentially by using only simple comparison arguments. First, we establish Theorem 3.2.2.

*Proof of Theorem 3.2.2.* We give the proof in case $\overline{\varsigma} \in (0,\infty)$, the case $\overline{\varsigma} = \infty$ being similar and even simpler. By assumption $\gamma_\varepsilon/\varepsilon \to +\infty$, therefore for every $j \in \mathbb{N}$ and for $\varepsilon$ sufficiently small depending on $j$ we deduce the pointwise estimate $\widetilde{\mathcal{F}}_\varepsilon \geq \mathcal{F}_\varepsilon^{(j)}$, where the latter functionals are defined as $\widetilde{\mathcal{F}}_\varepsilon$ in (3.2.6) with $\varphi(\gamma_\varepsilon f^2)$ substituted by $\varphi(j\varepsilon f^2)$. Thus, we deduce that $\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{\mathcal{F}}_\varepsilon \geq \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{(j)} =: F_{\overline{\varsigma}}^{(j)}$ for every $j \in \mathbb{N}$. In particular, if $\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{\mathcal{F}}_\varepsilon(u,v)$ is finite, then $u \in GBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. $x \in O$. Furthermore, Theorem 3.2.1 yields that the energy densities of $F_{\overline{\varsigma}}^{(j)}$ are given by the convex envelope of $h_{\overline{\varsigma}}^{(j)}(t) = \inf_{\tau\in[0,\infty)} \{\varphi(\frac{1}{\tau})t^2 + \frac{\varsigma^2}{4} j\tau\}$ for every $t \geq 0$ for the bulk term (cf. (3.1.9) and use a reparametrization), $g^{(j)}(s) = g_1\big(j^{1/2}s\big)$ for every $s \geq 0$ for the jump term (cf. (3.2.9) and (3.2.11)), and $(j\varphi'(0^+))^{1/2}\overline{\varsigma}t$ for every $t \geq 0$ for the Cantor term. From the latter, by letting $j \to \infty$ we immediately deduce that $u \in GSBV(O)$. Moreover, from the equality $g^{(j)} = g_1(j^{1/2}s)$ (cf. (3.2.11)), it is clear that $g^{(j)}(s) \to 2\Psi(1)\chi_{(0,\infty)}(s)$ as $j \to \infty$ thanks to (3.2.15) in Proposition 3.2.4. Therefore, $u \in SBV(O)$. Finally, let $t > 0$ and $\tau_j \in [0,\infty)$ be such that $\varphi(\frac{1}{\tau_j})t^2 + \frac{\varsigma^2}{4} j\tau_j \leq h_{\overline{\varsigma}}^{(j)}(t) + 1/j$. Then, as $h_{\overline{\varsigma}}^{(j)}(t) \leq \varphi(\infty)t^2$ for all $j$, $(\tau_j)_j$ is infinitesimal as $j \to \infty$, so that $h_{\overline{\varsigma}}^{(j)}(t) \to \varphi(\infty)t^2$ as $j \to \infty$. Thus, we have shown that $\Gamma\text{-}\liminf_{\varepsilon\to 0} \widetilde{\mathcal{F}}_\varepsilon \geq \widetilde{F}$.

The upper bound inequality easily follows from the estimate $\varphi(\gamma_\varepsilon f^2(t)) \leq \chi_{(0,\infty)}(t)$ for every $t > 0$, and the construction in Proposition 3.2.15. □

Note that if $\gamma_\varepsilon = 1$ for every $\varepsilon > 0$ and $Q$ is strictly increasing, we recover exactly the Ambrosio and Tortorelli model with any assigned continuous degradation function $\psi$ choosing $\varphi$ appropriately.

Now we turn to address the case $\overline{\varsigma} = 0$.

**Proposition 3.2.17.** *Assume (Hp 1′)-(Hp 4′) (see the introduction of Section 3.2) , with $\overline{\varsigma} = 0$. Then*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) = 0 \tag{3.2.73}$$

*if $u \in L^1(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, and $\infty$ otherwise on $L^1(O,\mathbb{R}^2)$.*

*Proof.* By a standard density argument and the $L^1(O)$ lower semicontinuity of the $\Gamma$-limsup, to prove the result it is sufficient to establish the upper bound inequality for $u \in BV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$.

With this aim, fix $j \in \mathbb{N}$ and define for every $t \in [0,1)$

$$f^{(j)}(t) := \frac{l(t)}{(j^2\omega(1-t)) \wedge Q(1-t)}\,.$$

Notice that, $f^{(j)}(t) \geq f(t)$ for every $t \in [0,1)$ and $j \in \mathbb{N}$; and moreover as $\overline{\varsigma} = 0$ there is $t_j$ such that $(j^2\omega(1-t)) \wedge Q(1-t) = j^2\omega(1-t)$ for all $t \in [t_j,1)$, and $f^{(j)}(t) = f(t)$ for all $t \in [0,t_j]$. In particular, if $\mathcal{F}_\varepsilon^{(j)}$ denotes the functional defined as $\mathcal{F}_\varepsilon$ with $f^{(j)}$ in place of $f$, we have that $\mathcal{F}_\varepsilon(u,v) \leq \mathcal{F}_\varepsilon^{(j)}(u,v)$, for every $(u,v) \in L^1(O,\mathbb{R}^2)$, and $j \in \mathbb{N}$. Thus, by Proposition 3.2.10 we get that

$$\Gamma(L^1)\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,1) \leq \int_O h^{**}_{1/j}(|u'|)\,\mathrm{d}x + \frac{1}{j}|D^c u|(O) + \int_O g^{(1/j)}(|[u]|)\mathrm{d}\mathcal{H}^0,$$

where $g^{(1/j)}$ is defined as $g$ in (3.2.9), with $f^{(j)}$ in place of $f$ in the definition of $\theta$ in (3.2.10).

Finally, we conclude by dominated convergence by taking into account that $(h^{**}_{1/j})_{j\in\mathbb{N}}$ and $(g^{(1/j)})_{j\in\mathbb{N}}$ are decreasing in $j$, with $h^{**}_{1/j} \to h_0 = 0$ by (3.1.23) in Proposition 3.1.3, and $g^{(1/j)}(s) \to 0$ for every $s \in [0,\infty)$ as $g^{(1/j)}(s) \leq s/j \wedge 2\Psi(1)$ by item (ii) in Proposition 3.2.4. □

Finally, we discuss the role of the assumptions $\{\varphi(\infty), \varphi'(0^+)\} \in (0,\infty)$. Clearly, $\varphi(\infty) > 0$ not to have a trivial limit, the other alternatives are dealt with in the next result. Recall that with $\varphi'(0^+) = \infty$ we mean that the limit of the difference quotient of $\varphi$ in $t = 0$ exists and it is not finite, and that in such a case the corresponding function $h^{**}_{\overline{\varsigma}}$ is superlinear at infinity (cf. (3.1.19)). Instead, if $\varphi(\infty) = \infty$, $h^{**}_{\overline{\varsigma}}$ is subquadratic in the origin (cf. (3.1.21)).

**Corollary 3.2.18.** *Assume (Hp 1′) and (Hp 2′).*

*(a) If (Hp 4′) holds, and $\varphi$ is continuous, increasing with $\varphi(\infty) = \infty$, then $\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = F_{\overline{\varsigma}}$ (cf. (3.2.3)) if $\overline{\varsigma} \in (0,\infty)$. Instead, if $\overline{\varsigma} = \infty$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u,v) = \int_{J_u} g(|[u]|)\mathrm{d}\mathcal{H}^0 \tag{3.2.74}$$

*if $u \in GSBV(O)$ with $u' = 0$ $\mathcal{L}^1$-a.e. on $O$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, $\infty$ otherwise on $L^1$, and $g$ is defined in (3.2.9);*

*(b) if (Hp 3) holds, $\varphi^{-1}(0) = 0$, $\varphi'(0^+) = 0$, then for every $\overline{\varsigma} \in (0, \infty]$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) = 0 \tag{3.2.75}$$

*if $u \in L^1(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, and $\infty$ otherwise on $L^1$;*

*(c) if (Hp 3) holds, $\varphi^{-1}(0) = 0$, $\varphi'(0^+) = \infty$, then for every $\overline{\varsigma} \in (0, \infty]$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) = \int_O h^{**}_{\overline{\varsigma}}(|u'|)\,dx + 2\Psi(1)\mathcal{H}^0(J_u) \tag{3.2.76}$$

*if $u \in SBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, and $\infty$ otherwise on $L^1(O, \mathbb{R}^2)$;*

*(d) if $\varphi$ is continuous, increasing, with $\varphi^{-1}(0) = 0$, and $\varphi(\infty) = \varphi'(0^+) = \infty$, then the equality in (3.2.76) holds for every $\overline{\varsigma} \in (0, \infty)$, while for $\overline{\varsigma} = \infty$*

$$\Gamma(L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) = 2\Psi(1)\mathcal{H}^0(J_u) \tag{3.2.77}$$

*if $u \in SBV(O)$ with $u' = 0$ $\mathcal{L}^1$-a.e. on $O$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, $\infty$ otherwise on $L^1$.*

*Proof.* The general strategy is to compare each $\mathcal{F}_\varepsilon$ with an auxiliary functional $\mathcal{F}^{(j)}_\varepsilon$ either from below or from above according to the case. For every $j \in \mathbb{N}$, we may apply Theorem 3.2.1 to $(\mathcal{F}^{(j)}_\varepsilon)_\varepsilon$, letting $F^{(j)}_{\overline{\varsigma}}$ be the corresponding $\Gamma$-limit, we then pass to the limit as $j \to \infty$ to obtain an estimate from above of the $\Gamma$-limsup or from below for the $\Gamma$-liminf with the functional in the corresponding statement.

**Proof of (a).** If $\overline{\varsigma} \in (0, \infty)$, we can argue as in Theorem 3.2.1. Indeed, the only difference is that the bulk energy density $h^{**}_{\overline{\varsigma}}$ is sub-quadratic close to the origin (cf. (3.1.21)). Instead, if $\overline{\varsigma} = \infty$ let $\mathcal{F}^{(j)}_\varepsilon$ be obtained substituting $\varphi$ with $j \wedge \varphi(t)$, then $\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) \geq F^{(j)}_\infty$. Note that the surface energy densities are given by $g$ in (3.1.14) for every $j \in \mathbb{N}$, instead the bulk energy densities equal to $jt^2$ by item (ii) in Lemma 3.1.3. The lower bound then follows. The upper bound is a consequence of Proposition 3.2.15.

**Proof of (b).** Let $\mathcal{F}^{(j)}_\varepsilon$ be obtained by substituting $\varphi$ in the definition of $\mathcal{F}_\varepsilon$ with the function given by ${}^t\!/\!_j$ on the connected component of the set $\{t \in (0, \infty) : \varphi(t) < {}^t\!/\!_j\}$ whose closure contains the origin, and equal to $\varphi$ otherwise. Then, $\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) \leq F^{(j)}_{\overline{\varsigma}}$. In view of $g^{(j)}(s) = g_1(j^{1/2}s)$ (cf. (3.2.11)), we can infer that $g^{(j)}(s) \to 0$ as $j \to \infty$, for every $s \in [0, \infty)$. Hence, for every $\overline{\varsigma} \in (0, \infty]$ a rough upper bound for $\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon$ is given by

$$\widetilde{F}_{\overline{\varsigma}}(u, v) = \varphi(\infty) \int_O |u'|^2\,dx$$

if $u \in GBV(O)$ and $v = 1$ $\mathcal{L}^1$-a.e. on $O$, and $\infty$ otherwise on $L^1$. The $L^1$ lower semicontinuous envelope of $\widetilde{F}_{\overline{\varsigma}}$ coincides with the functional on the right hand side of (3.2.75), as the class of piecewise constant functions with a finite number of jumps is dense in $L^1$.

**Proof of (c).** Let $\widetilde{\mathscr{F}}_{\overline{\varsigma}}$ be the functional on the right hand side of (3.2.76), and let $\mathcal{F}^{(j)}_\varepsilon$ be obtained substituting $\varphi$ with $jt \wedge \varphi(t)$ for every $j \in \mathbb{N}$. Then $\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, v) \geq F^{(j)}_{\overline{\varsigma}}$, with $g^{(j)}(s) = g_1(js) \to$

$2\Psi(1)\chi_{(0,\infty)}(s)$, and $h^{**}_{\overline{\varsigma},j} \leq h^{**}_{\overline{\varsigma},j+1}$ with $h^{**}_{\overline{\varsigma},j} \to h^{**}_{\overline{\varsigma}}$ for every $j \in \mathbb{N}$. Indeed, the latter assertion is trivial if $\overline{\varsigma} = \infty$ thanks to the identity $h^{**}_{\infty,j}(t) = \varphi(\infty)t^2$ for every $j \in \mathbb{N}$ and $t \geq 0$ (cf. item (iii) in Proposition 3.1.3). Instead, if $\overline{\varsigma} \in (0,\infty)$ as $h_{\overline{\varsigma},j}(t) = \inf_{[0,\infty)}\{(\frac{1}{\tau} \wedge \varphi(\frac{1}{j\tau}))t^2 + \frac{\overline{\varsigma}^2}{4}j\tau\} \leq h_{\overline{\varsigma}}(t) \leq \varphi(\infty)t^2$, a minimum point $\tau_j$ satisfy $j\tau_j \leq \frac{4}{\overline{\varsigma}^2}\varphi(\infty)t^2$. Thus, being $\varphi$ bounded, we have $h_{\overline{\varsigma},j}(t) = \varphi(\frac{1}{j\tau_j})t^2 + \frac{\overline{\varsigma}^2}{4}j\tau_j \geq h_{\overline{\varsigma}}(t)$ for $j$ sufficiently large. More precisely, for every $M > 0$, $h_{\overline{\varsigma},j} = h_{\overline{\varsigma}}$ on $[0,M]$ for $j$ sufficiently large. The conclusion then follows arguing as to establish (3.1.22) in Proposition 3.1.3. Thus, $\Gamma\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon \geq \widetilde{\mathscr{F}_{\overline{\varsigma}}}$, and the upper bound follows as in Proposition 3.2.15 if $\overline{\varsigma} = \infty$, and Proposition 3.2.10 otherwise (in the latter case $h^{**}_{\overline{\varsigma}}$ has superlinear growth at infinity, cf. (3.1.19) in Proposition 3.1.3).

**Proof of (d).** If $\overline{\varsigma} = \infty$ we consider $j \wedge \varphi(t)$ and use the approximation argument in item (a) to conclude. Instead, if $\overline{\varsigma} \in (0,\infty)$ we consider $jt \wedge \varphi(t)$ and use the approximation argument in item (c) to conclude. $\square$

We point out that in case $\varphi'(0^+) = \infty$ and $\overline{\varsigma} \in (0,\infty)$, $h^{**}_{\overline{\varsigma}}$ does not necessarily coincide with $\varphi(\infty)t^2$. Indeed, taking $\varphi(t) = t^\alpha \wedge 1$, $\alpha \in (0,1)$, explicit calculations give $h^{**}_{\overline{\varsigma}}(t) = (t^2 \wedge (1+\alpha)(\frac{\overline{\varsigma}^2}{4\alpha})^{\frac{\alpha}{1+\alpha}} t^{\frac{2}{1+\alpha}})^{**}$, which is sub-quadratic for large values of $t$.

## 3.3 The vectorial case

In this section we examine the vectorial and multidimensional case of Theorem 3.1.1, considering the case $n \geq 2$, $N \geq 1$, with $O \subset \mathbb{R}^n$ bounded open set with Lipschitz boundary. The analysis follows the approach of [CFI24a] within the geometrically nonlinear setting and is carried out under assumptions (Hp 1)–(Hp 5) stated in Section 3.1, which ensure the structural properties required for the $\Gamma$–convergence analysis.

Throughout this section we restrict our attention to the linear cohesive regime, corresponding to

$$\overline{\varsigma} \in (0,\infty), \tag{3.3.1}$$

which represents the physically most relevant situation in this framework. In particular, the family of phase-field functionals $\mathcal{F}_\varepsilon : L^1(O,\mathbb{R}^{N+1}) \times \mathcal{A}(O) \to [0,\infty]$ has the following general form

$$\mathcal{F}_\varepsilon(u,v,A) = \begin{cases} \displaystyle\int_A \Big(\varphi(\varepsilon f^2(v))\Phi(\nabla u) + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x, & \text{if } (u,v) \in H^1(O,\mathbb{R}^N \times [0,1]), \\ \infty, & \text{otherwise.} \end{cases}$$

We recall also that the $\Gamma$–limit functional $\mathscr{F}_{\overline{\varsigma}} : L^1(O,\mathbb{R}^N) \times \mathcal{A}(O) \to [0,\infty]$ in Theorem 3.1.1 is given by

$$\mathscr{F}_{\overline{\varsigma}}(u,A) := \begin{cases} \displaystyle\int_A h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u)\,\mathrm{d}x + \int_A h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\mathrm{d}D^c u) + \int_{J_u\cap A} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}, & \\ & \text{if } u \in (GBV(O))^N, \\ \infty, & \text{otherwise,} \end{cases}$$

where $h^{\mathrm{qc}}_{\overline{\varsigma}}$ denotes the quasiconvex envelope of $h_{\overline{\varsigma}}$ (cf. (3.1.9)),

$$h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\xi) := \limsup_{t\to\infty} \frac{h^{\mathrm{qc}}_{\overline{\varsigma}}(t\xi)}{t} \tag{3.3.2}$$

is the *linear* recession function of $h^{\mathrm{qc}}_{\bar{\varsigma}}$, and the surface energy density $g : \mathbb{R}^N \times \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$g(\zeta,\nu) := \inf\left\{\liminf_{j\to\infty} \mathcal{F}^{\infty}_{\varepsilon_j}(u_j, v_j, Q^\nu) \ : \ \|u_j - u_{\zeta,\nu}\|_{L^1(Q^\nu,\mathbb{R}^N)} \to 0,\ \varepsilon_j \to 0\right\},$$

where $u_{\zeta,\nu}$ is defined in (2.0.3),

$$\mathcal{F}^{\infty}_{\varepsilon}(u,v,A) := \int_A \Big(\varphi'(0)\varepsilon f^2(v)\Phi_\infty(\nabla u) + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x,$$

and $\Phi_\infty(\xi) = \lim_{t\to\infty}\frac{\Phi(t\xi)}{t^2}$ is the *quadratic* recession function of $\Phi$ (cf. Section 3.1).

We also point out that, in the definition of $g(\zeta,\nu)$, the class of admissible test functions can be restricted to sequences $(u_j, v_j) \to (u_{\zeta,\nu}, 1)$ strongly in $L^2(Q^\nu, \mathbb{R}^{N+1})$ (see Proposition 3.3.3).

### 3.3.1 Properties of the surface energy densities

In this section we collect a number of properties of the surface energy density $g$, which illustrate the cohesive nature of the limit functional $\mathscr{F}_{\bar{\varsigma}}$. The discussion is divided into two parts.

In the first one, we show that the definition of $g(\zeta,\nu)$ (see (3.1.11)) can be equivalently restricted to $L^2$–convergent sequences satisfying periodic boundary conditions in $(n-1)$ mutually orthogonal directions orthogonal to $\nu$. This reformulation will be particularly useful in Section 3.3.3, where it plays a key role in the proof of the upper bound inequality (Step 3 of Theorem 3.3.17), and it also provides a convenient framework for establishing an alternative formulation of $g$ (see, e.g., Proposition 3.3.4).

The second part is then devoted to a detailed analysis of the structural properties of $g$.

#### Alternative formulation for $g(\zeta,\nu)$

As a first step, we establish the $L^\infty$-boundness of $g$ which is the content of the next lemma.

**Lemma 3.3.1.** *Under the assumptions of Theorem 3.1.1, there is a constant $L > 0$ such that*

$$g(\zeta,\nu) \le L \qquad \text{for all } (\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}. \tag{3.3.3}$$

*Proof.* Fix $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$. Then define

$$u_j(x) := \hat{u}_j(x\cdot\nu), \qquad v_j(x) := \hat{v}_j(x\cdot\nu),$$

where the profiles $\hat{u}_j, \hat{v}_j$ are given by the affine interpolation between the prescribed boundary values:

$$\hat{u}_j(t) := \begin{cases} 0, & \text{if } t \le -\varepsilon_j, \\ \zeta, & \text{if } t \ge \varepsilon_j, \\ \text{Aff. interp.}, & \text{if } -\varepsilon_j < t < \varepsilon_j, \end{cases} \qquad \hat{v}_j(t) := \begin{cases} 0, & \text{if } |t| \le \varepsilon_j, \\ 1, & \text{if } |t| \ge 2\varepsilon_j, \\ \text{Aff. interp.}, & \text{if } |t| \in (\varepsilon_j, 2\varepsilon_j). \end{cases}$$

In particular, since the term $\varphi'(0)\varepsilon_j f^2(\hat{v}_j)\Phi_\infty(\nabla\hat{u}_j)$ is identically 0 and $\omega$ is non-decreasing (cf. (Hp 2)), we obtain

$$\mathcal{F}^{\infty}_{\varepsilon_j}(u_j,v_j;Q^\nu) = 0 + \int_{Q^\nu\cap\{|x\cdot\nu|\le 2\varepsilon_j\}} \frac{\omega(1-\hat{v}_j)}{4\varepsilon_j}\,\mathrm{d}x + \int_{Q^\nu\cap\{\varepsilon_j\le|x\cdot\nu|\le 2\varepsilon_j\}} \varepsilon_j|\nabla\hat{v}_j|^2\,\mathrm{d}x$$
$$\le 0 + 4\varepsilon_j\frac{\omega(1)}{4\varepsilon_j} + 2\varepsilon_j\varepsilon_j\frac{1}{\varepsilon_j^2} = \omega(1)+2,$$

concluding the proof. □

The next two propositions are devoted to proving the $L^2$-periodic reduction of the class of competitors, previously mentioned, related to the definition of $g$ (cf. (3.1.11)).

With this aim, we fix a mollifier kernel $\phi_1 \in C^\infty_c(B_1)$, with $\int_{B_1}\phi_1\,\mathrm{d}x = 1$, and set $\phi_\varepsilon(x) := \varepsilon^{-n}\phi_1(x/\varepsilon) \in C^\infty_c(B_\varepsilon)$.

**Proposition 3.3.2.** *Let $(\zeta,\nu)\in\mathbb{R}^N\times\mathbb{S}^{n-1}$ and assume that an optimal sequence in (3.1.11) converges in $L^2(Q^\nu,\mathbb{R}^{N+1})$. Then, under the assumptions of Theorem 3.1.1, there exist $\varepsilon_j\to 0$ and $(\tilde{u}_j,\tilde{v}_j)\to(u_{\zeta,\nu},1)$ in $L^2(Q^\nu,\mathbb{R}^{N+1})$, with $\tilde{v}_j\in[0,1]$ $\mathcal{L}^n$-a.e. in $Q^\nu$, such that*

$$\lim_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(\tilde{u}_j,\tilde{v}_j;Q^\nu) = g(\zeta,\nu)$$

*where*

$$\tilde{u}_j = u_{\zeta,\nu}*\phi_{\varepsilon_j}\,, \qquad \tilde{v}_j = \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j\}}*\phi_{\varepsilon_j} \qquad \text{on } \partial Q^\nu. \tag{3.3.4}$$

*Proof.* Throughout the proof, $c>0$ denotes a generic constant whose value may change from line to line. **Step 1. Construction of $\tilde{u}_j$ and $\tilde{v}_j$.** By hypothesis there exist $\varepsilon_j\to 0$, together with $v_j$ and $u_j\to u_{\zeta,\nu}$ in $L^2(Q^\nu;\mathbb{R}^N)$, such that

$$g(\zeta,\nu) = \lim_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu). \tag{3.3.5}$$

For convenience, set

$$U_j := (u_{\zeta,\nu})*\phi_{\varepsilon_j}, \qquad V_j := \chi_{\{|x\cdot\nu|\ge 2\varepsilon_j\}}*\phi_{\varepsilon_j}. \tag{3.3.6}$$

Then $\|u_j-U_j\|_{L^2(Q^\nu,\mathbb{R}^N)}\to 0$. Moreover, by construction, $U_j\equiv u_{\zeta,\nu}$ on $\{|x\cdot\nu|\ge\varepsilon_j\}$, $V_j\equiv 0$ on $\{|x\cdot\nu|\le\varepsilon_j\}$, and $V_j\equiv 1$ on $\{|x\cdot\nu|\ge 3\varepsilon_j\}$. Hence, since $\Phi_\infty(0)=0$ and $f(0)=0$ (cf. Section 3.1),

$$\mathcal{F}^\infty_{\varepsilon_j}(U_j,V_j;Q^\nu) = \mathcal{F}^\infty_{\varepsilon_j}(0,V_j;Q^\nu)\le c\,,$$

since $\|\nabla V_j\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\le\frac{c}{\varepsilon_j}$. Next, even though $f(t)$ diverges as $t\to 1$, thanks to its continuity we can choose $\eta_j\to 0$ so that

$$f(1-\eta_j)\|u_j-U_j\|^2_{L^2(Q^\nu)}\to 0. \tag{3.3.7}$$

Then set $K_j := \lfloor\frac{f(1-\eta_j)}{\varepsilon_j}\rfloor$, where $\lfloor\cdot\rfloor$ is the integer part (cf. Chapter 2) and define $R^j_k := Q^\nu_{1-k\varepsilon_j}\setminus Q^\nu_{1-(k+1)\varepsilon_j}$. By (3.3.5) and an averaging argument, one can pick $k_j\in\{K_j+1,\dots,2K_j\}$ so that, setting $R_j := R^j_{k_j}$,

$$\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;R_j) + \mathcal{F}^\infty_{\varepsilon_j}(U_j,V_j;R_j) + \frac{f^2(1-\eta_j)}{\varepsilon_j}\|u_j-U_j\|^2_{L^2(R_j)} \le \frac{c}{K_j} + \frac{f^2(1-\eta_j)}{K_j\varepsilon_j}\|u_j-U_j\|^2_{L^2(Q^\nu)}. \tag{3.3.8}$$

Let $\theta_j \in C_c^1(Q^\nu_{1-k_j\varepsilon_j})$ satisfy $\theta_j = 1$ on $Q^\nu_{1-(k_j+1)\varepsilon_j}$ and $|\nabla\theta_j| \le 3/\varepsilon_j$, and define

$$\tilde u_j := \theta_j u_j + (1-\theta_j)U_j.$$

We now construct $\tilde v_j$. In the core region it should coincide with $v_j$, outside with $V_j$, and on the transition layer it must lie below both and also below $1-\eta_j$. Set first

$$\hat v_j(x) := \min\{1, 1-\eta_j + \varepsilon_j^{-1}\mathrm{dist}(x, R_j)\}, \tag{3.3.9}$$

which equals $1-\eta_j$ on $R_j$ and 1 at distance larger than $\eta_j\varepsilon_j$. Then define

$$\hat V_j(x) := \min\{1, V_j(x) + \varepsilon_j^{-1}\mathrm{dist}(x, Q^\nu \setminus Q^\nu_{1-(k_j+1)\varepsilon_j})\}, \tag{3.3.10}$$

which agrees with $V_j$ outside $Q^\nu_{1-(k_j+1)\varepsilon_j}$ and equals 1 inside $Q^\nu_{1-(k_j+3)\varepsilon_j}$ as well as for $|x\cdot\nu| \ge 3\varepsilon_j$. Finally, set

$$v_j^* := \min\{1, v_j + 2(k_j\varepsilon_j)^{-1}\mathrm{dist}(x, Q^\nu_{1-k_j\varepsilon_j})\}\,,$$

we define $\tilde v_j := \min\{v_j^*, \hat V_j, \hat v_j\}$. On $\partial Q^\nu$ the first and last terms equal 1, hence $\tilde v_j = \hat V_j = V_j$.

**Step 2. Estimate of the bulk term.** By construction, $|\nabla \tilde u_j| \le |\nabla u_j| + |\nabla U_j| + \frac{3}{\varepsilon_j}|u_j - U_j|$ and therefore, using (3.1.7), $\Phi_\infty(\nabla\tilde u_j) \le c\Phi_\infty(\nabla u_j) + c\Phi_\infty(\nabla U_j) + \frac{c}{\varepsilon_j^2}|u_j-U_j|^2$ on $R_j$. Moreover, since $\tilde v_j \le \min\{v_j, V_j, 1-\eta_j\}$ on $R_j$ and $\tilde v_j = V_j = 0$ on $\{\nabla U_j \neq 0\} \cap R_j$, Lemma 3.1.2 yields

$$\varepsilon_j f^2(\tilde v_j)\Phi_\infty(\nabla\tilde u_j) \le c f^2(v_j)\Phi_\infty(\nabla u_j) + c\varepsilon_j f^2(1-\eta_j)\frac{|u_j - U_j|^2}{\varepsilon_j^2} \quad \text{on } R_j.$$

Integrating over $R_j$ and invoking (3.3.8), we obtain

$$\int_{R_j} \varphi'(0)\varepsilon_j f^2(\tilde v_j)\Phi_\infty(\nabla\tilde u_j)\,\mathrm{d}x \le \frac{c}{K_j} + c f^2(1-\eta_j)\frac{\|u_j - U_j\|^2_{L^2(Q^\nu)}}{K_j\varepsilon_j}.$$

In particular, by definition of $K_j$ and (3.3.7), we obtain

$$\limsup_{j\to\infty} f^2(1-\eta_j)\frac{\|u_j - U_j\|^2_{L^2(Q^\nu)}}{K_j\varepsilon_j} = \limsup_{j\to\infty} f(1-\eta_j)\|u_j - U_j\|^2_{L^2(Q^\nu)} = 0,$$

and thus

$$\limsup_{j\to\infty}\int_{R_j} \varphi'(0)\varepsilon_j f^2(\tilde v_j)\Phi_\infty(\nabla\tilde u_j)\,\mathrm{d}x = 0.$$

Using again that the supports of $\nabla U_j$ and $V_j$ are disjoint, we obtain

$$\int_{Q^\nu\setminus Q^\nu_{1-k_j\varepsilon_j}} \varphi'(0)\varepsilon_j f^2(V_j)\Phi_\infty(\nabla U_j)\,\mathrm{d}x = 0.$$

Consequently,

$$\limsup_{j\to\infty}\int_{Q^\nu} \varphi'(0)\varepsilon_j f^2(\tilde v_j)\Phi_\infty(\nabla\tilde u_j)\,\mathrm{d}x \le \limsup_{j\to\infty}\int_{Q^\nu} \varphi'(0)\varepsilon_j f^2(v_j)\Phi_\infty(\nabla u_j)\,\mathrm{d}x. \tag{3.3.11}$$

**Step 3. Estimate of the phase-field energy.** By construction of $\tilde{v}_j$,

$$\mathcal{F}_{\varepsilon_j}^{\infty}(0,\tilde{v}_j;Q^\nu) \le \mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j^*;Q^\nu) + \mathcal{F}_{\varepsilon_j}^{\infty}(0,\hat{V}_j;Q^\nu) + \mathcal{F}_{\varepsilon_j}^{\infty}(0,\hat{v}_j;Q^\nu). \tag{3.3.12}$$

From (3.3.9) we have $|1-\hat{v}_j| \le \eta_j$, $\mathcal{L}^n(\{\hat{v}_j \neq 1\}) \le c\varepsilon_j$, and $\varepsilon_j|\nabla\hat{v}_j| \le c$ with $\mathcal{L}^n(\{\nabla\hat{v}_j \neq 0\}) \le c\varepsilon_j\eta_j$, hence

$$\mathcal{F}_{\varepsilon_j}^{\infty}(0,\hat{v}_j;Q^\nu) = \int_{Q^\nu}\Big(\frac{\omega(1-\hat{v}_j)}{4\varepsilon_j} + \varepsilon_j|\nabla\hat{v}_j|^2\Big)\,\mathrm{d}x \le c(\tau_j+\eta_j), \tag{3.3.13}$$

where $\tau_j := \max_{t\in[0,\eta_j]}\omega(t)$. From the definitions of $V_j$ and $\hat{V}_j$ we infer $\mathcal{L}^n(\{\hat{V}_j \neq 1\}) \le c\eta_j\varepsilon_j$ and $\varepsilon_j|\nabla\hat{V}_j| \le c$, so that

$$\mathcal{F}_{\varepsilon_j}^{\infty}(0,\hat{V}_j;Q^\nu) \le c\eta_j.$$

Finally, since $v_j^* = v_j$ in $Q^\nu_{1-k_j\varepsilon_j}$, $1-v_j^* \le 1-v_j$, and $|\nabla v_j^*| \le |\nabla v_j| + 2/(k_j\varepsilon_j)$ in $Q^\nu \setminus Q^\nu_{1-k_j\varepsilon_j}$, the monotonicity of $\omega$ (cf. (Hp 2)) yields

$$\begin{aligned}\mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j^*;Q^\nu) &\le \mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j;Q^\nu) + \frac{4\varepsilon_j\mathcal{L}^n(Q^\nu\setminus Q^\nu_{1-k_j\varepsilon_j})}{k_j^2\varepsilon_j^2} \\ &\quad + \frac{4}{k_j\varepsilon_j^{1/2}}\mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j;Q^\nu)^{1/2}\mathcal{L}^n(Q^\nu\setminus Q^\nu_{1-k_j\varepsilon_j})^{1/2} \le \mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j;Q^\nu) + \frac{c}{k_j^{1/2}}.\end{aligned}$$

Since $k_j \ge K_j+1 \to \infty$, $\eta_j \to 0$ and $\tau_j \to 0$, from (3.3.12) and (3.3.13) we deduce

$$\limsup_{j\to\infty}\mathcal{F}_{\varepsilon_j}^{\infty}(0,\tilde{v}_j;Q^\nu) \le \limsup_{j\to\infty}\mathcal{F}_{\varepsilon_j}^{\infty}(0,v_j;Q^\nu).$$

Together with (3.3.11), this completes the proof. □

We are now in a position to carry out the reduction on the class of test sequences appearing in the definition of $g(\cdot,\nu)$ in (3.1.11). Inspired by De Giorgi's slicing (or averaging) argument on the codomain, the family $\mathcal{T}_k$, defined in (2.0.10), replace sequences converging in $L^1$ with new ones sharing the same $L^1$ limit but uniformly bounded in $L^\infty$. This modification can be achieved at the expense of an arbitrarily small increase in the energy.

**Proposition 3.3.3.** *Let $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$. Then, under the assumptions of Theorem 3.1.1, for any sequence $\tilde{\varepsilon}_j \downarrow 0$, there exist $(\tilde{u}_j,\tilde{v}_j) \to (u_{\zeta,\nu},1)$ in $L^2(Q^\nu;\mathbb{R}^{N+1})$, with $\tilde{v}_j(x) \in [0,1]$ for $\mathcal{L}^n$-a.e. $x \in Q^\nu$, such that*

$$\lim_{j\to\infty}\mathcal{F}_{\tilde{\varepsilon}_j}^{\infty}(\tilde{u}_j,\tilde{v}_j;Q^\nu)=g(\zeta,\nu) \tag{3.3.14}$$

*and*

$$\tilde{u}_j = (u_{\zeta,\nu}) * \phi_{\tilde{\varepsilon}_j}\,, \qquad \tilde{v}_j = \chi_{\{|x\cdot\nu|\ge 2\tilde{\varepsilon}_j\}} * \phi_{\tilde{\varepsilon}_j} \qquad \text{on } \partial Q^\nu.$$

*Proof.* **Step 1. Reduction to an $L^2$–convergent minimizing sequence.** Let $\varepsilon_j \to 0$ and $(u_j,v_j)$ be such that $u_j \to u_{\zeta,\nu}$ in $L^1(Q^\nu,\mathbb{R}^N)$ and

$$g(\zeta,\nu) = \lim_{j\to\infty}\mathcal{F}_{\varepsilon_j}^{\infty}(u_j,v_j;Q^\nu).$$

By Lemma 3.3.1 we have $g(\zeta,\nu)<\infty$; hence, owing to the term ${}^{\omega(1-v_j)}/_{\varepsilon_j}$, we get $v_j\to 1$ in measure. Using the uniform $L^\infty$–bound on $v_j$, and possibly extracting a subsequence, it follows that $v_j\to 1$ in $L^2(Q^\nu)$.

We now claim that for every $j,M\in\mathbb{N}$ there exists $k_{M,j}\in\{M+1,\dots,2M\}$ such that

$$\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_{k_{M,j}}(u_j),v_j;Q^\nu)\le\Big(1+\frac{C^2}{M}\Big)\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)\,. \tag{3.3.15}$$

If $a_M>1+|\zeta|=1+\|u_{\zeta,\nu}\|_{L^\infty(Q^\nu)}$, where $(a_k)_k$ is the sequence defined in (2.0.10), then $\mathcal{T}_{k_M,j}(u_j)\to u_{\zeta,\nu}$ in $L^2(Q^\nu;\mathbb{R}^N)$, and by (3.3.15),

$$\limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_{k_M,j}(u_j),v_j;Q^\nu)\le\Big(1+\frac{C^2}{M}\Big)g(\zeta,\nu)\,.$$

In particular, letting $M\to\infty$, we infer that

$$g(\zeta,\nu)=\inf\{\liminf_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu):\|u_j-u_{\zeta,\nu}\|_{L^2(Q^\nu)}\to 0,\ \varepsilon_j\to 0\},$$

thereby completing the argument of this step.

It remains to establish (3.3.15). For each $k\in\{M+1,\dots,2M\}$, consider $\mathcal{T}_k(u_j)$ and observe that

$$\begin{aligned}\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j),v_j;Q^\nu)&=\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;\{|u_j|\le a_k\})\\&\quad+\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j),v_j;\{a_k<|u_j|<a_{k+1}\})+\mathcal{F}^\infty_{\varepsilon_j}(0,v_j;\{|u_j|\ge a_{k+1}\})\,.\end{aligned} \tag{3.3.16}$$

For the second term in (3.3.16), the growth bounds on $\Phi_\infty$ (cf. (3.1.7)) and the fact that $Lip(\mathcal{T}_k)\le 1$ imply

$$\begin{aligned}&\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j),v_j;\{a_k<|u_j|<a_{k+1}\})\\&\quad\le C^2\int_{\{a_k<|u_j|<a_{k+1}\}}\varphi'(0)\varepsilon_j f^2(v_j)\Phi_\infty(\nabla u_j)\,\mathrm{d}x+\mathcal{F}^\infty_{\varepsilon_j}(0,v_j;\{a_k<|u_j|<a_{k+1}\})\,.\end{aligned} \tag{3.3.17}$$

Combining (3.3.16) and (3.3.17), and using that $\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;A)+\mathcal{F}^\infty_{\varepsilon_j}(0,v_j;B)\le\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;A\cup B)$ for disjoint $A,B$, we deduce

$$\mathcal{F}^\infty_{\varepsilon_j}(\mathcal{T}_k(u_j),v_j;Q^\nu)\le\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)+C^2\int_{\{a_k<|u_j|<a_{k+1}\}}\varphi'(0)\varepsilon_j f^2(v_j)\Phi_\infty(\nabla u_j)\,\mathrm{d}x\,.$$

By averaging over $k\in\{M+1,\dots,2M\}$, one finds $k_{M,j}$ satisfying (3.3.15).

**Step 2. Conclusion.** By Step 1, we may assume that an optimal sequence for $g(\zeta,\nu)$ in (3.1.11) converges in $L^2(Q^\nu;\mathbb{R}^{N+1})$. Let $(\varepsilon_k,u_k,v_k)$ be the sequence given by Proposition 3.3.2. Since $\lim_{k\to\infty}\lim_{j\to 0}\tilde\varepsilon_j/\varepsilon_k=0$, we may choose a nondecreasing sequence $k(j)\to\infty$ such that $\lambda_j:=\tilde\varepsilon_j/\varepsilon_{k(j)}\to 0$. For any $r>0$, set $\tilde Q^\nu_r:=\Pi_\nu\cap Q^\nu_r$, where $\Pi_\nu$ is the hyperplane relative to $\nu$ (cf. (2.0.2)), and select points $x_1,\dots,x_{I_j}\in\tilde Q^\nu$, with $I_j:=\lfloor 1/\lambda_j\rfloor^{n-1}$, so that the sets $x_i+\tilde Q^\nu_{\lambda_j}$ are pairwise disjoint in $\tilde Q^\nu$. Define

$$\tilde u_j(x):=\begin{cases}u_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x-x_i\in Q^\nu_{\lambda_j}\text{ for some } i,\\ \tilde U_j(x), & \text{otherwise in } Q^\nu,\end{cases}$$

and

$$\tilde{v}_j(x) := \begin{cases} v_{k(j)}(\frac{x-x_i}{\lambda_j}), & \text{if } x-x_i \in Q^\nu_{\lambda_j} \text{ for some } i,\\ \tilde{V}_j(x), & \text{otherwise in } Q^\nu, \end{cases}$$

where $\tilde{U}_j := u_{\zeta,\nu} * \varphi_{\tilde{\varepsilon}_j}$ and $\tilde{V}_j := \chi_{\{|x\cdot\nu|\geq 2\tilde{\varepsilon}_j\}} * \varphi_{\tilde{\varepsilon}_j}$. One easily checks that $\tilde{U}_j(x) = U_{k(j)}(\frac{x-y}{\lambda_j})$ and $\tilde{V}_j(x) = V_{k(j)}(\frac{x-y}{\lambda_j})$ for all $y \in \Pi_\nu$, with $U_{k(j)} := u_{\zeta,\nu} * \phi_{\varepsilon_{k(j)}}$ and $V_{k(j)} := \chi_{\{|x\cdot\nu|\geq 2\varepsilon_{k(j)}\}} * \phi_{\varepsilon_{k(j)}}$. From the boundary conditions (3.3.4), it follows that $(u_j^*, v_j^*) \in W^{1,2}(Q^\nu;\mathbb{R}^{N+1})$. By construction, $\tilde{U}_j \equiv u_{\zeta,\nu}$ on $\{|x\cdot\nu| \geq \tilde{\varepsilon}_j\}$, while $\tilde{V}_j \equiv 0$ on $\{|x\cdot\nu| \leq \tilde{\varepsilon}_j\}$ and $\tilde{V}_j \equiv 1$ on $\{|x\cdot\nu| \geq 3\tilde{\varepsilon}_j\}$. Hence, using $\Phi_\infty(0) = 0$ and $f(0) = 0$, we have

$$\mathcal{F}^\infty_{\tilde{\varepsilon}_j}(\tilde{u}_j, \tilde{v}_j; Q^\nu) \leq I_j \lambda_j^{n-1} \mathcal{F}^\infty_{\varepsilon_{k(j)}}(u_{k(j)}, v_{k(j)}; Q^\nu) + c\mathcal{H}^{n-1}(\tilde{Q}^\nu \setminus \cup_i(x_i + \tilde{Q}^\nu_{\lambda_j})),$$

where $c > 0$ depends only on the mollifier kernel $\phi_1$ and on $\omega$. Since $\limsup_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_{k(j)}}(u_{k(j)}, v_{k(j)}; Q^\nu) = g(\zeta,\nu)$, letting $j \to \infty$ completes the proof. □

We next establish an equivalent expression for the surface energy $g$, analogous in structure to (3.2.17).

**Proposition 3.3.4.** *Under the assumptions of Theorem 3.1.1, for any $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$ one has*

$$g(\zeta,\nu) = \lim_{T\to\infty} \inf_{(u,v)\in\mathcal{U}^T_{\zeta,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u, v; Q^\nu_T)\,, \tag{3.3.18}$$

*where*

$$\mathcal{U}^T_{\zeta,\nu} := \Big\{(u,v) \in H^1(Q^\nu_T, \mathbb{R}^N \times [0,1])\colon v = \chi_{\{|x\cdot\nu|\geq 2\}} * \phi_1 \text{ and } u = u_{\zeta,\nu} * \phi_1 \text{ on } \partial Q^\nu_T\Big\}. \tag{3.3.19}$$

*Proof.* For every $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$ and $T > 0$, we define

$$g_T(\zeta,\nu) := \inf_{(u,v)\in\mathcal{U}^T_{\zeta,\nu}} \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u, v; Q^\nu_T)\,.$$

We first establish that

$$\limsup_{T\to\infty} g_T(\zeta,\nu) \leq g(\zeta,\nu) \quad \text{for every } (\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}. \tag{3.3.20}$$

Fix $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$ and let $T_j \uparrow \infty$ be a sequence realizing the upper limit on the left-hand side. By Proposition 3.3.3, we may consider $(u_j, v_j) \in H^1(Q^\nu;\mathbb{R}^{N+1})$ with $0 \leq v_j \leq 1$ and $(u_j, v_j) \to (u_{\zeta,\nu}, 1)$ in $L^2(Q^\nu;\mathbb{R}^{N+1})$, such that

$$u_j = u_{\zeta,\nu} * \phi_{\frac{1}{T_j}}\,, \qquad v_j = \chi_{\{|x\cdot\nu|\geq \frac{2}{T_j}\}} * \phi_{\frac{1}{T_j}} \quad \text{on } \partial Q^\nu, \tag{3.3.21}$$

and

$$\lim_{j\to\infty} \mathcal{F}^\infty_{\frac{1}{T_j}}(u_j, v_j; Q^\nu) = g(\zeta,\nu). \tag{3.3.22}$$

Define $(\tilde{u}_j(y), \tilde{v}_j(y)) := \big(u_j(\frac{y}{T_j}), v_j(\frac{y}{T_j})\big)$ for $y \in Q^\nu_{T_j}$. A simple change of variables yields

$$\frac{1}{T_j^{n-1}} \mathcal{F}^\infty_1(\tilde{u}_j, \tilde{v}_j; Q^\nu_{T_j}) = \mathcal{F}^\infty_{\frac{1}{T_j}}(u_j, v_j; Q^\nu)\,,$$

and $(\tilde{u}_j, \tilde{v}_j) \in \mathcal{U}^{T_j}_{\zeta,\nu}$ by (3.3.21). Then, combining (3.3.22) with the definition of $g_T(\zeta,\nu)$, we readily obtain (3.3.20).

To prove the converse inequality,

$$\liminf_{T\to\infty} g_T(\zeta,\nu) \geq g(\zeta,\nu)\,, \tag{3.3.23}$$

we assume, without loss of generality, that $\nu = e_n$. Fix $\rho > 0$ and take $T > 6$, depending on $\rho$, together with $(u_T, v_T) \in \mathcal{U}^T_{\zeta,e_n}$ satisfying

$$\frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(\zeta, e_n) + \rho\,. \tag{3.3.24}$$

Let $\varepsilon_j \to 0$ be such that $(\varepsilon_j T)^{-1} \in \mathbb{N}$ for every $j$ and $I_{\varepsilon_j} \subseteq \mathbb{R}^{n-1}\times\{0\}$ be the unique set with $Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d) \subseteq Q^{e_n}$ for any $d \in I_{\varepsilon_j}$, $Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d_1) \cap Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d_2) = \emptyset$ for every $d_1, d_2 \in I_{\varepsilon_j}$ with $d_1 \neq d_2$ and

$$\mathcal{H}^{n-1}\Big((Q^{e_n} \cap \Pi_{e_n}) \setminus \bigcup_{d\in I_{\varepsilon_j}} Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d)\Big) = 0.$$

Then define

$$(u_j(y), v_j(y)) := \begin{cases} \Big(u_T\Big(\frac{y}{\varepsilon_j} - d\Big), v_T\Big(\frac{y}{\varepsilon_j} - d\Big)\Big) & \text{if } y \in Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d) \text{ for some } d \in I_{\varepsilon_j}, \\ (\tilde{U}_j(y), \tilde{V}_j(y)), & \text{otherwise in } Q^{e_n}, \end{cases}$$

where $\tilde{U}_j(y) = (u_{\zeta,e_n} * \phi_1)(\frac{y}{\varepsilon_j})$, $\tilde{V}_j(y) = (\chi_{\{|x\cdot e_n|>2\}} * \phi_1)(\frac{y}{\varepsilon_j})$. Due to the boundary conditions of $(u_T, v_T)$ (cf. (3.3.19)), we have $(u_j, v_j) \in H^1(Q^{e_n}, \mathbb{R}^N \times [0,1])$. Moreover, by the Dominated Convergence Theorem, $(u_j, v_j) \to (u_{\zeta,e_n}, 1)$ in $L^1(Q^{e_n}; \mathbb{R}^{N+1})$. Since $\tilde{U}_j \equiv u_{\zeta,e_n}$ on $|x\cdot\nu| \geq \varepsilon_j$, $\tilde{V}_j \equiv 0$ on $|x\cdot\nu| \leq \varepsilon_j$, and $\tilde{V}_j \equiv 1$ on $|x\cdot\nu| \geq 3\varepsilon_j$, using $\Phi_\infty(0) = 0$, $f(0) = 0$ and $T > 6$, together with a change of variable, (3.3.24), and the fact that $\#I_{\varepsilon_j} = (\varepsilon_j T)^{1-n}$, we get

$$\begin{aligned} g(\zeta, e_n) &\leq \limsup_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^{e_n}) \\ &\leq \limsup_{j\to\infty} \Big(\sum_{d\in I_{\varepsilon_j}} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d))\Big) = \limsup_{j\to\infty} \varepsilon_j^{n-1} (\#I_{\varepsilon_j})\, \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \\ &\leq \frac{1}{T^{n-1}} \mathcal{F}^\infty_1(u_T, v_T; Q^{e_n}_T) \leq \liminf_{T\to\infty} g_T(\zeta, e_n) + \rho, \end{aligned}$$

Letting $\rho \to 0$ gives (3.3.23). Combining (3.3.20) and (3.3.23), we deduce that the limit $\lim_{T\to\infty} g_T(\zeta,\nu)$ exists and satisfies (3.3.18).

□

Building on this representation of $g$, we derive a modified version of Proposition 3.3.3, which includes an additional regularization term $\eta_\varepsilon \int \Phi(\nabla u)\, dx$. This term, which ensures the coercivity of the functionals $\mathcal{F}_\varepsilon$, is crucial for the analysis in Section 3.3.3.

**Proposition 3.3.5.** *For any sequences $\varepsilon_j \downarrow 0$ and $\eta_j \downarrow 0$ satisfying $\eta_j/\varepsilon_j \to 0$, and for every $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$, then under the assumptions of Theorem 3.1.1, there exist functions $(u_j, v_j) \to (u_{\zeta,\nu}, 1)$ in $L^2(Q^\nu;\mathbb{R}^{N+1})$, with $v_j(x) \in [0,1]$ for $\mathcal{L}^n$-a.e. $x \in Q^\nu$, such that*

$$\lim_{j\to\infty} \mathcal{F}^\infty_{\varepsilon_j}(u_j, v_j; Q^\nu) = g(\zeta,\nu)\,,$$

$$\lim_{j\to\infty} \eta_j \int_{Q^\nu} |\nabla u_j|^2 \,\mathrm{d}x = 0\,,$$

*and*

$$u_j = u_{\zeta,\nu} * \phi_{\varepsilon_j}\,, \qquad v_j = \chi_{\{|x\cdot\nu| \geq 2\varepsilon_j\}} * \phi_{\varepsilon_j} \qquad \text{on } \partial Q^\nu.$$

*Proof.* The construction follows closely the one presented in the second part of Proposition 3.3.4 (without loss of generality, we explicitly detail the case $\nu = e_n$). By the very definition of $(u_j, v_j)$ we have

$$\begin{aligned}\|\nabla u_j\|^2_{L^2(Q^{e_n})} &\leq \sum_{d\in I_{\varepsilon_j}} \|\nabla u_j\|^2_{L^2(Q^{e_n}_{\varepsilon_j T}(\varepsilon_j d))} + \frac{c}{\varepsilon_j^2}\mathcal{L}^n(Q^{e_n} \cap \{|x_n| \leq \varepsilon_j\}) \\ &= \varepsilon_j^{n-1}(\# I_{\varepsilon_j})\,\|\nabla u_T\|^2_{L^2(Q_T)} + \frac{c}{\varepsilon_j} = \frac{\|\nabla u_T\|^2_{L^2(Q_T)}}{T^{n-1}} + \frac{c}{\varepsilon_j} \leq \frac{c_T}{\varepsilon_j}\end{aligned}$$

where $c_T > 0$ is a constant depending only on the mollifier kernel $\phi_1$ and $T$. The proof is concluded by choosing the sequence $T_j \to \infty$ sufficiently slowly to ensure that $\frac{\eta_j c_{T_j}}{\varepsilon_j} \to 0$. $\square$

The next corollary shows that, in the isotropic setting, the surface energy density $g$ does not depend on the orientation $\nu$, but only on the amplitude of the jump. In this case, it can be expressed in terms of the scalar function $g_{\text{scal}}$ (see (3.1.14)).

**Corollary 3.3.6.** *If $\Phi_\infty(\xi) = |\xi|^2$, then under the assumptions of Theorem 3.1.1, it holds that $g(\zeta,\nu) = g_{\text{scal}}(|\zeta|)$ for every $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$, where $g_{\text{scal}}$ is defined in (3.1.14).*

*Proof.* From (3.2.17) and a translation argument, one can derive the following representation formula for $g_{\text{scal}}$:

$$g_{\text{scal}}(s) = \lim_{T\uparrow\infty} \inf_{(\alpha,\beta)\in\mathfrak{U}_s(-T/2,T/2)} \mathscr{F}^\infty_1(\alpha,\beta;(-T/2,T/2)),$$

where

$$\mathscr{F}^\infty_1(\alpha,\beta;(-T/2,T/2)) := \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\varphi'(0)f^2(\beta)|\alpha'|^2 + \tfrac{(1-\beta)^2}{4} + |\beta'|^2\right)\mathrm{d}x$$

and $\mathfrak{U}_s(-T/2,T/2)$ is given in (3.2.8). Let $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$ with $\zeta \neq 0$. We first establish that

$$g(\zeta,\nu) \geq g_{\text{scal}}(|\zeta|). \tag{3.3.25}$$

For any $T > 0$ and $(u,v) \in \mathcal{U}^T_{\zeta,\nu}$ (cf. Proposition 3.3.4), Fubini's theorem together with the definition of the slices

$$u^\nu_y(t) := \frac{\zeta}{|\zeta|}\cdot u(y+t\nu), \quad v^\nu_y(t) := v(y+t\nu),$$

valid for $\mathcal{H}^{n-1}$-a.e. $y \in \tilde{Q}^\nu_T := Q^\nu_T \cap \Pi_\nu$, implies that $(u^\nu_y, v^\nu_y) \in \mathfrak{U}_{|\zeta|}(-{}^T\!/2, {}^T\!/2)$ and

$$\frac{1}{T^{n-1}}\mathcal{F}^\infty_1(u,v;Q^\nu_T) \;\geq\; \frac{1}{T^{n-1}}\int_{\tilde{Q}^\nu_T} \mathscr{F}^\infty_1(u^\nu_y, v^\nu_y; (-{}^T\!/2, {}^T\!/2))\,\mathrm{d}\mathcal{H}^{n-1}(y) \;\geq\; \inf_{(\alpha,\beta)\in\mathcal{U}^T_{|\zeta|}} \mathscr{F}^\infty_1(\alpha,\beta;(-{}^T\!/2, {}^T\!/2)).$$

Taking the infimum over all $(u,v) \in \mathcal{U}^T_{\zeta,\nu}$ and then letting $T \to \infty$ yields (3.3.25).

We now turn to the reverse inequality,

$$g(\zeta,\nu) \leq g_{\mathrm{scal}}(|\zeta|). \tag{3.3.26}$$

Fix $T>0$ and $(\alpha,\beta) \in \mathfrak{U}_{|\zeta|}(-{}^T\!/2, {}^T\!/2)$. Let $\varepsilon_j \to 0$ and construct a sequence $(u_j, v_j)$, for the problem (3.1.11) defining $g$, as follows:

$$u_j(x) := \begin{cases} \alpha\Big(\dfrac{T}{\varepsilon_j}x\cdot\nu\Big)\dfrac{\zeta}{|\zeta|}, & \text{if } |x\cdot\nu| \leq \frac{\varepsilon_j}{2},\ x \in Q^\nu, \\ u_{\zeta,\nu}, & \text{otherwise in } Q^\nu, \end{cases}$$

$$v_j(x) := \begin{cases} \beta\Big(\dfrac{T}{\varepsilon_j}x\cdot\nu\Big), & \text{if } |x\cdot\nu| \leq \frac{\varepsilon_j}{2},\ x \in Q^\nu, \\ 1, & \text{otherwise in } Q^\nu. \end{cases}$$

By a change of variables, $\|u_j - u_{\zeta,\nu}\|_{L^1(Q^\nu)} \to 0$, and moreover

$$\mathcal{F}^\infty_{\varepsilon_j/T}(u_j, v_j; Q^\nu) = \mathscr{F}^\infty_1(\alpha,\beta;(-{}^T\!/2, {}^T\!/2)).$$

Hence

$$g(\zeta,\nu) \leq \mathscr{F}^\infty_1(\alpha,\beta;(-{}^T\!/2, {}^T\!/2)).$$

Letting $(\alpha,\beta) \in \mathfrak{U}_{|\zeta|}(-{}^T\!/2, {}^T\!/2)$ vary, we obtain (3.3.26). □

### Structural properties of $g$

We start by establishing a growth estimate for $g$, illustrating its asymptotically linear behavior for small jumps.

**Proposition 3.3.7.** *Under the assumptions of Theorem 3.1.1, then for every $(\zeta,\nu) \in \mathbb{R}^N \times \mathbb{S}^{n-1}$*

$$\frac{1}{K}(1 \wedge |\zeta|) \leq g(\zeta,\nu) \leq K(1 \wedge |\zeta|) \tag{3.3.27}$$

*for some constant $K>0$.*

*Proof.* Let $C$ be the constant appearing in (3.1.7), and define

$$\Phi^a_\infty(\xi) := \frac{1}{C}|\xi|^2, \qquad \Phi^b_\infty(\xi) := C|\xi|^2,$$

for every $\xi \in \mathbb{R}^{N\times n}$. Let now $g^a$ and $g^b$ denote the two surface energy densities obtained by replacing $\Phi_\infty$ in (3.1.11) with $\Phi^a_\infty$ and $\Phi^b_\infty$, respectively. In particular, by (3.1.7) and Corollary 3.3.6, for every

$(\zeta,\nu)\in\mathbb{R}^N\times\mathbb{S}^{n-1}$ we have

$$g^a_{\rm scal}(|\zeta|)=g^a(\zeta,\nu)\le g(\zeta,\nu)\le g^b(\zeta,\nu)=g^b_{\rm scal}(|\zeta|),$$

where $g^a_{\rm scal}$ and $g^b_{\rm scal}$ are defined as in (3.1.14), with $\varphi'(0)$ replaced by $C^{-1}\varphi'(0)$ and $C\varphi'(0)$, respectively. Hence, combining (3.2.16) with item (ii) of Proposition 3.2.4, we obtain the desired conclusion. $\square$

We now establish the subadditivity and continuity properties of the surface energy density $g$.

**Lemma 3.3.8.** *Under the assumptions of Theorem 3.1.1, the following hold:*

*(i) For every $\nu\in\mathbb{S}^{n-1}$ and $\zeta_1,\zeta_2\in\mathbb{R}^N$ one has*

$$g(\zeta_1+\zeta_2,\nu)\le g(\zeta_1,\nu)+g(\zeta_2,\nu);$$

*(ii) The function $g$ is continuous on $\mathbb{R}^N\times\mathbb{S}^{n-1}$.*

*Proof. (i) Subadditivity.* Fix $\zeta_1,\zeta_2\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$. Let $(u^i_j,v^i_j)$ be the sequences given in Proposition 3.3.3 corresponding to $\varepsilon_j:=1/j$ and to the pairs $(\zeta_i,\nu)$, for $i=1,2$. We extend these functions periodically along the directions tangent to $\Pi_\nu\cap Q^\nu$, and constantly along $\nu$. In particular, for each $i\in\{1,2\}$ and all $j$, we have $u^i_j=\zeta_i$ and $v^i_j=1$ in $\{x\cdot\nu\ge\frac12\}$, and $u^i_j=0$, $v^i_j=1$ in $\{x\cdot\nu\le-\frac12\}$.

We now use a rescaling analogous to that used in Proposition 3.3.3. Let $M_j\in\mathbb{N}$ with $M_j\to\infty$, and define $(u_j,v_j)$ by

$$u_j(x):=\begin{cases}u^1_j(M_jx+\frac12\nu), & \text{if } x\cdot\nu<0,\\ \zeta_1+u^2_j(M_jx-\frac12\nu), & \text{if } x\cdot\nu\ge0,\end{cases}\qquad v_j(x):=\begin{cases}v^1_j(M_jx+\frac12\nu), & \text{if } x\cdot\nu<0,\\ v^2_j(M_jx-\frac12\nu), & \text{if } x\cdot\nu\ge0.\end{cases}$$

Because of periodicity in the tangential directions, $(u_j,v_j)$ belongs to $H^1(Q^\nu;\mathbb{R}^{N+1})$. Moreover, $u_j=0$ and $v_j=1$ on $\{x\cdot\nu\le-\frac{1}{M_j}\}$, while $u_j=\zeta_1+\zeta_2$ and $v_j=1$ on $\{x\cdot\nu\ge\frac{1}{M_j}\}$, and the pair is $\frac{1}{M_j}$–periodic along the directions of $\Pi_\nu\cap Q^\nu$.

By a change of variables,

$$\begin{aligned}\|u_j-u_{\zeta_1+\zeta_2,\nu}\|_{L^1(Q^\nu)}&=\|u_j-u_{\zeta_1+\zeta_2,\nu}\|_{L^1(Q^\nu\cap\{|x\cdot\nu|\le\frac{1}{M_j}\})}\\&=\frac{1}{M_j}\|u^1_j\|_{L^1(Q^\nu)}+\frac{1}{M_j}\|u^2_j-\zeta_2\|_{L^1(Q^\nu)}\\&\le\frac{1}{M_j}\Big(\|u^1_j-u_{\zeta_1,\nu}\|_{L^1(Q^\nu)}+\|u^2_j-u_{\zeta_2,\nu}\|_{L^1(Q^\nu)}\Big)+\frac{|\zeta_1|+|\zeta_2|}{2M_j},\end{aligned}$$

hence $u_j\to u_{\zeta_1+\zeta_2,\nu}$ in $L^1(Q^\nu;\mathbb{R}^N)$. Furthermore, a straightforward computation yields

$$\mathcal{F}^\infty_{\varepsilon_j/M_j}(u_j,v_j;Q^\nu)=\mathcal{F}^\infty_{\varepsilon_j}(u^1_j,v^1_j;Q^\nu)+\mathcal{F}^\infty_{\varepsilon_j}(u^2_j,v^2_j;Q^\nu).$$

Taking the limit as $j\to\infty$ gives the desired inequality in (i).

*(ii) Continuity.* From (i) and Proposition 3.3.7, we deduce that $g(\zeta,\nu)\le g(\zeta',\nu)+K|\zeta-\zeta'|$, where $K$(cf. (3.3.27)) do not depend on $\nu$. Therefore, it remains to show only the continuity with respect to $\nu$.

Since $\Phi_\infty$ is continuous and strictly positive on the compact unit sphere $\mathbb{S}^{N\times n-1}$, there exists a non decreasing modulus of continuity $\vartheta$, such that

$$\Phi_\infty(\xi) \le (1+\vartheta(|\xi-\xi'|))\Phi_\infty(\xi') \quad \text{whenever } |\xi| = |\xi'| = 1.$$

This in turn implies that

$$\Phi_\infty(\eta) \le (1+\vartheta(|R-\mathrm{Id}|))\Phi_\infty(\eta R) \quad \text{for all } \eta \in \mathbb{R}^{N\times n},\ R\in O(n). \tag{3.3.28}$$

Fix $\nu \in \mathbb{S}^{n-1}$ and a sequence $\varepsilon_j \to 0$, and let $(u_j, v_j)$ be the sequence given by Proposition 3.3.3, extended periodically along $\Pi_\nu$ and constant in the $\nu$–direction as in part (i). Let $\tilde\nu \in \mathbb{S}^{n-1}$, $\tilde\nu \neq \nu$, and choose $R \in O(n)$ such that $\nu = R\tilde\nu$ and $|R-\mathrm{Id}| \le c|\nu-\tilde\nu|$ (for instance, $R$ acts as the identity on the subspace orthogonal to $\mathrm{span}\{\nu,\tilde\nu\}$ and rotates $\tilde\nu$ to $\nu$ in that plane).

Let $M_j \to \infty$ and define

$$\tilde u_j(x) := u_j(M_jRx), \qquad \tilde v_j(x) := v_j(M_jRx).$$

Since $u_j \to u_{\zeta,\nu}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$, it follows that $\tilde u_j \to u_{\zeta,\nu}$. Moreover, $\nabla\tilde u_j(x) = M_j\nabla u_j(M_jRx)R$, and using (3.3.28) we obtain

$$\Phi_\infty(\nabla\tilde u_j)(x) = M_j^2\Phi_\infty(\nabla u_jR)(M_jRx) \le M_j^2(1+\vartheta(|R-\mathrm{Id}|))\Phi_\infty(\nabla u_j)(M_jRx).$$

Substituting this into the definition of $\mathcal{F}^\infty_{\varepsilon_j}(\tilde u_j,\tilde v_j;Q^{\tilde\nu})$ and performing a change of variables yields

$$\mathcal{F}^\infty_{\varepsilon_j/M_j}(\tilde u_j,\tilde v_j;Q^{\tilde\nu}) \le (1+\vartheta(|R-\mathrm{Id}|))M_j^{1-n}\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;M_jRQ^{\tilde\nu}).$$

Since $(u_j,v_j)$ are periodic in the directions orthogonal to $\nu$, the set $\Pi_\nu\cap M_jRQ^{\tilde\nu}$ can be covered by at most $M_j^{n-1}+cM_j^{n-2}$ disjoint copies of $\Pi_\nu\cap Q^\nu$. Hence,

$$\begin{aligned}
g(\zeta,\tilde\nu) &\le \limsup_{j\to\infty}\mathcal{F}^\infty_{\varepsilon_j/M_j}(\tilde u_j,\tilde v_j;Q^{\tilde\nu})\\
&\le (1+\vartheta(|R-\mathrm{Id}|))\limsup_{j\to\infty}(1+\tfrac{c}{M_j})\mathcal{F}^\infty_{\varepsilon_j}(u_j,v_j;Q^\nu)\\
&= (1+\vartheta(|R-\mathrm{Id}|))g(\zeta,\nu) \le (1+\vartheta(c|\nu-\tilde\nu|))g(\zeta,\nu),
\end{aligned}$$

which proves the continuity of $g$ with respect to $\nu$. □

We conclude this section by proving, under the lower semicontinuity assumption of the limit functional $\mathscr{F}_{\overline{\varsigma}}$, that the surface energy density $g$ exhibits the same asymptotic behaviour as the recession function $h^{\mathrm{qc},\infty}_{\overline{\varsigma}}$.

**Proposition 3.3.9.** *Assume that the limit functional $\mathscr{F}_{\overline{\varsigma}}$ (see (3.1.8)) is lower semicontinuous in $L^1(O;\mathbb{R}^N)$. Then, under the assumptions of Theorem 3.1.1, for every $\nu\in\mathbb{S}^{n-1}$ it holds that*

$$\lim_{\zeta\to 0}\frac{g(\zeta,\nu)}{h^{\mathrm{qc},\infty}(\zeta\otimes\nu)} = 1\,.$$

*Proof.* Fix $\nu\in\mathbb{S}^{n-1}$ and let $x_0\in O$ and $\rho>0$ be such that $Q^\nu_\rho(x_0)\subset O$. Without loss of generality, by translation and scaling, we may assume that $x_0 = 0$ and $\rho = 1$. For every $\zeta \in \mathbb{R}^N$, define

$$w_j(x) := r(jx \cdot \nu)\zeta\,, \qquad x \in Q^\nu, \tag{3.3.29}$$

where $r(t) := (t \wedge 1) \vee 0$ for all $t \in \mathbb{R}$. Clearly, $w_j \to u_{\zeta,\nu}$ in $L^1(Q^\nu;\mathbb{R}^N)$, and thus, by the $L^1$–lower semicontinuity of $\mathscr{F}_{\overline{\varsigma}}$, we obtain

$$\begin{aligned} g(\zeta,\nu) = \mathscr{F}_{\overline{\varsigma}}(u_{\zeta,\nu};Q^\nu) &\le \liminf_{j\to\infty} \mathscr{F}_{\overline{\varsigma}}(w_j;Q^\nu) = \liminf_{j\to\infty} \int_{Q^\nu} h_{\overline{\varsigma}}^{\mathrm{qc}}(\nabla w_j)\,dx \\ &= \liminf_{j\to\infty} \int_{\{x\in Q^\nu:\,0\le x\cdot\nu\le 1/j\}} h_{\overline{\varsigma}}^{\mathrm{qc}}(j\zeta\otimes\nu)\,dx = \liminf_{j\to\infty} \frac{h_{\overline{\varsigma}}^{\mathrm{qc}}(j\zeta\otimes\nu)}{j} \\ &\le h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\zeta\otimes\nu)\,. \end{aligned} \tag{3.3.30}$$

On the other hand, given $\zeta \in \mathbb{R}^N$ and any sequences $\zeta_j \to \zeta$ and $t_j \to 0^+$, denote by $M_j$ the integer part of $t_j^{-1}$ and, for each $k \ge 3$, define

$$u_{j,k}(x) := \sum_{i=0}^{M_j-1} it_j\zeta_j\chi_{[\frac{i}{kM_j},\frac{i+1}{kM_j})}(x\cdot\nu) + \zeta\,\chi_{[\frac{1}{k},\frac{1}{2}]}(x\cdot\nu)\,.$$

We claim that $u_{j,k} \to w_k$ (as defined in (3.3.29)) uniformly on $Q^\nu$ for each fixed $k \ge 3$. Indeed, for $s := x\cdot\nu \in [\frac{i}{kM_j},\frac{i+1}{kM_j}) \subseteq [0,\frac{1}{k})$ we have

$$\begin{aligned} |it_j\zeta_j - \zeta ks| &\le |\zeta-\zeta_j| + |\zeta_j|\,|it_j - ks| \\ &\le |\zeta-\zeta_j| + |\zeta_j|\left(\frac{i}{M_j}|M_jt_j-1| + \frac{1}{M_j}\right) \le |\zeta-\zeta_j| + |\zeta_j|\left(|M_jt_j-1| + \frac{1}{M_j}\right) \to 0, \end{aligned}$$

uniformly in $i$. Hence, $\|w_k - u_{j,k}\|_{L^\infty(Q^\nu;\mathbb{R}^N)} \to 0$ as $j\to\infty$.

Moreover, one computes

$$\begin{aligned} Du_{j,k} = D^j u_{j,k} = (t_j\zeta_j\otimes\nu)\,\mathcal{H}^{n-1}\llcorner \bigcup_{i=1}^{M_j-1}\{x\in Q^\nu : x\cdot\nu = \tfrac{i}{kM_j}\} \\ + ((\zeta-(M_j-1)t_j\zeta_j)\otimes\nu)\,\mathcal{H}^{n-1}\llcorner\{x\in Q^\nu : x\cdot\nu = \tfrac{1}{k}\}\,. \end{aligned}$$

Therefore, by the $L^1$–lower semicontinuity of $\mathscr{F}_{\overline{\varsigma}}$,

$$\begin{aligned} \frac{1}{k}\,h_{\overline{\varsigma}}^{\mathrm{qc}}(k\zeta\otimes\nu) &= \mathscr{F}_{\overline{\varsigma}}(w_k;Q^\nu) \le \liminf_{j\to\infty}\mathscr{F}_{\overline{\varsigma}}(u_{j,k};Q^\nu) \\ &= \liminf_{j\to\infty}\int_{J_{u_{j,k}}} g([u_{j,k}](x),\nu)\,d\mathcal{H}^{n-1}(x) = \liminf_{j\to\infty}(M_j-1)g(t_j\zeta_j,\nu) = \liminf_{j\to\infty}\frac{g(t_j\zeta_j,\nu)}{t_j}\,. \end{aligned}$$

Since this holds for all such sequences, we deduce that

$$h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\zeta\otimes\nu) \le \liminf_{(t,\zeta')\to(0,\zeta)}\frac{g(t\zeta',\nu)}{t}\,. \tag{3.3.31}$$

Indeed, the superior limit in the definition of $h^{\mathrm{qc},\infty}_{\overline{\varsigma}}$ is an actual limit along rank–one directions, since $h^{\mathrm{qc}}_{\overline{\varsigma}}$ is convex on such directions.

Now, let $\widetilde{\zeta}_j \to 0$ be a sequence such that

$$\liminf_{\zeta\to 0} \frac{g(\zeta,\nu)}{h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\zeta\otimes\nu)} = \lim_{j\to\infty} \frac{g(\widetilde{\zeta}_j,\nu)}{h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\widetilde{\zeta}_j\otimes\nu)} \,.$$

Set $\zeta_j := \frac{\widetilde{\zeta}_j}{|\widetilde{\zeta}_j|}$; up to subsequences we may assume that $\zeta_j \to \zeta_\infty \in \mathbb{S}^{n-1}$, and let $t_j := |\widetilde{\zeta}_j| \to 0$. By the one–homogeneity of $h^{\mathrm{qc},\infty}_{\overline{\varsigma}}$,

$$\frac{g(\widetilde{\zeta}_j,\nu)}{h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\widetilde{\zeta}_j\otimes\nu)} = \frac{g(t_j\zeta_j,\nu)}{t_j}\,\frac{1}{h^{\mathrm{qc},\infty}_{\overline{\varsigma}}(\zeta_j\otimes\nu)} \,.$$

Combining this identity with (3.3.31) and the continuity of $h^{\mathrm{qc},\infty}$, we infer that

$$\liminf_{\zeta\to 0} \frac{g(\zeta,\nu)}{h^{\mathrm{qc},\infty}(\zeta\otimes\nu)} \geq 1 \,. \tag{3.3.32}$$

The desired conclusion follows immediately by combining (3.3.30) and (3.3.32). □

### 3.3.2 Lower Bound

In this subsection we establish the $\Gamma$–$\liminf$ inequality for the sequence of functionals $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$. The proof is divided into two parts, corresponding to the diffuse and surface contributions of the limit functional. The surface part relies on a blow-up argument around jump points, while the diffuse part is handled by means of the truncation Lemma 3.1.7 and Proposition 3.1.9. Finally, we characterize the effective domain of the $\Gamma$–limit, showing that the limit energy is infinite for functions in $(L^1 \setminus GBV(O))^N$ (see Proposition 3.3.13).

We start by dealing with the surface contribution in $BV$.

**Proposition 3.3.10.** *Let $u \in BV(O,\mathbb{R}^N)$ and $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(O,\mathbb{R}^{N+1})$, then under the assumptions of Theorem 3.1.1, we have*

$$\int_{J_u\cap A} g([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A) \,, \tag{3.3.33}$$

*for every $A \in \mathcal{A}(O)$, where $g$ is the function defined in (3.1.14).*

*Proof.* Fix $A \in \mathcal{A}(O)$. We first assume that the inferior limit on the right-hand side of (3.3.33) is finite, since otherwise the claim is trivial. Up to a subsequence, we may therefore suppose that $(u_\varepsilon, v_\varepsilon) \in H^1(O;\mathbb{R}^N \times [0,1])$, and that the above $\liminf$ is in fact a limit. For every $\varepsilon > 0$ we define a Radon measure $\mu_\varepsilon$ on $A$ by

$$\mu_\varepsilon := \left(\varphi(\varepsilon f^2(v_\varepsilon))\Phi(\nabla u_\varepsilon) + \frac{\omega(1-v_\varepsilon)}{4\varepsilon} + \varepsilon|\nabla v_\varepsilon|^2\right)\mathcal{L}^n \llcorner A.$$

Up to extracting a further subsequence, we have $\mu_\varepsilon \rightharpoonup \mu$ for some Radon measure $\mu$ on $A$, and hence

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon,A) \geq \mu(A).$$

To prove (3.3.33), it thus suffices to show that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \geq g([u](x_0), \nu_u(x_0)) \tag{3.3.34}$$

for $\mathcal{H}^{n-1}$-a.e. $x_0 \in J_u \cap A$. Fix $x_0 \in J_u \cap A$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u \cap A}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_\rho^\nu(x_0))}{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))} < \infty,$$

$$\lim_{\rho\to 0} \frac{\mathcal{H}^{n-1}(J_u \cap Q_\rho^\nu(x_0))}{\rho^{n-1}} = 1,$$

and

$$\lim_{\rho\to 0} \frac{1}{\rho^n} \int_{Q_\rho^\nu(x_0)} |u(y) - u_{x_0,\zeta,\nu}(y)| \mathrm{d}y = 0,$$

where $\nu := \nu_u(x_0)$ and $\zeta := [u](x_0)$. We remark that these conditions identify a set of full $\mathcal{H}^{n-1}$-measure in $J_u \cap A$.

