Università di Firenze, Università di Perugia, INdAM consorziate nel CIAFM

DOTTORATO DI RICERCA
IN MATEMATICA, INFORMATICA, STATISTICA

CURRICULUM IN MATEMATICA
CICLO XXXIV

Sede amministrativa Università degli Studi di Firenze
Coordinatore Prof. Matteo Focardi

Prime graphs of Finite Groups

Settore Scientifico Disciplinare MAT/02

Dottorando:
Lorenzo Bonazzi

Tutore
Prof. Silvio Dolfi

Coordinatore
Prof. Matteo Focardi

Anni 2018/2021

Contents

Chapter 1

Introduction

Group theorists place the beginning of the study of finite groups the day before the death of Evariste Galois, in 1832. Since then, the theory has begun to be a very important topic in modern algebra. In the last century, Frobenius introduced a new tool for studying finite groups, character theory, that consisted in viewing a finite group G as a group of matrices and considering a function χ that associates to every element g the trace $\chi(g)$ of its matrix form. Many properties of the algebraic structure of G and the geometric structure of its representation in matrix form, are encoded in the character χ, which is merely a complex function from G.

1.1 Background

All the groups treated in this thesis are finite.

The idea of studying groups with an associated geometrical structure of a certain shape, has been the leading strategy of this work. Suppose that X is a set of positive integers and $\pi(X)$ is the set of prime divisors that divide some element of X. We associate to X a graph, called *prime graph* on X, in the following way. The vertices are all the primes in $\pi(X)$ and two vertices p, q are adjacent if pq divides some element of X. So, we obtained a geometrical structure from a set of integers. In our work, X is obtained from G in some canonical way and we investigate the structure of the group G when we know the geometry of $\Delta(X)$.

If G is a finite group, there are several ways to obtain a set of integers related to the structure of G. Perhaps, the first and most classical example to consider, is $X = \mathrm{cd}(G)$, the set of degrees of irreducible characters of G. This set has a great influence on the structure of G and problems related to character degrees have been considered by many authors in the last decades. Many results in this context

can be formulated in terms of $\Delta(G)$, the prime graph on character degrees. For example, the celebrated Ito-Michler Theorem, perhaps the most famous result in the study of character degrees, can be considered as regarding the prime graph on character degrees: a prime p that is not a vertex of $\Delta(G)$. In general, many authors have studied the connections between the structure of a group and the properties of $\Delta(G)$, like the number of connected components (see [44]) or the diameter (see [40]). See [39] for a survey on the prime graph $\Delta(G)$.
Another very interesting choice for the set X, is $\mathrm{cs}(G)$, the set of all conjugacy class sizes. In this case, we denote the prime graph on conjugacy class sizes with $\Delta^*(G)$. Between the contexts of character degrees and conjugacy class sizes there are many analogies. For example, the Ito-Michler Theorem admits a version for class sizes, see [21, Theorem 6.1]. As happened for $\Delta(G)$, the prime graph on character degrees, it has been a common strategy to study the groups G such that $\Delta^*(G)$ has some particular shape. For further information on conjugacy classes, see [10] and [39].

In recent years, authors started to investigate whether the same results on character degrees and class sizes hold if we consider just the subsets of $\mathrm{cd}(G)$ and $\mathrm{cs}(G)$ that consist of the real irreducible character degrees $\mathrm{cd}_{\mathbb{R}}(G)$ and real conjugacy class sizes $\mathrm{cs}_{\mathbb{R}}(G)$. As before, we construct the prime graphs on real character degrees $\Delta_{\mathbb{R}}(G)$ and real class sizes $\Delta^*_{\mathbb{R}}(G)$. We recall that a $\chi \in \mathrm{Irr}(G)$ is *real* if $\chi(g) \in \mathbb{R}$ for all $g \in G$ and a conjugacy class $\mathcal{C}$ of G is *real* if $g^{-1} \in \mathcal{C}$ for all $g \in \mathcal{C}$. It is well known that real classes are exactly those on which every character takes real values. Reality in finite groups played a central role for a long time. Concepts like the Schur-Frobenius indicator were developed many decades ago. Is remarkable the work of Gow [24], proving that a real irreducible real character of odd degree of a solvable group G is afforded by a real representation and, in fact, it is induced by a real linear character of determinantal order 2 of some subgroup H of G. For example, Ito-Michler and Thompson Theorems hold for real characters for the prime $p = 2$. Nevertheless, there are some limits when we look only to real characters or real conjugacy classes. For example, real versions for p odd of Ito-Michler and Thompson Theorems are only partially true. Indeed, the prime 2 plays a special role in the theory of real characters. Also, there is a version of Ito-Michler theorem for real classes at the prime $p = 2$ (but not the Thomspon theorem).
In the present thesis are studied groups G such that $\Delta_{\mathbb{R}}(G)$ has no edges and groups G such that $\Delta^*_{\mathbb{R}}(G)$ has no edges. In the study of prime graphs, non-adjacency has a great impact on the algebraic structure of the underlying group. Therefore, in Chapters 3 and 4, are studied groups G in which the prime graph has no edges at all. Nevertheless, for what concerns characters and conjugacy classes, any direct reference to the concept of prime graphs is not used in order to maintain a light

notation. So, we study groups such that $\mathrm{cd}_{\mathbb{R}}(G)$ and $\mathrm{cs}_{\mathbb{R}}(G)$ *consist of prime power numbers.*

The last concept we treat in the present work, is the Gruenberg-Kegel graph $\Gamma(G)$ of a group G. In our previous notation, let be $X = \omega(G)$ the spectrum of G, namely the set of all orders of elements of G. Even before the concept of prime graph appeared, the first result in this context is the theorem of Higman [30], that classified the groups G such that $\Gamma(G)$ has no edges. One other remarkable result is the celebrated Gruenberg-Kegel Theorem, which characterized the groups G such that $\Gamma(G)$ is disconnected, see [58]. Also, if G is solvable, the number of connected components of $\Gamma(G)$ is at most 2. In the same spirit of the other chapters, in the present work are studied groups G such that $\Gamma(G)$ has some geometric properties. In particular, we give a structural description of groups G such that $\Gamma(G)$ has a cut-set.

1.2 Structure of the Thesis

In Chapter 2 we include some technical results about orbits lengths in modules V under the action of a solvable group G, under some conditions regarding the primes that appear as divisors of the orbit lengths. . Sections 2.2 and 2.3 are dedicated to the study of the case when the G-orbits of V have lengths that are q-powers. The last section is dedicated to the study of the case when $\mathbf{C}_G(v)$ contains a Hall π-subgroup of G as a normal subgroup, where π is a set of primes. The main results in this chapter will be used in Chapters 3 and 4.

Chapter 3 is dedicated to degrees of real irreducible characters.
The main aim is to describe the groups in which $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime power numbers. The non-solvable case is treated in Section 3.3 and such groups are described in Theorem 3.3.8; the proof of this result uses the classification of finite simple groups. As a consequence, there is good control on character degrees, see Corollary 3.3.9. The main result of Section 3.4 is Theorem 3.4.4, where it is proved that if G is solvable and $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime powers, then the primes that appear as a divisor of some real degree are contained in $\{2, p\}$, with p odd prime. The chapter ends with a section of general remarks in the study of real characters.

Chapter 4 is about real conjugacy class sizes. In the main Theorem 4.2.10 of this chapter, are characterised groups G in which $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-power number, under the assumption $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$. This additional hypothesis is due to a difficulty that is not new in literature, which is treated in Section 4.3. The chapter ends with a section containing examples and remarks.

In Chapter 5 are treated questions related to the Grunberg-Kegel graph $\Gamma(G)$ of a solvable group G. The main result is Theorem 5.3.7, where we describe the structure of a solvable group G for which σ is a cut-set for $\Gamma(G)$. As a consequence, we have an upper bound of the σ-length of a solvable group when σ is a cut-set of $\Gamma(G)$. In Section **??** it is shown that the bounds obtained are the best possible.

Chapter 2

Actions with large centralizers

The techniques that we present here may be considered as a consequence of the results that appear in [45, Chapter III].
Section 2.1 covers some preliminary number theoretic results. Sections 2.2 and 2.3 describe the situation where the G-orbits of M have size powers of a fixed prime. Proposition 2.2.6 is a description of the semi-linear case.
In Section 2.4 is treated the property $\mathcal{N}_\pi$. If π is a set of primes and $\pi \cap \pi(G) \neq \emptyset$, where $\pi(G)$ is the set of primes that divide $|G|$, we say that (G, M) satisfies that property $\mathcal{N}_\pi$ if $C_G(v)$ contains one non-trivial Hall π-subgroup of G as a normal subgroup. The case where (G, M) satisfies the property $\mathcal{N}_p$ can be found in [11]. The same techniques of that article are adapted to treat the case where (G, M) satisfies $\mathcal{N}_\pi$. In the following Chapters are used the properties $\mathcal{N}_2$ and $\mathcal{N}_{2'}$.

2.1 Some number theoretic lemmas

In this section we present a list of useful tools. All the following lemmas belong to the branch of number theory and are often used to study semi-linear groups, that are treated in the next section.

Definition 2.1.1. Let p be a prime.

1. If we can write $p = 2^n + 1$ for some integer n, then n is a 2-power and we call p a *Fermat prime.*

2. If we can write $p = 2^n - 1$ for some integer n, then n is a prime and we call p a *Mersenne prime.*

3. Let $a > 1$ and n be positive integers. Then p is called a *Zsigmondy prime divisor* for $a^n - 1$ if p divides $a^n - 1$ but p does not divide $a^j - 1$ for $j < n$.

(Note that this depends upon a and n and not just on $a^n - 1$.) If p is a Zsigmondy prime divisor for $a^n - 1$, then n is the order of a modulo p, hence n divides $p - 1$.

Recall that (n, m) denotes the *greatest common divisor* of the integers m, n.

Lemma 2.1.2. *Let a, b, n, m be positive integers.*

1. $(n^a - 1, n^b - 1) = n^{(a,b)} - 1$.
2. *Suppose that $q^m - 1 = p^n$ for primes p and q. Then one of the following holds.*
   (a) $m = 1$, $p = 2$ *and q is a Fermat prime;*
   (b) $n = 1$, $q = 2$ *and p is a Mersenne prime;*
   (c) $m = 2$, $n = q = 3$ *and* $p = 2$.
3. *Suppose that $q^m - 1 = 2^n \cdot 3$ for a prime q. Then one of the following holds*
   (a) $m = 1$;
   (b) $m = 2$ *and* $q \in \{5, 7\}$.
4. *If $a > 1$, then there exists a Zsigmondy prime divisor for $a^n - 1$ unless*
   (a) $n = 2$ *and* $a = 2^k - 1$ *for some* $k \in \mathbb{N}$, *or*
   (b) $n = 6$ *and* $a = 2$.

