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Simple non-rational convex polytopes via symplectic geometry

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Abstract

In this article we consider a generalization of manifolds and orbifolds which we call *quasifolds*; quasifolds of dimension k are locally isomorphic to the quotient of the space $\mathbb{R}^k$ by the action of a discrete group — typically they are not Hausdorff topological spaces. The analogue of a torus in this geometry is a *quasitorus*. We define Hamiltonian actions of quasitori on symplectic quasifolds and we show that *any* simple convex polytope, rational or not, is the image of the moment mapping for a family of effective Hamiltonian actions on symplectic quasifolds having twice the dimension of the corresponding quasitorus. © 2001 Elsevier Science Ltd. All rights reserved.

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Introduction

The convexity theorem of Atiyah [2] and Guillemin–Sternberg [7] says that if T is a torus acting in a Hamiltonian fashion on a compact, connected symplectic manifold M, then the image of the corresponding moment mapping is a *rational* convex polytope. One of the most interesting applications of this theorem is a classification theorem of Delzant [5], which states that if $\dim M = 2\dim T$ and the action is effective, then the space is completely characterized by the image of the moment mapping, which is a simple rational convex polytope satisfying a special integrality condition. One of the features of Delzant's result is that it provides an explicit construction for associating to each polytope the corresponding space; this construction involves the technique of symplectic reduction. The results of Atiyah, Guillemin–Sternberg and Delzant have subsequently been extended by Lerman–Tolman [11] to the case of torus actions on

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symplectic orbifolds; the image of the moment mapping in this case is still a rational polytope, and the extension of Delzant's theorem involves simple rational polytopes.

However, it is very natural to ask oneself whether a simple convex polytope that is *not* rational can also be viewed as the image of the moment mapping for a suitable symplectic space. Answering affirmatively to this question amounts to being able to perform symplectic reduction under rather general assumptions, thus allowing the resulting space to be pathological. This has lead us to consider a new class of spaces which we call *quasifolds*. Roughly speaking, a quasifold of dimension k is a space that is locally modeled on orbit spaces of discrete group actions on open subsets of the space $\mathbb{R}^k$. Manifolds and orbifolds are special cases of quasifolds, but quasifolds in general are not Hausdorff topological spaces. Just as for orbifolds, geometric objects on quasifolds may be thought of as collections of objects on the open sets of the space $\mathbb{R}^k$ that are invariant under the discrete group actions, and that behave correctly under coordinate changes. The natural analogue of a torus in this geometry is a *quasitorus*, which is the quotient of a vector space by a *quasilattice*. It is then possible to define Hamiltonian quasitorus actions on symplectic quasifolds and to extend the Delzant construction to show that every simple convex polytope Δ is the image of the moment mapping for quasitorus actions on a family, $\mathscr{M}_\Delta$, of quasifolds.

We remark that the initial motivation for this article came from a discussion with Traynor on the role of non-rational polytopes in the study of symplectic packings [12,15]. Orbit spaces of discrete group actions have been studied by Connes in the context of *noncommutative geometry* [4, Chapter II]; our approach is different and we do not fully understand the connection. Quasitori of dimension one have been studied by Donato, Iglesias and Lachaud [6,8,9] in the framework of the theory of *diffeological spaces*; on this occasion Iglesias introduced the terminology *irrational tori*. On the other hand Weinstein considered quasitori of dimension one to prequantize arbitrary symplectic manifolds [16,17]; he introduced the term *infracircles*. The subject of this article is also related to the *geometry of quasicrystals* [1,14]; for example the regular pentagon is not only a celebrated quasicrystal but is also a simple non-rational convex polytope. This is the reason underlying our choice of the terms quasifold and quasitorus; the term quasilattice on the other hand had already been introduced by quasicrystallographers.

The article is structured as follows: in Section 1 we define quasifolds and the essentials of their geometry, in Section 2 we define quasitori and Hamiltonian actions, in Section 3 we prove a symplectic reduction theorem and the extension of Delzant's construction to this setting. A brief appendix recalls the definitions of rational and simple convex polyhedral sets. All definitions and results are illustrated by examples.

The contents of this article have been announced in [13]. In the sequel we will give a more thorough treatment of the convexity theorem and of the failure of the uniqueness part of Delzant's theorem. In an article in collaboration with Battaglia [3] we introduce complex and Kähler structures on quasifolds, and see how the spaces in the family $\mathscr{M}_\Delta$ can be viewed as natural generalizations of the toric varieties that are usually associated to those simple convex polytopes that are rational.

1. Quasifolds

We begin by introducing the local model for quasifolds.

Definition 1.1 (Model). Let $\tilde{U}$ be a connected, simply connected manifold of dimension k and let Γ be a discrete group acting smoothly on the manifold $\tilde{U}$ so that the set of points, $\tilde{U}_0$, where the action is free, is connected and dense. Consider the space of orbits, $\tilde{U}/\Gamma$, of the action of the group Γ on the manifold $\tilde{U}$, endowed with the quotient topology, and the canonical projection $p : \tilde{U} \to \tilde{U}/\Gamma$. A *model* of dimension k is the triple $(\tilde{U}/\Gamma, p, \tilde{U})$, shortly $\tilde{U}/\Gamma$.

Remark 1.2. We remark that the assumption in Definition 1.1 that the manifold $\tilde{U}$ be simply connected could be omitted, at the expense of the definitions of smooth mapping, diffeomorphism, vector field and form, which would then become more complicated. This assumption happens to be very natural in our setting and, in practice, is not as strong as one may think. Assume in fact that the manifold $\tilde{U}$ is connected, but not simply connected; consider its universal cover, $\pi : U^{\#} \to \tilde{U}$, and its fundamental group, Π. The manifold $U^{\#}$ is connected and simply connected, the mapping π is smooth, the discrete group Π acts smoothly, freely and properly on the manifold $\tilde{U}$ and $\tilde{U} = U^{\#}/\Pi$. Consider the extension of the group Γ by the group Π, $1 \to \Pi \to \Lambda \to \Gamma \to 1$, defined as follows:

$$\Lambda = \{\lambda \in \mathrm{Diff}(U^{\#}) \mid \exists \gamma \in \Gamma \text{ s.t. } \pi(\lambda(u^{\#})) = \gamma \cdot \pi(u^{\#}) \ \forall u^{\#} \in U^{\#}\}.$$

It is easy to verify that Λ is a discrete group, that it acts on the manifold $U^{\#}$ according to the assumptions of Definition 1.1 and that $\tilde{U}/\Gamma = U^{\#}/\Lambda$.