From the weak* convergence of $\mu_\varepsilon$, we deduce that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_\rho^\nu(x_0))}{\rho^{n-1}} = \lim_{\rho\to 0, \rho\in I(x_0)} \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(Q_\rho^\nu(x_0))}{\rho^{n-1}}, \tag{3.3.35}$$

where

$$I(x_0) := \left\{ \rho \in \left(0, \frac{2}{\sqrt{n}} \mathrm{dist}(x_0, \partial A)\right) \;:\; \mu(Q_\rho^\nu(x_0)) = 0 \right\}.$$

Fix $\delta \in (0, 1)$ and, for all $\varepsilon > 0$, define

$$\delta_\varepsilon := \sup\{t \in [0, 1) : \varepsilon f^2(t) \leq \delta\}.$$

Using the same argument of Proposition 3.2.8 we have $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$ and, if $\varepsilon$ is small enough

$$\varepsilon f^2(t) \leq \delta, \tag{3.3.36}$$

if and only if $t \in [0, \delta_\varepsilon]$. Let $\hat{v}_\varepsilon := v_\varepsilon \wedge \delta_\varepsilon$ and we compute the following

$$\begin{aligned} \mathcal{F}_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0)) &= \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, Q_\rho^\nu(x_0) \setminus \{v_\varepsilon > \delta_\varepsilon\}) \\ &\quad + \int_{Q_\rho^\nu(x_0)\cap\{v_\varepsilon > \delta_\varepsilon\}} \varphi(\delta)\Phi(\nabla u_\varepsilon)\,\mathrm{d}x + \frac{\omega(1-\delta_\varepsilon)}{4\varepsilon} \mathcal{L}^n\left(Q_\rho^\nu(x_0) \cap \{v_\varepsilon > \delta_\varepsilon\}\right). \end{aligned}$$

Since $\varphi$ is non-decreasing and (3.3.36) holds, we have $\varphi(\delta) \leq \varphi(\varepsilon f^2(v_\varepsilon))$ on $\{v_\varepsilon > \delta_\varepsilon\}$, and hence

$$\mathcal{F}_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0)) \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, Q_\rho^\nu(x_0)) + \frac{\omega(1-\delta_\varepsilon)}{4\varepsilon} \mathcal{L}^n\left(Q_\rho^\nu(x_0) \cap \{v_\varepsilon > \delta_\varepsilon\}\right).$$

By definition of $\delta_\varepsilon$, (3.1.2), $\ell \leq 1$, and $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$, we deduce for $\varepsilon$ small enough (depending on $\delta$) that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0)) \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, Q_\rho^\nu(x_0)) + \frac{(\overline{\varsigma}^2 + \delta)\rho^n}{4\delta},$$

and, from (3.3.35),

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \geq \limsup_{\rho\to 0,\rho\in I(x_0)} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0))}{\rho^{n-1}}. \tag{3.3.37}$$

For every $\eta \in (0,1)$, from (3.1.6) and (3.1.7), we have $\Phi(\xi) \geq (1-\eta)\Phi_\infty(\xi)$ for $|\xi|$ sufficiently large, and by continuity of $\Phi_\infty$ there exists $C(\eta) > 0$ such that

$$\Phi(\xi) + C(\eta) \geq (1-\eta)\Phi_\infty(\xi) \qquad \text{for all } \xi \in \mathbb{R}^{N\times n}.$$

For $\rho \in I(x_0)$ let $\eta_\rho \in (0,1)$ be such that $\eta_\rho, \rho C(\eta_\rho) \to 0$ as $\rho \to 0$. Using (3.3.36) and (Hp 4) we obtain

$$\varphi(\varepsilon f^2(\hat{v}_\varepsilon)) \geq (1-\tilde{\vartheta}(\delta))\varphi'(0^+)\varepsilon f^2(\hat{v}_\varepsilon),$$

with $\tilde{\vartheta}(\delta) \to 0^+$ as $\delta \to 0^+$. Hence,

$$\varphi(\varepsilon f^2(\hat{v}_\varepsilon))\Phi(\nabla u_\varepsilon) \geq (1-\tilde{\vartheta}(\delta))\varphi'(0^+)\varepsilon f^2(\hat{v}_\varepsilon)\big((1-\eta_\rho)\Phi_\infty(\nabla u_\varepsilon) - C(\eta_\rho)\big),$$

and, recalling (3.3.36),

$$\frac{\mathcal{F}_\varepsilon(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0))}{\rho^{n-1}} \geq \frac{(1-\tilde{\vartheta}(\delta))(1-\eta_\rho)\mathcal{F}_\varepsilon^\infty(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0))}{\rho^{n-1}} - \frac{C(\eta_\rho)\mathcal{L}^n(Q_\rho^\nu(x_0))(1-\tilde{\vartheta}(\delta))\varphi'(0^+)\delta}{\rho^{n-1}},$$

where $\mathcal{F}_\varepsilon^\infty$ is defined in (3.1.12). Thus,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \geq (1-\tilde{\vartheta}(\delta)) \limsup_{\rho\to 0,\rho\in I(x_0)} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon^\infty(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0))}{\rho^{n-1}}. \tag{3.3.38}$$

Define the rescaled functions $u_\varepsilon^\rho(y) := u_\varepsilon(\rho y + x_0)$ and $\hat{v}_\varepsilon^\rho(y) := \hat{v}_\varepsilon(\rho y + x_0)$. Since $u_\varepsilon \to u$ in $L^1(O;\mathbb{R}^N)$, we can find sequences $(\rho_j)$ and $(\varepsilon_j)$ such that $u_{\varepsilon_j}^{\rho_j} \to u_{\zeta,\nu}$ in $L^1(Q^\nu;\mathbb{R}^N)$ and

$$\limsup_{\rho\to 0,\rho\in I(x_0)} \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon^\infty(u_\varepsilon, \hat{v}_\varepsilon, Q_\rho^\nu(x_0))}{\rho^{n-1}} = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}^\infty(u_{\varepsilon_j}^{\rho_j}, \hat{v}_{\varepsilon_j}^{\rho_j}, Q^\nu). \tag{3.3.39}$$

Combining (3.3.38), (3.3.39), and (3.1.14), we conclude that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x_0) \geq (1-\tilde{\vartheta}(\delta))g(\zeta,\nu),$$

and letting $\delta \to 0$ yields (3.3.34), completing the proof. □

Next, we establish the lower bound for the diffuse part on $BV \cap L^\infty$, making use of the truncation Lemma 3.1.7 and of the partial relaxation result proved in [CFI24a], stated in Proposition 3.1.9.

**Proposition 3.3.11.** *Let $u \in BV \cap L^\infty(O,\mathbb{R}^N)$ and $(u_\varepsilon, v_\varepsilon) \in L^1(O,\mathbb{R}^{N+1})$ with $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(O,\mathbb{R}^{N+1})$, then under the assumption of Theorem 3.1.1*

$$\int_A h_{\overline{\varsigma}}^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x + \int_A h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\mathrm{d}D^c u) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,, \tag{3.3.40}$$

*for every* $A \in \mathcal{A}(O)$.

*Proof.* Fix $A \in \mathcal{A}(O)$ and assume that the inferior limit on the right-hand side of (3.3.40) is finite; otherwise, the claim is trivial. In particular, we may suppose that $(u_\varepsilon, v_\varepsilon) \in H^1(A;\mathbb{R}^N \times [0,1])$ for all $\varepsilon > 0$. Let $\gamma \in (0, 1 \wedge \overline{\varsigma})$. By applying the notation and results of Lemma 3.1.7, we find $\delta_\gamma \in (0,1)$ such that, for every $\delta \in (0, \delta_\gamma)$, the function $u_\varepsilon^\delta$ belongs to $SBV(O;\mathbb{R}^N)$, since $u \in L^\infty(O;\mathbb{R}^N)$. Then,

$$\zeta_1(\delta)\int_A h^{\mathrm{qc}}_{\overline{\varsigma}_\gamma}(\nabla \tilde{u}_\varepsilon^\delta)\,\mathrm{d}x + \zeta_2(\delta)\mathcal{H}^{n-1}(A \cap J_{\tilde{u}_\varepsilon^\delta}) = H_\delta(\tilde{u}_\varepsilon^\delta, A) \leq \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A) + h_{\overline{\varsigma}_\gamma}(0)\mathcal{L}^n(A \cap \{v_\varepsilon \leq 1-\delta\})\,,$$

where $\overline{\varsigma}_\gamma = (\overline{\varsigma} - \gamma)(1-\overline{\varsigma})$. Hence, taking the $\liminf$ as $\varepsilon \to 0$, and using Proposition 3.1.9 we obtain

$$\zeta_1(\delta)\int_A h^{\mathrm{qc}}_{\overline{\varsigma}_\gamma}(\nabla u)\,\mathrm{d}x + \zeta_1(\delta)\int_A h^{\mathrm{qc},\infty}_{\overline{\varsigma}_\gamma}(\mathrm{d}D^c u) \leq \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, A)\,.$$

Finally, letting first $\delta \to 0$ and then $\gamma \to 0$, we deduce (3.3.40) in view of (3.1.28), (3.1.17), and by applying Beppo Levi's theorem. $\square$

As a consequence of Propositions 3.3.10 and 3.3.11, we now establish the $\Gamma$–$\liminf$ inequality in $BV \cap L^\infty$.

**Theorem 3.3.12.** *Let* $u \in BV \cap L^\infty(O;\mathbb{R}^N)$*, then under the assumptions of Theorem 3.1.1*

$$\mathscr{F}_{\overline{\varsigma}}(u, O) \leq \Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u, 1; O). \tag{3.3.41}$$

*Proof.* We proceed by localization. Assume that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(O;\mathbb{R}^{N+1})$, with $u \in BV \cap L^\infty(O;\mathbb{R}^N)$, and that

$$\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) < \infty.$$

For every $A \in \mathcal{A}(O)$, define

$$\begin{gathered}\mu(A) := \liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; A),\\ \lambda := \mathcal{L}^n \llcorner O + \mathcal{H}^{n-1}\llcorner J_u + |D^c u|,\\ \psi_1 := g([u], \nu_u), \qquad \psi_2 := h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u) + h^{\mathrm{qc},\infty}_{\overline{\varsigma}}\left(\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\right).\end{gathered}$$

Observe that $\mu$ is a monotone set function, superadditive on disjoint open subsets of $O$, $\lambda$ is a positive Borel measure, and $\psi_i$ are nonnegative Borel functions satisfying

$$\mu(A) \geq \int_A \psi_i\,\mathrm{d}\lambda, \qquad \text{for } i = 1,2 \text{ and all } A \in \mathcal{A}(O),$$

as a consequence of Propositions 3.3.10 and 3.3.11. Applying [Bra98, Proposition 1.16], we then obtain

$$\mu(O) \geq \int_O (\psi_1 \vee \psi_2)\,\mathrm{d}\lambda,$$

which yields the desired conclusion. $\square$

In the following proposition, we characterize the domain of the $\Gamma$–$\limsup$. More precisely, we show that it is contained in the space $GBV_\star(O;\mathbb{R}^N)$, introduced in [DMT22, Don25]. This space consists of all functions

$u \in (GBV(O))^N$ such that, for every $i \in \{1, \ldots, N\}$ and every $M > 0$, one has

$$\int_O |\nabla \tau_M(u_i)| \, \mathrm{d}x + |D^c(\tau_M(u_i))|(O) + \int_{J_{\tau_M(u_i)}} (|[\tau_M(u_i)]| \wedge 1) \, \mathrm{d}\mathcal{H}^{n-1} \leq L,$$

for some constant $L > 0$. Here $u_i$ denotes the $i$-th component of $u$, and $\tau_M(s) := (s \wedge M) \vee (-M)$ is the $M$-truncation function (cf. Section 2.0.1).

**Proposition 3.3.13.** *Let $(u, v) \in L^1(O; \mathbb{R}^{N+1})$ be such that $\Big(\Gamma\text{-}\limsup_{\varepsilon \to 0} \mathcal{F}_\varepsilon\Big)(u, v, O) < \infty$, then under the assumptions of Theorem 3.1.1, it follows that $v = 1$ $\mathcal{L}^n$–a.e. in $O$ and $u \in GBV_\star(O; \mathbb{R}^N)$ with $\mathscr{F}_{\overline{\varsigma}}(u, O) < \infty$.*

*Proof.* Consider a sequence $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(O; \mathbb{R}^{N+1})$ such that

$$\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) < \infty. \tag{3.3.42}$$

Due to the presence of the term $\frac{\omega(1-v_\varepsilon)}{\varepsilon}$, it necessarily follows that $v = 1$ $\mathcal{L}^n$-a.e. in $O$. Fix $\delta \in (0, 1)$ and, using the notation introduced in Lemma 3.1.7, exploit the growth conditions on $h^{\mathrm{qc}}_{\varsigma}$ (see (3.1.25)) to obtain

$$\int_O |\nabla u^\delta_\varepsilon| \, \mathrm{d}x + \mathcal{H}^{n-1}(J_{u^\delta_\varepsilon}) \leq c\big(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + 1\big),$$

for some constant $c > 0$ depending on $\delta$ and $\mathcal{L}^n(O)$. In particular, for each component $(u^\delta_\varepsilon)_i$ of $u^\delta_\varepsilon$, with $i \in \{1, \ldots, N\}$, we have $(u^\delta_\varepsilon)_i \in GSBV(O)$ and

$$\int_O |\nabla (u^\delta_\varepsilon)_i| \, \mathrm{d}x + \mathcal{H}^{n-1}(J_{(u^\delta_\varepsilon)_i}) \leq c\big(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) + 1\big).$$

Next, for $M > 0$, consider the truncation function $\tau_M(s) := (s \vee M) \wedge (-M)$. Then $\tau_M((u^\delta_\varepsilon)_i) \in BV_{\mathrm{loc}}(O)$ for every $\varepsilon > 0$, and from the previous estimate we deduce that

$$|D(\tau_M((u^\delta_\varepsilon)_i))|(O) \leq C_M,$$

for some constant $C_M > 0$ depending on $M$ but independent of $\varepsilon$. Hence, up to a subsequence, $\tau_M((u^\delta_\varepsilon)_i)$ converges weakly in $BV(O)$. From the convergence $v_\varepsilon \to 1$ and Lemma 3.1.7, it follows that $u^\delta_\varepsilon \to u$ in $L^1(O; \mathbb{R}^N)$ as $\varepsilon \to 0$, which in turn implies that $\tau_M(u_i) \in BV(O)$ for every $M > 0$. Consequently, $u_i \in GBV(O)$ for each $i \in \{1, \ldots, N\}$, and therefore $u \in (GBV(O))^N$.

Furthermore, by (3.1.4), there exists a constant $L \in (0, \infty)$ such that

$$\sup_{M>0} \sup_{\varepsilon>0} \mathcal{F}_\varepsilon(\tilde{\tau}_M(u_\varepsilon), v_\varepsilon, O) \leq L,$$

where $\tilde{\tau}_M(w) := (\tau_M(w_1), \ldots, \tau_M(w_N))$ for every $w \in \mathbb{R}^N$. Hence, for every $M > 0$, we have $\tilde{\tau}_M(u_\varepsilon) \to \tilde{\tau}_M(u)$ in $L^1(O; \mathbb{R}^N)$, and using (3.1.25), (3.3.27), and Theorem 3.3.12, we obtain that, for all $i \in \{1, \ldots, N\}$ and $M > 0$,

$$\int_O |\nabla \tau_M(u_i)| \, \mathrm{d}x + |D^c(\tau_M(u_i))|(O) + \int_{J_{\tau_M(u_i)}} (|[\tau_M(u_i)]| \wedge 1) \mathrm{d}\mathcal{H}^{n-1} \leq L.$$

This shows that $u \in GBV_\star(O, \mathbb{R}^N)$. By exploiting the structural properties of $GBV$-functions (see Section 2.0.1) and applying the Monotone Convergence Theorem, we conclude that

$$\int_O |\nabla u_i|\,\mathrm{d}x + |D^c u_i|(O) + \int_{J_{u_i}} (|[u_i]| \wedge 1)\mathrm{d}\mathcal{H}^{n-1} < \infty, \tag{3.3.43}$$

for every $i \in \{1, \dots, N\}$ and in particular, by combining (3.1.25), (3.1.26), and (3.3.27), we deduce that $\mathscr{F}_{\overline{\varsigma}}(u, O) < \infty$.

□

Now we extend the validity of the lower bound Theorem 3.3.12 to every $u \in (GBV(O))^N$. We first prove that the functional $\mathscr{F}_{\overline{\varsigma}}$ is continuous under truncations.

**Proposition 3.3.14.** *Let $\mathscr{F}_{\overline{\varsigma}}$ and $\mathcal{T}_k$ be defined as in (3.1.8) and (2.0.10), respectively. Then under the assumptions of Theorem 3.1.1, for all $u \in (GBV(O))^N$ with $\mathscr{F}_{\overline{\varsigma}}(u, O) < \infty$ ($u \in GBV_\star(O)^N$) we have*

$$\lim_{k\to\infty} \mathscr{F}_{\overline{\varsigma}}(\mathcal{T}_k(u), O) = \mathscr{F}_{\overline{\varsigma}}(u, O)\,.$$

*Proof.* We prove the convergence of the volume, Cantor, and surface contributions separately.

For the volume part, note that (3.1.25) ensures that $|\nabla u| \in L^1(O)$. Since $\nabla(\mathcal{T}_k(u)) = \nabla u$ for $\mathcal{L}^n$-a.e. $x \in O_k := \{|u| \le a_k\}$, by (3.1.25) we obtain

$$\Big| \int_O h^{\mathrm{qc}}_{\overline{\varsigma}}\big(\nabla(\mathcal{T}_k(u))\big)\,\mathrm{d}x - \int_O h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u)\,\mathrm{d}x \Big| \le K_{\overline{\varsigma}} \int_{O\setminus O_k} (1 + |\nabla u|)\,\mathrm{d}x.$$

Hence, since $a_k \to \infty$ as $k \to \infty$, it follows that

$$\lim_{k\to\infty} \int_O h^{\mathrm{qc}}_{\overline{\varsigma}}\big(\nabla(\mathcal{T}_k(u))\big)\,\mathrm{d}x = \int_O h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u)\,\mathrm{d}x.$$

For the surface contribution, recall that $J_{\mathcal{T}_k(u)} \subseteq J_u$ for every $k \in \mathbb{N}$, and that $\nu_{\mathcal{T}_k(u)} = \nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in J_{\mathcal{T}_k(u)}$. Since $\mathscr{F}_{\overline{\varsigma}}(u, O) < \infty$, we get

$$\int_O |\nabla u_i|\,\mathrm{d}x + |D^c u_i|(O) + \int_{J_{u_i}} (|[u_i]| \wedge 1)\mathrm{d}\mathcal{H}^{n-1} < \infty,$$

for all $i \in \{1, \dots, N\}$, and in particular (2.0.11) holds (cf. Section 2.0.1). Consequently, we have $(\mathcal{T}_k(u))^\pm \to u^\pm$ and $\chi_{J_{\mathcal{T}_k(u)}} \to \chi_{J_u}$ as $k \to \infty$, with $|[\mathcal{T}_k(u)]| \le |[u]|$ $\mathcal{H}^{n-1}$-a.e. in $J_u$. Hence,

$$\begin{aligned}\lim_{k\to\infty} \int_{J_{\mathcal{T}_k(u)}} g([\mathcal{T}_k(u)], \nu_{\mathcal{T}_k(u)})\mathrm{d}\mathcal{H}^{n-1} &= \lim_{k\to\infty} \int_{J_u} g([\mathcal{T}_k(u)], \nu_u)\chi_{J_{\mathcal{T}_k(u)}}\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{J_u} g([u], \nu_u)\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

by Lemmata 3.3.1 and 3.3.8(ii), together with the Dominated Convergence Theorem.

Regarding the Cantor part of the energy, the definitions of $\mathcal{T}_k$ and $D^c u$ give

$$D^c(\mathcal{T}_k(u)) \llcorner O_k = D^c u \llcorner O_k, \qquad |D^c(\mathcal{T}_k(u))| \ll |D^c u|, \qquad \frac{\mathrm{d}|D^c(\mathcal{T}_k(u))|}{\mathrm{d}|D^c u|} \le 1.$$

Then, applying (3.1.26), we obtain

$$\Big| \int_O h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\mathrm{d}D^c(\mathcal{T}_k(u))) - \int_O h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\mathrm{d}D^c u)\Big| \leq K_{\overline{\varsigma}} \int_{O\setminus O_k} \mathrm{d}|D^c u| = K_{\overline{\varsigma}} |D^c u|(O \setminus O_k).$$

Therefore,

$$\lim_{k\to\infty} \int_O h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\mathrm{d}D^c(\mathcal{T}_k(u))) = \int_O h_{\overline{\varsigma}}^{\mathrm{qc},\infty}(\mathrm{d}D^c u),$$

which completes the proof.

□

To conclude the section, we prove the lower bound for generalized functions of bounded variations.

**Theorem 3.3.15.** *Let $u \in (GBV(O))^N$. Then, under the assumptions of Theorem 3.1.1*

$$\mathscr{F}_{\overline{\varsigma}}(u, O) \leq \Gamma(L^1)\text{-}\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u, 1, O). \tag{3.3.44}$$

*Proof.* Let $u \in (GBV(O))^N$ and let $(u_\varepsilon, v_\varepsilon) \in H^1(O; \mathbb{R}^N \times [0,1])$ be such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(O, \mathbb{R}^{N+1})$. Without loss of generality, we may assume that $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) < \infty$, that the latter is actually a limit (up to a non–relabeled subsequence), and that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ $\mathcal{L}^n$–a.e. in $O$. In particular, by Proposition 3.3.13, we deduce that $u \in (GBV(O))^N$ and $\mathscr{F}_{\overline{\varsigma}}(u, O) < \infty$. Recalling the truncation operator $\mathcal{T}_k$ defined in (2.0.10), we have $\mathcal{T}_k(u_\varepsilon) \to \mathcal{T}_k(u)$ in $L^1(O, \mathbb{R}^N)$ for any $k$, and $\mathcal{T}_k(u) \in BV(O, \mathbb{R}^N)$ since $u \in (GBV(O))^N$. Hence, by Theorem 3.3.12 we obtain

$$\mathscr{F}_{\overline{\varsigma}}(\mathcal{T}_{k_M}(u), O) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon, O), \tag{3.3.45}$$

for every $k \in \mathbb{N}$.

We now claim that for each $M \in \mathbb{N}$ there exists $k_M \in \{M+1, \ldots, 2M\}$, independent of $\varepsilon$, such that, possibly after extracting another subsequence,

$$\mathcal{F}_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon, O) \leq \Big(1 + \frac{c}{M}\Big)\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) + c\,\mathcal{L}^n(\{|u_\varepsilon| > a_M\}), \tag{3.3.46}$$

for some constant $c > 0$ independent of both $\varepsilon$ and $M$. Assuming this, combining (3.3.45) and (3.3.46), and using the convergence $u_\varepsilon \to u$ in measure, we deduce that

$$\limsup_{M\to\infty} \mathscr{F}_{\overline{\varsigma}}(\mathcal{T}_{k_M}(u), O) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O).$$

Finally, applying the continuity under truncation of $\mathscr{F}_{\overline{\varsigma}}$ established in Proposition 3.3.14, we conclude that

$$\mathscr{F}_{\overline{\varsigma}}(u, O) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O),$$

and therefore (3.3.44) holds.

It remains to show (3.3.46). To this end, we employ De Giorgi's averaging–slicing argument on the range, already used in Step 1 of Proposition 3.3.3. For each $k \in \mathbb{N}$, we decompose the energy as

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon, O) = \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon; \{|u_\varepsilon| \le a_k\})& \\ + \mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}) + \mathcal{F}_\varepsilon(0, v_\varepsilon; \{|u_\varepsilon| \ge a_{k+1}\}).& \quad (3.3.47)\end{aligned}$$

By (3.1.4) and by the definition of $\mathcal{T}_k$, the middle term in the previous identity can be estimated as

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}) \le c \int_{\{a_k < |u_\varepsilon| < a_{k+1}\}} \varphi(\varepsilon f^2(v_\varepsilon)) \Phi(\nabla u_\varepsilon)\, \mathrm{d}x& \\ + c\mathcal{L}^n(\{a_k < |u_\varepsilon| < a_{k+1}\}) + \mathcal{F}_\varepsilon(0, v_\varepsilon; \{a_k < |u_\varepsilon| < a_{k+1}\}),& \quad (3.3.48)\end{aligned}$$

for some constant $c > 0$. Summing (3.3.47) and (3.3.48) and averaging over $k$, we obtain that there exists $k_{M,\varepsilon} \in \{M+1, \dots, 2M\}$ such that

$$\begin{aligned}\mathcal{F}_\varepsilon(\mathcal{T}_{k_{M,\varepsilon}}(u_\varepsilon), v_\varepsilon, O) &\le \frac{1}{M} \sum_{k=M+1}^{2M} \mathcal{F}_\varepsilon(\mathcal{T}_k(u_\varepsilon), v_\varepsilon, O) \\ &\le \Big(1 + \frac{c}{M}\Big) \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) + c\, \mathcal{L}^n(\{|u_\varepsilon| > a_M\}),\end{aligned}$$

for some constant $c > 0$. As $\varepsilon \to 0$, we can extract a subsequence such that $\{k_{M,\varepsilon}\}$ is independent of $\varepsilon$, which yields (3.3.46) and completes the proof. □

### 3.3.3 Upper bound

In this section we prove the $\Gamma$–lim sup inequality stated of Theorem 3.1.1. Following the approach of [CFI24a, Section 5], we introduce a perturbed version of the functional $\mathcal{F}_\varepsilon$, denoted by $\widehat{\mathcal{F}}_\varepsilon : L^1(O; \mathbb{R}^{N+1}) \times \mathcal{A}(O) \to [0, \infty]$, and defined as

$$\widehat{\mathcal{F}}_\varepsilon(u, v; A) := \begin{cases} \mathcal{F}_\varepsilon(u, v; A) + \eta_\varepsilon \displaystyle\int_A \Phi(\nabla u)\, \mathrm{d}x, & \text{if } u \in H^1(O; \mathbb{R}^N), \\ \infty, & \text{otherwise,} \end{cases} \quad (3.3.49)$$

where $\eta_\varepsilon > 0$ and $\eta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$.

The additional regularization term $\eta_\varepsilon \int \Phi(\nabla u)\, \mathrm{d}x$ guarantees sufficient coercivity of $\widehat{\mathcal{F}}_\varepsilon$ and facilitates the analysis of the $\Gamma$–limit. In particular, we will establish the upper bound directly for the modified functional, relying on the integral representation formula from [BFM98, Theorem 3.12].

As a preliminary step, we show that $\overline{\Gamma}\text{-}\lim_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(u, 1; \cdot)$ defines a Borel measure by proving the weak subadditivity of $\Gamma\text{–}\limsup_{\varepsilon\to\infty} \widehat{\mathcal{F}}_\varepsilon$.

**Lemma 3.3.16.** *Let $u \in L^1(O; \mathbb{R}^N)$, let $A', A, B \in \mathcal{A}(O)$ with $A' \subset\subset A$, then under the assumptions of Theorem 3.1.1,*

$$\widehat{\mathcal{F}}''(u, 1, A' \cup B) \le \widehat{\mathcal{F}}''(u, 1, A) + \widehat{\mathcal{F}}''(u, 1, B), \quad (3.3.50)$$

*where $\widehat{\mathcal{F}}''(u, 1; \cdot) := \Gamma\text{-}\limsup_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(u, 1; \cdot)$.*

*Proof.* It is not restrictive to assume that the right-hand side of (3.3.50) is finite. Hence, by Proposition 3.3.13, we may suppose that $u \in (GBV \cap L^1(A \cup B))^N$. Let $(u_\varepsilon^A, v_\varepsilon^A)$ and $(u_\varepsilon^B, v_\varepsilon^B) \in H^1(O; \mathbb{R}^{N+1})$ be such that

$$(u_\varepsilon^J, v_\varepsilon^J) \to (u, 1) \text{ in } L^1(O; \mathbb{R}^{N+1}), \tag{3.3.51}$$

and

$$\limsup_{\varepsilon \to 0} \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^J, v_\varepsilon^J; J) = \widehat{\mathcal{F}}''(u, 1; J), \tag{3.3.52}$$

for $J \in \{A, B\}$.

**Step 1. Validity of (3.3.50) for $u \in BV \cap L^2(A \cup B; \mathbb{R}^N)$ and $L^2$-convergent sequences.** Let $\delta := \operatorname{dist}(A', \partial A) > 0$ and fix $M \in \mathbb{N}$. For each $i \in \{1, \ldots, M\}$ we define

$$A_i := \{x \in O : \operatorname{dist}(x, A') < (\delta/M)i\}, \qquad A_0 := A',$$

so that $A_{i-1} \subset\subset A_i \subset A$. Let $\tilde{\phi}_i \in C_c^1(\mathbb{R}^n)$ be a cut-off function between $A_{i-1}$ and $A_i$ satisfying $\tilde{\phi}_i \equiv 1$ on $A_{i-1}$, $\operatorname{supp}(\tilde{\phi}_i) \subseteq A_i$, and $\|\nabla \tilde{\phi}_i\|_{L^\infty(O, \mathbb{R}^n)} \leq 2M/\delta$. We then define

$$u_\varepsilon^i := \tilde{\phi}_i \, u_\varepsilon^A + (1 - \tilde{\phi}_i) u_\varepsilon^B, \tag{3.3.53}$$

and

$$v_\varepsilon^i := \begin{cases} \tilde{\phi}_{i-1} \, v_\varepsilon^A + (1 - \tilde{\phi}_{i-1})(v_\varepsilon^A \wedge v_\varepsilon^B), & \text{on } A_{i-1}, \\ v_\varepsilon^A \wedge v_\varepsilon^B, & \text{on } A_i \setminus A_{i-1}, \\ \tilde{\phi}_{i+1}(v_\varepsilon^A \wedge v_\varepsilon^B) + (1 - \tilde{\phi}_{i+1}) v_\varepsilon^B, & \text{on } O \setminus A_i. \end{cases} \tag{3.3.54}$$

For brevity, let us introduce the auxiliary functional $\mathcal{G}_\varepsilon : L^1(O) \times \mathcal{A}(O) \to [0, \infty]$ defined by

$$\mathcal{G}_\varepsilon(v, A) := \begin{cases} \displaystyle\int_A \frac{\omega(1 - v)}{4\varepsilon} + \varepsilon |\nabla v|^2 \, \mathrm{d}x, & \text{if } v \in H^1(O, [0, 1]), \\ \infty, & \text{otherwise.} \end{cases}$$

From (3.3.53) and (3.3.54), we have $(u_\varepsilon^i, v_\varepsilon^i) \in H^1(O; \mathbb{R}^N \times [0, 1])$ for $i \in \{2, \ldots, M - 1\}$, and

$$\widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^i, v_\varepsilon^i, A' \cup B) \leq \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^A, v_\varepsilon^A, A_{i-2}) + \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^B, v_\varepsilon^B, B \setminus A_{i+1}) + \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^i, v_\varepsilon^i, B \cap (A_{i+1} \setminus A_{i-2})). \tag{3.3.55}$$

We now estimate the last term on the right-hand side of (3.3.55) by splitting it into three contributions. Starting with $\widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^i, v_\varepsilon^i; B \cap (A_{i-1} \setminus A_{i-2}))$, and using (3.1.4) together with Lemma 3.1.2, we obtain

$$\begin{aligned} \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon^i, v_\varepsilon^i; B \cap (A_{i-1} \setminus A_{i-2})) \leq KC \int_{B \cap (A_{i-1} \setminus A_{i-2})} \varphi(\varepsilon f^2(v_\varepsilon^A)) |\nabla u_\varepsilon^A|^2 \, \mathrm{d}x + \mathcal{G}_\varepsilon(v_\varepsilon^i; B \cap (A_{i-1} \setminus A_{i-2})) \\ + C\varphi(\infty) \mathcal{L}^n(B \cap (A_{i-1} \setminus A_{i-2})) + \frac{4M^2 \varepsilon}{\delta^2} \int_{B \cap (A_{i-1} \setminus A_{i-2})} |v_\varepsilon^A - v_\varepsilon^B|^2 \, \mathrm{d}x, \end{aligned} \tag{3.3.56}$$

where $K$ is as in Lemma 3.1.2. Similarly, for the intermediate region we get

$$\begin{aligned}&\widehat{\mathcal{F}}_\varepsilon(u^i_\varepsilon, v^i_\varepsilon; B\cap(A_i\setminus A_{i-1}))\\&\le 2C\int_{B\cap(A_i\setminus A_{i-1})}\varphi(\varepsilon f^2(v^A_\varepsilon\wedge v^B_\varepsilon))\Big(|\nabla u^A_\varepsilon|^2+|\nabla u^B_\varepsilon|^2+\frac{4M^2}{\delta^2}|u^A_\varepsilon-u^B_\varepsilon|^2\Big)\,\mathrm{d}x+\mathcal{G}_\varepsilon(v^A_\varepsilon\wedge v^B_\varepsilon; B\cap(A_i\setminus A_{i-1}))\\&\qquad+C\varphi(\infty)\mathcal{L}^n(B\cap(A_i\setminus A_{i-1})),\end{aligned}\tag{3.3.57}$$

and for the third region, a symmetric estimate as in (3.3.56):

$$\begin{aligned}\widehat{\mathcal{F}}_\varepsilon(u^i_\varepsilon, v^i_\varepsilon; B\cap(A_{i+1}\setminus A_i))&\le KC\int_{B\cap(A_{i+1}\setminus A_i)}\varphi(\varepsilon f^2(v^B_\varepsilon))|\nabla u^B_\varepsilon|^2\,\mathrm{d}x+\mathcal{G}_\varepsilon(v^i_\varepsilon; B\cap(A_{i+1}\setminus A_i))\\&\quad+C\varphi(\infty)\mathcal{L}^n(B\cap(A_{i+1}\setminus A_i))+\frac{4M^2\varepsilon}{\delta^2}\int_{B\cap(A_{i+1}\setminus A_i)}|v^A_\varepsilon-v^B_\varepsilon|^2\,\mathrm{d}x.\end{aligned}\tag{3.3.58}$$

Summing up (3.3.55)–(3.3.58) and averaging over $i$, we find an index $i_\varepsilon$ such that

$$\begin{aligned}\widehat{\mathcal{F}}_\varepsilon(u^{i_\varepsilon}_\varepsilon, v^{i_\varepsilon}_\varepsilon; A'\cup B)&\le\widehat{\mathcal{F}}_\varepsilon(u^A_\varepsilon, v^A_\varepsilon; A)+\widehat{\mathcal{F}}_\varepsilon(u^B_\varepsilon, v^B_\varepsilon; B)+\frac{3C\varphi(\infty)}{M-2}\mathcal{L}^n(B\cap(A\setminus A'))\\&\quad+\frac{24C\,M^2}{\delta^2(M-2)}\int_{B\cap(A\setminus A')}(|u^A_\varepsilon-u^B_\varepsilon|^2+\varepsilon|v^A_\varepsilon-v^B_\varepsilon|^2)\,\mathrm{d}x.\end{aligned}$$

Since a subsequence of $\{i_\varepsilon\}$ can be taken constant, say $i_\varepsilon=i'$, we have $(u^{i'}_\varepsilon, v^{i'}_\varepsilon)\to(u,1)$ in $L^1(O;\mathbb{R}^{N+1})$, and by taking the limits $\varepsilon\to 0$ and $M\to\infty$ we recover (3.3.50) under the stated assumptions.

**Step 2. Extension to $u\in(GBV\cap L^1(A\cup B))^N$.**

Using exactly the same argument in Theorem 3.3.15, if $u\in(GBV(A\cup B))^N$ then for every $M\in\mathbb{N}$ there exists $k_M\in\{M+1,\dots,2M\}$ such that

$$\begin{aligned}\widehat{\mathcal{F}}_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u^A_\varepsilon), v^A_\varepsilon; A)+\widehat{\mathcal{F}}_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u^B_\varepsilon), v^B_\varepsilon; B)&\le\Big(1+\frac{c}{M}\Big)\Big(\widehat{\mathcal{F}}_\varepsilon(u^A_\varepsilon, v^A_\varepsilon; A)+\widehat{\mathcal{F}}_\varepsilon(u^B_\varepsilon, v^B_\varepsilon; B)\Big)\\&\quad+c\mathcal{L}^n(\{|u^A_\varepsilon|\ge a_{M+1}\}\cup\{|u^B_\varepsilon|\ge a_{M+1}\}).\end{aligned}\tag{3.3.59}$$

for some contant $c$ independent of $\varepsilon$ and $M$. Moreover, p to a subsequence, we may assume $k_{\varepsilon,M}=k_M$ independent of $\varepsilon$. Passing to the limit $\varepsilon\to 0$, using (3.3.51), (3.3.52), and Step 1, we deduce

$$\widehat{\mathcal{F}}''(\mathcal{T}_{k_M}(u), 1; A'\cup B)\le\Big(1+\frac{c}{M}\Big)\Big(\widehat{\mathcal{F}}''(u,1;A)+\widehat{\mathcal{F}}''(u,1;B)\Big)+c\mathcal{L}^n(\{|u|\ge a_{M+1}\}).\tag{3.3.60}$$

Finally, since $\mathcal{T}_{k_M}(u)\to u$ in $L^1(O;\mathbb{R}^N)$ as $M\uparrow\infty$, the lower semicontinuity of $\widehat{\mathcal{F}}''$, with respect to the $L^1$-convergence, yields (3.3.50).

□

We prove now the upper bound inequality.

**Theorem 3.3.17.** *Under the assumptions of Theorem 3.1.1, we have that for every $(u,v)\in L^1(O;\mathbb{R}^{N+1})$ it holds*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0}\widehat{\mathcal{F}}_\varepsilon(u,v,O)\le F_{\overline{\varsigma}}(u,v),\tag{3.3.61}$$

*where $F_{\overline{\varsigma}}$ is defined in* (3.1.15).

*Proof.* Given a subsequence $(\widehat{\mathcal{F}}_{\varepsilon_k})$ of $(\widehat{\mathcal{F}}_\varepsilon)$, we can extract (without relabeling) a further subsequence that $\overline{\Gamma}$–converges to some functional $\overline{\mathcal{F}}$. In particular,

$$\overline{\mathcal{F}} = (\mathcal{F}')_- = (\mathcal{F}'')_-, \tag{3.3.62}$$

where $\mathcal{F}'$ and $\mathcal{F}''$ denote the $\Gamma(L^1)$–lower and upper limits of $\widehat{\mathcal{F}}_{\varepsilon_k}$, and the subscript $_-$ stands for the inner regular envelope of the corresponding functional (see [DM93, Definition 16.2 and Theorem 16.9]).

We observe that $\overline{\mathcal{F}}(u, v; \cdot)$ is the restriction to open sets of a Borel measure by [DM93, Theorem 14.23]. Indeed, by construction it is increasing and inner regular; additivity follows from (3.3.62) once one verifies that $(\mathcal{F}')_-$ is superadditive and $(\mathcal{F}'')_-$ is subadditive. The former is a direct consequence of the additivity of $\widehat{\mathcal{F}}_\varepsilon(u, v; \cdot)$ together with [DM93, Proposition 16.12]. The latter is obtained from Lemma 3.3.16, along the same lines as [DM93, Proposition 18.4], replacing [DM93, (18.6)] with Lemma 3.3.16.

We split the proof of (3.3.61) into several steps. Note first that it suffices to treat the case $v = 1$ $\mathcal{L}^n$–a.e. in $O$.

**Step 1. Diffuse estimate for $u \in BV(O; \mathbb{R}^N)$.** We first derive a global upper bound for $\mathcal{F}''$ which is sharp on the diffuse part when $u \in BV(O; \mathbb{R}^N)$. Define $H : L^1(O; \mathbb{R}^N) \times \mathcal{A}(O) \to [0, \infty]$ by

$$H(u; A) := \int_A h_{\overline{\varsigma}}(\nabla u) \, \mathrm{d}x \tag{3.3.63}$$

for $u \in W^{1,1}(O; \mathbb{R}^N)$, and $H(u; A) = \infty$ otherwise, where $h_{\overline{\varsigma}}$ is given by (3.1.9). We claim that

$$\mathcal{F}''(u, 1; A) \le H(u; A) \tag{3.3.64}$$

for $u \in W^{1,1}(O; \mathbb{R}^N)$ and $A \in \mathcal{A}(O)$. Assuming (3.3.64), set $H^* : L^1(O; \mathbb{R}^N) \times \mathcal{A}(O) \to [0, \infty]$ as

$$H^*(u; A) := \int_A h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u) \, \mathrm{d}x \tag{3.3.65}$$

for $u \in W^{1,1}(O; \mathbb{R}^N)$, and $H^*(u; A) = \infty$ otherwise. By the $L^1$–lower semicontinuity of $\mathcal{F}''$ and the relaxation theorem in [AF84, Statement III.7] (whose assumptions follow from Corollary 3.1.4), we deduce

$$\mathcal{F}''(u, 1; A) \le H^*(u; A).$$

Then [ADM92, Theorem 4.1] yields, for every $u \in BV(O; \mathbb{R}^N)$,

$$\mathcal{H}(u; A) := \mathrm{sc}^-(L^1)\text{-}H^*(u; A) = \int_A h^{\mathrm{qc}}_{\overline{\varsigma}}(\nabla u) \, \mathrm{d}x + \int_A h^{\mathrm{qc},\infty}_{\overline{\varsigma}}\left(\mathrm{d}D^s u\right), \tag{3.3.66}$$

and hence

$$\mathcal{F}''(u, 1; A) \le \mathcal{H}(u; A) \tag{3.3.67}$$

for all $u \in BV(O; \mathbb{R}^N)$ and $A \in \mathcal{A}(O)$.

To verify (3.3.64), consider first an affine map $u(x) = \xi x + b$ with $\xi \in \mathbb{R}^{N \times n}$ and $b \in \mathbb{R}^N$. By (3.1.9) there exists $\tau_\xi \in [0,\infty)$ such that

$$h_{\overline{\varsigma}}(\xi) = \varphi\Big(\frac{1}{\tau_\xi}\Big)\Phi(\xi) + \frac{\overline{\varsigma}^2}{4}\tau_\xi, \tag{3.3.68}$$

with the convention that $\varphi(1/\tau)$ is extended continuously at $\tau = 0$ by $\varphi(\infty) \in (0,\infty)$. Define $(u_k, v_k) := (u, \lambda_k)$ with

$$\varepsilon_k f^2(\lambda_k) = \varepsilon_k \frac{l(\lambda_k)}{Q(1-\lambda_k)} = \frac{1}{\tau_\xi}.$$

Then $\lambda_k \to 1$ as $k \to \infty$, thus $(u_k, v_k) \to (u, 1)$ in $L^1(O; \mathbb{R}^{N+1})$, and

$$\begin{aligned} \widehat{\mathcal{F}}_{\varepsilon_k}(u_k, v_k, A) &= \Big[\varphi\Big(\frac{1}{\tau_\xi}\Big)\Phi(\xi) + \frac{\omega(1-\lambda_k)}{4\varepsilon_k} + \eta_{\varepsilon_k}\Phi(\xi)\Big]\mathcal{L}^n(A) \\ &= \Big[\varphi\Big(\frac{1}{\tau_\xi}\Big)\Phi(\xi) + \frac{\omega(1-\lambda_k)}{4Q(1-\lambda_k)} l(\lambda_k)\tau_\xi + \eta_{\varepsilon_k}\Phi(\xi)\Big]\mathcal{L}^n(A). \end{aligned} \tag{3.3.69}$$

Using (3.1.2) and $\eta_\varepsilon = o(\varepsilon)$, we get

$$\limsup_{k\to\infty} \widehat{\mathcal{F}}_{\varepsilon_k}(u_k, v_k; A) = F_{\overline{\varsigma}}(u, 1),$$

which proves (3.3.64) for affine $u$ by (3.3.68)–(3.3.69).

Next, let $u \in C^0(O; \mathbb{R}^N)$ be piecewise affine, say $u(x) = \sum_{i=1}^M (\xi_i x + b_i)\chi_{O_i}(x)$ with pairwise disjoint $O_i \in \mathcal{A}(O)$ of Lipschitz boundary and $\mathcal{L}^n(O \setminus \cup_i O_i) = 0$. Set

$$u_k := u\ , \qquad v_k := \sum_{i=1}^M \tilde{\phi}_i \lambda_k^i,$$

where for each $i$ and $k$,

$$\varepsilon_k f^2(\lambda_k^i) = \frac{1}{\tau_{\xi_i}},$$

and $\{\tilde{\phi}_i\}$ is a partition of unity subordinate to the cover $\{O_i^\delta\}$, with the usual properties. Since $\sum_i \tilde{\phi}_i = 1$, we have

$$1 - v^k = \sum_{i=1}^M (1-\lambda_k^i)\tilde{\phi}_i \leq 1 - \lambda_k^{i_k}$$

for some $i_k$. Arguing as in (3.3.69) and by monotonicity of $\omega$,

$$\begin{aligned} \limsup_{k\to\infty} \widehat{\mathcal{F}}_{\varepsilon_k}(u_k, v_k; A) &\leq \sum_{i=1}^N \mathcal{L}^n(O_i^{-\delta} \cap A) h_{\overline{\varsigma}}(\xi_i) \\ &\quad + \limsup_{k\to\infty} \Big[\Big(l(\lambda_k^{i_k})\tau_{\xi_{i_k}} \frac{\omega(1-\lambda_k^{i_k})}{Q(1-\lambda_k^{i_k})} + \frac{\varepsilon_k M K}{\delta}\Big) \sum_{i=1}^M \mathcal{L}^n((O_i^\delta \setminus O_i^{-\delta}) \cap A)\Big]. \end{aligned} \tag{3.3.70}$$

Since $\lambda_k^{i_k} \to 1$, $(u_k, v_k) \to (u, 1)$ in $L^1(O; \mathbb{R}^{N+1})$, and letting $\delta \downarrow 0$ in (3.3.70) gives

$$\mathcal{F}''(u, 1; A) \leq \sum_{i=1}^N \mathcal{L}^n(O_i \cap A) h_{\overline{\varsigma}}(\xi_i) = H(u; A).$$

Finally, for general $u \in W^{1,1}(O;\mathbb{R}^N)$, extend $u$ (still denoted by $u$) to $W_0^{1,1}(O';\mathbb{R}^N)$ for some bounded $O' \supset\supset O$ (using that $O$ is Lipschitz), approximate by piecewise affine $u_k \to u$ in $W^{1,1}(O';\mathbb{R}^N)$ (see [ET99, Chapter X, Prop. 2.1]), and conclude by the continuity of $H$ in $W^{1,1}$ and the $L^1$–lower semicontinuity of $\mathcal{F}''$.

**Step 2. Inner regularity of $\mathcal{F}''(u,1;\cdot)$ and existence of the $\Gamma(L^1)$–limit for $u \in BV(O;\mathbb{R}^N)$.** We claim that for $u \in BV(O;\mathbb{R}^N)$,

$$\mathcal{F}''(u,1;\cdot) = (\mathcal{F}'')_-(u,1;\cdot). \tag{3.3.71}$$

Given $A$ open and $\delta > 0$, choose $A' \subset\subset A'' \subset\subset A$ and an open set $C$ with $A \setminus A' \subset C$ so that $\mathcal{H}(u;C) \le \delta$, where $\mathcal{H}$ is defined in (3.3.66). By Lemma 3.3.16 and (3.3.67),

$$\mathcal{F}''(u,1;A) \le \mathcal{F}''(u,1;A' \cup C) \le \mathcal{F}''(u,1;A'') + \mathcal{H}(u;C) \le \mathcal{F}''(u,1;A'') + \delta,$$

which gives (3.3.71). Hence, from (3.3.62),

$$\overline{\mathcal{F}}(u,1;\cdot) \le \mathcal{F}'(u,1;\cdot) \le \mathcal{F}''(u,1;\cdot) = \overline{\mathcal{F}}(u,1;\cdot), \tag{3.3.72}$$

so the $\Gamma$–limit of $\widehat{\mathcal{F}}_{\varepsilon_k}(u,1;\cdot)$ exists and coincides with $\overline{\mathcal{F}}(u,1;\cdot)$ for every $u \in BV(O;\mathbb{R}^N)$.

**Step 3. Integral representation of the $\Gamma(L^1)$–limit on $BV(O;\mathbb{R}^N) \times \{1\}$.** To represent $\overline{\mathcal{F}}$ in integral form and to estimate its densities, we consider the coercive perturbation

$$\overline{\mathcal{F}}_\lambda(u,1,A) := \overline{\mathcal{F}}(u,1,A) + \lambda|Du|(A),$$

for $u \in BV(O;\mathbb{R}^N)$, $A \in \mathcal{A}(O)$ and $\lambda \in (0,1]$. By (3.1.25), (3.1.26), (3.3.44), (3.3.67), and (3.3.72),

$$\lambda|Du|(O) - c\mathcal{L}^n(O) \le \overline{\mathcal{F}}_\lambda(u,1,O) \le c(|Du|(O) + \mathcal{L}^n(O)),$$

for some $c > 0$ and all $u \in BV(O;\mathbb{R}^N)$. Moreover, translation invariance holds:

$$\widehat{\mathcal{F}}_{\varepsilon_k}(u(\cdot - z), v(\cdot - z); z + A) = \widehat{\mathcal{F}}_{\varepsilon_k}(u,v;A), \qquad \widehat{\mathcal{F}}_{\varepsilon_k}(u + b, v; A) = \widehat{\mathcal{F}}_{\varepsilon_k}(u,v;A),$$

whence the analogous properties for $\overline{\mathcal{F}}$. By [BFM98, Theorem 3.12], for $u \in BV(O;\mathbb{R}^N)$ and $A \in \mathcal{A}(O)$,

$$\overline{\mathcal{F}}_\lambda(u,1;A) = \int_A h_\lambda(\nabla u)\,\mathrm{d}x + \int_A h_\lambda^\infty(\mathrm{d}D^c u) + \int_{J_u \cap A} g_\lambda([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1},$$

with

$$h_\lambda(\xi) := \limsup_{\rho \downarrow 0} \frac{1}{\rho^n} \inf\Big\{\overline{\mathcal{F}}_\lambda(w,1;Q_\rho) : \, w \in BV(Q_\rho;\mathbb{R}^N),\ w(x) = \xi x \text{ on } \partial Q_\rho\Big\}, \tag{3.3.73}$$

for $\xi \in \mathbb{R}^{N \times n}$,

$$g_\lambda(\zeta,\nu) := \limsup_{\rho \downarrow 0} \frac{1}{\rho^{n-1}} \inf\Big\{\overline{\mathcal{F}}_\lambda(w,1;Q_\rho^\nu) : \, w \in BV(Q_\rho^\nu;\mathbb{R}^N),\ w = u_{\zeta,\nu} \text{ on } \partial Q_\rho^\nu\Big\}, \tag{3.3.74}$$

for $\zeta \in \mathbb{R}^N$, $\nu \in \mathbb{S}^{n-1}$, and

$$h_\lambda^\infty(\xi) := \limsup_{t\to\infty} \frac{h_\lambda(t\xi)}{t}.$$

Using (3.3.67),

$$h_\lambda(\xi) \le \frac{1}{\rho^n}\overline{\mathcal{F}}_\lambda(\xi x, 1; Q_\rho) \le h^{\mathrm{qc}}(\xi) + \lambda|\xi|, \tag{3.3.75}$$

hence

$$h_\lambda^\infty(\xi) \le h^{\mathrm{qc},\infty}(\xi) + \lambda|\xi|. \tag{3.3.76}$$

We now prove

$$g_\lambda(\zeta,\nu) \le g(\zeta,\nu) + \lambda|\zeta|, \tag{3.3.77}$$

for all $\zeta \in \mathbb{R}^N$, $\nu \in \mathbb{S}^{n-1}$. From (3.3.74),

$$g_\lambda(\zeta,\nu) \le \limsup_{\rho\downarrow 0} \frac{1}{\rho^{n-1}}\overline{\mathcal{F}}_\lambda(u_{\zeta,\nu}, 1; Q_\rho^\nu) = \limsup_{\rho\downarrow 0} \frac{1}{\rho^{n-1}}\overline{\mathcal{F}}(u_{\zeta,\nu}, 1; Q_\rho^\nu) + \lambda|\zeta|. \tag{3.3.78}$$

By the definition of $\overline{\mathcal{F}}$, for any $(\tilde{u}_k, \tilde{v}_k) \to (u_{\zeta,\nu}, 1)$ in $L^1(Q_\rho^\nu; \mathbb{R}^{N+1})$,

$$\overline{\mathcal{F}}(u_{\zeta,\nu}, 1; Q_\rho^\nu) \le \limsup_{k\to\infty} \widehat{\mathcal{F}}_{\varepsilon_k}(\tilde{u}_k, \tilde{v}_k; Q_\rho^\nu). \tag{3.3.79}$$

Thus, it suffices to construct a suitable recovery sequence $(\tilde{u}_k, \tilde{v}_k)$ depending on $\rho$, $\zeta$, and $\nu$. By applying Proposition 3.3.5 with $\varepsilon_k^* := \varepsilon_k/\rho$ and $\eta_k^* := \eta_{\varepsilon_k}$, we obtain sequences $(u_k^*, v_k^*) \to (u_{\zeta,\nu}, 1)$ in $L^2(Q^\nu; \mathbb{R}^{N+1})$ such that

$$\lim_{k\to\infty} \mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu) = g(\zeta,\nu) \quad \text{and} \quad \lim_{k\to\infty} \eta_k^* \|\nabla u_k^*\|^2_{L^2(Q^\nu)} = 0. \tag{3.3.80}$$

Define, on $Q_\rho^\nu$,

$$\tilde{u}_k(y) := u_k^*\left(\frac{y}{\rho}\right),$$

so that $\tilde{u}_k \to u_{\zeta,\nu}$ in $L^2(Q_\rho^\nu; \mathbb{R}^N)$.

Fix $\tau > 0$ and $\delta \in (0,1)$. By (3.1.6), we have $\Phi(\xi) \le (1+\tau)\Phi_\infty(\xi)$ for $|\xi|$ large, and hence, for all $\xi$,

$$\Phi(\xi) \le (1+\tau)\Phi_\infty(\xi) + C(\tau). \tag{3.3.81}$$

Moreover, by the same argument used in the proof of Proposition 3.2.8, for every $\varepsilon > 0$ small enough (depending on $\delta$), there exists $\delta_\varepsilon \in (0,1)$ such that $\varepsilon f^2(\delta_\varepsilon) = \delta$ (in particular $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$), and

$$\varepsilon f^2(t) \le \delta \tag{3.3.82}$$

if and only if $t \in [0, \delta_\varepsilon]$.

Setting $v_{k,\delta}^* := v_k^* \wedge \delta_{\varepsilon_k}$, by (3.3.80) and the definition of $\delta_\varepsilon$, there exists a constant $c$ (depending only on $\rho$, $\zeta$, and $\nu$) such that

$$\mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_{k,\delta}^*; Q^\nu) \le \mathcal{F}^\infty_{\varepsilon_k^*}(u_k^*, v_k^*; Q^\nu) + c\,\varphi\left(\frac{\delta}{\rho}\right). \tag{3.3.83}$$

Now let

$$\tilde{v}_{k,\delta}(y) := v^*_{k,\delta}\left(\frac{y}{\rho}\right).$$

Since $\varphi$ is differentiable at 0 (see (Hp 4)), there exists a modulus of continuity $\vartheta$ such that

$$\varphi(\varepsilon_k f^2(\tilde{v}_{k,\delta})) \leq (1+\vartheta(\delta))\varphi'(0^+)\varepsilon_k f^2(\tilde{v}_{k,\delta}), \tag{3.3.84}$$

for every $k$. Changing variables, we obtain

$$\mathcal{F}^\infty_{\varepsilon_k}(\tilde{u}_k, \tilde{v}_{k,\delta}; Q^\nu_\rho) = \rho^{n-1}\mathcal{F}^\infty_{\varepsilon^*_k}(u^*_k, v^*_{k,\delta}; Q^\nu), \quad \|\nabla\tilde{u}_k\|^2_{L^2(Q^\nu_\rho)} = \rho^{n-2}\|\nabla u^*_k\|^2_{L^2(Q^\nu)}. \tag{3.3.85}$$

Combining (3.3.85), (3.3.84), (3.3.83), and (3.3.81), we deduce

$$\begin{aligned}
\mathcal{F}_{\varepsilon_k}(\tilde{u}_k, \tilde{v}_{k,\delta}; Q^\nu_\rho) &\leq (1+\vartheta(\delta))\int_{Q^\nu_\rho}\Big(\varphi'(0^+)\varepsilon_k f^2(\tilde{v}_{k,\delta})\Phi(\nabla\tilde{u}_k) + \frac{\omega(1-\tilde{v}_{k,\delta})}{4\varepsilon_k} + \varepsilon_k|\nabla\tilde{v}_{k,\delta}|\Big)\,dx \\
&\leq (1+\vartheta(\delta))\Big(\rho^{n-1}(1+\tau)\mathcal{F}^\infty_{\varepsilon^*_k}(u^*_k, v^*_{k,\delta}; Q^\nu) + \rho^n C(\tau)\Big) \\
&\leq (1+\vartheta(\delta))\Big(\rho^{n-1}(1+\tau)\mathcal{F}^\infty_{\varepsilon^*_k}(u^*_k, v^*_k; Q^\nu) + c\,\varphi\left(\frac{\delta}{\rho}\right) + \rho^n C(\tau)\Big).
\end{aligned}$$

Moreover, from (3.1.4) and (3.3.85),

$$\eta_{\varepsilon_k}\int_{Q^\nu_\rho}\Phi(\nabla\tilde{u}_k)\,dx \leq C\eta_{\varepsilon_k}\rho^{n-2}\|\nabla u^*_k\|^2_{L^2(Q^\nu)} + C\eta_{\varepsilon_k}\rho^n.$$

Up to extracting a subsequence, there exists an infinitesimal sequence $(\delta_k)_{k\in\mathbb{N}}$ such that $\delta^k_{\varepsilon_k} \to 1$ as $k\to\infty$ (cf. (3.3.82)). Setting $\tilde{v}_k := \tilde{v}_{k,\delta^k}$, we have $(\tilde{u}_k, \tilde{v}_k) \to (u_{\zeta,\nu}, 1)$ in $L^2(Q^\nu_\rho; \mathbb{R}^{N+1})$. Summing up and passing to the limit as $k\to\infty$, by (3.3.79) and (3.3.80), we obtain

$$\overline{\mathcal{F}}(u_{\zeta,\nu}, 1; Q^\nu_\rho) \leq \rho^{n-1}(1+\tau)g(\zeta,\nu) + C(\tau)\rho^n.$$

Dividing by $\rho^{n-1}$ and letting $\rho\downarrow 0$, in view of (3.3.78),

$$g_\lambda(\zeta,\nu) \leq (1+\tau)g(\zeta,\nu) + \lambda|\zeta|.$$

Since $\tau > 0$ is arbitrary, inequality (3.3.77) follows.

Finally, letting $\lambda\downarrow 0$, inequalities (3.3.75), (3.3.76), and (3.3.77) yield, for every $u\in BV(O;\mathbb{R}^m)$,

$$\overline{\mathcal{F}}(u, 1, O) \leq \mathscr{F}_{\overline{\varsigma}}(u, O) = F_{\overline{\varsigma}}(u, 1).$$

Combined with the lower bound (Theorem 3.3.12), this identifies the $\Gamma$–limit of the subsequence $\widehat{\mathcal{F}}_{\varepsilon_k}$. By Urysohn's property [DM93, Proposition 8.3], the result extends to the entire family $\widehat{\mathcal{F}}_\varepsilon$.

**Step 4. Extension to $(GBV(O))^N\times\{1\}$.** To extend (3.3.61) to $u\in(GBV(O))^N$, we argue by truncation. For $k\in\mathbb{N}$, let $\mathcal{T}_k$ be as in (2.0.10). By Steps 1–3,

$$\mathcal{F}''(\mathcal{T}_k(u), 1, O) \leq \mathscr{F}_{\overline{\varsigma}}(\mathcal{T}_k(u), O) = F_{\overline{\varsigma}}(\mathcal{T}_k(u), 1).$$

The claim follows by the $L^1$–lower semicontinuity of $\mathcal{F}''$ and Proposition 3.3.14.

$\square$

We are ready to prove the vectorial part of Theorem 3.1.1.

*Proof of Theorem 3.1.1 (vectorial part).* The lower bound has been proven in Theorem 3.3.15. The upper bound follows by Theorem 3.3.17 and $\mathcal{F}_\varepsilon \leq \widehat{\mathcal{F}}_\varepsilon$. $\square$

### 3.3.4 Compactness and convergence of minimizers

Next theorem establishes the compactness of sequences equibounded in energy and in $L^1$.

**Theorem 3.3.18.** *If $(u_\varepsilon, v_\varepsilon) \in H^1(O;\mathbb{R}^{N+1})$ is such that*

$$\sup_\varepsilon \left(\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon, O) + \|u_\varepsilon\|_{L^1(O,\mathbb{R}^N)}\right) < \infty, \tag{3.3.86}$$

*then, under the assumptions of Theorem 3.1.1, there exists a subsequence $(u_j, v_j)$ of $(u_\varepsilon, v_\varepsilon)$ and a function $u \in (GBV \cap L^1(O))^N$ such that $u_j \to u$ $\mathcal{L}^n$-a.e. and $v_j \to 1$ in $L^1(O)$.*

*Proof.* First, we observe that $v_\varepsilon \to 1$ in measure. Hence, since $0 \le v_\varepsilon \le 1$ and $\mathcal{L}^n(O) < \infty$, we may extract a subsequence (not relabeled) such that $v_\varepsilon \to 1$ in $L^1(O)$.

Next, using the notation introduced in Lemma 3.1.7, and fixing $\delta \in (0,1)$, we have that $(u^\delta_\varepsilon)_i \in GSBV(O)$ and

$$\sup_\varepsilon \left(\int_O |\nabla (u^\delta_\varepsilon)_i| \,\mathrm{d}x + \mathcal{H}^{n-1}(J_{(u^\delta_\varepsilon)_i})\right) < \infty, \tag{3.3.87}$$

for every $i = 1, \dots, N$, where $(\cdot)_i$ denotes the $i$-th component. In particular, by (3.3.86), the compactness theorem in $BV$, and a diagonal argument, there exist a subsequence $(\varepsilon_j)$ and a function $u \in GBV(O)^N$ such that, for every $i = 1, \dots, N$ and every $M > 0$,

$$\tau_M((u^\delta_{\varepsilon_j})_i) \to \tau_M(u_i) \quad \text{in } L^1(O) \text{ as } j \to \infty, \tag{3.3.88}$$

where $\tau_M(s) = (s \wedge M) \vee (-M)$.

Moreover, $u \in L^1(O,\mathbb{R}^N)$ since, for each $i = 1, \dots, N$,

$$\begin{aligned}\|u_i\|_{L^1(O)} &\le \liminf_{M\to\infty} \|\tau_M(u_i)\|_{L^1(O)} = \liminf_{M\to\infty} \lim_{j\to\infty} \|\tau_M((u^\delta_{\varepsilon_j})_i)\|_{L^1(O)} \\ &\le \sup_\varepsilon \|u_\varepsilon\|_{L^1(O,\mathbb{R}^N)} < \infty.\end{aligned}$$

Finally, since $v_\varepsilon \to 1$ in $L^1(O)$, arguing as in the proof of Theorem 3.2.6, we obtain that $\tau_M((u_{\varepsilon_j})_i) \to \tau_M(u_i)$ in measure for every $i = 1, \dots, N$ and $M > 0$. As $\mathcal{L}^n(O) < \infty$, a further diagonal argument yields the desired convergence.

□

Convergence of minimizers and of minimum values now follows in a standard way from Theorems 3.1.1 and 3.3.18. Let $\widehat{\mathcal{F}}_\varepsilon$ be as in (3.3.49), and fix $w \in L^q(O;\mathbb{R}^N)$ with $q > 1$. Define the functionals $\mathcal{C}_\varepsilon, \mathcal{C}_0 : L^q(O;\mathbb{R}^N) \times L^1(O) \to [0,\infty]$ by

$$\mathcal{C}_\varepsilon(u,v) := \begin{cases} \widehat{\mathcal{F}}_\varepsilon(u,v,O) + \displaystyle\int_O |u-w|^q \,\mathrm{d}x\,, & \text{if } (u,v) \in H^1(O;\mathbb{R}^N \times [0,1]), \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{C}_0(u,v) := F_{\overline{\varsigma}}(u,v) + \int_O |u-w|^q \,\mathrm{d}x,$$

where $F_{\overline{\varsigma}}$ is defined in (3.1.15).

The assumption ${}^{\eta_\varepsilon}/_{\varepsilon} \to 0$ as $\varepsilon \to 0$ is required to prevent the additional term $\eta_\varepsilon \Phi(\nabla u)$ from competing with the penalization term $(1-v)^2/\varepsilon$, which would otherwise affect the limit behaviour. Indeed, if $\eta_\varepsilon \sim \varepsilon$, one would recover a control on $|[u]|$, thereby losing the cohesive effect in the limit (see, e.g., [FI14]).

Conversely, the introduction of the perturbation term $\eta_\varepsilon \Phi(\nabla w)$ ensures the existence of minimizers for $\mathcal{C}_\varepsilon$, provided $\Phi$ is quasiconvex. In general, the coercivity of $\mathcal{C}_\varepsilon$ only yields the existence of minimizing sequences $(u_\varepsilon^j)_j$ converging weakly in $H^1(O;\mathbb{R}^N)$ to some $\bar{u}_\varepsilon$ that minimizes the relaxation of $\mathcal{C}_\varepsilon$. Since existence at fixed $\varepsilon$ does not interfere with the $\Gamma$-convergence argument, we state the following result for asymptotically minimizing sequences.

**Corollary 3.3.19.** *Assume the hypothesis of Theorem 3.1.1. Let $(u_\varepsilon, v_\varepsilon) \in H^1(O;\mathbb{R}^{N+1})$ be such that*

$$\limsup_{\varepsilon\to 0}\big(\mathcal{C}_\varepsilon(u_\varepsilon, v_\varepsilon) - m_\varepsilon\big) = 0,$$

*where $m_\varepsilon := \inf_{(u,v)\in H^1(O,\mathbb{R}^{N+1})} \mathcal{C}_\varepsilon(u,v)$. Then $v_\varepsilon \to 1$ in $L^1(O)$, and a subsequence of $u_\varepsilon$ converges in $L^q(O;\mathbb{R}^N)$ to a minimizer of*

$$\min_{u\in(GBV(O))^N} \mathcal{C}_0(u,1).$$

*Moreover, $m_\varepsilon$ converges to the minimum value of $\mathcal{C}_0$.*

*Proof.* The proof proceeds in three steps.

*Step 1. $\Gamma$-limit of $\widehat{\mathcal{F}}_\varepsilon$ in $L^q \times L^1$.* We first show that when passing from the $L^1\times L^1$ to the $L^q \times L^1$ topology, the expression of the $\Gamma$-limit of $\widehat{\mathcal{F}}_\varepsilon$ remains unchanged:

$$\Gamma(L^q\times L^1)\text{-}\lim_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(u,v,O) = F_{\overline{\varsigma}}(u,v).$$

The lower bound is immediate from the $L^1\times L^1$ case (Theorem 3.3.15), since convergence in $L^q(O;\mathbb{R}^N)$ is stronger than convergence in $L^1(O;\mathbb{R}^N)$ when $\mathcal{L}^n(O) < \infty$.

For the upper bound, we use a truncation argument. Fix $u \in BV \cap L^\infty(O;\mathbb{R}^N)$ with $F_{\overline{\varsigma}}(u,1) < \infty$. By Theorem 3.3.17, there exists a sequence $(u_\varepsilon, v_\varepsilon) \in H^1(O;\mathbb{R}^{N+1})$ such that $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(O;\mathbb{R}^{N+1})$ and

$$\limsup_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon, O) \le F_{\overline{\varsigma}}(u,1).$$

Let $M \in \mathbb{N}$ be large enough so that $a_M > \|u\|_{L^\infty(O;\mathbb{R}^N)}$ (see (2.0.10)), and for every $\varepsilon > 0$ choose $k_{\varepsilon,M} \in \{M+1, \dots, 2M\}$ satisfying

$$\int_{\{a_{k_{\varepsilon,M}} < |u_\varepsilon| < a_{k_{\varepsilon,M}+1}\}} (\eta_\varepsilon + \varphi(\varepsilon f^2(v_\varepsilon)))\Phi(\nabla u_\varepsilon)\,\mathrm{d}x \le \frac{1}{M}\int_O (\eta_\varepsilon + \varphi(\varepsilon f^2(v_\varepsilon)))\Phi(\nabla u_\varepsilon)\,\mathrm{d}x.$$

This yields

$$\widehat{\mathcal{F}}_\varepsilon(\mathcal{T}_{k_{\varepsilon,M}}(u_\varepsilon), v_\varepsilon) \le \Big(1 + \frac{C^2}{M}\Big)\widehat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) + C\,\mathcal{L}^n(\{a_{M+1} < |u_\varepsilon|\}),$$

where $\mathcal{T}_{k_{\varepsilon,M}}$ is defined in (2.0.10) and $C$ is the constant in (3.1.4). As in Theorem 3.3.15, we can extract a subsequence such that $k_{\varepsilon,M} = k_M$ is independent of $\varepsilon$. Since $(\mathcal{T}_{k_M}(u_\varepsilon))_\varepsilon$ is uniformly bounded in $L^\infty$ and

$M$ is large enough, we have $\mathcal{T}_{k_M}(u_\varepsilon) \to \mathcal{T}_{k_M}(u) = u$ in $L^q(O;\mathbb{R}^N)$, and $\mathcal{L}^n(\{a_{M+1} < |u_\varepsilon|\}) \to 0$ as $\varepsilon \to 0$. Hence,

$$\limsup_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(\mathcal{T}_{k_M}(u_\varepsilon), v_\varepsilon) \le \Big(1 + \frac{C^2}{M}\Big) F_{\overline{\varsigma}}(u,1).$$

Diagonalizing with respect to $M$ and using the lower bound, we conclude that every subsequence of $\{\widehat{\mathcal{F}}_\varepsilon\}_\varepsilon$ has a further subsequence that $\Gamma(L^q \times L^1)$-converges to $F_{\overline{\varsigma}}$ in $L^\infty(O;\mathbb{R}^N) \times L^1(O)$. Urysohn's lemma then guarantees convergence of the whole sequence.

For general $u \in (GBV \cap L^q(O))^N$, since $\mathcal{T}_k(u) \in (BV \cap L^\infty(O))^N$, we have

$$\Gamma(L^q \times L^1)\text{-}\limsup_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(\mathcal{T}_k(u), 1) \le F_{\overline{\varsigma}}(\mathcal{T}_k(u), 1),$$

and the conclusion follows letting $k \to \infty$ and using the continuity of $\mathscr{F}_{\overline{\varsigma}}$ (Proposition 3.3.14).

*Step 2. $\Gamma$-limit of $\mathcal{C}_\varepsilon$ in $L^1 \times L^1$.* We now prove that

$$\Gamma(L^1 \times L^1)\text{-}\lim_{\varepsilon\to 0} \mathcal{C}_\varepsilon(u,v) = \mathcal{C}_0(u,v).$$

The lower bound follows from Theorem 3.3.15, since $\eta_\varepsilon \ge 0$ and $\int_O |u-w|^q \,\mathrm{d}x$ is lower semicontinuous under $L^1$ convergence. In particular, if $\Gamma(L^1 \times L^1)\text{-}\liminf_{\varepsilon\to 0} \mathcal{C}_\varepsilon(u,v) < \infty$, then $u \in (GBV \cap L^q(O))^N$ and $v = 1$ $\mathcal{L}^n$-a.e. in $O$.

As for the upper bound, Step 1 provides a recovery sequence for $\widehat{\mathcal{F}}_\varepsilon$ in $L^q \times L^1$, which is also a recovery sequence for $\mathcal{C}_\varepsilon$ in $L^1 \times L^1$.

*Step 3. Convergence of minimizers.* Let $(u_\varepsilon, v_\varepsilon) \in H^1 \cap L^q(O;\mathbb{R}^{N+1})$ be a minimizing sequence for $\mathcal{C}_\varepsilon$. Since

$$\sup_{\varepsilon>0} \big(\widehat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon) + \|u_\varepsilon\|_{L^q(O;\mathbb{R}^N)}\big) < \infty,$$

Theorem 3.3.18 yields the existence of $u \in (GBV \cap L^q(O))^N$ and a subsequence (not relabeled) such that $u_\varepsilon \to u$ $\mathcal{L}^n$-a.e. in $O$ and $v_\varepsilon \to 1$ in $L^1(O)$. By Hölder's inequality,

$$\begin{aligned}\int_{\{|u_\varepsilon - u|>1\}} |u_\varepsilon - u| \,\mathrm{d}x &\le \|u_\varepsilon - u\|_{L^q(O;\mathbb{R}^N)} \big(\mathcal{L}^n(\{|u_\varepsilon - u| > 1\})\big)^{1-\frac{1}{q}} \\ &\le c\,\big(\mathcal{L}^n(\{|u_\varepsilon - u| > 1\})\big)^{1-\frac{1}{q}},\end{aligned}$$

and the right-hand side tends to 0 since $u_\varepsilon \to u$ in measure. Moreover, $(u_\varepsilon - u)\chi_{\{|u_\varepsilon - u|\le 1\}} \to 0$ in $L^1(O;\mathbb{R}^N)$ by dominated convergence, so $u_\varepsilon \to u$ in $L^1(O;\mathbb{R}^N)$.

By Step 2 and the general properties of $\Gamma$-convergence (see [DM93, Corollary 7.20]), we deduce that $(u,1)$ is a minimizer of $\mathcal{C}_0$ and that $\mathcal{C}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathcal{C}_0(u,1)$. Finally, we show that $u_\varepsilon \to u$ in $L^q(O;\mathbb{R}^N)$. From the previous steps,

$$\mathcal{C}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathcal{C}_0(u,1), \qquad \int_O |u-w|^q \,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \int_O |u_\varepsilon - w|^q \,\mathrm{d}x,$$

and

$$F_{\overline{\varsigma}}(u,1) \le \liminf_{\varepsilon\to 0} \widehat{\mathcal{F}}_\varepsilon(u_\varepsilon, v_\varepsilon, O),$$

so that

$$\int_O |u_\varepsilon - w|^q \,\mathrm{d}x \to \int_O |u - w|^q \,\mathrm{d}x.$$

Combined with the pointwise convergence, this implies $u_\varepsilon \to u$ in $L^q(O;\mathbb{R}^N)$ by the generalized dominated convergence theorem. $\square$

# Chapter 4

# Assigning the cohesive law

The main objective of this chapter is to rigorously address the problem of assigning or reconstructing the cohesive law $g$ (cf. (3.2.9)) arising as the surface energy density in the $\Gamma$–limit of the phase–field approximation considered in Section 3.2.