Proof. Part 1 is well known. Parts 2, 3 and 4 are respectively Propositions 3.1 and 3.2 and Theorem 6.2 of [45]. $\square$

2.2 Semi-linear groups

In this section, we discuss some problems in the context of the semi-linear groups (see the definition below). These groups play an important role in the study of solvable linear groups. They appear in Propositions 2.4.4 and 4.2.2.
We recall some basic definitions from [45, pages 37-38]. Let V be the field $\mathrm{GF}(q^n)$ for a prime-power q. Of course V is a vector space over $\mathrm{GF}(q)$ of dimension n. Fix $a \in V \setminus \{0\} = V^{\#}$, $w \in V$ and $\sigma \in \mathcal{G} := \mathrm{Gal}(\mathrm{GF}(q^n)/\mathrm{GF}(q))$. We define a mapping $T\colon V \to V$ by

$$T(x) = ax^{\sigma} + w.$$

Then T is a permutation on V and T is trivial if and only if $a = 1$, $\sigma = 1$ and $w = 0$. Thus we have the following subgroups of $\mathrm{Sym}(V)$:

i) $A(V) = \{x \mapsto x + w \mid w \in W\}$ consisting of translations.

ii) The *semi-linear* group

$$\Gamma(V) = \{x \mapsto ax^\sigma \mid a \in \mathrm{GF}(q^n)^\#, \sigma \in \mathcal{G}\}.$$

iii) The subgroup $\Gamma_0(V) = \{x \mapsto ax \mid a \in \mathcal{G}^\#\}$ of $\Gamma(V)$, consisting of multiplications.

iv) The *affine semi-linear group*

$$A\Gamma(V) = \{x \mapsto ax^\sigma + w \mid a \in \mathrm{GF}(q^n)^\#, \sigma \in \mathcal{G}, w \in V\}.$$

Clearly, $A(V)$ acts regularly on V and $A(V) \simeq V$ as a vector space over $\mathrm{GF}(q)$. Now $A(V)$ and V are $\Gamma(V)$-modules, where $\Gamma(V)$ acts on the normal subgroup $A(V)$ by conjugation and on V by semi-linear mappings. Hence, as is easily checked, $A(V) \simeq V$ as $\mathrm{GF}(q)[\Gamma(V)]$-modules. Observe that $\Gamma(V)$ and even $\Gamma_0(V)$ act transitively on the non-zero elements of V and $A(V)$. In fact, $A\Gamma(V)$ is the semi-direct product of $A(V)$ and $\Gamma(V)$. Also $\Gamma(V)$ is a point-stabilizer (for zero) in the doubly transitive permutation group $A\Gamma(V)$. Note that $\Gamma_0(V)$ is cyclic of order $q^n - 1$ and $\Gamma(V)/\Gamma_0(V) \simeq \mathcal{G}$ is cyclic of order n. If $\sigma \in \mathcal{G}$ has order m, then $|\mathbf{C}_V(\sigma)| = |\mathbf{C}_{A(V)}(\sigma)| = q^{n/m}$ and $|\mathbf{C}_{\Gamma_0(V)}(\sigma)| = q^{n/m} - 1$.
We will also write $\Gamma(q^n)$ for $\Gamma(V)$, etc.. If V is an n-dimensional vector space over $\mathrm{GF}(q)$, then V can be identified with the additive group of a field of order q^n, and in this sense we write $\Gamma(V)$, $\Gamma_0(V)$ etc. for $\Gamma(q^n)$, $\Gamma_0(q^n)$ etc.. Observe that e.g. $\Gamma(8^2)$ and $\Gamma(4^3)$ are distinct proper subgroups of $\Gamma(2^6)$. For the most part, we will assume that the base field $\mathrm{GF}(q)$ is the prime field.

As remarked earlier in the section, semi-linear groups appear quite often in the study of representation theory of solvable groups. For example, if H is a group and V is an H-module, suppose that, for every $v \in V$, the centralizer $\mathbf{C}_H(v)$ contains a Sylow p-subgroup of H, with p a prime divisor of $|H|$. Then Theorem 2.3.3 tells us that, apart from a few exceptions, we have that H is isomorphic to a subgroup of $\Gamma(V)$.

The following theorem is often useful. Recall that if G is a group, V a G-module and A is a subgroup of G, we denote with V_A the space V viewed as A-module.

Theorem 2.2.1. *[45, Theorem 2.1] Suppose that G acts faithfully on a $\mathrm{GF}(q)$-vector space V of order q^n, q a prime power. Assume that G has a normal abelian subgroup A for which V_A is irreducible. Then G may be identified as a subgroup of $\Gamma(q^n)$ (i.e. the points of V may be labeled as elements of $\mathrm{GF}(q^n)$ in such a way that G is permutationally isomorphic to a subgroup of $\Gamma(q^n)$) and $A \leq \Gamma_0(q^n)$.*

We next proceed through two technical results concerning semi-linear groups. These are used in the proof of Proposition 2.2.6.

Lemma 2.2.2. *[16, Lemma 3] Let $\mathbb{K}$ be a field with $|\mathbb{K}| = s^n$, s a prime and $g \in \Gamma(\mathbb{K})$.*

1. $|\mathbf{C}_{\Gamma_0(\mathbb{K})}(g)| = s^{n/d} - 1$ *for some divisor d of $|g|$.*
2. *If $\langle g \rangle \cap \Gamma_0(\mathbb{K}) = 1$, then $|\mathbf{C}_{\mathbb{K}}(g)| = s^{n/|g|}$.*

Lemma 2.2.3. *[12, Lemma 3.5] Let r be a prime, W a vector space of order r^n, and L a subgroup of $\Gamma(V)$. Also, setting $L_0 = L \cap \Gamma_0(W)$, let δ be a set of primes in $\pi(L) \setminus \pi(L_0)$, and let $D \in \mathrm{Hall}_\delta(L)$. Then $|D|$ divides n and, defining*

$$k = \frac{r^n - 1}{r^{n/|D|} - 1},$$

the following facts are equivalent.

1. *for every $w \in W$, $\mathbf{C}_L(w)$ contains a suitable conjugate of D.*
2. $|\{D^h | h \in L\}| = k$.
3. *k divides $|L_0|$ and $|D|$ is coprime to $r^n - 1$.*

We include two easy, but useful, lemmas about orbits of completely reducible actions of abelian group.

Lemma 2.2.4. *Let A be an abelian group and W a faithful and completely reducible A-module. Then W contains a regular A-orbit.*

Proof. We proceed by induction on $|W|$. If W is irreducible, by Theorem 2.2.1 we have that $A \leq \Gamma_0(W)$ and therefore A has a regular orbit on W. Suppose that $W = W_1 \oplus W_2$ is a decomposition in non-trivial A-submodules. Then $A/\mathbf{C}_A(W_i)$ has a regular orbit with generator w_i for $i = 1, 2$. so $w_1 + w_2$ is a representative of a regular A-orbit because $\mathbf{C}_A(W_1) \cap \mathbf{C}_A(W_2) = 1$. □

Lemma 2.2.5. *Let $L \leq \Gamma(V)$ that acts irreducibly on V. Assume that $|L| = 18$ and the L-orbits are 3-powers. Then L is dihedral of order 18.*

Proof. Let $P \in \mathrm{Syl}_3(L)$, observe that $P \trianglelefteq L$ since it has index 2 in L. Assume by contradiction that V_P is not irreducible. Then, by Clifford's Theorem, there are $V_1, V_2 \leq V$, irreducible under the action of P and such that $V = V_1 \oplus V_2$. Moreover, there is an involution $x \in L$ such that $L = P\langle x \rangle$ and $V_1^x = V_2$. By [45, Theorem 4.8], there is $v \in V_1$ a representative of a regular P-orbit of V. Suppose that v is centralized by a x^g with $g \in P$. Then, $w^x = w$ for $w = v^{g^{-1}} \in V_1$.

So, $w^x \in V_1 \cap V_2 = 0$ and this is impossible. This implies that there isn't any L-conjugate of x that centralizes v and therefore 2 divides $[L : \mathbf{C}_L(x)]$. But v generates a P-regular orbit, so v is the representative of a regular L-orbit and this is against the assumptions. It follows that V_P is irreducible and P is cyclic by [45, Lemma 0.5]. Note that L can't be abelian by Lemma 2.2.4. It follows that L is dihedral. □

We are now ready to prove one of the main results of this section. The following proposition describes the structure of a group L acting irreducibly on a module W as a subgroup of the semi-linear group $\Gamma(W)$, assuming that the lengths of L-orbits of W are q-power numbers, for a fixed prime q.
We recall that if m is an integer that is divisible by two different primes, we say that m is a *composite number.*

Proposition 2.2.6. *Let $L \leq \Gamma(W)$ and assume that W is a faithful irreducible L-module. Write $|W| = r^n$ for a prime r. Suppose that there exists a prime q such that $[L : \mathbf{C}_L(v)]$ is a non-trivial q-power, for all $v \in W^\#$, and that L is not a q-group. Then*

1. *q is odd, $\mathbf{F}(L)$ is a q-group and if $L_0 = L \cap \Gamma_0(W)$ and $D \in \mathrm{Hall}_{q'}(L)$, then $|L_0| = q^s$ with*
$$q^s = \frac{r^n - 1}{r^{n/|D|} - 1};$$
2. *If $(r^n, q) \neq (2^6, 3)$, then q is a Zsigmondy prime divisor of $r^n - 1$, $[L : L_0] = |D|$, where $D \in \mathrm{Hall}_{q'}(L)$ and $[L : \mathbf{C}_L(v)] = |L_0| = q^s$ for all $v \in W^\#$;*
3. *If 2 divides $|L|$ then $r = 2$ and L is a dihedral group. Moreover $|L| = 2 \cdot q$ with q a Fermat prime, except when $r^n = 2^6$ and $|L| = 2 \cdot 9$.*

Proof. Observe that $\mathbf{O}_{q'}(L) \leq \mathbf{C}_L(w)$ for every $w \in W$. Then $\mathbf{O}_{q'}(L) \leq \mathbf{C}_L(W) = 1$ and this implies that $\mathbf{F}(L)$ is a q-group. Call $\Gamma_0 = \Gamma_0(W)$. Let $L_0 = L \cap \Gamma_0$; note that $L_0 \trianglelefteq L$ and that L_0 acts semi-regularly on $W^\#$. Since $L_0 \leq \mathbf{F}(L)$, there exists an integer $s > 1$ such that $|L_0| = q^s$. Let $D \in \mathrm{Hall}_{q'}(L)$, then $D > 1$ because L is not a q-group. Moreover $D \cap \Gamma_0 \leq D \cap L_0 = 1$ and hence $D \simeq D\Gamma_0/\Gamma_0 \leq \Gamma(W)/\Gamma_0$, that has order n. We deduce that $|D|$ divides n and D is cyclic. Let $g \in L$ a generator of D. It follows from Lemma 2.2.2 that that $\mathbf{C}_{\Gamma_0}(D) = \mathbf{C}_{\Gamma_0}(\langle g \rangle) = r^{n/|D|} - 1$.

We now prove by contradiction that $\mathbf{C}_{L_0}(D) = L_0 \cap \mathbf{C}_{\Gamma_0}(D) = 1$. Let $K = \mathbf{C}_{L_0}(D)$ and assume that $K > 1$. The group D acts coprimely on L_0, so $L_0 = \mathbf{C}_{L_0}(D) \times [L_0, D]$. Since L_0 is a cyclic q-group and $1 < K$, we have that D centralizes L_0 and hence L_0D is cyclic. Moreover L/L_0 is cyclic and therefore $L_0D \trianglelefteq L$. Note that W is a completely reducible L_0D-module by Clifford's

Theorem. By Lemma 2.2.4, W contains a regular orbit $\mathcal{O}$ under the action of L_0D. Since L_0D is not a a group of prime power order, $|\mathcal{O}|$ is a composite number. If $w \in \mathcal{O}$, then $|\mathcal{O}|$ divides $|w^L|$, that therefore is a composite number; this is against the assumptions. Hence $L_0 \cap \mathbf{C}_{\Gamma_0}(D) = \mathbf{C}_{L_0}(D) = 1$.
This means that L_0 is isomorphic to a subgroup of $\Gamma_0/\mathbf{C}_{\Gamma_0}(D)$ and $|L_0| = q^s$ divides

$$\frac{r^n - 1}{|\mathbf{C}_{\Gamma_0}(D)|} = \frac{r^n - 1}{r^{n/|D|} - 1}.$$

Let $w \in W^{\#}$; then $\mathbf{C}_L(w)$ is cyclic since it is isomorphic to a subgroup of Γ/Γ_0. This means that $\mathbf{C}_L(w)$ contains at most one conjugate of D. On the other hand, the L-orbits have q-power lengths, thus $\mathbf{C}_L(w)$ contains exactly one conjugate of D, for every $w \in W^{\#}$. By Lemma 2.2.3, $r^n - 1/r^{n/|D|} - 1$ divides $|L_0|$, hence

$$q^s = |L_0| = \frac{r^n - 1}{r^{n/|D|} - 1}.$$

It remains to prove that $q > 2$. Suppose by contradiction that $q = 2$. Then r is odd and D is a Hall $2'$-subgroup of L. Therefore we have that

$$2^s = 1 + r^{n/|D|} + \cdots + r^{(|D|-1)n/|D|},$$

and this is an odd number. This concludes the proof of part 1.
We now prove part 2. Firstly, we prove that r^n is not of the form $2^6, (2^k - 1)^2$ for every $k \geq 1$.
If $r^n = 2^6$ then, by hypothesis, q is a prime divisor of $2^6 - 1$ different from 3. This means that $q = 7$, but this is impossible since each divisor t of 6 is not a solution of the equation $7 = (2^6 - 1)/(2^t - 1)$. Suppose that $r^n = (2^k - 1)^2$ for some $k \geq 1$, then $2^k - 1 = r^{n/2}$ and hence $n = 2$ by Lemma 2.1.2. Since $D > 1$ and, as remarked above, $|D|$ divides n, we have that $|D| = n = 2$ and $r = 2^k - 1$. Therefore, by part 1 we have

$$q^s = \frac{r^n - 1}{r^{n/|D|} - 1} = \frac{(2^k - 1)^2 - 1}{2^k - 2} = 2^k.$$

But q is odd for part 1 and this is impossible.
Assume that $r^n \neq 2^6, (2^k - 1)^2$; by part 4 of Lemma 2.1.2, there is a Zsigmondy prime divisor t of $r^n - 1$. Since t does not divide $r^{n/|D|} - 1$, we have that t divides $(r^n - 1)/(r^{n/|D|}) - 1 = q^s$, so $t = q$. Therefore q is the only Zsigmondy prime divisor of $r^n - 1$. Suppose that q divides n and write $n = bq$ with $b \geq 1$. Since $D > 1$ and $|D|$ divides n, we have that $b > 1$. Now

$$\begin{aligned}\frac{r^n - 1}{r^{n/b} - 1} &= 1 + r^q + \cdots + r^{q(b-1)} \equiv \\ &\equiv 1 + r + \cdots + r^{b-1} \pmod{q}\end{aligned}$$

because $r^q \equiv r \pmod q$. Now, $q = t$ divides $r^n - 1/r^{n/b} - 1$, then q divides also $1+r+\cdots+r^{b-1}$, that is a divisor of $r^b - 1$. Since q is a Zsigmondy prime divisor of $r^n - 1$ this is impossible. Hence q does not divide n and it follows that $L = L_0 D$.