Definition 1.3 (Tangent space). Consider a model $(\tilde{U}/\Gamma, p, \tilde{U})$. For any point $\tilde{u}$ in $\tilde{U}$ the group $\Gamma_{\tilde{u}} = \mathrm{Stab}(\tilde{u}, \Gamma)$ acts on the vector space $T_{\tilde{u}}\tilde{U}$. We define the *tangent space* of the model $\tilde{U}/\Gamma$ at the point $u = p(\tilde{u})$, denoted $T_u(\tilde{U}/\Gamma)$, to be the space of orbits $(T_{\tilde{u}}\tilde{U})/\Gamma_{\tilde{u}}$.

Remark 1.4. We remark that $T_u(\tilde{U}/\Gamma)$ itself defines a model and that it is a true vector space for all points u in $p(\tilde{U}_0)$.

Definition 1.5 (Smooth mapping, diffeomorphism of models). A *smooth mapping* of the models $(\tilde{U}/\Gamma, p, \tilde{U})$ and $(\tilde{V}/\Delta, q, \tilde{V})$ is a mapping $f : \tilde{U}/\Gamma \to \tilde{V}/\Delta$ having the property that there exists a smooth mapping $\tilde{f} : \tilde{U} \to \tilde{V}$ such that $q \circ \tilde{f} = f \circ p$; we will say that $\tilde{f}$ is a *lift* of f. We will say that the smooth mapping f is a *diffeomorphism of models* if it is bijective and if the lift $\tilde{f}$ is a diffeomorphism.

If the mapping $\tilde{f}$ is a lift of a smooth mapping of models $f : \tilde{U}/\Gamma \to \tilde{V}/\Delta$ so are the mappings $\tilde{f}^{\gamma}(-) = \tilde{f}(\gamma \cdot -)$, for all elements γ in Γ and ${}^{\delta}\tilde{f}(-) = \delta \cdot \tilde{f}(-)$, for all elements δ in Δ. We are about to show that if the mapping f is a diffeomorphism, then these are the only other possible lifts.

Lemma 1.6 (The orange lemma). *Consider two models, $\tilde{U}/\Gamma$ and $\tilde{V}/\Delta$, and let $f : \tilde{U}/\Gamma \to \tilde{V}/\Delta$ be a diffeomorphism of models. For any two lifts, $\tilde{f}$ and $\bar{f}$, of the diffeomorphism f there exists a unique element δ in Δ such that $\bar{f} = {}^{\delta}\tilde{f}$.*

Proof. Let $\tilde{V}_0$ be the connected and dense set of points in the manifold $\tilde{V}$ where the action of the group Δ is free, and consider a point $\tilde{v}$ in $\tilde{V}_0$, and the corresponding point $\tilde{u} = \tilde{f}^{-1}(\tilde{v})$. Then there is a unique element $\delta(\tilde{v})$ in Δ such that $\bar{f}(\tilde{u}) = \delta(\tilde{v}) \cdot \tilde{f}(\tilde{u})$. Since the group Δ is discrete, and the set $\tilde{V}_0$ is connected and dense, there exists a unique element δ in Δ such that $\bar{f} = {}^{\delta}\tilde{f}$. □

Lemma 1.7 (The green lemma). *Consider two models,* $\tilde{U}/\Gamma$ and $\tilde{V}/\Delta$, *and a diffeomorphism* $f: \tilde{U}/\Gamma \to \tilde{V}/\Delta$. *Then, for a given lift,* $\tilde{f}$, *of the diffeomorphism f, there exists a group isomorphism* $F: \Gamma \to \Delta$ *such that* $\tilde{f}^{\gamma} = {}^{F(\gamma)}\tilde{f}$, *for all elements* γ *in* Γ.

Proof. Take an element γ in Γ. Apply the orange lemma to the lifts $\tilde{f}, \bar{f} = \tilde{f}^{\gamma}$, and define $F(\gamma) = \delta$. Repeat for all elements γ in Γ and check that F is an isomorphism with the required property. □

Definition 1.8 (Vector field, *h*-form on a model). A *vector field*, X (respectively *h-form*, ω,) on a model $\tilde{U}/\Gamma$ is the assignment of a Γ-invariant vector field, $\tilde{X}$ (respectively *h*-form, $\tilde{\omega}$,) on the manifold $\tilde{U}$.

Definition 1.9 (Pushforward of a vector field). Consider two models, $\tilde{U}/\Gamma$ and $\tilde{V}/\Delta$, and a diffeomorphism $f: \tilde{U}/\Gamma \to \tilde{V}/\Delta$. Let X be a smooth vector field on the model $\tilde{U}/\Gamma$; we define the *pushforward* of X via *f*, denoted $f_* X$, to be the vector field on the model $\tilde{V}/\Delta$ that corresponds to the assignment of the Δ-invariant vector field $\tilde{f}_* \tilde{X}$, for any lift $\tilde{f}$ of the diffeomorphism *f*.

The notions of differential and pullback of a form, and the notion of interior product of a form with a vector field are defined in an analogous way.

Definition 1.10 (Symplectic form on a model). A *symplectic form*, ω, on a model $\tilde{U}/\Gamma$ is the assignment of a Γ-invariant symplectic form, $\tilde{\omega}$, on the manifold $\tilde{U}$.

We are now ready to define quasifolds.

Definition 1.11 (Quasifold). A dimension *k* *quasifold structure* on a topological space *M* is the assignment of an *atlas*, or collection of *charts*, $\mathscr{A} = \{(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha) \mid \alpha \in A\}$ having the following properties:

1. The collection $\{U_\alpha \mid \alpha \in A\}$ is a cover of *M*.
2. For each index α in $\mathscr{A}$, the set U_α is open, the space $\tilde{U}_\alpha/\Gamma_\alpha$ defines a model, where the set $\tilde{U}_\alpha$ is an open, connected and simply connected subset of the space $\mathbb{R}^k$, and the mapping ϕ_α is a homeomorphism of the space $\tilde{U}_\alpha/\Gamma_\alpha$ onto the set U_α.
3. For all indices α, β in A such that $U_\alpha \cap U_\beta \neq \emptyset$, the sets $\phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\phi_\beta^{-1}(U_\alpha \cap U_\beta)$ define models and the mapping
$$g_{\alpha\beta} = \phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \phi_\beta^{-1}(U_\alpha \cap U_\beta)$$
is a diffeomorphism. We will then say that the mapping $g_{\alpha\beta}$ is a *change of charts* and that the corresponding charts are *compatible*.
4. The atlas $\mathscr{A}$ is *maximal*, that is: if the triple $(U, \phi, \tilde{U}/\Gamma)$ satisfies property 2, and is compatible with all the charts in $\mathscr{A}$, then $(U, \phi, \tilde{U}/\Gamma)$ belongs to $\mathscr{A}$.