Starting from the $\Gamma$–convergence result stated in Theorem 3.2.1, whose generality proves to be crucial for this purpose, we develop a systematic strategy to construct a phase–field model capable of reproducing a prescribed traction–separation law. This approach provides a rigorous theoretical framework that links the analytical properties of the potentials defining the phase–field energy to the mechanical behaviour of the resulting cohesive response. More in particular, in Section 4.2 we take advantage of Theorem 3.2.1 to solve the latter problem for a large class of concave functions with either a linear or a superlinear behaviour for small jump amplitudes, including several examples which are typically used in the engineering community for numerical simulations (cf. Section 4.2.3). The latter problem has already been discussed from a numerical and engineering perspective in [Wu17, WN18, FFL21, LCM25, Wu24b].

Section 4.1 is devoted to provide an alternative characterization of the surface energy densities in the phase–field model. Our approach is different from the one introduced in [BCI21, BI24] (cf. the comments before Proposition 4.1.3 for a more detailed comparison). One of the main ideas we introduce is rewriting in a convenient way the minimization formula that defines the surface energy density $g(s)$ for every jump amplitude $s \geq 0$ (cf. (3.2.9)). Indeed, the latter entails the solution of a vector–valued, one–dimensional variational problem with Dirichlet boundary conditions. Via a change of variable we are able to reduce it to a minimum problem for a family of scalar one–dimensional functionals $G_{m,s}$, defined in (4.1.9), depending on the deformation gradient, parametrized by the minimum value $m$ of the phase–field variable and finite on $W^{1,1}$ functions satisfying a suitable Dirichlet boundary condition depending on $s$ (cf. Proposition 4.1.3). The new minimum problem is obtained by minimizing first each functional $G_{m,s}$ on $L^1$, and then minimizing the corresponding minimimum value $\mathfrak{G}_s(m)$ on the interval $(0,1)$ (cf. (4.1.11) and (4.1.19)), namely $g(s) = \inf_{m\in(0,1)} \mathfrak{G}_s(m)$ where $\mathfrak{G}_s(m) = \inf_{L^1} G_{m,s}$.

Another key step in our approach is the variational analysis of the energies $G_{m,s}$ introduced in (4.1.9). In particular, for each $m$ we determine the relaxed functional with respect to the $L^1$ topology of $G_{m,s}$, which turns out to be finite on $BV$, and to have a unique minimizer that is either in $W^{1,1}$ or in $SBV \setminus W^{1,1}$, with the singular part of the distributional derivative concentrated at the point $t = m$ in the latter case (cf. Theorem 4.1.5 and Corollary 4.1.6). The analysis of the function $\mathfrak{G}_s$ exploits an auxiliary key function

$\Lambda$, defined in (4.1.24), determining its monotonicity properties. More precisely, $\mathfrak{G}_s$ turns out to be increasing on the intervals where the minimizer $w_{m,s}$ of the relaxation of $G_{m,s}$ belongs to $SBV \setminus W^{1,1}$ and decreasing otherwise (for the precise statement see Lemma 4.1.11). In addition, the regularity of the minimizer $w_{m,s}$ depends on whether $m$ belongs or not to the $s$–sublevel set of $\Lambda$. Given this, we are able to show that $g(s)$ reduces either to $\inf\{\mathfrak{G}_s(m) : \Lambda(m) = s\}$ or to its behaviour for $m = 0$ and $m = 1$, for every $s$ in the range of $\Lambda$ (see Proposition 4.1.12 for the precise statement). In particular, when $\Lambda$ is strictly decreasing, $g$ turns out to be concave and of class $C^1$, with derivative expressed explicitly in terms of $\Lambda^{-1}(s)$ (cf. Theorem 4.1.13). This information is then used in Theorems 4.2.1 and 4.2.2 for the linear case, and in Theorems 4.2.4 and 4.2.5 for the superlinear one, to define a phase–field model by selecting either the damage potential while fixing the degradation function, or vice versa, so that the corresponding $\Gamma$–limit possesses a surface energy density exactly equal to the assigned function $g$. To this end, we are led to solve the Abel integral equation corresponding to the identity $g(s) = \mathfrak{G}_s(\Lambda^{-1}(s))$ (cf. (4.2.4)). A similar use of the Abel integral equation was also proposed in [FFL21, Section 3.2] for the heuristic calibration of the damage potential in phase–field models aimed at numerical convergence. In contrast, in the linear case, if $\Lambda$ is non–decreasing, the function $g$ always corresponds to the Dugdale cohesive law (cf. Theorem 4.1.16).

Finally, Section 4.3 presents several computational examples that complement and validate the theoretical results established in Section 4.2. In particular, we numerically reconstruct different cohesive laws by selecting suitable combinations of the phase–field potentials and comparing the resulting traction–separation profiles with classical cohesive models widely employed in computational fracture mechanics. These simulations not only confirm the consistency of the theoretical predictions but also highlight the flexibility of the proposed variational framework in reproducing a broad spectrum of material behaviours, from perfectly brittle to strongly ductile responses. This connection between rigorous analysis and computational implementation demonstrates the potential of our approach as a practical tool for model calibration and material design.

In the final part of the chapter, namely Section 4.3, we aim to bridge the mathematical theory with a more engineering–oriented perspective. To this end, we summarize the key quantities and well–established results describing the mechanical response of one–dimensional tensile tests for variational phase–field and gradient–damage models, and we outline the identification procedure for determining the phase–field potentials corresponding to a prescribed traction–separation law. The mechanical responses associated with different cohesive behaviours are then discussed in detail in Section 4.3.4, providing a direct interpretation of the analytical findings and illustrating how the theoretical construction of the cohesive law translates into observable mechanical phenomena.

## 4.1 Analysis of the surface energy densities arising in the phase-field approximation

In this section we study in depth the surface energy densities $g$ obtained via the phase-field approximation in Theorem 3.2.1. For solving the reconstruction problem it is key to provide an equivalent and more maneageable characterization with respect to its definition in formula (3.1.14).

With this aim, throughout the whole section $l$, $Q$, $\omega$ and $\varphi$ are assumed to satisfy (Hp 1′)-(Hp 4′) (cf. introduction of Section 3.2) even if not explicitly mentioned.

We start with the case in which $g$ is linear for small jump amplitudes. For the sake of notational simplicity we also assume that the function $\varphi$ in the definition of the degradation function obeys $\varphi'(0^+) = 1$ (clearly, this restriction can be easily dropped). Finally, we also assume that $Q$, $l$ and $\omega$ satify

(Hp 5$'$) $\theta : [0,1) \to [0,\infty)$ is strictly increasing, where

$$\theta(t) := \frac{\omega(1-t)}{Q(1-t)} l(t)\,.$$

We recall that, necessarily, $\theta \in C^0([0,1))$, with $\theta(0) = 0$, $\theta(1^-) = \bar{\varsigma}^2 \in (0,\infty]$ (cf. (3.2.5) and $l(1) = 1$). For some results we will need to strengthen (Hp 5$'$) (cf. assumption (Hp 5$''$) below).

We introduce the functional $\mathcal{G} : W^{1,1}((0,1);\mathbb{R}^2) \to [0,\infty)$, defined by

$$\mathcal{G}(u,v) := \int_0^1 \Big(\theta(v)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x\,. \tag{4.1.1}$$

With this notation at hand, formula (3.2.9) rewrites as

$$g(s) = \inf_{(u,v)\in\mathfrak{U}_s} \mathcal{G}(u,v) \tag{4.1.2}$$

for every $s \in [0,\infty)$, where $\mathfrak{U}_s$ is defined in (3.2.8).

We remark that the infimum defining $g$ can be equivalently taken on $\mathfrak{U}_{s,1}$, where

$$\mathfrak{U}_{s,1} := \{u \in W^{1,1}((0,1)), v \in H^1((0,1)) : u(0) = 0\,, u(1) = s\,,\ 0 \le v \le 1\,, v(0) = v(1) = 1\}\,, \tag{4.1.3}$$

as the functional $\mathcal{G}$ is continuous in the $W^{1,1}$ topology.

### 4.1.1 An equivalent characterization of $g$

In this section we introduce a different minimization problem defining $g$ which will be more suitable for our purposes. With this aim, in order to reduce first the space of competitors for $v$ to those satisfying suitable monotonicity properties we first prove a technical lemma.

**Lemma 4.1.1.** *Let $v \in W^{1,p}((0,1)) \cap C^0([0,1])$ with $p \in [1,\infty]$, and for every $x \in [0,1]$ set*

$$T_0 v(x) := \inf_{t\in[0,x]} v(t), \quad \textit{and} \quad T_1 v(x) := \inf_{t\in[x,1]} v(t)\,.$$

*Then,*

*(i) $T_0 v$, $T_1 v \le v$ on $[0,1]$, $T_0 v$ is non-increasing with $T_0 v(0) = v(0)$, $T_1 v$ is non-decreasing with $T_1 v(1) = v(1)$;*

*(ii) $T_0 v$, $T_1 v \in W^{1,p}((0,1)) \cap C^0([0,1])$, with $|(T_0 v)'|$, $|(T_1 v)'| \le |v'|$ $\mathcal{L}^1$ a.e on $(0,1)$;*

*(iii) $(T_i v)' = 0$ $\mathcal{L}^1$ a.e on $\{T_i v \ne v\}$, $i \in \{0,1\}$.*

*Proof.* We give the proof of the statements for $T_0 v$, that for $T_1 v$ being analogous.

The monotonicity of $T_0 v$ follows straightforwardly from the very definition, as well as the fact that $T_0 v \le v$.

Let $x_1, x_2 \in (0,1)$, with $x_1 < x_2$, if $T_0 v(x_2) < T_0 v(x_1)$ there exist $y_1 \in [0, x_1]$ and $y_2 \in (x_1, x_2]$ such that

$$v(x_1) \geq v(y_1) = T_0 v(x_1) > T_0 v(x_2) = v(y_2) .$$

In particular, it follows that

$$|T_0 v(x_1) - T_0 v(x_2)| \leq |v(x_1) - v(y_2)| \leq \int_{x_1}^{x_2} |v'(t)| \mathrm{d}t .$$

Therefore, $T_0 v$ is absolutely continuous and the statements in item (ii) holds.

Finally, note that if $T_0 v(x) = v(y)$ for some $y \in [0, x]$, then $T_0 v(t) = v(y)$ for every $t \in [y, x]$ by item (i). Thus, if $I = (x_1, x_2)$ is a connected component of the open set $\{T_0 v < v\}$ (by continuity of both $v$ and $T_0 v$), then $T_0 v(x) = v(x_1)$ for every $x \in I$, and finally $T_0 v(x_2) = v(x_1)$ by continuity of $T_0 v$. $\square$

**Proposition 4.1.2.** *Assume (Hp $1'$)-(Hp $5'$). For every $s \in [0, \infty)$*

$$g(s) = \inf_{(u,v) \in \mathfrak{U}_{s,1}} \mathcal{G}(u, v). \tag{4.1.4}$$

*Moreover, for every $s \in (0, \infty)$, $\eta > 0$, and $(u, v) \in \mathfrak{U}_{s,1}$ there exists $\overline{v} \in H^1((0,1))$ such that $\overline{v}(0) = \overline{v}(1) = 1$, $\overline{v}$ is strictly positive, strictly decreasing on $[0, x_0]$, strictly increasing on $[x_0, 1]$, for some $x_0 \in (0,1)$, and with Lipschitz inverse when restricted both to $[0, x_0]$ and to $[x_0, 1]$, and such that*

$$\mathcal{G}(u, \overline{v}) \leq \mathcal{G}(u, v) + \eta . \tag{4.1.5}$$

*Proof.* Fix $(u, v) \in \mathfrak{U}_{s,1}$ such that $\mathcal{G}(u, v) < \infty$. Let $(z_k)$ be an increasing sequence of positive functions in $L^\infty((0,1))$ such that $z_k(x) \to |u'(x)|$ for $\mathcal{L}^1$-a.e. $x \in (0,1)$. In particular setting

$$w_k := \frac{s}{\int_0^1 z_k \, \mathrm{d}x} z_k$$

we have that $(u_k, v) \in \mathfrak{U}_{s,1}$ with $u_k(t) := \int_0^t w_k \, \mathrm{d}x \in W^{1,\infty}((0,1))$ and

$$\lim_{k \to \infty} \mathcal{G}(u_k, v) \leq \mathcal{G}(u, v) ,$$

so that (4.1.4) easily follows.

Let $v$ be as in the statement above, without loss of generality we assume that $v \in C^0([0,1])$. Let $v_k := v \vee \frac{1}{k}$ then $v_k(x) \to v(x)$, $\omega(1 - v_k(x))|v_k'(x)|^2 = 0$ $\mathcal{L}^1$-a.e. on $\{v \leq \frac{1}{k}\}$, and $v_k = v$ otherwise, so that $\omega(1 - v_k(x))|v_k'(x)|^2 \leq \omega(1 - v(x))|v'(x)|^2$ $\mathcal{L}^1$-a.e. $(0,1)$. In particular, $\mathcal{G}(u, v_k) \to \mathcal{G}(u, v)$ for $k \to \infty$, as for every $k \geq 2$

$$\begin{aligned}
\mathcal{G}(u, v_k) &\leq \int_0^1 \Big(\theta(v_k)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x \\
&= \int_{\{v \leq \frac{1}{2}\}} \Big(\theta(v_k)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x + \int_{\{v > \frac{1}{2}\}} \Big(\theta(v)|u'|^2 + \omega(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x.
\end{aligned}$$

Hence, without loss of generality we can suppose $v$ strictly positive. Let then $x_0 \in (0,1)$ be a global minimum point of $v$, and define $\hat{v} : [0,1] \to [0,1]$ by

$$\hat{v}(x) := T_0 v(x)\chi_{[0,x_0]}(x) + T_1 v(x)\chi_{[x_0,1]}(x) \,.$$

Lemma 4.1.1 yields that $\hat{v} \in H^1((0,1))$, $\hat{v}(0) = \hat{v}(1) = 1$, $\hat{v}$ is non-increasing on $[0,x_0]$ and non-decreasing on $[x_0,1]$. Moreover, by item (iii) there

$$\mathcal{G}(u,\hat{v}) \leq \mathcal{G}(u,v)$$

as $\theta$ is non-decreasing and $\hat{v}' = 0$ $\mathcal{L}^1$-a.e. on $\{\hat{v} \neq v\}$. Next, consider the piecewise affine function

$$\zeta(x) := \begin{cases} \frac{x}{x_0} & \text{if } 0 \leq x \leq x_0 \\ -\frac{x-x_0}{1-x_0} + 1 & \text{if } x_0 \leq x \leq 1, \end{cases}$$

observe that $\hat{v} - \delta\zeta$ is strictly positive for $\delta$ small enough, and converges to $\hat{v}$ monotonically as $\delta \to 0$, so that $\mathcal{G}(u,\hat{v}-\delta\zeta) \to \mathcal{G}(u,\hat{v})$ as $\delta \to 0$. Therefore, (4.1.5) is satisfied. Moreover, the restrictions of $\hat{v} - \delta\zeta$ to $[0,x_0]$ and $[x_0,1]$ are invertible with, respectively, derivative less than or equal to $-\delta$ and bigger than or equal to $\delta$, and thus they are Lipschitz continuos. ☐

We are now ready to give a new characterization of $g$ as a minimimum problem obtained by first minimizing over the variable $u$ (belonging to a suitable admissible set) a family of functionals labeled through the minimimum value of $v$, and then minimizing on the latter. This is alternative to the approach in [BCI21, BI24] where $g$ is expressed as a minimum problem of a single functional depending on $v$ loosely speaking using $u$ to change variables. With this aim it is convenient to denote by $\zeta_{m,s}$ the unique affine function with $\zeta_{m,s}(2m-1) = 0$ and $\zeta_{m,s}(1) = s$.

**Proposition 4.1.3.** *Assume (Hp 1′)-(Hp 5′). For every $m \in (0,1)$ let $G_m : W^{1,1}((2m-1,1)) \to [0,\infty]$ be given by*

$$G_m(w) := \int_{2m-1}^{1} \Big(\theta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t \tag{4.1.6}$$

*where $\theta_m$ and $\omega_m$ are, respectively, the even extensions to $[2m-1,m]$ with respect to $t = m$ of $\theta$ and $\omega(1-\cdot)$ restricted to $[m,1]$. Then, for every $s \in [0,\infty)$*

$$g(s) = \inf_{m\in(0,1)} \inf\{G_m(w) \,:\, w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1))\} \,. \tag{4.1.7}$$

*Proof.* Let $s \in (0,\infty)$ and $m \in (0,1)$. In view of Proposition 4.1.2 we may take $v \in H^1((0,1)) \cap C^0([0,1])$ such that $0 \leq v \leq 1$, $v(0) = v(1) = 1$, $v(x_0) = \min_{[0,1]} v = m$, for some $x_0 \in (0,1)$, and $v|_{[0,x_0]}$ and $v|_{[x_0,1]}$ invertible with Lipschitz continuous inverses. For every $u \in \zeta_{1/2,s} + W_0^{1,1}((0,1))$, define $w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1))$ by

$$w(t) = \begin{cases} u \circ (v|_{[0,x_0]})^{-1}(2m-t) & t \in [2m-1,m] \\ u \circ (v|_{[x_0,1]})^{-1}(t) & t \in [m,1]. \end{cases} \tag{4.1.8}$$

By changing variable on the intervals $[0,x_0]$ and $[x_0,1]$ by means of the one-dimensional area formula [AT04, Theorem 3.4.6] we obtain the equality $\mathcal{G}(u,v) = G_m(w)$. Vice versa, being $v$ absolutely continuous, given $w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1))$ one can define $u \in \zeta_{1/2,s} + W_0^{1,1}((0,1))$ via (4.1.8) in a way that $\mathcal{G}(u,v) = G_m(w)$.

In particular, by equality (4.1.4) in Proposition 4.1.2 we can conclude that (4.1.7) holds. □

### 4.1.2 Analysis of the equivalent characterization

In this section we study the new characterization of $g$ provided in Proposition 4.1.3. With this aim, we recall first a relaxation result (cf. [GMS79, Section 2], [BB90]).

**Proposition 4.1.4.** *Assume (Hp 1′)-(Hp 5′). Let $m \in (0,1)$, $s \in (0,\infty)$, and $G_{m,s} : L^1((2m-1,1)) \to [0,\infty]$ be the functional given by*

$$G_{m,s}(w) := \begin{cases} G_m(w) & \text{if } w \in \zeta_{m,s} + W_0^{1,1}((2m-1,1)), \\ \infty & \text{otherwise,} \end{cases} \tag{4.1.9}$$

*where $G_m$ is defined in (4.1.6). Then the $L^1$-lower semicontinuous envelope of $G_{m,s}$ is given by*

$$\begin{aligned} sc^- G_{m,s}(w) = &\int_{2m-1}^{1} \Big(\theta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t + \int_{2m-1}^{1} \theta_m^{1/2}(t)\mathrm{d}|D^s w| \\ &+ \theta^{1/2}(1^-)|w((2m-1)^+)| + \theta^{1/2}(1^-)|w(1^-) - s|, \end{aligned} \tag{4.1.10}$$

*for every $w \in BV((2m-1,1))$, and $+\infty$ otherwise on $L^1((2m-1,1))$. In particular, $w((2m-1)^+) = w(1^-) - s = 0$ if $\theta(1^-) = \infty$.*

*Moreover, the following holds*

$$g(s) = \inf_{m\in(0,1)} \inf_{w \in L^1((2m-1,1))} sc^- G_{m,s}(w). \tag{4.1.11}$$

*Proof.* If $\theta(1^-) < \infty$ the claim is a direct consequence of [GMS79, Section 2].

Instead, if $\theta(1^-) = \infty$, we first prove a lower bound for the relaxed functional by approximation. Indeed, fix $s \in (0,\infty)$, $m \in (0,1)$ and denote by $\mathscr{G}_{m,s}$ the functional on the right hand side of (4.1.10). Let $\theta^{(j)}(t) := \theta(t) \wedge j$, $t \in [0,1]$, and apply the equality in (4.1.10) to the corresponding functional in (4.1.9) obtained by substituting $\theta_m$ with $\theta_m^{(j)}$. By monotone convergence it is then easy to conclude that $sc^- G_{m,s}(w) \geq \mathscr{G}_{m,s}(w)$ for every $w \in BV((2m-1,1))$ with $w(2m-1) = 0$ and $w(1) = s$, and that $sc^- G_{m,s}(w) = \infty$ otherwise.

For the reverse inequality, first choose $w \in BV((2m-1,1))$ with $w \equiv 0$ on $(2m-1, 2m-1+\lambda)$ and $w \equiv s$ on $(1-\lambda, 1)$, for some $\lambda$ sufficiently small. Thanks to the result in the case $\theta(1^-) < \infty$, which actually holds for every interval $(a,b) \subset (0,1)$, there exists e sequence $w_j \in H^1((2m-1+\lambda, 1-\lambda))$ such that $w_j \to w$ in $L^1((2m-1+\lambda, 1-\lambda))$ as $j \to \infty$ and

$$\begin{aligned} &\lim_{j\to\infty} \int_{2m-1+\lambda}^{1-\lambda} \Big(\theta_m(t)|w_j'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t \\ &\quad = \int_{2m-1+\lambda}^{1-\lambda} \Big(\theta_m(t)|w'|^2 + \omega_m(1-t)\Big)^{1/2} \mathrm{d}t + \int_{2m-1+\lambda}^{1-\lambda} \theta_m^{1/2}(t)\mathrm{d}|D^s w| \,. \end{aligned} \tag{4.1.12}$$

Clearly, extending $w_j$ to 0 on $(2m-1, 2m-1+\lambda)$, and to $s$ on $(1-\lambda, 1)$ implies the $L^1((2m-1,1))$ convergence of the extensions, still denoted by $w_j$ with a slight abuse of notation, to $w$. Moreover, using

(4.1.12) a simple calculation shows that

$$sc^{-}G_{m,s}(w) \leq \lim_{j\to\infty} G_{m,s}(w_j) = \mathscr{G}_{m,s}(w)\,,$$

and thus the identity in (4.1.10) follows for $w$ as above. To establish the latter in general, let $w \in BV((2m-1,1))$ with $w(2m-1)=0$, $w(1)=s$ and $\mathscr{G}_{m,s}(w)<\infty$, otherwise the proof is completed by the lower bound. For every $\lambda$ sufficiently small let $\varphi_\lambda : [2m-1,1] \to [2m-1+\lambda, 1-\lambda]$ given by $\varphi_\lambda(t) := m_\lambda(t-m)+m$, with $m_\lambda := \frac{1-m-\delta}{1-m}$. Then define the function $w_\lambda \in BV((2m-1,1))$ by

$$w_\lambda(\tau) := \begin{cases} 0 & \tau \in [2m-1, 2m-1+\lambda] \\ w(\varphi_\lambda^{-1}(\tau)) & \tau \in [2m-1+\lambda, 1-\lambda] \\ s & \tau \in [1-\lambda, 1] \end{cases}. \tag{4.1.13}$$

Note that $w_\lambda \to w$ in $L^1((2m-1,1))$ as $\lambda \to 0$, and moreover that $Dw_\lambda = (\varphi_\lambda)_\#(Dw)$ (the push-forward measure through $\varphi_\lambda$). Thus, we compute as follows

$$\begin{aligned}
&sc^{-}G_{m,s}(w_\lambda) \\
&\quad= \int_{2m-1+\lambda}^{1-\lambda} \Big(\theta_m(\tau)|w_\lambda'|^2 + \omega_m(1-\tau)\Big)^{1/2} \mathrm{d}\tau + \int_{2m-1-\lambda}^{1-\lambda} \theta_m^{1/2}(\tau)\mathrm{d}|D^s w_\lambda| \\
&\qquad+ \int_{2m-1}^{2m-1+\lambda} \omega_m^{1/2}(1-\tau)\mathrm{d}\tau + \int_{1-\lambda}^{1} \omega_m^{1/2}(1-t)\mathrm{d}\tau \\
&\quad= \int_{2m-1}^{1} \Big(\theta_m(\varphi_\lambda(t))|w'|^2 + \omega_m(1-\varphi_\lambda(t))\, m_\lambda^2\Big)^{1/2} \mathrm{d}t \\
&\qquad+ \int_{2m-1}^{1} \theta_m^{1/2}(\varphi_\lambda(t))\mathrm{d}|D^s w| + 2\int_{1-\lambda}^{1} \omega^{1/2}(1-t)\mathrm{d}t \\
&\quad\leq \int_{2m-1}^{1} \Big(\theta_m(t)|w'|^2 + \omega_m(1-\varphi_\lambda(t))\Big)^{1/2} \mathrm{d}t \\
&\qquad+ \int_{2m-1}^{1} \theta_m^{1/2}(t)\mathrm{d}|D^s w| + 2\int_{1-\lambda}^{1} \omega^{1/2}(1-t)\mathrm{d}t\,,
\end{aligned}$$

where in the last inequality we have used that $m_\lambda < 1$, and that $\theta_m(\varphi_\lambda(t)) \leq \theta_m(t)$ on $[2m-1,1]$ by taking into account that $\theta_m$ is decreasing on $[2m-1,m]$ together with the inequality $\varphi_\lambda(t) \geq t$ on the same interval, and increasing on $[m,1]$ and the inequality $\varphi_\lambda(t) \leq t$ on the same interval. The conclusion the follows at once by Dominated Convergence Theorem and then $L^1((2m-1,1))$ lower semicontinuity of $sc^{-}G_{m,s}$. $\square$

Next, we investigate several properties of the minimizers of $G_{m,s}$ at $m \in (0,1)$ and $s \in (0,\infty)$ fixed.

**Theorem 4.1.5.** *Assume (Hp 1′)-(Hp 5′). For every $m \in (0,1)$ and $s \in (0,\infty)$, let $G_{m,s}$ be defined in (4.1.9). Then there exists a function $w_{m,s}$ such that*

$$sc^{-}G_{m,s}(w_{m,s}) = \inf_{w\in L^1((2m-1,1))} sc^{-}G_{m,s}(w). \tag{4.1.14}$$

*Moreover, $w_{m,s}$ is the unique minimizer of $sc^{-}G_{m,s}$ on $L^1((2m-1,1))$, $w_{m,s} \in SBV((2m-1,1))$, is strictly increasing and odd symmetric with respect to $t=m$, $w_{m,s}(2m-1)=0$, $w_{m,s}(1)=s$, and for (a unique)*

$\lambda_{m,s} \in (0, \theta^{1/2}(m)]$

$$w'_{m,s}(t) = \left( \frac{\lambda^2_{m,s} \omega_m(1-t)}{\theta_m(t)(\theta_m(t) - \lambda^2_{m,s})} \right)^{1/2} \tag{4.1.15}$$

*for $\mathcal{L}^1$-a.e $t \in (2m-1, 1)$, and either*

*(1)* $w_{m,s} \in W^{1,1}((2m-1,1))$, *with* $w_{m,s}(m) = \frac{s}{2}$*;*

*or*

*(2)* $w_{m,s} \in SBV((2m-1,1))$, *with* $D^j w_{m,s}$ *concentrated on* $t = m$, *and* $\lambda_{m,s} = \theta^{1/2}(m)$.

*Proof.* **Step 1. Existence of a minimizer.** We note that $\inf_{[2m-1,1]} \theta_m = \theta_m(m) > 0$ thanks to the condition $m > 0$. Therefore, $sc^- G_{m,s}$ is a coercive functional on $BV((2m-1,1))$ (cf. (4.1.10)). The existence of a minimizer $w_{m,s}$ of $sc^- G_{m,s}$ follows from the Direct Method of the Calculus of Variations and the BV compactness theorem.

**Step 2. Regularity of minimizers.** We prove that any minimizer $w_{m,s}$ attains the boundary data and the singular part of its distributional derivative is concentrated in $t = m$. With this aim the crucial observation is that the function $\theta_m$ assumes its unique minimum value in $t = m$. Then, consider

$$\hat{w}(t) := \int_{2m-1}^{t} w'_{m,s}(\tau) \mathrm{d}\tau + D^s w_{m,s}((2m-1,1)) \chi_{(m,1)}(t) \,,$$

and define the test function

$$w(t) := \begin{cases} \hat{w}(t) & t \in (2m-1, m] \\ \hat{w}(t) + (s - \hat{w}(1)) & t \in (m, 1) \end{cases} .$$

Clearly, $w \in BV((2m-1,1))$, with

$$Dw = w'_{m,s} \mathcal{L}^1 \llcorner (2m-1,1) + (D^s w_{m,s}((2m-1,1)) + s - \hat{w}(1)) \delta_{\{t=m\}},$$

$w(2m-1) = 0$, and $w(1) = s$. Since $\hat{w}(1) = Dw_{m,s}((2m-1,1)) = w_{m,s}(1) - w_{m,s}(2m-1)$ we have that $sc^- G_{m,s}(w) < sc^- G_{m,s}(w_{m,s})$ if either the boundary data are not attained by $w_{m,s}$ or $D^s w_{m,s}$ is not concentrated on $t = m$. Indeed, computing the energy for $w$ shows that the contributions for the energy of $w_{m,s}$ corresponding either to the boundary terms or to the singular part of the distributional derivative, which are both multiplied by $\theta_m^{1/2}(t)$ for some $t \neq m$, have been either moved to or concentrated in $t = m$, which is the unique minimum point of the same function $\theta_m^{1/2}(t)$.

**Step 3. Euler-Lagrange equation and structural properties.** By Step 2 any minimizer $w_{m,s}$ is $SBV((2m-1,1))$, $w_{m,s}(2m-1) = 0$, $w_{m,s}(1) = s$, and $D^s w_{m,s}$ is at most concentrated on $t = m$. Note also that by truncation necessarily $w_{m,s} \in [0, s]$ $\mathcal{L}^1$-a.e. on $(2m-1, 1)$.

Assume first that $w_{m,s} \in W^{1,1}((2m-1,1))$, then by using smooth outer variations, i.e. $w_{m,s} + \varepsilon\phi$ with $\phi \in C^1_c((2m-1,1))$, we obtain the following Euler-Lagrange equation

$$\frac{\theta_m(t) w'_{m,s}(t)}{\left(\theta_m(t)|w'_{m,s}(t)|^2 + \omega_m(1-t)\right)^{1/2}} = \lambda_{m,s} \tag{4.1.16}$$

for $\mathcal{L}^1$-a.e. $t \in (2m-1,1)$ and for some $\lambda_{m,s} \in \mathbb{R}$. Actually, $\lambda_{m,s} \in (0,\infty)$ as $w_{m,s} \in [0,s]$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$ and attains the boundary data, in turn implying that $w'_{m,s} > 0$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$, and thus $\lambda_{m,s} \le \theta^{1/2}(m)$ (cf. (4.1.16)). Furthermore, (4.1.16) shows that $w_{m,s}$ is strictly increasing and odd symmetric with respect to $m$. Solving $w'_{m,s}$ in (4.1.16) and integrating on $(2m-1,1)$ gives

$$s = 2\int_m^1 \Big(\frac{\lambda_{m,s}^2 \omega(1-t)}{\theta(t)(\theta(t)-\lambda_{m,s}^2)}\Big)^{1/2} \mathrm{d}t.$$

Therefore, we conclude the uniqueness of $\lambda_{m,s}$ that satisfies (4.1.15), and in turn that $w_{m,s}$ is the only possible minimizer of $sc^- G_{m,s}$ among $W^{1,1}$ functions, using the strict monotonicity of

$$\lambda \mapsto \int_m^1 \Big(\frac{\lambda^2 \omega(1-t)}{\theta(t)(\theta(t)-\lambda^2)}\Big)^{1/2} \mathrm{d}t\,. \tag{4.1.17}$$

Instead, if $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$, we consider again smooth outer variations separately on $(2m-1,m)$ and $(m,1)$ to infer the Euler-Lagrange equation in (4.1.16) with constants $\lambda^-_{m,s}$ and $\lambda^+_{m,s}$ on each interval, respectively. Since, $w_{m,s}$ takes value in $[0,s]$ $\mathcal{L}^1$-a.e. on $(2m-1,1)$, necessarily $\lambda^\pm_{m,s} \ge 0$.

Next, we use variations on the whole interval that move the value of the jump point in $t=m$, i.e. $w_{m,s}+\varepsilon\phi$ with $\phi \in C^1((2m-1,m)\cup(m,1)) \cap SBV((2m-1,1))$, and $\phi(2m-1)=\phi(1)=0$. Note that for $\varepsilon$ sufficiently small $[w_{m,s}+\varepsilon\phi](m)$ has the same sign of $[w_{m,s}](m)$, therefore we get

$$\lambda^-_{m,s}\phi(m^-) - \lambda^+_{m,s}\phi(m^+) \pm \theta^{1/2}(m)(\phi(m^+)-\phi(m^-)) = 0,$$

where the sign $+$ corresponds to the case $[w_{m,s}](m) > 0$ and $-$ to the other case. In particular, it follows that $\lambda^+_{m,s} = \lambda^-_{m,s} = \pm\theta^{1/2}(m)$, and since they are both non negative we conclude (4.1.15). Hence, $w_{m,s}$ is odd symmetric with respect to $t=m$. Moreover, $w_{m,s}(m^-) < \frac{s}{2}$ since otherwise having assumed $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ then $w_{m,s}(m^-) > \frac{s}{2}$ and the function $\hat{w} = w_{m,s}\chi_{(2m-1,m)} + (w_{m,s} \vee w_{m,s}(m^-))\chi_{[m,1)}$ would be $W^{1,1}((2m-1,1))$ with $sc^-G_{m,s}(\hat{w}) < sc^-G_{m,s}(w_{m,s})$. Therefore, $w_{m,s}$ is strictly increasing and it is the only possible minimizer of $sc^-G_{m,s}$ belonging to $SBV \setminus W^{1,1}((2m-1,1))$ in view of (4.1.15). $\square$

By taking into account the odd symmetry of the minimizer $w_{m,s}$ with respect to $t=m$ we rewrite the minimal value of the energy $sc^-G_{m,s}$ in an alternative and more efficient form. With this aim it is convenient to introduce the function $\mathscr{G} : \{(m,\lambda) \in (0,1)^2 : m \in (0,1),\ \lambda \in (0,\theta^{1/2}(m)]\} \to [0,\infty]$ defined by

$$\begin{aligned} \mathscr{G}(m,\lambda) &:= 2\int_m^1 \Big(\frac{\theta(t)-\lambda^2}{\theta(t)}\omega(1-t)\Big)^{1/2} \mathrm{d}t + 2\lambda \int_m^1 \Big(\frac{\lambda^2\omega(1-t)}{\theta(t)(\theta(t)-\lambda^2)}\Big)^{1/2} \mathrm{d}t \\ &=: A(m,\lambda) + \lambda B(m,\lambda)\,. \end{aligned} \tag{4.1.18}$$

Note that $\mathscr{G}$ is infinite if and only if $B$ is infinite, and the only points in which this can happen are of the type $(m,\theta^{1/2}(m))$. In addition we set $\mathscr{G}(0,0) := 2\Psi(1)$.

**Corollary 4.1.6.** *Assume (Hp 1′)-(Hp 5′). For every $m \in (0,1)$, and $s \in (0,\infty)$, if $w_{m,s}$ denotes the*

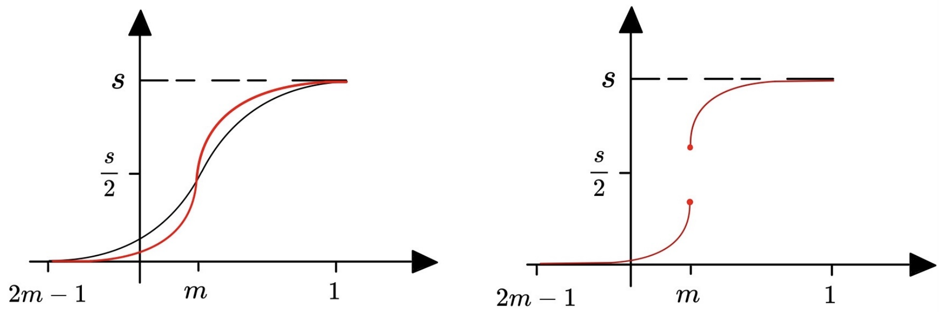



Figure 4.1: Optimal profiles $w_{m,s}$ of $\mathrm{sc}^{-}G_{m,s}$: *left*, two regular profiles (black and red), where the red one has Lagrange multiplier exactly $\theta^{\frac{1}{2}}(m)$; *right*, a discontinuous profile.

*minimizer of $sc^{-}G_{m,s}$ in Theorem 4.1.5, we have*

$$\begin{aligned}\mathfrak{G}_s(m) &:= sc^{-}G_{m,s}(w_{m,s})\\ &= 2\int_m^1 \Big(\theta(t)|w'_{m,s}|^2 + \omega(1-t)\Big)^{1/2}\mathrm{d}t + 2\theta^{1/2}(m)(w_{m,s}(m^+) - \frac{s}{2})\,,\end{aligned} \tag{4.1.19}$$

*and either*

$$w_{m,s} \in W^{1,1}((2m-1,1)) \Longleftrightarrow s = B(m,\lambda_{m,s})\,, \tag{4.1.20}$$

*where $\lambda_{m,s} \in (0, \theta^{1/2}(m)]$ is as in Theorem 4.1.5, and*

$$\mathfrak{G}_s(m) = 2\int_m^1 \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2}\mathrm{d}t = \mathscr{G}(m,\lambda_{m,s}) = A(m,\lambda_{m,s}) + \lambda_{m,s}s\,, \tag{4.1.21}$$

*or*

$$w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1)) \Longleftrightarrow s > B(m,\theta^{1/2}(m))\,, \tag{4.1.22}$$

*and*

$$w_{m,s}(m^+) = s - \frac{1}{2}B(m,\theta^{1/2}(m))\,,$$

*and*

$$\begin{aligned}\mathfrak{G}_s(m) &= 2\int_m^1 \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\theta(m)}\Big)^{1/2}\mathrm{d}t + \theta^{1/2}(m)(s - B(m,\theta^{1/2}(m)))\\ &= \mathscr{G}(m,\theta^{1/2}(m)) + \theta^{1/2}(m)(s - B(m,\theta^{1/2}(m))) = A(m,\theta^{1/2}(m)) + \theta^{1/2}(m)s\,.\end{aligned} \tag{4.1.23}$$

**Remark 4.1.7.** *We stress that the value $\mathscr{G}(m,\theta^{1/2}(m))$ in (4.1.23) is necessarily finite due to (4.1.22), analogously due to (4.1.21) if $\lambda_{m,s} = \theta^{1/2}(m)$.*

*Proof of Corollary 4.1.6.* The proof of (4.1.19) follows from a simple calculation using the explicit form of $sc^{-}G_{m,s}$ in (4.1.10), the odd symmetry of $w_{m,s}$ and the fact that it is increasing. Moreover, the same properties of $w_{m,s}$ yield that

$$B(m,\lambda_{m,s}) = 2\int_m^1 \Big(\frac{\lambda_{m,s}^2\omega(1-t)}{\theta(t)(\theta(t)-\lambda_{m,s}^2)}\Big)^{1/2}\mathrm{d}t = \int_{2m-1}^1 w'_{m,s}\mathrm{d}t\,,$$

implying (4.1.20) and (4.1.22).

Finally, formulas (4.1.21) and (4.1.23) follows by direct computations using (4.1.15) and (4.1.19). $\square$

### 4.1.3 A key auxiliary function

Under suitable assumptions on $s$, $\theta$ and $\omega$ we will show that there is a unique value $m_s$ such that $g(s) = sc^- G_{m_s,s}(w_{m_s,s})$. With this aim we define the auxiliary function $\Lambda : (0,1) \to [0,\infty]$ as follows

$$\Lambda(x) := B(x, \theta^{1/2}(x)) . \tag{4.1.24}$$

The importance of $\Lambda$ is related to the fact that (4.1.20) rewrites equivalently as $\Lambda(m) \geq s$, and (4.1.22) as $\Lambda(m) < s$. In the next proposition we determine several properties of $\Lambda$, as well as that its definition is well posed provided that the following assumption on $\theta$ which strengthens (Hp 5′) holds:

(Hp 5″) $\theta \in C^1((0,1))$ with $\theta'(t) > 0$ for every $t \in (0,1)$.

Recall that $\Psi$ is the function introduced in (3.1.27).

**Proposition 4.1.8.** *Assume (Hp 1′)-(Hp 4′), and (Hp 5″). Then*

*(i)* $\Lambda \in C^0((0,1))$;

*(ii) if the following limit exists*

$$\lim_{x\to 0^+} (\theta^{1/2})'(x) =: (\theta^{1/2})'(0^+) \in [0,\infty] , \tag{4.1.25}$$

*then $\Lambda$ has right limit in $x = 0$ given by*

$$\lim_{x\to 0^+} \Lambda(x) = \frac{\pi \omega^{1/2}(1)}{2(\theta^{1/2})'(0^+)} ; \tag{4.1.26}$$

*(iii) if $\theta(1^-) = \overline{\varsigma}^2 \in (0,\infty)$, and there is $p > 2$ such that for some $z \in (0, \overline{\varsigma}^2)$*

$$(0, \overline{\varsigma}^2) \ni t \mapsto F(t) := \frac{(\Psi \circ \theta^{-1})'(\overline{\varsigma}^2 - t)}{(\overline{\varsigma}^2 - t)^{1/2}} \in L^p((0,z)) , \tag{4.1.27}$$

*then*

$$\lim_{x\to 1^-} \Lambda(x) = 0 . \tag{4.1.28}$$

*If, in addition, (4.1.27) holds for every $z \in (0,1)$ then $\Lambda \in C^{1/2-1/p}_{\mathrm{loc}}([0,1))$;*

*(iv) if $[0,1] \ni x \mapsto (x(1-x))^{1/2} F(xt)$ is strictly decreasing (non-increasing, strictly increasing, non-decreasing) for $\mathcal{L}^1$-a.e. $t \in (0,1)$, then $\Lambda$ is strictly increasing (non-decreasing, strictly decreasing, non-increasing);*

*(v) if $(\theta \circ \Psi^{-1})^{1/2}$ is convex, then $\Phi$ is strictly decreasing.*

*Proof.* Let $\gamma \in (0, 1/2)$, then there is $\delta \in (0, \gamma/2)$ such that $\theta(t) - \theta(y) \geq \frac{\theta'(y)}{2}(t-y)$ for every $y \in [\gamma, 1-\gamma]$ and $t$ such that $|t - y| \leq \delta$. Therefore, recalling that $\theta'(t) > 0$ for every $t \in (0,1)$, $\Lambda$ is well defined for every $x \in (0,1)$.

In addition, if $x_j \to x$ we have for every $\varepsilon > 0$ sufficiently small

$$\Lambda(x_j) = 2\int_{x_j}^{x_j+\varepsilon} \Big(\frac{\theta(x_j)\omega(1-t)}{\theta(t)(\theta(t)-\theta(x_j))}\Big)^{1/2} \mathrm{d}t + 2\int_{x_j+\varepsilon}^{1} \Big(\frac{\theta(x_j)\omega(1-t)}{\theta(t)(\theta(t)-\theta(x_j))}\Big)^{1/2} \mathrm{d}t =: I_j + K_j\,.$$

It is clear from the continuity assumption on $\theta$ that

$$\lim_j K_j = 2\int_{x+\varepsilon}^{1} \Big(\frac{\theta(x)\omega(1-t)}{\theta(t)(\theta(t)-\theta(x))}\Big)^{1/2} \mathrm{d}t\,.$$

Moreover, we estimate $I_j$ as follows

$$0 \le I_j \le C \sup_{(x-\varepsilon,x+2\varepsilon)} (\theta\theta')^{-1/2} \int_{x_j}^{x_j+\varepsilon} (t-x_j)^{-1/2}\mathrm{d}t = 2C \sup_{(x-\varepsilon,x+2\varepsilon)} (\theta\theta')^{-1/2}\varepsilon^{1/2}\,,$$

and the conclusion follows.

Next, we prove that the right limit of $\Lambda$ in 0 exists and the equality in (4.1.26) holds. With this aim, note that for every $\delta > 0$ being $\theta$ and $\omega$ continuous on $[0,1]$ and $\theta(0) = 0$, we have

$$\Lambda(x) = 2(\omega^{1/2}(1) + o(1)) \int_{x}^{\delta} \Big(\frac{\theta(x)}{\theta(t)(\theta(t)-\theta(x))}\Big)^{1/2} \mathrm{d}t + o(1)$$

as $x \to 0^+$. Changing variable with $\tau = (\theta(t)/\theta(x))^{1/2}$ and applying the Mean Value Theorem gives some $\bar{t} \in [x,\delta]$ such that

$$\begin{aligned}\Lambda(x) &= 4(\omega^{1/2}(1) + o(1)) \frac{\theta^{1/2}(\bar{t})}{\theta'(\bar{t})} \int_1^{(\frac{\theta(\delta)}{\theta(x)})^{1/2}} \frac{1}{\tau(\tau^2-1)^{1/2}}\mathrm{d}\tau + o(1)\\ &= \frac{2(\omega^{1/2}(1)+o(1))}{(\theta^{1/2})'(\bar{t})}\Big(\arctan\Big(\Big(\frac{\theta(\delta)}{\theta(x)}\Big)^{1/2} + \Big(\frac{\theta(\delta)}{\theta(x)} - 1\Big)^{1/2}\Big) - \frac{\pi}{4}\Big) + o(1)\end{aligned}$$

as $x \to 0^+$, where the last equality follows from an elementary integration argument, and the fact that $\theta^{1/2} \in C^1((0,1))$ (cf. the assumption on $\theta$). Eventually (4.1.26) follows at once by letting first $x \to 0^+$ and then $\delta \to 0^+$, using (4.1.25).

Next, we prove (4.1.28). Recalling that $\theta(1^-) = \bar{\varsigma}^2 \in (0,\infty)$, changing variables with $t = \theta^{-1}(\bar{\varsigma}^2 - \tau)$ in the definition of $\Lambda$ (cf. (4.1.18) and (4.1.24)) and letting $\lambda = \bar{\varsigma}^2 - \theta(x)$, we deduce the following Abel integral equation involving the function $F$ defined in the statement

$$\frac{\Lambda(\theta^{-1}(\bar{\varsigma}^2-\lambda))}{2(\bar{\varsigma}^2-\lambda)^{1/2}} = \int_0^{\lambda} \frac{(\Psi\circ\theta^{-1})'(\bar{\varsigma}^2-\tau)}{(\bar{\varsigma}^2-\tau)^{1/2}(\bar{\varsigma}^2-\theta(x)-\tau)^{1/2}}\mathrm{d}\tau = \int_0^{\lambda} \frac{F(\tau)}{(\lambda-\tau)^{1/2}}\mathrm{d}\tau\,, \tag{4.1.29}$$

(see Section 2.0.2). By assumption $F \in L^p((0,z))$, $p > 2$, for some $z \in (0,\bar{\varsigma}^2)$, so that if $\frac{1}{q} + \frac{1}{p} = 1$, $q \in (1,2)$, and if $\lambda \in (0,z)$ we get

$$|\Lambda(\theta^{-1}(\bar{\varsigma}^2-\lambda))| \le 2(\bar{\varsigma}^2-\lambda)^{1/2}\|F\|_{L^p((0,z))}\Big(\frac{2}{2-q}\lambda^{\frac{2-q}{2}}\Big)^{1/q}\,,$$

and thus the limit of $\Lambda(\theta^{-1}(\bar{\varsigma}^2-\lambda))$ as $\lambda\to 0^+$ is infinitesimal, corresponding to the left limit of $\Lambda$ in 1. In addition, if $F\in L^p((0,z))$, $p>2$, for every $z\in(0,\bar{\varsigma}^2)$ the left hand side of (4.1.29) is $C^{0,1/2-1/p}_{\mathrm{loc}}([0,\bar{\varsigma}^2))$ by Theorem 2.0.4.

Next, we prove the monotonicity assertions. We start with the claim in item (iv), focusing on the strictly increasing case, the other cases being analogous. With this aim we use the change of variable $\tau=tz$ in (4.1.29) to get

$$\Lambda(\theta^{-1}(1-z))=2(z(1-z))^{1/2}\int_0^1\Big(\frac{t}{1-t}\Big)^{1/2}F(tz)\mathrm{d}t,$$

and the conclusion follows being $[0,1]\ni z\mapsto (z(1-z))^{1/2}F(tz)$ strictly decreasing for $\mathcal{L}^1$-a.e. $t\in(0,1)$, and $\theta$ strictly increasing.

To establish the claim in item (v) we first use the change of variable $\tau=\frac{\theta(t)}{\theta(x)}$ in the definition of $\Lambda$ (cf. (4.1.24)) to deduce that

$$\Lambda(x)=2\int_1^{\frac{1}{\theta(x)}}\Big(\frac{\theta(x)}{\tau(\tau-1)}\Big)^{1/2}\frac{\Psi'(\theta^{-1}(\theta(x)\tau))}{\theta'(\theta^{-1}(\theta(x)\tau))}\mathrm{d}\tau=\int_1^{\frac{1}{\theta(x)}}\frac{\zeta'\big((\theta(x)\tau)^{1/2}\big)}{\tau(\tau-1)^{1/2}}\mathrm{d}\tau, \tag{4.1.30}$$

where $\zeta$ denotes the inverse of $(\theta\circ\Psi^{-1})^{1/2}$, namely $\zeta(t)=\Psi(\theta^{-1}(t^2))$. We conclude using the fact that $\theta$ is strictly increasing, and $(\theta\circ\Psi^{-1})^{1/2}$ is convex and increasing. □

**Remark 4.1.9.** *More generally, using the one-dimensional area formula (see [AT04, Theorem 3.4.6]), the definition of $\Lambda$ is well posed and the conclusions in item (iii) hold assuming $\theta^{-1}\in W^{1,1}((0,1))$.*

We note that formula (4.1.26) corresponds to [BI24, (3.11) in Proposition 3.7]
in case $\omega(t)=t^2$ (there it is equivalently stated in terms of the differentiability in $t=0$ of the function $f$ in (3.1.1)).

The monotonicity of $\Lambda$ is a key tool in the analysis below of the variational problem defining $g$ as it implies that, for every $s\ge 0$, the energies $\mathfrak{G}_s$ are minimized either in the interior of $[0,1]$ in the strictly decreasing case, or at the boundary in the increasing case. In this respect, the convexity assumption on $(\theta\circ\Psi^{-1})^{1/2}$ is equivalent to the assumption used in [BCI21, BI24] (with $Q(t)=\omega(t)=t^2$) which in our notation reads as the convexity of $(\theta\circ(2(\Psi(1)-\Psi))^{-1})^{1/2}$. We remark that the concavity of $(\theta\circ(2(\Psi(1)-\Psi))(t^{1/2})$ has been used in [LCM25] (with $Q(t)=\omega(t)=t^2$) to infer the concavity of the corresponding energy. We have not been able to show that such a condition implies neither that $\Lambda$ is increasing nor that the energy $\mathfrak{G}_s$ is concave. Despite this, using item (iv) simple calculations show that the explicit example given in [LCM25] (namely, $\theta(t)=1-(1-t)^4$ and $\omega(t)=t^2$) yields that $\Lambda$ is increasing.

**Remark 4.1.10.** *The fact that $\Lambda$ is strictly decreasing is also satisfied, for instance, in the following particular cases for every non-decreasing damage potential $\omega$*

*(i)* $\theta(t):=t^\alpha$, *with* $\alpha\ge 2$;

*(ii)* $\theta(t):=\frac{e^t-1}{e-1}$;

*where $\Psi$ is defined in (3.1.27). For instance, it suffices to take $Q=\omega$ and $l(t)=\theta(t)=t^\alpha$ in case (i), analogously in case (ii). To prove the claim we use the change of variable $\frac{t}{x}=\tau$, for every $x\in(0,1)$, to get*

$$\Lambda(x)=2\int_1^{\frac{1}{x}}x\Big(\frac{\theta(x)\omega(1-\tau x)}{\theta(\tau x)(\theta(\tau x)-\theta(x))}\Big)^{1/2}\mathrm{d}\tau\,.$$

*In case $\theta(t) = t^\alpha$, we obtain for every $x \in (0,1)$*

$$\Lambda(x) = 2\int_1^{\frac{1}{x}} x^{1-\frac{\alpha}{2}}\Big(\frac{\omega(1-\tau x)}{\tau^\alpha(\tau^\alpha - 1)}\Big)^{1/2} \mathrm{d}\tau\,,$$

*which is strictly decreasing, being $\alpha \geq 2$ and $\omega$ non-decreasing by assumption. Similarly, one can argue for the exponential. Note also that the result in item (iii) of Proposition 4.1.8 applies to $\theta(t) = t^\alpha$ for $\alpha \geq 2$.*

### 4.1.4 Dependence of $g$ on the parameters of the phase-field model

In this section we show the explicit dependence of $g$ on the damage potential and the degradation function. With this aim, we prove a technical result which will be instrumental in what follows.

**Lemma 4.1.11.** *Assume (Hp 1′)-(Hp 5′). Let $s \in (0,\infty)$ and $0 < m_0 < m_1 < 1$, consider $\lambda_{m,s} \in (0, \theta^{1/2}(m)]$ in Theorem 4.1.5, and $\mathfrak{G}_s$ in (4.1.19). Then*

*(i) if $\lambda_{m,s} \in (0, \theta^{1/2}(m))$ for every $m \in (m_0, m_1)$, then $\mathfrak{G}_s \in C^1((m_0, m_1))$ and is strictly decreasing. Moreover, $(m_0, m_1) \mapsto \lambda_{m,s} \in C^1((m_0, m_1))$ and is strictly increasing;*

*(ii) if $\lambda_{m,s} = \theta^{1/2}(m)$ for every $m \in (m_0, m_1)$ and $\theta \in C^1((m_0, m_1))$ with $\theta'(t) > 0$ for every $t \in (0,1)$, then $\mathfrak{G}_s \in C^1((m_0, m_1))$ and increasing.*

*More precisely, let $w_{m,s}$ be the function defined in Theorem 4.1.5 then*

*(a) if $w_{m,s} \in SBV \setminus W^{1,1}((2m-1, 1))$ for every $m \in (m_0, m_1)$ then $\mathfrak{G}_s$ is strictly increasing on $(m_0, m_1)$.*

*(b) if $w_{m,s} \in W^{1,1}((2m-1, 1))$ for some $m \in (m_0, m_1)$ then $\mathfrak{G}'_s(m) = 0$.*

*Proof.* We start with the first case noting that in particular (4.1.20) necessarily holds, and thus $w_{m,s} \in W^{1,1}((2m-1, 1))$ in view of Corollary 4.1.6. Moreover, (4.1.21) yields that

$$\mathfrak{G}_s(m) = 2\int_m^1 \Big(\frac{\theta(t)\omega(1-t)}{\theta(t) - \lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t\,. \tag{4.1.31}$$

Next, we claim that $(m_0, m_1) \mapsto \lambda_{m,s}$ is continuously differentiable and strictly increasing thanks to the implicit function theorem applied to the function $B$ defined in (4.1.18) defined on the open set

$$T := \Big\{(m, \lambda) \;:\; m \in (m_0, m_1),\; \lambda \in (0, \theta^{1/2}(m))\Big\}\,.$$

Indeed, item (i) in Theorem 4.1.5 ensures that for every $m \in (m_0, m_1)$ the unique value of $\lambda$ such that $B(m, \lambda) = s$ is $\lambda_{m,s} \in (0, \theta^{1/2}(m))$, and

$$\begin{aligned}\frac{\partial B}{\partial \lambda}(m,\lambda) &= 2\int_m^1 \Big(\frac{\omega(1-t)}{\theta(t)(\theta(t) - \lambda^2)}\Big)^{1/2} \mathrm{d}t + 2\lambda^2 \int_m^1 \Big(\frac{\omega(1-t)}{\theta(t)(\theta(t) - \lambda^2)^3}\Big)^{1/2} \mathrm{d}t \\ &= 2\int_m^1 \Big(\frac{\theta(t)\omega(1-t)}{(\theta(t) - \lambda^2)^3}\Big)^{1/2} \mathrm{d}t > 0\,,\end{aligned} \tag{4.1.32}$$

Thus, by the implicit function theorem

$$\begin{aligned}\lambda'_{m,s} &= -\frac{\partial B}{\partial m}(m,\lambda_{m,s})\left(\frac{\partial B}{\partial \lambda}(m,\lambda_{m,s})\right)^{-1} \\ &= 2\left(\frac{\lambda^2_{m,s}\omega(1-m)}{\theta(m)(\theta(m)-\lambda^2_{m,s})}\right)^{1/2}\left(\frac{\partial B}{\partial \lambda}(m,\lambda_{m,s})\right)^{-1} > 0\,.\end{aligned}$$

Therefore, $(m_0,m_1)\mapsto\lambda_{m,s}$ is strictly increasing. In turn, this implies that $\mathfrak{G}_s$ in (4.1.31) is differentiable on $(m_0,m_1)$. Indeed, using (4.1.21), namely $\mathfrak{G}_s(m)=A(m,\lambda_{m,s})+\lambda_{m,s}s$, (4.1.20) and (4.1.32) we get

$$\mathfrak{G}'_s(m)=\left(\frac{\partial A}{\partial \lambda}(m,\lambda_{m,s})+s\right)\lambda'_{m,s}+\frac{\partial A}{\partial m}(m,\lambda_{m,s})=\frac{\partial A}{\partial m}(m,\lambda_{m,s})<0 \tag{4.1.33}$$

as from the very definition of $A$ in (4.1.18) easy calculations lead to

$$\frac{\partial A}{\partial \lambda}(m,\lambda)=-B(m,\lambda)\,,\qquad \frac{\partial A}{\partial m}(m,\lambda)=-2\left(\frac{\theta(m)-\lambda^2}{\theta(m)}\omega(1-m)\right)^{1/2}.$$

Next, we deal with the second case. In particular note that $w_{m,s}$ can be either $W^{1,1}$ or $SBV\setminus W^{1,1}$, and in both cases $\lambda_{m,s}=\theta^{1/2}(m)$. We claim that $\mathfrak{G}_s\in C^1((m_0,m_1))$ with

$$\mathfrak{G}'_s(m)=\frac{\theta'(m)}{2\theta^{1/2}(m)}(s-\Lambda(m)) \tag{4.1.34}$$

for every $m\in(m_0,m_1)$. Given this for granted, the right hand side in (4.1.34) is either null if $w_{m,s}\in W^{1,1}$ thanks to $s=B(m,\theta^{1/2}(m))=\Lambda(m)$ (cf. (4.1.20)) and (4.1.24)), or is strictly positive if $w_{m,s}\in SBV\setminus W^{1,1}$ $s>B(m,\theta^{1/2}(m))=\Lambda(m)$ (cf. (4.1.22)), and thus we conclude the monotonicity of $\mathfrak{G}_s$.

To prove (4.1.34) define the map $L:T'\to\mathbb{R}$ by

$$L(m,\rho)=2\int_m^1\left(\frac{(\theta(t)-\theta(\rho))\omega(1-t)}{\theta(t)}\right)^{1/2}\mathrm{d}t+s\theta^{1/2}(m),$$

where $T':=\{(m,\rho)\,:\,m\in(m_0,m_1),\ \rho\in(0,m)\}$. Note that $L\in C^0(\overline{T'})$, and $\mathfrak{G}_s(m)=L(m,m)$ on $(m_0,m_1)$. Thanks to the assumption $\theta\in C^1((m_0,m_1))$, then $L\in C^1(T')$ with

$$\frac{\partial L}{\partial m}(m,\rho)=-2\left(\frac{(\theta(m)-\theta(\rho))\omega(1-m)}{\theta(m)}\right)^{1/2}+\frac{s\theta'(m)}{2\theta^{1/2}(m)}\,, \tag{4.1.35}$$

and

$$\frac{\partial L}{\partial \rho}(m,\rho)=-\theta'(\rho)\int_m^1\left(\frac{\omega(1-t)}{\theta(t)(\theta(t)-\theta(\rho))}\right)^{1/2}\mathrm{d}t=-\theta'(\rho)\frac{B(m,\theta^{1/2}(\rho))}{2\theta^{1/2}(\rho)}. \tag{4.1.36}$$

Being $\theta^{-1}$ locally Lipschitz on $(m_0,m_1)$, we can extend $\nabla L$ to $\overline{T}$ continuously. Therefore, an elementary argument shows that $\mathfrak{G}_s\in C^1((m_0,m_1))$ and

$$\mathfrak{G}'_s(m)=\frac{\partial L}{\partial m}(m,m)+\frac{\partial L}{\partial \rho}(m,m)=\frac{\theta'(m)}{2\theta^{1/2}(m)}(s-\Lambda(m))\,,$$

where we have used (4.1.35), (4.1.36), and (4.1.24) to deduce (4.1.34). $\square$

We are now ready to infer some interesting consequences of the previous statement for the minimum problem defining $g$. With this aim, we introduce

$$s_{\mathrm{frac}} := \sup\{s \in [0,\infty) \mid g(s) < 2\Psi(1)\} \in (0,\infty], \tag{4.1.37}$$

where $\Psi$ is defined in (3.1.27). The value $s_{\mathrm{frac}}$ is well defined in view of either item (ii) in Proposition 3.2.4 in case $\theta(1^-) < \infty$ or item (iii) in Proposition 3.2.5 otherwise. Moreover, $g(s) < 2\Psi(1)$ for every $s \in [0, s_{\mathrm{frac}})$ by taking into account that $g$ is non-decreasing.

**Proposition 4.1.12.** *Assume (Hp 1′)-(Hp 4′), and (Hp 5″). Then*

*(i)* $g(s) = \overline{\varsigma} s$ *for every* $s \in [0, \inf \Lambda]$*. In particular,* $\inf \Lambda = 0$ *if* $\theta(1^-) = \infty$*;*

*(ii)* $g(s) = (\inf_{\Lambda^{-1}(s)} \mathfrak{G}_s) \wedge (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$ *for every* $s \in (\inf \Lambda, \sup \Lambda)$*;*

*(iii)* $g(s) = 2\Psi(1)$ *for every* $s \in [\sup \Lambda, \infty)$*.*

*In particular,* $s_{\mathrm{frac}} \in [\inf \Lambda, \sup \Lambda]$*.*

*Proof.* **Step 1.** Assume $\inf \Lambda > 0$ otherwise the statement is trivial. Let $s \in [0, \inf \Lambda)$, then $w_{m,s} \in W^{1,1}((2m-1, 1))$ for every $m \in (0,1)$. Therefore, the energy $\mathfrak{G}_s$ is strictly decreasing on $(0,1)$ by Lemma 4.1.11, so that we may use formula (4.1.19) and the monotonicity of $\theta$ to bound the energy from below as follows:

$$\mathfrak{G}_s(m) \geq 2\theta^{1/2}(m) \int_m^1 w'_{m,s} \mathrm{d}t = \theta^{1/2}(m) s\,.$$

Therefore, $\theta(1^-) < \infty$ and

$$\lim_{m\to 1^-} \mathfrak{G}_s(m) \geq \overline{\varsigma} s\,. \tag{4.1.38}$$

Hence, we deduce that $g(s) \geq \overline{\varsigma} s$ for every $s \in (s_{\mathrm{frac}}, \inf \Lambda)$. On the other hand, $g(s) \leq \overline{\varsigma} s \wedge (2\Psi(1))$ by item (ii) in Proposition 3.2.4. Hence, $s_{\mathrm{frac}} \geq \inf \Lambda$ and $g(s) = \overline{\varsigma} s$ for $s \in (s_{\mathrm{frac}}, \inf \Lambda)$. By continuity of $g$ we have that the last equality extends to $s = \inf \Lambda$.

**Step 2.** Let $s \in (\inf \Lambda, \sup \Lambda)$, then consider the open sub-level set $\{m \in (0,1) : \Lambda(m) < s\}$ and the open super-level set $\{m \in (0,1) : \Lambda(m) > s\}$. Let $(m_0, m_1) \subset \{m \in (0,1) : \Lambda(m) < s\}$ be a connected component, then $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (m_0, m_1)$. By case (ii)-(a) in Lemma 4.1.11 the energy is increasing on $(m_0, m_1)$ so that

$$\inf_{(m_0,m_1)} \mathfrak{G}_s = \lim_{m\to m_0^+} \mathfrak{G}_s(m).$$

In particular, Lemma 4.1.11 itself implies the continuity of $\mathfrak{G}_s$ on $(m_0, m_1)$, and if $m_0 > 0$ we claim that $\mathfrak{G}_s$ is right continuous in $m_0$. Indeed, using (4.1.23) we obtain

$$\lim_{m\to m_0^+} \mathfrak{G}_s(m) = \lim_{m\to m_0^+} A(m, \theta^{1/2}(m)) + \theta^{1/2}(m_0) s = \mathfrak{G}_s(m_0)\,, \tag{4.1.39}$$

where in the last equality we have used that $A \in C^0(\{(m,\lambda) \in (0,1)^2 : m \in (0,1),\, \lambda \in (0, \theta^{1/2}(m)]\})$. Therefore, it follows that $\mathfrak{G}_s$ is continuous on $[m_0, 1)$. Note that in this case $\Lambda(m_0) = s$.

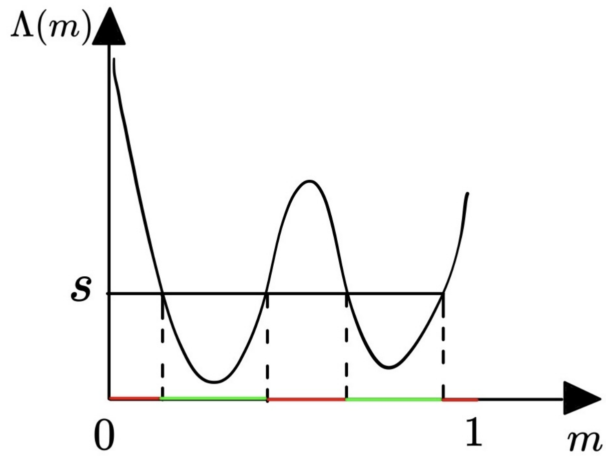


Figure 4.2: Example of the partition of the optimal profiles $w_{m,s}$ induced by the level $s$ and the function $\Lambda$: on the green subintervals the map $\mathfrak{G}_s$ is non-decreasing, whereas on the red subintervals it is strictly decreasing.

Now let $(m_0, m_1) \subset \{m \in (0,1) : \Lambda(m) > s\}$ be a connected component, then $w_{m,s} \in W^{1,1}((2m-1,1))$ for every $m \in (m_0, m_1)$. By case (i) in Lemma 4.1.11 the energy is strictly decreasing on $(m_0, m_1)$ so that

$$\inf_{(m_0,m_1)} \mathfrak{G}_s = \lim_{m \to m_1^-} \mathfrak{G}_s(m)\,,$$

with $\mathfrak{G}_s$ continuous on $(m_0, m_1)$. We claim that if $m_1 < 1$ then $\mathfrak{G}_s$ is left-continuous in $m_1$ as well. Indeed, we note that $(m_0, m_1) \mapsto \lambda_{m,s}$ is continuously differentiable and strictly increasing (cf. Lemma 4.1.11). Then necessarily $\lim\limits_{m \to m_1^-} \lambda_{m,s} = \theta^{1/2}(m_1)$ by (4.1.20) as $\Lambda(m_1) = s$ by continuity of $\Lambda$ in $(0,1)$. To conclude we argue as follows:

$$\begin{aligned}\mathfrak{G}_s(m) &= 2\int_m^1 \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t \\ &= 2\int_m^{m_1} \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t + 2\int_{m_1}^1 \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t\,.\end{aligned}$$

The first summand above is infinitesimal by the local Lipschitz continuity of $\theta^{-1}$. Indeed, if $K^2$ is the maximum value of $\frac{\theta(t)\omega(1-t)}{(\theta^{-1})'(t)}$ on $[\frac{1}{2}\theta^{1/2}(m_1), \theta^{1/2}(m_1)]$ then for $m$ sufficiently close to $m_1$ we get

$$\int_m^{m_1} \Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2} \mathrm{d}t \le K \int_m^{m_1} \frac{1}{\big(t-\theta^{-1}(\lambda_{m,s}^2)\big)^{1/2}} \mathrm{d}t \le 2K(m_1 - \theta^{-1}(\lambda_{m,s}^2))^{1/2}\,,$$

and the conclusion follows by the continuity as $\lim_{m\to m_1^-}\lambda_{m,s}=\theta^{1/2}(m_1)$. Therefore, we have

$$\begin{aligned}\lim_{m\to m_1^-}\mathfrak{G}_s(m)&=\lim_{m\to m_1^-}2\int_{m_1}^1\Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\lambda_{m,s}^2}\Big)^{1/2}\mathrm{d}t\\&=2\int_{m_1}^1\Big(\frac{\theta(t)\omega(1-t)}{\theta(t)-\theta(m_1)}\Big)^{1/2}\mathrm{d}t=\mathfrak{G}_s(m_1),\end{aligned}$$

by the monotone convergence theorem.

Therefore, we have shown that $g(s)\ge(\inf_{\Lambda^{-1}(s)}\mathfrak{G}_s)\wedge(\liminf_{m\to0^+}\mathfrak{G}_s)\wedge(\liminf_{m\to1^-}\mathfrak{G}_s)$. The opposite inequality is trivial as $g(s)=\inf_{(0,1)}\mathfrak{G}_s$.

**Step 3.** Next, consider any $s>\sup\Lambda$, then by (4.1.22) necessarily $w_{m,s}\in SBV\setminus W^{1,1}((2m-1,1))$ for every $m\in(0,1)$. Therefore, $\mathfrak{G}_s$ is strictly increasing on $(0,1)$ by case 2 in Lemma 4.1.11, and thus

$$g(s)=\inf_{(0,1)}\mathfrak{G}_s=\lim_{m\to0^+}\mathfrak{G}_s(m)=\lim_{m\to0^+}A(m,\theta^{1/2}(m))$$

where in the last inequality we have used the explicit formula (4.1.23) for $\mathfrak{G}_s$ and the fact that the second summand in there is infinitesimal. On the other hand, by using Fatou's lemma we conclude the continuity of $[0,1)\ni m\mapsto A(m,\theta^{1/2}(m))$ in $m=0$ and the thesis. Indeed, by Fatou's lemma we have

$$\begin{aligned}g(s)&=\lim_{m\to0^+}A(m,\theta^{1/2}(m))\\&=2\lim_{m\to0^+}\int_0^1\Big(\frac{(\theta(t)-\theta(m))\vee0}{\theta(t)}\,\omega(1-t)\Big)^{1/2}\mathrm{d}t\ge2\Psi(1)\,.\end{aligned}\tag{4.1.40}$$

The conclusion for $s>\sup\Lambda$ follows from the estimate $g(s)\le\bar{\varsigma}s\wedge(2\Psi(1))$ if $\theta(1^-)<\infty$ (cf. item (ii) in Proposition 3.2.4), and the estimate $g(s)\le g_0(s)\le2\Psi(1)$ if $\theta(1^-)=\infty$ (cf. item (ii) in Proposition 3.2.5). By continuity of $g$ we obtain the equality $g(\sup\Lambda)=2\Psi(1)$. Therefore, $s_{\mathrm{frac}}\le\sup\Lambda$. □

Next, we assume $\Lambda$ to be strictly decreasing, as discussed for instance in Remark 4.1.10, to identify a unique minimizing value (defined in terms of $\Lambda$ itself) for the problem defining $g(s)$ introduced in (4.1.11).

**Theorem 4.1.13.** *Assume (Hp 1′)-(Hp 4′), and (Hp 5″), and that $\Lambda:(0,1)\to(0,\infty)$ in (4.1.24) is strictly decreasing. Then, the following hold*

*(i)* $s_{\mathrm{frac}}=\sup\Lambda=\Lambda(0^+)$;

*(ii)* $g(s)=\bar{\varsigma}s$ *for every* $s\in[0,\inf\Lambda]$. *Moreover, for every* $s\in(\inf\Lambda,s_{\mathrm{frac}})$ *let* $m_s:=\Lambda^{-1}(s)$, *then* $w_{m_s,s}\in W^{1,1}((2m_s-1,1))$, $\lambda_{m_s,s}=\theta^{1/2}(m_s)$ *and*

$$g(s)=\mathfrak{G}_s(m_s)\,,\tag{4.1.41}$$

*and $m_s$ is the unique value $m$ such that $g(s)=\mathfrak{G}_s(m)$;*

*(iii)* $g\in C^1([0,\infty))$ *if* $\theta(1^-)<\infty$, *and* $g\in C^1((0,\infty))\cap C^0([0,\infty))$ *if* $\theta(1^-)=\infty$, *with*

$$g'(s)=\theta^{1/2}(m_s)\,,\tag{4.1.42}$$

*for $s \in [\inf \Lambda, s_{\mathrm{frac}}]$, $g'(s) = \overline{\varsigma}$ for $s \in [0, \inf \Lambda]$, and $g'(s) = 0$ for $s \geq s_{\mathrm{frac}}$. In particular, $g$ is concave (actually strictly concave on $[\inf \Lambda, s_{\mathrm{frac}}]$). Moreover, $\overline{\varsigma} s_{\mathrm{frac}} \geq 2\Psi(1)$ if $\theta(1^-) < \infty$.*

*Proof.* Being $\Lambda$ strictly decreasing $\inf \Lambda = \Lambda(1^-)$ and $\sup \Lambda = \Lambda(0^+)$, then Proposition 4.1.12 yields that $s_{\mathrm{frac}} \in [\Lambda(1^-), \Lambda(0^+)]$, and $\Lambda((0,1)) = (\Lambda(1^-), \Lambda(0^+))$ by continuity of $\Lambda$ (cf. Proposition 4.1.8).

**Step 1: minimality of the relaxed energy at $m = m_s$.** Fix $s \in (\Lambda(1^-), \Lambda(0^+))$, and let $m_s := \Lambda^{-1}(s) \in (0,1)$. Then $w_s := w_{m_s,s} \in W^{1,1}((2m_s - 1, 1))$ by (4.1.20), with $\lambda_{m_s,s} = \theta^{1/2}(m_s)$, $w_s(2m_s - 1) = 0$, $w_s(1) = s$, and moreover in view of (4.1.15)

$$w_s(t) = \frac{s}{2} + \int_{m_s}^{t} \Big( \frac{\theta(m_s)\omega(1-\tau)}{\theta(\tau)(\theta(\tau) - \theta(m_s))} \Big)^{1/2} \mathrm{d}\tau \,.$$

In addition, the following claims hold due to the strict monotonicity of $\Lambda$, using that $B(m, \cdot)$ is strictly increasing for all $m$, and $B(\cdot, \lambda)$ is strictly decreasing for all $\lambda$

(i) $w_{m,s} \in W^{1,1}((2m-1,1))$ and $\lambda_{m,s} \in (0, \theta^{1/2}(m))$ for every $m \in (0, m_s)$,

(ii) $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (m_s, 1)$.

To conclude (4.1.41) we apply case (ii) in Proposition 4.1.12 noting that $\mathfrak{G}_s(m_s) < (\liminf_{m \to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m \to 1^-} \mathfrak{G}_s)$ (the latter inferior limits are actually limits being $\Lambda$ monotone) thanks to Lemma 4.1.11 and items (i) and (ii) above.

The equality $g(s) = \overline{\varsigma} s$ for $s \in [0, \Lambda(1^-)]$ follows directly from case (i) in Proposition 4.1.12.