Finally, we prove part 3. Assume that 2 divides $|L|$. Recall that L_0 is a q-group by part 1. We already proved that the case $(r^n, q) = (2^6, 7)$ cannot appear. Assume that $(r^n, q) = (2^6, 3)$. Then $|L| = 6$ or 18, because 2 divides $|L|$. If $|L| = 6$, then L is isomorphic to the symmetric group S_3 and hence W can't be irreducible as L-module, because every irreducible representation of S_3 on every field has dimension at most 2 (see Theorem 4.12 and Corollary 8.7 of [36]). By Lemma 2.2.5, L is dihedral of order 18 and the thesis follows. Assume that $r^n \neq 2^6$. By part 2, we have $L = L_0 D$. Since 2 divides $|L| = |D| \cdot |L_0|$ by hypothesis, we have that 2 divides $|D|$, because $|L_0|$ is odd. Thus

$$q^s = \frac{r^n - 1}{r^{n/|D|} - 1} = \frac{r^n - 1}{r^{n/2} - 1} \frac{r^{n/2} - 1}{r^{n/|D|} - 1}.$$

Since $n/|D|$ divides $n/2$, the second term in the product is an integer, so it is a q-power and q divides $r^{n/2} - 1$. But q is a Zsigmondy prime divisor of $r^n - 1$, and this implies that the second term is equal to 1 , in other words $|D| = 2$. Now, note that we have

$$q^s = \frac{r^n - 1}{r^{n/2} - 1} = r^{n/2} + 1,$$

hence $q^s - 1 = r^{n/2}$. Since $r^n \neq 2^6$, by part 2 of Lemma 2.1.2, we have that $s = 1$, q is a Fermat prime and $r = 2$. $\square$

The group $\Gamma(2^6)$ has an orbit of length 27 on the vector space W of order 2^6, hence it is not true that all the non-trivial orbits have the same length. Thus, the assumption $(r^n, q) \neq (2^6, 3)$ is necessary in part 2 of Proposition 2.2.6.

2.3 Primitive and imprimitive modules

Let H be a group. An irreducible H-module V is called *imprimitive* if V can be written $V = V_1 \oplus \cdots \oplus V_n$ for $n > 1$ subspaces (not-submodules) V_i that are transitively permuted by H. If $T = \mathbf{N}_H(V_1)$, then $V \simeq V_1^H$ is induced from T (see Chapter 5 of [34] for the notion of induced module). We say that V is *primitive*, if V is not imprimitive, or equivalently V is not induced from a submodule of a proper subgroup of H.
An irreducible H-module V is called *quasi-primitive* if V_N is homogeneous for all $N \trianglelefteq H$. It is a consequence of Clifford's Theorem that a primitive module V is quasi-primitive. At times, it is more convenient to weaken the quasi-primitive condition to V_N homogeneous for all characteristic subgroups N of H. We then call

V a *pseudo-primitive* H-module. If H is a solvable group and V is a H-module, then V is primitive if and only if V is quasi-primitive (see [6]); this is not true if H is not solvable. If V is pseudo-primitive, then V may not be quasi-primitive, even if H is solvable, as Example 2.3.1 shows.

Example 2.3.1. Suppose H is a group such that $\mathbf{F}(H) = F$ is extra-special of order 3^3 and exponent 3, $[H : F] = 2$, $\mathbf{Z}(H) = \mathbf{Z}(F)$ and $\mathbf{O}^{2'}(H) = H$. Then H has a unique faithful pseudo-primitive irreducible module V over GF(2). Furthermore

1. $|V| = 2^6$ and, for all $v \in V^{\#}$, 2 divides $|\mathbf{C}_H(v)|$ (and therefore $[H : \mathbf{C}_H(v)]$ is a 3-power);
2. There exists $v \in V$ such that $|\mathbf{C}_H(v)| = 2$;
3. Suppose that H is characteristic in $H_0 \leq \mathrm{GL}(V)$, with H_0 solvable, V a pseudo-primitive H_0-module and $2 \nmid [H : \mathbf{C}_H(v)]$ for every $v \in V$. Then $H = H_0$.
4. The module V is not quasi-primitive.

Parts 1-3 of previous example appear as Example 10.3 in [45]. In the proof is pointed out that there is $C \trianglelefteq G$ elementary abelian of order 3^2 such that V_C is not homogeneous. Therefore part 4 follows.

Definition 2.3.2. Let V be a vector space. Then $\mathrm{GL}(V)$, the general linear group of V, acts on V and we call *affine general linear group* the semi-direct product $\mathrm{AGL}(V) = V \rtimes \mathrm{GL}(V)$. As the special linear group $\mathrm{SL}(V)$ is a normal subgroup of $\mathrm{GL}(V)$, then $\mathrm{ASL}(V) = V \rtimes \mathrm{SL}(V)$ is a normal subgroup of $\mathrm{AGL}(V)$, that we call *affine special linear group.* If V is a m-dimensional vector space over a field with p^n elements, we denote $\mathrm{AGL}(V)$ and $\mathrm{ASL}(V)$ with $\mathrm{AGL}_m(p^n)$ and $\mathrm{ASL}_m(p^n)$.

Let H be a finite group acting faithfully on a module V and let be p a prime divisor of the order of H. An orbit condition of relevance in the representation theory of finite solvable groups is that the length of every H-orbit on V is not divisible by p, or in other words that the centralizer in H of every element in V contains a Sylow p-subgroup of H. The following theorem describes the structure of H in this case, when the action of H on V is primitive.

Theorem 2.3.3. *[45, Theorem 10.5] Let H be a solvable group and W a pseudo-primitive faithful H-module. Suppose that there is a prime $p \in \pi(H)$ such that $p \nmid [H : \mathbf{C}_H(v)]$ for every $v \in W^{\#}$. Then W is irreducible and one of the following occurs.*

1. *$\mathbf{O}^{p',p}(H)$ is cyclic p'-group and $H \leq \Gamma(W)$;*

2. $\mathrm{ASL}_2(3) \leq H \leq \mathrm{AGL}_2(3)$*;*

3. $|W| = 2^6$, $p = 2$ *and* H *is the group in the Example 2.3.1.*

The next proposition can be considered as the extension of Theorem 2.3.3 to the imprimitive case, in the special situation where all the H-orbits of V have lengths that are non-trivial q-powers, for a fixed odd prime q.

Proposition 2.3.4. *Let* H *be a group of odd order and* V *be a faithful irreducible* $\mathrm{GF}(r)[H]$*-module, with* r *a prime and* $|V| = r^n$*. Suppose that, for all* $v \in V^{\#}$, $[H : \mathbf{C}_H(v)]$ *is a non-trivial* q*-power for an odd prime* q*. Then, either* H *is a* q*-group or the followings hold:*

1. *There is* $W \leq V$ *such that* $V = W^H$*, where* H *is isomorphic to a subgroup of the wreath product* $L \wr Q$ *with* Q *a* q*-group and* W *is a primitive* L*-module of order* r^m*;*

2. $L \leq \Gamma(W)$*; moreover, if* $D \in \mathrm{Hall}_{q'}(L)$ *and* $L_0 = L \cap \Gamma_0(W)$*, then* $[L : L_0] = |D|$*, all the* L*-orbits have length* $|L_0|$ *and*

$$|L_0| = \frac{r^m - 1}{r^{m/|D|} - 1};$$

Proof. Suppose that H is not a q-group. Let $W \leq V$ minimal such that $W^H = V$ and suppose that $|W| = r^m$. Write $V = W_1 \oplus \cdots \oplus W_t$, where the subspaces W_i are transitively permuted by H. Let $W = W_1$ and $W_i \simeq W$ for every $i \geq 2$. Let

$$N = \bigcap_{i=1}^{t} \mathbf{N}_H(W_i).$$

Then $N \trianglelefteq H$ and there is an induced transitive action of H on the set $\Omega = \{1 \dots t\}$. The kernel of this action is N. It is easy to see that H acts also on $\mathcal{P}(\Omega)$ and N is the kernel of this action. Call $Q = H/N$. Since Q has odd order, by [45, Corollary 5.7b)] there is $\Delta \subseteq \mathcal{P}(\Omega)$ such that $\mathrm{Stab}_H(\Delta) = N$. Let $w_i \in W_i^{\#}$ for every $i \in \Delta$ and let $v = \sum_{i \in \Delta} w_i$. Suppose that there is $g \in H$ such that $v^g = v$. Since g acts also on $\mathcal{P}(\Omega)$, the power-set of Ω, we have $\Delta^g = \Delta$; it follows that $g \in \mathrm{Stab}_H(\Delta) = N$. We have proved that $\mathbf{C}_H(v) \leq N$. Since $[H : \mathbf{C}_H(v)]$ is a q-power, we have that Q is a q-group. Call $L = \mathbf{N}_H(W)/\mathbf{C}_H(W)$. Let $1 = g_1, g_2, \dots, g_t$ be a set of representatives for the cosets of $\mathbf{N}_H(W)$, so that $W^{g_i} = W_i$ and $L^{g_i} = \mathbf{N}_H(W_i)/\mathbf{C}_H(W_i)$. There exists an immersion $i \colon H \to L \wr Q$ such that the action of H on V is isomorphic to the action of Hi on $W \times \{1, \dots, t\}$. Note that by minimality, W is a primitive L-module. Therefore, part 1 follows.
Let $v \in W$. Then $\mathbf{C}_H(v) \leq \mathbf{N}_H(W)$ and $|v^L| = [\mathbf{N}_H(W) : \mathbf{C}_H(v)]$. Therefore

$|v^L|$ divides $|v^H|$. It follows that the lengths of the L-orbits in W are q-powers. It remains to show that every non-trivial element $w \in W$ generates a non-trivial L-orbit. In fact, if there is $w \in W^{\#}$ such that $w^L = w$, consider $v = \sum_i w^{g_i}$. Since H is isomorphic to a subgroup of $L \wr Q$ as a permutation group of V, then $0 \neq v \in \mathbf{C}_V(H)$, that is against the assumptions. So the action of L on W inherits the hypotheses. Since H is not a q-group, a Hall q'-subgroup of H is non-trivial and it is contained in the base group of $L \wr Q$. This means that L is not a q-group. Since W a primitive L-module, L is not the group in Example 2.3.1. Moreover, q is odd and by Theorem 2.3.3, we can assume $L \leq \Gamma(W)$. Thus, Proposition 2.2.6 applies and $\mathbf{F}(L)$ is a q-group. So $L_0 = L \cap \Gamma_0(W)$ is a q-group and if $D \in \mathrm{Hall}_{q'}(L)$, then

$$|L_0| = \frac{r^m - 1}{r^{m/|D|} - 1},$$

where $r^m = |W|$. Suppose, by contradiction, that $(r^m, q) = (2^6, 3)$. Then $L_0 \leq \mathbf{F}(L)$ and L_0 is a 3-group. This means that L is a $\{2, 3\}$-group and this is impossible since L is not a 3-group and has odd order. Therefore, by part 2 of Proposition 2.2.6, part 2 of this Proposition follows. $\square$

Definition 2.3.5. Let Σ be a finite non-empty set, and let G be a subgroup of $\mathrm{Sym}(\Sigma)$. Also, let $\mathcal{O}$ be an orbit of the action of G on Σ, and π a set of prime numbers. We say that the orbit $\mathcal{O}$ is π-deranged if there exists a π-element of G which does not fix any element in $\mathcal{O}$.