We will say that a space *M* with a quasifold structure is a *quasifold*.

Remark 1.12. A quasifold where all the groups Γ_α are trivial is a manifold, one where all the groups Γ_α are finite is an orbifold.

Example 1.13 (The quasisphere). Let s, t be two positive real numbers such that $s/t \notin \mathbb{Q}$. Consider the space $\mathbb{C}^2$ with the standard symplectic form $\omega_0 = (1/2\pi i)(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ and with the $\mathbb{R}$-action: $(\theta,(z_1, z_2)) = (e^{2\pi i\theta}z_1, e^{2\pi i\theta s/t}z_2)$ of moment mapping

$$\begin{aligned} \Psi\colon \mathbb{C}^2 &\to \mathbb{R} \\ (z_1, z_2) &\mapsto |z_1|^2 + \tfrac{s}{t}|z_2|^2 - s. \end{aligned}$$

Consider the level set $\Psi^{-1}(0)$; this space is an ellipsoid of dimension 3 with center the origin and radii $(\sqrt{s},\sqrt{t})$. Consider now the space of orbits $M = \Psi^{-1}(0)/\mathbb{R}$. We want to show that it is a quasifold of dimension 2. We cover it with two open sets, $U_S = \{[z_1{:}z_2] \in M \mid z_2 \neq 0\}$ and $U_N = \{[z_1{:}z_2] \in M \mid z_1 \neq 0\}$. Denote by $B(r)$, for any $r > 0$, the open ball in the space $\mathbb{C}$ of center the origin and radius $\sqrt{r}$. Then the discrete group $\Gamma_S = \mathbb{Z}$ acts on the open set $\tilde{U}_S = B(s)$ by the rule $(k, z) \mapsto e^{2\pi i k t/s}z$; this action is free on the connected, dense subset $\tilde{U}_S - \{0\}$ and the mapping

$$\begin{aligned} \phi_S : \tilde{U}_S/\Gamma_S &\to U_S \\ [z] &\mapsto \left[z{:}\sqrt{t - \tfrac{t}{s}|z^2|}\right] \end{aligned}$$

is a homeomorphism. Similarly, the group $\Gamma_N = \mathbb{Z}$ acts on the open set $\tilde{U}_N = B(t)$ by the rule $(m, w) \mapsto e^{2\pi i m s/t}w$; this action is free on the connected, dense subset $\tilde{U}_N - \{0\}$ and the mapping

$$\begin{aligned} \phi_N : \tilde{U}_N/\Gamma_N &\to U_N \\ [w] &\mapsto \left[\sqrt{s - \tfrac{s}{t}|w|^2}{:}w\right] \end{aligned}$$

is a homeomorphism. Let us check that these two charts are compatible. The set $\phi_S^{-1}(U_S \cap U_N)$ defines a model: it is the quotient of $\mathbb{R} \times (0, \sqrt{s})$ by the following action of $\mathbb{Z}^2$: $((h, k),(\sigma, \rho)) \mapsto (\sigma + h + kt/s, \rho)$. Similarly, the set $\phi_N^{-1}(U_S \cap U_N)$ is the quotient of $\mathbb{R} \times (0, \sqrt{t})$ by the following action of $\mathbb{Z}^2$: $((l, m),(\tau, \upsilon)) \mapsto (\tau + l + ms/t, \upsilon)$. Remark that

$$\begin{aligned} g_{SN} = \phi_N^{-1} \circ \phi_S : \phi_S^{-1}(U_S \cap U_N) &\to \phi_N^{-1}(U_S \cap U_N) \\ [z = e^{2\pi i\sigma}\rho] &\mapsto \left[w = e^{-2\pi i\sigma s/t}\sqrt{t - \tfrac{t}{s}\rho^2}\right] \end{aligned}$$

is a diffeomorphism of models: its lift is given by $(\sigma,\rho) \mapsto (-\sigma s/t, \sqrt{t - (t/s)\rho^2})$. Now complete this collection with all other compatible charts.

We now proceed to give quasifolds all the necessary geometrical structure.

Definition 1.14 (Smooth mapping, diffeomorphism of quasifolds). Let M and N be two quasifolds. A continuous mapping $f: M \to N$ is said to be a *smooth mapping of quasifolds* if there exists a chart $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ around each point m in the space M, a chart $(V_\alpha, \psi_\alpha, \tilde{V}_\alpha/\Delta_a)$ around the point $f(m)$, and a smooth mapping of models $f_\alpha : \tilde{U}_\alpha/\Gamma_\alpha \to \tilde{V}_\alpha/\Delta_\alpha$ such that $\psi_\alpha \circ f_\alpha = f \circ \phi_\alpha$. If the smooth mapping f is bijective, and if its inverse is smooth, we will say that it is a *diffeomorphism of quasifolds*.

Let us say a word about the definition of smooth mapping. Consider Definition 1.14 and denote by $\tilde{f}_\alpha$ a lift of the smooth mapping of models f_α, by p_α the canonical projection $\tilde{U}_\alpha \to \tilde{U}_\alpha/\Gamma_\alpha$, and by q_α the canonical projection $\tilde{V}_\alpha \to \tilde{V}_\alpha/\Delta_\alpha$. Then, by combining Definitions 1.5 and 1.14, we get that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{U}_\alpha & \xrightarrow{\tilde{f}_\alpha} & \tilde{V}_\alpha \\
{\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle q_\alpha} \\
\tilde{U}_\alpha/\Gamma_\alpha & \xrightarrow{f_\alpha} & \tilde{V}_\alpha/\Delta_\alpha \\
{\scriptstyle \phi_\alpha}\downarrow & & \downarrow{\scriptstyle \psi_\alpha} \\
U_\alpha & \xrightarrow{f} & V_\alpha
\end{array}
$$

Let us look at the special case $N = V$, a vector space (this includes all moment maps; see Definition 2.8). The space V is a smooth quasifold of one chart so a mapping $f: M \to V$ is smooth if, and only if, there exists a chart $\phi_\alpha : \tilde{U}_\alpha/\Gamma_\alpha \to U_\alpha$ around each point m in the space M, such that the mapping $\tilde{f}_\alpha = f \circ \phi_\alpha \circ p_\alpha : \tilde{U}_\alpha \to V$ is smooth (here p_α still denotes the canonical projection $\tilde{U}_\alpha \to \tilde{U}_\alpha/\Gamma_\alpha$).