**Step 2. Differentiability of $g$.** To establish the differentiability of $g$ and prove (4.1.42), fix $s_1, s_2 \in (\Lambda(1^-), \Lambda(0^+))$. Thanks to (4.1.41), using the competitor $w = \frac{s_1}{s_2} w_{s_2} \in W^{1,1}((2m_2 - 1, 1))$, where $m_2 := m_{s_2}$ and $w_{s_2} = w_{s_2, m_2}$, we obtain that

$$\begin{aligned} g(s_1) &= \inf_{m \in (0,1)} \inf_{L^1((2m-1,1))} sc^- G_{m,s_1} \leq sc^- G_{m_2,s_1}(w) \\ &= 2 \int_{m_2}^{1} \Big( \theta(t)|w'|^2 + \omega(1-t) \Big)^{1/2} \mathrm{d}t = g(s_2) \\ &+ 2 \left( \frac{s_1^2}{s_2^2} - 1 \right) \int_{m_2}^{1} \frac{\theta(t)|w'_{s_2}|^2}{\Big((\frac{s_1}{s_2})^2 \theta(t)|w'_{s_2}|^2 + \omega(1-t)\Big)^{1/2} + \Big(\theta(t)|w'_{s_2}|^2 + \omega(1-t)\Big)^{1/2}} \mathrm{d}t. \end{aligned} \tag{4.1.43}$$

In particular, if $s_1 < s_2$ we get

$$\begin{aligned} &\frac{g(s_1) - g(s_2)}{s_1 - s_2} \\ &\geq 2 \frac{s_2 + s_1}{s_2^2} \int_{m_2}^{1} \frac{\theta(t)|w'_{s_2}|^2}{\Big((\frac{s_1}{s_2})^2 \theta(t)|w'_{s_2}|^2 + \omega(1-t)\Big)^{1/2} + \Big(\theta(t)|w'_{s_2}|^2 + \omega(1-t)\Big)^{1/2}} \mathrm{d}t, \end{aligned} \tag{4.1.44}$$

therefore using (4.1.15) (see also (4.1.16)), we infer

$$\begin{aligned}\liminf_{s_1\to s_2^-}\frac{g(s_1)-g(s_2)}{s_1-s_2} &\geq \frac{2}{s_2}\int_{m_2}^{1}\frac{\theta(t)|w'_{s_2}|^2}{\big(\theta(t)|w'_{s_2}|^2+\omega(1-t)\big)^{1/2}}\mathrm{d}t\\ &=\frac{2}{s_2}\int_{m_2}^{1}\theta^{1/2}(m_2)w'_2\mathrm{d}t=\theta^{1/2}(m_2).\end{aligned}$$

Using (4.1.43) with $s_1>s_2$ we obtain

$$\begin{aligned}&\limsup_{s_2\to s_1^-}\frac{g(s_2)-g(s_1)}{s_2-s_1}\leq\limsup_{s_2\to s_1^-}\frac{s_1+s_2}{s_2^2}\int_{m_2}^{1}\frac{\theta(t)|w'_{s_2}|^2}{\big(\theta(t)|w'_{s_2}|^2+\omega(1-t)\big)^{1/2}}\mathrm{d}t\\ &=\limsup_{s_2\to s_1^-}\frac{s_2+s_1}{s_2^2}\int_{m_2}^{1}\theta^{1/2}(m_2)w'_2\mathrm{d}t=\limsup_{s_2\to s_1^-}\frac{s_2+s_1}{2s_2}\theta^{1/2}(m_2)=\theta^{1/2}(m_1),\end{aligned}\tag{4.1.45}$$

where in the last equality we have used the continuity of $\Lambda^{-1}$. In particular, we have shown that at each point $s\in(\Lambda(1^-),\Lambda(0^+))$ the left derivative of $g$ exists and equals $\theta^{1/2}(m_s)$.

The conclusion for the right derivative can be deduced analogously by passing to the inferior limit as $s_2\to s_1^+$ in (4.1.44), and to the superior limit as $s_1\to s_2^-$ in (4.1.45).

To conclude, we need only to prove the differentiability of $g$ in $\Lambda(1^-)$ if $\theta(1^-)<\infty$, and in $\Lambda(0^+)$ in general assuming the latter finite. Indeed, $g'(s)=\overline{\varsigma}$ for $s\in[0,\Lambda(1^-))$ if $\theta(1^-)<\infty$, and $g'(s)=0$ if $s>\Lambda(0^+)$. Thus, the formula for $g'$ in (4.1.42) and the continuity of $\theta$ respectively in $\Lambda(1^-)$ if $\theta(1^-)<\infty$ and in $\Lambda(0^+)$ in general, imply that

$$\lim_{s\to\Lambda(1^-)^+}g'(s)=\theta^{1/2}(1)=\overline{\varsigma},\qquad\lim_{s\to\Lambda(0^+)^-}g'(s)=\theta^{1/2}(0)=0\,.$$

Furthermore, we deduce that $s_{\text{frac}}=\Lambda(0^+)$ thanks to (4.1.42) and as by definition $g(s)=2\Psi(1)$ for every $s\geq s_{\text{frac}}$.

Moreover, $(\Lambda(1^-),\Lambda(0^+))\ni s\mapsto m_s$ is non-increasing as $\Lambda$ is so, in turn implying the concavity of $g$ by (4.1.42). Eventually, if $\theta(1^-)<\infty$, necessarily $\overline{\varsigma}s_{\text{frac}}\geq 2\Psi(1)$ as $g'(0)=\overline{\varsigma}$ and $g$ is concave. □

In some cases under a more restrictive set of assumptions an expression for $g$ similar to that in (4.1.41), and the differentiability of $g$, have been first established in [BI24, Proposition 3.3] and [BI24, Proposition 3.6], respectively.

**Remark 4.1.14.** *We point out that for $s\in(\inf\Lambda,\sup\Lambda)$ the energy $\mathfrak{G}_s$ in Theorem 4.1.13 turns out to be $C^1((0,1))$ with $\mathfrak{G}'_s(m_s)=0$ thanks to item (ii) case (b) in Lemma 4.1.11.*

**Remark 4.1.15.** *If $\Lambda$ is non-increasing on $(0,1)$, one cannot expect $g$ to be of class $C^1$. Indeed, within the hypothesis of Proposition 4.1.12, assume that $\Lambda(m)=s_0$ if $m\in[m_0,m_1]$, for some $0<m_0<m_1<1$, and strictly decreasing otherwise. Then, from the condition $\Lambda(m)=s_0$ for every $m\in[m_0,m_1]$ it follows that the corresponding minimizers $w_{m,s}\in W^{1,1}$. Defining $\overline{m}:=\inf\{m\in[m_0,m_1]:\lambda_{m,s}=\theta^{1/2}(m)\}$, and setting $\overline{m}=m_1$ if the latter set is empty, we have that $g(s_0)=\mathfrak{G}_{s_0}(m)$ for every $m\in[\overline{m},m_1]$. Indeed, item (ii) case (b) in Lemma 4.1.11 yields that $\mathfrak{G}'_s(m)=0$ on the interval $(\overline{m},m_1)$, if $\overline{m}<m_1$.*

*Otherwise, on $(0,1)\setminus[m_0,m_1]$ we may argue as in Theorem 4.1.13 to obtain formula (4.1.41). Moreover (4.1.42) hold for every $s\in(0,s_0)\cup(s_0,1)$, where $s_{\text{frac}}=\Lambda(0^+)$. Therefore, $g\in C^1((0,s_0)\cup(s_0,1))$, and $g$*

*is not differentiable in* $s_0$ *as*

$$\lim_{s\to s_0^-} g'(s) = \theta^{1/2}(m_1) \neq \lim_{s\to s_0^+} g'(s) = \theta^{1/2}(m_0)\,.$$

Next, we deal with the case $\Lambda$ increasing and $\theta(1^-) < \infty$, we show that the cohesive law $g$ always corresponds to that in the Dugdale cohesive model.

**Theorem 4.1.16.** *Assume (Hp 1′)-(Hp 4′), and (Hp 5″),* $\theta(1^-) < \infty$*, and that* $\Lambda : (0,1) \to (0,\infty)$ *in* (4.1.24) *is non-decreasing. Then,* $g(s) = \overline{\varsigma} s \wedge (2\Psi(1))$ *for every* $s \geq 0$*.*

*Proof.* The proof is similar to that of Theorem 4.1.13. Recall that by continuity $\Lambda((0,1)) = (\Lambda(0^+), \Lambda(1^-))$. Next, we use Proposition 4.1.12 to infer that $g(s) = \overline{\varsigma} s$ for all $s \in [0, \Lambda(0^+)]$ by case (i) there, and $g(s) = 2\Psi(1)$ for all $s \geq \Lambda(1^-)$ by case (iii) there. Thus, we are left with considering the case $\Lambda(0^+) < \Lambda(1^-)$ and $s \in (\Lambda(0^+), \Lambda(1^-))$. Let $m \in [m_0, m_1] = \Lambda^{-1}(s) \in (0,1)$ and note that $w_{m,s} \in W^{1,1}((2m-1,1))$ by (4.1.20), with $\lambda_{m,s} = \theta^{1/2}(m)$, and moreover in view of (4.1.15)

$$w_{m,s}(t) = \frac{s}{2} + \int_m^t \Big(\frac{\theta(m)\omega(1-\tau)}{\theta(\tau)(\theta(\tau)-\theta(m))}\Big)^{1/2} \mathrm{d}\tau\,.$$

In addition, the following claims hold due to the monotonicity of $\Lambda$

(i) $w_{m,s} \in SBV \setminus W^{1,1}((2m-1,1))$ for every $m \in (0, m_0)$

(ii) $w_{m,s} \in W^{1,1}((2m-1,1))$ and $\lambda_{m,s} \in (0, \theta^{1/2}(m))$ for every $m \in (m_1, 1)$.

To conclude $g(s) = \overline{\varsigma} s \wedge (2\Psi(1))$ we apply case (ii) in Proposition 4.1.12 noting that $\mathfrak{G}_s(m_s) > (\liminf_{m\to 0^+} \mathfrak{G}_s) \wedge (\liminf_{m\to 1^-} \mathfrak{G}_s)$ (the latter inferior limits are actually limits by monotonicity) thanks to Lemma 4.1.11 and items (i) and (ii) above. Therefore, we have that

$$g(s) = \inf_{(0,1)} \mathfrak{G}_s(m) = (\lim_{m\to 0^+} \mathfrak{G}_s(m)) \wedge (\lim_{m\to 1^-} \mathfrak{G}_s(m))\,.$$

The first limit is easily seen to be bigger than $2\Psi(1)$ arguing as to deduce (4.1.40) in Proposition 4.1.12). In addition, the second limit above is bigger than $\overline{\varsigma} s$ arguing as to deduce (4.1.38) in Proposition 4.1.12). Therefore, $g(s) \geq \overline{\varsigma} s \wedge (2\Psi(1))$ for every $s \in (\Lambda(0^+), \Lambda(1^-))$, the reverse inequality is contained in item (ii) in Proposition 3.2.4. □

## 4.2 Reconstructing the cohesive law $g$

In this section we show how to assign a given cohesive law $g_0$, either with a linear or a superlinear behaviour for small amplitudes of the jump, by appropriately choosing the parameters of the phase-field model in (3.2.1). More precisely, either fixing the damage potential and choosing appropriately the degradation function or vice versa, we consider the corresponding functionals $\mathcal{F}_\varepsilon$ and show that $g_0$ equals the surface energy density of their $\Gamma$-limit according to Theorem 3.2.1.

A scaling argument shows that the assumption $\varphi'(0^+) = 1$ in Theorems 4.2.1, 4.2.2, 4.2.4, and 4.2.5 below can be easily dispensed with. The functions in the conclusions of those statements would then be depending on the value $\varphi'(0^+) \neq 1$.

### 4.2.1 The linear case

We start with reconstructing cohesive laws that behave linearly for small jump amplitudes. More precisely, let

(Hp 6)$_{\mathrm{lin}}$ $g_0 \in C^1([0,\infty))$, $g_0^{-1}(0) = \{0\}$, $g_0$ bounded and non-decreasing;

(Hp 7)$_{\mathrm{lin}}$ $g_0$ concave, $g_0'(0) = \overline{\varsigma} \in (0,\infty)$, $g_0'$ is strictly decreasing on $[0,(s_{\mathrm{frac}})_0)$ where

$$(s_{\mathrm{frac}})_0 := \sup\{s \in [0,\infty) : \, g_0(s) < \lim_{s\to\infty} g_0(s)\} \in (0,\infty] \, .$$

In addition, it is convenient to define the auxiliary function $R : [0,1] \to [0, g_0(\infty)]$ by

$$R(t) := \begin{cases} g_0\big((g_0')^{-1}((\overline{\varsigma}^2 - t)^{1/2})\big) & \text{if } t \in [0,\overline{\varsigma}^2) \\ g_0((s_{\mathrm{frac}})_0) & \text{if } t = \overline{\varsigma}^2. \end{cases} \tag{4.2.1}$$

As a consequence of (Hp 6)$_{\mathrm{lin}}$, (Hp 7)$_{\mathrm{lin}}$ and the definition of $(s_{\mathrm{frac}})_0$, $R$ is strictly increasing. Moreover, $g_0'(s) \to 0$ for $s \to (s_{\mathrm{frac}})_0^-$ as $g_0 \in C^1([0,\infty))$ is concave and strictly decreasing, so that $g_0(\infty) < \infty$. Therefore, $R$ is continuous. We further assume that

(Hp 8)$_{\mathrm{lin}}$ $R$ is convex on $[0,\overline{\varsigma}^2]$ and $W^{1,p}((0,\overline{\varsigma}^2))$ for some $p > 2$.

Having introduced $R$, for every $\tau \in [0,\overline{\varsigma}^2]$ consider

$$\phi(\tau) := \frac{1}{\pi}\int_0^\tau \frac{R'(t)}{(\tau - t)^{1/2}}\mathrm{d}t \, . \tag{4.2.2}$$

By Abel's Inversion Formula (cf. Theorem 2.0.5), $\phi$ is the unique $L^1((0,1))$ function satisfying the following Abel integral equation

$$R(t) = \int_0^t \frac{\phi(\tau)}{(t-\tau)^{1/2}}\mathrm{d}\tau \, ,$$

for every $t \in [0,\overline{\varsigma}^2]$. This is a consequence of the smoothness of $R$ and of $R(0) = 0$. Moreover, the regularity, the strict monotonicity and the convexity of $R$ in (Hp 8)$_{\mathrm{sup}}$ yield that $\phi \in C^0([0,\overline{\varsigma}^2])$, $\phi^{-1}(0) = \{0\}$, and $\phi$ is strictly increasing (cf. Theorem 2.0.4). More generally, by changing variables in the very definition of $\phi$ above (i.e., $t = \tau y$), the strict monotonicity of $\phi$ follows provided $R'(t)t^{1/2}$ is non-decreasing and not constant on $(0,\overline{\varsigma}^2)$.

We are now ready to reconstruct $g_0$. In the ensuing two results we will always choose $\omega = \overline{\varsigma}^2 Q$, so that $\theta = \overline{\varsigma}^2 l$. We start fixing $\theta$ with $\theta'(t) > 0$ for every $t \in (0,1)$, and find the appropriate $\omega$. In particular, $\theta(t) = \overline{\varsigma}^2 t^2$ satisfies all the conditions below.

**Theorem 4.2.1.** *Assume that $g_0$ satisfy (Hp 6)$_{\mathrm{lin}}$-(Hp 7)$_{\mathrm{lin}}$, $R$ satisfy (Hp 8)$_{\mathrm{lin}}$, and $\theta_0$ is such that $\theta_0(0) = 0$, $\theta_0(1^-) = \overline{\varsigma}^2$, $\theta_0^{1/2} \in C^1([0,1])$ with $\theta_0'(t) > 0$ for every $t \in (0,1)$, and concave in a left neighborhood of $t = 1$ (in particular, $\theta_0$ satisfies (Hp 5$''$)).*

*Then there exists $\omega_0 \in C^0([0,1])$ such that $l(t) := \overline{\varsigma}^{-2}\theta_0(t)$, $\overline{\varsigma}^2 Q(t) = \omega(t) := \omega_0(t)$ on $[0,1]$ satisfy (Hp 1)-(Hp 2), and for any $\varphi$ satisfying (Hp 3$'$)-(Hp 4$'$) with $\varphi'(0^+) = 1$ the corresponding functionals $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_{\overline{\varsigma}}$ in (3.2.3) with $g = g_0$.*

*Proof.* Let us first assume $\bar\varsigma = 1$. Let $\phi$ be defined in (4.2.2), and then set for every $t \in [0,1]$

$$\omega_0(1-t) := \big((\theta_0^{1/2}(t))'\phi(1-\theta_0(t))\big)^2 . \tag{4.2.3}$$

Clearly, $\omega_0 \in C^0([0,1])$ and with the choices $l := \theta_0$ and $Q = \omega := \omega_0$, (Hp 1′) and (Hp 2′) are satisfied as $\phi$ is strictly increasing and $\theta_0^{1/2}$ is concave in a left neighborhood of $t = 1$. Moreover, (3.1.2) holds with $\bar\varsigma = 1$. In particular, choosing any $\varphi$ that satisfies (Hp 3′) and (Hp 4′) with $\varphi'(0^+) = 1$, by Theorem 3.2.1 $(\mathcal{F}_\varepsilon)_\varepsilon$ $\Gamma$-converges to $F_1$, where the surface energy density $g$ is given by

$$g(s) := \inf_{(u,v)\in\mathfrak{U}_s} \int_0^1 \Big(\theta_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\Big)^{1/2} \mathrm{d}x.$$

Note that $\Phi(1^-) = 0$ by (4.1.28) in Proposition 4.1.8. Indeed, using (4.2.3) we have

$$F(t) = \frac{(\Psi_0 \circ \theta_0^{-1})'(1-t)}{(1-t)^{1/2}} = \frac{\phi(t)}{2(1-t)} \in C^0([0,z])$$

for every $z \in (0,1)$ being $\phi \in C^0([0,1])$. In addition, $\Lambda$ is strictly decreasing as (4.2.3) implies that for every $t \in [0,1]$

$$\big(\Psi_0(\theta_0^{-1}(t^2))\big)' = 2t\big(\Psi_0 \circ \theta_0^{-1}\big)'(t^2) = \phi(1-t^2)\,,$$

the claim thus follows from item (v) in Proposition 4.1.8, and the fact that $\phi$ is strictly increasing. Therefore, we may apply Theorem 4.1.13 to deduce for every $s \in (0, s_{\mathrm{frac}})$

$$\begin{aligned} g(s) &= 2\int_{\theta_0^{-1}((g'(s))^2)}^1 \Big(\frac{\theta_0(t)\omega_0(1-t)}{\theta_0(t)-(g'(s))^2}\Big)^{1/2} \mathrm{d}t \\ &= 2\int_0^{1-(g'(s))^2} \Big((1-r)\frac{\omega_0(1-\theta_0^{-1}(1-r))}{1-(g'(s))^2-r}\Big)^{1/2} \frac{1}{\theta_0'(\theta_0^{-1}(1-r))}\mathrm{d}r\,, \end{aligned} \tag{4.2.4}$$

having used the change of variables $r = 1-\theta_0(t)$. Therefore, on setting $\lambda = 1-(g'(s))^2$ and using the very definition of $\omega_0$ we conclude for every $\lambda \in [0,1)$

$$g((g')^{-1})\big((1-\lambda)^{1/2}\big) = \int_0^\lambda \frac{\phi(r)}{(\lambda-r)^{1/2}}\mathrm{d}r = R(\lambda) = g_0\big((g_0')^{-1}((1-\lambda)^{1/2}\big)\,,$$

where we have used that $g'$ is invertible in view of (4.1.42). Inverting $g\circ(g')^{-1}$ on $[0, g(s_{\mathrm{frac}}))$ and $g_0\circ(g_0')^{-1}$ on $[0, g_0((s_{\mathrm{frac}})_0))$ we conclude that $(g^{-1})' = (g_0^{-1})'$ on $[0, g_0((s_{\mathrm{frac}})_0) \wedge g(s_{\mathrm{frac}}))$. From this, using the very definitions of $(s_{\mathrm{frac}})_0$ and $s_{\mathrm{frac}}$, together with $g_0'((s_{\mathrm{frac}})_0) = g'(s_{\mathrm{frac}}) = 0$, $g_0' > 0$ on $(0,(s_{\mathrm{frac}})_0)$, and $g' > 0$ on $(0, s_{\mathrm{frac}})$, we deduce that $(s_{\mathrm{frac}})_0 = s_{\mathrm{frac}}$. Being $g^{-1}(0) = g_0^{-1}(0) = 0$, we conclude that $g(s) = g_0(s)$ for every $s \in [0,\infty)$.

If $\bar\varsigma \neq 1$, apply the previous argument to $\widetilde{g}_0 := \bar\varsigma^{-1} g_0$ and $\widetilde{l}_0(t) = t^2$, to find $\widetilde{\omega}_0$ such that $\widetilde{g}_0$ is the surface energy density of the $\Gamma$-limit of the corresponding functionals $\mathcal{F}_\varepsilon$ with $\widetilde{Q} = \widetilde{\omega} := \widetilde{\omega}_0$ and $\widetilde{l} := \widetilde{l}_0$. The conclusion for $g_0$ then follows by taking $l(t) := t^2$, $\omega := \bar\varsigma^2\widetilde{\omega}_0$, and $Q := \widetilde{\omega}_0$. □

Next, given $g_0$, we fix $\omega$ (and $Q = \bar\varsigma^{-2}\omega$), and determine $\theta$, equivalently $l$.

**Theorem 4.2.2.** *Let $g_0$ satisfy (Hp 6)$_{\mathrm{lin}}$-(Hp 7)$_{\mathrm{lin}}$, $R$ satisfy (Hp 8)$_{\mathrm{lin}}$, and $\omega_0 \in C^0([0,1],[0,\infty))$ , $\omega_0^{-1}(\{0\}) = \{0\}$, $\omega_0$ increasing in a right neighbourhood of $0$ and such that $2\Psi_0(1) = g_0(\infty)$, where $\Psi_0$ is defined as in (3.1.27) integrating $\omega_0$.*

*Then, there exists $l_0 \in C^0([0,1]) \cap C^1((0,1))$ with $l_0' > 0$ on $(0,1)$, such that $l := l_0$, $\bar{\varsigma}^2 Q = \omega := \omega_0$ satisfy (Hp 1$'$), (Hp 2$'$), $\theta := \bar{\varsigma}^2 l$ satisfies (Hp 5$''$), and for any $\varphi$ satisfying (Hp 3$'$)-(Hp 4$'$) with $\varphi'(0^+) = 1$, the functionals $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_{\bar{\varsigma}}$ in (3.2.3) with $g = g_0$.*

*Proof.* Let us first assume $\bar{\varsigma} = 1$. With $\phi$ defined in (4.2.2), let $l_0 : [0,1] \to [0,1]$ be for every $t \in [0,1]$

$$l_0^{-1}(1-t) := \Psi_0^{-1}\left(\frac{g_0(\infty)}{2} - \frac{1}{2}\int_0^t \frac{\phi(\tau)}{(1-\tau)^{1/2}}\mathrm{d}\tau\right).$$

Then $l_0 \in C^0([0,1])$, $l_0$ is strictly increasing, $l_0(0) = 0$ as $R(1) = g_0(\infty)$, $l_0(1) = 1$ as $2\Psi_0(1) = g_0(\infty)$, $l_0 \in C^1((0,1))$ with $l_0' > 0$ in $(0,1)$, $l_0^{-1} \in W^{1,1}((0,1))$ and for every $t \in (0,1)$

$$2\big((1-t)\omega_0(1 - l_0^{-1}(1-t))\big)^{1/2}(l_0^{-1})'(1-t) = \phi(t). \tag{4.2.5}$$

We observe that setting $l = \theta := l_0$ and $Q = \omega := \omega_0$, (Hp 1$'$) and (Hp 2$'$) are satisfied. In particular choosing any $\varphi$ that satisfies (Hp 3$'$) and (Hp 4$'$) with $\varphi'(0^+) = 1$, Theorem 3.2.1 implies that $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_1$ (cf. (3.2.3)) where $g$ is given by

$$g(s) = \inf_{(u,v)\in\mathfrak{U}_s}\int_0^1 \Big(l_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\Big)^{1/2}\mathrm{d}x. \tag{4.2.6}$$

Moreover $(\theta \circ \Psi_0^{-1})^{1/2}$ is convex, because it is non-decreasing and its inverse $\Psi_0(\theta^{-1}(t^2))$ is concave. Indeed, (4.2.5) yields that the derivative of the latter function is given by

$$2t\omega_0^{1/2}(1 - l_0^{-1}(t^2))(l_0^{-1})'(t^2) = \phi(1-t^2)$$

for every $t \in (0,1)$. Hence, the corresponding function $\Lambda$ is strictly decreasing, and $\Lambda(1^-) = 0$ by (4.1.28) in Proposition 4.1.8. Moreover, the condition in (4.1.27) is satisfied as $F(t) = \frac{\phi(t)}{2(1-t)} \in C^0([0,1))$, so that $\Phi(1^-) = 0$. Hence, Theorem 4.1.13 yields that for every $s \in (0, s_{\mathrm{frac}})$

$$g(s) = 2\int_{l_0^{-1}((g'(s))^2)}^1 \Big(\frac{l_0(t)\omega_0(1-t)}{l_0(t) - (g'(s))^2}\Big)^{1/2}\mathrm{d}t\,.$$

Setting $\lambda = 1 - (g'(s))^2$, and changing variable with $\tau = 1 - l_0(t)$ we find

$$g\big((g')^{-1}((1-\lambda)^{1/2})\big) = \int_0^\lambda \frac{\phi(t)}{(\lambda - t)^{1/2}}\mathrm{d}t = R(\lambda) = g_0\big((g_0')^{-1}((1-\lambda)^{1/2})\big)\,,$$

for every $\lambda \in [0,1)$. Arguing as in Theorem 4.2.1 the conclusion $g = g_0$ follows at once.

If $\bar{\varsigma} \neq 1$, apply the previous argument to $\widetilde{g}_0 := \bar{\varsigma}^{-1}g_0$ and $\widetilde{\omega}_0 := \bar{\varsigma}^{-2}\omega_0$ to find $\widetilde{l}_0$ such that $\widetilde{g}_0$ is the surface energy density of the $\Gamma$-limit of the corresponding functionals $\mathcal{F}_\varepsilon$ with $\widetilde{Q} = \widetilde{\omega} := \widetilde{\omega}_0$ and $\widetilde{l} := \widetilde{l}_0$. The conclusion for $g_0$ then follows by taking $l := \widetilde{l}_0$, $\omega := \omega_0$, and $Q := \bar{\varsigma}^{-2}\omega_0$. $\square$

### 4.2.2 The superlinear case

In this section we show how to reconstruct suitable cohesive laws with superlinear behaviour for small jump amplitudes. Let

(Hp 6)$_{\text{sup}}$ $g_0 \in C^1((0,\infty)) \cap C^0([0,\infty))$, $g_0^{-1}(0) = \{0\}$, $g_0$ is bounded and non-decreasing;

(Hp 7)$_{\text{sup}}$ $g'(0^+) = \infty$, $g_0'$ is convex on $(0, (s_{\text{frac}})_0]$ where

$$(s_{\text{frac}})_0 := \sup\{s \in [0,\infty) : \, g_0(s) < \lim_{s\to\infty} g_0(s)\} < \infty \,,$$

and $g_0 \in C^2((0,(s_{\text{frac}})_0))$ with $g_0''((s_{\text{frac}})_0^-) < 0$.

In particular, $g_0$ is concave on $[0,\infty)$, and $g_0'$ is strictly decreasing on $(0,(s_{\text{frac}})_0]$. Next, define the auxiliary function $R : [0,\infty) \to (0, g_0(\infty)]$ by

$$R(t) := g_0((g_0')^{-1}(t^{1/2})) \tag{4.2.7}$$

Observe that $R(0^+) = g_0(\infty)$, $R(\infty) = 0$, $R$ is strictly decreasing and convex, $R \in C^1([0,\infty))$ with

$$R'(t) = \begin{cases} (2g_0''((g_0')^{-1}(t^{1/2})))^{-1} & \text{if } t \in (0,\infty) \\ (2g_0''((s_{\text{frac}})_0^-))^{-1} & \text{if } t = 0. \end{cases}$$

For every $\tau \in [0,\infty)$ then define

$$\phi(\tau) := -\frac{1}{\pi}\int_\tau^\infty \frac{R'(t)}{(t-\tau)^{1/2}} \mathrm{d}t \,. \tag{4.2.8}$$

In the following proposition we establish some basic properties of the function $\phi$ as a solution of an Abel equation with datum $R$.

**Proposition 4.2.3.** *Assume (Hp 6)$_{\text{sup}}$ and (Hp 7)$_{\text{sup}}$, and let $\phi$ be defined by (4.2.8). Then, $\phi \in C^0([0,\infty))$ is strictly positive, non-increasing and satisfies the Abel equation*

$$R(t) = \int_t^\infty \frac{\phi(\tau)}{(\tau - t)^{1/2}} \mathrm{d}\tau \tag{4.2.9}$$

*for every $t \in [0,\infty)$.*

*Proof.* It suffices to observe that, by the change of variables $\tau = (g')^{-1}(t^{1/2})$, we obtain

$$\phi(0) = -\int_0^\infty \frac{R'(t)}{t^{1/2}}\, dt = -\int_0^\infty \frac{1}{2g_0''\big((g_0')^{-1}(t^{1/2})\big)t^{1/2}}\, dt = (s_{\text{frac}})_0 < \infty,$$

and therefore Theorem 2.0.6 applies. $\square$

Next, we fix $\omega$ and determine both $l$ and $Q$, and thus $\theta$.

**Theorem 4.2.4.** *Let $g_0$ satisfy (Hp 6)$_{\text{sup}}$-(Hp 7)$_{\text{sup}}$, and let $\omega_0 \in C^0([0,1],[0,\infty))$, $\omega_0^{-1}(\{0\}) = \{0\}$, $2\Psi_0(1) = g_0(\infty)$, $\Psi_0$ defined as in (3.1.27) integrating $\omega_0$.*

*Then, there exist $l_0$ and $Q_0 \in C^0([0,1])$ such that $l := l_0$, $Q := Q_0$, $\omega := \omega_0$ satisfy (Hp 1′), (Hp 2′) with $\overline{\varsigma} = \infty$, $\theta := \overline{\varsigma}^2 l$ satisfies (Hp 5″), and for any $\varphi$ satisfying (Hp 3′)-(Hp 4′) with $\varphi'(0^+) = 1$ the functionals $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_\infty$ in (3.2.3) with $g = g_0$.*

*Proof.* Let $\phi$ be defined in (4.2.8), then let $\theta_0^{-1} : [0,\infty) \to [0,1]$ be

$$\theta_0^{-1}(t) := \Psi_0^{-1}\left(\frac{g_0(\infty)}{2} - \frac{1}{2}\int_t^\infty \frac{\phi(\tau)}{\tau^{1/2}}\mathrm{d}\tau\right)$$

Then $\theta_0 \in C^0([0,1)) \cap C^1((0,1))$ by Proposition 4.2.3. Moreover, $\theta_0(0) = 0$ as $R(0) = g_0(\infty)$, $\theta_0(t) \to \infty$ as $t \to 1$ as $2\Psi_0(1) = g_0(\infty)$, $\theta_0'(t) > 0$ for every $t \in (0,1)$ and

$$2(\theta_0^{-1})'(t)\big(t\omega_0(1-\theta_0^{-1}(t))\big)^{1/2} = \phi(t)$$

for every $t \in (0,1)$. We observe that setting $l(t) := \theta_0(t) \wedge 1$, $Q(0) := 0$, $Q(t) := \frac{\omega_0(t)l(1-t)}{\theta_0(1-t)}$ if $t \in (0,1]$ and $\omega := \omega_0$, (Hp $1'$) and (Hp $2'$) are satisfied. In particular, for every $\varphi$ as in the statement, Theorem 3.2.1 implies that $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_\infty$ where $g$ is given by

$$g(s) = \inf_{(u,v)\in\mathfrak{U}_s} \int_0^1 \Big(\theta_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\Big)^{1/2}\mathrm{d}x. \tag{4.2.10}$$

Moreover $(\theta_0 \circ \Psi^{-1})^{1/2}$ is convex, because it is non-decreasing and its inverse $\Psi(\theta_0^{-1}(t^2))$ is concave. Indeed, the derivative of the latter function is

$$(\omega_0(1-\theta_0^{-1}(t^2)))^{1/2}(\theta_0^{-1})'(t^2)\, 2t = \phi(t^2).$$

Therefore, by Theorem 4.1.13 we infer that

$$g(s) = 2\int_{\theta_0^{-1}((g'(s))^2)}^1 \Big(\frac{\theta_0(t)\omega_0(1-t)}{\theta_0(t)-(g'(s))^2}\Big)^{1/2}\mathrm{d}t$$

and changing variable with $\theta_0(t) = \tau$ and setting $\lambda = (g'(s))^2$, we find

$$\begin{aligned} g\big((g')^{-1}(\lambda^{1/2})\big) &= 2\int_\lambda^\infty \Big(\frac{\tau\,\omega_0(1-\theta_0^{-1}(\tau))}{\tau-\lambda}\Big)^{1/2}(\theta_0^{-1})'(\tau)\mathrm{d}\tau \\ &= \int_\lambda^\infty \frac{\phi(\tau)}{(\tau-\lambda)^{1/2}}\mathrm{d}\tau = R(\lambda) = g_0\big((g_0')^{-1}(\lambda^{1/2})\big)\,, \end{aligned}$$

for every $\lambda \in [0,\infty)$. Arguing as in Theorem 4.2.1 the conclusion $g = g_0$ follows at once. □

Finally, with fixed $\theta$ we determine $l$, $Q$ and $\omega$.

**Theorem 4.2.5.** *Assume that $g_0$ satisfy (Hp 6)$_{\sup}$-(Hp 7)$_{\sup}$, and $\theta_0$ is such that $\theta_0(0) = 0$, $\theta_0(1^-) = \infty$, $\theta_0^{1/2} \in C^1([0,1))$ with $\theta_0'(t) > 0$ for every $t \in (0,1)$, concave in a left neighborhood of $t = 1$, and such that $(\theta_0^{1/2}(t))'\phi(\theta_0(t))$ is decreasing and infinitesimal as $t \to 1$, where $\phi$ is defined in (4.2.8) in particular $\theta_0$ satisfies (Hp $5''$).*

*Then there exist $\omega_0$ and $Q_0 \in C^0([0,1])$ such that $l(t) := \theta_0(t) \wedge 1$, $Q := Q_0$, $\omega := \omega_0$ satisfy (Hp $1'$)-(Hp $2'$) with $\bar\varsigma = \infty$; and for any $\varphi$ satisfying (Hp $3'$)-(Hp $4'$) with $\varphi'(0^+) = 1$ the functionals $\mathcal{F}_\varepsilon$ in (3.2.1) $\Gamma$-converge to $F_\infty$ in (3.2.3) with $g = g_0$.*

*Proof.* Let $\phi$ be defined in (4.2.8), and then set for every $t \in [0,1)$

$$\omega_0(1-t) := \left((\theta_0^{1/2}(t))'\phi(\theta_0(t))\right)^2, \tag{4.2.11}$$

and $\omega_0(0) := 0$. Note that $\omega_0 \in C^0([0,1])$ by the assumptions on $\theta_0$. With the choices $l(t) := \theta_0(t) \wedge 1$, $Q(0) := 0$ and $Q(1-t) := \frac{\omega_0(1-t)l(t)}{\theta_0(t)}$ for $t \in [0,1)$, (Hp 1) and (Hp 2) are satisfied as $\phi$ is strictly increasing and $\theta_0^{1/2}$ is concave in a left neighborhood of $t = 1$. Moreover, (3.1.2) holds with $\overline{\varsigma} = \infty$. In particular, choosing any $\varphi$ that satisfies (Hp 3) and (Hp 4) with $\varphi'(0^+) = 1$, by Theorem 3.2.1 $(\mathcal{F}_\varepsilon)_\varepsilon$ $\Gamma$-converges to $F_\infty$, where the surface energy density $g$ is given by

$$g(s) := \inf_{(u,v)\in\mathfrak{U}_s} \int_0^1 \left(\theta_0(v)|u'|^2 + \omega_0(1-v)|v'|^2\right)^{1/2} \mathrm{d}x.$$

The rest of the proof follows exactly as in Theorem 4.2.4. $\square$

### 4.2.3 Examples

In this section, we apply Theorems 4.1.16, 4.2.1, 4.2.2 and 4.2.4 to find closed form solutions for phase-field models whose $\Gamma$-limit have as surface energy density assigned cohesive laws $g$ frequently used in applications, Fig. 4.4. From a numerical point of view the problem has already been discussed extensively in the papers [Wu17, WN18, FFL21, LCM25, Wu24b]. We note that the reconstruction problem for the Dugdale cohesive law has been addressed in [CFI16, Section 7.1] and [LCM25] using a truncated degradation function of the form $\varphi(t) = 1 \wedge t$. In contrast, we obtain here Dugdale cohesive law by using a generic degradation function $\varphi(t) = \frac{t}{1+t}$. Consequently, in a one-dimensional setup, the mechanical global response corresponding to the Dugdale cohesive law with the truncated degradation function is obtained only in the limit as $\varepsilon \to 0$, since the critical stress also depends on the regularizing length $\varepsilon$ [LCM25, Fig. 2]. By employing the smooth degradation function, as done here, the regularizing length $\varepsilon$ only influences the phase-field profile, ensuring that the mechanical global response is immediately and correctly described. Further details on this point will be discussed in Section 4.3.

We will use the notation of Section 4.2. Recall that we have assumed (Hp 1′)-(Hp 4′), and moreover (Hp 5″). To be consistent with Section 4.2, we suppose that the function $\varphi$ satisfies $\varphi'(0^+) = 1$. For the sake of simplicity, if $\theta(1^-) < \infty$ we shall choose $Q = \omega$ so that $g'(0^+) = 1$. This is clearly not a limitation using the rescaling argument employed in the proofs of Theorems 4.2.1 and 4.2.2.

**Dugdale's model.**

Consider the Dugdale's model given by the cohesive law

$$g(s) = \begin{cases} s & \text{if } s \in [0,k] \\ k & \text{if } s \in (k,\infty) \end{cases} \tag{4.2.12}$$

where $k \in (0,\infty)$. We observe that if $l$ and $\omega$ are such that $l \in C^1((0,1))$ with $l'(t) > 0$ for every $t \in (0,1)$ and

$$\Psi(l^{-1}(\tau^2)) = \frac{k}{\pi}(\sin^{-1}(\tau) - \tau(1-\tau^2)^{1/2})$$

for every $\tau \in [0,1]$, then an easy computation gives $\Lambda(x) = 2kl^{1/2}(x)$ for every $x \in (0,1)$, where $\Psi$ and $\Lambda$ are respectively defined in (3.1.27) and (4.1.24). Therefore, if $\mathcal{F}_\varepsilon$ are the functionals defined in (3.2.1) with $Q = \omega$, then Theorem 4.1.16 implies that $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon$ is a functional as in (3.1.15) with $g$ as surface energy density. In particular we obtain the following

- if $\omega(1-t) = k^2(1-t)^2$ then $l^{-1}(t) = 1 - \left[1 - \frac{2}{\pi}\left(\sin^{-1}(t^{\frac{1}{2}}) - (t-t^2)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$
- if $\omega(1-t) = \frac{9k^2}{16}(1-t)$ then $l^{-1}(t) = 1 - \left[1 - \frac{2}{\pi}\left(\sin^{-1}(t^{\frac{1}{2}}) - (t-t^2)^{\frac{1}{2}}\right)\right]^{\frac{2}{3}}$

**Linear softening.**

Let $k \in (0,\infty)$, and consider softening laws of the form

$$g'(s) = \begin{cases} 1 - ks & \text{if } s \in [0, \frac{1}{k}] \\ 0 & \text{if } s \in (\frac{1}{k}, \infty) \end{cases}$$

One can easily verify that the corresponding $g$ satisfies (Hp 6)-(Hp 7), $R(t) = \frac{t}{2k}$ fulfills (Hp 8), and that $\phi(t) = \frac{t^{1/2}}{k\pi}$ by (4.2.2). In particular, we deduce that

- if $l(t) = t^2$ then $\omega(1-t) = \frac{1-t^2}{k^2\pi^2}$,
- if $\omega(t) = \frac{t^2}{4k^2}$ then $l^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}((t-t^2)^{1/2} + \cos^{-1}((1-t)^{1/2}))\right)^{1/2}$,
- if $\omega(t) = \frac{9t}{64k^2}$ then $l^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}((t-t^2)^{1/2} + \cos^{-1}((1-t)^{1/2}))\right)^{2/3}$.

We remark that in case $l(t) = t^2$ we obtain exactly the same damage potential $\omega$ as in [FFL21, Section 4.1]

**Bilinear softening**

In this case we fix $a, b, k_1, k_2 \in (0,\infty)$ such that $a < b$, $k_2 < k_1$ and

$$g'(s) = \begin{cases} 1 - k_1 s & \text{if } s \in [0,a] \\ (1 - k_1 a + k_2 a) - k_2 s & \text{if } s \in (a,b] \\ 0 & \text{if } s \in (b,\infty) \end{cases}$$

is continuous. Therefore, the corresponding $g$ satisfies (Hp 6)-(Hp 7) and simple calculations yield $R \in W^{1,\infty}((0,1))$ with

$$R'(t) = \begin{cases} \frac{1}{2k_1} & \text{if } t \in [0, 1-(1-k_1 a)^2] \\ \frac{1}{2k_2} & \text{if } t \in (1-(1-k_1 a)^2, 1]. \end{cases}$$

In particular, $R$ fulfills (Hp 8), and if $l(t) = t^2$ then

$$\omega(1-t) = \begin{cases} \left(\frac{(1-t^2)^{1/2}}{k_1\pi} + (\frac{1}{k_2\pi} - \frac{1}{k_1\pi})((1-k_1 a)^2 - t^2)^{1/2}\right)^2 & \text{if } t \in [0, 1-k_1 a] \\ \frac{1-t^2}{k_1^2\pi^2} & \text{if } t \in (1-k_1 a, 1]. \end{cases}$$

**Hyperbolic softening**

Let $k \in (0, \infty)$ and consider hyperbolic softening laws of the form

$$g'(s) = \begin{cases} \frac{2}{1+ks} - 1 & \text{if } s \in [0, \frac{1}{k}] \\ 0 & \text{if } s \in (\frac{1}{k}, \infty). \end{cases}$$

Then

$$g(s) = \begin{cases} \frac{2}{k}\log(1+ks) - s & \text{if } s \in [0, \frac{1}{k}] \\ \frac{1}{k}(2\log(2) - 1) & \text{if } s \in (\frac{1}{k}, \infty) \end{cases}$$

satisfies (Hp 6)-(Hp 7) and the corresponding function $R$ fulfills (Hp 8) with

$$R'(t) = \frac{1}{k(1 + (1-t)^{\frac{1}{2}})^2}$$

for every $t \in [0, 1]$. In particular, by an easy computation and Theorem 4.2.1, we obtain, with $l(t) = t^2$, that

$$\omega(1-t) = \begin{cases} \frac{(2t^2\log(t)+1-t^2)^2}{k^2\pi^2(1-t^2)^3} & t \in [0, 1) \\ 0 & t = 1 \end{cases}.$$

**Quadratic hyperbolic softening**

Let $k \in (0, \infty)$ and consider quadratic hyperbolic softening laws of the form $g'(s) = \frac{1}{(1+ks)^2}$. Then $g$ satisfies (Hp 6)-(Hp 7) with

$$R'(t) = \frac{4k}{(1-t)^{\frac{1}{4}}} \tag{4.2.13}$$

for every $t \in [0, 1)$. In particular $R' \in L^p((0, 1))$ for every $p \in [1, 4)$ and thus $R$ fulfills (Hp 8). From a simple computation and Theorem 4.2.1 we can infer that if $l(t) = t^2$ then $\omega(1-t) = \phi(1-t^2)^2$ where

$$\phi(t) = \frac{16k}{3\pi(1-t)^{\frac{1}{2}}}\left[(1-t)^{\frac{3}{4}}\,{}_2F_1(1/2, 3/4; 7/4; 1) - {}_2F_1(1/2, 3/4; 7/4; 1/(1-t))\right] \tag{4.2.14}$$

and ${}_2F_1(\cdot\,,\cdot\,;\cdot\,;\cdot)$ is the Hypergeometric function.

**Exponential softening**

Let $k \in (0, \infty)$ and consider exponential softening laws of the form $g'(s) = e^{-ks}$, namely $g(s) = \frac{1}{k}(1 - e^{-ks})$ for $s \geq 0$. It is easy to verify that $g$ fulfills (Hp 6)-(Hp 7) with $R(t) = \frac{1}{k}(1 - (1-t)^{1/2})$. Note that $R' \notin L^p((0, 1))$ for any $p > 2$, and thus $R$ does not satisfy (Hp 8) so that we can apply neither Theorem 4.2.1 nor 4.2.2. Assumption (Hp 8) ensures the continuity of the function $\phi$ on $[0, 1]$, which indeed fails in this particular case as $\phi(t) = \frac{1}{k\pi}\cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right)$. Therefore, using the constructions in Theorem 4.2.1 and 4.2.2 we would get that either $l$ or $\omega$ is not in $C^0([0, 1])$. On the other hand, the continuity of both $l$ and $\omega$ is a crucial assumption in Theorem 3.2.1 (cf. (Hp 1′)-(Hp 2′)).

To overcome this problem, we slightly modify $g$ as follows: let $\delta > 0$ and $s_\delta \to \infty$ as $\delta \to 0^+$, we linearize

$g'$ in $(s_\delta, \infty)$ to obtain a new cohesive law $g_\delta$ given by

$$g'_\delta(s) := \begin{cases} e^{-ks} & \text{if } s \in [0, s_\delta] \\ (e^{-ks_\delta} - ke^{-ks_\delta}(s - s_\delta)) \vee 0 & \text{if } s \in (s_\delta, \infty). \end{cases}$$

For every $\delta \in (0, 1)$, $g_\delta$ satisfies (Hp 6)-(Hp 7) with

$$R_\delta(t) = \begin{cases} \frac{1}{k}(1 - (1-t)^{1/2}) & \text{if } t \in [0, \hat{s}_\delta] \\ \frac{1}{k}(1 - (1-\hat{s}_\delta)^{1/2}) & \text{if } t \in (\hat{s}_\delta, 1], \end{cases}$$

for some $\hat{s}_\delta$ such that $\hat{s}_\delta \to 1$ as $\delta \to 0^+$. Moreover, $R_\delta$ fulfills (Hp 8) and $g_\delta \to g$ uniformly and monotonically on $[0, \infty)$. In particular, one may choose $\hat{s}_\delta = 1 - \delta$ to obtain

$$\phi_\delta(t) = \begin{cases} \frac{1}{k\pi} \cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right) & \text{if } t \in [0, 1-\delta] \\ \frac{1}{k\pi}\left(\cosh^{-1}\left(\frac{1}{(1-t)^{1/2}}\right) - \cosh^{-1}\left((\frac{\delta}{1-t})^{1/2}\right) + \left(\frac{\delta-(1-t)}{\delta}\right)^{1/2}\right) & \\ & \text{if } t \in (1-\delta, 1]\,, \end{cases}$$

and hence get from Theorem 4.2.1 for $l(t) = t^2$

$$\omega_\delta(1-t) = \begin{cases} \frac{1}{k^2\pi^2}\left(\cosh^{-1}(\frac{1}{t}) - \cosh^{-1}(\frac{\delta^{1/2}}{t}) + \left(\frac{\delta - t^2}{\delta}\right)^{1/2}\right)^2 & \text{if } t \in [t, \delta^{1/2}] \\ \frac{1}{k^2\pi^2}(\cosh^{-1})^2(\frac{1}{t}) & \text{if } t \in (\delta^{1/2}, 1]. \end{cases}$$

Let then $\mathcal{F}_{\varepsilon,\delta}$ be defined as in (3.2.1) with $l(t) = t^2$ and $Q = \omega = \omega_\delta$. Theorem 3.2.1 implies that $\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,\delta}$ is a functional as in (3.1.15) with $g_\delta$ as surface energy density, and diffuse energy densities independent from $\delta$. Noting that $g_{\delta_2} \geq g_{\delta_1}$ for $0 < \delta_1 < \delta_2 < 1$, [DM93, Proposition 5.4] yields that $\Gamma\text{-}\lim_{\delta\to 0}\left(\Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,\delta}\right)$ is a functional as in (3.1.15) with surface energy density $g(s) = \frac{1}{k}(1 - e^{-ks})$, $s \geq 0$.

In conclusion, we observe that $\omega_\delta$ coincides with the damage potential found in [FFL21, Section 4.2] on $[\delta^{1/2}, 1]$ for every $\delta \in (0, 1)$, using the identity $\cosh^{-1}(1/t) = \tanh^{-1}((1-t^2)^{1/2})$.

### Logarithmic softening

In this section we provide an application of Theorem 4.2.4 in order to approximate cohesive laws with logarithmic behaviour for small jump amplitudes. Let $k \in (0, \infty)$ and consider

$$g'(s) = \begin{cases} -\log(ks) & \text{if } s \in (0, 1/k] \\ 0 & \text{if } s \in (1/k, \infty), \end{cases}$$

then

$$g(s) = \begin{cases} s(1 - \log(ks)) & \text{if } s \in [0, \frac{1}{k}] \\ \frac{1}{k} & \text{if } s \in (\frac{1}{k}, \infty) \end{cases}$$

satisfies (Hp 6′)-(Hp 7′) and the corresponding function $R$ is given by

$$R'(t) = -\frac{1}{2k} e^{-t^{1/2}}$$

for every $t \in [0, \infty)$. In particular, by (4.2.8), we have that

$$\phi(t) = \frac{1}{2k\pi} \int_t^\infty \frac{e^{-x^{1/2}}}{(x-t)^{1/2}}\, \mathrm{d}x$$

and from Theorem 4.2.4, if $\omega(t) = \frac{9}{16k} t$, then we obtain

$$f^2(t) := \frac{l(t)}{Q(1-t)} = \frac{16k\theta(t)}{9(1-t)}$$

where

$$\theta^{-1}(t) = 1 - \left[ k \int_t^\infty \frac{\phi(x)}{x^{1/2}}\, \mathrm{d}x \right]^{\frac{2}{3}}.$$

## 4.3 Computational results

In this section we consider the standard one-dimensional traction bar problem of Fig. 4.3a. The bar, initially sound, is assumed to have reference length $L$ and to be made of a homogeneous material with a linear behaviour in the elastic regime and a stress-softening response in the inelastic regime (brittle material). The left end ($x = 0$) is kept fixed, while on the right end ($x = L$) a monotonically increasing displacement $U$ is imposed by a hard device, which induces tensile states. Consequently, only positive crack-opening displacements are considered ($\delta \geq 0$).

This paradigmatic problem allows us to illustrate, with closed-form results, the main mechanical features of the phase-field cohesive fracture response of models associated with different traction-separation laws, as discussed in Section 4.2. For each considered model we highlight the evolution of key mechanical quantities, including phase-field and displacement profiles as shown in Fig. 4.3c.

### 4.3.1 The mathematical and engineering frameworks

In this section, we aim to present the mathematical setting of Section 3.2 from an engineering point of view.

**Mathematical framework** The general energy functional of Section 3.2, specified for the considered one-dimensional problem of Fig. 4.3a reads

$$\mathcal{F}_\varepsilon(u, v, [0,1]) := \int_0^1 \left( \varphi(\varepsilon f^2(v))|u'|^2 + \frac{\omega(1-v)}{4\varepsilon} + \varepsilon |v'|^2 \right) \mathrm{d}x \tag{4.3.1}$$

with $u(0) = 0$ and $u(1) = U/L$. In (4.3.1), $u$ and $v$ are respectively the displacement and phase-field variable fields on the normalized support $[0,1]$, with $v$ decreasing from 1 to 0 as the crack evolves, $\varphi(\varepsilon f^2(v))$ a generic degradation function, encompassing the elastic properties of the material, with $\varphi$ satisfying hypotheses (Hp 3'-4') and $f(v) := (l(v)/Q(1-v))^{1/2}$ satisfying (Hp 1'), $\omega(1-v)$ the local phase-field dissipation potential

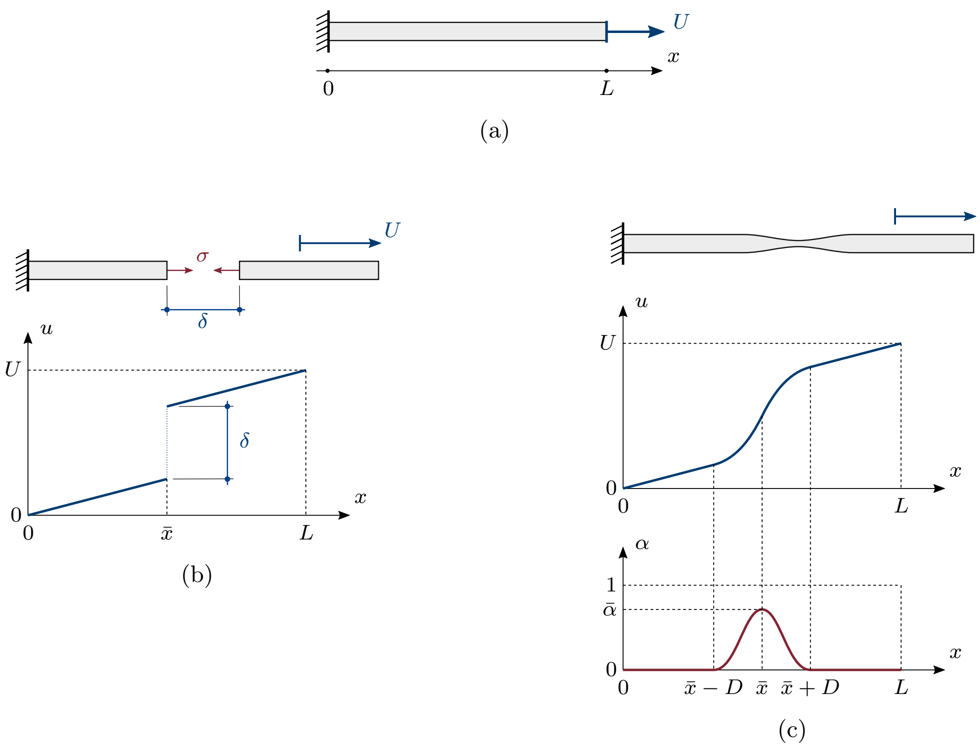



Figure 4.3: The one-dimensional tensile problem (a): An example of evolution with a sharp-crack (b) and the corresponding phase-field regularization (c).

satisfying hypothesis (Hp 1') and $\varepsilon$ an internal length ((Hp 1'-4') are reported in the introduction of Section 3.2).

We have already pointed out in the introduction of Section 3 that the energy functional (4.3.1) includes, as particular cases, the elastic degradation functions $\varphi(\varepsilon f^2(v)) = \widetilde{f}_\varepsilon^2(v)$ and $\varphi(\varepsilon f^2(v)) = f_\varepsilon^2(v)$, where

$$\widetilde{f}_\varepsilon(v) := 1 \wedge \varepsilon^{1/2} f(v) \tag{4.3.2}$$

and

$$f_\varepsilon(v) := \left( \frac{\varepsilon l(v)}{\varepsilon l(v) + Q(1-v)} \right)^{1/2}. \tag{4.3.3}$$

For instance, the degradation function (4.3.2) has been considered in [FI17, LCM23, LCM25], whereas (4.3.3) has been adopted in [Wu17, WN18, FFL21] with specific choices of the functions $l$ and $Q$.

Throughout this section, and in accordance with the motivations discussed in Chapter 1, we restrict our attention to degradation functions of the form (4.3.3).

**Engineering framework** From an engineering and mechanical perspective, the emphasis is on the physical interpretation of variables, functions, and parameters, with a focus on material behavior and practical applications. This viewpoint, widely adopted in the engineering community following [PM10a, PM10b, MMP16], typically expresses the cohesive phase-field energy functional for the problem in Fig. 4.3c as

$$\mathcal{E}(u,\alpha) = \int_0^L \Bigg( \underbrace{\frac{1}{2}\mathsf{g}_\ell(\alpha) E\,\varepsilon^2}_{=:\,\psi(\varepsilon,\alpha)} + \underbrace{G_\mathrm{c}\left(\frac{\mathsf{w}(\alpha)}{\ell} + \ell\,(\alpha')^2\right)}_{=:\,G_\mathrm{c}\gamma(\alpha,\alpha')} \Bigg)\,\mathrm{d}x\,, \tag{4.3.4}$$

where $u$, $\varepsilon := u'$, and $\alpha$ are, respectively, the displacement, the total strain, and the phase-field variable fields with $\alpha$ increasing from 0 to 1 as the crack evolves, $\mathsf{g}_\ell(\alpha)$ is the decreasing material degradation function with the subscript emphasizing its key dependence on the material internal length $\ell$ that enables cohesive fracture responses, $E$ denotes the one-dimensional axial stiffness (Young's modulus times the cross section area), $G_\mathrm{c}$ is the one-dimensional fracture toughness and $\mathsf{w}(\alpha)$ is the local phase-field dissipation function, an increasing function with respect to $\alpha$. The first term in the integrand, $\psi(\varepsilon,\alpha)$, represents the internal potential energy whereas the second term represents the dissipated surface fracture energy density with $\gamma(\alpha,\alpha')$ being the crack density function [FFL21]. Their sum, integrated over the entire domain, then represents the total internal energy. Mechanically, the phase-field variable can be considered as a damage variable and the phase-field model as a gradient-damage model, [MMP16]. In this view, the following irreversibility condition is often assumed:

$$\dot{\alpha} \geq 0\,, \tag{4.3.5}$$

aiming at ensuring that the dissipated fracture energy is never decreasing in order to fulfill the second law of thermodynamics and preventing healing effects. More details on the irreversibility condition are discussed in Sec. 4.3.2. For a fully developed crack, the integral over the domain of the crack density function must be equal to one. This requirement is reflected into the phase-field function $\mathsf{w}(\alpha)$ by the following condition

$$4\int_0^1 \sqrt{\mathsf{w}(\alpha)}\,\mathrm{d}\alpha = 1\,. \tag{4.3.6}$$

In standard gradient-damage models, where an arbitrary damage function is usually assumed, condition (4.3.6) is ensured by dividing the fracture toughness by the constant $c_\mathsf{w} := 4\int_0^1 \sqrt{\mathsf{w}(\alpha)}$.

**Link between the mathematical and engineering frameworks** The displacement variable $u$ has the same meaning in both perspectives, apart from its normalization with respect to the bar length $L$, whereas for the phase-field variable, the following relation holds:

$$\alpha = 1 - v. \tag{4.3.7}$$

By normalizing lengths by $L$ and dividing by $E\,L/2$ in (4.3.4), a link between the energy functionals (4.3.1) and (4.3.4) is established. The local phase-field potentials are related by

$$\omega(1-v) = 4\,k^2\,\mathsf{w}(\alpha) \tag{4.3.8}$$

and the internal lengths are related by

$$\varepsilon = k\frac{\ell}{L}\,, \tag{4.3.9}$$

with $k = 2G_c\, L/E$ being a material constant. The degradation function in (4.3.4) consequently reads

$$\mathsf{g}_\ell(\alpha) = \frac{1}{1 + \frac{E}{2G_c \ell} \frac{Q(\alpha)}{\mathsf{l}(\alpha)}}\,, \tag{4.3.10}$$

with

$$l(v) = \mathsf{l}(\alpha). \tag{4.3.11}$$

Condition (4.3.6) in terms of $\omega$ then becomes

$$\int_0^1 \sqrt{\omega(v)}\,\mathrm{d}v = \frac{k}{2}\,. \tag{4.3.12}$$

### 4.3.2 Features of the mechanical model: 1D analytical results

In this section, we summarize well-established results and key quantities concerning the mechanical response of the one-dimensional tensile problem for variational phase-field and gradient-damage models, as presented in [PM11, MMP16, Wu17, CDB21], to which we refer the reader for further details. These results arise from a variational analysis within the energetic formulation framework, where the evolution of a rate-independent system is assumed to be governed by two fundamental principles: a stability condition and an energy balance [Mie06, MR15]. Most of these results are parameterized with respect to the maximum phase-field state $\bar{\alpha} \in [0, 1]$ within the bar. Here, we assume without loss of generality that a single localization occurs at $\bar{x} = L/2$, with $\alpha(\bar{x}) = \bar{\alpha}$. The localization is assumed to remain entirely within the bar during its evolution, avoiding interactions with boundaries. This is commonly achieved by enforcing the phase-field variable to vanish at the boundaries. Mechanically, such localization behavior is typically ensured by the design of standard uniaxial test specimens (e.g., dog-bone shapes), in which the ends are thickened and widened to prevent cracking in the grip regions.

**Equilibrium, constitutive equation, and phase-field evolution laws** A first consequence of the variational energetic analysis is that the uniaxial stress is always constant along the bar due to equilibrium, that is,

$$\sigma' = 0 \qquad \forall x \in (0, L)\;, \tag{4.3.13}$$

where $\sigma$ is linked to the total strain by the constitutive equation

$$\sigma = \mathsf{g}_\ell(\alpha) E \varepsilon\,. \tag{4.3.14}$$

Another consequence is the set of necessary Kuhn-Tucker conditions in the bulk for the phase-field evolution:

$$\frac{1}{2}\mathsf{g}_\ell'(\alpha) E\,\varepsilon^2 + G_c \left( \frac{\mathsf{w}'(\alpha)}{\ell} - 2\ell\,\alpha'' \right) \geq 0\,, \tag{4.3.15a}$$

$$\dot{\alpha} \geq 0\,, \tag{4.3.15b}$$

$$\left( \frac{1}{2}\mathsf{g}_\ell'(\alpha) E\,\varepsilon^2 + G_c \left( \frac{\mathsf{w}'(\alpha)}{\ell} - 2\ell\,\alpha'' \right) \right) \dot{\alpha} = 0\,, \tag{4.3.15c}$$

$\forall x \in (0, L)$. Equations (4.3.13) and (4.3.15) serve as the fundamental basis for deriving the results hereafter presented and used to construct the mechanical responses in Sec. 4.3.4.

**Limit and critical stresses** The *limit stress* is the stress level associated with the evolving phase-field variable. Assuming a stress-softening response, the limit stress is a non-increasing function of $\bar{\alpha}$, and reads

$$\sigma(\bar{\alpha}) \coloneqq \sqrt{K \frac{\mathsf{w}(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})}}\,, \tag{4.3.16}$$

with $K = 2E\,G_c/\ell$ being a material constant and $\mathsf{h}_\ell(\alpha) = 1/\mathsf{g}_\ell(\alpha) - 1$ being the complementary degradation function.

The existence of the following limit is assumed, as stated in (Hp 2′) (cf. Section 3.2)

$$\bar{\sigma} \coloneqq \lim_{\alpha \to 0^-} \sigma(\bar{\alpha}), \tag{4.3.17}$$

with $\bar{\sigma}$ being the *critical stress*, that is the stress at which the phase-field evolution is triggered and marking the end of the initial elastic response. The corresponding critical strain then reads $\bar{\varepsilon} \coloneqq \bar{\sigma}/E$. The limit value $\bar{\varsigma}$ (cf. (3.2.5)) is linked to the critical stress by the following relation

$$\bar{\varsigma} = \frac{2L}{E}\bar{\sigma}\,. \tag{4.3.18}$$

**Phase-field profile evolution** The symmetric phase-field profiles are described by the following implicit relation:

$$|x - \bar{x}| = \ell \int_\alpha^{\bar{\alpha}} \sqrt{\frac{\mathsf{h}_\ell(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})\mathsf{w}(\beta) - \mathsf{w}(\bar{\alpha})\mathsf{h}_\ell(\beta)}}\,\mathrm{d}\beta \tag{4.3.19}$$

which scales linearly with the internal length $\ell$. Accordingly, half-width of the support of the phase-field localization is given by

$$D(\bar{\alpha}) = \ell \int_0^{\bar{\alpha}} \sqrt{\frac{\mathsf{h}_\ell(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})\mathsf{w}(\beta) - \mathsf{w}(\bar{\alpha})\mathsf{h}_\ell(\beta)}}\,\mathrm{d}\beta\,. \tag{4.3.20}$$

**Displacement jump (crack opening)** For a one-dimensional tensile problem, the displacement jump amplitude associated with the evolving cohesive crack, that is the crack opening, can easily be determined. From the constitutive relation (4.3.14), the total strain reads

$$\varepsilon = \frac{\sigma}{\mathsf{g}_\ell(\alpha)\,E} = \frac{\sigma}{E}\left(1 + \mathsf{h}_\ell(\alpha)\right)\,. \tag{4.3.21}$$

Integrating (4.3.21) over the entire bar, the right-end global displacement is then given by

$$U = \frac{\sigma}{E}\int_0^L \left(1 + \mathsf{h}_\ell(\alpha(x))\right)\mathrm{d}x = \frac{\sigma}{E}L + \underbrace{2\frac{\sigma}{E}\int_{\bar{x}}^{\bar{x}+D} \mathsf{h}_\ell(\alpha(x))\,\mathrm{d}x}_{=:\,\delta\ \text{(crack opening)}}\,. \tag{4.3.22}$$

The first term in the last expression clearly represents the elastic contribution to the overall displacement. Consequently, the second term corresponds to the crack displacement jump $\delta$. By changing the variable

$x \to \alpha$ and using (4.3.17), the displacement jump as a function of the maximum phase-field value is given by

$$\begin{aligned}
\delta(\bar{\alpha}) &= \frac{2\,\sigma}{E}\int_0^D \mathsf{h}_\ell(\alpha(x-\bar{x}))\,\mathrm{d}x \\
&= \frac{2\,\sigma\,\ell}{E}\int_0^{\bar{\alpha}} \sqrt{\frac{\mathsf{h}_\ell(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})\mathsf{w}(\beta)-\mathsf{w}(\bar{\alpha})\mathsf{h}_\ell(\beta)}}\,\mathsf{h}_\ell(\beta)\,\mathrm{d}\beta \\
&= \frac{4\,G_\mathrm{c}}{c_\mathsf{w}\bar{\sigma}}\sqrt{\frac{\mathsf{w}'(0)}{\mathsf{h}_\ell'(0)}}\int_0^{\bar{\alpha}} \sqrt{\frac{\mathsf{w}(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})\mathsf{w}(\beta)-\mathsf{w}(\bar{\alpha})\mathsf{h}_\ell(\beta)}}\,\mathsf{h}_\ell(\beta)\,\mathrm{d}\beta\,.
\end{aligned} \tag{4.3.23}$$

The ultimate crack opening, that may be finite or infinite, is given by

$$\bar{\delta} \coloneqq \lim_{\alpha\to 1}\delta(\alpha)\,. \tag{4.3.24}$$

**Traction-separation cohesive law** Combining (4.3.16) and (4.3.23) yields the traction-separation law parameterized with respect to the maximum phase-field state:

$$\sigma(\delta)\;:\;\bar{\alpha}\mapsto(\delta(\bar{\alpha}),\sigma(\bar{\alpha}))\;. \tag{4.3.25}$$

Notably, neither $\sigma$ nor $\delta$ depend on $\ell$, unlike models based on (4.3.2).

In Section 4.2, where the construction of a phase-field model associated with a specific cohesive law is explored, the assumption $Q = \omega/\varsigma$ is made throughout, without loss of generality. By applying this assumption to the mechanical model using the relations (4.3.8) and (4.3.18), the degradation function (4.3.10) becomes:

$$\mathsf{g}_\ell(\alpha) = \frac{1}{1+\frac{2\,G_\mathrm{c}\,E}{\ell\,\bar{\sigma}^2}\frac{\mathsf{w}(\alpha)}{\mathsf{l}(\alpha)}}\,. \tag{4.3.26}$$

Then, (4.3.26) explicitly shows how (4.3.4) depends on the critical stress $\bar{\sigma}$ and the fracture toughness $G_\mathrm{c}$, two material parameters with clear mechanical significance that characterize the failure behavior and can be easily identified through standard experiments.

**Dissipated fracture energy** The fracture toughness $G_\mathrm{c}$ is equal to the area of the region of the $(\delta,\sigma)$ plane delimited by the $\sigma$ axis and the curve $\sigma\mapsto\delta(\sigma)$. Indeed

$$\begin{aligned}
\int_0^{\bar{\sigma}}\delta(\sigma)\,\mathrm{d}\sigma &= \int_0^{\bar{\sigma}}\frac{2\,\sigma\,\ell}{E}\int_0^{\bar{\alpha}(\sigma)} \sqrt{\frac{\mathsf{h}_\ell(\bar{\alpha})}{\mathsf{h}_\ell(\bar{\alpha})\mathsf{w}(\beta)-\mathsf{w}(\bar{\alpha})\mathsf{h}_\ell(\beta)}}\,\mathsf{h}_\ell(\beta)\,\mathrm{d}\beta\,\mathrm{d}\sigma \\
&= \frac{2\,\ell}{E}\int_0^1\int_0^{\sigma(\beta)}\frac{\sigma\,\mathsf{h}_\ell(\beta)}{\sqrt{\mathsf{w}(\beta)-\frac{\sigma^2}{K}\mathsf{h}_\ell(\beta)}}\,\mathrm{d}\sigma\,\mathrm{d}\beta \\
&= -\frac{2\,\ell\,K}{E}\int_0^1\left[\sqrt{\mathsf{w}(\beta)-\frac{\sigma^2}{K}\mathsf{h}_\ell(\beta)}\right]_0^{\sigma(\beta)}\mathrm{d}\beta \\
&= \frac{2\,\ell\,K}{E}\int_0^1\sqrt{\mathsf{w}(\beta)}\,\mathrm{d}\beta \\
&= G_\mathrm{c}\,.
\end{aligned} \tag{4.3.27}$$

For all models except Dugdale's, presented in the next section, the fracture toughness corresponds to the dissipated energy. Indeed, in Dugdale's cohesive phase-field model, that will be discussed in Section 4.3.9, a snap-back response occurs during the evolution, causing the dissipated energy to be greater than $G_c$, the dissipated fracture energy. It is worth remarking that even in this case, (4.3.27) remains valid. The surface fracture energy as a function of the crack opening, that is the fracture energy dissipated up to the a given crack opening, will be denoted as $G = G(\delta)$. Within the mathematical framework, the crack opening, the surface fracture energy and the limit stress are respectively denoted as $s$, $g_0(s)$ and $g_0'(s)$.

**Irreversibility** The issue of modeling irreversibility in phase-field models approximating sharp fracture problems is still an open problem. When the phase-field variable is treated as a damage variable, the irreversibility condition (4.3.5) is typically imposed [MMP16]. However, phase-field profiles constructed from (4.3.19) do not always guarantee the fulfillment of the irreversibility condition (4.3.5), as discussed in [Wu24a].

In sharp fracture models, irreversibility applies to the fracture set $\Gamma$, approximated by the region where $\alpha = 1$, ensuring that $\Gamma(t_i) \subseteq \Gamma(t_j), \forall t_i \leq t_j$, where $t$ is the time-evolution parameter (cf. [BFM08]). From this perspective, (4.3.5) may be overly restrictive. Moreover, in cohesive fracture models, irreversibility also influences the unloading behavior in traction-separation laws, an aspect considered even more rarely, [CLO18].

Here, we interpret the phase-field variable primarily as a mathematical regularization tool rather than a strict damage variable, and thus do not enforce (4.3.5) on its evolution. However, this issue will be further discussed in the next section, where different models yielding the same cohesive response suggests the possibility of finding models satisfying (4.3.5).

### 4.3.3 Identifying the phase-field model associated with a prescribed cohesive fracture law

In Section 4.2, we showed how to assign a given cohesive law, either linear or superlinear for small amplitude of the displacement jump, by appropriately selecting the parameters of the phase-field model in (4.3.1). More precisely, by either fixing the local phase-field dissipation potential $\omega$ and choosing an appropriate degradation function $f_\varepsilon(v)$ or viceversa, we define the corresponding functionals $\mathcal{F}_\varepsilon$ and demonstrate that $g_0$ equals the surface energy density of their $\Gamma$-limit $g$, as established in Theorem 3.2.1. The rigorous procedure to construct a phase-field model associated with a prescribed cohesive law is developed in detail in Section 4.2 under the assumptions (Hp 6-8)$_{\mathrm{lin}}$, together with (Hp 1'-4') for the validity of Theorem 3.2.1.

Operatively, the steps required to deduce the phase-field model corresponding to a given cohesive law are:

(i) Consider the physical traction-separation law $\sigma(\delta)$ for a given cohesive law;

(ii) Define the non-dimensional cohesive law $g_0(s)$, adopted within the mathematical framework in Section 4.2, from the relation

$$g_0'(s) \coloneqq \frac{\sigma(k_0\, s)}{\sigma(0)}\,, \tag{4.3.28}$$

with $g_0(0) = 0$ and $k_0 > 0$ being a scaling factor for the non-dimensional displacement jump $s$ which encompasses a characteristic crack opening reference length. Of course, an admissible cohesive law

$g_0(s)$ and the corresponding traction-separation law $g_0'(s)$ must satisfy $(\text{Hp } 6)_{\text{lin}}$ and $(\text{Hp } 7)_{\text{lin}}$;

(iii) Assuming, without loss of generality, $Q(t) = \omega(t)$, determine the functions $l$ and $\omega$, fully characterizing the phase-field regularizing energy functional (4.3.1), by adopting the construction in Theorems 4.2.1 and 4.2.2;

(iv) Fix $k_0$ through (4.3.12);

(v) Reconstruct the material functions $\mathsf{w}$, $\mathsf{l}$ and $\mathsf{g}_\ell$ of the engineering phase-field energy functional through (4.3.8), (4.3.11) and (4.3.26), respectively.

It is worth emphasizing that, for a given admissible cohesive law, multiple combinations of $\{l, \omega\}$ may be deduced. This aspect and its implications will be highlighted throughout the examples in Sec. 4.3.4.

### 4.3.4 Examples of the mechanical response of different cohesive laws

In this section, we present the mechanical response of the uniaxial tensile bar test of Fig. 4.3a for different cohesive-laws considered in Section 4.2.3. The phase-field model associated with each assigned cohesive law is obtained through the construction summarized in Sec. 4.3.3. Specifically we consider throughout this section the following softening cohesive laws: Linear (Fig. 4.4a and Sec. 4.3.5), bi-linear (Fig. 4.4b and Sec. 4.3.6), exponential (Barenblatt like, Fig. 4.4c and Sec. 4.3.7), hyperbolic (Fig.s 4.4d and 4.4e, and Sec. 4.3.8) and Dugdale (Fig. 4.4f and Sec. 4.3.9).

These laws are widely used by the engineering community for describing the fracture process of materials exhibiting a quasi-brittle failure behaviour as, for instance, concrete, fibre-reinforced composites or geological materials [GPE94, RPPG07, PP11, DDMDMP12]. Such quasi-brittle failures often involves a combination of progressive micro-cracking and crack bridging or aggregate interlocking. The geometric and constitutive parameters adopted throughout this section are reported in Tab. 4.1 and are the same as in [FFL21, Section 7.3].

| | |
|---|---|
| $L$ | 200 mm |
| $E$ | 30000 MPa |
| $G_c$ | 0.12 $\frac{\text{N}}{\text{mm}}$ |
| $\bar{\sigma}$ | 3 MPa |
| $\ell$ | variable |

Table 4.1: Geometric and constitutive parameters.