If B is the base group of $H \wr K$, then $B = \prod_{i=1..|K|} H$ and if $x \in B$, then $x = (x_1, \ldots, x_{|K|})$.

Definition 2.3.6. Let H, K groups. An element x of the wreath product $H \wr K$ is called *diagonal* if $x \in B$, the base group of $H \wr K$, and it is of the form $x = (x_i)_{i=1,\ldots,|K|}$, where $x_i = x_j$ for all i, j.

Theorem 2.3.7. *Let G be a solvable group, q a prime number and V a faithful irreducible* $\mathrm{GF}(q)[H]$*-module. Assume that H is not a 2-group and that there are no $2'$-deranged orbits for the action of G on V. Then $q = 3$ and H is isomorphic to a subgroup of $H \wr K$, where $H = \mathbf{N}_G(W)/\mathbf{C}_G(W)$ for some subspace W of V such that $V = W^G$, W is a primitive L-module of order 3^2, and $K = G/N$ with $N = \cap_{g \in G} \mathbf{N}_G(W^g)$. Moreover, the following holds.*

1. *The action of G on V is isomorphic to the action of the image of G, as a subgroup of $H \wr K$, on $W^{\oplus r}$, where $r = [G : \mathbf{N}_G(W)]$.*

2. *Q is isomorphic to quotient G/N.*

3. *L isomorphic to either* $\mathrm{GL}_2(3)$ *or to* $\mathrm{SL}_2(3)$ *and Q is a (possibly trivial) 2-group.*

4. *If $x \in \mathbf{N}_G(W)$ and x is centralized by a Sylow 2-subgroup of G, then the image of x in $H \wr K$ is diagonal, i.e. all its entries are identical and it is invariant under the action of K.*

Proof. The first part of the theorem follows from the proof of [19, Theorem 2.2], where $T = \mathbf{N}_G(W)$. By [19, Theorem 2.1], W is a module of dimension 2 over GF(3).
Part 1 and 2 follow respectively from [18, Remark 2.3] and [18, Remark 2.1].
Part 3 is contained in the statement of [19, Theorem 2.2].
For part 4, use the notations of [18, Remark 2.1]. If $T = \mathbf{N}_G(W)$, then T has 2-power index in G. Let $S \in \mathrm{Syl}_2(G)$ that centralizes x. We can choose a right transversal $\{t_1, \ldots, t_r\} \subseteq S$ for T in H. Therefore, for all i, $t_i x = x t_i$ and hence $t_{x,i} = t_{x,j} = x$ for all i, j. So, $\phi(x)$ lies in the base group and it is diagonal. Hence the image of x in $H \wr K$ lies in the base group and it is diagonal, in particular is invariant under the action of K. $\square$

Suppose H is a solvable group and V is a H-module of odd order. Suppose that $\mathbf{C}_H(v)$ contains a Hall $2'$-subgroup of H for every $v \in V^{\#}$. Then the action of H on V has no $2'$-deranged orbit and Theorem 2.3.7 applies.

2.4 The condition $\mathcal{N}_\pi$

If π is a set of primes and n is an integer, we say that n is a π-number if all the primes that divide n belong to π. If G is a group, $\pi(G)$ denotes the set of primes that divide $|G|$.

Definition 2.4.1. Let π be a non-empty set of primes. Let G be a solvable group acting faithfully on the finite module M. Assume that $\pi \cap \pi(G) \neq \emptyset$. We say that the pair (G, M) satisfies the condition $\mathcal{N}_\pi$ if for every $v \in M^{\#}$, $\mathbf{C}_G(v)$ contains a Hall π-subgroup of G as a *normal* subgroup.

If $\pi = \{p\}$, then the condition $\mathcal{N}_\pi = \mathcal{N}_p$ can be defined for all groups (see [11]).

Definition 2.4.2. If G is a group and π is a set of primes, then $\mathbf{O}^\pi(G)$ is the smallest normal subgroup whose index in G is a π-number. We denote $\mathbf{O}^{\pi,\pi'}(G) = \mathbf{O}^{\pi'}(\mathbf{O}^\pi(G))$.

Lemma 2.4.3. *Let G be a solvable group and M a faithful G-module. Suppose that (G, M) satisfies the condition $\mathcal{N}_p$ (where p is a fixed prime). Then M is irreducible and one of the following holds:*

1. *$G \leq \Gamma(M)$ and $\mathbf{O}^{p',p}(G)$ is cyclic p'-group;*

2. *$\mathrm{ASL}_2(3) \leq GM \leq \mathrm{AGL}_2(3)$ and $p = 3$.*

Lemma 2.4.3 is Corollary 10 in [11]. With more or less the same technique, we can prove a generalization of the previous result.

Proposition 2.4.4. *Let G be a solvable group and M a faithful G-module. Suppose that (G, M) satisfies the condition $\mathcal{N}_{2'}$. Then M is irreducible and one of the following holds:*

1. *$G \leq \Gamma(M)$ and $\mathbf{O}^{q',q}(G)$ is cyclic q'-group for some odd prime q;*
2. *$\mathrm{ASL}_2(3) \leq MG \leq \mathrm{AGL}_2(3)$ and $\pi(MG) = \{2, 3\}$.*

For the proof of Proposition 2.4.4, we need some preliminary results. The next lemma is similar to Proposition 8 of [11].

Lemma 2.4.5. *Let π be a set of primes and suppose that (G, M) satisfies the condition $\mathcal{N}_\pi$. If $N \trianglelefteq G$ and N is not a π'-group, then (N, M) satisfies the condition $\mathcal{N}_\pi$*

Proof. Let $N \trianglelefteq G$. Then, for every $x \in M^\#$, $[N : \mathbf{C}_N(x)] = [N : \mathbf{C}_G(x) \cap N]$ and this divides $[G : \mathbf{C}_G(x)]$, that is a π'-number. Therefore, there is $H_0 \in \mathrm{Hall}_\pi(N)$ such that $H_0 \leq \mathbf{C}_N(x)$. By hypothesis, there is $H \in \mathrm{Hall}_\pi(G)$ such that $H \trianglelefteq \mathbf{C}_G(x)$. Clearly, $H_0 \leq H$ and $H_0 = H \cap \mathbf{C}_N(x)$. This means that $H_0 \trianglelefteq \mathbf{C}_N(x)$. If N is not a π'-group, then $\pi \cap \pi(N) \neq \emptyset$ and hence (N, M) satisfies the condition $\mathcal{N}_\pi$. □

The following proposition is similar to [60, Lemma 4].

Lemma 2.4.6. *Let G be a solvable group and π a set of primes. If (G, M) satisfies the condition $\mathcal{N}_\pi$, then M is an irreducible G-module.*

Proof. The group M is a G-module and let p be its characteristic. We firstly prove that $\mathcal{P}_M = \{\mathbf{C}_M(H)^\# \mid H \in \mathrm{Hall}_\pi(G)\}$ is a partition of $M^\#$. Let $H_1, H_2 \in \mathrm{Hall}_\pi(G)$ and suppose that there exists a non-trivial $v \in \mathbf{C}_M(H_1) \cap \mathbf{C}_M(H_2)$. Then $H_1, H_2 \leq \mathbf{C}_G(v)$ and since (G, M) satisfies the condition $\mathcal{N}_\pi$, we have that H_1 and H_2 are normal in $\mathbf{C}_G(v)$ and hence they coincide. Moreover, if $v \in M^\#$, then there exists $H \in \mathrm{Hall}_\pi(G)$ such that $H \trianglelefteq \mathbf{C}_G(v)$; hence $v \in \mathbf{C}_M(H)$. It follows that $\mathcal{P}_M$ is a partition of $M^\#$. If $p^n = |M|$ and $p^a = |\mathbf{C}_M(H)|$, we have that

$$\frac{p^n - 1}{p^a - 1} = |\mathcal{P}_M| = |\mathrm{Hall}_\pi(G)| = [G : \mathbf{N}_G(H)]$$

is an integer, and hence, by Lemma 2.1.2, we have that a divides n and we can write

$$[G : \mathbf{N}_G(H)] = 1 + p^a + \cdots + p^{(t-1)a}$$

with $t = n/a$. Suppose now that $1 < U$ is a G-submodule of M. If $|U| = p^m$ and $|\mathbf{C}_U(H)| = p^b$, applying to $\mathcal{P}_U$ the same counting argument as above we have that

$$1 + p^a + \cdots + p^{(t-1)a} = [G : \mathbf{N}_G(H)] = 1 + p^b + \cdots + p^{(s-1)b}$$

with $s = m/b$. Since the representation of $[G : \mathbf{N}_G(H)]$ in base p is unique, we have that $a = b$, $t = s$ and $m = n$, hence $U = M$. $\square$

Proposition 2.4.7. *Let G be a solvable group and M a faithful G-module. Suppose that (G, M) satisfies the condition $\mathcal{N}_{2'}$ and that $G = \mathbf{O}^{q'}(G)$ for some odd prime q. Then M is irreducible and quasi-primitive.*

Proof. The irreducibility of M follows from Lemma 2.4.6. The proof of the fact that M is quasi-primitive is by contradiction. So, assume that M is not quasi-primitive. By [45, Proposition 0.2] there is $C \trianglelefteq N$ such that M_C is not homogeneous and if $M_C = M_1 \oplus \cdots \oplus M_n$ is the decomposition in homogeneous components, then G/C faithfully and primitively permutes $\{M_1 \dots M_n\}$. By hypothesis, there is a prime $q > 2$ such that $G = \mathbf{O}^{q'}(G)$, hence q divides $|\bar{G}|$, where $\bar{G} = G/C$. Since q is odd, from [45, Proposition 9.3] it follows that $q = 3$, $n = 8$, $\bar{G} \simeq A\Gamma(2^3)$ and $C/\mathbf{C}_C(M_i)$ acts transitively on $M_i^{\#}$. Let $v \in M_1$; if $r^a = |M_1|$ we have that $[C : \mathbf{C}_C(v)] = r^a - 1$. Moreover, if $Q \in \mathrm{Hall}_{2'}(G)$ and $r^s = |\mathbf{C}_M(Q)|$, the same argument as in Lemma 2.4.6 gives

$$[G : \mathbf{N}_G(Q)] = \frac{r^{8a} - 1}{r^s - 1} = 1 + r^s + \cdots + r^{(\frac{8a}{s} - 1)s} \geq r^{4a}.$$

Note that s is a divisor of $8a$ by Lemma 2.1.2, hence we get $s \leq 4a$ and this implies that $(8a/s - 1)s \geq 4a$.
Observe that C is a 2-group. Indeed,if there exists an odd prime p such that p divides $|C|$, then (C, M) satisfies the condition $\mathcal{N}_{2'}$ and by Lemma 2.4.6 the module M is irreducible under the action of C and this is a contradiction. Hence C is a 2-group and G is 2-closed since $\bar{G} \simeq A\Gamma(2^3)$. Let $S \in \mathrm{Syl}_2(G)$, so $S \trianglelefteq G$. If $Q \leq \mathbf{C}_G(v)$ for some non-trivial $v \in M$, then Q and $\mathbf{C}_C(v)$ have coprime orders and are normal in $\mathbf{C}_G(v)$. Hence $\mathbf{C}_C(v) \leq \mathbf{C}_S(Q)$. Note furthermore that $[\mathbf{N}_S(Q), Q] \leq S \cap Q = 1$ and thus $\mathbf{N}_S(Q) = \mathbf{C}_S(Q)$. This means that

$$[G : \mathbf{N}_G(Q)] = [S : \mathbf{C}_S(Q)] \leq [S : \mathbf{C}_C(v)] = [S : C][C : \mathbf{C}_C(v)] = 8 \cdot (r^a - 1).$$

In conclusion, we have the inequality

$$r^{4a} \leq 8(r^a - 1),$$

that is equivalent to $r^a(8 - r^{3a}) \geq 8 > 0$, and this is impossible since $(8 - r^{3a}) \leq 0$ for every integer $a \geq 1$ and prime r. $\square$

Proof. (of Proposition 2.4.4) Let q be an odd prime that divides $|G|$ and consider $H = \mathbf{O}^{q'}(G)$ and $K = \mathbf{O}^{q}(H)$. Then q divides $|H|$ and by Proposition 2.4.7 we have that M is an irreducible and quasi-primitive H-module. Hence, by Theorem 2.3.3, one of the following occurs:

1. $K = \mathbf{O}^{q}(H)$ is a cyclic q'-group and $H \leq \Gamma(M)$;

2. MH is isomorphic to $\mathrm{ASL}_2(3)$ or $\mathrm{AGL}_2(3)$;

3. $|M| = 2^6$ and H is the group in the Example 2.3.1.