Definition 1.15 (Vector field, h-form on a quasifold). A *vector field*, X, (respectively *h-form*, ω), on a quasifold M is the assignment of a chart $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ around each point m in the space M and of a vector field, X_α, (respectively h-form, ω_α,) on the model $\tilde{U}_\alpha/\Gamma_\alpha$. We require that whenever we have two such charts, $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ and $(U_\beta, \phi_\beta, \tilde{U}_\beta/\Gamma_\beta)$, with the property that $U_\alpha \cap U_\beta \neq \emptyset$, then $(g_{\alpha\beta})_* X_\alpha = X_\beta$ (respectively $(g_{\alpha\beta})^* \omega_\beta = \omega_\alpha$) for the corresponding change of charts $g_{\alpha\beta}$.

Definition 1.16 (Pushforward of a vector field). Let M and N be two quasifolds, let X be a vector field on the quasifold M, and let $f: M \to N$ be a diffeomorphism; then there exists a chart $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ around any given point m in the space M, a chart $(V_\alpha, \psi_\alpha, \tilde{V}_\alpha/\Delta_\alpha)$ around the point $n = f(m)$, a vector field X_α on the model $\tilde{U}_\alpha/\Gamma_\alpha$, and a smooth mapping $f_\alpha : U_\alpha \to V_\alpha$ such that $\psi_\alpha \circ f_\alpha = f \circ \phi_\alpha$. We define the *pushforward* of X via f, denoted $f_* X$, to be the vector field on the quasifold N given by the assignment of the chart $(V_\alpha, \psi_\alpha, \tilde{V}_\alpha/\Delta_\alpha)$ around the point n and of the vector field $f_{\alpha *} X_\alpha$ on the model $\tilde{V}_\alpha/\Delta_\alpha$.

Completely analogous definitions hold for the notions of differential and pullback of a form, and for the notion of interior product of a form with a vector field.

Definition 1.17 (Symplectic form-structure-quasifold, symplectomorphism). A *symplectic form* on a quasifold M is a 2-form, ω, such that each form ω_α (see Definition 1.15) is symplectic. A *symplectic structure* on a quasifold M is the assignment of a symplectic form ω, and we will say that (M, ω), or shortly M, is a *symplectic quasifold*. A *symplectomorphism* between two symplectic quasifolds (M, ω) and (N, σ) is a diffeomorphism $f: M \to N$ such that $f^* \sigma = \omega$.

Example 1.18 (Quasilinear model). Let V be a symplectic vector space with a linear, effective and symplectic action of a torus T. Take any discrete subgroup $\Gamma \subset T$, and consider its induced action on the space V. The group Γ acts freely on a connected, dense subset of the space V, thus the space of orbits $V_\Gamma = V/\Gamma$ is a symplectic quasifold of dimension $2l = \dim V$.

The quasisphere in Example 1.13 can also be endowed with a symplectic structure.

Example 1.19 (Quasisphere). Consider the quasisphere of Example 1.13 and define a symplectic form by assigning the Γ_S-invariant symplectic form $\tilde{\omega}_S = (1/2\pi i)\, dz \wedge d\bar{z}$ to the set $\tilde{U}_S$ and the Γ_N-invariant symplectic form $\tilde{\omega}_N = (1/2\pi i)\, dw \wedge d\bar{w}$ to the set $\tilde{U}_N$.

2. Quasitori and their actions on quasifolds

We devote this section to quasitori and their Hamiltonian actions on symplectic quasifolds. We start with a number of definitions and properties and we end with some crucial examples.

Definition 2.1 (Quasilattice, quasitorus). Let $\mathfrak{d}$ be a vector space of dimension n. A *quasilattice* in $\mathfrak{d}$ is the $\mathbb{Z}$-span, Q, of a set of $\mathbb{R}$-spanning vectors $X_1, \ldots, X_d$ in $\mathfrak{d}$. We call *quasitorus* of dimension n the group and quasifold of one chart $D = \mathfrak{d}/Q$.

Notice that in the previous definition $d \geqslant n$ and that if $d = n$, then the quasilattice Q is a lattice and the quasitorus D is a honest torus. A quasitorus is compact, connected and abelian, and the group operations of multiplication and inversion are smooth quasifold mappings.

Example 2.2 (A quasicircle). The first example of a (non-smooth) quasitorus is the quasitorus of dimension 1 (*quasicircle*) $D^1 = \mathbb{R}/Q$, where $Q = s\mathbb{Z} + t\mathbb{Z}$, $s/t \notin \mathbb{Q}$. To discover everything about this innocuous-looking group we refer the reader to Donato and Iglesias [6], Iglesias [8] and Iglesias and Lachaud [9].

The quasifold tangent space at the identity of a quasitorus $D = \mathfrak{d}/Q$ is always the vector space $\mathfrak{d}$. By analogy with the smooth case we make the following

Definition 2.3 (Quasi-Lie algebra, exponential mapping). Let $D = \mathfrak{d}/Q$ be a quasitorus. We define the *quasi-Lie algebra* of D to be the vector space $\mathfrak{d}$. The natural projection of $\mathfrak{d}$ onto D is called *exponential mapping*, denoted $\exp_D$, or simply exp.

Definition 2.4 (Quasitorus homomorphism, isomorphism and epimorphism). A group homomorphism (respectively epimorphism and isomorphism) between quasitori that is a smooth quasifold map is called *quasitorus homomorphism* (respectively *epimorphism* and *isomorphism*).

Given two quasitori, $D_1 = \mathfrak{d}_1/Q_1$ and $D_2 = \mathfrak{d}_2/Q_2$, and a quasitorus homomorphism $f: D_1 \to D_2$, it is easy to check that the unique lift, $\tilde{f}$, of the homomorphism f satisfying $\tilde{f}(0) = 0$ is a linear mapping $\tilde{f}: (\mathfrak{d}_1, Q_1) \to (\mathfrak{d}_2, Q_2)$, and is an epimorphism, respectively isomorphism, whenever the homomorphism f is. Again by analogy with honest tori, we will call this lift the quasi-Lie algebra homomorphism associated to the quasitorus homomorphism f. The following proposition explains why we are interested in quasitori.

Proposition 2.5. *Let T be a torus and N a Lie subgroup.*[1] *Then T/N is a quasitorus of dimension* $n = \dim T - \dim N$.