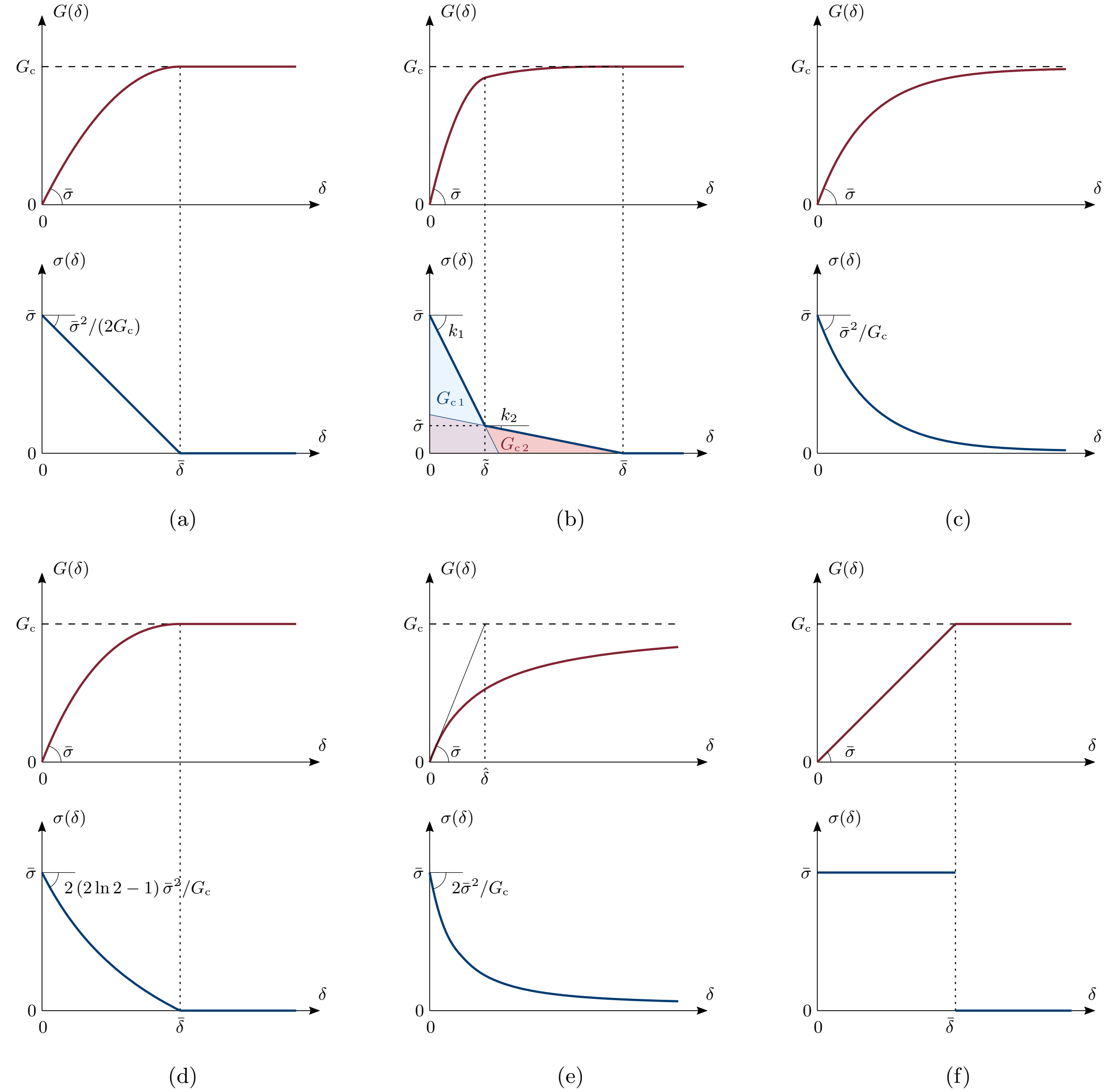



Figure 4.4: Qualitative trends of the considered softening cohesive laws: (a) linear, (b) bilinear, (c) exponential (Barenblatt like), hyperbolic ((d) linear and (e) quadratic) and (f) Dugdale.

### 4.3.5 Linear softening cohesive law

The linear softening cohesive law is completely defined once two of the following three parameters are assigned: the tensile strength $\bar{\sigma}$, the fracture toughness $G_c$ and the ultimate crack opening $\bar{\bar{\delta}}$, Fig. 4.4a. The linear traction-separation law reads

$$\sigma(\delta) = \begin{cases} \bar{\sigma}\left(1 - \dfrac{\delta}{\bar{\bar{\delta}}}\right), & \text{for } \delta \leq \bar{\bar{\delta}} \\ 0, & \text{for } \delta > \bar{\bar{\delta}} \end{cases}, \tag{4.3.29}$$

where the ultimate crack opening is given by $\bar{\delta} = 2\,G_c/\bar{\sigma}$.

Replacing $\delta$ by $k_\delta\, s$, with $k_\delta$ being a generic constant, and fixing $k_s = k_\delta/\bar{\delta}$, (4.3.29) yields, according to (4.3.28), the following linear traction-separation law within the mathematical framework

$$g'(s) = \begin{cases} 1 - k_0 s, & \text{for } s \le \frac{1}{k_0} \\ 0, & \text{for } s > \frac{1}{k_0} \end{cases} . \tag{4.3.30}$$

Thus, $k_s$ encompasses here the characteristic crack opening length $\bar{\delta}$. A similar rescaling procedure will be adopted to all subsequent examples.

In Section 4.2.3, three different models, all describing the same traction-separation law (4.3.30), are derived, sharing the same crack opening scaling factor $k_s = 1/(2k)$. The corresponding engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for these models read:

$$\begin{cases} \mathsf{l}^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}\left(\arccos(\sqrt{t}) + \sqrt{t - t^2}\right)\right)^{2/3} \\ \mathsf{w}(\alpha) = \dfrac{9}{64}\alpha \end{cases} , \tag{$\mathtt{L}_1$}$$

$$\begin{cases} \mathsf{l}^{-1}(t) = 1 - \left(1 - \frac{2}{\pi}\left(\arccos(\sqrt{t}) + \sqrt{t - t^2}\right)\right)^{1/2} \\ \mathsf{w}(\alpha) = \dfrac{\alpha^2}{4} \end{cases} , \tag{$\mathtt{L}_2$}$$

and

$$\begin{cases} \mathsf{l}(\alpha) = (1-\alpha)^2 \\ \mathsf{w}(\alpha) = \dfrac{2\alpha - \alpha^2}{\pi^2} \end{cases} . \tag{$\mathtt{L}_{12}$}$$

The functions $\mathsf{l}(\alpha)$ for models ($\mathtt{L}_1$) and ($\mathtt{L}_2$) do not admit an explicit analytical inverse. As a consequence, interpolation functions to reconstruct $\mathsf{l}(\alpha)$ are hereafter adopted. Models ($\mathtt{L}_1$) and ($\mathtt{L}_2$) are derived by fixing $l$ and deducing, by means of Theorem 4.2.1, $\omega$ whereas, for model ($\mathtt{L}_{12}$), $\omega$ is first fixed and (the inverse of) $l$ deduced by means of Theorem 4.2.2. Model ($\mathtt{L}_{12}$) corresponds to the linear model considered in [FFL21, Section 4.1]. The degradation function $\mathsf{g}_\ell$ and the local material dissipation function $\mathsf{w}$ trends for the three linear models are depicted in Fig. 4.5. A major difference between these three models is the function behaviour of $\mathsf{w}$, namely: linear for ($\mathtt{L}_1$), convex for ($\mathtt{L}_2$) and concave for ($\mathtt{L}_{12}$). It is worth also noting that the local phase-field dissipation potential $\mathsf{w}$ for ($\mathtt{L}_2$), which is essentially the same of the $\mathtt{AT}_2$ phase-field model for brittle fracture [TLB+18], lacks of a linear term. Nevertheless, this model is still able to describe an explicit elastic response associated with a non-vanishing critical stress. Therefore, the presence of a linear term in the local material dissipation function is not a necessary but only sufficient condition for observing an explicit elastic response.

Fig.s 4.6a and 4.6b respectively represent the surface fracture energy and the traction-separation law as a function of the displacement jump for all the considered linear models. As also evident from the global response in Fig. 4.7, which exhibits a linear elastic response followed by a linear softening damage response, the tree models ($\mathtt{L}_1$)-($\mathtt{L}_{12}$) are describing the same cohesive fracture response.

To highlight the role of $\ell$, three different values for the internal length have been considered, namely:

$$\ell = \{10, 5, 1\} \quad (\text{mm}) . \tag{4.3.31}$$

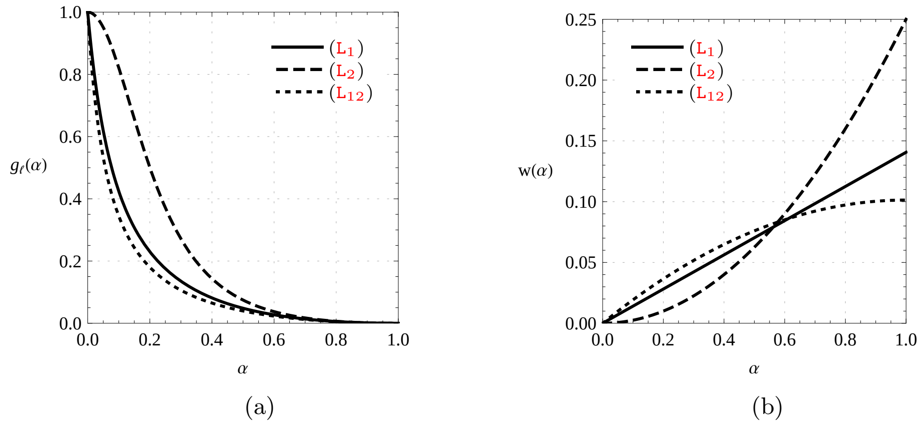



Figure 4.5: (a) Degradation and (b) local dissipation functions for the linear cohesive fracture models ($\mathtt{L}_1$), ($\mathtt{L}_2$) and ($\mathtt{L}_{12}$).

It turns out, as predicted, that the global response is independent of the value of the internal-length, provided it is sufficiently small compared to the domain size, Fig. 4.7. The internal length has instead a significant impact on the phase-field localization size, as evident in Fig. 4.8, which reports the phase-field profiles evolutions, obtained from (4.3.19), for the linear models together with the ultimate phase-field profiles associated with the three internal lengths (4.3.31). The trends of the half-width localization support (4.3.20) with respect to the fracture evolution are highlighted in Fig. 4.9. It is then immediately clear that although the three linear models describe the same cohesive fracture response and global response, the behaviour of their corresponding phase-field profiles is very different. For instance, the phase-field profile for ($\mathtt{L}_{12}$) evolves with a constant and finite support, therefore automatically satisfying the irreversibility condition (4.3.5), whereas for ($\mathtt{L}_2$) the support in unbounded. Clearly, for these examples, the phase-field at the boundaries has been allowed to assume values different from zero. The phase-field profiles for model ($\mathtt{L}_1$) instead have a finite support that slightly shrinks during the evolution.

We also report the evolution of the displacement profile for ($\mathtt{L}_{12}$) where the convergence of the smooth transition towards a sharp is noticeable as the internal length decreases, Fig. 4.10.

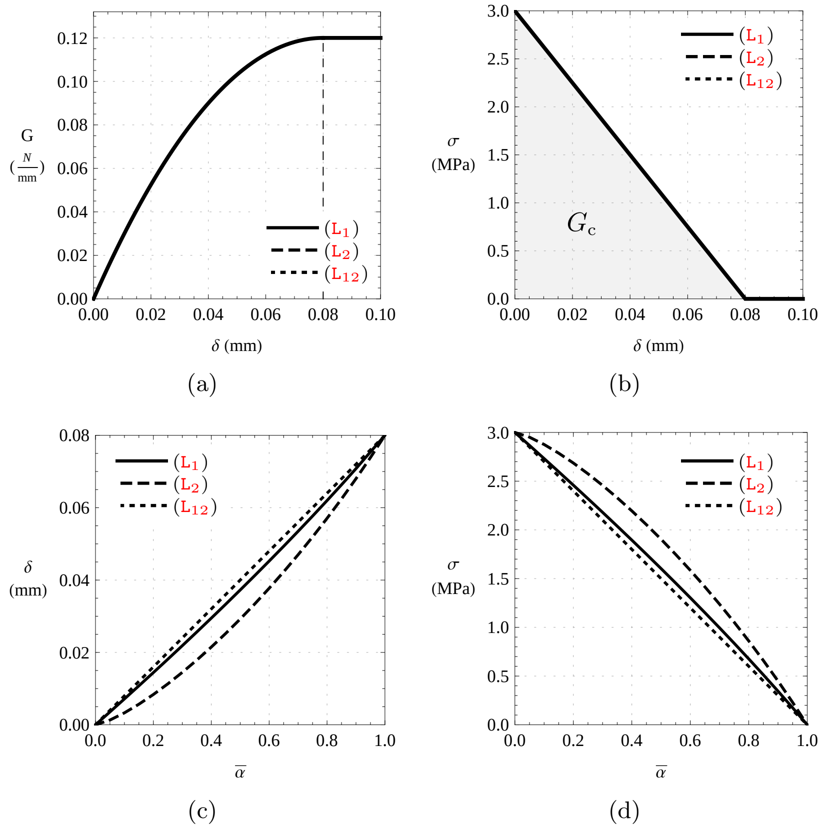



Figure 4.6: (a) Surface fracture energy and (b) traction-separation law as a function of the displacement jump, obtained using the parametrisation (4.3.25) of the displacement jump (c) and the critical stress (d) with respect to the maximum phase-field value for all considered linear models ($\mathtt{L}_1$)–($\mathtt{L}_{12}$).

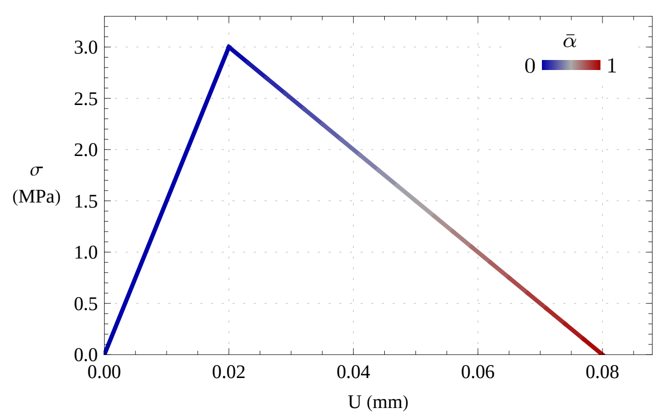
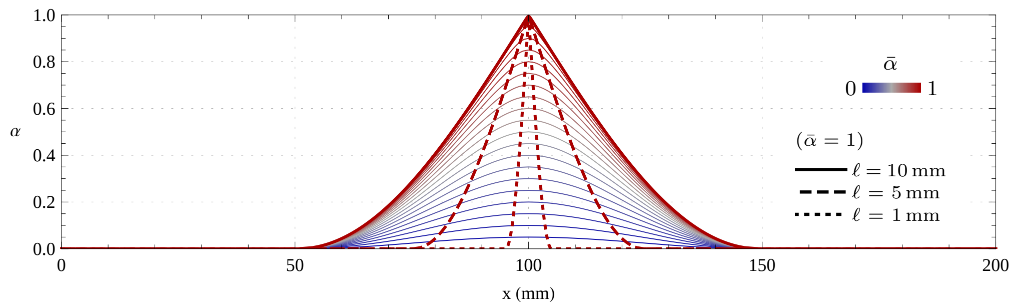
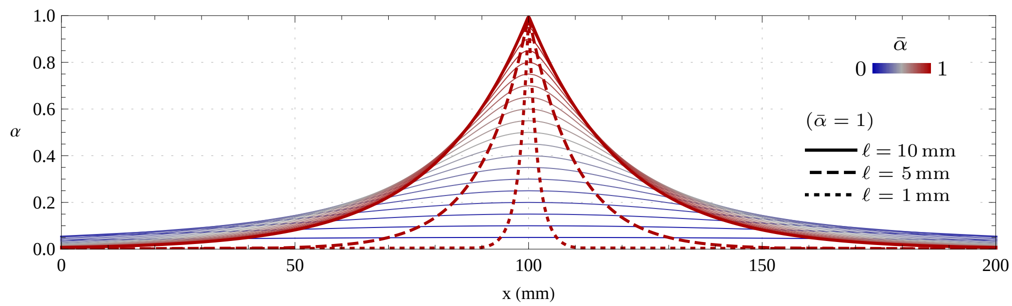


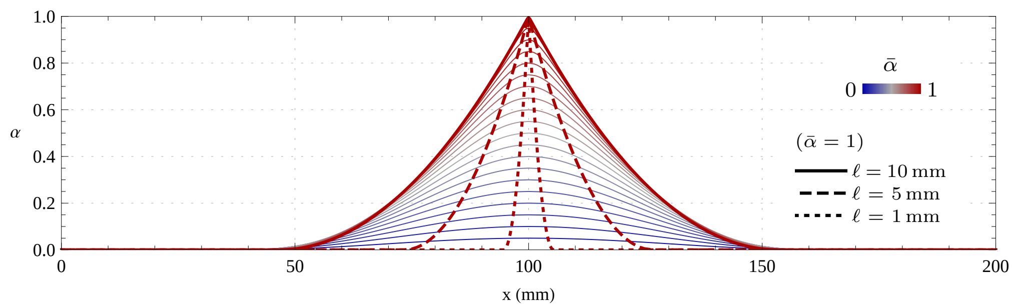

Figure 4.7: Global response of the bar, exhibiting an independent response both with respect to the linear models (($\mathtt{L}_1$), ($\mathtt{L}_2$) and ($\mathtt{L}_{12}$)) and the different internal lengths (4.3.31). The initial elastic response is shown in black. The dependence on the maximum phase-field state is also highlighted.




(a)




(b)




(c)

Figure 4.8: Phase-field profiles at given maximum phase-field states for the linear models: (a) for ($\mathtt{L}_1$), (b) for ($\mathtt{L}_2$) and (c) for ($\mathtt{L}_{12}$). The ultimate phase-field profiles associated with the three internal lengths (4.3.31) are also highlighted.

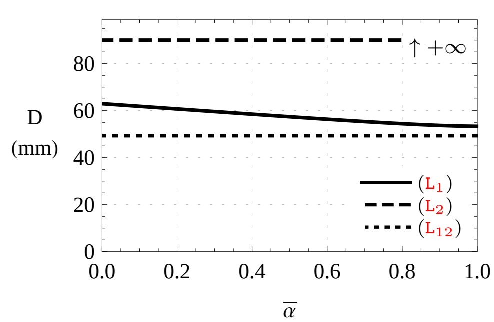



Figure 4.9: Half-width of the support of the phase-field profiles for the three linear models ($\mathtt{L}_1$), ($\mathtt{L}_2$) and ($\mathtt{L}_{12}$) with respect to the fracture evolution ($\ell = 10\,\mathrm{mm}$).

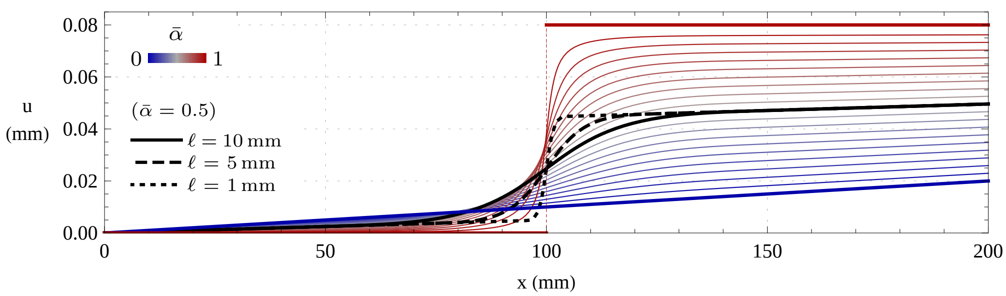



Figure 4.10: Displacement profiles at given maximum phase-field states for ($\mathtt{L}_{12}$) and $\ell = 10\,\mathrm{mm}$. The displacement profiles at $\bar{\alpha} = 0.5$ for smaller internal lengths are also highlighted.

### 4.3.6 Bilinear softening cohesive law

To define the bilinear softening cohesive law of Fig. 4.4b, four parameters are necessary, as, for instance, the tensile strength $\bar{\sigma}$, the fracture toughness $G_{\rm c}$, the ratio between the initial critical stress and the intermediate junction stress, and the ratio between the total fracture toughness and the fracture toughness associated with the first linear behaviour. Accordingly we assign

$$\bar{\sigma}, \qquad G_{\rm c}, \qquad \beta \left(= \frac{\bar{\sigma}}{\tilde{\sigma}}\right), \qquad \gamma \left(= \frac{G_{\rm c}}{G_{\rm c\,1}}\right), \tag{4.3.32}$$

and deduce the rest of the parameter as

$$\begin{gathered} G_{\rm c\,1} = \gamma\, G_{\rm c}, \qquad \tilde{\sigma} = \beta\,\bar{\sigma}, \qquad \tilde{\delta} = (1-\beta)\frac{2\,G_{\rm c\,1}}{\bar{\sigma}}, \qquad \bar{\delta} = 2\frac{(G_{\rm c} - G_{\rm c\,1}(1-\beta))}{\beta\bar{\sigma}}, \\ k_1 = \frac{\bar{\sigma}^2}{2\,G_{\rm c\,1}}, \qquad k_2 = \frac{\beta^2\bar{\sigma}^2}{2\,(G_{\rm c} - G_{\rm c\,1}(1-\beta^2))}, \qquad G_{\rm c\,2} = G_{\rm c} - G_{\rm c\,1}(1-\beta^2)\,. \end{gathered} \tag{4.3.33}$$

The bilinear traction-separation law reads

$$\sigma(\delta) = \begin{cases} \bar{\sigma} - k_1\delta, & \text{for } \delta \leq \tilde{\delta} \\ \tilde{\sigma} - k_2\left(\delta - \tilde{\delta}\right), & \text{for } \tilde{\delta} < \delta \leq \bar{\delta}\,. \\ 0, & \text{for } \delta > \bar{\delta} \end{cases} \tag{4.3.34}$$

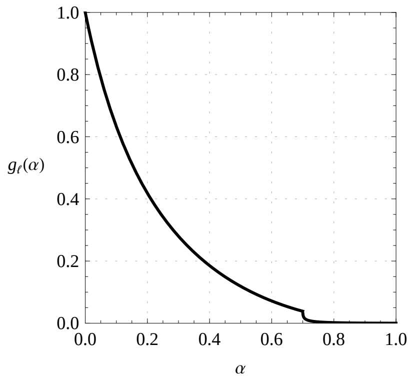
For the third and fourth parameter in (4.3.32) we respectively assume $\beta = 0.3$ and $\gamma = 2.5$, this last being a common value adopted for concrete, [BYZ02]. According to (4.3.28), the bilinear traction-separation law (4.3.34) within the mathematical framework becomes

$$g'(s) = \begin{cases} 1 - k_0\, s, & \text{for } s \leq \tilde{s} \\ 1 - k_0\tilde{s}\,(1-\eta) - \eta k_0\, s, & \text{for } \tilde{s} < s \leq \bar{s}\,. \\ 0, & \text{for } \delta > \bar{\delta} \end{cases} \tag{4.3.35}$$

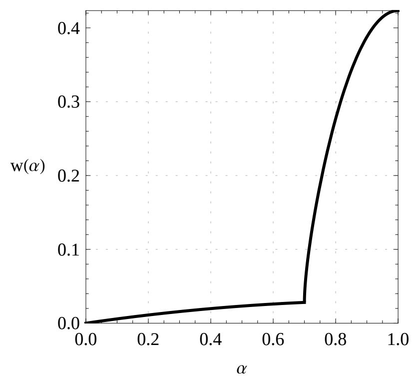
with $\tilde{s} = (1 - \beta/k_0)$, $\bar{s} = (1 - k_0\tilde{s}\,(1-\eta))\,/(\eta k_0)$ and $\eta = k_1/k_2$. In Section 4.2.3, an example of material functions $\{l, \omega\}$ for (4.3.35) is derived by fixing $l$, as ($\mathrm{L}_{12}$), and deducing, by means of Theorem 4.2.1, $\omega$. For the present case, condition (4.3.12) provides as scaling factor $k_0 \simeq 0.91/k$. The corresponding engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for the considered bilinear model read:

$$\begin{cases} \mathsf{l}(\alpha) = (1-\alpha)^2 \\ \mathsf{w}(\alpha) = \dfrac{1}{4k^2} \begin{cases} \dfrac{2\alpha - \alpha^2}{\pi^2 k_0^2}, & \text{for } 0 \leq \alpha < 1-\beta \\ \dfrac{\left(\sqrt{2\alpha - \alpha^2} + (\eta - 1)\sqrt{2\alpha - \alpha^2 + \beta^2 - 1}\right)^2}{\pi^2 k_0^2}, & \text{for } 1-\beta \leq \alpha \leq 1 \end{cases} \end{cases}. \tag{B}$$

The degradation function $\mathsf{g}_\ell$ and the local material dissipation function $\mathsf{w}$ trends for the bilinear model are represented in Fig. 4.11. The surface fracture energy and the traction-separation law as a function of




(a)




(b)

Figure 4.11: (a) Degradation and (b) local dissipation functions for the bilinear cohesive fracture model (B).

the displacement jump for the considered bilinear cohesive model are represented in Fig.s 4.12a and 4.12b, respectively. The global response of Fig. 4.13 clearly reflects the bilinear cohesive fracture behaviour. The initial linear elastic response is indeed followed by a bilinear softening damage response. Also for the bilinear model, the global response does not change with the internal length. The internal length, if sufficiently small compared to the bar length, only affects the phase-field localization size, as evident from the phase-field and displacement field profiles evolutions represented, respectively, in Fig. 4.14 and Fig. 4.16, where the same three internal lengths (4.3.31) of the linear model have been considered. The trend of the half-width phase-field localization support (4.3.20) with respect to the fracture evolution is highlighted in Fig. 4.15. To the best of the author's knowledge, no phase-field model capable of capturing a bilinear softening cohesive response has yet been presented in the literature.

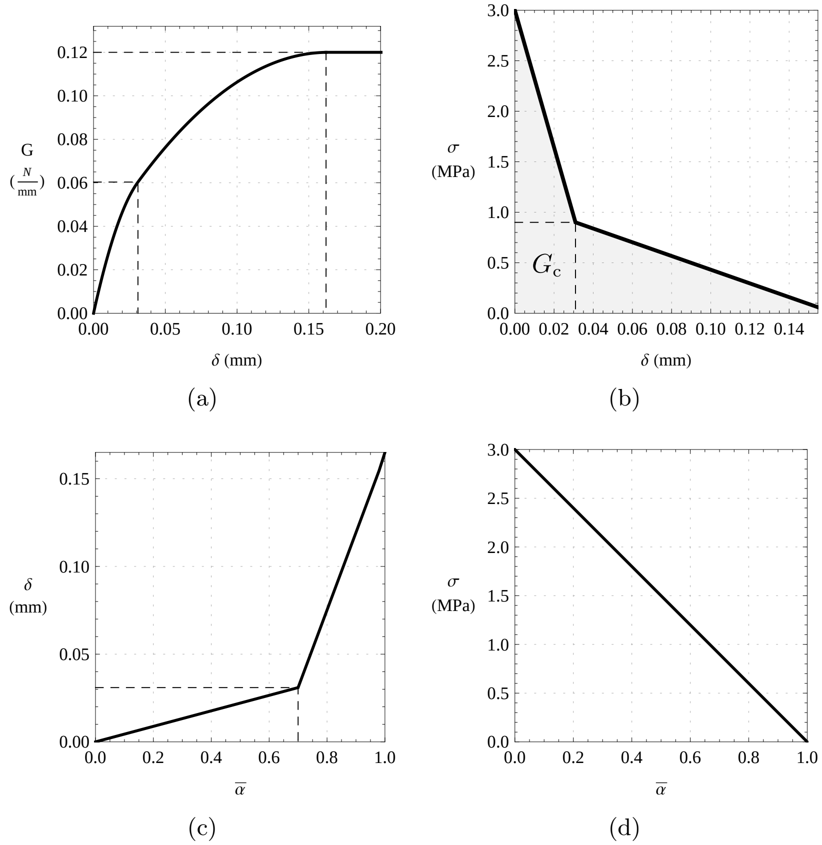



Figure 4.12: (a) Surface fracture energy and (b) traction-separation law of the bilinear model (B) as a function of the displacement jump, obtained using the parametrization (4.3.25) of the displacement jump (c) and the critical stress (d) with respect to the maximum phase-field value.

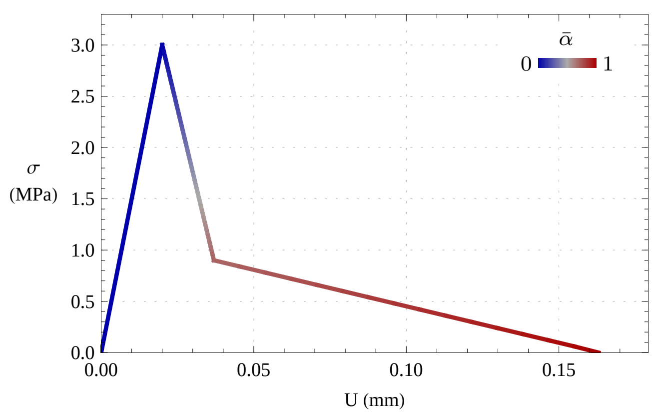



Figure 4.13: Global response for the bilinear model (B), independent of the internal length. The initial elastic response is shown in black. The dependence on the maximum phase-field state is also highlighted.

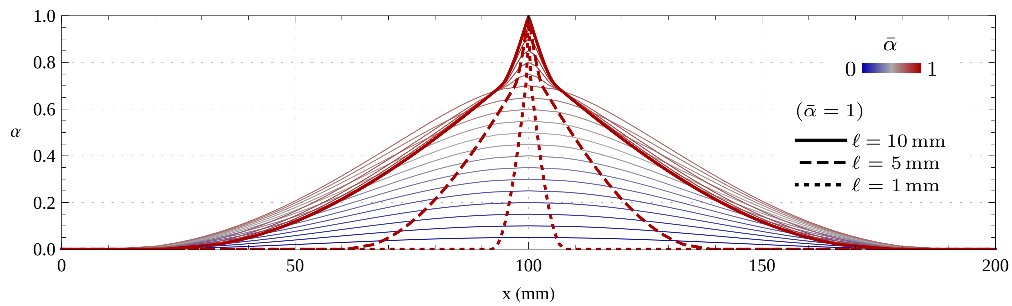



Figure 4.14: Phase-field profiles at given maximum phase-field states for the bilinear model (B). The ultimate phase-field profiles associated with the three internal lengths (4.3.31) are also highlighted.

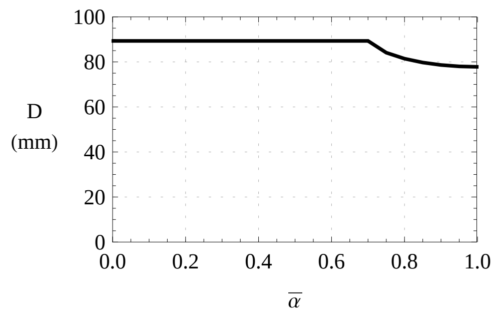



Figure 4.15: Half-width of the support of the phase-field profile for the bilinear model (B) with respect to the fracture evolution ($\ell = 10\,\mathrm{mm}$).

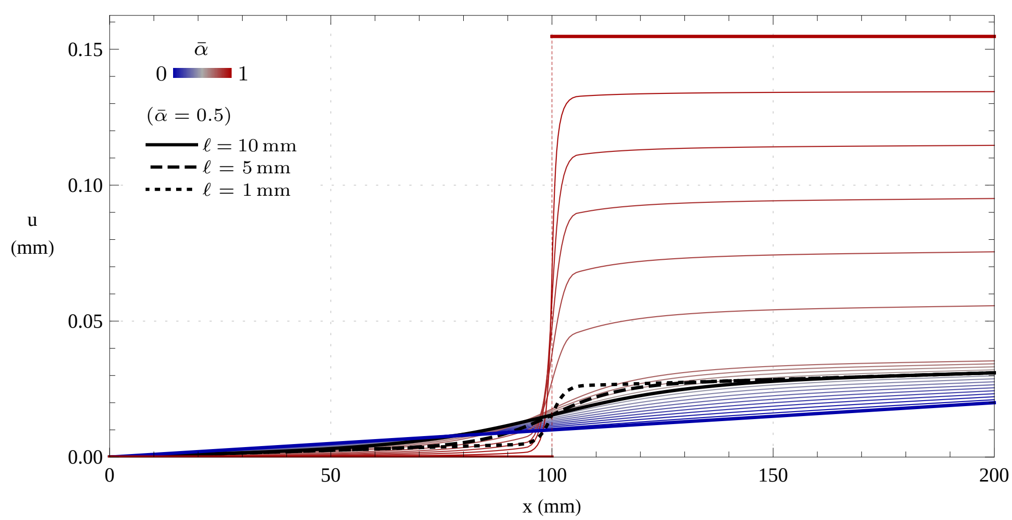



Figure 4.16: Displacement profiles at given maximum phase-field states for the bilinear model (B) with $\ell = 10\,\mathrm{mm}$. The displacement profiles at $\bar{\alpha} = 0.5$ for smaller internal lengths are also highlighted.

### 4.3.7 Exponential softening cohesive law

The exponential or Barenblatt like cohesive law, widely adopted in applications and also considered in [Wu17], is given by the following traction-separation law

$$\sigma(\delta) = \bar{\sigma}\,\exp\left(-\frac{\bar{\sigma}}{G_c}\delta\right). \tag{4.3.36}$$

Its behaviour is fully determined by two parameters, the critical stress and the fracture toughness, and is characterized by an infinite ultimate crack opening (4.3.24), namely $\bar{\bar{\delta}} = +\infty$. According to (4.3.28), the linear traction-separation law within the mathematical framework (4.3.36) becomes

$$g'(s) = \exp\left(-k_0 s\right). \tag{4.3.37}$$

In Section 4.2.3, an example of material functions $\{l, \omega\}$ for (4.3.37) is derived by fixing $l$ and deducing, by means of Theorem 4.2.1, $\omega$, with condition (4.3.12) providing as scaling factor $k_0 = 1/k$. In fact, a slightly modified version of (4.3.37) is considered, to ensure that all the assumptions of Theorem 3.2.1 are satisfied, specifically the continuity of the pair $\{l, \omega\}$. This modified traction-separation law introduces a small parameter that influences the fracture behavior only at very large crack opening displacements. Neglecting this modification, which affects only the asymptotic behavior, the corresponding engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for the exponential model (4.3.36) simplify to:

$$\begin{cases} \mathsf{l}(\alpha) = (1-\alpha)^2 \\ \mathsf{w}(\alpha) = \dfrac{\left(\cosh^{-1}\left(\frac{1}{1-\alpha}\right)\right)^2}{4\,\pi^2} \end{cases}. \tag{E}$$

The following analysis indicates that the impact on practical applications of the modified version of the model is negligible and that the exponential cohesive fracture response (4.3.36) based on the material functions (E) is well captured.

Unlike in [Wu17], where the exponential cohesive law was derived by approximation with fitted polynomial functions, (E) provides an analytical closed-form expression. Being $\lim_{\alpha\to 1} \mathsf{w}(\alpha) = +\infty$, cohesive forces never vanish across the crack.

The degradation function $\mathsf{g}_\ell$ and the local material dissipation function $\mathsf{w}$ trends for the exponential model (E) are represented in Fig. 4.17.

Fig.s 4.18a and 4.18b respectively represent the surface fracture energy and the traction-separation law as a function of the displacement jump for the considered exponential cohesive model.

The global response of Fig. 4.19 clearly reflects the exponential cohesive fracture behaviour. The initial linear elastic response is indeed followed by a concave softening damage response approaching asymptotically zero. Also for the exponential model, the global response does not change with the internal length. The internal length, if sufficiently small compared to the bar length, only affects the phase-field localization size, as evident from the phase-field and displacement field profiles evolutions represented, respectively, in Fig.s 4.20 and 4.22, where the three internal lengths (4.3.31) have been considered. The trend of the half-width phase-field localization support (4.3.20) with respect to the fracture evolution is highlighted in Fig. 4.21.

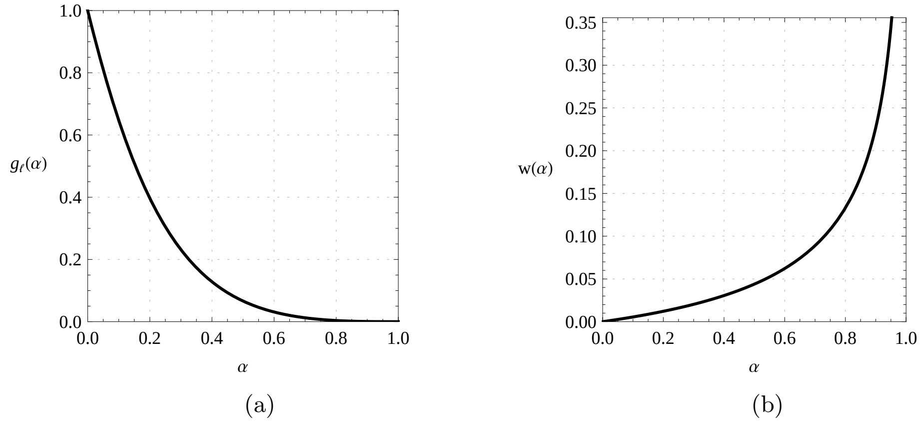



Figure 4.17: (a) Degradation and (b) local dissipation functions for the exponential cohesive fracture model (E).

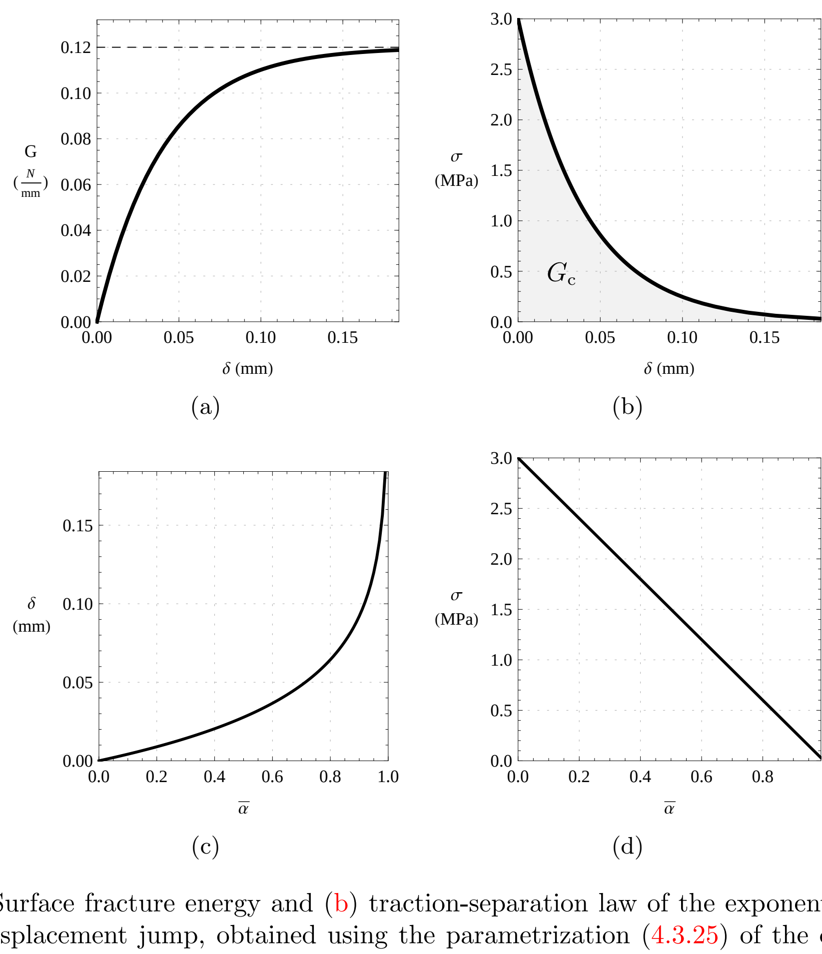



Figure 4.18: (a) Surface fracture energy and (b) traction-separation law of the exponential model (E) as a function of the displacement jump, obtained using the parametrization (4.3.25) of the displacement jump (c) and the critical stress (d) with respect to the maximum phase-field value.

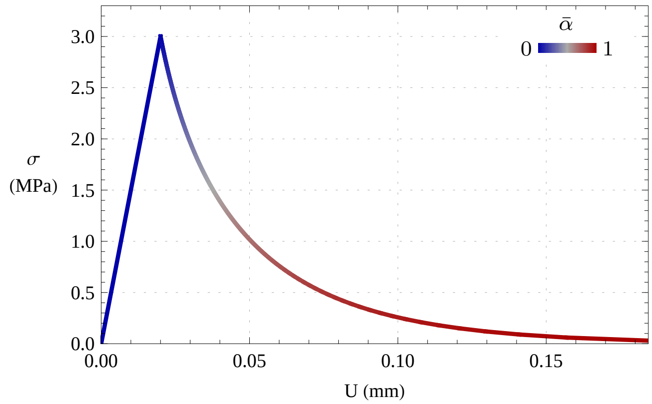



Figure 4.19: Global response for the exponential model (E), independent of the internal length. The initial elastic response is shown in black. The dependence on the maximum phase-field state is also highlighted.

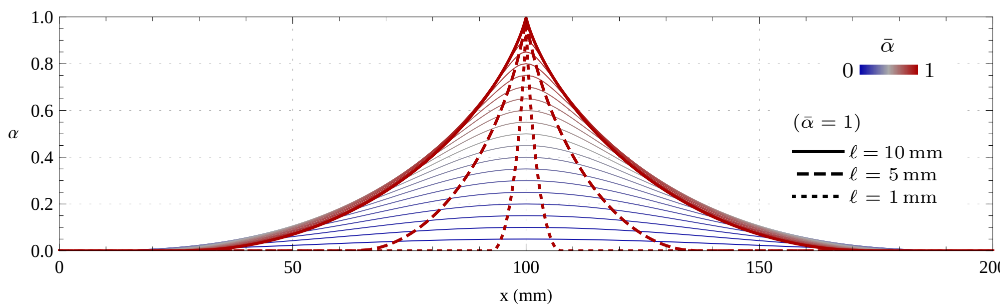



Figure 4.20: Phase-field profiles at given maximum phase-field states for the exponential model (E). The ultimate phase-field profiles associated with the three internal lengths (4.3.31) are also highlighted.

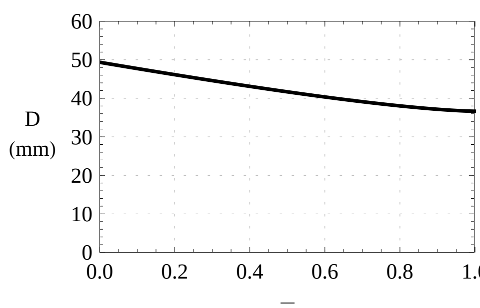



Figure 4.21: Half-width of the support of the phase-field profile for the exponential model (E) with respect to the fracture evolution ($\ell = 5\,\text{mm}$).

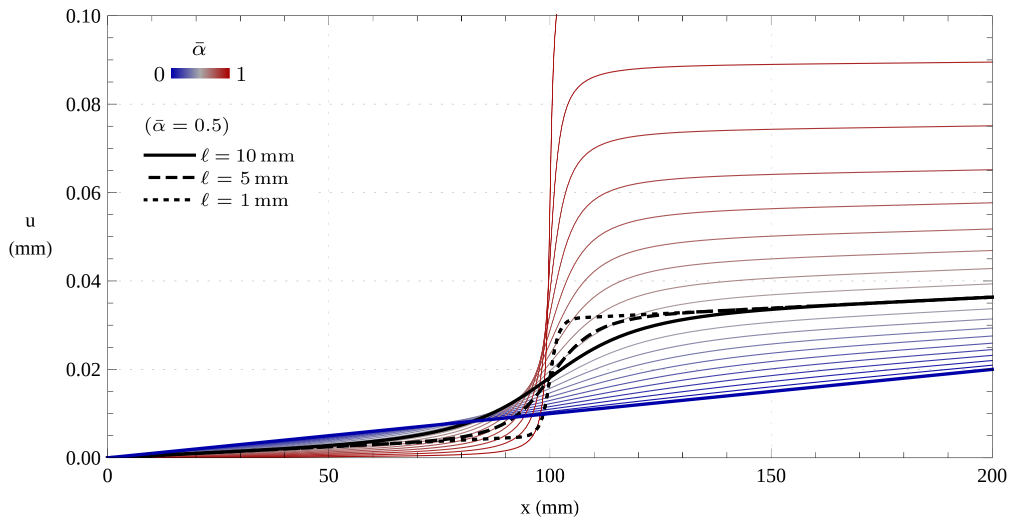



Figure 4.22: Displacement profiles at given maximum phase-field states for the exponential model (E) with $\ell = 5\,\mathrm{mm}$. The displacement profiles at $\bar{\alpha} = 0.5$ for smaller internal lengths are also highlighted.

### 4.3.8 Hyperbolic softening cohesive law

In this section we consider two hyperbolic cohesive fracture models, both defined through only two parameters, which mainly differs for their limit behaviour.

**Linear hyperbolic function** The engineering traction-separation law of the first hyperbolic model is given by

$$\sigma(\delta) = \begin{cases} \bar{\sigma}\left(\dfrac{2}{1+\delta/\bar{\bar{\delta}}} - 1\right), & \text{for } \delta \leq \bar{\bar{\delta}} \\ 0, & \text{for } \delta > \bar{\bar{\delta}} \end{cases}, \tag{4.3.38}$$

with $\bar{\bar{\delta}} = G_c/\left(\bar{\sigma}(2\ln 2 - 1)\right)$ being the ultimate crack opening. According to (4.3.28), the traction-separation law within the mathematical framework for (4.3.38) becomes

$$g'(s) = \begin{cases} \dfrac{2}{1+k_0\, s} - 1, & \text{for } s \leq \frac{1}{k_0} \\ 0, & \text{for } s > \frac{1}{k_0} \end{cases}. \tag{4.3.39}$$

In Section 4.2.3, an example of material functions $\{l, \omega\}$ for (4.3.39) is derived by fixing $l$ and deducing, by means of Theorem 4.2.1, $\omega$, with condition (4.3.12) providing as scaling factor $k_0 \simeq 0.386/k$. The corresponding engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for the considered hyperbolic model read:

$$\begin{cases} \mathsf{l}(\alpha) = (1-\alpha)^2 \\ \mathsf{w}(\alpha) = \dfrac{0.170\,\left(2\alpha - \alpha^2 + 2(1-\alpha)^2 \log\left(1-\alpha\right)\right)^2}{(2-\alpha)^3\alpha^3} \end{cases}. \tag{H1}$$

**Quadratic hyperbolic function** The engineering traction-separation law of the second hyperbolic model is given by

$$\sigma(\delta) = \frac{\bar{\sigma}}{\left(1+\delta/\hat{\delta}\right)^2}, \tag{4.3.40}$$

with $\hat{\delta} = G_c/\bar{\sigma}$. According to (4.3.28), the traction-separation law within the mathematical framework for (4.3.40) becomes

$$g'(s) = \frac{1}{(1+k_0\, s)^2}. \tag{4.3.41}$$

In Section 4.2.3, an example of material functions $\{l, \omega\}$ for (4.3.41) is derived by fixing $l$ and deducing, by means of Theorem 4.2.1, $\omega$, with condition (4.3.12) providing as scaling factor $k_0 \simeq 1/k$. The corresponding engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for the considered hyperbolic model read:

$$\begin{cases} \mathsf{l}(\alpha) = (1-\alpha)^2 \\ \mathsf{w}(\alpha) = \dfrac{\left({}_2F_1\left(-1/4, 1, 1/2, 2\alpha - \alpha^2\right) - (1-\alpha)^2\, {}_2F_1\left(3/4, 1, 1/2, 2\alpha - \alpha^2\right)\right)^2}{(4\pi)^2\left(2\alpha - \alpha^2\right)} \end{cases}, \tag{H2}$$

with ${}_2F_1$ being the hypergeometric function.

The main difference between the two hyperbolic models is that model (4.3.38) owns a finite ultimate crack opening whereas (4.3.40) does not. The second model has also been considered in [Wu17], although with material functions obtained by a fitting procedure. None of the two hyperbolic models have been instead considered in [FFL21].

In Fig. 4.23, the degradation function $\mathsf{g}_\ell$ and the local material dissipation function $\mathsf{w}$ trends for the two hyperbolic models are depicted. Fig.s 4.24a and 4.24b respectively represent the surface fracture energy and

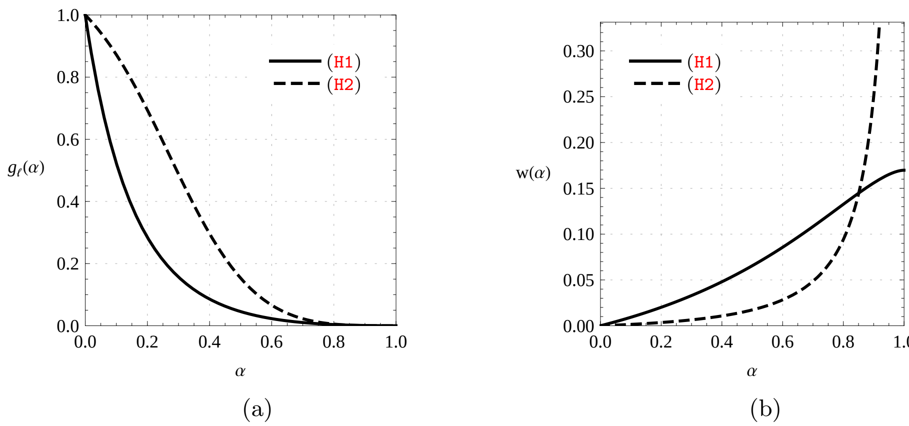



Figure 4.23: (a) Degradation and (b) local dissipation functions for the hyperbolic cohesive fracture models.

the traction-separation law as a function of the displacement jump for all the considered hyperbolic models. Their global responses, which again are independent from the internal length, are compared in Fig. 4.25.

To highlight the influence of the $\ell$, a set of three different values for the internal length for the two hyperbolic models have been considered:

$$(\texttt{H1}):\ \ell = \{10, 5, 1\} \quad (\mathrm{mm})\,, \tag{4.3.42a}$$

$$(\texttt{H2}):\ \ell = \{5, 2.5, 0.5\} \quad (\mathrm{mm})\,. \tag{4.3.42b}$$

Fig. 4.27 reports the phase-field profiles evolutions, obtained from (4.3.19), together with the ultimate phase-field profiles associated with the internal lengths (4.3.42). The trends of the half-width localization support (4.3.20) with respect to the fracture evolution are highlighted in Fig. 4.26.

The evolution of the displacement profiles for (H1) and (H2) are reported in Fig. 4.28. Therein, the convergence of the smooth displacement profiles towards profiles with a discontinuity (the crack) is noticeable as the internal length decreases.

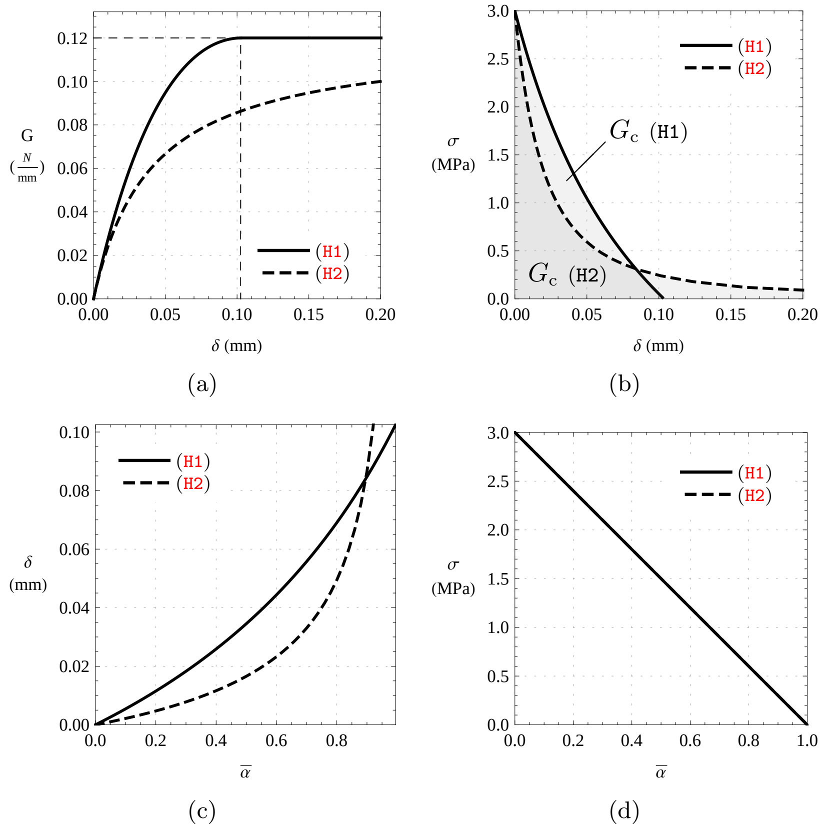



Figure 4.24: (a) Surface fracture energy and (b) traction-separation law as a function of the displacement jump, obtained using the parametrisation (4.3.25) of the displacement jump (c) and the critical stress (d) with respect to the maximum phase-field value for the considered hyperbolic models (H1) and (H2).

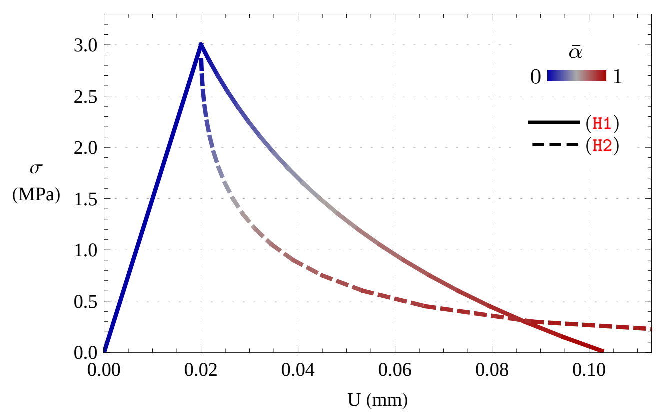



Figure 4.25: Global response of the bar for the hyperbolic models ((H1) and (H2)), independent of the internal length. The initial elastic response is shown in black. The dependence on the maximum phase-field state is also highlighted.

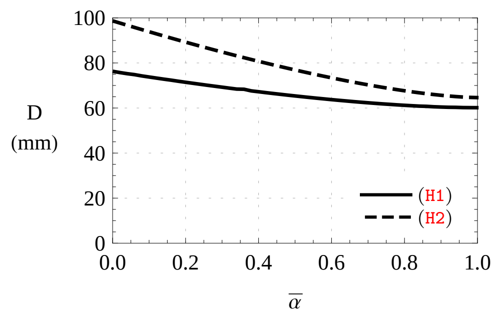
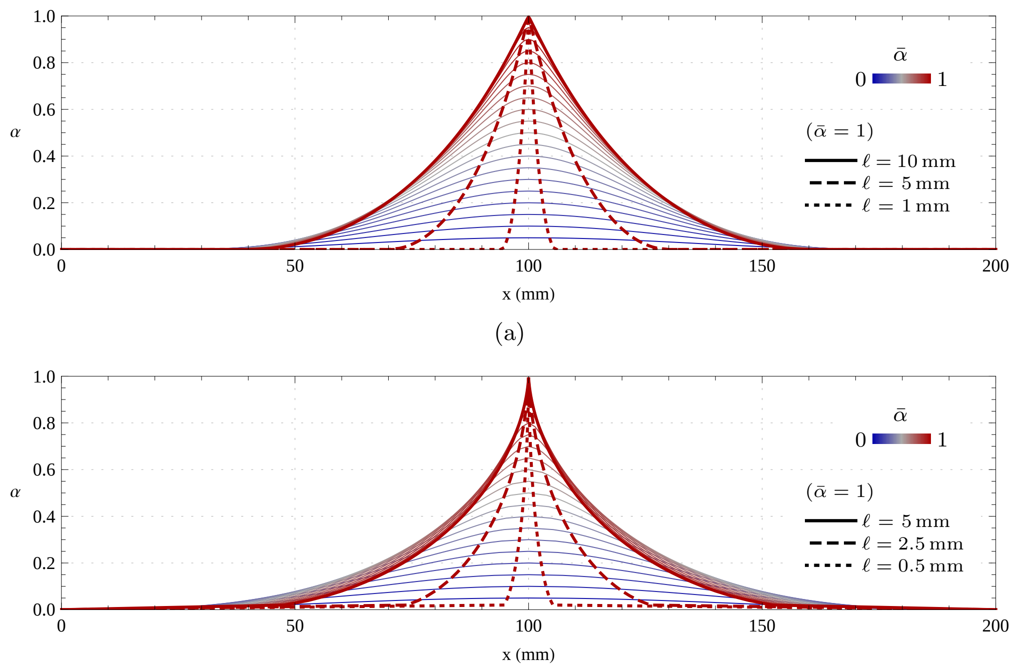


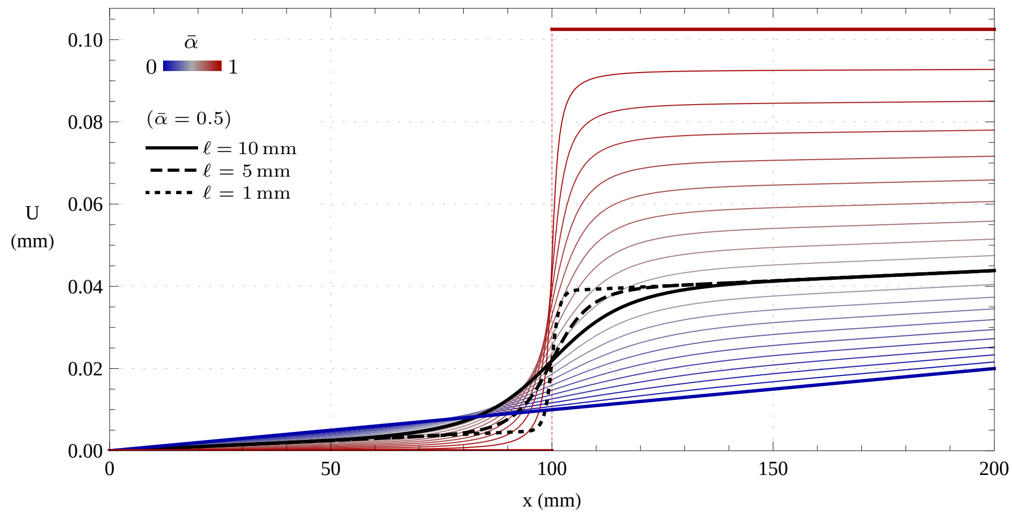

Figure 4.26: Half-width of the support of the phase-field profiles for the hyperbolic models with respect to the fracture evolution ((H1): $\ell = 10\,\mathrm{mm}$ and (H2): $\ell = 5\,\mathrm{mm}$).



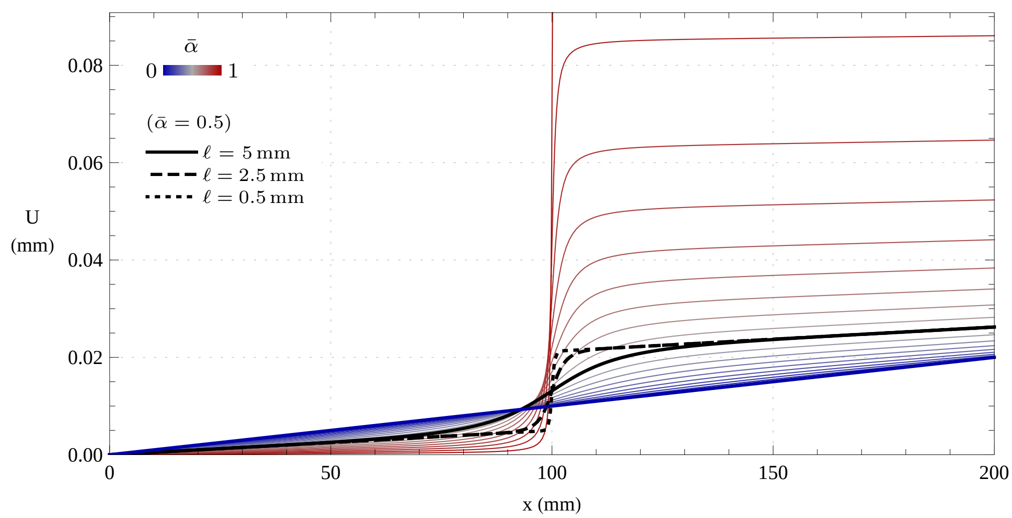

(a)

(b)

Figure 4.27: Phase-field profiles at given maximum phase-field states for the hyperbolic models: (a) for (H1) and (b) for (H2). The ultimate phase-field profiles associated with different internal lengths are also highlighted.




(a)




(b)

Figure 4.28: Displacement profiles at given maximum phase-field states for the hyperbolic models: (a) for (H1) with $\ell = 10\,\mathrm{mm}$ and (b) for (H2) with $\ell = 5\,\mathrm{mm}$. The displacement profiles at $\bar{\alpha} = 0.5$ for smaller internal lengths are also highlighted.

### 4.3.9 Dugdale's cohesive law

As the linear model, Dugdale cohesive law is completely defined once two of the following three parameters are defined: the tensile strength $\bar{\sigma}$, the fracture toughness $G_c$ and the ultimate crack opening $\bar{\delta}$ since $G_c = \bar{\delta}\,\bar{\sigma}$, Fig. 4.4f. The Dugdale traction-separation law reads

$$\sigma(\delta) = \begin{cases} \bar{\sigma}, & \text{for } \delta \leq \bar{\delta} \\ 0, & \text{for } \delta > \bar{\delta} \end{cases}, \tag{4.3.43}$$

where the ultimate crack opening is given by $\bar{\delta} = G_c/\bar{\sigma} = 0.04\,\text{mm}$ with the constitutive parameters of Tab. 4.1. According to (4.3.28), the linear traction-separation law within the mathematical framework (4.3.43) becomes

$$g'(s) = \begin{cases} 1, & \text{for } s \leq k_0 \\ 0, & \text{for } s > k_0 \end{cases}. \tag{4.3.44}$$

Taking advantage of the results of Theorem 4.1.16, to which Dugdale model applies, two different models have been derived, both describing the same Dugdale cohesive behaviour and sharing the same scaling factor $k_0 = k$. Compared to Theorems 4.2.1 and 4.2.2, provides a slightly less constructive procedure for the deduction of the material functions $l$ and $\omega$ of the phase-field model. The engineering material functions $\mathsf{l}$ and $\mathsf{w}$ for the present two models read:

$$\begin{cases} \mathsf{l}^{-1}(t) = 1 - \left(1 - \dfrac{2}{\pi}\left(\arcsin(\sqrt{t}) - \sqrt{t - t^2}\right)\right)^{2/3} \\ \mathsf{w}(\alpha) = \dfrac{9}{64}\alpha \end{cases}, \tag{$\mathtt{D}_1$}$$

and

$$\begin{cases} \mathsf{l}^{-1}(t) = 1 - \left(1 - \dfrac{2}{\pi}\left(\arcsin(\sqrt{t}) - \sqrt{t - t^2}\right)\right)^{1/2} \\ \mathsf{w}(\alpha) = \dfrac{\alpha^2}{4} \end{cases}. \tag{$\mathtt{D}_2$}$$

Both models are derived by prescribing $\omega$, linear in ($\mathtt{D}_1$) and quadratic in ($\mathtt{D}_2$), and assuming $\Psi$, from which (the inverse of) $l$ was obtained (cf. Section 4.2.3). The function $\Psi$, defined in (3.1.27), is chosen so that the auxiliary function $\Lambda$ in (4.1.24) corresponds to $\Lambda(x) = 2\,k_0\,l^{1/2}(x)$ (cf. Theorem 4.1.16).

As for the linear models, the presence or absence of a linear term in the phase-field local dissipation function is not a necessary condition for an explicit elastic response. The function $\mathsf{l}(\alpha)$ for models ($\mathtt{D}_1$) and ($\mathtt{D}_2$) does not admit an explicit analytical inverse. As a consequence, an interpolation function to reconstruct $\mathsf{l}(\alpha)$ is hereafter adopted, similar to what have been done for ($\mathtt{L}_1$) and ($\mathtt{L}_2$). The degradation function $\mathsf{g}_\ell$ and the local material dissipation function $\mathsf{w}$ trends for the two models are depicted in Fig. 4.29. Therein the influence of the internal length is also highlighted.

Fig.s 4.30a and 4.30b respectively represent the traction-separation laws as a function of the displacement jump for the two considered Dugdale models. The link between these responses with the maximum value of the phase-field is also provided, deduced from the parametrisation (4.3.25) of the displacement jump (Fig. 4.30c) and the critical stress (Fig. 4.30d).

Differently from the previous cohesive fracture responses, the displacement jump is not monotonically

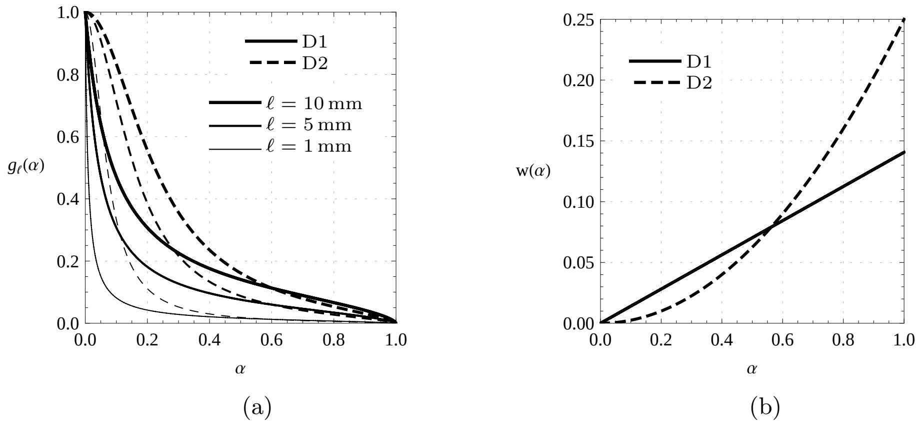



Figure 4.29: (a) Degradation and (b) local dissipation functions for the Dugdale cohesive fracture models ($\mathtt{D}_1$) and ($\mathtt{D}_2$), for different internal lengths.

increasing with respect to the maximum phase-field value. Let us identify the critical crack opening, that is the maximum observable crack opening associated with non-vanishing cohesive forces, and the associated critical phase-field value as

$$\delta^* \coloneqq \max_\alpha \delta(\alpha) \qquad \text{and} \qquad \alpha^* \coloneqq \arg\max_\alpha \delta(\alpha)\,. \tag{4.3.45}$$

As a consequence, the energy dissipated within the bar is not equal to the fracture toughness, although (4.3.27) continues to hold, but rather given by $G = G(\delta^*) = G_\mathrm{c}\,\delta^*$. Indeed, the global responses in Fig.s 4.31a and 4.31b highlights the occurrence of a snap back response. At the critical state $\{U^*, \delta^*, \alpha^*\} \simeq \{0.101\,\mathrm{mm}, 0.081\,\mathrm{mm}, 0.186\}$ for ($\mathtt{D}_1$) and $\{U^*, \delta^*, \alpha^*\} \simeq \{0.120\,\mathrm{mm}, 0.100\,\mathrm{mm}, 0.120\}$ for ($\mathtt{D}_2$), the phase-field variable suddenly localizes to its optimal profile for $\bar{\alpha} = 1$. Except for the critical displacement value, the two models describe the same global response. The phenomenology is essentially equal to that one observed in the same problem for $\mathtt{AT}_1$ and $\mathtt{AT}_2$ phase-field brittle fracture models, [TLB+18], where an instable brittle fracture occurs at the peak load. Nevertheless, the present phase-field approximations correctly describe the sought cohesive response in terms of crack opening–limit stress evolution and global response.

To further understand the model behaviour, let us highlight in Fig. 4.32 the energetic contributions to the crack density function during the crack evolution for ($\mathtt{D}_1$). As noticeable from Fig. 4.32a, the dissipated fracture energy is initially linearly increasing with respect to the crack opening, consistent with the expected Dugdale cohesive response up to the critical crack opening. During this stage, the local term is the dominant contribution. Fig. 4.32b highlights the initial negligible contribution of the gradient term to the crack density function. For a further phase-field evolution up to $\bar{\alpha} = 1$, the analytical crack-opening computed in (4.3.23) decreases to zero, with a decrease of the local term contribution and a simultaneous significant increase of the gradient term such that the total dissipated fracture energy remains constant. It is worth remarking that this second stage, although analytically well defined, is not observable for a monotonically increasing bar-end displacement, since, after the critical crack opening, only the occurrence of a brutal crack is compatible through a snap-back response. It is worth noting that the dissipated fracture energy is always non decreasing, therefore satisfying the second law of thermodynamics, although its single contributions, as the local term, are not. This fact allow us to remark that the gradient phase-field term must be considered as a contribution

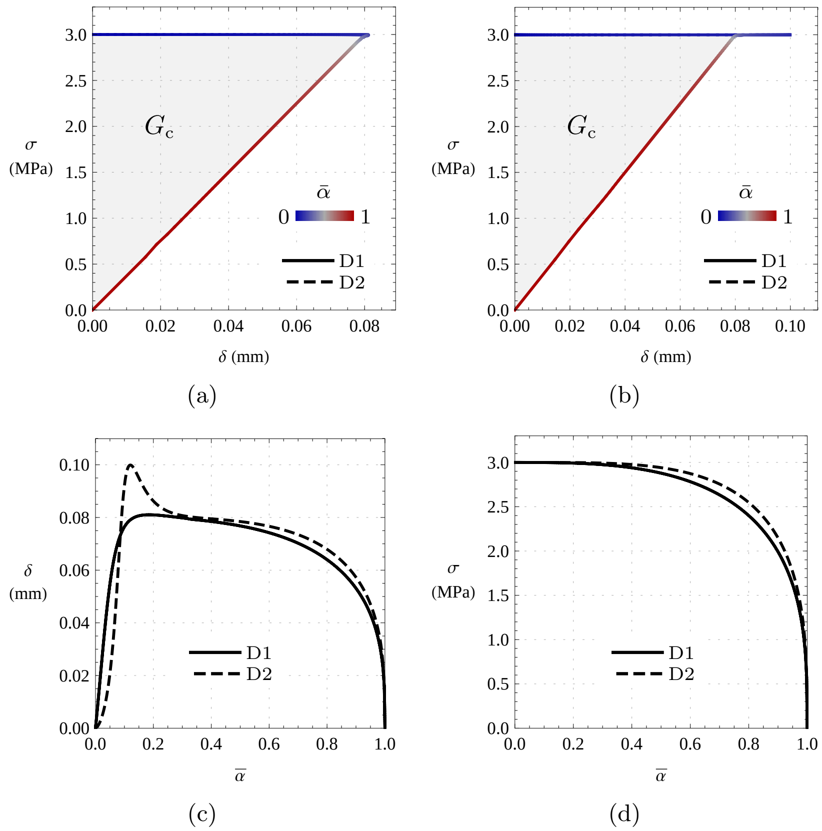



Figure 4.30: Traction-separation laws (a) and (b) for, respectively ($\mathtt{D}_1$) and ($\mathtt{D}_2$) as a function of the displacement jump, obtained using the parametrisation (4.3.25) of the displacement jump (c) and the critical stress (d) with respect to the maximum phase-field value. In (a) and (b) the dependence on the maximum phase-field state is also highlighted.

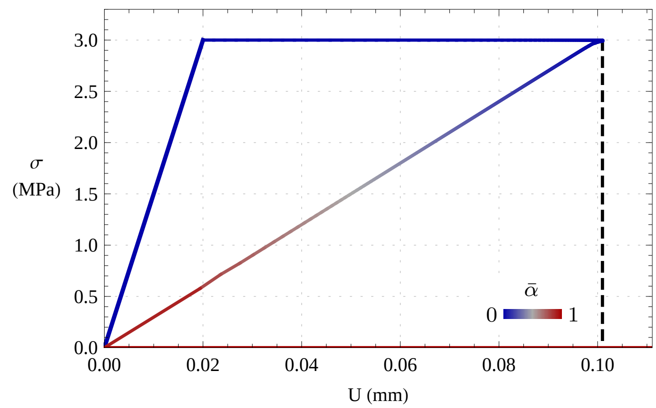
to the dissipated energy and not to the free energy, as instead done in many works.

Fig. 4.33 reports the phase-field profiles evolutions, obtained from (4.3.19), for the two Dugdale models by considering the following internal lengths

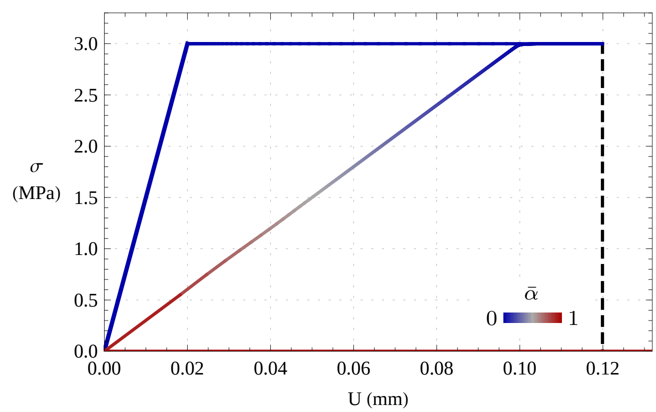
$$\ell = \{1, 0.5, 0.1\} \quad (\mathrm{mm})\,. \tag{4.3.46}$$

Specifically Fig.s 4.33a and 4.33b report the phase-field evolution for the largest internal length together with the ultimate phase-field profiles associated with the smaller internal lengths. Fig. 4.33c highlights the phase-field evolution for the smallest internal length considered in (4.3.46). In all three Fig.s 4.33a-4.33c, the phase-field profile associated with the critical crack opening displacement is also highlighted by a thick curve. The trends of the half-width localization support (4.3.20) with respect to the fracture evolution are highlighted in Fig. 4.34. Two different values of the internal length have been considered, one for each model, in order to let the dimensions be comparable. Indeed, ($\mathtt{D}_1$) admits a finite value for the ultimate crack opening whereas ($\mathtt{D}_2$) does not.

It is then immediately clear that although the two Dugdale models describe the same cohesive fracture response and global response, apart from the critical opening displacement values, the behaviour of their




(a)



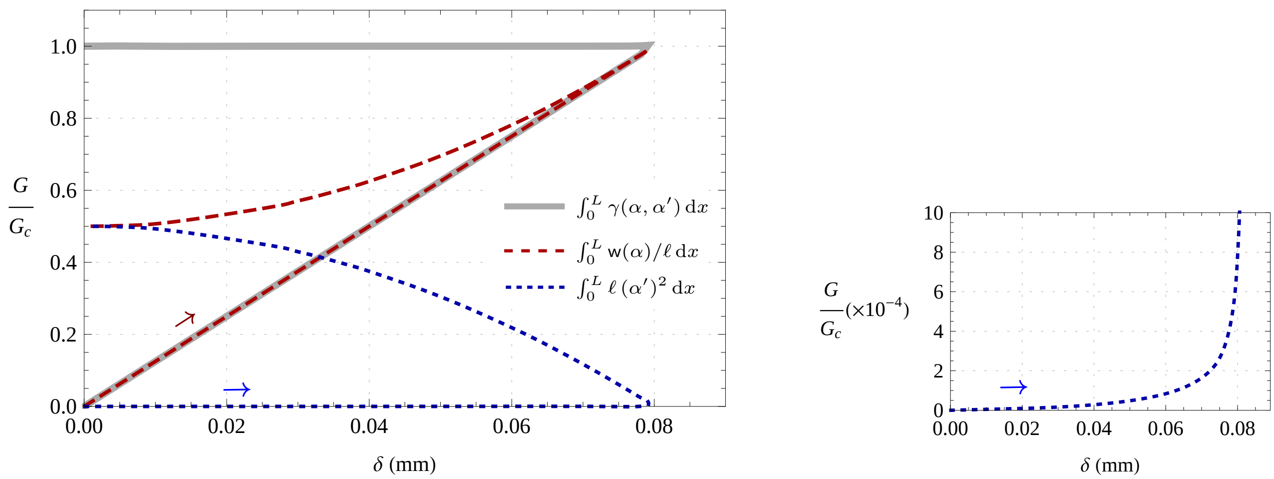

(b)

Figure 4.31: Global responses (a) and (b) for respectively ($\mathtt{D}_1$) and ($\mathtt{D}_2$), independent of the internal length. The initial elastic response is shown in black. The dependence on the maximum phase-field state is also highlighted. At the critical displacement $U^*$, a snap back response occur with the critical stress suddenly dropping from the limit stress value to zero.

corresponding phase-field profiles is not the same. Clearly, for ($\mathtt{D}_2$), the phase-field at the boundaries has been allowed to assume values different from zero. The phase-field profiles for model ($\mathtt{D}_1$) instead have a finite support. For both models we notice that the localization support significantly shrinks after the critical phase-field value as $\bar{\alpha}$ increases. This also explains the behaviour of the crack density function observed in Fig. 4.32a.