If 2 holds, then there is nothing to prove. Note that the case 3 cannot occur, since the group in the Example 2.3.1 is not quasi-primitive. Suppose that case 1 holds. If $v \in M^{\#}$, then there exists a Sylow q-subgroup Q such that $Q \leq \mathbf{C}_G(v)$ and $H = QK$. Since M is irreducible, we have that $M = \langle v^H \rangle = \langle v^K \rangle$ and it follows that M is irreducible as K-module. Since K is cyclic, $G \leq \Gamma(M)$ by Theorem 2.2.1. $\square$

Chapter 3

Prime graphs on real degrees

In [42] and [43], Manz classified the groups G in which $\mathrm{cd}(G)$, the set of character degrees of a group G, consists of prime powers. In this chapter, we deal with the same problem, but for real characters.
In Section 3.1 we include some results of the representation theory that takes into account the fields of values.
In Section 3.2 we give some background and technical results in the context of real characters.
In Section 3.3 we discuss non-solvable groups G such that $\mathrm{cd}_{\mathbb{R}}(G)$, the set of degrees of real irreducible characters, consists of prime power numbers. The main Theorem 3.4.4 gives a description of such groups. As a consequence, we get a control on the real degrees of this type of groups, see Corollary 3.3.9. We remark that in this section we use the classification of finite simple groups.
In Section 3.4 we prove that if G is a solvable group such that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime power numbers, then the primes involved are contained in $\{2, p\}$, with p an odd prime. In Section 3.5 we discuss some limits of the study of real characters.

3.1 Character values and Schur-Frobenius indicator

We begin with a brief introduction of the role played by the underlying field in representation theory.

Definition 3.1.1. Let $\mathbb{E}$ be a field and $\mathbb{F}$ a subfield of $\mathbb{E}$. If $\mathfrak{X}$ is a representation over $\mathbb{E}$ of a group G, then we call $\mathfrak{X}$ an $\mathbb{E}$*-representation of* G. If χ is the character of $\mathfrak{X}$ and $\chi(g) \in \mathbb{F}$ for all $g \in G$, we say that χ is an *irreducible* $\mathbb{F}$*-character*. When the ground field is not specified, we assume that $\mathbb{E} = \mathbb{C}$, the field of complex numbers. So, a representation of G is over the complex numbers and we denote $\mathrm{Irr}(G)$ the set of characters of irreducible representations over complex numbers.

Definition 3.1.2. Let G be a group and $\chi \in \mathrm{Irr}(G)$.

1. We say that χ is *real* if χ is an $\mathbb{R}$-character.

2. We say that χ is *rational* if χ is an $\mathbb{Q}$-character.

If G is a group, we denote with $\mathrm{Irr}_{\mathbb{R}}(G)$ the set of real irreducible characters of G. Let $\mathbb{K} = \mathbb{Q}(\varepsilon)$, where ε is a $|G|$-th root of unity. It is known that every irreducible character of G can be afforded by a representation over $\mathbb{K}$. Call $\mathcal{G}$ the Galois group of the field extension $\mathbb{K}|\mathbb{Q}$. Then $\mathcal{G}$ acts on $\mathrm{Irr}(G)$ and the rational characters are the $\mathcal{G}$-fixed points of the action of $\mathrm{Irr}(G)$. Analogously, $\mathrm{Irr}_{\mathbb{R}}(G)$ is the set of irreducible characters fixed the complex conjugation on $\mathrm{Irr}(G)$.
If $\mathbb{F}$ is a subfield of a field $\mathbb{E}$ and $\mathfrak{Y}$ is a $\mathbb{F}$-representation of a group G, we may view $\mathfrak{Y}$ as an $\mathbb{E}$-representation. As such, we denote it with $\mathfrak{Y}^{\mathbb{E}}$.

Definition 3.1.3. [34, Definition 10.1] Let G be a group and $\mathbb{F}$ a subfield of a splitting field (see [34, Definition 9.3]) $\mathbb{E}$ of G. Let χ an irreducible $\mathbb{E}$-character and $\mathfrak{X}$ an $\mathbb{E}$-representation which affords G. If $\mathfrak{Y}$ is an $\mathbb{F}$-representation of G such that $\mathfrak{X}$ is a constituent of $\mathfrak{Y}^{\mathbb{E}}$, the multiplicity of $\mathfrak{X}$ as a constituent of $\mathfrak{Y}^{\mathbb{E}}$ is the *Schur index* of χ over $\mathbb{F}$. It is denoted by $m_{\mathbb{F}}(\chi)$.

Let $\chi \in \mathrm{Irr}(G)$ and $\mathbb{F}$ be a subfield of the complex numbers. Suppose that χ is an $\mathbb{F}$-character. Then, it may happen that χ is not afforded by any $\mathbb{F}$-representation defined on $\mathbb{F}$. A useful tool that helps us with this problem, in the case where $\mathbb{F} = \mathbb{R}$, is the *Frobenius-Schur indicator* ν_2, that is defined for a irreducible character χ as follows:

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

The Schur-Frobenius indicator is a function $\nu_2 \colon G \to \{-1, 0, 1\}$ that can be computed from the character table and, as the following theorem shows, allows us to understand which real irreducible character of G is afforded by a real representation.

Theorem 3.1.4. *[34, pag. 58] Let G be a group and $\chi \in \mathrm{Irr}(G)$. Then*

$$\nu_2(\chi) = \begin{cases} 1 & \textit{iff } \chi \textit{ is real and afforded by a real representation;} \\ -1 & \textit{iff } \chi \textit{ is real but not afforded by any real representation;} \\ 0 & \textit{iff } \chi \textit{ is not real.} \end{cases}$$

Consider $G = Q_8$ the quaternion group with 8 elements and $\chi \in \mathrm{Irr}(G)$ the only irreducible character of degree 2. As the complex conjugation acts on $\mathrm{Irr}(G)$, we have that χ is real. Nevertheless, it is not afforded by any real representation. Indeed, by direct calculation we have $\nu_2(\chi) = -1$.

We remark that this problem exists only for characters defined on fields of characteristic 0. Suppose $\mathbb{E}$ is a field of prime characteristic and $\mathfrak{X}$ is an absolutely irreducible (see Definition 9.1 of [34]) $\mathbb{E}$-representation of G. If the character of $\mathfrak{X}$ takes values on a subfield $\mathbb{F} \subseteq \mathbb{E}$ then the representation $\mathfrak{X}$ can be realized over $\mathbb{F}$, see [34, Theorem 9.14].

3.2 Preliminaries

Definition 3.2.1. Let G be a group. For character theory, we assume the notation of [34]. In particular, $\mathrm{Lin}(G) = \mathrm{Irr}(G/G')$ is the set of linear irreducible characters of G. Moreover, if π is a set of primes, $\mathrm{Irr}_\pi(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) \text{ is a } \pi\text{-number } \}$ and $\mathrm{cd}_\pi(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_\pi(G)\}$. We add $\mathbb{R}$ as a subscript to indicate that we are considering only real characters. So, for example,

1. $\mathrm{Irr}_\mathbb{R}(G \mid H) = \{\chi \in \mathrm{Irr}_\mathbb{R}(G) \mid [\chi_H, \phi] \neq 0 \text{ for some } \phi \in \mathrm{Irr}(H)\}$,
2. $\mathrm{cd}_\mathbb{R}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_\mathbb{R}(G)\}$,
3. $\mathrm{Irr}_{\mathbb{R},\pi}(G) = \mathrm{Irr}_\mathbb{R}(G) \cap \mathrm{Irr}_\pi(G)$,
4. $\mathrm{cd}_{\mathbb{R},\pi}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_{\mathbb{R},\pi}(G)\}$.

One of the first results that appeared in the context of real character degrees, is due to Chillag and Mann [13], in which are classified the groups where all real irreducible characters are linear.

Theorem 3.2.2. *[13, Theorem 1.1] Let G be a group. Suppose that $\mathrm{cd}_\mathbb{R}(G) = \{1\}$. Then $G = O \times T$ where O is a group of odd order and T is a 2-group.*

In [13], groups such that $\mathrm{cd}_\mathbb{R}(G) = \{1\}$ are called "nearly odd". Due to the importance of their classification, nowadays these groups are known as of *Chillag-Mann type.*

If G is a group of Chillag-Mann type, then $G = O \times T$ where $O \in \mathrm{Hall}_{2'}(G)$ and $T \in \mathrm{Syl}_2(G)$. Since T is a direct factor of G, also T is a group of Chillag-Mann type. So, since $\mathrm{Irr}_\mathbb{R}(O) = \{1_O\}$ as O has odd order, the study of groups of Chillag-Mann type reduces to the study of 2-groups of Chillag-Mann type. If T is a 2-group of Chillag-Mann type, one important feature is that $\mathrm{Irr}_\mathbb{R}(T) = \mathrm{Irr}_\mathbb{R}(T/\Phi(T))$.

Lemma 3.2.3. *Let T be a 2-group and λ a linear real irreducible character of T. Then $\Phi(T) \leq \ker \lambda$.*

Proof. If λ is a linear irreducible real character of T, λ is a faithful character of $T/\ker\lambda$ that is a cyclic group of order n where $n = o(\lambda)$. Hence the values of λ coincide with the complex numbers that are complex n-roots of unity and these are real only if $n = 2$. This means that $[T : \ker\lambda] = 2$ and $\Phi(T) \leq \ker\lambda$. $\square$

Corollary 3.2.4. *Let T be a 2-group of Chillag-Mann type, then*

$$\mathrm{Irr}_{\mathbb{R}}(T) = \mathrm{Irr}_{\mathbb{R}}(T/\Phi(T))$$

The 2-groups of Chillag-Mann type are characterized in [13, Proposition 4.1]. Let T be a group and $t \in T$. If there is $s \in T$ such that $s^2 = t$, we say that t is a *square* and s is a *square root* of t. Suppose that T is a 2-group and T^2 the subgroup of T generated by squares of the elements of T. Then, if T is a 2-group of Chillag-Mann type, then every element of T^2 is a square and the number of square roots is constant.

We recall the definition of the prime graph on real degrees.

Definition 3.2.5. Let G be a group. Let $\Delta_{\mathbb{R}}(G)$ be the graph whose vertex-set is $\mathcal{V}_G = \pi(\mathrm{cd}_{\mathbb{R}}(G))$, the set of primes that divide some element in $\mathrm{cd}_{\mathbb{R}}(G)$, and two vertices $p, q \in \mathcal{V}_G$ are adjacent if and only if $pq4$ divide some element in $\mathrm{cd}_{\mathbb{R}}(G)$. The graph $\Delta_{\mathbb{R}}(G)$ is called *the prime graph of* $\mathrm{cd}_{\mathbb{R}}(G)$.

If Γ is a graph, we denote $n(\Gamma)$ the number of connected components of Γ.

Theorem 3.2.6. *[17, Theorem 5.1] Let G be a group. Then*

1. *$n(\Delta_{\mathbb{R}}(G)) \leq 3$;*
2. *$n(\Delta_{\mathbb{R}}(G)) \leq 2$ if G is solvable.*

We include two results that are real versions of the Ito-Michler Theorem. The first is for the prime $p = 2$.

Theorem 3.2.7. *[17, Theorem A] Let G be a group and $T \in \mathrm{Syl}_2(G)$. Then $\mathrm{cd}_{\mathbb{R}}(G)$ consists of odd numbers if and only if $T \trianglelefteq G$ and T is of Chillag-Mann type.*

We mention that a version of Theorem 3.2.7 for rational characters appears in [56].
For p odd, Theorem 3.2.7 is no longer true. Nevertheless, there is the following partial result.

Theorem 3.2.8. *[55, Theorem A] Let G be a group and p be a prime. Suppose that $p \nmid \chi(1)$ for every $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ with $\nu_2(\chi) = 1$. Then $\mathbf{O}^{p'}(G)$ is solvable; in particular, G is p-solvable.*

Note that in [55] appears also a version of Theorem 3.2.8 for real Brauer characters.