Proof. Choose a complement, $\mathfrak{d}$, of the vector subspace $\mathfrak{n} = \mathrm{Lie}(N)$ in the vector space $\mathfrak{t} = \mathrm{Lie}(T)$; consider the surjective mapping $p_{\mathfrak{d}} = \Pi \circ \exp_T|_{\mathfrak{d}} : \mathfrak{d} \to T/N$ where $\Pi : T \to T/N$ denotes the canonical projection. Then the set $Q = \ker p_{\mathfrak{d}}$ is a quasilattice (a lattice if the group N is compact) and the mapping $p_{\mathfrak{d}}$ induces a group isomorphism $\mathfrak{d}/Q \simeq T/N$. Notice that two different choices of a complement $\mathfrak{d}$ yield isomorphic quasitori; the group T/N thus inherits a well-defined structure of quasitorus. □

We remark that the subspace $\mathfrak{d}$ of the preceding proof is the quasi-Lie algebra of the quasitorus $D \simeq T/N$ and that $p_{\mathfrak{d}} = \exp_D$. One important special case is the quotient of a torus T by any of its discrete subgroups, Γ. In this case we have $T/\Gamma = \mathfrak{d}/Q$, where $\mathfrak{d} \simeq \mathfrak{t}$. Another example is the quotient of a two-dimensional torus by an immersed line of slope $s/t \notin \mathbb{Q}$ (*Kronecker foliation*); the corresponding quasitorus is the quasicircle of Example 2.2.

Definition 2.6 (Smooth action). A *smooth action* of a quasitorus D on a quasifold M is a smooth mapping $\tau : D \times M \to M$ such that $\tau(d_1 \cdot d_2, m) = \tau(d_1, \tau(d_2, m))$ and $\tau(1_D, m) = m$ for all elements d_1, d_2 in the quasitorus D and for each point m in the space M.

According to this definition, there exist charts $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ and $(V_\alpha, \psi_\alpha, \tilde{V}_\alpha/\Delta_\alpha)$ around each point m in the space M, and smooth mappings $\tilde{\tau}_\alpha, \tau_\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc}
\delta \times \tilde{U}_\alpha & \xrightarrow{\tilde{\tau}_\alpha} & \tilde{V}_\alpha \\
\downarrow & & \downarrow \\
D \times (\tilde{U}_\alpha/\Gamma_\alpha) & \xrightarrow{\tau_\alpha} & \tilde{V}_\alpha/\Delta_\alpha \\
\downarrow & & \downarrow \\
D \times U_\alpha & \xrightarrow{\tau} & V_\alpha
\end{array}$$

Notice that, since $\tau(1_D, p) = p$ for each point p in the space M, we have that the set U_α is contained in the set V_α; it is therefore possible to assume that $\tilde{\tau}_\alpha(0, \tilde{u}) = \tilde{u}$ for each point $\tilde{u}$ in the set $\tilde{U}_\alpha$, and that the set $\tilde{U}_\alpha$ is contained in the set $\tilde{V}_\alpha$. Now fix an element X in the space $\mathfrak{d}$; then, for small enough real numbers t, the points $\tilde{\tau}_\alpha(tX, \tilde{u})$ belong to the set $\tilde{U}_\alpha$ whenever the point $\tilde{u}$ does. These data allow us to define the fundamental vector field of the smooth action τ.

Definition 2.7 (Fundamental vector field). Consider a smooth action, τ, of a quasitorus $D = \mathfrak{d}/Q$ on a quasifold M. For any element X in the space $\mathfrak{d}$ we define a vector field X_M on the space M, called *fundamental vector field of the action* corresponding to X, which is given by the assignment, for each point m in the space M, of the chart $(U_\alpha, \phi_\alpha, \tilde{U}_\alpha/\Gamma_\alpha)$ (see discussion above) and of the Γ_α-invariant

[1] We allow and actually prefer immersed subgroups.

vector field on the set $\tilde{U}_\alpha$ given by

$$\tilde{X}_M(\tilde{u}) = \left.\frac{d}{dt}\right|_0 \tilde{\tau}_\alpha(tX,\tilde{u}), \quad \tilde{u} \in \tilde{U}_\alpha.$$

Notice that, for a fixed element d in the quasitorus D, the mapping $\tau_d(\,-\,) = \tau(d,\,-\,)$ is a diffeomorphism of the quasifold M.

Definition 2.8 (Hamiltonian action, moment mapping). A smooth action, τ, of a quasitorus $D = \mathfrak{d}/Q$ on a symplectic quasifold (M,ω) is *Hamiltonian* if it preserves the symplectic form ($\tau_d^\sharp\omega = \omega$ for all d in the quasitorus D) and if there exists a smooth D-invariant mapping $\Phi: M \to \mathfrak{d}^*$, which we call *moment mapping*, such that $\imath(X_M)\omega = d\langle\Phi, X\rangle$, for each element X in the space $\mathfrak{d}$.

Example 2.9 (The quasilinear model). Consider the quasilinear model V_Γ of Example 1.18. The linear, effective and symplectic action of the torus T on the space V is Hamiltonian and it can be described as follows. Write $T = \mathfrak{t}/L$, where $\mathfrak{t}$ denotes the Lie algebra of the torus T and L is the lattice $\ker \exp_T$, and consider the corresponding weight lattice

$$L^* = \{\mu \in \mathfrak{t}^* \mid \mu(X) \in \mathbb{Z} \quad \forall X \in L\}.$$

The space V decomposes into l complex one-dimensional T-invariant subspaces V_j and there exist weights α_j in the lattice L^*, $j = 1,\ldots,l$, such that the action is given by

$$\begin{array}{rccccl} \hat{\tau}: & T & \times & V & \to & V \\ & (\exp_T(X) & , & v) & \mapsto & (e^{2\pi i\alpha_1(X)}v_1,\ldots,e^{2\pi i\alpha_l(X)}v_l), \end{array}$$

and the moment mapping is given by

$$\begin{array}{rcl} \hat{\Phi}: V & \to & \mathfrak{t}^* \\ v & \mapsto & \sum_{j=1}^{l} |v_j|^2\alpha_j. \end{array}$$

The image of $\hat{\Phi}$ is the rational convex polyhedral cone $\hat{\mathscr{C}}$ of vertex O and spanned by the weights α_j. Denote by p the projection $V \to V_\Gamma$, by D the quasitorus $\mathfrak{d}/Q \simeq T/\Gamma$, by Π the projection $T \to D$, and by π: $(\mathfrak{t}, L) \to (\mathfrak{d}, Q)$ the corresponding quasi-Lie algebra isomorphism. The action of the torus T on the vector space V induces an action, τ, of the quasitorus D on the space V_Γ as follows:

$$\begin{array}{ccc} T \times V & \xrightarrow{\hat{\tau}} & V \\ {\scriptstyle \Pi \times p}\downarrow & & \downarrow{\scriptstyle p} \\ D \times V_\Gamma & \xrightarrow{\tau} & V_\Gamma \end{array}$$

This action is Hamiltonian and the corresponding moment mapping Φ is given by

$$\begin{array}{ccc} V & \xrightarrow{\hat{\Phi}} & t^* \\ {\scriptstyle p}\downarrow & & \uparrow{\scriptstyle \pi^*} \\ V_\Gamma & \xrightarrow{\Phi} & \delta^* \end{array}$$

Notice that the image of the mapping Φ is the convex polyhedral cone $\mathscr{C} = (\pi^*)^{-1}(\hat{\mathscr{C}})$, which is spanned by the elements $\beta_j = (\pi^*)^{-1}(\alpha_j)$ in the space $\mathfrak{d}^*$.