The evolution of the displacement profiles for ($\mathtt{D}_1$) and ($\mathtt{D}_2$) are reported in Fig. 4.35. Therein, the convergence of the smooth displacement profiles towards profiles with a discontinuity (the crack) is noticeable as the internal length decreases.

We note that the reconstruction problem for the Dugdale cohesive law was first addressed in [CFI16, Section 7.1] and further investigated in [LCM25] in a finite strain setting for phase-field models based on the truncated degradation function (4.3.2). For such models, a further regularization of the truncated degradation function is required for numerical implementation and analyses [FI17, LCM25]. Moreover, the critical stress depends on the internal regularization length, and the global mechanical response characteristic of a Dugdale-type cohesive law is recovered only in the limit as this length tends to zero (see, for instance,

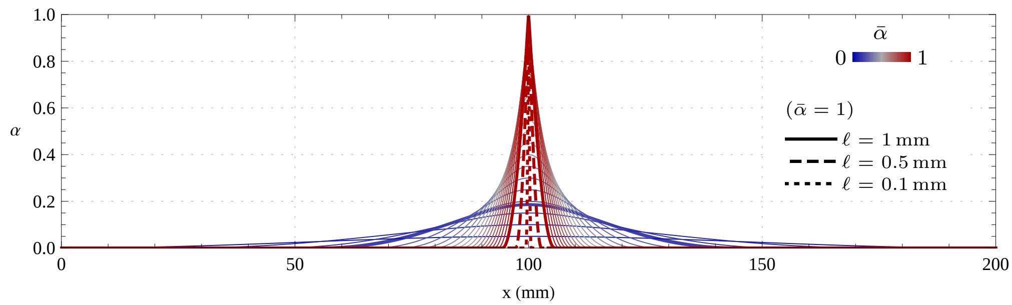



(a) $\ell_1 = 0.05$, $\ell_2 = 0.025$ (b)

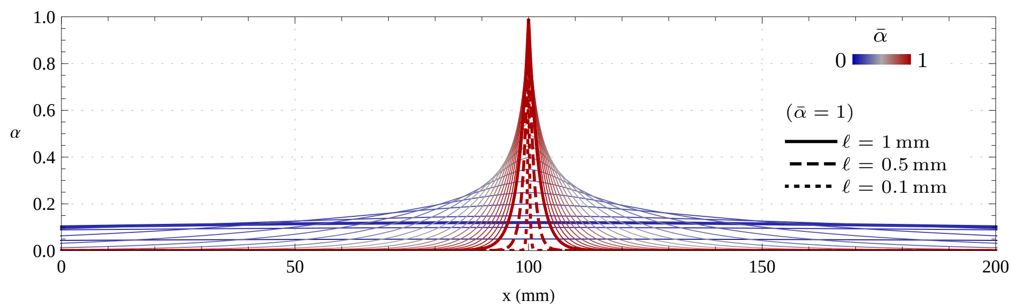
Figure 4.32: (a) crack density function decomposition between the local and gradient contributions for ($\mathtt{D}_1$) and $\ell = 1\,\mathrm{mm}$. (b) zoom on the gradient term. The arrows denote the evolution direction.

Fig. 2 in [LCM25]).

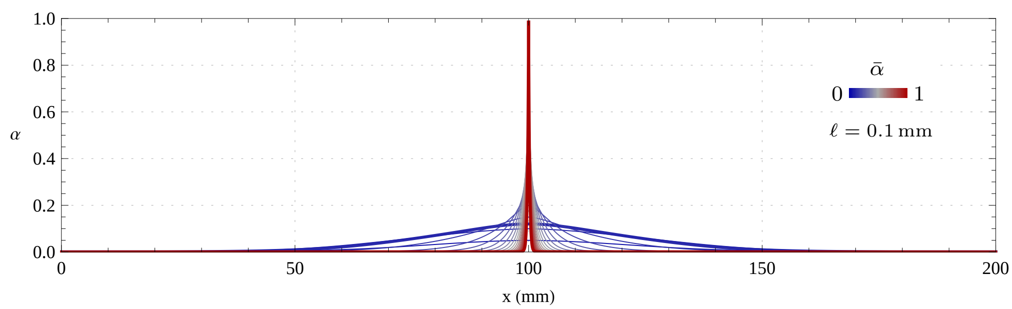
Differently, here we adopt the smooth degradation function (4.3.3), which is directly implementable for numerical simulations. As a result, the (sufficiently small) internal regularization length of the phase-field model does not affect the global mechanical response, but only influences the shape of the phase-field and displacement profiles.




(a)




(b)




(c) $\ell = 0.1$

Figure 4.33: Phase-field profiles at given maximum phase-field states for the Dugdale models: (a) for ($\mathtt{D}_1$) with $\ell = 1\,\mathrm{mm}$, (b) for ($\mathtt{D}_2$) with $\ell = 1\,\mathrm{mm}$ and (c) for ($\mathtt{D}_2$) with $\ell = 0.1\,\mathrm{mm}$. In all subfigures the phase-field profile associated with the critical crack opening displacement is highglighted by a thick line. In (a) and (b) the ultimate phase-field profiles associated with the three internal lengths (4.3.46) are also highlighted.

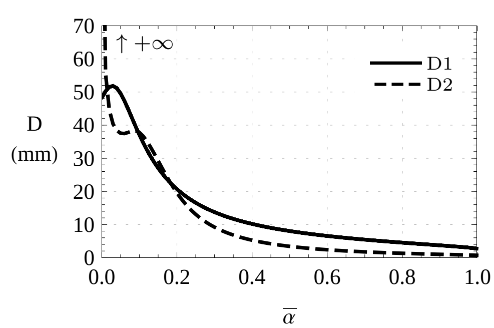
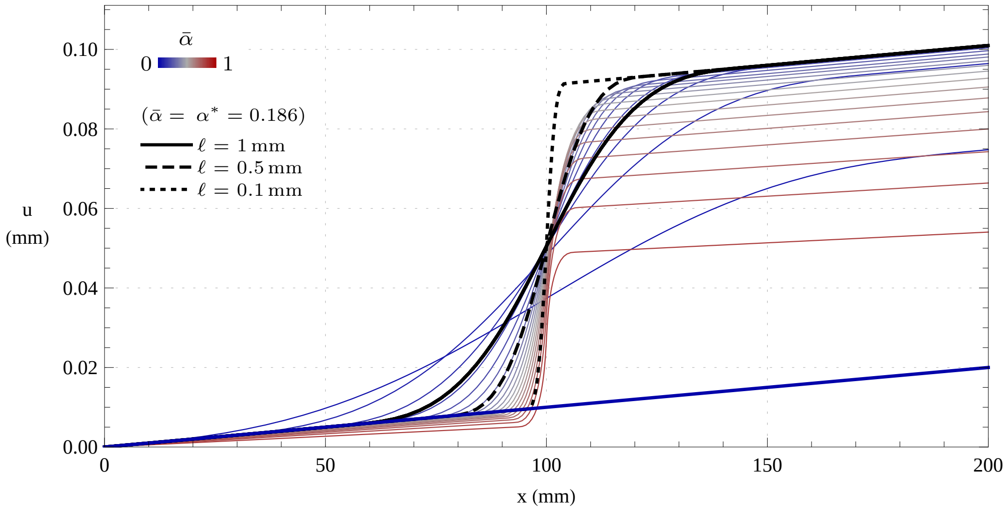
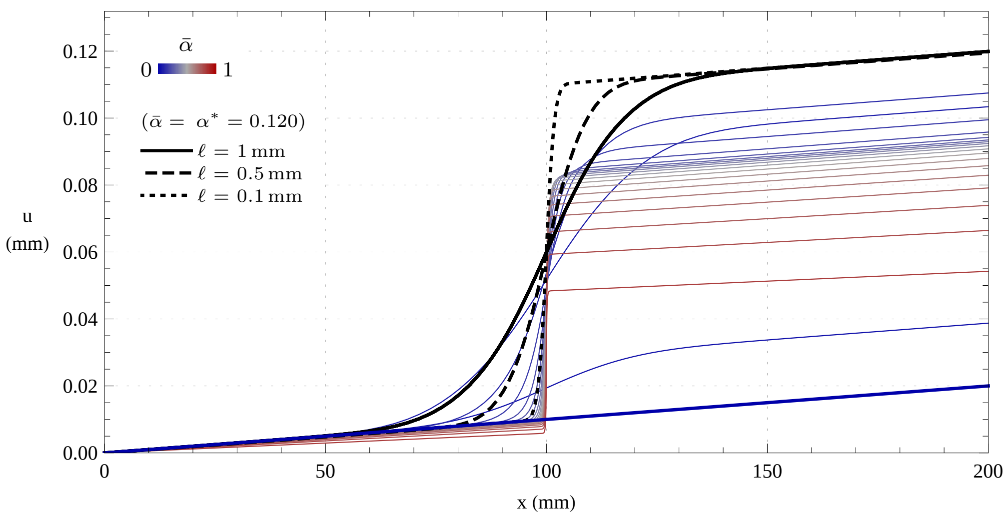



Figure 4.34: Half-width of the support of the phase-field profiles for the two Dugdale models with respect to the fracture evolution ($\ell = 0.5\,\mathrm{mm}$ for ($\mathtt{D}_1$) and $\ell = 0.025\,\mathrm{mm}$ for ($\mathtt{D}_2$)).




(a)




(b)

Figure 4.35: Displacement profiles (a) and (b) at given maximum phase-field states for, respectively, ($\mathtt{D}_1$) and ($\mathtt{D}_2$) with $\ell = 1\,\mathrm{mm}$. The critical displacement profiles at $\bar{\alpha} = \alpha^* = 0.186$ for ($\mathtt{D}_1$) and $\bar{\alpha} = \alpha^* = 0.120$ for ($\mathtt{D}_2$) for smaller internal lengths are also highlighted corresponding to the critical state, where a snap-back of the response occurs. The displacement profiles corresponding to the elastic limit are highlighted with a thick blue line.

# Chapter 5

# Stochastic Homogenization

This chapter is primarily concerned with the stochastic homogenisation (cf. Theorem 5.1.7) of a family of random phase–field functionals $\mathcal{H}_\varepsilon : \Omega \times L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \to [0,\infty]$ defined by

$$\mathcal{H}_\varepsilon(\omega)(u,v;A) = \begin{cases} \displaystyle\int_A \Big(v^2\, f\big(\omega, \tfrac{x}{\varepsilon}, \nabla u\big) + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x, & \text{if } (u,v) \in W^{1,1}(A;\mathbb{R}^N) \times W^{1,2}(A), \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.0.1}$$

where $(\Omega, \mathcal{M}, P)$ denotes the underlying probability space, $\omega \in \Omega$, and the integrand $f$ satisfies the assumptions stated in Definition 5.1.3.

The overall strategy of the chapter is twofold. First, we address a purely deterministic homogenisation problem, establishing a set of sufficient conditions on $f$ that ensure the $\Gamma$–convergence of $\mathcal{H}_\varepsilon(\omega)$ for $P$-a.e. $\omega \in \Omega$. In the second step, we move to the stochastic framework and prove that these deterministic assumptions are automatically satisfied when $f$ is a stationary random integrand. By combining the two parts, we obtain a stochastic homogenisation theorem as a direct consequence of the deterministic analysis.

**Deterministic homogenisation** Here we consider the deterministic phase-field functionals $\mathcal{H}_\varepsilon : L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \to [0,\infty]$ given by

$$\mathcal{H}_\varepsilon(u,v;A) = \begin{cases} \displaystyle\int_A \Big(v^2\, f\big(\tfrac{x}{\varepsilon}, \nabla u\big) + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\Big)\mathrm{d}x, & \text{if } (u,v) \in W^{1,1}(A;\mathbb{R}^N) \times W^{1,2}(A), \\ +\infty, & \text{otherwise,} \end{cases} \tag{5.0.2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,\infty)$ satisfies the assumptions listed in Definition 5.1.1, and we stress here that we do not require any continuity of in the spatial variable, since this would be unnatural for the applications.

Under these general assumptions, using the localisation method of $\Gamma$-convergence [DM93], we can prove the existence of a subsequence $(\varepsilon_j)$ such that, for every $A \subset \mathbb{R}^n$ open and bounded, the functionals $\mathcal{H}_{\varepsilon_j}(\cdot,\cdot,A)$ $\Gamma$-converge to an abstract functional $\widehat{\mathcal{H}}(\cdot,\cdot,A)$. Furthermore, the latter has the property that for every $u \in BV_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ the set function $A \mapsto \widehat{\mathcal{H}}(u,1,A)$ is the restriction to the open subsets of $\mathbb{R}^n$ of a Borel measure (cf. Theorem 5.3.2). We observe that since we do not assume any spatial periodicity of $f$, the continuity of $z \mapsto \widehat{\mathcal{H}}(u(\cdot - z), 1, A+z)$ may fail and therefore we cannot directly use the integral representation result in $BV$ [BFM98] to derive the form of $\widehat{\mathcal{H}}$. Our integral representation result is then obtained under some additional assumptions, which are however more general than periodicity. We require that the limits of some scaled minimisation problems, defined in terms of $f$ and $f^\infty$, exist and are independent of the spatial variable. These limits will then define the volume and surface integrands of $\widehat{\mathcal{H}}$. Eventually, the Cantor integrand will be automatically identified due to the lower semicontinuity of $\widehat{\mathcal{H}}$.

Specifically, we make the two following assumptions. If $Q_r(rx)$ denotes the open cube with side-length $r$ centred at $rx$ (cf. (2.0.4)) and $\ell_\xi(x) := \xi x$, the first assumption amounts to asking that for every $\xi \in \mathbb{R}^{N\times n}$ the limit

$$\lim_{r\to\infty} \frac{1}{r^n} \inf\left\{\int_{Q_r(rx)} f(y,\nabla u)\mathrm{d}y \colon u \in W^{1,1}(Q_r(rx),\mathbb{R}^N),\ u = \ell_\xi \text{ on } \partial Q_r(rx)\right\} \tag{5.0.3}$$

exists and it is independent of $x \in \mathbb{R}^n$. The value of (5.0.3) is denoted by $f_{\mathrm{hom}}(\xi)$.

Moreover, if $Q_r^\nu(rx)$ denotes the open cube with side-length $r$ centred at $rx$ and one side orthogonal to $\nu \in \mathbb{S}^{n-1}$ (cf. (2.0.4)) and $u_{rx,\zeta,\nu}$ is the jump-function defined in (2.0.3), we also require that for every $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ the limit

$$\begin{aligned}\lim_{r\to+\infty} \frac{1}{r^{n-1}} \inf\Big\{ \int_{Q_r^\nu(rx)} \big(v^2 f^\infty(y,\nabla u) + (1-v)^2 + |\nabla v|^2\big)\mathrm{d}y \colon u \in W^{1,1}(Q_r^\nu(rx),\mathbb{R}^N),\\ v \in W^{1,2}(Q_r^\nu(rx)) \text{ and } (u,v) = (u_{rx,\zeta,\nu},1) \text{ on } \partial Q_r^\nu(rx)\Big\}\end{aligned} \tag{5.0.4}$$

exists and is independent of $x \in \mathbb{R}^N$. The value of (5.0.4) is denoted by $g_{\mathrm{hom}}(\zeta,\nu)$.

It is worth mentioning here that $f_{\mathrm{hom}}$ and $g_{\mathrm{hom}}$ satisfy a number of properties (cf. Section 5.2) which ensure, in particular, that they are Borel measurable.

Then, assuming (5.0.3) and (5.0.4) we resort to the blow-up technique in $BV$ [BFM98] to show that for every $u \in BV(A,\mathbb{R}^N)$ the following identities hold true

$$\begin{aligned}
\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) &= f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A\\
\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) &= f^\infty_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } |D^c u|\text{-a.e. } x \in A\\
\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}}(x) &= g_{\mathrm{hom}}([u](x),\nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap A.
\end{aligned}$$

In their turn, these allow us to represent $\widehat{\mathcal{H}}$ in an integral form first on $BV$, and then by standard truncation arguments on $GBV$ as well. Furthermore, since in the equalities above the right-hand side does not depend on the subsequence $(\varepsilon_j)$, under assumptions (5.0.3) and (5.0.4) we obtain a homogenisation result for the whole sequence $(\mathcal{H}_\varepsilon)_\varepsilon$ (see Theorem 5.3.1).

**Stochastic homogenisation** Here we consider an underlying probability space $(\Omega,\mathcal{M},P)$ and allow the integrand $f$ to additionally depend on $\omega \in \Omega$ in a suitable measurable way. Then, if $f$ is a *stationary random integrand* in the sense of Definition 5.1.4, we show that assumptions (5.0.3) and (5.0.4) are automatically satisfied for $P$-a.e. $\omega \in \Omega$, that is *almost surely*. As it is by-now costumery (see [CDMSZ19b, CDMSZ22]) this is done by appealing to the Ackoglu and Krengel Subadditive Ergodic Theorem [AK81].

More specifically, the proof that (5.0.3) holds is standard and follows as in [RZ23]. On the other hand, the verification of (5.0.4) is highly non trivial, as it is always the case when working with "surface terms" where there is a dimensional mismatch between the domain of integration and the scaling, and, moreover, a boundary datum which is inherently inhomogeneous (cf. 5.0.4).

Once (5.0.3) and (5.0.4) are shown to hold we can immediately resort to the deterministic analysis to deduce that the random functionals $\mathcal{H}_\varepsilon(\omega)$ (cf. (5.0.1)) homogenise, almost surely, to the random, autonomous free-discontinuity functional

$$\mathcal{H}_{\text{hom}}(\omega)(u,v,A) = \int_A f_{\text{hom}}(\omega,\nabla u)\,\mathrm{d}x + \int_A f^\infty_{\text{hom}}(\omega,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u| + \int_{J_u\cap A} g_{\text{hom}}(\omega,[u],\nu_u)\mathrm{d}\mathcal{H}^{n-1},$$

if $u \in (GBV(A))^N$ and $v = 1$ $\mathcal{L}^n$-a.e in $A$, where $f_{\text{hom}}$ and $g_{\text{hom}}$ are defined, respectively, by (5.0.3) and (5.0.4) while $f^\infty_{\text{hom}}$ is the recession function of $f_{\text{hom}}$ (see Theroem 5.1.7).

## 5.1 Setting and Main Result

In this section we introduce the class of the admissible random integrands for the definition of the rigorous definition of the random phase-field functionals $\mathcal{H}_\varepsilon$ (cf. (5.0.1)).

**Definition 5.1.1** (Admissible integrand)**.** *Let $C \geq 1$ and $\alpha \in (0,1)$ be given, then $\mathcal{AI}(C,\alpha)$ denotes the collection of all functions $f : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,+\infty)$ with the following properties:*

*(f1)* (measurability) *$f$ is $\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\mathbb{R}^{N\times n})$-measurable;*

*(f2)* (linear growth) *for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{N\times n}$*

$$C^{-1}|\xi| \leq f(x,\xi) \leq C(|\xi|+1);$$

*(f3)* (continuity) *for every $x \in \mathbb{R}^n$ the maps $\xi \mapsto f(x,\xi)$ and $\xi \mapsto f^\infty(x,\xi)$ are continuous, where $f^\infty$ is the recession function of $f$, defined by*

$$f^\infty(x,\xi) := \limsup_{t\to\infty} \frac{f(x,t\xi)}{t}, \tag{5.1.1}$$

*for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}$;*

*(f4)* (recession function) *for every $x \in \mathbb{R}^n$ every $\xi \in \mathbb{R}^{N\times n}$ and every $t > 0$*

$$\Big| f^\infty(x,\xi) - \frac{f(x,t\xi)}{t}\Big| < \frac{C}{t}(1+f(x,t\xi)^{1-\alpha}).$$

**Remark 5.1.2.** *Let $f \in \mathcal{AI}(C,\alpha)$, then thanks to (f2) and (f4), for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{N\times n}$ we have that there exists the limit*

$$\lim_{t\to+\infty} \frac{f(x,t\xi)}{t} = f^\infty(x,\xi),$$

*and*

$$C^{-1}|\xi| \leq f^\infty(x,\xi) \leq C|\xi|. \tag{5.1.2}$$

*Moreover, for every $L > 0$ there exists $M > 0$ such that for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$ with $|\xi| = 1$ and $t > L$ we have that*

$$\left| f^\infty(x,\xi) - \frac{f(x,t\xi)}{t} \right| \leq \frac{M}{t^\alpha}. \tag{5.1.3}$$

**Definition 5.1.3** (Random integrand)**.** *A function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,+\infty)$ is called a **random integrand** if*

*(s-f1) $f$ is $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable;*

*(s-f2) $f(\omega,\cdot,\cdot) \in \mathcal{AI}(C,\alpha)$ for every $\omega \in \Omega$, where $\mathcal{AI}(C,\alpha)$ is as in Definition 5.1.1.*

If $f$ is a random integrand then its recission functio $f^\infty : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,+\infty)$ is given by

$$f^\infty(\omega,x,\xi) = \lim_{t\to+\infty} \frac{f(\omega,x,t\xi)}{t},$$

where the existence of the limit above is ensured by the very definition of random integrand together with Remark 5.1.2.

**Definition 5.1.4** (Stationary random integrand)**.** *A random integrand $f$ is **stationary** if there exists $\tau = (\tau_z)_{z\in\mathbb{Z}^n}$ $n$-dimensional group of $P$-preserving transformation on $(\Omega,\mathcal{M},P)$ (cf. Section 2.0.3) such that*

$$f(\omega,x+z,\xi) = f(\tau_z\omega,x,\xi)$$

*for every $\omega \in \Omega$, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^{N\times n}$.*

*If in addition $\tau$ is ergodic (cf. Section 2.0.3) we call $f$ an ergodic random integrand.*

### 5.1.1 Statements of the main results

Let $f$ be a given stationary random integrand and for $\varepsilon > 0$ let $\mathcal{H}_\varepsilon : \Omega \times L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0,+\infty]$ the random phase-field functional defined in (5.0.1).

**Remark 5.1.5.** *For $v \in W^{1,2}(A)$ set $\tilde{v} := (v \vee 0) \wedge 1$. We notice that for every $\varepsilon > 0$, $\omega \in \Omega$ there holds*

$$\mathcal{H}_\varepsilon(\omega)(u,\tilde{v},A) \leq \mathcal{H}_\varepsilon(\omega)(u,v,A),$$

*for every $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A)$ and $A \in \mathcal{A}(\mathbb{R}^n)$. Therefore it is not restrictive to assume that the phase-filed variable $v$ satisfies the pointwise bounds $0 \leq v \leq 1$ for $\mathcal{L}^n$-a.e. $x \in A$.*

Before stating the homogenisation results we need some additional notation. Let $h : \mathbb{R}^n \times \mathbb{R}^{n\times N} \to [0,\infty)$ satisfy (f2) and (f1). For $A \in \mathcal{A}_\infty(\mathbb{R}^n)$ and $(u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1])$ consider the following auxiliary integral functionals

$$E^h(u,A) := \int_A h(x,\nabla u)\,\mathrm{d}x, \tag{5.1.4}$$

and

$$S^h(u,v,A) := \int_A (v^2 h(x,\nabla u) + (1-v)^2 + |\nabla v|^2)\,\mathrm{d}x. \tag{5.1.5}$$

Moreover, let $w \in BV_{\rm loc}(\mathbb{R}^n, \mathbb{R}^N)$ and define the minimisation problems

$$m^h_{\rm b}(w, A) := \inf\{E^h(u, A) \,:\, u \in W^{1,1}(A, \mathbb{R}^N),\; u = w \text{ on } \partial A\} \tag{5.1.6}$$

and

$$m^h_{\rm s}(w, A) := \inf\{S^h(u, v, A) \colon u \in W^{1,1}(A, \mathbb{R}^N),\; v \in W^{1,2}(A, [0, 1]),\; (u, v) = (w, 1) \text{ on } \partial A\}, \tag{5.1.7}$$

where $u = w$ on $\partial A$ has to be intended in the sense of traces and inner traces for $u$ and $w$, respectively. If $A \subseteq \mathbb{R}^n$ is a set such that $\mathrm{int}A \in \mathcal{A}_\infty(\mathbb{R}^n)$ then we use the following convention $m^h_{\rm b}(w, A) := m^h_{\rm b}(w, \mathrm{int}A)$ and $m^h_{\rm s}(w, A) := m^h_{\rm s}(w, \mathrm{int}A)$.

The main result of this chapter is contained in Theorem 5.1.7, below, and provides an *almost sure* $\Gamma$-convergence result for the functionals $\mathcal{H}_\varepsilon$ defined in (5.0.1). In order to state this result we preliminarily need to state a theorem which guarantees the almost sure existence of the integrands of the $\Gamma$-limit. Namely, the next theorem establishes the existence and spatial homogeneity of the limits defining the asymptotic cell formulas appearing in Theorem 5.1.7 below.

Throughout this chapter, for every $\xi \in \mathbb{R}^{N\times n}$, we denote by $\ell_\xi$ the linear map given by

$$\ell_\xi(x) := \xi x. \tag{5.1.8}$$

We also adopt the following shorthand notation:

$$m^{f_\omega}_{\rm b} := m^{f(\omega,\cdot,\cdot)}_{\rm b} \quad \text{and} \quad m^{f^\infty_\omega}_{\rm s} := m^{f^\infty(\omega,\cdot,\cdot)}_{\rm s}.$$

**Theorem 5.1.6** (Homogenisation formulas)**.** *Let $f$ be a stationary random integrand. Then there exists $\Omega' \in \mathcal{M}$ with $P(\Omega') = 1$, such that for every $\omega \in \Omega'$*

*(i) every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$, and $\xi \in \mathbb{R}^{N\times n}$, the limit*

$$\lim_{r\to+\infty} \frac{m^{f_\omega}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n}$$

*exists and it is independent of $x, \nu$ and $k$ ($\ell_\xi$ is defined in (5.1.8));*

*(ii) every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$, the limit*

$$\lim_{r\to+\infty} \frac{m^{f^\infty_\omega}_{\rm s}(u_{rx,\zeta,\nu}, Q^\nu_r(rx))}{r^{n-1}}$$

*exists and it is independent of $x$ ($u_{rx,\zeta,\nu}$ is defined in (2.0.3)).*

*More precisely there exist a $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable function $f_{\rm hom} : \Omega \times \mathbb{R}^{N\times n} \to [0, \infty)$ and a $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{S}^{n-1})$-measurable function $g_{\rm hom} : \Omega \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$,*

$\xi \in \mathbb{R}^{N\times n}$, $\zeta \in \mathbb{R}^N$, *and* $\nu \in \mathbb{S}^{n-1}$

$$f_{\rm hom}(\omega,\xi) = \lim_{r\to+\infty} \frac{m_{\rm b}^{f_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n}$$
$$= \lim_{r\to+\infty} \frac{m_{\rm b}^{f_\omega}(\ell_\xi, Q_r(rx))}{r^n} = \lim_{r\to+\infty} \frac{m_{\rm b}^{f_\omega}(\ell_\xi, Q_r)}{r^n},$$

$$f^\infty_{\rm hom}(\omega,\xi) = \lim_{r\to+\infty} \frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n}$$
$$= \lim_{r\to+\infty} \frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r(rx))}{r^n} = \lim_{r\to+\infty} \frac{m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r)}{r^n},$$

$$g_{\rm hom}(\omega,\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty_\omega}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}},$$

*where* $f^\infty_{\rm hom}(\omega,\cdot)$ *denotes the recession function of* $f_{\rm hom}(\omega,\cdot)$ *(cf. (3.3.2)).*

*If we additionally assume that* $f$ *is ergodic, then* $f_{\rm hom}$ *and* $g_{\rm hom}$ *are independent of* $\omega$ *and*

$$f_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n}\int_\Omega m_{\rm b}^{f_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega),$$

$$f^\infty_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n}\int_\Omega m_{\rm b}^{f^\infty_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega),$$

$$g_{\rm hom}(\zeta,\nu) = \lim_{r\to+\infty} \frac{1}{r^n}\int_\Omega m_{\rm s}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r)\mathrm{d}P(\omega).$$

We are now in a position to state the main result of this chapter.

**Theorem 5.1.7** (Almost sure Γ-convergence)**.** *Let* $f$ *be a stationary random integrand. For* $\varepsilon > 0$ *and* $\omega \in \Omega$ *let* $\mathcal{H}_\varepsilon(\omega)$ *be the functionals defined in (5.0.1). Then, there exists* $\Omega' \in \mathcal{M}$ *with* $P(\Omega') = 1$ *such that for every* $\omega \in \Omega'$, $A \in \mathcal{A}(\mathbb{R}^n)$, *and* $(u,v) \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$ *we have*

$$\Gamma(L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{\varepsilon\to0} \mathcal{H}_\varepsilon(\omega)(u,v,A) = \mathcal{H}_{\rm hom}(\omega)(u,v,A),$$

*where* $\mathcal{H}_{\rm hom}(\omega) : L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0,\infty]$ *is defined as*

$$\mathcal{H}_{\rm hom}(\omega)(u,v,A) := \begin{cases} \displaystyle\int_A f_{\rm hom}(\omega,\nabla u)\,\mathrm{d}x + \int_A f^\infty_{\rm hom}(\omega,\tfrac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|})\mathrm{d}|D^c u| + \int_{J_u\cap A} g_{\rm hom}(\omega,[u],\nu_u)\mathrm{d}\mathcal{H}^{n-1} \\ \qquad\qquad \text{if } u \in (GBV(A))^N \text{ and } v = 1 \text{ for } \mathcal{L}^n\text{-a.e. } x \in A \\ +\infty \qquad\qquad \text{otherwise} \end{cases}$$

*with* $f_{\rm hom}$ *and* $g_{\rm hom}$ *as in Theorem 5.1.6. Moreover, if in addition* $f$ *is ergodic (cf. Definition 5.1.4), then the functional* $\mathcal{H}_{\rm hom}$ *is deterministic.*

The proof of Theorem 5.1.7 will be carried out in a number of steps in the next sections.

## 5.2 Properties of the homogenized integrands

In this section we prove a number of structural properties of the homogenized integrands $f_{\hom}$ and $g_{\hom}$.

For later use, it is convenient to work in a deterministic framework where the dependence of $f_{\hom}$ and $g_{\hom}$ on $\omega$ is not taken into account. Then, as a consequence, we need to assume that the limits defining $f_{\hom}$ and $g_{\hom}$ exist and are spatially homogeneous.

We start with $f_{\hom}$.

**Proposition 5.2.1.** *Let $f \in \mathcal{AI}(C, \alpha)$ and assume that for every $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{N\times n}$ the limit*

$$\lim_{r\to+\infty} \frac{m_{\mathrm{b}}^f(\ell_\xi, Q_r(rx))}{r^n} =: f_{\hom}(\xi) \tag{5.2.1}$$

*exists and is independent of $x$ ($\ell_\xi$ is defined in (5.1.8)). Then, $f_{\hom}$ satisfies the following properties:*

*(i)* $f_{\hom}$ *is quasiconvex;*

*(ii) for every $\xi_1, \xi_2 \in \mathbb{R}^{N\times n}$*

$$|f_{\hom}(\xi_1) - f_{\hom}(\xi_2)| \le K|\xi_1 - \xi_2|,$$

*where $K$ is a constant that depends only on $n, N$ and $C$;*

*(iii) for every $\xi \in \mathbb{R}^{N\times n}$*

$$C^{-1}|\xi| \le f_{\hom}(\xi) \le C(|\xi| + 1). \tag{5.2.2}$$

*Proof.* $(i)$ The quasiconvexity of $f_{\hom}$ defined as in (5.2.1) is shown in [RZ23, Proposition 5.5 Step 2].

$(ii)$ Let $\xi_1, \xi_2 \in \mathbb{R}^{N\times n}$ and $r > 0$ be fixed. For every $u \in BV(Q_r, \mathbb{R}^N)$ and $A \in \mathcal{A}(Q_r)$ consider the auxiliary functional defined as

$$J(u, A) := \begin{cases} \displaystyle\int_A f(y, \nabla u)\mathrm{d}y & \text{if } u \in W^{1,1}(Q_r, \mathbb{R}^N) \\ +\infty & \text{in } BV(Q_r, \mathbb{R}^N) \setminus W^{1,1}(Q_r, \mathbb{R}^N), \end{cases}$$

as well as $\overline{J}(\cdot, A) := sc^-(L^1)J(\cdot, A)$ (its relaxation in the $L^1$-topology)

By (f2) we have that $\overline{J}(u, A) \le C(|Du|(A) + \mathcal{L}^n(A))$, therefore thanks to [BFM98, Lemma 3.1 and Lemma 4.1.2] we get

$$\begin{aligned} |m_{\overline{J}}(\ell_{\xi_1}, Q_r) - m_{\overline{J}}(\ell_{\xi_2}, Q_r)| &\le C\|\ell_{\xi_1} - \ell_{\xi_2}\|_{L^1(\partial Q_r^\nu(rx))} \\ &\le C\widehat{K}|\xi_1 - \xi_2| \int_{\partial Q_r} |y|\mathrm{d}\mathcal{H}^{n-1}(y) \le \frac{C\widehat{K}n^{1/2}}{2}\mathcal{H}^{n-1}(\partial Q_1)|\xi_1 - \xi_2|r^n \end{aligned} \tag{5.2.3}$$

where $\widehat{K}$ depends only on $n$ and $N$, while

$$m_{\overline{J}}(\ell_\xi, Q_r) := \inf\{\overline{J}(u, Q_r) \;:\; u \in BV(Q_r, \mathbb{R}^N) \text{ with } u = \ell_\xi \text{ on } \partial Q_r\}.$$

Appealing to [BFM98, Lemma 4.1.3] we deduce that

$$m_{\overline{J}}(\ell_{\xi_1}, Q_r) = m_{\mathrm{b}}^f(\xi_1, Q_r) \quad \text{and} \quad m_{\overline{J}}(\ell_{\xi_2}, Q_r) = m_{\mathrm{b}}^f(\xi_2, Q_r)$$

therefore, combining (5.2.1) and (5.2.3) readily gives

$$|f_{\rm hom}(\xi_1) - f_{\rm hom}(\xi_2)| \leq K|\xi_1 - \xi_2|,$$

with $K := \frac{C\widehat{K}\sqrt{n}}{2}\mathcal{H}^{n-1}(\partial Q_1)$.

*(iii)* Let $\xi \in \mathbb{R}^{N\times n}$, $r > 0$, and $u \in W^{1,1}(Q_r, \mathbb{R}^N)$ with $u = \ell_\xi$ on $\partial Q_r$ be arbitrary and fixed. By (f2) we have

$$C^{-1}|\xi|r^n = C^{-1}\Big|\int_{Q_r} \nabla u \;\mathrm{d}x\Big| \leq C^{-1}\int_{Q_r} |\nabla u|\,\mathrm{d}x \leq \int_{Q_r} f(x, \nabla u)\,\mathrm{d}x,$$

therefore passing to the inf on $u$ we immediately get

$$C^{-1}|\xi|r^n \leq m^f_{\rm b}(\ell_\xi, Q_r)$$

for every $\xi \in \mathbb{R}^{N\times n}$ and $r > 0$. Hence the second inequality in (5.2.2) follows by (5.2.1).

The second inequality in (5.2.2) is a consequence of the trivial inequality

$$m^f_{\rm b}(\ell_\xi, Q_r) \leq \int_{Q_r} f(x, \nabla \ell_\xi)\,\mathrm{d}x \leq C(|\xi| + 1)r^n$$

which, in turn, is implied by (f2). □

Below we prove that $f^\infty_{\rm hom}$ can be equivalently expressed as the limit of suitable (scaled) minimisation problems. To prove it we make use of the following lemma.

**Lemma 5.2.2.** *Let $g \in \mathcal{AI}(C, \alpha)$, $A \in \mathcal{A}(\mathbb{R}^n)$, $(u, v) \in W^{1,1}(A, \mathbb{R}^N) \times W^{1,2}(A, [0, 1])$, then for every $t > 0$ we have that*

$$\int_A \Big|v^2 g^\infty(y, \nabla u) - v^2\frac{g(y, t\nabla u)}{t}\Big|\mathrm{d}y \leq \frac{K}{t}\mathcal{L}^n(A) + \frac{K}{t^\alpha}\mathcal{L}^n(A)^\alpha\Big(\int_A v^2|\nabla u|\mathrm{d}y\Big)^{1-\alpha},$$

*where $K$ is a positive constant depending only on $C$ and $\alpha$.*

*Proof.* Thanks to (f4), $0 \leq v \leq 1$ and $\alpha \in (0, 1)$ we have that

$$\int_A \Big|v^2 g^\infty(y, \nabla u) - v^2\frac{g(y, t\nabla u)}{t}\Big|\mathrm{d}y \leq \frac{C}{t}\mathcal{L}^n(A) + \frac{C}{t}\int_A v^{2(1-\alpha)} g(y, t\nabla u)^{1-\alpha}\mathrm{d}y\,,$$

thus by Jensen's Inequality we deduce that

$$\frac{C}{t}\int_A v^{2(1-\alpha)} g(y, t\nabla u)^{1-\alpha}\mathrm{d}y \leq \frac{C}{t}\mathcal{L}^n(A)^\alpha\Big(\int_A v^2 g(y, t\nabla u)\mathrm{d}y\Big)^{1-\alpha}.$$

Eventually, we conclude by (f2). □

**Proposition 5.2.3.** *Let $f \in \mathcal{AI}(C, \alpha)$ and assume that for every $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$, and $\xi \in \mathbb{R}^{N\times n}$*

$$\lim_{r\to+\infty} \frac{m^f_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} = f_{\rm hom}(\xi) \tag{5.2.4}$$

*where $f_{\rm hom}$ is as in (5.2.1). Let $f^\infty_{\rm hom}$ be the recession function of $f_{\rm hom}$ (cf. (3.3.2)), then for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{n\times N}$, $\nu \in \mathbb{S}^{n-1}$ and $k \in \mathbb{N}$ we have*

$$f^\infty_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n},$$

*hence, in particular, $f^\infty_{\rm hom} = (f^\infty)_{\rm hom}$.*

*Proof.* Let $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{n\times N}$, $\nu \in \mathbb{S}^{n-1}$, $k \in \mathbb{N}$ and $\eta \in (0,1)$ be fixed. By (5.1.6), for every $r > 0$ there exists $u_r \in W^{1,1}(Q^{\nu,k}_r(rx))$ with $u_r = \ell_\xi$ on $\partial Q^{\nu,k}_r(rx)$, such that

$$E^{f^\infty}(u_r, Q^{\nu,k}_r(rx)) \le m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx)) + \eta k^{n-1}r^n, \tag{5.2.5}$$

and

$$C^{-1}\int_{Q^{\nu,k}_r(rx)} |\nabla u_r| {\rm d}y \le m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx)) + \eta k^{n-1}r^n \le C(|\xi|+1)k^{n-1}r^n.$$

In particular, by Lemma 5.2.2, for every $t \ge 1$ we obtain

$$\begin{aligned}\int_{Q^{\nu,k}_r(rx)} &|f^\infty(y,\nabla u_r) - \frac{1}{t}f(y,t\nabla u_r)|{\rm d}y \\ &\le \frac{K}{t}k^{n-1}r^n + \frac{K}{t^\alpha}(k^{n-1}r^n)^\alpha\Big(\int_{Q^{\nu,k}_r(rx)}|\nabla u_r|{\rm d}y\Big)^{1-\alpha} \le \frac{1}{t^\alpha}\hat{K}k^{n-1}r^n\,,\end{aligned}$$

where $\hat{K}$ depends only on $C$, $\alpha$ and $\xi$. Hence, for $t \ge 1$,

$$E^{f_t}(u_r, Q^{\nu,k}_r(rx)) \le E^{f^\infty}(u_r, Q^{\nu,k}_r(rx)) + \frac{1}{t^\alpha}\hat{K}k^{n-1}r^n,$$

where $f_t(y,\xi) := \frac{f(y,t\xi)}{t}$ and consequently, by (5.2.5),

$$\frac{m^{f_t}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} \le \frac{m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} + \eta + \frac{\hat{K}}{t^\alpha}. \tag{5.2.6}$$

Observing that $m^{f_t}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx)) = \frac{1}{t}m^{f}_{\rm b}(\ell_{t\xi}, Q^{\nu,k}_r(rx))$, thanks to the linearity of $\xi \mapsto \ell_\xi$, we get

$$\lim_{r\to\infty}\frac{m^{f_t}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} = \lim_{r\to\infty}\frac{m^{f}_{\rm b}(\ell_{t\xi}, Q^{\nu,k}_r(rx))}{tk^{n-1}r^n} = \frac{f_{\rm hom}(t\xi)}{t}, \tag{5.2.7}$$

by (5.2.4). Hence, from (5.2.6) and (5.2.7), letting $\eta \to 0$, we have

$$\limsup_{t\to+\infty}\frac{f_{\rm hom}(t\xi)}{t} \le \liminf_{r\to+\infty}\frac{m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n}.$$

Exchanging the role of $f_t$ and $f^\infty$ and arguing analogously, we obtain

$$\limsup_{r\to+\infty}\frac{m^{f^\infty}_{\rm b}(\ell_\xi, Q^{\nu,k}_r(rx))}{k^{n-1}r^n} \le \liminf_{t\to+\infty}\frac{f_{\rm hom}(t\xi)}{t}.$$

□

To prove the properties satisfied by $g_{\text{hom}}$, we first establish some technical results in the spirit of [BFM98, Section 3].

**Lemma 5.2.4.** *Let $x \in \mathbb{R}^n$, $r > 1$, $\nu \in \mathbb{S}^{n-1}$, and $w_1, w_2 \in BV_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, then we have that*

$$|m_{\text{s}}^{f^\infty}(w_1, Q_r^\nu(x)) - m_{\text{s}}^{f^\infty}(w_2, Q_r^\nu(x))| \leq C \int_{\partial Q_r^\nu(x)} |w_1 - w_2| \mathcal{H}^{n-1}.$$

*Proof.* First we observe that for every $w \in BV_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, $x \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$, and $r > 1$ there holds $m_{\text{s}}^{f^\infty}(w, Q_r^\nu(rx)) = (m_{\text{s}}^{f^\infty})^*(w, Q_r^\nu(rx))$, where

$$\begin{aligned}(m_{\text{s}}^{f^\infty})^*(w, Q_r^\nu(rx)) := \inf\{S^{f^\infty}(u, v, Q_r^\nu(rx)) \ : \ & u \in W^{1,1}(Q_r^\nu(rx), \mathbb{R}^N), \ v \in W^{1,2}(Q_r^\nu(rx), [0,1]), \\ & (u, v) = (w, 1) \text{ on } \partial Q_r^\nu(rx), \ \ v \geq \eta \text{ for some } \eta \in (0,1)\}, \end{aligned} \tag{5.2.8}$$

with $S^{f^\infty}$ defined in (5.1.5). In fact, given $\eta \in (0,1)$, $u \in W^{1,1}(Q_r^\nu(x), \mathbb{R}^N)$ and $v \in W^{1,2}(Q_r^\nu(x), [0,1])$, $v_\eta := v \vee \eta \in W^{1,2}(Q_r^\nu(x), [0,1])$ with $v_\eta = 1$ on $\partial Q_r^\nu(x)$, and

$$\int_{Q_r^\nu(x)} (1 - v_\eta)^2 + |\nabla v_\eta|^2 \mathrm{d}y \leq \int_{Q_r^\nu(x)} (1 - v)^2 + |\nabla v|^2 \mathrm{d}y,$$

and

$$\lim_{\eta \to 0^+} \int_{Q_r^\nu(x)} v_\eta^2 f^\infty(y, \nabla u) \mathrm{d}y = \int_{Q_r^\nu(x)} v^2 f^\infty(y, \nabla u) \mathrm{d}y.$$

Let $v \in W^{1,2}(Q_r^\nu(x), [0,1])$ with $v = 1$ on $\partial Q_r^\nu(x)$ and $v \geq \eta$ for some $\eta \in (0,1)$. Define the functional $\mathcal{H}_v : BV(Q_r^\nu(x), \mathbb{R}^N) \times \mathcal{A}(Q_r^\nu(x)) \longrightarrow [0, +\infty]$ as

$$\mathcal{H}_v(u, B) := \begin{cases} \displaystyle\int_B v^2 f^\infty(y, \nabla u) \mathrm{d}y & \text{if } u \in W^{1,1}(Q_r^\nu(x), \mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

Consider its relaxation $\overline{\mathcal{H}}_v := sc^-(L^1)\mathcal{H}_v : BV(Q_r^\nu(x), \mathbb{R}^N) \times \mathcal{A}(Q_r^\nu(x)) \to [0, +\infty]$. [BFM98, Lemma 3.1 and Lemma 4.1.2] and $\overline{\mathcal{H}}_v(u, B) \leq C|Du|(B)$ imply that

$$|m_{\overline{\mathcal{H}}_v}(w_1, Q_r^\nu(x)) - m_{\overline{\mathcal{H}}_v}(w_2, Q_r^\nu(x))| \leq C \int_{\partial Q_r^\nu(x)} |w_1 - w_2| \mathrm{d}\mathcal{H}^{n-1}, \tag{5.2.9}$$

where we recall that $m_{\overline{\mathcal{H}}_v}(w, Q_r^\nu(x))$ is defined in (5.3.9) using the functional $\overline{\mathcal{H}}_v$.

In addition, $m_{\overline{\mathcal{H}}_v}(w_i, Q_r^\nu(x)) = m_{\mathcal{H}_v}(w_i, Q_r^\nu(x))$, $i \in \{1, 2\}$, by [BFM98, Lemma 4.1.3]. Therefore, using (5.2.8) we can rewrite $m_{\text{s}}^{f^\infty}(w, Q_r^\nu(rx))$ as

$$\begin{aligned} m_{\text{s}}^{f^\infty}(w, Q_r^\nu(rx)) = \inf\{ m_{\overline{\mathcal{H}}_v}(w, Q_r^\nu(rx)) + & \textstyle\int_{Q_r^\nu(rx)} ((1 - v)^2 + |\nabla v|^2) \mathrm{d}y : \\ & v \in W^{1,2}(Q_r^\nu(rx), [0,1]) \ v \geq \eta \ \text{ for some } \eta \in (0,1)\}, \end{aligned}$$

and thus we can conclude thanks to (5.2.9). $\square$

The next result corollary follows directly from Lemma 5.2.4. Before stating it, we introduce the following notation: for $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, $\nu \in \mathbb{S}^{n-1}$, and $\varepsilon > 0$, we define the functions $\overline{u}^{\varepsilon}_{x,\zeta,\nu} : \mathbb{R}^n \to \mathbb{R}^N$ by

$$\overline{u}^{\varepsilon}_{x,\zeta,\nu}(y) := \zeta\, \overline{u}\left(\frac{1}{\varepsilon}(y-x)\cdot\nu\right), \tag{5.2.10}$$

where $\overline{u} : \mathbb{R} \to [0,1]$ is a fixed smooth cut-off function such that $\overline{u} \equiv 1$ on $[\frac{1}{2}, +\infty)$ and $\overline{u} \equiv 0$ on $(-\infty, -\frac{1}{2}]$; for convenience, we also set $\overline{u}_{x,\zeta,\nu} := \overline{u}^{1}_{x,\zeta,\nu}$ and $\overline{u}_{\zeta,\nu} := \overline{u}_{0,\zeta,\nu}$.

**Corollary 5.2.5.** *Let $x \in \mathbb{R}^n$, $r > 1$, $\nu \in \mathbb{S}^{n-1}$, and $\zeta \in \mathbb{R}^N$ then we have that*

$$|m_{\mathrm{s}}^{f^\infty}(u_{x,\zeta,\nu}, Q^{\nu}_r(x)) - m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu}, Q^{\nu}_r(x))| \le 2C|\zeta| r^{n-2},$$

*where $u_{x,\zeta,\nu}$ and $\overline{u}_{x,\zeta,\nu}$ are defined in 2.0.3 and (5.2.10), respectively.*

The next lemma will be widely used in Section 5.6.

**Lemma 5.2.6.** *Let $x, z \in \mathbb{R}^n$, $\nu_1, \nu_2 \in \mathbb{S}^{n-1}$, and $r_2 > r_1 > 2R \ge 1$ be such that $Q^{\nu_1}_{r_1}(x) \subset\subset Q^{\nu_2}_{r_2}(z)$ and $|(y-x)\cdot\nu_1| \le \frac{1}{2}$ imply*

$$|(y-z)\cdot\nu_2| \le R \qquad \text{for every } y \in Q^{\nu_2}_{r_2}(z). \tag{5.2.11}$$

*Then, for every $\zeta \in \mathbb{R}^N$ and $\eta > 0$ the following statements hold true: for every $r_3 \ge r_2$ such that $Q^{\nu_2}_{r_2}(z) \subset\subset Q^{\nu_1}_{r_3}(x)$, then*

$$m_{\mathrm{s}}^{f^\infty}(\overline{u}_{z,\zeta,\nu_2}, Q^{\nu_2}_{r_2}(z)) - m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q^{\nu_1}_{r_1}(x)) \le \eta r_1^{n-1} + \tilde{K}((r_3 - r_1)r_3^{n-2} + R|\zeta| r_2^{n-2});$$

*with $\tilde{K}$ depending only on $n$ and $C$, and $\overline{u}_{x,\zeta,\nu}$ is defined in (5.2.10).*

*Proof.* Fix $\zeta \in \mathbb{R}^N$, $\eta > 0$ and let $(u,v) \in W^{1,1}(Q^{\nu_1}_{r_1}(x), \mathbb{R}^N) \times W^{1,2}(Q^{\nu_1}_{r_1}(x), [0,1])$, with $(u,v) = (\overline{u}_{x,\zeta,\nu}, 1)$ on $\partial Q^{\nu_1}_{r_1}(x)$, such that

$$S^{f^\infty}(u,v,Q^{\nu_1}_{r_1}(x)) = \int_{Q^{\nu_1}_{r_1}(x)} v^2(f^\infty(y,\nabla u) + (1-v)^2 + |\nabla v|^2)\mathrm{d}y \le m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu}, Q^{\nu_1}_{r_1}(x)) + \eta r_1^{n-1}. \tag{5.2.12}$$

Define $(\hat{u}, \hat{v})$ by

$$(\hat{u}(y), \hat{v}(y)) := \begin{cases} (u(y), v(y)) & \text{if } y \in Q^{\nu_1}_{r_1}(x) \\ (\overline{u}_{x,\zeta,\nu}, 1) & \text{if } y \in Q^{\nu_2}_{r_2}(z) \setminus Q^{\nu_1}_{r_1}(x)\,, \end{cases}$$

and note that $(\hat{u}, \hat{v}) \in W^{1,1}(Q^{\nu_2}_{r_2}(z), \mathbb{R}^N) \times W^{1,2}(Q^{\nu_2}_{r_2}(z), [0,1])$ with $(\hat{u}, \hat{v}) = (\overline{u}_{x,\zeta,\nu}, 1)$ on $\partial Q^{\nu_2}_{r_2}(z)$.

From (5.2.11) we have that $\overline{u}_{z,\zeta,\nu_2}(y) = \overline{u}_{x,\zeta,\nu_1}(y)$ for every $y \in Q^{\nu_2}_{r_2}(z)$ such that $|(y-z)\cdot\nu_2| > R$; in particular Lemma 5.2.4 yields that

$$\begin{aligned} |m_{\mathrm{s}}^{f^\infty}(\overline{u}_{z,\zeta,\nu_2}, Q^{\nu_2}_{r_2}(z)) - m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q^{\nu_2}_{r_2}(z))| &\le \int_{\partial Q^{\nu_2}_{r_2}(z)} |\overline{u}_{z,\zeta,\nu_2} - \overline{u}_{x,\zeta,\nu_1}| \mathrm{d}\mathcal{H}^{n-1} \\ = \int_{\partial Q^{\nu_2}_{r_2}(z)\cap\Sigma_{\hat{K}}} |\overline{u}_{z,\zeta,\nu_2} - \overline{u}_{x,\zeta,\nu_1}| \mathrm{d}\mathcal{H}^{n-1} &\le 8R(n-1)|\zeta| r_2^{n-2}, \end{aligned} \tag{5.2.13}$$

where $\Sigma_{\nu_2,R} := \{|(y-z)\cdot\nu_2| \le R\}$. Furthermore, setting $\Sigma_{\nu_1,1/2} := \{|(y-x)\cdot\nu_1| \le \frac{1}{2}\}$, from 5.2.10, (5.1.2) and (5.2.12) we get

$$
\begin{aligned}
& m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_2}^{\nu_2}(z)) \leq S^{f^\infty}(\hat{u}, \hat{v}, Q_{r_2}^{\nu_2}(z)) \\
& \leq S^{f^\infty}(u, v, Q_{r_1}^{\nu_1}(x)) + \int_{Q_{r_2}^{\nu_2}(z)\setminus Q_{r_1}^{\nu_1}(x)} f^\infty(\nabla \hat{u})\mathrm{d}y \\
& \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\mathcal{L}^n((Q_{r_2}^{\nu_2}(z) \setminus Q_{r_1}^{\nu_1}(x)) \cap \Sigma_{\nu_1}) \\
& \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\mathcal{L}^n((Q_{r_3}^{\nu_1}(z) \setminus Q_{r_1}^{\nu_1}(x)) \cap \Sigma_{\nu_1}) \\
& \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}(r_3^{n-1} - r_1^{n-1}) \\
& \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C(n-1)\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}(r_3 - r_1)r_3^{n-2}.
\end{aligned} \tag{5.2.14}
$$

Therefore, recollecting (5.2.13) and (5.2.14), we deduce

$$
\begin{aligned}
& m_{\mathrm{s}}^{f^\infty}(\overline{u}_{z,\zeta,\nu_2}, Q_{r_2}^{\nu_2}(z)) \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_2}^{\nu_2}(z)) + 8R(n-1)|\zeta| r_2^{n-2} \\
& \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{x,\zeta,\nu_1}, Q_{r_1}^{\nu_1}(x)) + \eta r_1^{n-1} + C(n-1)\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}(r_3 - r_1)r_3^{n-2} + 8R(n-1)|\zeta| r_2^{n-2}
\end{aligned}
$$

and therefore the claim. □

We are now in a position to establish some properties satisfied by $g_{\mathrm{hom}}$.

**Proposition 5.2.7.** *Let $f \in \mathcal{AI}(C, \alpha)$ and assume that for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$ the limit*

$$
\lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} =: g_{\mathrm{hom}}(\zeta, \nu) \tag{5.2.15}
$$

*exists and is independent of $x$, where $u_{rx,\zeta,\nu}$ is defined in (2.0.3). Then, $g_{\mathrm{hom}}$ satisfies the following properties:*

*(i) for every $\zeta_1, \zeta_2 \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$*

$$
|g_{\mathrm{hom}}(\zeta_1, \nu) - g_{\mathrm{hom}}(\zeta_2, \nu)| \leq C\mathcal{H}^{n-1}(\partial Q_1)|\zeta_1 - \zeta_2|; \tag{5.2.16}
$$

*(ii) $g_{\mathrm{hom}} : \mathbb{R}^N \times \hat{\mathbb{S}}_\pm^{n-1} \to [0, +\infty)$ is continuous ($\hat{\mathbb{S}}_\pm^{n-1}$ is defined in (2.0.1));*

*(iii) for every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$*

$$
\frac{2|\zeta|}{C(|\zeta|+2)} \leq g_{\mathrm{hom}}(\zeta, \nu) \leq \frac{2C|\zeta|}{|\zeta|+2}; \tag{5.2.17}
$$

*(iv) for every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$*

$$
g_{\mathrm{hom}}(\zeta, \nu) = g_{\mathrm{hom}}(-\zeta, -\nu).
$$

*Proof.* To prove (i) fix $\nu \in \mathbb{S}^{n-1}$ and $\zeta_1, \zeta_2 \in \mathbb{R}^{N\times n}$. Thanks to Lemma 5.2.4 we get

$$
|m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\zeta_1,\nu}, Q_r^\nu) - m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\zeta_2,\nu}, Q_r^\nu)| \leq C \int_{\partial Q_r^\nu} |\overline{u}_{\zeta_1,\nu} - \overline{u}_{\zeta_2,\nu}| \mathrm{d}\mathcal{H}^{n-1}.
$$

By the definition of $\overline{u}_{\zeta,\nu}$ (see (5.2.10)), using Lemma 5.2.4 we have

$$\int_{\partial Q_r^\nu} |\overline{u}_{\zeta_1,\nu} - \overline{u}_{\zeta_2,\nu}| \mathrm{d}\mathcal{H}^{n-1} = \int_{\partial Q_r^\nu} |\zeta_1 - \zeta_2| \overline{\mathrm{u}}\,(y\cdot\nu)\, \mathrm{d}\mathcal{H}^{n-1}(y) \leq \mathcal{H}^{n-1}(\partial Q_1)|\zeta_1 - \zeta_2| r^{n-1}.$$

Then we conclude by (5.2.15) also noticing that by Corollary 5.2.5 we have

$$g_{\mathrm{hom}}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty}(\overline{u}_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}},$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$.

To prove (ii) we preliminarily show that $g_{\mathrm{hom}}(\zeta,\cdot) : \hat{\mathbb{S}}^{n-1}_{\pm} \to [0,+\infty)$ (cf. (2.0.1)) is continuous for every $\zeta \in \mathbb{R}^N$. Fix $\zeta \in \mathbb{R}^N$, $\nu \in \hat{\mathbb{S}}^{n-1}_{\pm}$ and a sequence $(\nu_j)_{j\in\mathbb{N}}$ in $\hat{\mathbb{S}}^{n-1}_{\pm}$ such that $\nu_j \to \nu$ as $j \to +\infty$. For every $\delta \in (0,1/2)$, by the continuity of the map $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_{\pm}$ (cf. Remark 2.0.1), there exists $j_\delta$ such that for every $r > 0$ and every $j \geq j_\delta$

$$Q_r^\nu \subset\subset Q^{\nu_j}_{(1+\delta)r} \subset\subset Q^\nu_{(1+2\delta)r}\,. \tag{5.2.18}$$

Setting $\kappa_j := \max\{|R_{\nu_j}(e_i)\cdot\nu| \ : \ i = 1,\dots,n-1\}$, we have that $\kappa_j \to 0$ as $j \to +\infty$, by the continuity of the map $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_{\pm}$. Letting $y \in \overline{Q^{\nu_j}_{r(1+\delta)}}$, then $y = y' + (y\cdot\nu_j)\nu_j$ where

$$y' \in R_{\nu_j}\Big(\big[-\frac{r}{2}(1+\delta), \frac{r}{2}(1+\delta)\big]^{n-1} \times \{0\}\Big).$$

In particular $(y\cdot\nu_j)(\nu\cdot\nu_j) = y\cdot\nu - y'\cdot\nu$ and thus, if $|y\cdot\nu| \leq \frac{1}{2}$ and $j$ is large enough, we get

$$|y\cdot\nu_j| \leq \tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|} + \tfrac{1}{2\nu_j\cdot\nu} \leq \tfrac{(n-1)\kappa_j r(1+\delta)+1}{2(1-\delta)} = K(\delta) r\kappa_j + 1, \tag{5.2.19}$$

where $K(\delta) := \frac{(n-1)(1+\delta)}{2(1-\delta)}$. Applying Lemma 5.2.6 with $R = K(\delta)r\kappa_j + 1$, we deduce

$$\begin{aligned} &m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\zeta,\nu_j}, Q^{\nu_j}_{r(1+\delta)}) \leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) + \eta r^{n-1} \\ &\quad + \tilde{K}(2\delta(1+2\delta)^{n-2} r^{n-1} + (K(\delta)r\kappa_j + 1)|\zeta|(1+\delta)^{n-2} r^{n-2}), \end{aligned}$$

where $\tilde{K}$ depends only on $n$ and $C$. Consequently, letting the $r \to +\infty$, appealing to (5.2.15) and to Lemma 5.2.4, we get

$$(1+\delta)^{n-1} g_{\mathrm{hom}}(\zeta,\nu_j) \leq g_{\mathrm{hom}}(\zeta,\nu) + \eta + \tilde{K}(1+2\delta)^{n-2}(2\delta + K(\delta)\kappa_j|\zeta|).$$

Taking the $\limsup$ for $j \to +\infty$ we have

$$(1+\delta)^{n-1} \limsup_{j\to+\infty} g_{\mathrm{hom}}(\zeta,\nu_j) \leq g_{\mathrm{hom}}(\zeta,\nu) + \eta + 2\tilde{K}\delta(1+2\delta)^{n-2}$$

thus letting $\eta,\delta \to 0$ we obtain

$$\limsup_{j\to+\infty} g_{\mathrm{hom}}(\zeta,\nu_j) \leq g_{\mathrm{hom}}(\zeta,\nu).$$

An analogous argument, using the cube $Q^{\nu_j}_{(1-\delta)r}$, shows that

$$g_{\mathrm{hom}}(\zeta,\nu) \le \liminf_{j\to+\infty} g_{\mathrm{hom}}(\zeta,\nu_j)$$

and hence the claim.

To establish the continuity with respect to both variables, consider a sequence $(\zeta_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^{N\times n}$ such that $\zeta_j \to \zeta$. Thanks to (5.2.16), we have that

$$\begin{aligned}|g_{\mathrm{hom}}(\zeta,\nu) - g_{\mathrm{hom}}(\zeta_j,\nu_j)| &\le |g_{\mathrm{hom}}(\zeta,\nu) - g_{\mathrm{hom}}(\zeta,\nu_j)| + |g_{\mathrm{hom}}(\zeta,\nu_j) - g_{\mathrm{hom}}(\zeta_j,\nu_j)| \\ &\le |g_{\mathrm{hom}}(\zeta,\nu) - g_{\mathrm{hom}}(\zeta,\nu_j)| + C\mathcal{H}^{n-1}(\partial Q_1)|\zeta-\zeta_j|\end{aligned}$$

and therefore we get (ii).

To prove (iii) fix $\zeta\in\mathbb{R}^N$ and $\nu\in\mathbb{S}^{n-1}$, and recall that by (5.2.15) and the spatial homogeneity of $g_{\mathrm{hom}}$ we have that

$$g_{\mathrm{hom}}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,. \tag{5.2.20}$$

We notice that for every $r>0$ and $M\in\mathbb{N}$ we have

$$\frac{m_{\mathrm{s}}^{f^\infty}(u_{\zeta,\nu},Q_{Mr}^\nu)}{Mr^{n-1}} \le \frac{m_{\mathrm{s}}^{f^\infty}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,.$$

Indeed, assume for simplicity $\nu=e_n$, then if $(u,v)$ is a competitor for $m_{\mathrm{s}}^{f^\infty}(u_{\zeta,e_n},Q_r^{e_n})$ then $(u_M,v_M)$ defined by $(u,v)(x-r\mathtt{i})$ for $x\in r\mathtt{i}+Q_r^{e_n}$, for $\mathtt{i}\in\mathbb{Z}^{n-1}\times\{0\}$ with components in $[-M+1,M-1]$, and equal to $u_{\zeta,e_n}$ otherwise on $Q_{Mr}^{e_n}$ is a competitor for $m_{\mathrm{s}}^{f^\infty}(u_{\zeta,e_n},Q_{Mr}^{e_n})$ with $S^{f^\infty}(u_M,v_M,Q_{Mr}^{e_n}) = M^{n-1}S^{f^\infty}(u,v,Q_r^{e_n})$. Thus, we infer that

$$\lim_{r\to+\infty}\frac{m_{\mathrm{s}}^{f^\infty}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}} = \inf_{r>0}\frac{m_{\mathrm{s}}^{f^\infty}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,.$$

Moreover, by (f2) and $C\ge 1$, we have that

$$C^{-1}(AF)_\varepsilon(u,v,Q_r^\nu) \le \mathcal{H}_\varepsilon(u,v,Q_r^\nu) \le C(AF)_\varepsilon(u,v,Q_r^\nu)\,,$$

where $(AF)_\varepsilon : L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})\times\mathcal{A}(\mathbb{R}^n)\to[0,+\infty]$ is given by

$$(AF)_\varepsilon(u,v,B) := \begin{cases}\displaystyle\int_B (v^2|\nabla u| + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2)\mathrm{d}y & \text{if } (u,v)\in W^{1,1}(B,\mathbb{R}^N)\times W^{1,2}(B,[0,1]) \\ +\infty & \text{otherwise.}\end{cases}$$

Therefore, we conclude that

$$C^{-1}\inf_{r>0}\frac{m_{\mathrm{s}}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}} \le g_{\mathrm{hom}}(\zeta,\nu) \le C\inf_{r>0}\frac{m_{\mathrm{s}}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,.$$

Finally, [AF02, Lemma 3.8 and Remark 3.9] yield that

$$\inf_{r>0}\frac{m_{\mathrm{s}}^{|\cdot|}(u_{\zeta,\nu},Q_r^\nu)}{r^{n-1}} = g(\zeta) := \frac{2|\zeta|}{|\zeta|+2}.$$

where using the same notation as in [AF02]

$$g(s) := \min_{[0,1]}\{t^2 s + 4\int_t^1 (1-\lambda)\mathrm{d}\lambda\} = \frac{2s}{s+2}\,.$$

Eventually, (iv) is a direct consequence of the identity $R_\nu(Q_1) = R_{-\nu}(Q_1)$ (see Remark 2.0.1) and of the fact that $u = u_{-\zeta,-\nu}$ on $\partial Q_r^\nu$ if and only if $u + \zeta = u_{\zeta,\nu}$ on $\partial Q_r^\nu$ for every $\zeta \in \mathbb{R}^N$, $\nu \in \mathbb{S}^{n-1}$, $r > 0$, and $u \in W^{1,1}(Q_r^\nu, \mathbb{R}^N)$. $\square$

## 5.3 Deterministic homogenisation

To prove the stochastic homogenisation result in Theorem 5.1.7 we follow the same proof strategy as in [CDMSZ19b, CDMSZ22]. To this end, we preliminarily work in a deterministic framework (where $\omega \in \Omega$ is regarded as fixed) and prove a homogenisation result without assuming any periodicity of the integrand. Then, in Section 5.6, the deterministic homogenisation result at fixed $\omega$ will be used in combination with the Subadditive Ergodic Theorem 2.0.9, to derive an almost sure $\Gamma$-convergence result for the random functionals $\mathcal{H}_\varepsilon(\omega)$ (cf. (5.0.1)).

The main result of this section is stated in the following theorem.

**Theorem 5.3.1** (Deterministic homogenisation)**.** *Let $f \in \mathcal{AI}(C,\alpha)$ and consider the phase-field functionals $\mathcal{H}_\varepsilon : L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0,+\infty]$ given by*

$$\mathcal{H}_\varepsilon(u,v,A) := \begin{cases} \displaystyle\int_A \big(v^2 f(\tfrac{x}{\varepsilon}, \nabla u) + \tfrac{(1-v)^2}{\varepsilon} + \varepsilon|\nabla v|^2\big)\,\mathrm{d}x, & (u,v) \in W^{1,1}(A,\mathbb{R}^N) \times W^{1,2}(A,[0,1]) \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3.1}$$

*Assume that*

*(i) for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$, $\nu \in \mathbb{S}^{n-1}$ and $k \in \mathbb{N}$ the limit*

$$\lim_{r\to+\infty} \frac{m_{\mathrm{b}}^f(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} =: f_{\mathrm{hom}}(\xi) \tag{5.3.2}$$

*exists and is independent of $x, \nu$ and $k$ ($\ell_\xi$ is defined in (5.1.8));*

*(ii) for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ the limit*

$$\lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} =: g_{\mathrm{hom}}(\zeta,\nu) \tag{5.3.3}$$

*exists and is independent of $x$ ($u_{rx,\zeta,\nu}$ is defined in (5.2.10)).*

*Let, moreover, $f^\infty_{\mathrm{hom}}$ be the recession function of $f_{\mathrm{hom}}$ (cf. (3.3.2)). Then, for every $A \in \mathcal{A}(\mathbb{R}^n)$ and every $(u,v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ we have*

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}))\text{-}\lim_{\varepsilon\to 0} \mathcal{H}_\varepsilon(u,v,A) = \mathcal{H}_{\mathrm{hom}}(u,v,A),$$

*where* $\mathcal{H}_{\hom}: L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})\times\mathcal{A}(\mathbb{R}^n)\longrightarrow[0,+\infty]$ *is the functional defined by*

$$\mathcal{H}_{\hom}(u,v,A):=\int_A f_{\hom}(\nabla u)\,\mathrm{d}x+\int_A f^\infty_{\hom}(\tfrac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|})\mathrm{d}|D^cu|+\int_{J_u\cap A} g_{\hom}([u],\nu_u)\mathrm{d}\mathcal{H}^{n-1}, \tag{5.3.4}$$

*if* $u\in(GBV(A))^N$ *and* $v=1$ $\mathcal{L}^n$*-a.e in* $A$*,* $\mathcal{H}_{\hom}(u,v,A)=+\infty$*, otherwise.*

To prove Theorem 5.3.1 we use a standard approach in homogenisation theory based on the compactness of $\Gamma$-convergence and on the so-called localization method (cf. [BD98, DM93]). As in Section 3.3.3, we first show that for every infinitesimal sequence $(\varepsilon_j)_{j\in\mathbb{N}}$, up to a subsequence, the functionals $\mathcal{H}_{\varepsilon_j}$, defined in (5.3.1), $\Gamma$-converge to some abstract functional $\widehat{\mathcal{H}}$. Then, we prove that $\widehat{\mathcal{H}}$ admits an integral representation as in (5.3.4) on $BV(A,\mathbb{R}^N)$, for every $A\in\mathcal{A}(\mathbb{R}^n)$. Eventually, thanks to (5.3.2) and (5.3.3) we deduce that $\widehat{\mathcal{H}}$ does not depend on the extracted subsequence, and hence the homogenisation result for $(\mathcal{H}_\varepsilon)_\varepsilon$ follows by the Uryshon property of $\Gamma$-convergence.

We start by proving the following abstract $\Gamma$-convergence result.

**Theorem 5.3.2** ($\Gamma$-convergence and properties of the $\Gamma$-limit)**.** *Let* $f\in\mathcal{AI}(C,\alpha)$ *and* $\mathcal{H}_\varepsilon$ *be as in* (5.3.1)*, then there exists a subsequence* $(\varepsilon_j)_{j\in\mathbb{N}}$ *and a functional* $\widehat{\mathcal{H}}: L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})\times\mathcal{A}(\mathbb{R}^n)\longrightarrow[0,+\infty]$ *such that, for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *and every* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ *with* $u\in BV(A,\mathbb{R}^N)$

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{j\to+\infty}\mathcal{H}_{\varepsilon_j}(u,1,A)=\widehat{\mathcal{H}}(u,1,A). \tag{5.3.5}$$

*Moreover* $\widehat{\mathcal{H}}$ *satisfies the following properties:*

*(i)* *(*locality*)* $\widehat{\mathcal{H}}(u_1,v_1,A)=\widehat{\mathcal{H}}(u_2,v_2,A)$ *for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *and every* $(u_1,v_1),(u_2,v_2)\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$ *such that* $(u_1,v_1)=(u_2,v_2)$ $\mathcal{L}^n$*-a.e in* $A$*;*

*(ii)* *(*semicontinuity*)* *for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *the functional* $\widehat{\mathcal{H}}(\cdot,1,A): L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)\longrightarrow[0,+\infty]$ *is lower semicontinuous;*

*(iii)* *(*upper bound*)* *for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *and every* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ *with* $u\in BV(A,\mathbb{R}^N)$ *there holds*

$$\widehat{\mathcal{H}}(u,1,A)\le C(\mathcal{L}^n(A)+|Du|(A)); \tag{5.3.6}$$

*(iv)* *(*lower bound*)* *for every* $M>0$ *there exists* $C_M>0$ *such that for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *and every* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ *with* $u\in BV(A,\mathbb{R}^N)$ *and* $\|u\|_{L^\infty(A,\mathbb{R}^N)}\le M$ *we have*

$$C_M|Du|(A)\le\widehat{\mathcal{H}}(u,1,A); \tag{5.3.7}$$

*(v)* *(*measure property*)* *for every* $A\in\mathcal{A}(\mathbb{R}^n)$*, every* $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ *such that* $u\in BV(A,\mathbb{R}^N)$*, the set function* $\widehat{\mathcal{H}}(u,1,\cdot):\mathcal{A}(A)\to[0,+\infty]$ *is the restriction of a finite Radon measure on* $A$*;*

*(vi)* *(*translation invariance in $u$*)* *for every* $A\in\mathcal{A}(\mathbb{R}^n)$ *and every* $(u,v)\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$ *we have*

$$\widehat{\mathcal{H}}(u+\zeta,v,1,A)=\widehat{\mathcal{H}}(u,v,1,A),$$

*for every* $\zeta \in \mathbb{R}^N$.

*Proof.* Given any sequence of positive real numbers decreasing to zero [DM93, Theorem 16.9] provides us with a subsequence $(\varepsilon_j)$ such that

$$\overline{\Gamma}\text{-}\lim_{j\to+\infty} \mathcal{H}_{\varepsilon_j} = \widehat{\mathcal{H}},$$

where $\widehat{\mathcal{H}} : L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0, +\infty]$ is increasing, inner regular, and superadditive as a set function and lower semicontinuous in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ as a functional. By definition of $\overline{\Gamma}$-convergence, we have

$$\widehat{\mathcal{H}}'_- = \widehat{\mathcal{H}} = \widehat{\mathcal{H}}''_-, \tag{5.3.8}$$

where

$$\widehat{\mathcal{H}}'(\cdot, A) = \Gamma\text{-}\liminf_{j\to\infty} \mathcal{H}_{\varepsilon_j}(\cdot, A), \quad \widehat{\mathcal{H}}''(\cdot, A) = \Gamma\text{-}\limsup_{j\to\infty} \mathcal{H}_{\varepsilon_j}(\cdot, A),$$

and

$$\widehat{\mathcal{H}}'_-(\cdot, A) := \sup_{A' \subset\subset A} \widehat{\mathcal{H}}'(\cdot, A'), \quad \widehat{\mathcal{H}}''_-(\cdot, A) := \sup_{A' \subset\subset A} \widehat{\mathcal{H}}''(\cdot, A').$$

The locality property and the translation invariance of $\widehat{\mathcal{H}}$ are direct consequences of (5.3.8), and of the locality and translation invariance of $\widehat{\mathcal{H}}'$ and $\widehat{\mathcal{H}}''$.

Arguing exactly as in [AF02, Lemma 5.1], for every $A \in \mathcal{A}(\mathbb{R}^n)$, every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ such that $u \in BV(A, \mathbb{R}^N)$, every $A', A'' \in \mathcal{A}(A)$ and every $B' \subset\subset A'$, with $B' \in \mathcal{A}(A)$ we can obtain that

$$\widehat{\mathcal{H}}''(u, 1, B' \cup A'') \leq \widehat{\mathcal{H}}''(u, 1, A') + \widehat{\mathcal{H}}''(u, 1, A''),$$

from which we can easily deduce that the inner regular envelope $\widehat{\mathcal{H}}''_-(u, 1, \cdot)$ is subadditive on $\mathcal{A}(A)$. Therefore, thanks to the De Giorgi-Letta Criterion, we infer that the set function $\widehat{\mathcal{H}}(u, 1, \cdot) : \mathcal{A}(A) \to [0, +\infty]$ is the restriction to the open sets of a Borel measure on $A$.