The following is a real version of Thompson's Theorem for the prime $p = 2$. For p odd, see the recent work [53].

Theorem 3.2.9. *[49, Theorem A] Let G be a group. Then every real non-linear irreducible character of G has even degree if and only if G has a normal 2-complement.*

A special case of the previous theorem is when $\mathrm{cd}_{\mathbb{R}}(G)$ consists of 2-powers.

Theorem 3.2.10. *[49, Theorem C] Let G be a group. Then $\mathrm{cd}_{\mathbb{R}}(G)$ consists in 2-powers if and only if G has a normal 2-complement K and $[K', T] = 1$ for $T \in \mathrm{Syl}_2(G)$.*

One important result that deals with fields of values is the following theorem.

Theorem 3.2.11. *Let G be a group and $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ of odd degree. Then χ is afforded by a real representation. If furthermore G is solvable, then there is $H \leq G$ and $\lambda \in \mathrm{Lin}(H)$ such that $o(\lambda) = 2$ and $\chi = \lambda^G$.*

Proof. Note that, by Brauer-Speiser Theorem (see page 171 of [34]), we have that $m_{\mathbb{Q}}(\chi) \leq 2$. It follows that $m_{\mathbb{Q}}(\chi) = 1$ as it divides $\chi(1)$ (see [34, 10.2h)]). This means that χ is afforded by a rational representation. If furthermore G is solvable, by Gow's Theorem [24], χ is induced by a linear character $\lambda \in \mathrm{Irr}(H)$, with $H \leq G$ and $o(\lambda) = 2$. □

3.2.1 Real characters and normal subgroups

This section is dedicated to the study of behaviour of real characters with respect to normal subgroups.

Lemma 3.2.12. *Let G be a group and $N \trianglelefteq G$. If $[G : N]$ is odd and $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$, then every irreducible constituent of χ_N is real.*

Proof. The complex conjugation acts on the irreducible constituents of χ_N. These are odd in number, because $[G : N]$ is odd. Therefore, there is one irreducible constituent of χ_N that is real. Since all the irreducible constituents of χ_N are G-conjugated, they are all real. □

If $\theta \in \mathrm{Irr}_{\mathbb{R}}(N)$, then, in general, it is not possible to find a real character $\chi \in \mathrm{Irr}(G)$ over θ. The next lemmas provide some conditions in which this problem is under control.

Lemma 3.2.13. *Let G be a group and $N \trianglelefteq G$. Suppose that $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ and $\theta \in \mathrm{Irr}_{\mathbb{R}}(N)$. Then the following hold.*

1. *Assume that $|N|$ is odd. If either $\chi(1)$ is odd or N centralizes a Sylow 2-subgroup of G, then $N \leq \ker(\chi)$.*

2. *If $[G : N]$ is odd, then θ has a unique real-valued extension $\hat{\theta} \in \mathrm{Irr}_{\mathbb{R}}(\mathrm{I}_G(\theta))$ and $\hat{\theta}^G \in \mathrm{Irr}_{\mathbb{R}}(G \mid \theta)$. Furthermore, $o(\theta) = o(\hat{\theta})$.*

3. *Suppose that $\theta(1)$ is odd and $o(\theta) = 1$. Then θ extends to $\hat{\theta} \in \mathrm{Irr}_{\mathbb{R}}(\mathrm{I}_G(\theta))$ and $\hat{\theta}^G \in \mathrm{Irr}_{\mathbb{R}}(G \mid \theta)$.*

Proof. Parts 2 and 3 are Lemma 2.1, 2.2 and 2.3 of [51]. Remains to prove part 1. If N has odd order and centralizes a Sylow 2-subgroup, then $N \leq \ker \chi$ by [20, Lemma 1.4]. Suppose that $\chi(1)|N|$ is odd. By Lemma 3.2.12, all the constituents of χ_N are real. Now, as N has odd order, $\mathrm{Irr}_{\mathbb{R}}(N) = \{1_N\}$, therefore $N \leq \ker \chi$. □

The following lemma is useful.

Lemma 3.2.14. *[20, Lemma 1.13] Let G be a group and suppose that $G = A \times B$, where $|A| = p$ is a prime. The number of complements for A in G is a power of p.*

Lemma 3.2.15. *Let G be a group that acts by automorphisms on the group M. For every involution $x\mathbf{C}_G(M) \in G/\mathbf{C}_G(M)$ there exists a non-trivial character $\mu \in \mathrm{Irr}(M)$ such that $\mu^x = \bar{\mu}$.*

Proof. Follow the proof of [20, Lemma 1.6]. □

Lemma 3.2.16. *[20, Lemma 1.7] Let G be a group, H a subgroup of G, ψ a character of H, and $x \in \mathbf{N}_G(H)$ such that $\psi^x = \bar{\psi}$. Then ψ^G is a real-valued character of G.*

Lemma 3.2.17. *Let G be a group and $M \trianglelefteq G$ of odd order. Suppose that there exists a 2-element $x \in G$ and a non-principal $\lambda \in \mathrm{Lin}(M)$ such that $\lambda^x = \bar{\lambda}$. If M has a complement in $I = \mathrm{I}_G(\lambda)$, then λ extends to $\hat{\lambda} \in \mathrm{Irr}_{\mathbb{R}}(I)$ such that $\hat{\lambda}^x = \bar{\hat{\lambda}}$. Moreover, either*

1. *I is 2-nilpotent and $[G : I]$ is a 2-power or*

2. *there exists $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ such that $\chi(1)$ is a multiple of $2r$, with $r > 2$ a prime number.*

Proof. Consider a 2-element $x \in G$ and non-principal $\lambda \in \mathrm{Lin}(M)$ such that $\lambda^x = \bar{\lambda}$. Let $\bar{I} = I/\ker \lambda$. From elementary character theory, we have that $\bar{M} \leq \mathbf{Z}(\bar{I})$. Let I_0 be a complement for M in I. It follows that $\bar{I} = \bar{M} \times \bar{I}_0$. By Lemma 3.2.14, the number of complements of $\bar{M}$ in $\bar{I}$ is odd and x permutes them, as x normalizes I, M and $\ker \lambda$. Therefore, since x is a 2-element, we can assume that $\bar{I}_0$ is normalized by x. We can see λ as a character of $\bar{M}$; so, consider $\hat{\lambda} = \lambda \times 1_{\bar{I}_0} \in \mathrm{Irr}(\bar{I} \mid \lambda)$. It follows that

$$\hat{\lambda}^x = \lambda^x \times 1_{\bar{I}_0^x} = \bar{\lambda} \times 1_{\bar{I}_0} = \bar{\hat{\lambda}}.$$

Assume that part 2 is not true. Note that $\hat{\lambda}^G \in \mathrm{Irr}_{\mathbb{R}}(G)$ and 2 divides $\chi(1)$ as 2 divides $[G : I]$. This means that $\chi(1)$ is a non-trivial 2-power and it follows that $[G : I]$ is a 2-power. Suppose by contradiction that I is not 2-nilpotent. Call $J = I\langle x\rangle$; if the group J is 2-nilpotent, then also I is 2-nilpotent, since x is a 2-element. Hence J is not 2-nilpotent. By Theorem 3.2.9, there is $\tau \in \mathrm{Irr}_{\mathbb{R}}(J)$ such that $2 \nmid \tau(1)$ and $\tau(1) > 1$; take r an odd prime number that divides $\tau(1)$. Note that $I \trianglelefteq J$ and if ϕ is an irreducible constituent of τ_I, then $\tau(1)/\phi(1)$ divides $[J : I]$ by Lemma 3.2.27, but $[J : I]$ is a 2-power, while $\tau(1)$ is odd. This means that $\tau_I = \phi \in \mathrm{Irr}_{\mathbb{R}}(I)$ and that θ is J-invariant. Since M has odd order and ϕ has odd degree, by part 1 of 3.2.13 Lemma we have that $M \leq \ker\phi$ and $\phi \in \mathrm{Irr}_{\mathbb{R}}(I/M)$. By Gallagher's Theorem we have that $\hat{\lambda}\phi \in \mathrm{Irr}(I \mid \lambda)$ and $(\hat{\lambda}\phi)^x = \hat{\lambda}^x\phi = \overline{\hat{\lambda}\phi}$. So, by Clifford's Theorem and Lemma 3.2.16, we have that $\psi = (\hat{\lambda}\phi)^G \in \mathrm{Irr}_{\mathbb{R}}(G)$. Now, 2 divides $\psi(1)$ because $x \notin I$ and 2 divides $[G : I]$. On the other hand $r \mid \phi(1)$ and $\phi(1)$ divides $\psi(1)$. A contradiction. □

Lemmas 3.2.15 and 3.2.17 are often used together.
The situation where M is a 2-group is considered in the following Lemma.

Lemma 3.2.18. *Let G be a group and $M \trianglelefteq G$. Let $\lambda \in \mathrm{Lin}(M)$ be such that $o(\lambda) = 2$. If M has a complement in $\mathrm{I}_G(\lambda) = I$, then λ extends to $\hat{\lambda} \in \mathrm{Irr}_{\mathbb{R}}(I)$.*

Proof. Firstly observe that λ is real. Let I_0 be a complement for M. Let $\bar{I} = I/\ker\lambda$, so $\hat{\lambda} = \lambda \times 1_{\bar{I}_0}$ is an irreducible real character of $\bar{I}$, where λ can be viewed as a faithful irreducible character of $\bar{M}$. Now we can lift $\hat{\lambda}$ to a character of I, which we call again $\hat{\lambda}$. □

For completeness, we include the following result, that uses the same techniques of the lemmas above.

Lemma 3.2.19. *[49, Lemma 2.2] Suppose that a cyclic 2-group $\langle\sigma\rangle$ acts by automorphisms on a group K and let $L \trianglelefteq K$ be $\langle\sigma\rangle$-invariant, with K/L of odd order. Let $\theta \in \mathrm{Irr}(L)$ such that $\theta^\sigma = \bar{\theta}$, then there exists $\mu \in \mathrm{Irr}(K)$ over θ such that $\mu^\sigma = \bar{\mu}$.*

Let G be a group and $N \trianglelefteq G$. A known technique that may be useful when we want to find a real irreducible character of G over $\theta \in \mathrm{Irr}_{\mathbb{R}}(N)$, is the *tensor induction*. See [33, Section 4] for further details.

Lemma 3.2.20. *Let N be a minimal normal subgroup of a group G, $N = S_1 \times \cdots \times S_n$ where $S \simeq S$ is a non-abelian simple group. Call $A = \mathrm{Aut}(S)$. Let $\sigma \in \mathrm{Irr}_{\mathbb{R}}(S)$ and suppose that σ extends to a real character of A. Then $\sigma \times \cdots \times \sigma$ extends to a real character of G.*

Proof. We follow the proof of [7, Lemma 5]. Let $\theta \in \mathrm{Irr}_{\mathbb{R}}(\mathrm{Aut}(S))$ that extends σ and T a transversal for N in G. If $g \in G$ and $t \in T$, then call $t \cdot g$ the element of T such that $tg \in Nt \cdot g$. This induces a permutation of g on T. Call $s(t)$ the size of the g-orbit containing t. Then

$$\chi(g) = \prod_{t \in T} \theta(tg^{s(t)}t^{-1})$$

is a real irreducible character of G. □

3.2.2 Central extensions

In this subsection we briefly discuss some results about the theory of central extensions, Schur multipliers and character triples.

Definition 3.2.21. If G is a group and M is a G-module, we denote $H^2(G, M)$ the second cohomology group.

Definition 3.2.22. Let G be a group. Then $\mathrm{M}(G) = H^2(G, \mathbb{C})$ is the *Schur multiplier* of G.

Lemma 3.2.23. *[34, Corollary 11.20] Let G, Γ be two groups and $\pi\colon \Gamma \to G$ a surjective homomorphism such that $A = \ker \pi \leq \mathbf{Z}(\Gamma)$. If $A \leq \Gamma'$ then A is isomorphic to a subgroup of $\mathrm{M}(G)$.*

Definition 3.2.24. Let G be a group and $N \trianglelefteq G$. Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then (G, N, θ) is called a *character triple*.

In the following lemma, we use the concept of *isomorphism of character triples* and we assume the notation of Definition 11.23 of [34].