Example 2.10 (The quasisphere). Let us go back to the quasisphere M of Examples 1.13 and 1.19. Consider now the quasilattice $Q = s\mathbb{Z} + t\mathbb{Z}$ and the quasicircle $D^1 = \mathbb{R}/Q$. The mapping

$$\begin{array}{llll} \tau: & D^1 \times M & \to & M \\ & ([\theta],[z{:}w]) & \mapsto & \left[\mathrm{e}^{2\pi \mathrm{i}\theta/s} z{:}w\right] \end{array}$$

defines a Hamiltonian action of the quasicircle D^1 (a quasirotation) on the quasifold M, with moment mapping

$$\begin{array}{llll} \Phi: & M & \to & \mathbb{R}^* \\ & [z{:}w] & \mapsto & \dfrac{|z|^2}{s} = 1 - \dfrac{|w|^2}{t}. \end{array}$$

Notice finally that $\Phi(M) = [0,1]$ just like for truly rotating spheres, teardrops, or rugby balls.

We conclude with an example of a honest torus acting on a quasifold. This example has a different flavor than all the others that we treat.

Example 2.11 (The horocycle foliation). Let us consider the upper half-plane $\mathscr{H} = \{(x, y) \in \mathbb{R}^2 | \text{s.t.} \quad y > 0\}$ with the standard symplectic form $dx \wedge dy$. We let the group $\mathbb{Z}$ act on the space $\mathscr{H}$ as follows: $(k,(x, y)) \mapsto (x + ky, y)$. This action is free and symplectic. We now consider the following free and Hamiltonian S^1-action on the quotient space $\mathscr{H}/\mathbb{Z} : (\mathrm{e}^{2\pi \mathrm{i}\theta},[x : y]) \mapsto [x + \theta y : y]$; the moment mapping is given by $[x : y] \mapsto \frac{1}{2}y^2$.

3. From simple polytopes to symplectic quasifolds

The aim of this section is to extend Delzant's construction and to show that any simple convex polytope is the image of the moment mapping for a family of effective Hamiltonian quasitorus actions on symplectic quasifolds of the appropriate dimension. This is a consequence of the following symplectic reduction theorem.

Theorem 3.1. *Let T be a torus of Lie algebra $\mathfrak{t}$, let $T \times X \to X$ be a Hamiltonian action of the torus T on a symplectic manifold X and assume that the moment mapping $J : X \to \mathfrak{t}^*$ is proper. Consider the induced action of any Lie subgroup N of T and suppose that 0 is a regular value of the corresponding moment mapping $\Psi : X \to \mathfrak{n}^*$ ($\mathfrak{n}$ denotes the Lie algebra of N). Then $M = \Psi^{-1}(0)/N$ is a symplectic quasifold of dimension $\dim X - 2\dim N$ and the induced (T/N)-action on the quasifold M is Hamiltonian.*

Proof. The slice theorem (see [10]) applied to the T-action on the manifold $\Psi^{-1}(0)$ gives invariant neighborhoods of the orbits $T \cdot x$ that are of the form $T \times_{T_x} B_x$, where $T_x = \mathrm{Stab}(x, T)$, and B_x is an open ball in the space $T_x(\psi^{-1}(0))/T_x(T \cdot x)$. The quotient $(T \times_{T_x} B_x)/N$ is a (T/N)-invariant neighborhood of the orbit $(T/N) \cdot [x]$ in the space M. Let us check that this neighborhood is a quasifold chart; the argument is quite similar to the one in the proof of Proposition 2.5. Denote by $\mathfrak{t}_x$ the Lie algebra of the group T_x. Since the value 0 is regular for the mapping Ψ, we have that $\mathfrak{t}_x \cap \mathfrak{n} = \{0\}$; choose a complement $\mathfrak{d}_x$ of the vector subspace $\mathfrak{t}_x \oplus \mathfrak{n}$ in the space $\mathfrak{t}$. Denote by Π_x the projection $T \times_{T_x} B_x \to (T \times_{T_x} B_x)/N$ and define a surjective mapping $p_x : \mathfrak{d}_x \times B_x \to (T \times_{T_x} B_x)/N$ according to the following rule: $p_x(Y,b) = \Pi_x([\exp_T Y{:}b])$, $(Y,b) \in \mathfrak{d}_x \times B_x$. Now consider the quasilattice Q of the proof of Proposition 2.5 chosen relatively to the complement $\mathfrak{d} = \mathfrak{d}_x \oplus \mathfrak{t}_x$ of the subspace $\mathfrak{n}$ in the space $\mathfrak{t}$. It is easy to check that the discrete group

$$\Lambda_x = \{(Y_Q, \exp_T T_Q) \in \mathfrak{d}_x \times T_x \,|\, Y_Q + T_Q \in Q\}$$

acts on the connected, simply connected open set $\mathfrak{d}_x \times B_x$ as follows:

$$\begin{array}{ccccc} \Lambda_x & \times & (\mathfrak{d}_x \times B_x) & \to & \mathfrak{d}_x \times B_x \\ ((Y_Q, \exp_T T_Q) & , & (Y,b)) & \mapsto & (Y + Y_Q, \exp_T T_Q \cdot b), \end{array}$$

and that the mapping p_x induces a homeomorphism $(\mathfrak{d}_x \times B_x)/\Lambda_x \simeq (T \times_{T_x} B_x)/N$. The remainder of the proof proceeds like the proof of the classical symplectic reduction theorem. The symplectic form on the manifold X induces a Λ_x-invariant symplectic form on the open set $\mathfrak{d}_x \times B_x$, thus a symplectic form on each chart $(T \times_{T_x} B_x)/N$; similarly the action of the torus T on the manifold X induces a Hamiltonian action of the quasitorus T/N on each chart, the corresponding moment mapping being induced by the one for the T-action on the manifold X. The required compatibility properties are satisfied. □

Remark 3.2 (Quasi-universal covers). We like to think of the manifolds $U^{\#}$ in Remark 1.2, $\mathfrak{d}$ in the proof of Proposition 2.5, and $\mathfrak{d}_x \times B_x$ in the proof of Theorem 3.1, as the *quasi-universal covers* of the quasifolds $\tilde{U}/\Gamma$, T/N and $(T \times_{T_x} B_x)/N$, respectively; the discrete groups Λ, Q and Λ_x would then be the corresponding fundamental groups. If the group Γ were finite and the group N were compact this would be in agreement with Thurston's notion of orbifold universal cover.