For every $A \in \mathcal{A}$, and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $u \in BV(A, \mathbb{R}^N)$, in view of (f2), we obtain

$$\widehat{\mathcal{H}}''(u, 1, A) \leq C(|Du|(A) + \mathcal{L}^n(A)).$$

Hence, in particular

$$\widehat{\mathcal{H}}(u, 1, A) \leq C(|Du|(A) + \mathcal{L}^n(A)),$$

so that using [DM93, Proposition 18.6] we get

$$\widehat{\mathcal{H}}'(u, 1, A) = \widehat{\mathcal{H}}(u, 1, A) = \widehat{\mathcal{H}}''(u, 1, A).$$

The latter eventually provides the $\Gamma$-convergence statement in (5.3.5). Eventually, the lower bound is a consequence of (f2), $C^{-1} \leq 1$ and [ABS99, Theorem 4.1 and Remark 4.2] and [AF02, Section 3.1 and Proposition 4.1]. $\square$

The next three subsections are devoted to the proof of Theorem 5.3.1. Namely, in subsections 5.3.1 - 5.5

we identify, respectively, the three measure derivatives

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}, \quad \frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}, \quad \text{and} \quad \frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^c u|}.$$

In fact, we will prove that under the assumptions of Theorem 5.3.1, for every $A \in \mathcal{A}(\mathbb{R}^n)$ and $u \in BV(A,\mathbb{R}^N)$ the following three equalities hold:

$$\begin{aligned}
&\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) = f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A,\\
&\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}}(x) = g_{\mathrm{hom}}([u](x),\nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap A,\\
&\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) = f^{\infty}_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } |D^c u|\text{-a.e. } x \in A.
\end{aligned}$$

Since in the equalities above the right-hand sides do not depend on the subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$, we will be able to conclude that $\widehat{H}$ is subsequence independent and therefore the Γ-convergence result holds for the whole sequence $(\mathcal{H}_\varepsilon)_\varepsilon$ (cf. Theorem 5.3.1).

The strategy to prove the identities above uses, on one hand, the global method for relaxation in $BV$ [BFM98] and, on the other hand, a direct (although involved) comparison argument.

For later use it is useful to recall the following notation: let $U \in \mathcal{A}_\infty(\mathbb{R}^n)$ and $G : BV(U,\mathbb{R}^N)\times\mathcal{A}(U) \longrightarrow [0,\infty)$; for every $(w,A) \in BV(U,\mathbb{R}^N)\times\mathcal{A}_\infty(U)$ set

$$m_G(w,A) := \inf\{G(u,A) \ : \ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N),\ u \in BV(A,\mathbb{R}^N),\ u = w \text{ on } \partial A\}. \tag{5.3.9}$$

In addition, we use the notation $sc^-(L^1)G$ for the relaxation of $G$ with respect to the $L^1$ convergence, namely $sc^-(L^1)G(u,A) := \Gamma(L^1)\text{-}\lim_j G(u;A)$ (cf. [DM93]).

In what follows we will use in several instances a truncation lemma that follows from De Giorgi's slicing and averaging argument on the codomain, already used in Section 3.3. We give a detailed proof since the statement is slightly different from the standard references. In particular, in Propositions 5.3.4 and 5.5.1 we will take $v \equiv 1$, while in Proposition 5.4.1 it is important that the constant $\gamma$ in (5.3.10) equals 0. We recall the notation $\mathcal{T}_k$ for the smooth truncation operators and $a_k$ for the related sequence introduced in 2.0.10.

**Lemma 5.3.3.** *Let $A \in \mathcal{A}(\mathbb{R}^n)$ and $\mathcal{G} : (GBV(A))^N \times L^1(A,[0,1]) \to [0,\infty]$ be the functional defined by*

$$\mathcal{G}(u,v) := \int_A v(x) g(x,\nabla u(x))\,\mathrm{d}x$$

*where $g : \mathbb{R}^n\times\mathbb{R}^{N\times n} \to [0,\infty)$ is a Borel function for which there exists $\gamma \in [0,\infty)$ such that*

$$c^{-1}|\xi| \le g(x,\xi) \le c(|\xi|+\gamma) \tag{5.3.10}$$

*for every $(x,\xi) \in \mathbb{R}^n\times\mathbb{R}^{N\times n}$, and for some $c > 0$.*

*Then for every $M \in \mathbb{N}$ and $(u,v) \in (GBV(A))^N \times L^1(A,[0,1])$ there exists $k \in \{M+1,\dots,2M\}$ such* *that* $\mathcal{T}_k(u) \in BV \cap L^\infty(A, \mathbb{R}^N)$ *with* $\|\mathcal{T}_k(u)\|_{L^\infty} \le a_{k+1}$*,* $\mathcal{H}^{n-1}(J_{\mathcal{T}_k(u)} \cap A) \le \mathcal{H}^{n-1}(J_u \cap A)$ *and*

$$\mathcal{G}(\mathcal{T}_k(u), v) \le \Big(1 + \frac{c^2}{M}\Big)\mathcal{G}(u, v) + c\gamma\mathcal{L}^n(\{|u| > a_M\})\,.$$

*Proof.* Let us fix $M \in \mathbb{N}$ and $(u, v) \in (GBV(A))^N \times L^1(A, [0, 1])$ with $\mathcal{G}(u, v) < \infty$, otherwise the claim follows trivially. By averaging, there exists $k \in \{M + 1, \dots, 2M\}$ such that

$$\int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)g(x, \nabla u(x))\,\mathrm{d}x \le \frac{1}{M}\mathcal{G}(u, v). \tag{5.3.11}$$

By the properties of $GBV$ functions (cf. Section 2.0.1) and the very definition of $\mathcal{T}_k$ (see (2.0.10)), we have that $\mathcal{T}_k(u) \in BV \cap L^\infty(A, \mathbb{R}^N)$ with $\|\mathcal{T}_k(u)\|_{L^\infty} \le a_{k+1}$, $J_{\mathcal{T}_k(u)} \cap A \subseteq J_u \cap A$ and $\nabla(\mathcal{T}_k(u))(x) = \nabla\mathcal{T}_k(u(x))\nabla u(x)$ for $\mathcal{L}^n$-a.e. $x \in A$. Furthermore, being $Lip(\mathcal{T}_k) \le 1$, we can check for every $y, v \in \mathbb{R}^N$ with $|v| = 1$ that $|(\nabla\mathcal{T}_k(y))v| = |\partial_v\mathcal{T}_k(y)| \le 1$ that provides $\|\nabla\mathcal{T}_k(y)\|_2 \le 1$ for every $y \in \mathbb{R}^N$ (here $\|\cdot\|_2$ stands for the matrix norm on $\mathbb{R}^{N\times N}$ induced by $|\cdot|$ on $\mathbb{R}^N$) and consequently

$$|\nabla(\mathcal{T}_k(u))(x)| \le |\nabla u(x)| \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in A. \tag{5.3.12}$$

In particular, in virtue of $\mathcal{T}_k(y) = u$ on $\{|y| < a_k\}$ and $\mathcal{T}_k(y) = 0$ on $\{|y| \ge a_{k+1}\}$, we obtain

$$\begin{aligned}
\mathcal{G}(\mathcal{T}_k(u), v) &= \int_{A\cap\{|u|<a_k\}} v(x)g(x, \nabla u)\,\mathrm{d}x + \int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)g(x, \nabla(\mathcal{T}_k(u)))\,\mathrm{d}x+\\
&+ \int_{A\cap\{|u|\ge a_{k+1}\}} v(x)g(x, 0)\,\mathrm{d}x \le \mathcal{G}(u) + c\int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)|\nabla(\mathcal{T}_k(u))|\,\mathrm{d}x\\
&+ c\gamma\mathcal{L}^n(|u| \ge a_k) \le \mathcal{G}(u) + c\int_{A\cap\{a_k\le|u|<a_{k+1}\}} v(x)|\nabla(u)|\,\mathrm{d}x + c\gamma\mathcal{L}^n(|u| > a_M)\\
&\le \Big(1 + \frac{c^2}{M}\Big)\mathcal{G}(u, v) + c\gamma\mathcal{L}^n(|u| > a_M),
\end{aligned}$$

where in the first inequality we used (5.3.10), in the second one (5.3.12) and finally in the last one (5.3.10) and (5.3.11). ☐

### 5.3.1 Identification of the volume term

This section is devoted to identify the measure derivative $\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}$ with $f_{\text{hom}}$.

**Proposition 5.3.4** (Homogenised volume integrand)**.** *Let* $f \in \mathcal{AI}(C, \alpha)$ *satisfy* (5.2.1)*. Let* $\widehat{\mathcal{H}}$ *be as in* (5.3.5)*. Then, for every* $A \in \mathcal{A}(\mathbb{R}^n)$ *and every* $u \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$*, with* $u \in BV(A, \mathbb{R}^N) \cap L^\infty(A, \mathbb{R}^N)$ *there holds*

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1, \cdot)}{\mathrm{d}\mathcal{L}^n}(x) = f_{\text{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A,$$

*where* $f_{\text{hom}}$ *is as in* (5.2.1)*.*

To prove Proposition 5.3.4, we need the two following technical lemmas.

**Lemma 5.3.5.** *Let $h \in \mathcal{AI}(C,\alpha)$ be given and define $\hat{h} : \mathbb{R}^n \times \mathbb{R}^{N\times n} \to [0,\infty)$ as*

$$\hat{h}(x,\xi) := \limsup_{\eta\to 0} \inf\left\{ \int_Q h(x+\eta z, \nabla w)\mathrm{d}z \ :\ w - \ell_\xi \in W_0^{1,1}(Q,\mathbb{R}^N)\right\}, \tag{5.3.13}$$

*Let $A \in \mathcal{A}_\infty(\mathbb{R}^n)$ and let $E^h(\cdot,A)$ and $E^{\hat{h}}(\cdot,A)$ be defined as in (5.1.4). Moreover, consider the functionals $F^h, F^{\hat{h}} : L^1(A,\mathbb{R}^N) \longrightarrow [0,\infty]$ given by*

$$F^h(u) := \begin{cases} E^h(u,A) & \text{if } u \in W^{1,1}(A,\mathbb{R}^N) \\ +\infty & \text{otherwise} \end{cases}, \quad F^{\hat{h}}(u) := \begin{cases} E^{\hat{h}}(u,A) & \text{if } u \in W^{1,1}(A,\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

*Then the following statements hold:*

*(i) if $h$ is 1-homogeneous in $\xi$, then the same holds for $\hat{h}$;*

*(ii) there exists $H \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(H) = 0$ such that for every $x \in \mathbb{R}^n \setminus H$ and every $\xi \in \mathbb{R}^{N\times n}$*

$$\hat{h}(x,\xi) \le h(x,\xi);$$

*(iii) for every $u \in W^{1,1}(A,\mathbb{R}^N)$*

$$sc^-(L^1)F^h(u) = F^{\hat{h}}(u);$$

*(iv) for every $u \in BV(A,\mathbb{R}^N)$*

$$\left| sc^-(L^1)F^h(u) - \int_A \hat{h}(x,\nabla u)\,\mathrm{d}x \right| \le C|D^s u|(A);$$

*(v) for every $u \in L^1(A,\mathbb{R}^N)$*

$$sc^-(L^1)F^h(u) = sc^-(L^1)F^{\hat{h}}(u);$$

*(vi) for every $\xi \in \mathbb{R}^{N\times n}$*

$$m_{\mathrm{b}}^h(\ell_\xi, A) = m_{\mathrm{b}}^{\hat{h}}(\ell_\xi, A).$$

*Proof.* Property $(i)$ readily follows from the definition of $\hat{h}$. Instead, $(iii)$ and $(iv)$ are a direct consequence of [BFM98, Theorem 4.1.4].

To prove $(ii)$ let $\xi \in \mathbb{Q}^{N\times n}$ be fixed, by definition we get

$$\hat{h}(x,\xi) \le \limsup_{\eta\to 0} \int_Q h(x+\eta z, \xi_j)\mathrm{d}z = \limsup_{\eta\to 0} \frac{1}{\eta^n}\int_{Q_\eta(x)} g(z,\xi)\mathrm{d}z$$

for every $x \in \mathbb{R}^n$. Then, the Lebesgue Differentiation Theorem provides us with a set $H_\xi \subset \mathbb{R}^n$ such that $\mathcal{L}^n(H_\xi) = 0$ and $\hat{h}(x,\xi) \le h(x,\xi)$ for every $x \in \mathbb{R}^n \setminus H_\xi$. Therefore, we conclude by setting

$$H := \bigcup_{\xi \in \mathbb{Q}^{N\times n}} H_\xi\,,$$

and invoking the continuity of $h$, (f3), and the lower semicontinuity of $\hat{h}$ as the bulk energy density of the functional $sc^{-}(L^1)F^{\hat{h}}$.

The proof of $(v)$ follows straightforwardly from $(ii)$ and $(iii)$.

To conclude the proof, we are left to show $(vi)$. We start noticing that in view of $(ii)$ we only need to prove that

$$m_{\mathrm{b}}^{h}(\ell_\xi, A) \le m_{\mathrm{b}}^{\hat{h}}(\ell_\xi, A).$$

for every $\xi \in \mathbb{R}^{N\times n}$. To prove the inequality above, fix $\xi \in \mathbb{R}^{N\times n}$ and let $u \in W^{1,1}(A,\mathbb{R}^N)$ satisfy $u = \ell_\xi$ on $\partial A$. By $(iii)$ we can infer the existence of a sequence $(u_j)_{j\in\mathbb{N}} \subset W^{1,1}(A,\mathbb{R}^N)$ such that $u_j \to u$ in $L^1(A,\mathbb{R}^N)$ as $j \to \infty$ and

$$\lim_{j\to\infty} E^h(u_j, A) = E^{\hat{h}}(u, A).$$

By [BFM98, Lemma 2.6 and Remark 2.7] we can find a sequence $(w_j)_{j\in\mathbb{N}} \subset W^{1,1}(A,\mathbb{R}^N)$ satisfying $w_j = \ell_\xi$ on $\partial A$ such that $w_j \to u$ in $L^1(A,\mathbb{R}^N)$ as $j \to \infty$ and

$$\limsup_{j\to\infty} E^h(w_j, A) \le \liminf_{j\to\infty} E^h(u_j, A) = E^{\hat{h}}(u, A),$$

therefore the claim follows by the arbitrariness of $u$. □

In the following lemma we state a truncation result in the same spirit as in Lemma 3.1.7.

**Lemma 5.3.6.** *Let $\mathcal{H}_\varepsilon$ be the functionals defined in (5.3.1). Then, for every $\delta \in (0,1)$, $A \in \mathcal{A}(\mathbb{R}^n)$, and $(u,v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$ with $u \in W^{1,1}(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$ and $W^{1,2}(A,[0,1])$, there exists $u^\delta \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N) \cap SBV(A,\mathbb{R}^N)$ (also depending on $A$) such that for every $\varepsilon > 0$*

$$H_\varepsilon^\delta(u^\delta, A) \le \mathcal{H}_\varepsilon(u,v,A) + C\mathcal{L}^n(\{v \le \delta\} \cap A), \tag{5.3.14}$$

*where $H_\varepsilon^\delta : L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0,+\infty]$ is the functional given by*

$$H_\varepsilon^\delta(w,A) := \begin{cases} \alpha_\delta \displaystyle\int_A f(\tfrac{x}{\varepsilon}, \nabla w)\,\mathrm{d}x + \beta_\delta \mathcal{H}^{n-1}(J_w \cap A) & \text{if } w \in SBV(A,\mathbb{R}^N) \\ +\infty & \text{otherwise,} \end{cases}$$

*with $\alpha_\delta, \beta_\delta > 0$ such that*

$$\lim_{\delta\to 1} \alpha_\delta = 1 \quad \text{and} \quad \lim_{\delta\to 1} \beta_\delta = 0.$$

*Moreover, if $(u_\varepsilon, v_\varepsilon) \to (u,1)$ in $L^1(A,\mathbb{R}^{N+1})$ as $\varepsilon \to 0$, then the corresponding $(u_\varepsilon^\delta)$ satisfies*

$$u_\varepsilon^\delta \to u \quad \text{in } L^1(A,\mathbb{R}^N) \quad \text{as } \varepsilon \to 0. \tag{5.3.15}$$

*Proof.* The argument proceeds along the same lines as in the proof of Lemma 3.1.7. □

We are now ready to identify the Radon-Nikodym derivative of $\widehat{\mathcal{H}}$ with respect to the Lebesgue measure with $f_{\mathrm{hom}}$.

*Proof of Proposition 5.3.4.* Fix $A \in \mathcal{A}(\mathbb{R}^n)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$. We divide the proof into two steps.

*Step 1:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) \leq f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A.$$

For every $A \in \mathcal{A}(\mathbb{R}^n)$ and $q \in \mathbb{Q} \cap (0,1)$ set $\widehat{\mathcal{H}}_q(u,1,A) := \widehat{\mathcal{H}}(u,1,A) + q|Du|(A)$. Thanks to [BFM98, Lemma 3.5] we obtain that for $\mathcal{L}^n$-a.e. $x \in A$

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) + q|\nabla u(x)| = \lim_{\rho\to 0} \frac{m_{\widehat{\mathcal{H}}_q}(\ell_{\nabla u(x)}, Q_\rho(x))}{\rho^n} \tag{5.3.16}$$

where where $m_{\widehat{\mathcal{H}}_q}$ is as in (5.3.9). Let $x \in A$ be that (5.3.16) holds, and set $\xi := \nabla u(x)$. In view of (5.2.1), for every $\rho > 0$ we have

$$f_{\mathrm{hom}}(\xi) = \lim_{r\to+\infty} \frac{m^f_{\mathrm{b}}(\ell_\xi, Q_r(\frac{r}{\rho}x))}{r^n}. \tag{5.3.17}$$

Fix $\eta \in (0,1)$. By (5.1.6), for every $\rho, r > 0$ there exists $w^\rho_r \in W^{1,1}(Q_r(\frac{r}{\rho}x), \mathbb{R}^N)$ with $w^\rho_r = \ell_\xi$ on $\partial Q_r(\frac{r}{\rho}x)$, such that

$$\int_{Q_r(\frac{r}{\rho}x)} f(y, \nabla w^\rho_r)\mathrm{d}y \leq m^f_{\mathrm{b}}(\ell_\xi, Q_r(\tfrac{r}{\rho}x)) + \eta r^n\,. \tag{5.3.18}$$

Thus, for every $\rho > 0$, (5.3.17) and (5.3.18) yield

$$\limsup_{r\to+\infty} \frac{1}{r^n} \int_{Q_r(\frac{r}{\rho}x)} f(y, \nabla w^\rho_r)\mathrm{d}y \leq f_{\mathrm{hom}}(\xi) + \eta.$$

Now, let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as in (5.3.5) and set $r = \frac{\rho}{\varepsilon_j}$. Define $u^\rho_{\varepsilon_j} : \mathbb{R}^n \to \mathbb{R}^N$ as

$$u^\rho_{\varepsilon_j}(y) := \begin{cases} \varepsilon_j w^\rho_r\left(\frac{y}{\varepsilon_j}\right) & \text{if } y \in Q_\rho(x) \\ \ell_\xi(y) & \text{if } y \in \mathbb{R}^n \setminus Q_\rho(x), \end{cases}$$

therefore $u^\rho_{\varepsilon_j} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $u^\rho_{\varepsilon_j} = \ell_\xi$ on $\mathbb{R}^n \setminus Q_\rho(x)$. Changing variables and again invoking (5.3.18), for every $\rho > 0$ we get

$$\limsup_{j\to+\infty} \frac{1}{\rho^n} \int_{Q_{\rho(1+\eta)}(x)} f(\tfrac{y}{\varepsilon_j}, \nabla u^\rho_{\varepsilon_j})\mathrm{d}y \leq f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1) \tag{5.3.19}$$

where we also used (f2), the fact that $\nabla u^\rho_{\varepsilon_j} = \xi$ on $Q_{\rho(1+\eta)}(x)\setminus\overline{Q_\rho(x)}$, and (5.3.5). Appealing to (5.3.19), (f2), and the Poincaré Inequality, for every $\rho$ we can find a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ (not relabeled) such that $u^\rho_{\varepsilon_j}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ to some $u^\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N) \cap BV(Q_{\rho(1+\eta)}(x), \mathbb{R}^N)$ with $u^\rho = \ell_\xi$ on $\partial Q_{\rho(1+\eta)}(x)$.

Moreover, by (5.3.5), (5.3.19), and (f2), for every $\rho > 0$, we have that

$$\begin{aligned}\frac{m_{\widehat{\mathcal{H}}}(\ell_\xi, Q_{\rho(1+\eta)}(x))}{\rho^n} &\leq \frac{\widehat{\mathcal{H}}(u^\rho, 1, Q_{\rho(1+\eta)}(x)) + q|Du^\rho|(Q_{\rho(1+\eta)}(x))}{\rho^n} \\ &\leq \liminf_{j\to+\infty}\Big(\tfrac{\mathcal{H}_{\varepsilon_j}(u^\rho_{\varepsilon_j}, 1, Q_{\rho(1+\eta)}(x))}{\rho^n} + \tfrac{q}{\rho^n}\int_{Q_{\rho(1+\eta)}(x)} |\nabla u^\rho_{\varepsilon_j}| \mathrm{d}y\Big) \\ &\leq \big(f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1))\big)(1+qC).\end{aligned}$$

Eventually, by (5.3.16) and taking the limit as $\rho \to 0$ we get

$$\begin{aligned}(1+\eta)^n\Big(\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) + q\xi\Big) &= \lim_{\rho\to 0}\frac{m_{\widehat{\mathcal{H}}}(\ell_\xi, Q_{\rho(1+\eta)}(x))}{\rho^n} \\ &\leq \big(f_{\mathrm{hom}}(\xi) + \eta + C(|\xi|+1)((1+\eta)^n - 1)\big)(1+qC),\end{aligned}$$

hence the claim follows by letting $\eta, q \to 0$.

*Step 2:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{L}^n}(x) \geq f_{\mathrm{hom}}(\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in A.$$

Let $A' \in \mathcal{A}(A)$, by Theorem 5.3.2, we can find a sequence $(u_j, v_j)_{j\in\mathbb{N}} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ such that $(u_j, v_j) \in W^{1,1}(A', \mathbb{R}^N) \times W^{1,2}(A', [0,1])$, $(u_j, v_j) \to (u, 1)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$, $v_j(x) \to 1$ for $\mathcal{L}^n$-a.e. $x \in A'$ as $j \to +\infty$ and

$$\lim_{j\to+\infty}\mathcal{H}_{\varepsilon_j}(u_j, v_j, A') = \widehat{\mathcal{H}}(u, 1, A'). \tag{5.3.20}$$

Let $\delta \in (0,1)$ be fixed; by Lemma 5.3.6 we have

$$H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq \mathcal{H}_{\varepsilon_j}(u_j, v_j, A') + C\mathcal{L}^n(\{v_j \leq \delta\} \cap A'),$$

where $(u^\delta_j) \subset SBV(A', \mathbb{R}^N)$ with $u^\delta_j \to u$ in $L^1(A', \mathbb{R}^N)$. Therefore by (5.3.20) we get

$$\liminf_{j\to+\infty} H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq \widehat{\mathcal{H}}(u, 1, A'), \tag{5.3.21}$$

since $(v_j)_{j\in\mathbb{N}}$ converges in measure to 1 on $A'$.

We now consider the measures $\mu^\delta_j$ defined on $A'$ as follows

$$\mu^\delta_j := \alpha_\delta f(\tfrac{x}{\varepsilon_j}, \nabla u^\delta_j)\mathcal{L}^n \llcorner A' + \beta_\delta \mathcal{H}^{n-1} \llcorner (J_{u^\delta_j} \cap A')\,.$$

Note that by (5.3.21), there is a subsequence (not relabeled) and a finite Radon measure $\mu^\delta$ on $A'$ such that $\mu^\delta_j \overset{*}{\rightharpoonup} \mu^\delta$ as $j \to +\infty$.

Now let $x_0 \in A'$ be a point of approximate differentiability of $u$, and additionally assume that

$$\lim_{\rho\to 0}\frac{\mu^\delta(Q_\rho(x_0))}{\rho^n} = \frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)\,. \tag{5.3.22}$$

Such conditions determine a subset of full measure in $A'$. Then, consider the rescaled function $u^\rho : Q_1 \to \mathbb{R}^N$ given by

$$u^\rho(y) := \frac{u(x_0+\rho y) - u(x_0)}{\rho},$$

thanks to [AFP00, Remark 3.72] we have $u^\rho \to \ell_\xi$ in $L^1(Q_1, \mathbb{R}^N)$, where $\xi := \nabla u(x_0)$.

By the weak*-convergence of $\mu_j^\delta$ towards $\mu^\delta$ we have

$$\begin{aligned} \frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) &= \lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho(x_0))}{\rho^n} = \lim_{\rho\to 0,\ \rho\in I(x_0)} \lim_{j\to+\infty} \frac{\mu_j^\delta(Q_\rho(x_0))}{\rho^n} \\ &= \lim_{\rho\to 0,\ \rho\in I(x_0)} \lim_{j\to+\infty} \rho^{-n}\Big(\alpha_\delta \int_{Q_\rho(x_0)} f(\tfrac{x}{\varepsilon_j}, \nabla u_j^\delta)\,\mathrm{d}x + \beta_\delta \mathcal{H}^{n-1}(J_{u_j^\delta} \cap Q_\rho(x_0))\Big) \end{aligned} \tag{5.3.23}$$

where $I(x_0) := \{\rho \in (0, \frac{2}{\sqrt{n}}\mathrm{dist}(x_0, \partial A'))\colon \mu^\delta(\partial Q_\rho(x_0)) = 0\}$.

For every $\rho$ and $j$, define the rescalings $u_j^\rho \in SBV(A', \mathbb{R}^N)$ by

$$u_j^\rho(y) := \frac{u_j^\delta(x_0+\rho y) - u(x_0)}{\rho},$$

then $u_j^\rho \to u^\rho$ in $L^1(Q_1, \mathbb{R}^N)$ as $j \to +\infty$. Furthermore, thanks to (5.3.23) we get

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) = \lim_{\rho\to 0,\ \rho\in I(x_0)} \lim_{j\to+\infty} \Big(\alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla u_j^\rho)\mathrm{d}y + \frac{\beta_\delta}{\rho}\mathcal{H}^{n-1}(J_{u_j^\rho} \cap Q_1)\Big). \tag{5.3.24}$$

Fix $M \in \mathbb{N}$, for every $\rho$ and $j$, we apply Lemma 5.3.3 with $v \equiv 1$ so that there is $k_{\rho,j} \in \{M+1, \dots, 2M\}$ such that $\hat{u}_j^\rho := \mathcal{T}_{k_{\rho,j}}(u_j^\rho) \in SBV(Q_1, \mathbb{R}^N)$,

$$\int_{Q_1} f(\tfrac{x_0+\rho_i y}{\varepsilon_i}, \nabla \hat{u}_j^\rho)\mathrm{d}y \le \Big(1 + \frac{C^2}{M}\Big) \int_{Q_1} f(\tfrac{x_0+\rho_i y}{\varepsilon_i}, \nabla u_j^\rho)\mathrm{d}y + C\mathcal{L}^n(\{|u_j^\rho| \ge a_M\}). \tag{5.3.25}$$

Up to subsequences (not relabeled) we can assume that $k_{\rho,j} \in \{M+1, \dots, 2M\}$ actually depends only on $\rho$. If we choose $a_M > \sup_{y\in Q_1} \ell_\xi(y)$ we get that

$$\lim_{\rho\to 0} \lim_{j\to+\infty} \hat{u}_j^\rho = \ell_\xi \quad \text{in } L^1(Q_1, \mathbb{R}^N)$$

and

$$\lim_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \mathcal{L}^n(\{|u_j^\rho| \ge a_M\}) = 0, \tag{5.3.26}$$

since we also have that $\lim_{\rho\to 0}\lim_{j\to+\infty} u_j^\rho = \ell_\xi$ in $L^1(Q_1, \mathbb{R}^N)$. In particular, for $M$ is large enough, by combining (5.3.24), (5.3.25), and (5.3.26) we can infer

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \ge \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{u}_j^\rho)\mathrm{d}y, \tag{5.3.27}$$

and

$$\lim_{\rho\to 0, \rho\in I(x_0)} \limsup_{j\to+\infty} \mathcal{H}^{n-1}(J_{\hat{u}_j^\rho} \cap Q_1) = 0, \tag{5.3.28}$$

since $\mathcal{T}_{k_{\rho,j}} \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)$ is finite, and $\beta_\delta > 0$. Set

$$\tau_{\rho,j} := \|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q_1,\mathbb{R}^N)} + \frac{\rho}{j};$$

then, $\lim\limits_{\rho\to 0}\lim\limits_{j\to+\infty} \tau_{\rho,j} = 0$. Thus, for every $\rho > 0$ small and every $j$ large (depending on $\rho$) we have $\tau_{\rho,j} \in (0,1)$. Therefore, thanks to the Coarea formula and to the properties of the traces of $BV$ functions on rectifiable sets (see [AFP00, Theorem 3.77]), there exists $\hat{r}_{\rho,j} \in (1 - \tau_{\rho,j}^{1/2}, 1)$ such that

$$\int_{\partial Q_{\hat{r}_{\rho,j}}} |(\hat{u}_j^\rho)^- - \ell_\xi| \mathrm{d}\mathcal{H}^{n-1} \le \tau_{\rho,j}^{-1/2} \|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q_1,\mathbb{R}^N)} \le \tau_{\rho,j}^{1/2}, \tag{5.3.29}$$

where $(\hat{u}_j^\rho)^-$ is the inner trace of $\hat{u}_j^\rho$ on $\partial Q_{\hat{r}_{\rho,j}}$. Therefore, defining the functions $w_j^\rho \in SBV(Q_1, \mathbb{R}^N)$ as

$$w_j^\rho(y) := \begin{cases} \hat{u}_j^\rho(y) & \text{if } y \in Q_{\hat{r}_{\rho,j}} \\ \ell_\xi(y) & \text{if } y \in Q_1 \setminus Q_{\hat{r}_{\rho,j}}, \end{cases}$$

thanks to (f2) we have that

$$\alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{u}_j^\rho) \mathrm{d}y + \alpha_\delta (C|\xi| + 1)\mathcal{L}^n(Q_1 \setminus Q_{\hat{r}_{\rho,j}}) \ge \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^\rho) \mathrm{d}y$$

and, since $\lim\limits_{\rho\to 0}\lim\limits_{j\to+\infty} \hat{r}_{\rho,j} = 1$, from (5.3.27) we obtain

$$\Big(1 + \frac{C^2}{M}\Big) \frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \ge \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to+\infty} \alpha_\delta \int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^\rho) \mathrm{d}y. \tag{5.3.30}$$

Furthermore, thanks to (5.3.29), (2.0.10), and to the definition of $\hat{u}_j^\rho$ we can estimate the singular part of $Dw_j^\rho$ as follows

$$|D^s w_j^\rho|(Q_1) \le \int_{J_{w_j^\rho} \cap Q_1} |[w_j^\rho]| \mathrm{d}\mathcal{H}^{n-1} \le \tau_{\rho,j}^{1/2} + 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho} \cap Q_1). \tag{5.3.31}$$

Now, for every $\rho >$ and $j \in \mathbb{N}$, consider functional $F_{\rho,j} : L^1(Q_1, \mathbb{R}^N) \longrightarrow [0, \infty]$ given by

$$F_{\rho,j}(w) := \begin{cases} \displaystyle\int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w) \mathrm{d}y & \text{if } w \in W^{1,1}(Q_1, \mathbb{R}^N) \\ \infty & \text{otherwise.} \end{cases}$$

In view of Lemma 5.3.5 (iv), for every $w \in SBV(Q_1, \mathbb{R}^N)$ we have

$$\Big| sc^-(L^1)F_{\rho,j}(w) - \int_{Q_1} f_{\rho,j}(y, \nabla w) \mathrm{d}y \Big| \le C \int_{J_w \cap Q_1} |[w]| \mathrm{d}\mathcal{H}^{n-1}, \tag{5.3.32}$$

where $f_{\rho,j} := \hat{h}$, with $h(x,\xi) := f\left(\frac{x_0+\rho x}{\varepsilon_j}, \xi\right)$ for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}$ (cf. (5.3.13)). By [BFM98, Lemma 2.6], for every $\rho$ and $j$ we can find $\hat{w}_j^\rho \in W^{1,1}(Q_1,\mathbb{R}^N)$ with $\hat{w}_j^\rho = \ell_\xi$ on $\partial Q_1$ and such that

$$\Big| sc^-(L^1)F_{\rho,j}(w_j^\rho) - \int_{Q_1} f_{\rho,j}(y,\nabla \hat{w}_j^\rho)\mathrm{d}y \Big| \le \frac{\rho}{j}. \tag{5.3.33}$$

In particular, from (5.3.31), (5.3.32) and (5.3.33) and the equality $f_{\rho,j}(y,\xi) = \hat{f}(\frac{x_0+\rho y}{\varepsilon_j},\xi)$ which follows from formula (5.3.13), we infer

$$\begin{aligned}
\int_{Q_1} f(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla w_j^\rho)\mathrm{d}y &\ge \int_{Q_1} f_{\rho,j}(y,\nabla w_j^\rho)\mathrm{d}y \ge sc^-(L^1)F_{\rho,j}(w_j^\rho) - C\int_{J_{w_j^\rho}\cap Q_1} |[w_j^\rho]|\mathrm{d}\mathcal{H}^{n-1} \\
&\ge \int_{Q_1} f_{\rho,j}(y,\nabla \hat{w}_j^\rho)\mathrm{d}y - \frac{\rho}{j} - C(\tau_{\rho,j}^{1/2} + 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho}\cap Q_1)) \\
&= \int_{Q_1} \hat{f}(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla \hat{w}_j^\rho)\mathrm{d}y - \tfrac{\rho}{j} - C(\tau_{\rho,j}^{1/2} + 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^\rho}\cap Q_1))\,.
\end{aligned} \tag{5.3.34}$$

Setting

$$r_{\rho,j} = \frac{\rho}{\varepsilon_j} \quad \text{and} \quad \overline{w}_j^\rho(x) := r_{\rho,j}\hat{w}_j^\rho(\tfrac{x}{r_{\rho,j}} - \tfrac{x_0}{\rho})$$

we have $\overline{w}_j^\rho \in W^{1,1}(Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0),\mathbb{R}^N)$ with $\overline{w}_j^\rho = \ell_\xi - \frac{1}{\varepsilon_j}x_0$ on $\partial Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)$ and

$$\int_{Q_1} \hat{f}(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla \hat{w}_j^\rho)\mathrm{d}y = \frac{1}{r_{\rho,j}^n}\int_{Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)} \hat{f}(x,\nabla \overline{w}_j^\rho)\,\mathrm{d}x.$$

In particular, Lemma 5.3.5 (vi) gives

$$\int_{Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0)} \hat{f}(x,\nabla \overline{w}_j^\rho)\,\mathrm{d}x \ge m_{\mathrm{b}}^{\hat{f}}(\ell_\xi, Q_{r_{\rho,j}}(\tfrac{r_{\rho,j}}{\rho}x_0)) = m_{\mathrm{b}}^{f}(\ell_\xi, Q_{r_{\rho,j}}(\tfrac{r_{\rho,j}}{\rho}x_0))\,.$$

Therefore, (5.3.30), (5.3.34) and (5.2.1) yield

$$\begin{aligned}
\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) &\ge \limsup_{\rho\to 0,\ \rho\in I(x_0)}\ \limsup_{j\to+\infty} \alpha_\delta \frac{m_{\mathrm{b}}^{f}(\ell_\xi, Q_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x_0))}{r_{\rho,j}^n} \\
&= \alpha_\delta f_{\mathrm{hom}}(\xi) = \alpha_\delta f_{\mathrm{hom}}(\nabla u(x_0))
\end{aligned}$$

and thus

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) \ge \alpha_\delta f_{\mathrm{hom}}(\nabla u(x_0))$$

by letting $M\to\infty$. Hence, recalling (5.3.21), we deduce that

$$\widehat{\mathcal{H}}(u,1,A') \ge \liminf_{j\to\infty} H_{\varepsilon_j}^\delta(u_j^\delta,A') = \liminf_{j\to\infty}\mu_j^\delta(A') \ge \mu^\delta(A') \ge \alpha_\delta\int_{A'} f_{\mathrm{hom}}(\nabla u)\,\mathrm{d}x\,.$$

Eventually, the claim follows by letting $\delta\to 0$ and by the arbitrariness of $A'\in\mathcal{A}(A)$. □

## 5.4 Identification of the surface term

In this subsection we show that the Radon Nikodym derivative of $\widehat{\mathcal{H}}$ with respect to $\mathcal{H}^{n-1}\llcorner J_u$ equals to $g_{\rm hom}$ for every $u \in BV$.

**Proposition 5.4.1** (Homogenised surface integrand)**.** *Let $f \in \mathcal{AI}(C,\alpha)$ satisfy (5.2.15). Let $\widehat{\mathcal{H}}$ be as in (5.3.5). Then, for every $A \in \mathcal{A}(\mathbb{R}^n)$ and every $u \in L^1_{\rm loc}(\mathbb{R}^n,\mathbb{R}^N)$, with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$ there holds*

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) = g_{\rm hom}([u](x),\nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap A,$$

*where $g_{\rm hom}$ is as in (5.2.15).*

To prove Proposition 5.4.1 we need a preliminary lemma which is an extension to our setting of some results contained in [BFM98].

**Lemma 5.4.2.** *Let $U \in \mathcal{A}(\mathbb{R}^n)$ be fixed and let $G : BV(U,\mathbb{R}^N) \times \mathcal{A}(U) \longrightarrow [0,\infty)$ be such that*

*(1) for every $u \in BV(U,\mathbb{R}^N)$ the set function $G(u,\cdot)$ is the restriction to $\mathcal{A}(U)$ of a finite Radon measure on $U$;*

*(2) for every $A \in \mathcal{A}(U)$ the functional $G(\cdot,A)$ is $L^1(A,\mathbb{R}^n)$-lower semicontinuous;*

*(3) there exists $K \in (0,\infty)$ such that*

$$G(u,A) \le K(\mathcal{L}^n(A) + |Du|(A))$$

*for every $u \in BV(U,\mathbb{R}^N)$ and every $A \in \mathcal{A}(U)$.*

*(4) For every $M \in (0,\infty)$ there exists $K_M \in (0,\infty)$ such that*

$$K_M|Du|(A) \le G(u,A)$$

*for every $u \in BV(U,\mathbb{R}^N)$ with $\|u\|_{L^\infty(U)} \le M$ and every $A \in \mathcal{A}(U)$.*

*Then, if $w \in BV(U,\mathbb{R}^N)$ is such that $2\|w\|_{L^\infty(U)} \le M$ we have that for $\mathcal{H}^{n-1}$-a.e. $x \in J_w$*

$$\frac{\mathrm{d}G(w,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_w}(x) = \lim_{\rho\to 0}\frac{m^M_G(w,Q^{\nu_w(x)}_\rho(x))}{\rho^{n-1}} = \lim_{\rho\to 0}\frac{m^M_G(u_{x,[w](x),\nu_w(x)},Q^{\nu_w(x)}_\rho(x))}{\rho^{n-1}} \tag{5.4.1}$$

*where*

$$m^M_G(w,A) := \inf\{G(v,A) :\ v \in BV(A,\mathbb{R}^N),\ v = w \text{ on } \partial A,\ \|v\|_{L^\infty(A,\mathbb{R}^N)} \le M\}. \tag{5.4.2}$$

*Proof.* The proof follows by combining a number of arguments from [BFM98, Section 3] which we briefly summarize. Appealing to [BFM98, Lemma 3.5 and formula (3.17) in Theorem 3.7] the equality in (5.4.1) can be established for functionals $G$ satisfying assumptions (1)-(3) above, and the stronger growth condition

$$C|Du|(A) \le G(u,A), \tag{5.4.3}$$

for every $u \in BV(A,\mathbb{R}^N)$.

In their turn, [BFM98, Lemma 3.5 and formula (3.17) in Theorem 3.7] are a consequence of [BFM98, Lemmata 3.1 and 3.3]. Namely, [BFM98, Lemmata 3.1] establishes the Lipschitz continuity of $m_G$ as in (5.3.9), with respect to the traces and is stated under the sole positivity of $G$. It is easy to check that an analogous result holds true for $m_G^M$ as in (5.4.2).

Moreover, (5.4.3) is used in [BFM98, Lemma 3.3] to prove the equality $G(u,A) = \sup_{\delta>0} m_{G,\delta}(u,A)$, where

$$m_{G,\delta}(u,A) := \inf\Big\{\sum_{i\in\mathbb{N}} m_G(u, Q_{r_i}^{\nu_i}(x_i)) : Q_{r_i}^{\nu_i}(x_i) \subset A, Q_{r_i}^{\nu_i}(x_i) \cap Q_{r_j}^{\nu_j}(x_j) = \emptyset, i \neq j,$$
$$\mathrm{diam} Q_{r_i}^{\nu_i}(x_i) < \delta,\, \mu(A \setminus \cup_{i\in\mathbb{N}} Q_{r_i}^{\nu_i}(x_i)) = 0\Big\},$$

with $\mu := \mathcal{L}^n + |D^s u|$.

Then to conclude we notice that the same identity holds true for $m_G^M$ under the assumptions (1)-(4). In fact, one inequality is trivial, while the other can be obtained by exhibiting a competitor with the same $L^\infty$ bound. $\square$

We are now ready to show Proposition 5.4.1.

*Proof of Proposition 5.4.1.* Let us fix $A \in \mathcal{A}(\mathbb{R}^n)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u \in BV(A,\mathbb{R}^N) \cap L^\infty(A,\mathbb{R}^N)$. We divide the proof into two steps.

*Step 1:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \leq g_{\mathrm{hom}}([u](x),\nu_u(x)) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_u \cap A.$$

Thanks to Theorem 5.3.2 and Lemma 5.4.2, we obtain for $\mathcal{H}^{n-1}$-a.e. $x \in J_u \cap A$ that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) = \lim_{\rho\to 0} \frac{m^L_{\widehat{\mathcal{H}}}(u, Q_\rho^{\nu_u(x)}(x))}{\rho^{n-1}} = \lim_{\rho\to 0} \frac{m^L_{\widehat{\mathcal{H}}}(u_{x,[u](x),\nu_u(x)}, Q_\rho^{\nu_u(x)}(x))}{\rho^{n-1}}, \tag{5.4.4}$$

for every $L > 0$ with $2\|u\|_{L^\infty(A)} \leq L$.

Fix $x \in A \cap J_u$ such that (5.4.4) holds and set $\zeta = [u](x)$ and $\nu_u(x) = \nu$. By combining (5.2.15) and Corollary 5.2.5, for every $\rho > 0$ we get

$$g_{\mathrm{hom}}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu}, Q_r^\nu(\frac{r}{\rho}x))}{r^{n-1}}. \tag{5.4.5}$$

Let $\eta \in (0,1)$ be fixed. By (5.4.5), for every $\rho$ and every $r$ large enough, we have that

$$m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu}, Q_r^\nu(\tfrac{r}{\rho}x)) \leq g_{\mathrm{hom}}(\zeta,\nu) r^{n-1} + \eta r^{n-1}.$$

Therefore there exist $w_r^\rho \in W^{1,1}(Q_r^\nu(\frac{r}{\rho}x),\mathbb{R}^N)$ and $v_r^\rho \in W^{1,2}(Q_r^\nu(\frac{r}{\rho}x),[0,1])$ with $(w_r^\rho, v_r^\rho) = (\overline{u}_{\frac{r}{\rho}x,\zeta,\nu}, 1)$ on

$\partial Q_r^\nu(\frac{r}{\rho}x)$, such that

$$S^{f^\infty}(w_r^\rho, v_r^\rho, Q_r^\nu(\tfrac{r}{\rho}x)) = \int_{Q_r^\nu(\frac{r}{\rho}x)} \big((v_r^\rho)^2 f^\infty(y, \nabla w_r^\rho) + (1-v_r^\rho)^2 + |\nabla v_r^\rho|^2\big)\,\mathrm{d}y \leq$$
$$\leq m_{\mathrm{s}}^{f^\infty}(\overline{u}_{\frac{r}{\rho}x,\zeta,\nu}, Q_r^\nu(\tfrac{r}{\rho}x)) + \eta r^{n-1} \leq g_{\mathrm{hom}}(\zeta,\nu) r^{n-1} + 2\eta r^{n-1}. \tag{5.4.6}$$

Next apply Lemma 5.3.3 with $\gamma = 0$ to infer that $\mathcal{T}_{k_r^\rho}(w_r^\rho) =: \hat{w}_r^\rho \in W^{1,1}(Q_r^\nu(\frac{r}{\rho}x), \mathbb{R}^N)$ satisfies

$$S^{f^\infty}(\hat{w}_r^\rho, v_r^\rho, Q_r^\nu(\tfrac{r}{\rho}x)) \leq \Big(1 + \frac{C^2}{M}\Big) S^{f^\infty}(w_r^\rho, v_r^\rho, Q_r^\nu(\tfrac{r}{\rho}x))$$
$$\leq \Big(1 + \frac{C^2}{M}\Big)\big(g_{\mathrm{hom}}(\zeta,\nu) r^{n-1} + 2\eta r^{n-1}\big). \tag{5.4.7}$$

Moreover, if $M \in \mathbb{N}$ is such that $(2|\zeta| + 1) + 2\|u\|_{L^\infty(A)} \leq a_{2M}$ then $\hat{w}_r^\rho = \overline{u}_{\frac{r}{\rho}x,\zeta,\nu}$ on $\partial Q_r^\nu(\frac{r}{\rho}x)$ and $\|\hat{w}_r^\rho\|_{L^\infty(Q_r^\nu(\frac{r}{\rho}x))} \leq a_{2M+1}$.

In particular, by (5.4.7) we have that

$$\frac{1}{r^{n-1}} \int_{Q_r^\nu(\frac{r}{\rho}x)} (v_r^\rho)^2 |\nabla \hat{w}_r^\rho| \mathrm{d}y \leq C\Big(1 + \frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu) + 2\eta). \tag{5.4.8}$$

By Lemma 5.2.2 and (5.4.8), given $\rho > 0$, for every $r$ large enough we have

$$\frac{1}{r^{n-1}} \int_{Q_r^\nu(\frac{r}{\rho}x)} |(v_r^\rho)^2 f^\infty(y, \nabla \hat{w}_r^\rho) - (v_r^\rho)^2 \tfrac{\rho}{r} f(y, \tfrac{r}{\rho}\nabla \hat{w}_r^\rho)| \mathrm{d}y$$
$$\leq K\rho + \frac{K\rho^\alpha}{r^{(n-1)(1-\alpha)}}\Big(\int_{Q_r^\nu(\frac{r}{\rho}x)} (v_r^\rho)^2 |\nabla \hat{w}_r^\rho| \mathrm{d}y\Big)^{1-\alpha}$$
$$\leq K\rho + K\rho^\alpha C^{1-\alpha}\Big(1 + \frac{C^2}{M}\Big)^{1-\alpha}(g_{\mathrm{hom}}(\zeta,\nu) + 2\eta)^{1-\alpha}. \tag{5.4.9}$$

Therefore, recollecting (5.4.5), (5.4.6) and (5.4.9), for every $\rho > 0$ we get the following

$$\limsup_{r\to+\infty} \frac{1}{r^{n-1}} \int_{Q_r^\nu(\frac{r}{\rho}x)} ((v_r^\rho)^2 \tfrac{\rho}{r} f(y, \tfrac{r}{\rho}\nabla \hat{w}_r^\rho) + (1-v_r^\rho)^2 + |\nabla v_r^\rho|^2)\mathrm{d}y$$
$$\leq \Big(1 + \frac{C^2}{M}\Big)(g_{\mathrm{hom}}(\zeta,\nu) + 2\eta) + \tilde{K}(M,\rho,\eta),$$

where $\tilde{K}(M,\rho,\eta) := K\rho + K\rho^\alpha C^{1-\alpha}\big(1 + \frac{C^2}{M}\big)^{1-\alpha}(g_{\mathrm{hom}}(\zeta,\nu) + 2\eta)^{1-\alpha}$.

Given $\varepsilon > 0$ and $\rho > 0$, we define $(\hat{u}_\varepsilon^\rho, \hat{v}_\varepsilon^\rho) : \mathbb{R}^n \to \mathbb{R}^{N+1}$ as follows

$$\hat{u}_\varepsilon^\rho(y) := \begin{cases} \hat{w}_r^\rho(\frac{ry}{\rho}) & \text{if } y \in Q_\rho^\nu(x) \\ \overline{u}_{x,\zeta,\nu}^\varepsilon & \text{if } y \in \mathbb{R}^n \setminus Q_\rho^\nu(x) \end{cases} \qquad \hat{v}_\varepsilon^\rho(y) := \begin{cases} \hat{v}_r^\rho(\frac{ry}{\rho}) & \text{if } y \in Q_\rho^\nu(x) \\ 1 & \text{if } y \in \mathbb{R}^n \setminus Q_\rho^\nu(x) \end{cases}$$

with $r = \frac{\rho}{\varepsilon}$. Thereby $\hat{u}_\varepsilon^\rho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $\|\hat{u}_\varepsilon^\rho\|_{L^\infty(\mathbb{R}^n)} \leq a_{2M+1}$ and $\hat{v}_\varepsilon^\rho \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n, [0,1])$. Changing variables it is immediate to get

$$\limsup_{j\to+\infty}\Big(\frac{1}{\rho^{n-1}}\int_{Q^\nu_\rho(x)}((\hat v^\rho_{\varepsilon_j})^2 f(\tfrac{y}{\varepsilon_j},\nabla\hat u^\rho_{\varepsilon_j})+\tfrac{(1-\hat v^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat v^\rho_{\varepsilon_j}|^2)\mathrm{d}y\Big)$$
$$\le\Big(1+\frac{C^2}{M}\Big)(g_{\hom}(\zeta,\nu)+2\eta)+\tilde K(M,\rho,\eta),$$

where $(\varepsilon_j)_{j\in\mathbb{N}}$ is the sequence in Theorem 5.3.2 along which the $\Gamma$-convergence of $(\mathcal{H})_{\varepsilon>0}$ holds. Moreover, we observe that

$$\frac{1}{\rho^{n-1}}\int_{Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x)}((\hat v^\rho_{\varepsilon_j})^2 f(\tfrac{y}{\varepsilon_j},\nabla\hat u^\rho_{\varepsilon_j})+\tfrac{(1-\hat v^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat v^\rho_{\varepsilon_j}|^2)\mathrm{d}y$$
$$\le\frac{C}{\rho^{n-1}}\int_{Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x)}(1+|\nabla\overline{u}^{\varepsilon_j}_{x,\zeta,\nu}|)\mathrm{d}y$$
$$\le C\rho\left((1+\eta)^n-1\right)+\frac{C}{\rho^{n-1}}\int_{(Q^\nu_{\rho(1+\eta)}(x)\setminus Q^\nu_\rho(x))\cap\{|(y-x)\cdot\nu|\le\varepsilon_j/2\}}|\nabla\overline{u}^{\varepsilon_j}_{x,\zeta,\nu}|\mathrm{d}y$$
$$\le C\rho\left((1+\eta)^n-1\right)+C|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\left((1+\eta)^{n-1}-1\right)\le C((1+\eta)^n-1)(\rho+|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}).$$

Therefore, for every $\rho$, we have that

$$\sup_{j\in\mathbb{N}}\int_{Q^\nu_{\rho(1+\eta)}(x)}((\hat v^\rho_{\varepsilon_j})^2|\nabla\hat u^\rho_{\varepsilon_j}|+\frac{(1-\hat v^\rho_{\varepsilon_j})^2}{\varepsilon_j}+\varepsilon_j|\nabla\hat v^\rho_{\varepsilon_j}|^2\mathrm{d}y)<+\infty.$$

From $\|\hat u^\rho_{\varepsilon_j}\|_{L^\infty(Q^\nu_\rho(x))}\le a_{2M+1}$ and [AF02, Lemma 7.1] there exists a subsequence (not relabeled) of $(\varepsilon_j)_{j\in\mathbb{N}}$ and $u^\rho\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ such that $(\hat u^\rho_{\varepsilon_j},\hat v^\rho_{\varepsilon_j})\to(u^\rho,1)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1})$, $u^\rho\in BV(Q^\nu_{\rho(1+\eta)}(x),\mathbb{R}^N)$, $u^\rho(y)=u_{x,\zeta,\nu}(y)$ for $\mathcal{L}^n$-a.e. $y\in\mathbb{R}^n\setminus Q^\nu_\rho(x)$. By assumption (ii) In Theorem 5.3.1 (cf. formula (5.3.2)), it follows that for every $\rho$

$$\frac{m^{a_{2M+1}}_{\widehat{\mathcal{H}}}(u_{x,\zeta,\nu},Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}\le\frac{\widehat{\mathcal{H}}(u^\rho,1,Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}\le\liminf_{j\to+\infty}\frac{\mathcal{H}_{\varepsilon_j}(\hat u^\rho_{\varepsilon_j},\hat v^\rho_{\varepsilon_j},Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}$$
$$\le\Big(1+\frac{C^2}{M}\Big)(g_{\hom}(\zeta,\nu)+2\eta)+\tilde K(M,\rho,\eta)+C((1+\eta)^n-1)(\rho+|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}).$$

In particular, thanks to (5.4.4) we have that

$$(1+\eta)^{n-1}\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)=\lim_{\rho\to0}\frac{m^{a_{2M+1}}_{\widehat{\mathcal{H}}}(u_{x,\zeta,\nu},Q^\nu_{(1+\eta)\rho}(x))}{\rho^{n-1}}$$
$$\le\Big(1+\frac{C^2}{M}\Big)(g_{\hom}(\zeta,\nu)+2\eta)+C|\zeta|\|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})}\left((1+\eta)^n-1\right)$$

and thereby, letting $M\to+\infty$ and $\eta\to0$, we can conclude.

*Step 2:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x)\ge g_{\hom}([u](x),\nu_u(x))\quad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in J_u\cap A.$$

By Theorem 5.3.2 there exists $(u_j,v_j)\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u_j\in W^{1,1}(A,\mathbb{R}^N)$ and $v_j\in W^{1,2}(A,[0,1])$, such that $v_j(x) \to 1$ for $\mathcal{L}^n$-a.e. $x \in A$ as $j \to +\infty$,

$$(u_j, v_j) \to (u, 1) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}) \quad \text{and} \quad \lim_{j \to +\infty} \mathcal{H}_{\varepsilon_j}(u_j, v_j, A) = \widehat{\mathcal{H}}(u, 1, A).$$

For $\mathcal{H}^n$-a.e. $x \in J_u \cap A$ (cf. [AFP00, Theorem 3.77 and Proposition 3.92]) we have

$$\lim_{\rho \to 0} \frac{1}{\rho^{n-1}} |Du|(Q_\rho^{\nu_u(x)}(x)) = |[u](x)| \neq 0, \tag{5.4.10}$$

$$\lim_{\rho \to 0} \frac{1}{\rho^n} \int_{Q_\rho^{\nu_u(x)}(x)} |u(y) - u_{x,[u](x),\nu_u(x)}(y)| \mathrm{d}y = 0, \tag{5.4.11}$$

$$\lim_{\rho \to 0} \frac{\widehat{\mathcal{H}}(u, 1, Q_\rho^{\nu_u(x)}(x))}{\rho^{n-1}} = \frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1, \cdot)}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x) < +\infty. \tag{5.4.12}$$

Let us fix $x \in J_u \cap A$ such that (5.4.10)-(5.4.12) are satisfied, and set $\zeta := [u](x)$ and $\nu := \nu_u(x)$. Using the lower bound inequality in the $\Gamma$-convergence of $(\mathcal{H}_\varepsilon)_\varepsilon$ on $Q_\rho^{\nu_u(x)}(x)$ and $A \setminus Q_\rho^{\nu_u(x)}(x)$, the super-additivity of the inferior limit operator implies that $\lim_{j \to +\infty} \mathcal{H}_{\varepsilon_j}(u_j, v_j, Q_\rho^\nu(x)) = \widehat{\mathcal{H}}(u, 1, Q_\rho^\nu(x))$ for every $\rho \in I(x) := \{\rho \in (0, \frac{2}{\sqrt{n}} \mathrm{dist}(x, \partial A)) \; : \; \widehat{\mathcal{H}}(u, 1, \partial Q_\rho^\nu(x)) = 0\}$. Hence, we deduce that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1 \cdot)}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x) = \lim_{I(x) \ni \rho \to 0} \lim_{j \to +\infty} \frac{1}{\rho^{n-1}} \int_{Q_\rho^\nu(x)} (v_j^2 f(\tfrac{y}{\varepsilon_j}, \nabla u_j) + \tfrac{(1-v_j)^2}{\varepsilon_j} + \varepsilon_j |\nabla v_j|^2) \mathrm{d}y. \tag{5.4.13}$$

Now, we consider the rescalings $(u_j^\rho, v_j^\rho), (u^\rho, v^\rho) : Q^\nu \to \mathbb{R}^{N+1}$ given by

$$(u_j^\rho(y), v_j^\rho(y)) := (u_j(x + \rho y), v_j(x + \rho y)) \quad \text{and} \quad (u^\rho(y), v^\rho(y)) := (u(x + \rho y), 1)$$

Then $u_j^\rho \in W^{1,1}(Q^\nu, \mathbb{R}^N)$, $v_j^\rho \in W^{1,2}(Q^\nu, [0,1])$, $u^\rho \in BV(Q^\nu, \mathbb{R}^N)$, and $(u_j^\rho, v_j^\rho) \to (u^\rho, 1)$ in $L^1(Q^\nu, \mathbb{R}^{N+1})$, $u^\rho \to u_{\zeta,\nu}$ in $L^1(Q^\nu, \mathbb{R}^N)$ by (5.4.11), and $v_j^\rho \to 1$ in $L^2(Q^\nu)$ for every $\rho$ by (5.4.13). Changing variables, formula (5.4.13) rewrites as

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1 \cdot)}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x) = \lim_{I(x) \ni \rho \to 0} \lim_{j \to +\infty} \int_{Q^\nu} (\rho (v_j^\rho)^2 f(\tfrac{x + \rho y}{\varepsilon_j}, \tfrac{1}{\rho} \nabla u_j^\rho) + \tfrac{\rho}{\varepsilon_j} (1 - v_j^\rho)^2 + \tfrac{\varepsilon_j}{\rho} |\nabla v_j^\rho|^2) \mathrm{d}y\,,$$

thus by (f2) we infer that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1 \cdot)}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u}(x) \geq C \limsup_{I(x) \ni \rho \to 0} \limsup_{j \to +\infty} \int_{Q^\nu} (v_j^\rho)^2 |\nabla u_j^\rho| \mathrm{d}y. \tag{5.4.14}$$

Lemma 5.2.2 implies that

$$\rho \int_{Q^\nu} (v_j^\rho)^2 f(\tfrac{x + \rho y}{\varepsilon_j}, \tfrac{1}{\rho} \nabla u_j^\rho) \mathrm{d}y \geq \int_{Q^\nu} (v_j^\rho)^2 f^\infty(\tfrac{x + \rho y}{\varepsilon_j}, \nabla u_j^\rho) \mathrm{d}y - K\rho - K\rho^\alpha \Big( \int_{Q^\nu} (v_j^\rho)^2 |\nabla u_j^\rho| \mathrm{d}y \Big)^{1-\alpha} \tag{5.4.15}$$

where $K$ is a constant that depends only on $C$ and $\alpha$. Thanks to (5.4.12), (5.4.14) and (5.4.15) we get

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \geq \limsup_{I(x)\ni\rho\to 0}\limsup_{j\to+\infty}\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j,v^\rho_j,Q^\nu), \tag{5.4.16}$$

where

$$\mathcal{H}^{\infty,x}_{\rho,\varepsilon}(u,v,A) := \int_A (v^2 f^\infty(\tfrac{x+\rho y}{\varepsilon},\nabla u) + \tfrac{\rho}{\varepsilon}(1-v)^2 + \tfrac{\varepsilon}{\rho}|\nabla v|^2)\mathrm{d}y.$$

Now, for every $\rho$ and $j$ we consider the sequences $(a^\rho_j)_{j\in\mathbb{N}}$, $(b^\rho_j)_{j\in\mathbb{N}}$ and $(s^\rho_j)_{j\in\mathbb{N}}$ given by

$$a^\rho_j := \rho + \|u^\rho_j - w^\rho\|^{\frac12}_{L^1(Q^\nu)} + \|v^\rho_j - 1\|^{\frac12}_{L^2(Q^\nu)}, \quad b^\rho_j := \left\lfloor \frac{\rho}{\varepsilon_j}\right\rfloor \quad \text{and} \quad s^\rho_j := \frac{a^\rho_j}{b^\rho_j}, \tag{5.4.17}$$

where $(w^\rho)$ is a sequence in $W^{1,1}(Q^\nu,\mathbb{R}^N)$ such that $w^\rho = u_{\zeta,\nu}$ on $\partial Q^\nu$ for every $\rho$,

$$|Dw^\rho|(Q^\nu) \to |Du_{\zeta,\nu}|(Q^\nu) \quad \text{and} \quad w^\rho \to u_{\zeta,\nu} \text{ in } L^1(Q^\nu,\mathbb{R}^N) \text{ as } \rho\to 0, \tag{5.4.18}$$

(see [BFM98, Lemma 2.5]), where $\lfloor s\rfloor$ denotes the integer part of $s\in\mathbb{R}$. Fix $\rho$ small enough, such that $\rho + \|u^\rho - w^\rho\|^{\frac12}_{L^1(Q^\nu)} < \frac14$ and then fix $j$ large enough such that $0 < a^\rho_j < \frac12$ and $2 < b^\rho_j$. For every $i = 0,\dots,b^\rho_j$ we define $Q^\nu_{\rho,j,i}$ as

$$Q^\nu_{\rho,j,i} := (1 - a^\rho_j + i s^\rho_j)Q^\nu,$$

while for every $i = 1,..,b^\rho_j$ we consider the cut-off function $\phi^\rho_{j,i}\in C^\infty_c(Q^\nu_{\rho,j,i})$ such that $0\leq\phi^\rho_{j,i}\leq 1$, $\phi^\rho_{j,i}\equiv 1$ on $Q^\nu_{\rho,j,i-1}$ and $\|\nabla\phi^\rho_{j,i}\|_{L^\infty(\mathbb{R}^n)} \leq 2(s^\rho_j)^{-1}$. Set for $i = 1,..,b^\rho_j$

$$u^\rho_{j,i} := \phi^\rho_{j,i-1}u^\rho_j + (1-\phi^\rho_{j,i-1})w^\rho \quad \text{and} \quad v^\rho_{j,i} := \phi^\rho_{j,i}v^\rho_j + (1-\phi^\rho_{j,i}).$$

Then $u^\rho_{j,i}\in W^{1,1}(Q^\nu,\mathbb{R}^N)$, $v^\rho_{j,i}\in W^{1,2}(Q^\nu,[0,1])$ with $(u^\rho_{j,i},v^\rho_{j,i}) = (u_{\zeta,\nu},1)$ on $\partial Q^\nu$. Moreover, for every $i = 2,..,b_{\rho,j}$ we have the following

$$\begin{aligned}
&\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i},v^\rho_{j,i},Q^\nu) \leq \mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j,v^\rho_j,Q^\nu_{\rho,j,i-2})\\
&+\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla u^\rho_{j,i})\mathrm{d}y + \int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (\tfrac{\rho}{\varepsilon_j}(1-v^\rho_j)^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_j|^2)\mathrm{d}y\\
&+\int_{Q^\nu_1\setminus Q^\nu_{\rho,j,i-1}} (v^\rho_{j,i})^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j},\nabla w^\rho)\mathrm{d}y + \int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} (\tfrac{\rho}{\varepsilon_j}(1-v^\rho_{j,i})^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_{j,i}|^2)\mathrm{d}y.
\end{aligned}$$

We estimate separately the terms appearing above. We start with

$$\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j,v^\rho_j,Q^\nu_{\rho,j,i-2}) + \int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (\tfrac{\rho}{\varepsilon_j}(1-v^\rho_j)^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_j|^2)\mathrm{d}y \leq \mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j,v^\rho_j,Q^\nu).$$

Moreover, since $\nabla u^\rho_{j,i} = \phi^\rho_{j,i-1}\nabla u^\rho_j + (1-\phi^\rho_{j,i-1})\nabla w^\rho + \nabla\phi^\rho_{j,i-1}\otimes(u^\rho_j - w^\rho)$, we have that

$$
\begin{aligned}
&\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j}, \nabla u^\rho_{j,i})\mathrm{d}y \\
&\le C\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2(|\nabla u^\rho_j| + |\nabla w^\rho| + |\nabla\phi^\rho_{j,i-1}||u^\rho_j - w^\rho|)\mathrm{d}y \\
&\le C^2\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} (v^\rho_j)^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j}, \nabla u^\rho_j)\mathrm{d}y + C\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} |\nabla w^\rho|\mathrm{d}y \\
&+ \frac{2C}{s^\rho_j}\int_{Q^\nu_{\rho,j,i-1}\setminus Q^\nu_{\rho,j,i-2}} |u^\rho_j - w^\rho|\mathrm{d}y.
\end{aligned}
$$

Analogously, we obtain

$$
\int_{Q^\nu\setminus Q^\nu_{\rho,j,i-1}} (v^\rho_{j,i})^2 f^\infty(\tfrac{x+\rho y}{\varepsilon_j}, \nabla w^\rho)\mathrm{d}y \le C\int_{Q^\nu\setminus Q^\nu_{\rho,j,i-1}} |\nabla w^\rho|\mathrm{d}y.
$$

Since $\nabla v^\rho_{j,i} = \phi^\rho_{j,i}\nabla v^\rho_j + (v^\rho_j - 1)\nabla\phi^\rho_{j,i}$, we have that

$$
\begin{aligned}
&\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} (\tfrac{\rho}{\varepsilon_j}(1-v^\rho_{j,i})^2 + \tfrac{\varepsilon_j}{\rho}|\nabla v^\rho_{j,i}|^2)\mathrm{d}y \\
&\le \frac{\rho}{\varepsilon_j}\mathcal{L}^n(Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + 2\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} \frac{\varepsilon_j}{\rho}|\nabla v^\rho_j|^2\mathrm{d}y + \frac{8\varepsilon_j}{\rho(s^\rho_j)^2}\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} |v^\rho_j - 1|^2\mathrm{d}y \\
&\le \frac{\rho}{\varepsilon_j}\mathcal{L}^n(Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + 2\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}) + \frac{8\varepsilon_j}{\rho(s^\rho_j)^2}\int_{Q^\nu_{\rho,j,i}\setminus Q^\nu_{\rho,j,i-1}} |v^\rho_j - 1|^2\mathrm{d}y\,.
\end{aligned}
$$

In particular, thanks to the previous calculations and and recalling the definition of $s^\rho_j$, there exists $i^\rho_j \in \{2,\dots,b^\rho_j\}$ such that

$$
\begin{aligned}
&\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) \le \frac{1}{b^\rho_j - 1}\sum_{i=2}^{b^\rho_j}\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i}, v^\rho_{j,i}, Q^\nu) \\
&\le \mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu) + \frac{2}{b^\rho_j - 1}\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu) + C\int_{Q^\nu\setminus Q^\nu_{0,\rho,j}} |\nabla w^\rho|\mathrm{d}y \\
&+ \frac{2Cb^\rho_j}{(b^\rho_j - 1)a^\rho_j}\int_{Q^\nu\setminus Q^\nu_{\rho,j,0}} |u^\rho_j - w^\rho|\mathrm{d}y + \frac{\rho}{\varepsilon_j(b^\rho_j - 1)}\mathcal{L}^n(Q^\nu\setminus Q^\nu_{\rho,j,0}) \\
&+ \frac{8\varepsilon_j(b^\rho_j)^2}{\rho(a^\rho_j)^2(b^\rho_j - 1)}\int_{Q^\nu\setminus Q^\nu_{\rho,j,0}} |v^\rho_j - 1|^2\mathrm{d}y.
\end{aligned}
$$

Hence, by the definition of $a^\rho_j$ and $b^\rho_j$ in (5.4.17) we deduce that

$$
\begin{aligned}
&\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) \le \Big(1 + \frac{2}{b^\rho_j - 1}\Big)\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_j, v^\rho_j, Q^\nu) \\
&+ C\int_{Q^\nu\setminus Q^\nu_{\rho,j,0}} |\nabla w^\rho|\mathrm{d}y + 4Ca^\rho_j + 3\mathcal{L}^n(Q^\nu\setminus Q^\nu_{\rho,j,0}) + 16(a^\rho_j)^2.
\end{aligned}
$$

Thus, setting where $\kappa_\rho = 1 - \|u^\rho - w^\rho\|_{L^1(Q^\nu)}$, from (5.4.16) and (5.4.17) we obtain

$$\limsup_{I(x)\ni\rho\to 0}\limsup_{j\to+\infty}\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) \le \frac{\mathrm{d}\widehat{\mathcal{H}}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) + C\limsup_{I(x)\ni\rho\to 0}|Dw^\rho|(Q^\nu\setminus Q^\nu_{\kappa_\rho}),$$

As $w^\rho \to u_{\zeta,\nu}$ strictly in $BV$ (cf. (5.4.18)), we have that $|Dw^\rho|(Q^\nu\setminus Q^\nu_{\kappa_\rho})\to 0$ as $\rho\to 0$, and thus

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \ge \limsup_{I(x)\ni\rho\to 0}\limsup_{j\to+\infty}\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu). \tag{5.4.19}$$

By the change of variable and the 1-homogeneity of $f^\infty$ , we have

$$\mathcal{H}^{\infty,x}_{\rho,\varepsilon_j}(u^\rho_{j,i^\rho_j}, v^\rho_{j,i^\rho_j}, Q^\nu) = \frac{1}{r^{n-1}_{\rho,j}}\int_{Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x)}\left(\overline{v}^2_{\rho,j}f^\infty(y,\nabla\overline{u}_{\rho,j}) + (1-\overline{v}_{\rho,j})^2 + |\nabla\overline{v}_{\rho,j}|^2\right)\mathrm{d}y,$$

where $r_{\rho,j} := \frac{\rho}{\varepsilon_j}$, $\overline{u}^\rho_j(y) := u^\rho_{j,i^\rho_j}(\frac{y}{r_{\rho,j}} - \frac{x}{\rho})$ and $\overline{v}^\rho_j(y) := v^\rho_{j,i^\rho_j}(\frac{y}{r_{\rho,j}} - \frac{x}{\rho})$. In this way $\overline{u}^\rho_j \in W^{1,1}(Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x),\mathbb{R}^N)$, $\overline{v}^\rho_j \in W^{1,2}(Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x),[0,1])$ with $(\overline{u}^\rho_j,\overline{v}^\rho_j) = (u_{\frac{r_{\rho,j}}{\rho}x,\zeta,\nu},1)$ on $\partial Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x)$. In particular, by (5.4.19), the definition of $m^{f^\infty}_{\mathrm{s}}$ in (5.1.7) and the assumption (b) of Theorem 5.3.1 we obtain

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1\cdot)}{\mathrm{d}\mathcal{H}^{n-1}\llcorner J_u}(x) \ge \limsup_{\tilde{I}(x)\ni\rho\to 0}\limsup_{j\to+\infty}\frac{m^{f^\infty}_{\mathrm{s}}(u_{\frac{r_{\rho,j}}{\rho}x,\zeta,\nu}, Q^\nu_{r_{\rho,j}}(\frac{r_{\rho,j}}{\rho}x))}{r^{n-1}_{\rho,j}} = g_{\mathrm{hom}}(\zeta,\nu),$$

deducing the claim. $\square$

## 5.5 Identification of the Cantor term

Eventually, in this subsection we identify the density of the Cantor part of the Γ-limit $\widehat{\mathcal{H}}$.

**Proposition 5.5.1** (Homogenised Cantor integrand)**.** *Let $f \in \mathcal{AI}(C,\alpha)$ satisfy (5.2.4). Let $\widehat{\mathcal{H}}$ be as in (5.3.5). Then for every $A\in\mathcal{A}(\mathbb{R}^n)$ and every $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$, with $u\in BV(A,\mathbb{R}^N)\cap L^\infty(A,\mathbb{R}^N)$, we have that*

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^cu|}(x) = f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\quad\text{for } |D^cu|\text{-a.e. } x\in A,$$

*where $f^\infty_{\mathrm{hom}}$ is the recession function of $f_{\mathrm{hom}}$ as in (5.2.1).*

*Proof.* Let us fix $A\in\mathcal{A}(\mathbb{R}^n)$ and $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ with $u\in BV\cap L^\infty(A,\mathbb{R}^N)$. We divide the proof into two steps.