Lemma 3.2.25. *[34, Lemma 11.24] Let $(\tau, \sigma)\colon (G, N, \theta) \to (\Gamma, M, \phi)$ an isomorphism of character triples. For all $N \leq H \leq G$ the map $\sigma_H\colon \mathrm{Ch}(H \mid \theta) \to \mathrm{Ch}(H^\tau \mid \phi)$ is a bijection and for all $\chi \in \mathrm{Ch}(H \mid \theta)$*

$$\frac{\chi(1)}{\theta(1)} = \frac{\chi^\tau(1)}{\phi(1)}$$

Theorem 3.2.26. *[34, Theorem 11.28] Let (G, N, θ) be a character triple and let (Γ, M) the Schur covering of G/N. There there is $\lambda \in \mathrm{Irr}(M)$ such that (G, N, θ) is isomorphic to (Γ, M, λ).*

Suppose that (G, N, θ) is a triple isomorphism. Suppose that $M = \mathrm{M}(G/N)$ is the Schur multiplier of G/N and assume that $|M| = 2$. Consider $(\tau, \sigma)\colon (G, N, \theta) \to (\Gamma, M, \lambda)$, where Γ is the Schur covering of G/N and M. In general, if $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ is real, then χ^τ is not necessarily real. Nevertheless, if χ is the unique character with a certain degree, by Lemma 3.2.25 χ^τ is real. A stronger result appears in [25, Theorem 3.6], under the assumption that G/N is perfect.
The next Lemma is quite useful in the following situation. Suppose that N is a normal subgroup of the group G and $\chi \in \mathrm{Irr}(G)$. If $(\chi(1), [G : N]) = 1$ then the restriction χ_N is an irreducible character of N and we are in the position to apply Gallagher's Theorem.

Lemma 3.2.27. *[34, Corollary 11.29] Let G be a group and N a normal subgroup. Suppose that $\chi \in \mathrm{Irr}(G)$ and θ an irreducible constituent of χ_N. Then*

$$\frac{\chi(1)}{\theta(1)} \mid [G : N]$$

We include one last Lemma in the context of central extensions.

Lemma 3.2.28. *[34, Exercise 11.18] Let $G = CH$ with C, H cyclic groups with $C \trianglelefteq G$ and $H \cap C = \mathbf{Z}(G)$. Then $\mathrm{M}(G) = 1$.*

For a modern exposition of the theory of character triples, see Chapter 5 of [47].

3.3 Non-solvable groups whose real degrees are prime powers

In this section, we describe non-solvable groups such that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime-power numbers. Non-solvable groups such that the degree of *all* character degrees are prime power numbers are characterized in the main result of [42], although with different techniques. The main Theorem is 3.3.8. The results of this section appear in [8] and [14].

Definition 3.3.1. If G is a group, we denote $\mathrm{Rad}(G)$ the *solvable radical* of G, namely the largest solvable normal subgroup of G.

We recall that $\pi(G)$ denotes the set of primes that divide $|G|$.

Theorem 3.3.2. *Let G be a group with $\mathrm{Rad}(G) = 1$. Suppose that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime-power numbers, then G is isomorphic to A_5 or $\mathrm{L}_2(8)$.*

Proof. Let M be a minimal normal subgroup of G. Then $M = S_1 \times \cdots \times S_n$ is the product of simple groups, all isomorphic to a simple group S. Since $\mathrm{Rad}(G) = 1$, the group S is non-abelian.

Step 1: S is isomorphic to one of the following groups

$$A_5, A_6, \mathrm{L}_2(8), \mathrm{L}_3(3), PSp_4(3), \mathrm{L}_2(7), PSU_3(3), \mathrm{L}_2(17).$$

Let $p \in \pi(M)$. If M is not contained in $\mathbf{O}^{p'}(G)$, then $M \cap \mathbf{O}^{p'}(G) = 1$ by minimality. In this case, p would divide $|G/\mathbf{O}^{p'}(G)|$ since p divides $|M|$, and this impossible. Therefore $M \leq \mathbf{O}^{p'}(G)$, so $\mathbf{O}^{p'}(G)$ is not solvable. By Theorem 3.2.8 there is a real irreducible character χ of G such that $p \mid \chi(1)$. By the hypothesis, $\chi(1)$ is a non-trivial p-power. This means that for every prime $p \in \pi(M)$, there is $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ such that $\chi(1)$ is a non-trivial p-power. By Theorem 3.2.6, the number of connected components of $\Delta_{\mathbb{R}}(G)$ is at most 3. In our hypothesis, $\Delta_{\mathbb{R}}(G)$ consists of isolated vertices and hence the number of primes that appear as divisors of some real irreducible character, is at most 3. It follows by Burnside Theorem that M, and hence S, is divisible by at most (and hence exactly) 3 primes. Now, by Lemma 2.1 in [61], the simple groups having order divided by exactly 3 distinct primes are those stated.

Step 2: S is isomorphic to one of the following groups: $A_5, \mathrm{L}_2(8), A_6$.

If $S \in \{PSp_4(3), \mathrm{L}_3(3), PSU_3(3)\}$ then there is a non-linear character $\sigma \in \mathrm{Irr}_{\mathbb{R}}(S)$ such that $\sigma(1)$ is an odd composite number. Let $\theta = \sigma \times \cdots \times \sigma \in \mathrm{Irr}_{\mathbb{R}}(M)$. Then $2 \nmid \theta(1)$ and $o(\theta) = 1$ since M is perfect. So, by part 3 of Lemma 3.2.13, there is $\chi \in \mathrm{Irr}_{\mathbb{R}}(G \mid \theta)$. As $\theta(1)$ divides $\chi(1)$, the degree of χ is a composite number, against the hypothesis.

Suppose that $S \in \{\mathrm{L}_2(7), \mathrm{L}_2(17)\}$. There is a real character $\sigma \in \mathrm{Irr}_{\mathbb{R}}(S)$ such that $\sigma(1)$ is a composite number and σ extends to a real character of $\mathrm{Aut}(S)$. By Lemma 3.2.20 the character $\theta = \sigma \times \cdots \times \sigma$ extends to a real character $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$. Again $\chi(1) = \theta(1) = \sigma(1)^n$ is a composite number.

Step 3: $n = 1$ and M is a simple group.

The only left possibilities are $S \in \{A_5, \mathrm{L}_2(8), A_6\}$. Checking the character table of these groups, we see that there are two non-linear characters $\sigma, \rho \in \mathrm{Irr}_{\mathbb{R}}(S)$ such that $\sigma(1)$ is a non-trivial p-power and $\rho(1)$ is a non-trivial q-power, where p, q are odd primes. Let $\theta = \sigma \times 1 \times \cdots \times 1 \in \mathrm{Irr}_{\mathbb{R}}(M)$. Since $o(\theta) = 1$ and $\theta(1)$ is odd, the character θ extends to a character $\varphi \in \mathrm{Irr}_{\mathbb{R}}(\mathrm{I}_G(\theta))$ by part 3 of Lemma 3.2.13 and $\chi = \varphi^G$ has p-power degree, hence $[G : \mathrm{I}_G(\theta)] = p^k$ for $k > 1$. Since

$\mathrm{I}_G(\theta) \leq \mathbf{N}_G(S_1)$, we have that

$$n = [G : \mathbf{N}_G(S_1)] \text{ divides } [G : \mathrm{I}_G(\theta)] \text{ that is a non-trivial } p\text{-power},$$

so n is a p-power. By the same argument with ρ in place of σ, n divides q^m for some $m > 1$. Therefore, we get that $n \mid (p^k, q^m) = 1$. Hence $M = S_1 = S$.

Step 4: $\mathbf{C}_G(M) = 1$.

Suppose, by contradiction, that $\mathbf{C}_G(M)$ is non-trivial and take N a minimal normal subgroup of G contained in $\mathbf{C}_G(M)$. For the same arguments used on M, we have that N is simple and is isomorphic to one of the following groups: $A_5, \mathrm{L}_2(8), A_6$. As before, take $\sigma \in \mathrm{Irr}_{\mathbb{R}}(M)$ with $\sigma(1)$ a p-power and $\rho \in \mathrm{Irr}_{\mathbb{R}}(N)$ with $\rho(1)$ a q-power for odd primes p, q. Note that $[M, N] \leq M \cap N \leq M \cap \mathbf{C}_G(M) = 1$ since M is simple. So $MN = M \times N$ is perfect and normal in G and $\sigma \times \rho \in \mathrm{Irr}_{\mathbb{R}}(MN)$ is a character such that $o(\sigma \times \rho) = 1$. Note that $2 \nmid (\sigma \times \rho)(1)$. By part 3 of Lemma 3.2.13 there is $\chi \in \mathrm{Irr}_{\mathbb{R}}(G \mid \sigma \times \rho)$, and this is impossible since $\chi(1)$ is a composite number.

Conclusion: we proved so far $S \leq G \leq \mathrm{Aut}(S)$ and

$$S \in \{A_5, A_6, \mathrm{L}_2(8)\}.$$

Now, if $S \simeq A_6$ then $\mathrm{Aut}(S)/S$ is isomorphic to the Klein group and every of the five subgroups between S and $Aut(S)$ have a rational character of degree 10, and this is impossible. So $S \in \{A_5, \mathrm{L}_2(8)\}$. In any of these cases, $[Aut(S) : S]$ is a prime number, this means that in $\mathrm{Aut}(S)$ there is only one subgroup strictly above S, namely $Aut(S)$ itself. But both $\mathrm{Aut}(A_5)$ and $\mathrm{Aut}(\mathrm{L}_2(8))$ have a real character with degree a non prime-power number. Hence $G = A_5$ or $G = \mathrm{L}_2(8)$. $\square$

The proof of the previous theorem uses the classification of finite simple group. Moreover, specific information about simple groups, their character degrees and automorphism groups is obtained using the software `GAP`.

Table 3.1: Maximal subgroups of A_5.

A_4	D_{10}	S_3
12	10	6
5	6	10

For completeness, we list the maximal subgroups of A_5 and $\mathrm{L}_2(8)$. In the first row there is the order of the maximal subgroup and in the second there is the

Table 3.2: Maximal subgroups of $\mathrm{L}_2(8)$.

F_{56}	D_{18}	D_{14}
56	18	14
9	28	72

index in the surrounding group.
The Schur multipliers of simple groups are known. In particular, $|\mathrm{M}(A_5)| = 2$ and $\mathrm{M}(\mathrm{L}_2(8)) = 1$.

Definition 3.3.3. Let G be a group. Then, $G^{(\infty)}$ indicates the last term of the derived series of a group G.

Theorem 3.3.4. *Let G be a non-solvable group such that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime-power numbers. Then $G = KR$ with $R = \mathrm{Rad}(G)$ and $K = G^{(\infty)}$. Moreover $K \cap R = L$ is a 2-group and K/L is isomorphic to A_5 or $\mathrm{L}_2(8)$.*

Proof. Let $K = G^{(\infty)}$ be the last term of the derived series of G and $\bar{G} = G/L$. Observe that the hypotheses are preserved under quotients. Hence, by Theorem 3.3.2, G/R is a simple group and since $1 < KR/R \trianglelefteq G/R$, we have that $G = KR$ and $\bar{K} \simeq G/R$ is isomorphic to A_5 or $\mathrm{L}_2(8)$. Moreover, $\bar{G} = \bar{K} \times \bar{R}$ because $[K, R] \leq L$.
Suppose by contradiction that $\bar{R}$ has a real character θ of non trivial degree. By Theorems 3.2.9 and 3.2.7, there are two non-linear characters $\phi_1, \phi_2 \in \mathrm{Irr}_{\mathbb{R}}(\bar{K})$ such that $\phi_1(1)$ is even and $\phi_2(1)$ is odd. If $\theta(1)$ is odd, consider $\chi = \theta\phi_1$ and if $\theta(1)$ is even, consider $\chi = \theta\phi_2$. In any case, χ can be lifted to a real character of G and $\chi(1) = (\theta\phi_i)(1) = \theta(1)\phi_i(1)$ is a composite number, but this is impossible. It follows that every real character of R/L is linear and by Theorem 3.2.2, $\bar{R} = \bar{O} \times \bar{H}$ where $O \in \mathrm{Hall}_{2'}(R)$ and $H \in \mathrm{Syl}_2(R)$. Write G_0 for the preimage in G of $\bar{KH}$. Note that G_0 is a normal subgroup of odd index. Note that $G_0 = LKH = KH$. If there is a real character $\phi \in \mathrm{Irr}_{\mathbb{R}}(G_0)$ such that $\phi(1)$ is a composite number, then, by part 2 of Lemma 3.2.13, there is a real character $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ above ϕ, since G_0 has odd index in G. This produces a contradiction, because $\phi(1)$ divides $\chi(1)$. Hence we can assume that $G = G_0$ and $\bar{O} = 1$.