Let us now apply Theorem 3.1 to extend Delzant's construction. Let $\mathfrak{d}$ be a vector space of dimension n. The key idea is the observation that any simple convex polytope in the dual space $\mathfrak{d}^*$ can be obtained by slicing a translate of the positive orthant of the space $(\mathbb{R}^d)^*$ with an appropriate subspace.

Theorem 3.3. *Let $\mathfrak{d}$ be a vector space of dimension n. For any simple convex polytope $\Delta \subset \mathfrak{d}^*$ there exists an n-dimensional quasitorus D of quasi-Lie algebra $\mathfrak{d}$, a $2n$-dimensional compact symplectic quasifold M, and an effective Hamiltonian action of the quasitorus D on the quasifold M such that the image of the corresponding moment mapping is the polytope Δ.*

Proof. Consider the space $\mathbb{C}^d$ endowed with the standard symplectic form $\omega_0 = (1/2\pi \mathrm{i})\sum_{j=1}^d \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ and the standard action of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$\begin{array}{llcccl} \tau\colon & T^d & \times & \mathbb{C}^d & \to & \mathbb{C}^d \\ & ((\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_d}) & , & \underline{z}) & \mapsto & (\mathrm{e}^{2\pi \mathrm{i}\theta_1} z_1, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_d} z_d). \end{array}$$

This action is effective and Hamiltonian and its moment mapping is given by

$$\begin{array}{rccl} J\colon & \mathbb{C}^d & \to & (\mathbb{R}^d)^* \\ & \underline{z} & \mapsto & \sum_{j=1}^{d} |z_j|^2 e_j^* + \lambda, \quad \lambda \in (\mathbb{R}^d)^* \text{ constant.} \end{array}$$

The mapping J is proper and its image is the cone $\mathscr{C}_\lambda = \lambda + \mathscr{C}_0$, where $\mathscr{C}_0$ denotes the positive orthant in the space $(\mathbb{R}^d)^*$. Write the polytope Δ as in the appendix, formula (A.1) and consider the surjective linear mapping

$$\begin{array}{rccl} \pi\colon & \mathbb{R}^d & \to & \mathfrak{d}, \\ & e_j & \mapsto & X_j. \end{array}$$

Let Q be any quasilattice in the vector space $\mathfrak{d}$ containing the vectors $X_1, \dots, X_d$ (for example $Q = \sum_{j=1}^d X_j \mathbb{Z}$), and consider the dimension n quasitorus $D = \mathfrak{d}/Q$. Then the linear mapping π induces a quasitorus epimorphism $\Pi\colon T^d \to D$. Now define N to be the kernel of the mapping Π and choose $\lambda = \sum_{j=1}^d \lambda_j e_j^*$. Then, according to Theorem 3.1, the quasitorus T^d/N acts in a Hamiltonian fashion on the symplectic quasifold $M = \Psi^{-1}(0)/N$. Denote by i the Lie algebra inclusion $\mathrm{Lie}(N) \to \mathbb{R}^d$. If we identify the quasitori D and T^d/N using the epimorphism Π, we get a Hamiltonian action of the quasitorus D whose moment mapping has image equal to $(\pi^*)^{-1}(\mathscr{C}_\lambda \cap \ker i^*) = (\pi^*)^{-1}(\mathscr{C}_\lambda \cap \mathrm{im}\, \pi^*) = (\pi^*)^{-1}(\pi^*(\Delta)) = \Delta$. This action is effective since the level set $\Psi^{-1}(0)$ contains points of the form $\underline{z} \in \mathbb{C}^d$, $z_j \neq 0$, $j = 1, \dots, d$, where the T^d-action is free. Notice finally that $\dim M = 2d - 2\dim N = 2d - 2(d-n) = 2n = 2\dim D$. $\square$

Remark 3.4 (Uniqueness?). Notice that we had many choices in this construction. To begin with, the pairs (X_j, λ_j) in (A.1) are far from being unique; moreover there are infinitely many quasilattices that contain a fixed choice of the vectors X_j. As a consequence, the quasitorus, quasifold and action are far from being unique (see Example 3.5 below), but we will return to this matter in future work. For the moment we just point out that if the polytope Δ is rational relatively to a lattice L, by choosing the elements X_j to be in the lattice L, and the quasilattice Q to be equal to the lattice L itself, we distinguish among our spaces a family of orbifolds, in accordance with Lerman—[11]; if the polytope Δ also satisfies Delzant's integrality condition, by taking the elements X_j to be primitive in the lattice L, we obtain a manifold, in accordance with Delzant [5].

We conclude this section with three telling examples, where we apply the construction described in Theorem 3.3 to three different polytopes.

Example 3.5 (The unit interval). As a first example we consider the unit interval $[0,1] \subset \mathbb{R}^*$. We apply the construction with the choice of vectors $X_1 = s$, $X_2 = -t$, $s, t \in \mathbb{R}^*_+$, and with the corresponding quasilattice $Q = X_1\mathbb{Z} + X_2\mathbb{Z}$. We leave it as an exercise to show that if $s/t \notin \mathbb{Q}$ we obtain the quasisphere of Examples 1.13, 1.19 and 2.10, while in the remaining cases we get the standard sphere, and its orbifold cousins, the teardrop and rugby ball.