*Step 1:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^cu|}(x) \le f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\Big)\quad\text{for } |D^cu|\text{-a.e. } x\in A.$$

By Alberti's Rank-one Theorem [Alb93] we know that for $|D^cu|$-a.e. $x\in A$ we have

$$\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}(x) = a(x)\otimes\nu(x) \tag{5.5.1}$$

where $(a(x),\nu(x))\in\mathbb{R}^N\times\mathbb{S}^{n-1}$. By Theorem 5.3.2 and by [BFM98, Lemma 3.9] we have that for $|D^cu|$-a.e. $x\in A$ there exists a doubly indexed positive sequence $(t_{\rho,k})$, with $\rho\in(0,\infty)$ and $k\in\mathbb{N}$, such that for every $k\in\mathbb{N}$

$$t_{\rho,k}\to+\infty\quad\text{ and }\quad\rho t_{k,\rho}\to 0\quad\text{ as }\rho\to 0,\tag{5.5.2}$$

and for every $q\in\mathbb{Q}\cap(0,1)$

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^cu|}(x)+qa(x)\otimes\nu(x)=\lim_{k\to+\infty}\limsup_{\rho\to 0}\frac{m_{\widehat{\mathcal{H}}_q}(\ell_{t_{\rho,k}a(x)\otimes\nu(x)},Q_r^{\nu(x),k}(\frac{r}{\rho}x)}{k^{n-1}\rho^n t_{\rho,k}},\tag{5.5.3}$$

where for every $A\in\mathcal{A}(\mathbb{R}^n)$ and $q\in\mathbb{Q}\cap(0,1)$ let $\widehat{\mathcal{H}}_q(u,1,A):=\widehat{\mathcal{H}}(u,1,A)+q|Du|(A)$, and $Q_r^{\nu,k}(z)$ is the parallelepiped defined in 2.0.5. Let $x\in A$ be such that (5.5.1)-(5.5.3) hold true, and set $a:=a(x)$ and $\nu:=\nu(x)$. Thanks to Proposition 5.2.3, for every $\rho>0$ and every $k\in\mathbb{N}$ we have

$$f^\infty_{\mathrm{hom}}(a\otimes\nu)=\lim_{r\to\infty}\frac{m_{\mathrm{b}}^{f^\infty}(\ell_{a\otimes\nu},Q_r^{\nu,k}(\frac{r}{\rho}x))}{k^{n-1}r^n}.\tag{5.5.4}$$

Let us fix $\eta\in(0,1)$. By the definition of $m_{\mathrm{b}}^{f^\infty}$ in (5.1.6), for every $k\in\mathbb{N}$, $\rho\in(0,1)$ and $r\in(0,\infty)$ there exists a function $\hat{u}_r^{\rho,k}\in W^{1,1}(Q_r^{\nu,k}(\frac{r}{\rho}x),\mathbb{R}^N)$ with $\hat{u}_r^{\rho,k}=\ell_{a\otimes\nu}$ on $\partial Q_r^{\nu,k}(\frac{r}{\rho}x)$ such that

$$E^{f^\infty}\left(\hat{u}_r^{\rho,k},Q_r^{\nu,k}(\tfrac{r}{\rho}x)\right)\le m_{\mathrm{b}}^{f^\infty}\left(\ell_{a\otimes\nu},Q_r^{\nu,k}(\tfrac{r}{\rho}x)\right)+\eta k^{n-1}r^n\le C|a|k^{n-1}r^n+\eta k^{n-1}r^n\tag{5.5.5}$$

and in particular

$$\int_{Q_r^{\nu,k}(\frac{r}{\rho}x)}|\nabla\hat{u}_r^{\rho,k}|\mathrm{d}y\le C^2(|a|+1)k^{n-1}r^n.$$

Therefore, by Lemma 5.2.2, we have that

$$\begin{aligned}&\frac{1}{k^{n-1}r^n}\int_{Q_r^{\nu,k}(\frac{r}{\rho}x)}\Big|f^\infty(y,\nabla\hat{u}_r^{\rho,k})-\frac{1}{t_{\rho,k}}f(y,t_{\rho,k}\nabla\hat{u}_r^{\rho,k})\Big|\mathrm{d}y\\&\le\frac{K}{t_{\rho,k}}k^{n-1}r^n+\frac{K}{t_{\rho,k}^\alpha}(k^{n-1}r^n)^\alpha\Big(\int_{Q_r^{\nu,k}(\frac{r}{\rho}x)}|\nabla u_r|\mathrm{d}y\Big)^{1-\alpha}\le\frac{\hat{K}}{t_{\rho,k}^\alpha}k^{n-1}r^n\end{aligned}\tag{5.5.6}$$

where $\hat{K}$ depends only on $C$, $\alpha$ and $a$. Collecting (5.5.4)-(5.5.6), we infer that

$$\limsup_{r\to+\infty}\frac{1}{k^{n-1}r^n t_{\rho,k}}\int_{Q_r^{\nu,k}(\frac{r}{\rho}x)}f(y,t_{\rho,k}\nabla\hat{u}_r^{\rho,k})\mathrm{d}y\le f^\infty_{\mathrm{hom}}(a\otimes\nu)+\eta+\frac{\hat{K}}{t_{\rho,k}^\alpha}.$$

For $k\in\mathbb{N}$, $\rho\in(0,1)$ and $\varepsilon\in(0,\infty)$ we define the function $u_\varepsilon^{\rho,k}:\mathbb{R}^n\to\mathbb{R}^N$ given by

$$u_\varepsilon^{\rho,k}(y):=\begin{cases}\varepsilon t_{\rho,k}\hat{u}_r^{\rho,k}(\frac{y}{\varepsilon}) & \text{if } y\in Q_\rho^{\nu,k}(x)\\ t_{\rho,k}\ell_{a\otimes\nu} & \text{if } y\in\mathbb{R}^n\setminus Q_\rho^{\nu,k}(x),\end{cases}$$

where $r := \frac{\varrho}{\varepsilon}$. Thus $u_\varepsilon^{\rho,k} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $u_\varepsilon^{\rho,k} = t_{\rho,k}\ell_{a\otimes\nu}$ on $\partial Q_\rho^{\nu,k}(x)$ and changing variable we obtain

$$\limsup_{\varepsilon\to 0} \frac{1}{k^{n-1}\rho^n t_{\rho,k}} \int_{Q^{\nu,k}_{\rho(1+\eta)}(x)} f\left(\tfrac{x}{\varepsilon}, \nabla u_\varepsilon^{\rho,k}\right) \mathrm{d}y$$
$$\leq f^\infty_{\mathrm{hom}}(a\otimes\nu) + \eta + \frac{\hat{K}}{t^\alpha_{\rho,k}} + C\left(\frac{1}{t_{\rho,k}} + |a|\right)((1+\eta)^n - 1), \tag{5.5.7}$$

since $u_\varepsilon^{\rho,k}$ coincides with $t_{\rho,k}\ell_{a\otimes\nu}$ on $Q^{\nu,k}_{\rho(1+\eta)}(x)\backslash Q^{\nu,k}_\rho(x)$. By (5.5.7) and Poincaré inequality, we can extract a subsequence (not relabelled) of $(\varepsilon_j)_{j\in\mathbb{N}}$, for every $\rho \in (0,1)$ and $k \in \mathbb{N}$, such that $u^{\rho,k}_{\varepsilon_j} \to u^{\rho,k}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, where $u^{\rho,k} \in BV(Q^{\nu,k}_{\rho(1+\eta)}(x), \mathbb{R}^N)$ with $u^{\rho,k} = t_{\rho,k}\ell_{a\otimes\nu}$ on $Q^{\nu,k}_{\rho(1+\eta)}(x) \setminus Q^{\nu,k}_\rho(x)$. As a consequence of the $\Gamma$-convergence stated in Theorem 5.3.2, of the superadditivity of the inferior limit operator, of ((f2)) and of estimate (5.5.7) we obtain

$$\begin{aligned}\frac{m_{\widehat{\mathcal{H}}_q}(\ell_{t_{\rho,k}a\otimes\nu}, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} &\leq \frac{\widehat{\mathcal{H}}_q(u^{\rho,k}, 1, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} \\ &\leq \liminf_{j\to+\infty}\Big(\frac{\mathcal{H}_{\varepsilon_j}(u^{\rho,k}_{\varepsilon_j}, 1, Q^{\nu,k}_{\rho(1+\eta)}(x))}{k^{n-1}\rho^n t_{\rho,k}} + \frac{q}{k^{n-1}\rho^n t_{\rho,k}}\int_{Q^{\nu,k}_{\rho(1+\eta)}(x)} |\nabla u^{\rho,k}_{\varepsilon_j}|\mathrm{d}y\Big) \\ &\leq (1+qC)\Big(f^\infty_{\mathrm{hom}}(a\otimes\nu) + \eta + \frac{\hat{K}}{t^\alpha_{\rho,k}} + C\left(\frac{1}{t_{\rho,k}} + |a|\right)((1+\eta)^n - 1)\Big).\end{aligned}$$

We can pass to the limit in the last inequality for $\rho \to 0$ and then for $k \to +\infty$, using (5.5.2) and (5.5.3) we arrive to

$$(1+\eta)^n\Big(\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) + qa\otimes\nu\Big) \leq (1+qC)(f^\infty_{\mathrm{hom}}(a\otimes\nu) + \eta + C|a|((1+\eta)^n - 1))\,.$$

The claim follows by letting $\eta$ and $q \to 0$.

*Step 2:* We claim that

$$\frac{\mathrm{d}\widehat{\mathcal{H}}(u,1,\cdot)}{\mathrm{d}|D^c u|}(x) \geq f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}\Big) \quad \text{for } |D^c u|\text{-a.e. } x \in A\,. \tag{5.5.8}$$

Let $A' \in \mathcal{A}(A)$, then by Theorem 5.3.2 we can find a sequence $(u_j, v_j)_{j\in\mathbb{N}} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ such that $(u_j, v_j) \in W^{1,1}(A', \mathbb{R}^N) \times W^{1,2}(A', [0,1])$, $(u_j, v_j) \to (u, 1)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$, $v_j \to 1$ for $\mathcal{L}^n$-a.e. $x \in A'$ as $j \to \infty$ and

$$\lim_{j\to\infty} \mathcal{H}_{\varepsilon_j}(u_j, v_j, A') = \widehat{\mathcal{H}}(u, 1, A').$$

Fix $\delta \in (0,1)$; by Lemma 5.3.6 we have

$$H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq \mathcal{H}_{\varepsilon_j}(u_j, v_j, A') + C\mathcal{L}^n(\{v_j \leq \delta\} \cap A'),$$

where $u^\delta_j \in SBV(A', \mathbb{R}^N)$ with $u^\delta_j \to u$ in $L^1(A', \mathbb{R}^N)$ as $j \to \infty$, and therefore

$$\liminf_{j\to\infty} H^\delta_{\varepsilon_j}(u^\delta_j, A') \leq \widehat{\mathcal{H}}(u, 1, A'). \tag{5.5.9}$$

Define on $A'$ the measures $\mu_j^\delta$ given by

$$\mu_j^\delta := \alpha_\delta f\left(\tfrac{x}{\varepsilon_j}, \nabla u_j^\delta\right) \mathcal{L}^n \llcorner A' + \beta_\delta \mathcal{H}^{n-1} \llcorner (J_{u_j^\delta} \cap A')\,.$$

By definition of $H_\varepsilon^\delta$ and the compactness of Radon measures, there exists subsequence (not relabeled) of $(\varepsilon_j)_{j\in\mathbb{N}}$ and a finite Radon measure $\mu^\delta$ on $A'$ such that $\mu_j^\delta \to \mu^\delta$ weakly* in the sense of measures on $A'$ as $j \to \infty$. For $|D^c u|$-a.e. $x \in A'$ (cf. [AFP00, Proposition 3.92 and Theorem 3.94]) there exists $a(x) \in \mathbb{R}^N$ and $\nu(x) \in \mathbb{S}^{n-1}$ such that for every $k \in \mathbb{N}$ we have

$$\lim_{\rho\to 0} \frac{Du(Q_\rho^{\nu(x),k}(x))}{|Du|(Q_\rho^{\nu(x),k}(x))} = \frac{\mathrm{d}D^c u}{\mathrm{d}|D^c u|}(x) = a(x) \otimes \nu(x) \tag{5.5.10}$$

$$\lim_{\rho\to 0} \frac{|Du|(Q_\rho^{\nu(x),k}(x))}{\rho^n} = \infty \tag{5.5.11}$$

$$\lim_{\rho\to 0} \frac{|Du|(Q_\rho^{\nu(x),k}(x))}{\rho^{n-1}} = 0 \tag{5.5.12}$$

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|Du|}(x) = \frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x). \tag{5.5.13}$$

Fix now $x_0 \in A'$ such that (5.5.10)-(5.5.13) hold true and set $a := a(x_0)$ and $\nu := \nu(x_0)$. For $k \in \mathbb{N}$ and $\rho$ set

$$t_{\rho,k} := \frac{|Du|(Q_\rho^{\nu,k}(x_0))}{k^{n-1}\rho^n},$$

therefore

$$t_{\rho,k} \to \infty \quad \text{and} \quad \rho t_{\rho,k} \to 0 \quad \text{as } \rho \to 0. \tag{5.5.14}$$

and

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) = \lim_{\rho\to 0} \frac{\mu^\delta(Q_\rho^{\nu,k}(x_0))}{t_{\rho,k} k^{n-1}\rho^n}$$

By the weak*-convergence of $(\mu_{\varepsilon_j}^\delta)_{j\in\mathbb{N}}$ to $\mu^\delta$ we infer that

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) = \lim_{\rho\to 0, \rho\in I(x_0)} \lim_{j\to\infty} \frac{\mu_j^\delta(Q_\rho^{\nu,k}(x_0))}{t_{\rho,k} k^{n-1}\rho^n}, \tag{5.5.15}$$

where $I(x_0) := \{\rho \in (0,1) :\ \mu^\delta(\partial Q_\rho^{\nu,k}(x_0)) = 0 \text{ for every } k \in \mathbb{N} \text{ s.t. } Q_\rho^{\nu,k}(x_0) \subset\subset A'\}$. Note that $I(x_0)$ has full measure in $(0,1)$. Fix $k \in \mathbb{N}$ and consider the rescaled functions $u_j^{\rho,k}$, $u^{\rho,k} : Q^{\nu,k} \to \mathbb{R}^N$

$$\begin{aligned} u_j^{\rho,k}(y) &= \frac{1}{k^{n-1}\rho t_{\rho_i,k}}\left(u_j^\delta(x_0 + \rho_i y) - \frac{1}{k^{n-1}\rho_i^n}\int_{Q_\rho^{\nu,k}(x_0)} u_j^\delta(z)\mathrm{d}z\right) \\ u^{\rho,k}(y) &= \frac{1}{k^{n-1}\rho t_{\rho,k}}\left(u(x_0 + \rho y) - \frac{1}{k^{n-1}\rho^n}\int_{Q_\rho^{\nu,k}(x_0)} u(z)\mathrm{d}z\right). \end{aligned}$$

From now on we work at $k \in \mathbb{N}$ fixed and this will tend to $\infty$ only at the very end of the proof. Therefore, for those parameters infinitesimal as $j \to \infty$ and $\rho \to 0$ the possible dependence on $k$ will not be highlighted. For every $\rho$ small enough, depending on $k$, $u_j^{\rho,k} \in SBV(Q^{\nu,k}, \mathbb{R}^N)$, $u^{\rho,k} \in BV(Q^{\nu,k}, \mathbb{R}^N)$, $u_j^{\rho,k} \to u^{\rho,k}$ in

$L^1(Q^{\nu,k},\mathbb{R}^N)$ as $j\to\infty$, and the function $u^{\rho,k}$ satisfies the following

$$\int_{Q^{\nu,k}} u^{\rho,k}(y)\mathrm{d}y = 0 \quad \text{with} \quad Du^{\rho,k}(Q^{\nu,k}) = \frac{Du(Q^{\nu,k}_\rho(x_0))}{|Du|(Q^{\nu,k}_\rho(x_0))} \to a\otimes\nu \quad \text{as } \rho\to 0.$$

Moreover, from (5.5.15) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|} = \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \frac{\mu^\delta_j(Q^{\nu,k}_\rho(x_0))}{k^{n-1}\rho^n t_{\rho,k}} \\
&= \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \Big(\frac{\alpha_\delta}{k^{n-1}\rho^n t_{\rho,k}} \int_{Q^{\nu,k}_\rho(x_0)} f(\tfrac{x}{\varepsilon_j},\nabla u^\delta_j)\,\mathrm{d}x + \frac{\beta_\delta}{k^{n-1}\rho^n t_{\rho,k}} \mathcal{H}^{n-1}(J_{u^\delta_j}\cap Q^{\nu,k}_\rho(x_0))\Big) = \\
&= \lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} \Big(\frac{\alpha_\delta}{k^{n-1} t_{\rho,k}} \int_{Q^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla u^{\rho,k}_j)\,\mathrm{d}x + \frac{\beta_\delta}{k^{n-1}\rho t_{\rho,k}} \mathcal{H}^{n-1}(J_{u^{\rho,k}_j}\cap Q^{\nu,k})\Big).
\end{aligned} \tag{5.5.16}$$

By [ADM92, Theorem 2.3] and [Lar98, Lemma 5.1] there exists a subsequence (not relabeled), depending on $k$, such that $u^{\rho,k}\to u^k$ in $L^1(Q^{\nu,k},\mathbb{R}^N)$ as $\rho\to 0$, where $u^k\in BV(Q^{\nu,k},\mathbb{R}^N)$, $u^k(y)=\chi_k(y\cdot\nu)a$ for every $y\in Q^{\nu,k}$, $\chi_k:[-1/2,1/2]\to\mathbb{R}$ is a nondecreasing function such that $D\chi_k((-1/2,1/2)) = \chi_k(1/2) - \chi_k(-1/2) = \frac{1}{k^{n-1}}$, $-\frac{1}{k^{n-1}} \le \chi_k(-1/2) \le 0 \le \chi_k(1/2) \le \frac{1}{k^{n-1}}$, and

$$\int_{-1/2}^{1/2} \chi_k(t)\mathrm{d}t = 0.$$

Furthermore, being $\chi_k$ continuous in $-1/2$ and $1/2$, thanks to the trace's properties of $BV$ functions, we have that the inner trace of $u^k$ satisfies $u^k=\ell^k$ on $\partial^\perp Q^{\nu,k}$ (cf. 2.0.6) where

$$\ell^k(y) := \frac{1}{k^{n-1}}\ell_{a\otimes\nu}(y) + \Big(\chi_k(1/2) - \frac{1}{2k^{n-1}}\Big)a.$$

To obtain a uniform $L^\infty$-bound on the rescalings, we let $M\in\mathbb{N}$ and use Lemma 5.3.3 with $v\equiv 1$ to get for every $k\in\mathbb{N}$, every $\rho$ small enough and every $j$ (up to a subsequence), $m_\rho\in\{M+1,\dots,2M\}$ such that

$$\Big(1+\frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \ge \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q^{\nu,k}} f\left(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla \hat{u}^{\rho,k}_j\right)\mathrm{d}y, \tag{5.5.17}$$

and

$$\mathcal{H}^{n-1}(J_{\hat{u}^{\rho,k}_j}\cap Q^{\nu,k}) + \|\hat{u}^{\rho,k}_j - u^k\|_{L^1(Q^{\nu,k})} \le \mathcal{H}^{n-1}(J_{u^\rho_j}\cap Q^{\nu,k}) + \|u^{\rho,k}_j - u^k\|_{L^1(Q^{\nu,k})}$$

where $\hat{u}^{\rho,k}_j := \mathcal{T}_{m_\rho}(u^{\rho,k}_j)\in SBV(Q^{\nu,k},\mathbb{R}^N)$. Therefore, choosing $M$ such that $a_M > |a|$, it follows from (5.5.14) and (5.5.16) that

$$\lim_{\rho\to 0,\rho\in I(x_0)} \lim_{j\to\infty} (\mathcal{H}^{n-1}(J_{\hat{u}^{\rho,k}_j}\cap Q^{\nu,k}) + \|\hat{u}^{\rho,k}_j - u^k\|_{L^1(Q^{\nu,k})}) = 0\,. \tag{5.5.18}$$

Next we change the boundary datum $\hat{u}^{\rho,k}_j$ on a neighborhood of $\partial^\perp Q^{\nu,k}$ with $u^k$. For every $\rho$ small enough and every $j$ large enough (depending on $\rho$), we have that

$$\|\hat{u}^{\rho,k}_j - u^k\|_{L^1(Q^{\nu,k})} + \frac{\rho}{j} =: \tau_{\rho,j}\in(0,1)$$

and thanks to Fubini's Theorem and the trace properties of $BV$ functions on rectifiable sets, there exists $q_{\rho,j} \in (1/2 - \tau_{\rho,j}^{1/2}, 1/2)$

$$\int_{\partial^\perp R_\nu(B^k_{\rho,j})} |(\hat{u}_j^{\rho,k})^- - u^k| \mathrm{d}\mathcal{H}^{n-1} \leq \frac{\|\hat{u}_j^\rho - \ell_\xi\|_{L^1(Q^{\nu,k})}}{\tau_{\rho,j}^{1/2}} \leq \tau_{\rho,j}^{1/2}, \tag{5.5.19}$$

where $(\hat{u}_j^{\rho,k})^-$ is the inner trace of $\hat{u}_j^{\rho,k}$ on $R_\nu(B^k_{\rho,j})$, where $B^k_{\rho,j} := (-\frac{k}{2}, \frac{k}{2})^{n-1} \times (-q_{\rho,j}, q_{\rho,j})$ and $R_\nu$ is the rotation in Remark 2.0.1. Defining the functions $w_j^{\rho,k} \in BV(Q^{\nu,k}, \mathbb{R}^N)$ as

$$w_j^{\rho,k}(y) = \begin{cases} \hat{u}_j^{\rho,k}(y) & \text{if } y \in R_\nu(B^k_{\rho,j}) \\ u^k & \text{if } y \in Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j}), \end{cases}$$

we have that

$$\begin{aligned} &\frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla \hat{u}_j^{\rho,k})\mathrm{d}y + \alpha_\delta C \|\nabla u^k\|_{L^1(Q^{\nu,k}\setminus R_\nu(B^k_{\rho,j}))} \\ &+ \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \mathcal{L}^n(Q^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) \geq \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla w_j^{\rho,k})\mathrm{d}y. \end{aligned}$$

Since $\lim\limits_{\rho\to 0}\lim\limits_{j\to\infty} \mathcal{L}^n(Q_1^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) = 0$, we get from (5.5.17)

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \frac{\alpha_\delta}{k^{n-1}t_{\rho,k}} \int_{Q^{\nu,k}} f(\tfrac{x_0+\rho y}{\varepsilon_j}, k^{n-1}t_{\rho,k}\nabla w_j^{\rho,k})\mathrm{d}y\,. \tag{5.5.20}$$

In particular, we infer that

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq C\alpha_\delta \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \int_{Q^{\nu,k}} |\nabla w_j^{\rho,k}|\mathrm{d}y\,, \tag{5.5.21}$$

and thus by (5.5.19) we conclude

$$\begin{aligned} |D^s w_j^{\rho,k}|(Q^{\nu,k}) &\leq |D^s \hat{u}_j^{\rho,k}|(R_\nu(B^k_{\rho,j})) + |D^s u^k|(Q^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) + \tau_{\rho,j}^{\frac{1}{2}} \\ &\leq 2a_{2M+1}\mathcal{H}^{n-1}(J_{\hat{u}_j^{\rho,k}} \cap Q^{\nu,k}) + |D^s u^k|(Q^{\nu,k} \setminus R_\nu(B^k_{\rho,j})) + \tau_{\rho,j}^{\frac{1}{2}} \end{aligned}$$

and therefore

$$\lim_{\rho\to 0, \rho\in I(x_0)} \lim_{j\to\infty} |D^s w_j^{\rho,k}|(Q^{\nu,k}) = 0. \tag{5.5.22}$$

In addition, thanks to Lemma 5.2.2 and (5.5.20) we deduce that

$$\Big(1 + \frac{C^2}{M}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^c u|}(x_0) \geq \limsup_{\rho\to 0,\ \rho\in I(x_0)} \limsup_{j\to\infty} \alpha_\delta \int_{Q^{\nu,k}} f^\infty(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla w_j^{\rho,k})\mathrm{d}y. \tag{5.5.23}$$

We now change the boundary datum $w_j^{\rho,k}$ on a neighborhood of $\partial^\| Q^{\nu,k}$ (cf. (2.0.7)) with $\ell_k$. Let $h_k \in (0, k)$ be such that $\mathcal{L}^n(Q^{\nu,k} \setminus Q^{\nu,h_k}) \to 0$ as $k \to \infty$, necessarily $h_k \to \infty$. Then, by Fubini's Theorem there exists $\lambda^k_{\rho,j} \in (k-h_k, k)$ such that

$$\int_{\partial^{\|} Q^{\nu,\lambda^k_{\rho,j}}} |(w_j^{\rho,k})^- - \ell_k| \mathcal{H}^{n-1} \leq \frac{2}{h_k} \int_{Q^{\nu,k}} |w_j^{\rho,k} - \ell_k| \mathrm{d}y, \tag{5.5.24}$$

where $(w_j^{\rho,k})^-$ is the inner trace of $w_j^{\rho,k}$ on $\partial^{\|} Q^{\nu,\lambda^k_{\rho,j}}$. Furthermore, since $w_j^{\rho,k} = u^k = \ell_k$ on $\partial^{\perp} Q^{\nu,k}$, using Poincarè inequality on the one-dimensional restrictions of $w_j^{\rho,k}$ in the $\nu$ direction, we obtain that

$$\int_{Q^{\nu,k}} |w_j^{\rho,k} - \ell_k| \mathrm{d}y \leq 2 \Big| D w_j^{\rho,k} - \frac{a \otimes \nu}{k^{n-1}} \Big| (Q^{\nu,k}) \leq 2 |D w_j^{\rho,k}|(Q^{\nu,k}) + 2|a| \,.$$

Therefore, by (5.5.24) we infer that

$$\int_{\partial^{\|} Q^{\nu,\lambda^k_{\rho,j}}} |(w_j^{\rho,k})^- - \ell_k| \mathcal{H}^{n-1} \leq \frac{4}{h_k} |D w_j^{\rho,k}|(Q^{\nu,k}) + \frac{4}{h_k}|a|. \tag{5.5.25}$$

Defining $\hat{w}_j^{\rho,k}$ as

$$\hat{w}_j^{\rho,k}(y) = \begin{cases} w_j^{\rho,k} & \text{if } y \in Q^{\nu,\lambda^k_{\rho,j}} \\ \ell_k(y) & \text{if } y \in Q^{\nu,k} \setminus Q^{\nu,\lambda^k_{\rho,j}} \end{cases}$$

we have that $\hat{w}_j^{\rho,k} \in BV(Q^{\nu,k}, \mathbb{R}^N)$, $\hat{w}_j^{\rho,k} = \ell_k$ on $\partial Q^{\nu,k}$, and by (5.5.25)

$$\begin{aligned} |D^s \hat{w}_j^{\rho,k}|(Q^{\nu,k}) &\leq |D^s w_j^{\rho,k}|(Q^{\nu,k}) + \frac{4}{h_k} |D w_j^{\rho,k}|(Q^{\nu,k}) + \frac{4}{h_k}|a| \\ &\leq 2|D^s w_j^{\rho,k}|(Q_1^{\nu,k}) + \frac{4}{h_k} \|\nabla w_j^{\rho,k}\|_{L^1(Q^{\nu,k})} + \frac{4}{h_k}|a|. \end{aligned}$$

In particular, by (5.5.22), we obtain

$$\limsup_{\rho \to 0, \rho \in I(x_0)} \limsup_{j \to \infty} |D^s \hat{w}_j^{\rho,k}|(Q^{\nu,k}) = \frac{4}{h_k} \limsup_{\rho \to 0, \rho \in I(x_0)} \limsup_{j \to \infty} \|\nabla w_j^{\rho,k}\|_{L^1(Q^{\nu,k})} + \frac{4|a|}{h_k} \tag{5.5.26}$$

and, from (5.5.23),

$$\begin{aligned} &\Big(1 + \frac{C^2}{M}\Big) \frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0) + \frac{C|a|\mathcal{L}^n(Q^{\nu,k} \setminus Q^{\nu,h_k})}{k^{n-1}} \\ &\qquad \geq \limsup_{\rho \to 0,\, \rho \in I(x_0)} \limsup_{j \to \infty} \alpha_\delta \int_{Q^{\nu,k}} f^\infty(\tfrac{x_0 + \rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k}) \mathrm{d}y. \end{aligned} \tag{5.5.27}$$

We now argue as in Proposition 5.3.4 (cf. (5.3.32), (5.3.33)), and for every $\rho$ and $j$ fixed we use Lemma 5.3.5 to infer the existence of $\overline{w}_j^{\rho,k} \in W^{1,1}(Q^{\nu,k}, \mathbb{R}^N)$ such that $\overline{w}_j^{\rho,k} = \ell_k$ on $\partial Q^{\nu,k}$ and

$$\begin{aligned} \int_{Q^{\nu,k}} f^\infty(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k}) \mathrm{d}y &\geq \int_{Q_1^{\nu,k}} \widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \hat{w}_j^{\rho,k}) \mathrm{d}y \\ &\geq sc^-(L^1) F^\infty_{\rho,j}(\hat{w}_j^{\rho,k}) - C|D^s \hat{w}_j^{\rho,k}|(Q^{\nu,k}) \\ &\geq \int_{Q^{\nu,k}} \widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j}, \nabla \overline{w}_j^{\rho,k}) \mathrm{d}y - C|D^s \hat{w}_j^{\rho,k}|(Q^{\nu,k}) - \frac{\rho}{j} \,. \end{aligned}$$

Therefore, from (5.5.21), (5.5.26), (5.5.27) we conclude form the last inequality above that

$$\Big(1+\frac{C^2}{M}-\frac{4}{h_k}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)+\frac{C|a|\mathcal{L}^n(Q_1^{\nu,k}\setminus Q_1^{\nu,h_k})}{k^{n-1}}+C\alpha_\delta\frac{4|a|}{h_k}$$
$$\geq \limsup_{\rho\to0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty}\alpha_\delta\int_{Q^{\nu,k}}\widehat{f^\infty}(\tfrac{x_0+\rho y}{\varepsilon_j},\nabla\overline{w}_j^{\rho,k})\mathrm{d}y\,. \tag{5.5.28}$$

Set $r_{\rho,j}:=\frac{\rho}{\varepsilon_j}$ and $\tilde{w}_j^{\rho,k}(x):=r_{\rho,j}\overline{w}_j^{\rho,k}\left(\frac{x}{r_{\rho,j}}-\frac{x_0}{\rho}\right)$, we have that $\tilde{w}_j^{\rho,k}\in W^{1,1}(Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right),\mathbb{R}^N)$ with $\tilde{w}_j^{\rho,k}=\ell_k-\frac{x_0}{\varepsilon_j}$ on $\partial Q_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)$ and, thanks to the 1-homogeneity of $\widehat{f^\infty}$ (cf. item (i) in Lemma 5.3.5), we infer that

$$\int_{Q^{\nu,k}}\widehat{f^\infty}\left(\frac{x_0+\rho y}{\varepsilon_j},\nabla\overline{w}_j^{\rho,k}\right)\mathrm{d}y=\frac{1}{k^{n-1}r^n_{\rho,j}}\int_{Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)}\widehat{f^\infty}(y,k^{n-1}\nabla\tilde{w}_j^{\rho,k})\mathrm{d}y.$$

In particular, since $k^{n-1}\tilde{w}_j^{\rho,k}=\ell_{a\otimes\nu}-\frac{k^{n-1}}{\varepsilon_j}x_0$ on $\partial Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)$, thanks to (5.5.28) we obtain

$$\begin{aligned}
&\Big(1+\frac{C^2}{M}-\frac{4}{h_k}\Big)\frac{\mathrm{d}\mu^\delta}{\mathrm{d}\mathcal{L}^n}(x_0)+\frac{C|a|\mathcal{L}^n(Q^{\nu,k}\setminus Q^{\nu,h_k})}{k^{n-1}}+C\alpha_\delta\frac{4|a|}{h_k}\\
&\qquad\geq \limsup_{\rho\to0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty}\frac{\alpha_\delta}{k^{n-1}r^n_{\rho,j}}\int_{Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)}\widehat{f^\infty}(y,k^{n-1}\nabla\tilde{w}_j^{\rho,k})\mathrm{d}y\\
&\qquad\geq \limsup_{\rho\to0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty}\alpha_\delta\frac{m_{\mathrm{b}}^{\widehat{f^\infty}}\left(\ell_{a\otimes\nu},Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right)\right)}{k^{n-1}r^n_{\rho,j}}\\
&\qquad= \limsup_{\rho\to0,\ \rho\in I(x_0)}\ \limsup_{j\to\infty}\alpha_\delta\frac{m_{\mathrm{b}}^{f^\infty}(\ell_{a\otimes\nu},Q^{\nu,k}_{r_{\rho,j}}\left(\frac{r_{\rho,j}}{\rho}x_0\right))}{k^{n-1}r^n_{\rho,j}}=\alpha_\delta f^\infty_{\mathrm{hom}}(a\otimes\nu)\,,
\end{aligned}$$

where the last-but-one equality follows from Lemma 5.3.5 (vi), and the last equality follow from Proposition 5.2.3. Then, taking $k\to\infty$ and $M\to\infty$ in this order, we infer that

$$\frac{\mathrm{d}\mu^\delta}{\mathrm{d}|D^cu|}(x_0)\geq\alpha_\delta f^\infty_{\mathrm{hom}}(a\otimes\nu)=\alpha_\delta f^\infty_{\mathrm{hom}}\Big(\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}(x_0)\Big).$$

Therefore, using (5.5.9) we conclude that

$$\widehat{\mathcal{H}}(u,1,A')\geq\liminf_{j\to\infty}H^\delta_{\varepsilon_j}(u_j^\delta,A')=\liminf_{j\to\infty}\mu_j^\delta(A')\geq\mu^\delta(A')\geq\alpha_\delta\int_{A'}f^\infty_{\mathrm{hom}}(\tfrac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}(x))\mathrm{d}|D^cu|,$$

thus (5.5.8) follows by letting $\delta\to0$ recalling that $\alpha_\delta\to1$ and by the arbitrariness of $A'\in\mathcal{A}(A)$. □

Finally, we are in a position to prove the deterministic homogenisation result Theorem 5.3.1.

*Proof of Theorem 5.3.1.* Theorem 5.3.2 implies that from any strictly positive infinitesimal sequence we can extract a subsequence $(\varepsilon_j)$ such that

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty}\mathcal{H}_{\varepsilon_j}(u,1,A)=\widehat{\mathcal{H}}(u,1,A)$$

with $\widehat{\mathcal{H}} : L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}) \times \mathcal{A}(\mathbb{R}^n) \longrightarrow [0, +\infty]$. Moreover, $\widehat{\mathcal{H}}(u, 1, \cdot)$ is the restriction to open sets of a finite Radon measure on $A$ and $\widehat{\mathcal{H}}(u, 1, A) \leq C(|Du|(A) + \mathcal{L}^n(A))$ for every $A \in \mathcal{A}(\mathbb{R}^n)$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ such that $u \in BV(A, \mathbb{R}^N)$. Therefore, $\widehat{\mathcal{H}}(u, 1, \cdot)$ is absolutely continuous respect to the measure $\mathcal{L}^n \llcorner A + |D^c u| \llcorner A + \mathcal{H}^{n-1} \llcorner J_u \cap A$. Since $\mathcal{L}^n \llcorner A, |D^c u| \llcorner A, \mathcal{H}^{n-1} \llcorner J_u \cap A$ are mutually singular, by the properties of Radon-Nikodym derivatives, for every $B \in \mathcal{A}(A)$ we have that

$$\widehat{\mathcal{H}}(u, 1, B) = \int_B \frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1, \cdot)}{\mathrm{d}\mathcal{L}^n} \,\mathrm{d}x + \int_B \frac{\mathrm{d}\widehat{\mathcal{H}}(u, 1, \cdot)}{\mathrm{d}|D^c u|} \mathrm{d}|D^c u| + \int_{J_u \cap B} \frac{\mathrm{d}\widehat{\mathcal{H}}}{\mathrm{d}\mathcal{H}^{n-1} \llcorner J_u} \mathrm{d}\mathcal{H}^{n-1} .$$

In particular, if $A \in \mathcal{A}(\mathbb{R}^n)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ with $u \in BV(A, \mathbb{R}^N) \cap L^\infty(A, \mathbb{R}^N)$, Propositions 5.3.4, 5.4.1 and 5.5.1 give that

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty} \mathcal{H}_{\varepsilon_j}(u, 1, A) = \widehat{\mathcal{H}}(u, 1, A) = \mathcal{H}_{\mathrm{hom}}(u, 1, A),$$

where $\mathcal{H}_{\mathrm{hom}}$ is as in (5.3.4). From (f2), for every $A \in \mathcal{A}(\mathbb{R}^n)$ and every $(u, v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ with $(u, v) \in W^{1,1}(A, \mathbb{R}^N) \times W^{1,2}(A, [0, 1])$ we have that

$$C^{-1} \int_A \Big( v^2 |\nabla u| + \frac{(1 - v)^2}{\varepsilon_j} + \varepsilon_j |\nabla v|^2 \Big) \,\mathrm{d}x \leq \mathcal{H}_{\varepsilon_j}(u, v, A),$$

hence, by [ABS99, Theorem 4.1] and [AF02, Remark 3.5], for every $(u, v) \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ such that $u \notin (GBV(A))^N$ or $v \neq 1$ on $A$ we get

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}))\text{-}\liminf_{j\to\infty} \mathcal{H}_{\varepsilon_j}(u, v, A) = \mathcal{H}_{\mathrm{hom}}(u, v, A) = +\infty .$$

Eventually, arguing exactly as in [AF02, Section 6] we obtain

$$\Gamma(L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1}))\text{-}\lim_{j\to\infty} \mathcal{H}_{\varepsilon_j}(u, 1, A) = \mathcal{H}_{\mathrm{hom}}(u, 1, A)$$

for every $A \in \mathcal{A}(\mathbb{R}^n)$ and every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ such that $u \in (GBV(A))^N$. Indeed, the lower bound inequality for general $GBV$ maps follows easily from Lemma 5.3.3 and the result in the $BV \cap L^\infty$-setting. Instead, the upper bound inequality is a consequence of the latter together with both the $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{N+1})$ lower semicontinuity of $\Gamma\text{-}\limsup_{j\to\infty} \mathcal{H}_{\varepsilon_j}(\cdot, 1, A)$ and the continuity of $\mathcal{H}_{\mathrm{hom}}$ along sequences of maps obtained via the smooth truncations $(\mathcal{T}_k)_{k\in\mathbb{N}}$, namely $\mathcal{H}_{\mathrm{hom}}(\mathcal{T}_k(u), 1, A) \to \mathcal{H}_{\mathrm{hom}}(u, 1, A)$ as $k \to \infty$ for every $u \in (GBV(A))^N$ (cf. [AF02, Lemma 6.1], Proposition 3.3.14 and Theorem 3.3.15).

Since the $\Gamma$-limit does not depend on the extracted subsequence Uryshon's property of $\Gamma$-convergence yields the claim. $\square$

## 5.6 Stochastic homogenisation

This section is devoted to the proof of the stochastic homogenisation result stated in Theorem 5.1.7. The proof will be achieved by showing that if $f$ is a stationary random integrand in the sense of Definition 5.1.4, then the assumptions of Theorem 5.3.1 are satisfied for $P$-a.e. $\omega \in \Omega$. Here a pivotal role is played by the

Subadditive Ergodic Theorem, Theorem 2.0.9.

The following proposition establishes the existence and spatial homogeneity of $f_{\mathrm{hom}}$. The proof can be found in [CDMSZ22, Proposition 9.1] and in [RZ23, Lemma 4.1].

**Proposition 5.6.1** (Homogenized random volume integrand)**.** *Let $f$ be a stationary random integrand. Then there exist $\Omega' \in \mathcal{M}$, with $P(\Omega') = 1$ and a $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable function $f_{\mathrm{hom}} : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$, $\nu \in \mathbb{S}^{n-1}$ and $k \in \mathbb{N}$*

$$f_{\mathrm{hom}}(\omega, \xi) = \lim_{r\to+\infty} \frac{m_{\mathrm{b}}^{f_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = \lim_{r\to+\infty} \frac{m_{\mathrm{b}}^{f_\omega}(\ell_\xi, Q_r)}{r^n}.$$

*If in addition $f$ is ergodic, then $f_{\mathrm{hom}}$ is independent of $\omega$ and*

$$f_{\mathrm{hom}}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n} \int_\Omega m_{\mathrm{b}}^{f_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega).$$

Propositions 5.2.3 and 5.6.1 readily imply the following result.

**Proposition 5.6.2** (Homogenized random Cantor integrand)**.** *Let $f$ be a stationary random integrand. Then there exist $\Omega' \in \mathcal{M}$, with $P(\Omega') = 1$ and a $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^{N\times n})$-measurable function $f^\infty_{\mathrm{hom}} : \Omega \times \mathbb{R}^{N\times n} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, every $\xi \in \mathbb{R}^{N\times n}$ every $k \in \mathbb{N}$, every $x \in \mathbb{R}^n$ and every $\nu \in \mathbb{S}^{n-1}$*

$$f^\infty_{\mathrm{hom}}(\omega, \xi) = \lim_{t\to+\infty} \frac{f_{\mathrm{hom}}(\omega, t\xi)}{t}$$

*and*

$$f^\infty_{\mathrm{hom}}(\omega, \xi) = \lim_{r\to+\infty} \frac{m_{\mathrm{b}}^{f^\infty_\omega}(\ell_\xi, Q_r^{\nu,k}(rx))}{k^{n-1}r^n} = \lim_{r\to+\infty} \frac{m_{\mathrm{b}}^{f^\infty_\omega}(\ell_\xi, Q_r)}{r^n}.$$

*If in addition $f$ is ergodic, then $f^\infty_{\mathrm{hom}}$ is independent of $\omega$ and*

$$f^\infty_{\mathrm{hom}}(\xi) = \lim_{r\to+\infty} \frac{1}{r^n} \int_\Omega m_{\mathrm{b}}^{f^\infty_\omega}(\ell_\xi, Q_r)\mathrm{d}P(\omega). \tag{5.6.1}$$

The analogous result for the surface integrand is more involved and requires a new proof.

**Proposition 5.6.3** (Homogenized random surface integrand)**.** *Let $f$ be a stationary random integrand. Then there exist $\Omega' \in \mathcal{M}$, with $P(\Omega') = 1$ and a $\mathcal{M}\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{S}^{n-1})$-measurable function $g_{\mathrm{hom}} : \Omega\times\mathbb{R}^N\times\mathbb{S}^{n-1} \to [0, +\infty)$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$*

$$g_{\mathrm{hom}}(\omega, \zeta, \nu) = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(u_{rx,\zeta,\nu}, Q_r^\nu(rx))}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}.$$

*If in addition $f$ is ergodic, then $g_{\mathrm{hom}}$ is independent of $\omega$ and*

$$g_{\mathrm{hom}}(\zeta, \nu) = \lim_{r\to+\infty} \frac{1}{r^{n-1}} \int_\Omega m_{\mathrm{s}}^{f^\infty_\omega}(u_{\zeta,\nu}, Q_r^\nu)\mathrm{d}P(\omega). \tag{5.6.2}$$

*Proof.* We divide the proof into a number of steps.

*Step 1:* Let $\overline{u}_{\zeta,\nu}$ be as in 2.0.3 of the notation list. In this step we prove that for every $\zeta \in \mathbb{Q}^N$ and

$\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and for $P$-a.e. $\omega \in \Omega$ there exists the limit

$$\lim_{r \to +\infty} \frac{m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}}$$

and defines an $x$-independent random variable.

To prove the claim let $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and $\zeta \in \mathbb{Q}^N$ be fixed, $R_\nu \in O(n) \cap \mathbb{Q}^{n\times n}$ be the orthogonal matrix as in Remark 2.0.1, and $M_\nu$ be a positive integer such that $M_\nu R_\nu \in \mathbb{Z}^{n\times n}$, so that $M_\nu R_\nu(z', 0) \in \Pi_0^\nu \cap \mathbb{Z}^n$. Given $A' = [a_1, b_1) \times \cdots \times [a_{n-1}, b_{n-1}) \in \mathcal{I}_{n-1}$ we define the $n$-dimensional interval $T_\nu(A')$ as

$$T_\nu(A') := M_\nu R_\nu(A' \times [-c, c)), \quad \text{with } c := \frac{1}{2} \max_{1\le j\le n-1}(b_j - a_j). \tag{5.6.3}$$

For every $\omega \in \Omega$ and every $A' \in \mathcal{I}_{n-1}$ we set

$$\mu_{\zeta,\nu}(\omega, A') := \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A')). \tag{5.6.4}$$

We now show that $\mu_{\zeta,\nu} : \Omega \times \mathcal{I}_{n-1} \to [0, +\infty)$ defines an $(n-1)$-dimensional subadditive process on $(\Omega, \mathcal{M}, P)$. The separability and completeness of $W^{1,1}(A, \mathbb{R}^N) \times W^{1,2}(A, [0, 1])$ for every $A \in \mathcal{A}(\mathbb{R}^n)$ combined with [RR23, Lemma C.2] and $(f2)$ in Definition 5.1.1 give the $\mathcal{M}$-measurability of the map $\omega \mapsto m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A'))$ for every $A' \in \mathcal{I}_{n-1}$.

Next, we prove that $\mu_{\zeta,\nu}$ is stationary with respect to an $(n-1)$-dimensional group of $P$-preserving transformations $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$. To this end, fix $z' \in \mathbb{Z}^{n-1}$ and $A' \in \mathcal{I}_{n-1}$. By (5.6.3) we have that

$$T_\nu(A' + z') = M_\nu R_\nu(A' \times [-c, c)) + M_\nu R_\nu(z', 0) = T_\nu(A') + z'_\nu,$$

where $z'_\nu := M_\nu R_\nu(z', 0) \in \Pi^\nu \cap \mathbb{Z}^n$. Thus by (5.6.4) we get

$$\mu_{\zeta,\nu}(\omega, A' + z') = \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A' + z')) = \frac{1}{M_\nu^{n-1}} m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A') + z'_\nu). \tag{5.6.5}$$

Now let $u, v$ be test functions in the definition of $m_{\rm s}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, T_\nu(A') + z'_\nu)$ and for $x \in T_\nu(A')$ set

$$\tilde{u}(x) := u(x + z'_\nu) \quad \text{and} \quad \tilde{v}(x) := v(x + z'_\nu).$$

Then, a change of variables together with the stationarity of $f$ yield

$$S^{f^\infty(\omega)}(u, v, T_\nu(A') + z'_\nu) = S^{f^\infty(\tau_{z'_\nu}\omega)}(\tilde{u}, \tilde{v}, T_\nu(A')).$$

Set $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}} := (\tau_{z'_\nu})_{z'\in\mathbb{Z}^{n-1}}$; we notice that $(\tau^\nu_{z'})_{z'\in\mathbb{Z}^{n-1}}$ is well defined since $z'_\nu \in \mathbb{Z}^n$ and it defines a group of $P$-preserving transformations on $(\Omega, \mathcal{M}, P)$. Then, the equality above can be rewritten as

$$S^{f^\infty(\omega)}(u, v, T_\nu(A') + z'_\nu) = S^{f^\infty(\tau^\nu_{z'}\omega)}(\tilde{u}, \tilde{v}, T_\nu(A')). \tag{5.6.6}$$

Moreover, since $z'_\nu \in \Pi_\nu \cap \mathbb{Z}^n$ we also have that $\tilde{u} = \bar{u}_{\zeta,\nu}$ on $\partial T_\nu(A')$. Thus gathering (5.6.5) and (5.6.6), by the arbitrariness of $\tilde{u}, \tilde{v}$ we infer

$$\mu_{\zeta,\nu}(\omega, A' + z') = \mu_{\zeta,\nu}(\tau^{\nu}_{z'}\omega, A'),$$

and hence the stationarity of $\mu_{\zeta,\nu}$ with respect to $(\tau^{\nu}_{z'})_{z'\in\mathbb{Z}^{n-1}}$.

To show that $\mu_{\zeta,\nu}$ is subadditive in $\mathcal{I}_{n-1}$, fix $\omega \in \Omega$ and $A' \in \mathcal{I}_{n-1}$ and let $(A'_i)_{1\le i\le M} \subset \mathcal{I}_{n-1}$ be a finite family of pairwise disjoint sets such that $A' = \cup_{i=1}^M A'_i$. For every $\eta > 0$ and $i \in \{1, \dots, M\}$, let $(u_i, v_i) \in W^{1,1}(T_\nu(A'_i), \mathbb{R}^N) \times W^{1,2}(T_\nu(A'_i), [0, 1])$ with $(u_i, v_i) = (\overline{u}_{\zeta,\nu}, 1)$ on $\partial T_\nu(A'_i)$ such that

$$\int_{T_\nu(A'_i)} \left(v_i^2 f^\infty(\omega, y, \nabla u_i) + (1 - v_i)^2 + |\nabla v_i|^2\right) \mathrm{d}y \le m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, T_\nu(A'_i)) + \eta.$$

Note that by construction we always have $\cup_{i=1}^M T_\nu(A'_i) \subseteq T_\nu(A')$, thus we define

$$(u(y), v(y)) := \begin{cases} (u_i(y), v_i(y)) & \text{if } y \in T_\nu(A'_i) \\ (\overline{u}_{\zeta,\nu}, 1) & \text{if } y \in T_\nu(A') \setminus \cup_i T_\nu(A'_i). \end{cases}$$

In particular, $(u, v) \in W^{1,1}(T_\nu(A'), \mathbb{R}^N) \times W^{1,2}(T_\nu(A'), [0, 1])$ with $(u, v) = (\overline{u}_{\zeta,\nu}, 1)$ on $\partial T_\nu(A')$. Hence, we get

$$\begin{aligned} m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, T_\nu(A')) &\le \int_{T_\nu(A')} \left(v^2 f^\infty(\omega, y, \nabla u) + (1 - v)^2 + |\nabla v|^2\right) \mathrm{d}y \\ &= \sum_{i=1}^M \int_{T_\nu(A'_i)} \left(v_i^2 f^\infty(\omega, y, \nabla u_i) + (1 - v_i)^2 + |\nabla v_i|^2\right) \mathrm{d}y \le \sum_{i=1}^M m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, T_\nu(A'_i)) + M\eta \end{aligned}$$

and the subadditivity follows by the arbitrariness of $\eta > 0$.

Finally, we show that $\mu_{\zeta,\nu}$ is bounded. To this end we observe that for every $A' \in \mathcal{I}_{n-1}$ and every $\omega \in \Omega$ we have

$$\mu_{\zeta,\nu}(\omega, A') = \frac{1}{M_\nu^{n-1}} m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, T_\nu(A')) \le \frac{C}{M_\nu^{n-1}} \int_{T_\nu(A')} |\nabla \overline{u}_{\zeta,\nu}| \mathrm{d}y \le C|\zeta| \|\overline{\mathrm{u}}'\|_{L^\infty(\mathbb{R})} \mathcal{L}^{n-1}(A'),$$

where we used $T_\nu(A') \cap \Pi^\nu = M_\nu R_\nu(A' \times \{0\})$ and $\{|\nabla(\overline{u}_{\zeta,\nu}(y)| > 0\} \subseteq \{|y \cdot \nu| \le 1/2\}$.

Therefore, for every fixed for every $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ defines a subadditive process. Then, we can apply Theorem 2.0.9 to deduce the existence of a $\mathcal{M}$-measurable function $\psi_{\nu,\zeta} : \Omega \to [0, +\infty)$, and a set $\Omega_{\zeta,\nu} \in \mathcal{M}$ with $P(\Omega_{\zeta,\nu}) = 1$, such that for every $\omega \in \Omega_{\zeta,\nu}$

$$M_\nu^{n-1} \psi_{\zeta,\nu}(\omega) = M_\nu^{n-1} \lim_{r\to+\infty} \frac{\mu_{\zeta,\nu}(\omega, Q'_r)}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}.$$

where $Q'_r := Q_r \cap \{x_n = 0\}$, $r > 0$.

*Step 2:* In this step we prove the existence of $\tilde{\Omega} \in \mathcal{M}$ with $P(\tilde{\Omega}) = 1$ such that for every $\omega \in \tilde{\Omega}$ and for every $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ the following limit exists

$$\lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}}$$

and defines an $x$-independent $\mathcal{M}\otimes\mathcal{B}(\mathbb{R}^N)\otimes\mathcal{B}(\mathbb{S}^{n-1})$-measurable function.

To prove the claim let $\tilde\Omega$ denote the intersection of the sets $\Omega_{\zeta,\nu}$, as in Step 1, for $\zeta\in\mathbb{Q}^N$ and $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$. Clearly, $\tilde\Omega\in\mathcal{M}$ and $P(\tilde\Omega)=1$. Let $\underline{g},\overline{g}:\tilde\Omega\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty]$ be the functions given by

$$\underline{g}(\omega,\zeta,\nu):=\liminf_{r\to+\infty}\frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}\,,\qquad \overline{g}(\omega,\zeta,\nu):=\limsup_{r\to+\infty}\frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu},Q_r^\nu)}{r^{n-1}}.$$

By Step 1, for every $\omega\in\tilde\Omega$, every $\zeta\in\mathbb{Q}^N$ and every $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$ we have that

$$\underline{g}(\omega,\zeta,\nu)=\overline{g}(\omega,\zeta,\nu)\,. \tag{5.6.7}$$

Furthermore, fixed $\omega\in\tilde\Omega$ and $\nu\in\mathbb{S}^{n-1}$, arguing as in Proposition 5.2.7 (i) we have

$$|\underline{g}(\omega,\zeta_1,\nu)-\underline{g}(\omega,\zeta_2,\nu)|+|\overline{g}(\omega,\zeta_1,\nu)-\overline{g}(\omega,\zeta_2,\nu)|\le 2C\mathcal{H}^{n-1}(\partial Q_1)|\zeta_1-\zeta_2| \tag{5.6.8}$$

for every $\zeta_1,\zeta_2\in\mathbb{R}^N$. From (5.6.7) and (5.6.8) we deduce that for every $\omega\in\tilde\Omega$, every $\zeta\in\mathbb{R}^N$ and every $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$

$$\underline{g}(\omega,\zeta,\nu)=\overline{g}(\omega,\zeta,\nu)\,, \tag{5.6.9}$$

and that $\underline{g}(\cdot,\zeta,\nu):\tilde\Omega\to[0,+\infty)$ is $\mathcal{M}$-measurable for every $\zeta\in\mathbb{R}^N$ and every $\nu\in\mathbb{S}^{n-1}\cap\mathbb{Q}^n$.

We now claim that for every $\omega\in\tilde\Omega$ and every $\zeta\in\mathbb{R}^N$, the restrictions of the functions $\nu\mapsto\underline{g}(\omega,\zeta,\nu)$ and $\nu\mapsto\overline{g}(\omega,\zeta,\nu)$ to the sets $\hat{\mathbb{S}}^{n-1}_\pm$ are continuous. We show only the continuity of $\underline{g}$ on $\hat{\mathbb{S}}^{n-1}_+$, the proof for $\overline{g}$ is analogous. To this end, let $\omega\in\tilde\Omega$, $\zeta\in\mathbb{R}^N$, $\nu\in\hat{\mathbb{S}}^{n-1}_+$, then by density let $(\nu_j)_{j\in\mathbb{N}}\subset\hat{\mathbb{S}}^{n-1}_+\cap\mathbb{Q}^n$ be such that $\nu_j\to\nu$ as $j\to+\infty$. By the continuity of $\nu\mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_+$, for every $\delta\in(0,1/2)$ there exists a $j_\delta\in\mathbb{N}$ such that

$$Q_r^\nu\subset\subset Q^{\nu_j}_{(1+\delta)r}\subset\subset Q^\nu_{(1+2\delta)r} \tag{5.6.10}$$

for every $j\ge j_\delta$ and every $r>0$.

Setting $\kappa_j:=\max\{|R_{\nu_j}(e_i)\cdot\nu|\ :\ i=1,\dots,n-1\}$ we have that $\kappa_j\to0$ as $j\to+\infty$, thanks to the continuity of $\nu\mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_\pm$. We observe that for every $y\in\overline{Q^\nu_{r(1+\delta)}}$, we have $y=y'+(y\cdot\nu_j)\nu_j$ where

$$y'\in R_{\nu_j}\Big(\big(-\frac{r}{2}(1+\delta),\frac{r}{2}(1+\delta)\big)^{n-1}\times\{0\}\Big)$$

and in particular, if in addition $|y\cdot\nu|\le\frac12$, for $j$ large enough depending only on $\delta$, we get

$$|y\cdot\nu_j|\le\tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|}+\tfrac{1}{2(1-\delta)}<\tfrac{(n-1)\kappa_j r(1+\delta)}{2(1-\delta)}+1=K(\delta)r\kappa_j+1,$$

where $K(\delta):=\frac{(n-1)(1+\delta)}{2(1-\delta)}$. Then, by applying Lemma 5.2.6, with $R=K(\delta)r\kappa_j+1$, we obtain

$$\begin{aligned}m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu_j},Q^{\nu_j}_{r(1+\delta)})&\le m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu},Q_r^\nu)+\eta r^{n-1}\\&\qquad+\tilde K(2\delta(1+2\delta)^{n-2}r^{n-1}+(K(\delta)r\kappa_j+1)|\zeta|(1+\delta)^{n-2}r^{n-2}).\end{aligned}$$

Therefore, dividing by $r^{n-1}$, passing to the liminf as $r\to+\infty$, and to the limsup as $j\to+\infty$, and finally letting $\eta, \delta \to 0$ we obtain

$$\limsup_{j\to+\infty} \underline{g}(\omega,\zeta,\nu_j) \le \underline{g}(\omega,\zeta,\nu).$$

An analogous argument using the cubes $Q^{\nu_j}_{(1-\delta)r}$ shows that

$$\underline{g}(\omega,\zeta,\nu) \le \liminf_{j\to+\infty} \underline{g}(\omega,\zeta,\nu_j)$$

implying the claim.

In particular, thanks to (5.6.9) we deduce that for every $\omega \in \tilde{\Omega}$, $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$

$$\underline{g}(\omega,\zeta,\nu) = \overline{g}(\omega,\zeta,\nu)\,. \tag{5.6.11}$$

The $\mathcal{M}$-measurability of $\underline{g}(\cdot,\zeta,\nu) : \tilde{\Omega} \to [0,+\infty)$ for every $\zeta \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ follows from the analogous property for $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Furthermore, the map $\underline{g}(\omega,\cdot,\cdot) : \mathbb{R}^N \times \hat{\mathbb{S}}^{n-1}_{\pm} \to [0,+\infty)$ is continuous for every $\omega \in \tilde{\Omega}$ thanks to (5.6.8).

Thus, defining $g_{\hom} : \Omega \times \mathbb{R}^N \times \mathbb{S}^{n-1} \to [0,+\infty)$ by

$$g_{\hom}(\omega,\zeta,\nu) := \begin{cases} \underline{g}(\omega,\zeta,\nu) & \text{if } \omega \in \tilde{\Omega} \\ \frac{2|\zeta|}{C(|\zeta|+2)} & \text{if } \omega \notin \tilde{\Omega}, \end{cases}$$

we have that $g_{\hom}$ is $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{S}^{n-1})$-measurable and, thanks to Corollary 5.2.5,

$$g_{\hom}(\omega,\zeta,\nu) = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(u_{\zeta,\nu}, Q^\nu_r)}{r^{n-1}} \tag{5.6.12}$$

for every $\omega \in \tilde{\Omega}$, every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$.

*Step 3:* In this step we show the existence of $\Omega' \in \mathcal{M}$ with $\Omega' \subseteq \tilde{\Omega}$ and $P(\Omega') = 1$, such that for every $\omega \in \Omega'$, $z \in \mathbb{Z}^n$, $\zeta \in \mathbb{Q}^N$, $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, and for every integer sequence $(r_k)$ with $r_k \ge k$ for every $k$

$$\lim_{k\to+\infty} \frac{m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{-kz,\zeta,\nu}, Q^\nu_{r_k}(-kz))}{r_k^{n-1}} = g_{\hom}(\omega,\nu,\zeta)\,. \tag{5.6.13}$$

Let $z \in \mathbb{Z}^n$, $\zeta \in \mathbb{Q}^N$, $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, $\eta > 0$ and $\delta \in (0, 1/4)$. Arguing exactly as in [CDMSZ19b, Theorem 6.1] we can prove the existence of a set $\Omega^{\zeta,\nu,\eta}_z \in \mathcal{M}$, with $\Omega^{\zeta,\nu,\eta}_z \subseteq \tilde{\Omega}$, $P(\Omega^{\zeta,\nu,\eta}_z) = 1$, and an integer $m_0 = m_0(\zeta,\nu,\eta,z,\omega,\delta) > \frac{1}{\delta}$ satisfying the following property: for every $\omega \in \Omega^{\zeta,\nu,\eta}_z$ and for every integer $m \ge m_0$ there exists $i = i(\zeta,\nu,\eta,z,\omega,\delta,m) \in \{m+1,\dots,m+\ell\}$, with $\ell := \lfloor 5m\delta \rfloor$, such that

$$\left| \frac{m_{\mathrm{s}}^{f^\infty_\omega}(\overline{u}_{-iz,\zeta,\nu}, Q^\nu_h(-iz))}{h^{n-1}} - g_{\hom}(\omega,\zeta,\nu) \right| \le \eta \quad \text{for every } h \in \mathbb{N} \text{ with } h \ge j_0, \tag{5.6.14}$$

where $j_0 = j_0(\zeta,\nu,\eta,z,\omega,\delta)$, and $\lfloor s \rfloor$ denotes the integer part of $s \in \mathbb{R}$.

Define $\Omega'$ as the intersection of the sets $\Omega^{\zeta,\nu,\eta}_z$ for $\zeta \in \mathbb{Q}^N$, $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, $\eta \in \mathbb{Q}$, with $\eta > 0$ and $z \in \mathbb{Z}^n$. Thus $\Omega' \subseteq \tilde{\Omega}$ and $P(\Omega') = 1$. Let $\omega \in \Omega'$ and $r_k$ be as required, $\delta > 0$ with $20\delta(|z|+1) < 1$ and $\eta \in \mathbb{Q}$ with

$\eta > 0$. For every $k \geq 2m_0(\zeta,\nu,\eta,z,\omega,\delta)$, let $\underline{r}_k, \overline{r}_k \in \mathbb{N}$ be defined as

$$\underline{r}_k := r_k - 2(i_k - k)\lfloor |z| + 1\rfloor \quad \text{and} \quad \overline{r}_k := r_k + 2(i_k - k)\lfloor |z| + 1\rfloor,$$

where

$$i_k = i(\zeta,\nu,\eta,z,\omega,\delta,k) \in \{k+1,\ldots,k+\lfloor 5k\delta\rfloor\}, \tag{5.6.15}$$

therefore, by construction, we have that $Q^\nu_{\underline{r}_k}(-i_k z) \subset\subset Q^\nu_{r_k}(-kz) \subset\subset Q^\nu_{\overline{r}_k}(-i_k z)$.

Since $20\delta(|z|+1) < 1$, $k \leq r_k$ and $i_k - k \leq 5k\delta$ by (5.6.15), for every $y \in Q^\nu_{\underline{r}_k}(-i_k z)$ such that $|(y+i_k z)\cdot\nu| \leq \frac{1}{2}$, we obtain that

$$\begin{aligned}|(y+kz)\cdot\nu| &= |(y+i_k z)\cdot\nu + (kz - i_k z)\cdot\nu| \leq \frac{1}{2} + |(kz-i_k z)\cdot\nu| \\ &\leq (i_k - k)|z| + \frac{1}{2} \leq 5k\delta|z| + \frac{1}{2} \leq 5r_k\delta|z| + \frac{1}{2},\end{aligned}$$

and $r_k - \underline{r}_k = 2(i_k-k)\lfloor |z|+1\rfloor \leq 10k\delta\lfloor |z|+1\rfloor \leq 10r_k\delta\lfloor |z|+1\rfloor < \frac{r_k}{2}$. Applying Lemma 5.2.6, with $R = 5r_k\delta|z| + \frac{1}{2}$, we obtain

$$\begin{aligned}&m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q^\nu_{r_k}(-kz)) \leq m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q^\nu_{\underline{r}_k}(-i_k z)) + \eta \underline{r}_k^{n-1} \\ &+ \frac{\tilde{K}}{2}(10\delta(|z|+1)r_k^{n-1} + (10 r_k\delta|z|+1)|\zeta| r_k^{n-2}).\end{aligned}$$

In particular, from the latter estimate, (5.6.14) and $\underline{r}_k \leq r_k$, for every $k$ large enough such that $\underline{r}_k \geq j_0(\zeta,\nu,\eta,z,\omega,\delta)$, we obtain

$$\begin{aligned}g_{\mathrm{hom}}(\omega,\zeta,\nu) + \eta &\geq \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q^\nu_{\underline{r}_k}(-i_k z))}{\underline{r}_k^{n-1}} \geq \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-i_k z,\zeta,\nu}, Q^\nu_{\underline{r}_k}(-i_k z))}{r_k^{n-1}} \\ &\geq \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q^\nu_{r_k}(-kz))}{r_k^{n-1}} - \eta - \frac{\tilde{K}}{2}(10\delta(|z|+1) + (10\delta|z| + \frac{1}{r_k})|\zeta|)\end{aligned}$$

and thus, taking the limsup for $k \to +\infty$ and letting $\eta,\delta \to 0$, we get

$$g_{\mathrm{hom}}(\omega,\zeta,\nu) \geq \limsup_{k\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q^\nu_{r_k}(-kz))}{r_k^{n-1}}.$$

Arguing analogously with the external cubes $Q^\nu_{\overline{r}_k}(-i_k z)$ we get

$$g_{\mathrm{hom}}(\omega,\zeta,\nu) \leq \liminf_{k\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{-kz,\zeta,\nu}, Q^\nu_{r_k}(-kz))}{r_k^{n-1}},$$

obtaining the claim.

*Step 4:* Let $\Omega'$ be the set introduced in Step 3, then for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{Q}^N$ and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ there holds

$$\lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{rx,\zeta,\nu}, Q^\nu_r(rx))}{r^{n-1}} = g_{\mathrm{hom}}(\omega,\zeta,\nu)\,. \tag{5.6.16}$$

Fix $\omega, x, \zeta, \nu$ as required, $\eta \in (0, \frac{1}{2})$, $q \in \mathbb{Q}^n$ with $|x-q| < \eta$, and $h \in \mathbb{Z}$ such that $z := hq \in \mathbb{Z}^n$. Consider a sequence of real numbers $t_k \to +\infty$ as $k \to +\infty$ and let $s_k := \frac{t_k}{h}$. Fixing an integer $j > 2|z|+1$ and setting $r_k := \lfloor t_k + 2\eta t_k \rfloor + j$ we have that $Q^\nu_{t_k}(t_k x) \subset\subset Q^\nu_{r_k}(\lfloor s_k \rfloor z)$. Since $|(t_k x - \lfloor s_k \rfloor z)\cdot \nu| \leq |t_k x - t_k q| + |s_k z - \lfloor s_k \rfloor z| \leq t_k \eta + |z|$, for every $y \in Q^\nu_{r_k}(\lfloor s_k \rfloor z)$ such that $|(y - t_k x)\cdot\nu| \leq \frac{1}{2}$ we have that $|(y - \lfloor s_k \rfloor z)\cdot \nu| \leq t_k\eta + |z| + \frac{1}{2}$. In particular, for $k$ large enough depending only on $z$, we can apply Lemma 5.2.6, with $R = t_k\eta + |z| + \frac{1}{2}$, to obtain

$$\begin{aligned} &m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\lfloor s_k \rfloor z, \zeta, \nu}, Q^\nu_{r_k}(\lfloor s_k \rfloor z)) \leq m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x)) + \eta r_k^{n-1} \\ &+ \tilde{K}((2\eta r_k + j) r_k^{n-2} + (r_k \eta + |z| + \frac{1}{2})|\zeta| r_k^{n-2}), \end{aligned} \tag{5.6.17}$$

where we used $r_k \geq t_k$. From (5.6.17), dividing by $t_k^{n-1}$ and recalling that $r_k \geq t_k \geq s_k \geq \lfloor s_k \rfloor$, we obtain that

$$\frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\lfloor s_k \rfloor z, \zeta, \nu}, Q^\nu_{r_k}(\lfloor s_k \rfloor z))}{r_k^{n-1}} - \eta - \tilde{K}\Big((2\eta + \frac{j}{r_k}) + (\eta + \frac{|z|}{r_k} + \frac{1}{2r_k})|\zeta|\Big) \leq \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

Since $\omega \in \Omega'$ and $r_k \geq \lfloor s_k \rfloor$, we can apply (5.6.13), taking the $\liminf$ as $k \to \infty$ and letting $\eta \to 0$ we obtain

$$g_{\mathrm{hom}}(\omega, \zeta, \nu) \leq \liminf_{k \to +\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

Arguing analogously we obtain

$$g_{\mathrm{hom}}(\omega, \zeta, \nu) \geq \limsup_{k \to +\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{t_k x, \zeta, \nu}, Q^\nu_{t_k}(t_k x))}{t_k^{n-1}}.$$

deducing the claim, thenks to the generality of the sequence $(t_k)_{k \in \mathbb{N}}$.

*Step 5:* Let $\Omega'$ be the set introduced in Step 3, then for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1}$

$$g_{\mathrm{hom}}(\omega, \zeta, \nu) = \lim_{r \to +\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}.$$

For $\omega$, $x$, $\zeta, \nu$ as above define

$$\underline{g}(\omega, x, \zeta, \nu) := \liminf_{r \to +\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}, \qquad \overline{g}(\omega, x, \zeta, \nu) := \limsup_{r \to +\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{rx, \zeta, \nu}, Q^\nu_r(rx))}{r^{n-1}}.$$

Arguing exactly as in Proposition 5.2.7 (i) and in Step 2, we obtain from Step 4 that

$$\underline{g}(\omega, x, \zeta, \nu) = g_{\mathrm{hom}}(\omega, \zeta, \nu) = \overline{g}(\omega, x, \zeta, \nu) \tag{5.6.18}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$, and $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.

Now let $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^N$ and $\nu \in \hat{\mathbb{S}}^{n-1}_+$, by density there is $(\nu_j)_{j \in \mathbb{N}}$ in $\hat{\mathbb{S}}^{n-1}_+ \cap \mathbb{Q}^n$ such that $\nu_j \to \nu$ as $j \to +\infty$. Thanks to the continuity on $\hat{\mathbb{S}}^{n-1}_+$ of the map $\nu \mapsto R_\nu$, for every $\delta \in (0, \frac{1}{2})$ there exists $j_\delta$, such that

$$Q_r^{\nu}(rx) \subset\subset Q^{\nu_j}_{(1+\delta)r}(rx) \subset\subset Q^{\nu}_{(1+2\delta)r}(rx) \tag{5.6.19}$$

for every $j \geq j_\delta$ and every $r > 0$. Let us fix $j \geq j_\delta$, $r > 0$ and $\eta > 0$. Setting $c_j := \max\{|R_{\nu_j}(e_i) \cdot \nu| : i = 1, \dots, n-1\}$ we have that $c_j \to 0$ as $j \to +\infty$, by continuity of $\nu \mapsto R_\nu$ on $\hat{\mathbb{S}}^{n-1}_{\pm}$, and recalling that $R_\nu$ is a rotation with $R_\nu e_n = \nu$ (cf. Remark 2.0.1). For every $y \in \overline{Q^{\nu}_{r(1+\delta)}(rx)}$ we have that $y - rx = y' + ((y-rx)\cdot\nu_j)\nu_j$ where

$$y' \in R_{\nu_j}\Big(\big(-\frac{r}{2}(1+\delta), \frac{r}{2}(1+\delta)\big)^{n-1} \times \{0\}\Big),$$

with, if $j$ is large enough depending only on $\delta$,

$$|(y-rx)\cdot\nu_j| \leq \tfrac{|y'\cdot\nu|}{|\nu_j\cdot\nu|} + \tfrac{1}{2(1-\delta)} < \tfrac{(n-1)c_j r(1+\delta)}{2(1-\delta)} + 1 = K(\delta) r c_j + 1,$$

where $K(\delta) := \frac{(n-1)(1+\delta)}{2(1-\delta)}$, if in addition $|(y-rx)\cdot\nu| \leq \frac{1}{2}$. Therefore, we can apply Lemma 5.2.6, with $R = K(\delta)rc_j + 1$, and we get

$$\begin{aligned} m_{\rm s}^{f^\infty_\omega}(\overline{u}_{rx,\zeta,\nu_j}, Q^{\nu_j}_{r(1+\delta)}(rx)) &\leq m_{\rm s}^{f^\infty_\omega}(\overline{u}_{rx,\zeta,\nu}, Q^{\nu}_{r}(rx)) + \eta r^{n-1} \\ &\quad + \tilde{K}(2\delta(1+2\delta)^{n-2} r^{n-1} + (K(\delta)rc_j + 1)|\zeta|(1+\delta)^{n-2} r^{n-2}). \end{aligned}$$

Dividing by $r^{n-1}$ and letting $r \to +\infty$, we obtain

$$\begin{aligned} (1+\delta)^{n-1} \underline{g}\big(\omega, \frac{x}{1+\delta}, \zeta, \nu_j\big) &\leq \underline{g}(\omega, x, \zeta, \nu) \\ &\quad + \eta + \tilde{K}(2\delta(1+2\delta)^{n-2} + K(\delta)c_j|\zeta|(1+\delta)^{n-2}). \end{aligned}$$

Hence, we may use (5.6.18) as $\nu_j \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and deduce by taking the superior limit as $j \to +\infty$ and letting $\eta \to 0$ in the latter estimate

$$\begin{aligned} (1+\delta)^{n-1} \limsup_{j\to+\infty} g_{\rm hom}(\omega, \zeta, \nu_j) &= (1+\delta)^{n-1} \limsup_{j\to+\infty} \underline{g}\big(\omega, \frac{x}{1+\delta}, \zeta, \nu_j\big) \\ &\leq \underline{g}(\omega, x, \zeta, \nu) + 2\tilde{K}\delta(1+2\delta)^{n-2}. \end{aligned}$$

Therefore, by the continuity of $g_{\rm hom}$ established in Step 2, letting $\delta \to 0$ we obtain

$$g_{\rm hom}(\omega, \zeta, \nu) \leq \underline{g}(\omega, x, \zeta, \nu).$$

Arguing analogously we have $\overline{g}(\omega, x, \zeta, \nu) \leq g_{\rm hom}(\omega, \zeta, \nu)$, and recalling Corollary 5.2.5 we conclude.

*Step 5:* In this step we show that if $f$ is ergodic then $g_{\rm hom}$ is deterministic.

Set $\hat{\Omega} = \bigcap_{z\in\mathbb{Z}^n} \tau_z(\tilde{\Omega})$; we clearly have that $\hat{\Omega} \in \mathcal{M}$, $\hat{\Omega} \subseteq \tilde{\Omega}$ and $\tau_z(\hat{\Omega}) = \hat{\Omega}$ for every $z \in \mathbb{Z}^n$. Moreover, since $\tau_z$ is a $P$-preserving transformation and $P(\tilde{\Omega}) = 1$, we have that $P(\hat{\Omega}) = 1$. We claim that

$$g_{\rm hom}(\tau_z\omega, \zeta, \nu) \leq g_{\rm hom}(\omega, \zeta, \nu) \tag{5.6.20}$$

for every $\omega \in \hat{\Omega}$, every $\zeta \in \mathbb{R}^N$ and every $\nu \in \mathbb{S}^{n-1}$. Fix $z \in \mathbb{Z}^n$, $\omega \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$. For every $r > 3|z|$, let

$(u_r, v_r) \in W^{1,1}(Q_r^\nu, \mathbb{R}^N) \times W^{1,2}(Q_r^\nu, [0,1])$, with $(u_r, v_r) = (\overline{u}_{\zeta,\nu}, 1)$ on $\partial Q_r^\nu$ such that

$$\int_{Q_r^\nu} \left(v_r^2 f^\infty(\omega, y, \nabla u_r) + (1-v_r)^2 + |\nabla v_r|^2\right) \mathrm{d}y \leq m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) + 1. \tag{5.6.21}$$

By the stationarity of $f$ (and hence of $f^\infty$) we infer that

$$m_{\mathrm{s}}^{f_{\tau_z\omega}^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) = m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q_r^\nu(z)). \tag{5.6.22}$$

Observe that $Q_r^\nu \subset\subset Q_{r+3|z|}^\nu(z)$ for every $r > 3|z|$, and for every $y \in Q_{r+3|z|}^\nu(z)$ such that $|y \cdot \nu| \leq \frac{1}{2}$ we have that

$$1 \geq \frac{1}{2} + |\nu \cdot z| \geq |y \cdot \nu| = |(y-z)\cdot\nu + z\cdot\nu| + |\nu \cdot z| \geq |(y-z)\cdot\nu|.$$

Then we can apply Lemma 5.2.6, with $R = 1$, and for every $\eta > 0$ we obtain

$$\begin{aligned} m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q_{r+3|z|}^\nu(z)) &\leq m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu) + \eta r^{n-1} \\ &\quad + \tilde{K}(r+3|z|)^{n-2}(3|z| + |\zeta|). \end{aligned}$$

Therefore, by definition of $g_{\mathrm{hom}}$, $\hat{\Omega} \subseteq \tilde{\Omega}$, and (5.6.22) we obtain

$$\begin{aligned} g_{\mathrm{hom}}(\tau_z\omega, \zeta, \nu) &= \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f_{\tau_z(\omega)}^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}} = \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q_r^\nu(z))}{r^{n-1}} \\ &= \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{z,\zeta,\nu}, Q_{r+3|z|}^\nu(z))}{r^{n-1}} \leq \lim_{r\to+\infty} \frac{m_{\mathrm{s}}^{f_\omega^\infty}(\overline{u}_{\zeta,\nu}, Q_r^\nu)}{r^{n-1}} \leq g_{\mathrm{hom}}(\omega, \zeta, \nu) \end{aligned}$$

thus deducing the claim.

By (5.6.20) and the properties of $(\tau_z)_{z\in\mathbb{Z}^n}$, we clearly infer that

$$g_{\mathrm{hom}}(\tau_z\omega, \zeta, \nu) = g_{\mathrm{hom}}(\omega, \zeta, \nu)$$

and hence, using the same argument as in [CDMSZ19b, Corollary 6.3], if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic we deduce that $g_{\mathrm{hom}}$ does not depend on $\omega$ and thus is deterministic. To conclude, we just observe that the representation of $g_{\mathrm{hom}}(\zeta, \nu)$ as in (5.6.2) is a direct consequence of (5.6.12), and the Dominated Convergence theorem (cf. (5.2.17)). $\square$

Finally, we are in a position to prove the main result of the chapter, Theorem 5.1.7.

*Proof of Theorem 5.1.7.* The proof readily follows by combining Theorem 5.3.1, Proposition 5.6.1, 5.6.2, and 5.6.3. $\square$

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