Example 3.6 (The right triangle). As a second example we consider the right triangle in $(\mathbb{R}^2)^*$ of vertices (0,0), $(s,0)$ and $(0,t)$, where s, t are two positive real numbers such that $s/t \notin \mathbb{Q}$. We apply the construction with the choice of vectors $X_1 = (1,0)$, $X_2 = (0,1)$, $X_3 = (-t, -s)$ and with the corresponding quasilattice $Q = X_1\mathbb{Z} + X_2\mathbb{Z} + X_3\mathbb{Z}$. Then we have $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -st$ and a linear mapping

$$\begin{aligned} \pi: \quad (\mathbb{R}^3, \mathbb{Z}^3) &\rightarrow (\mathbb{R}^2, Q) \\ (x, y, z) &\mapsto (x - tz, y - sz) \end{aligned}$$

that induces a quasitorus homomorphism $\Pi: T^3 \rightarrow D^2 = \mathbb{R}^2/Q$ whose kernel is given by

$$N = \{(e^{2\pi i \sigma t}, e^{2\pi i \sigma s}, e^{2\pi i \sigma}) \mid \sigma \in \mathbb{R}\}.$$

Consider now the standard action $\tau: T^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$ with moment mapping given by

$$\begin{aligned} J: \quad \mathbb{C}^3 &\rightarrow (\mathbb{R}^3)^*, \\ \underline{z} &\mapsto (|z_1|^2, |z_2|^2, |z_3|^2 - st). \end{aligned}$$

Then the N-moment mapping is given by

$$\begin{aligned} \Psi: \quad \mathbb{C}^3 &\rightarrow \mathbb{R}^* \\ \underline{z} &\mapsto t|z_1|^2 + s|z_2|^2 + |z_3|^2 - st \end{aligned}$$

and

$$\Psi^{-1}(0) = \{\, \underline{z} \in \mathbb{C}^3 \mid t|z_1|^2 + s|z_2|^2 + |z_3|^2 = st \,\}$$

is the dimension 5 ellipsoid of center the origin and of radii $(\sqrt{s}, \sqrt{t}, \sqrt{st})$. The quasitorus D^2 acts on the quasifold $M = \Psi^{-1}(0)/N$ with moment mapping

$$\begin{aligned} \Phi: \quad M &\rightarrow (\mathbb{R}^2)^* \\ [\underline{z}] &\mapsto (|z_1|^2, |z_2|^2) \end{aligned}$$

and $\Phi(M) = \Delta$. We call the quasifold M projective quasispace, by analogy with the case of the rational right triangle ($s/t \in \mathbb{Q}$), which gives either a weighted or an ordinary projective space.

The unit interval and the right triangle are actually rational (with respect to the appropriate choice of lattices). Here comes finally an example of a polytope that is not.

Example 3.7 (The regular pentagon). Let us take the regular pentagon in $(\mathbb{R}^2)^*$. We choose the vectors $X_1 = (1,0)$, $X_2 = (a, b)$, $X_3 = (c, d)$, $X_4 = (c, -d)$, $X_5 = (a, -b)$ and the corresponding quasilattice $Q = \sum_{j=1}^{5} X_j\mathbb{Z}$, where $a = \cos 2\pi/5$, $b = \sin 2\pi/5$, $c = \cos 4\pi/5$, $d = \sin 4\pi/5$. Then we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = c$ and a linear mapping

$$\begin{aligned} \pi: \quad (\mathbb{R}^5, \mathbb{Z}^5) &\rightarrow (\mathbb{R}^2, Q) \\ (x_1, x_2, x_3, x_4, x_5) &\mapsto (x_1 + a(x_2 + x_5) + c(x_3 + x_4), b(x_2 - x_5) + d(x_3 - x_4)) \end{aligned}$$

that induces a quasitorus homomorphism $\Pi: T^5 \rightarrow D^2 = \mathbb{R}^2/Q$ whose kernel is given by

$$N = \{(e^{2\pi i \phi}, e^{2\pi i \theta}, e^{2\pi i \sigma}, e^{2\pi i [2a(\theta - \sigma) + \phi)]}, e^{2\pi i [2a(\theta - \phi) + \sigma)]}) \mid (\phi, \theta, \sigma) \in \mathbb{R}^3\}.$$

Consider now the standard action $\tau: T^5 \times \mathbb{C}^5 \to \mathbb{C}^5$ with moment mapping given by

$$\begin{array}{rccl} J: & \mathbb{C}^5 & \to & (\mathbb{R}^5)^*, \\ & \underline{z} & \mapsto & (|z_1|^2 + c, |z_2|^2 + c, |z_3|^2 + c, |z_4|^2 + c, |z_5|^2 + c). \end{array}$$

Then the N-moment mapping is given, for $\underline{z} \in \mathbb{C}^5$, by

$$\Psi(\underline{z}) = -\left(\frac{\sqrt{5}}{2}, \sqrt{5}c, \frac{\sqrt{5}}{2}\right) + (|z_1|^2 + |z_4|^2 - 2a|z_5|^2, |z_2|^2 + 2a(|z_4|^2 + |z_5|^2), |z_3|^2 + |z_5|^2 - 2a|z_4|^2)$$

and

$$\Psi^{-1}(0) = \left\{ |z_1|^2 + |z_4|^2 - 2a|z_5|^2 = |z_3|^2 + |z_5|^2 - 2a|z_4|^2 = \frac{\sqrt{5}}{2}, |z_2|^2 + 2a(|z_4|^2 + |z_5|^2) = \sqrt{5}c \right\}.$$

The quasitorus D^2 acts on the quasifold $M = \Psi^{-1}(0)/N$ and $\Phi(M) = \Delta$.

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Appendix. A few generalities on convex polyhedral sets

In this appendix we just recall the few definitions that we need from the theory of convex polyhedral sets. Let $\mathfrak{d}$ be a vector space of dimension n.

Definition A.1 (Convex polyhedral set). We call *convex polyhedral set* in the dual space $\mathfrak{d}^*$ the intersection of a finite number of half-spaces, that is, a set $\Delta \subset \mathfrak{d}^*$ for which there exist elements $X_1, \ldots, X_d$ in $\mathfrak{d}$ and $\lambda_1, \ldots, \lambda_d$ in $\mathbb{R}$ such that

$$\Delta = \bigcap_{j=1}^{d} \{\mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geqslant \lambda_j\}. \tag{A.1}$$

We will always assume that our convex polyhedral sets have dimension[2] n. Convex polytopes and convex polyhedral cones are the examples of convex polyhedral sets that we are mostly concerned with.

[2] That is, the dimension of the affine subspace that they generate.

Definition A.2 (Rational convex polyhedral set). A convex polyhedral set $\Delta \subset \mathfrak{d}^*$ is said to be *rational* if there exists a lattice $L \subset \mathfrak{d}$ such that the elements X_j in (A.1) can be taken in the lattice L.

For example, the regular pentagon is not a rational polytope, or, in the words of a quasicrystal geometer, the group of symmetries of a regular pentagon is not a lattice-preserving group. We conclude with the definition of simple convex polyhedral set.

Definition A.3 (Simple convex polyhedral set). A convex polyhedral set $\Delta \subset \mathfrak{d}^*$ is said to be *simple* if there are exactly n edges stemming from each vertex.

For example, among the platonic solids the cube, the dodecahedron and the tetrahedron are simple polytopes, while the icosahedron and the octahedron are not.

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