Suppose, working by contradiction, that $O > 1$, namely L is not a 2-group. Consider M/M_0 the first term of a descending principal series of L that is not a 2-group, hence $M, M_0 \trianglelefteq G$, M/M_0 is an elementary abelian p-group for p odd and L/M is a 2-group. Replacing G with G/M_0, we can assume that $M_0 = 1$ and M is a minimal normal subgroup of G.
Since K/L is simple, $\mathbf{C}_K(M) = K$ or $\mathbf{C}_K(M) \leq L$. If $\mathbf{C}_K(M) = K$, then M has a

direct complement N in L and consider $\bar{K} = K/N$. Note that $1 < \bar{M} \leq \mathbf{Z}(\bar{K}) \cap \bar{K}'$, since $K = K'$ is perfect and hence $|M|$ divides $|\mathrm{M}(G)|$ by Lemma 3.2.23. This is impossible, since $|\mathrm{M}(A_5)| = 2$ and $\mathrm{M}(\mathrm{L}_2(8)) = 1$.

Hence $\mathbf{C}_K(M) \leq L$ and the action of K on M is non-trivial. Moreover K/L is a non-abelian simple group, so it has even order. By Lemma 3.2.15 there exists an element $\lambda \in \hat{M}$ and $x \in K$ such that $\lambda^x = \bar{\lambda}$. Let $I = \mathrm{I}_G(\lambda)$ and note that x is an element of even order that normalizes I but $x \notin I$, so 2 divides $[G : I]$.

Let $\tilde{I} = I/\ker\lambda$, observe that $\tilde{M} \leq \mathbf{Z}(\bar{I})$. Take $P \in \mathrm{Syl}_p(I)$; since the index of K in G is a 2-power, every subgroup of G with odd order is contained in K; it follows that $P \leq K$. Moreover, $\tilde{M} \leq \mathbf{Z}(\tilde{P})$, $\tilde{P} \in \mathrm{Syl}_p(\tilde{I})$ and PL/L is a p-subgroup of the simple group K/L, that is isomorphic to A_5 or $\mathrm{L}_2(8)$. Now, if p is an odd prime, every Sylow p-subgroup of A_5 or $\mathrm{L}_2(8)$ is cyclic (see tables 3.1 and 3.2). Hence, $P/M \simeq \bar{P}/\tilde{M} \simeq PL/L$ is cyclic and $\bar{P}$ is abelian.

Since $\bar{M} \leq \mathbf{Z}(\bar{I})$, we have that $\bar{M} \not\leq \bar{I}'$ by [35, Theorem 5.3]. In addition $\bar{M} \cap \bar{I}' = 1$ because $\bar{M}$ has order p. Write $\bar{I}/\bar{I}' = Q \times B$, where $B \in \mathrm{Hall}_{p'}(\bar{I}/\bar{I}')$ and $Q \in \mathrm{Syl}_p(\bar{I}/\bar{I}')$. Note that Q and B are x-invariant, as x normalizes I. By abuse of notation, we write $M \leq Q$ in place of $\bar{M}\bar{I}'/\bar{I}' \leq Q$. In this notation M is a group of order p and λ is a faithful character of M. The 2-group $\langle x \rangle$ acts on the abelian group Q, hence by Maschke's Theorem [38, 8.4.6] there is an $\langle x \rangle$-invariant complement T for the $\langle x \rangle$-invariant subgroup M, so $Q = M \times T$. Let $\hat{\lambda} = \lambda \times 1_T \in \mathrm{Irr}(Q)$ and $\delta = \hat{\lambda} \times 1_B \in \mathrm{Irr}(\bar{I}/\bar{I}')$, we have that

$$\delta^x = \hat{\lambda}^x \times 1_{B^x} = (\lambda^x \times 1_{T^x}) \times 1_B = (\bar{\lambda} \times 1_T) \times 1_B = \bar{\delta}.$$

We return to the previous notation, so δ lifts to a character of I, which we call again δ. Note that $I < G$ as 2 divides $[G : I]$.

If $IH < G$, then IH/H is a proper subgroup of G/H that is a simple group isomorphic to A_5 or $\mathrm{L}_2(8)$. The maximal subgroups of these two groups are known as well as their indexes, see tables 3.1 and 3.2. In particular, there always is an odd prime q such that q divides $[G : IH]$ and hence $2q$ divides $[G : I]$. Note that $\delta \in \mathrm{Irr}(I \mid \lambda)$, so $\chi = \delta^G \in \mathrm{Irr}(G)$. Moreover, by Lemma 3.2.16, we have that χ is real. So, $2q$ divides $\chi(1)$ as $2q$ divides $[G : I]$, and this is impossible.

Suppose now $IH = G$. In this case, $I/I \cap H \simeq G/H$ that is isomorphic to A_5 or $\mathrm{L}_2(8)$. Each of these two groups have a unique rational character ϕ of odd degree. The element x stabilizes the section $I/I \cap H$, hence by uniqueness $\phi^x = \phi$. By Gallagher Theorem [34, Corollary 6.17], $\phi\delta \in \mathrm{Irr}(I \mid \lambda)$ and by Clifford correspondence, $\chi = (\phi\delta)^G \in \mathrm{Irr}(G)$. Since ϕ is a real x-invariant character and $\delta^x = \bar{\delta}$, we have that $(\phi\delta)^x = \overline{\phi\delta}$. Hence, as before χ is a real irreducible character. Moreover $q \mid \chi(1)$ since $q \mid \phi(1)$ and $2 \mid \chi(1)$ since 2 divides $[G : I]$. So $\chi(1)$ is a composite number and this is impossible. $\square$

Lemma 3.3.5. *Let K be a perfect group and M a minimal normal subgroup of K that is an elementary abelian 2-group. Suppose that M is non-central in K and that K/M is isomorphic to $\mathrm{L}_2(8)$ or A_5. Then K has an irreducible non-linear real character with odd composite degree.*

Proof. Since K/M is simple we have that $\mathbf{C}_K(M) = M$. Suppose that K/M is isomorphic to A_5. There are two non-isomorphic irreducible A_5-modules W_1, W_2 of A_5 over $\mathrm{GF}(2)$. Both have dimension 4 and $H^2(A_5, W_1) = H^2(A_5, W_2) = 0$. Hence M has a complement S in K. It is easy to construct these groups and we see that $K = M \rtimes S = W_i \rtimes A_5$ has a real irreducible character of degree 15. Suppose now that $K/M \simeq \mathrm{L}_2(8)$. Let W_1, W_2, W_3 be the non-isomorphic irreducible $\mathrm{L}_2(8)$-modules over $\mathrm{GF}(2)$, where $\dim(W_1) = 6, \dim(W_2) = 8$ and $\dim(W_3) = 12$. If $M \simeq M_i$ with $i = 2, 3$, then $H^2(\mathrm{L}_2(8), W_i) = 0$ and hence M_i has a complement S in K. Then, as before, we conclude observing that $W_i \rtimes \mathrm{L}_2(8)$ has a real irreducible character of degree 63. Suppose that $M \simeq W_1$. Then, $\dim H^2(\mathrm{L}_2(8), W_1) = 3$. Nevertheless, there are just two perfect groups of order $2^6 \cdot |\mathrm{L}_2(8)|$. Both these groups have an irreducible real character of degree 63. $\square$

In the previous lemma, dimensions of cohomology groups and all the perfect groups of a given order is information that is accessible with the `GAP`'s packages `cohomolo` and `PerfectGroup`.

Proposition 3.3.6. *Let G be a non-solvable group and suppose that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime-power numbers. Let $K = G^{(\infty)}$ and $R = \mathrm{Rad}(G)$. Then $|K \cap R| \leq 2$ and if equality holds, then $K \simeq \mathrm{SL}_2(5)$.*

Proof. By Proposition 3.3.4, we have that $N = K \cap R$ is a 2-group. We prove that if $N > 1$ then $|N| = 2$ and K is isomorphic to $\mathrm{SL}_2(5)$. Let $V = N/\Phi(N)$, then V a normal elementary abelian 2-subgroup of $G/\Phi(N)$. Let $V > V_1 > \cdots > V_n$ a K-principal series of V. Let $N > N_1 > \cdots > N_n$ such that N_i the preimage in N of V_i. Then N/N_1 is an irreducible K/N-module and K/N is isomorphic A_5 or $\mathrm{L}_2(8)$ by Proposition 3.3.4. By Lemma 3.3.5 and part 3 of 3.2.13, N/N_1 is central in K/N_1. Since K is perfect, we have that N/N_1 is isomorphic to a subgroup of the Schur multiplier $\mathrm{M}(K/N)$ by Lemma 3.2.23. The only possibility is $|N/N_1| = 2$ and $K/N_1 \simeq \mathrm{L}_2(5)$, the Schur covering of A_5. Suppose by contradiction that $N_1/N_2 > 1$, write $\bar{K} = K/N_2$. Since $\mathrm{M}(\mathrm{SL}_2(5)) = 1$, $\bar{N}_1$ cannot be central in $\bar{K}$. Let $t \in K$ a 2-element such that $\langle tN_1 \rangle = \mathbf{Z}(K/N_1)$, namely the unique central involution in $\mathrm{SL}_2(5)$ and $\langle tN_1 \rangle = \mathbf{O}_2(K/N_1)$. Since N_1 is an irreducible module over GF(2), we have that t acts trivially on $\bar{N}_1$. Suppose that $\bar{t}^2 \neq 1$, then $\langle \bar{t}^2 \rangle$ would be a proper, non-trivial submodule of $\bar{N}_1$, against irreducibility. This means that $\bar{t}^2 = 1$ and hence $\langle \bar{t} \rangle$, that centralizes $\bar{N}_2$, is a minimal normal subgroup of $\bar{N}_2$. Observe that $\bar{K}/\langle t \rangle$ is a quotient of K that satisfies the hypotheses of Lemma 3.3.5. Hence by part 3 of Lemma 3.2.13 we derive a contradiction. □

Definition 3.3.7. Let G be a group and suppose that there are $H, K \trianglelefteq G$ such that $G = HK$ and $H \cap K \leq \mathbf{Z}(G)$. Then G is called the *central product* of H and K and we write $G = H \curlyvee K$.

Theorem 3.3.8. *[8, Theorem A] Let G be a non-solvable group and suppose that* $\mathrm{cd}_{\mathbb{R}}(G)$ *consists of prime-power numbers. Then* $\mathrm{Rad}(G) = H \times O$ *for a group O of odd order and a 2-group H of Chillag-Mann type. Furthermore, if $K = G^{(\infty)}$, then one of the following holds.*

1. *$G = K \times R$ and K is isomorphic to A_5 or* $\mathrm{L}_2(8)$*;*

2. *$G = (KH) \times O$ where $K \simeq$* $\mathrm{SL}_2(5)$*, $HK = H \curlyvee K$ and $K \cap H = \mathbf{Z}(K) < H$.*

Proof. By Proposition 3.3.6 and Proposition 3.3.4, if $K = G^{(\infty)}$ and $R = \mathrm{Rad}(G)$, then $G = KR$, and either $K \cap R = 1$ and K is simple isomorphic to A_5 or $\mathrm{L}_2(8)$ or $K \simeq \mathrm{SL}_2(5)$ and $K \cap R = \mathbf{Z}(K)$. In the first case, 1 follows. Suppose $K = \mathrm{SL}_2(5)$ and $K \cap R = \mathbf{Z}(K) = Z$. Note that Z is a normal subgroup of order 2, hence is central in R. Consider $\bar{G} = G/Z$. Then $\bar{G} = \bar{K} \times \bar{R}$ and hence $\bar{R}$ is a group of Chillag-Mann type, since $\bar{K}$ is simple and has irreducible real non-linear characters of both odd and even degree. This means that $\bar{R} = \bar{O} \times \bar{H}$ for $O \in \mathrm{Hall}_{2'}(R)$ and $H \in \mathrm{Syl}_2(R)$, note that $\bar{H}$ is of Chillag-Mann type. We have that R is 2-closed, hence $R = H \rtimes O$. Clearly, O acts trivially on H/Z, so $H = \mathbf{C}_H(O)Z \leq \mathbf{C}_H(O)\mathbf{Z}(R) \cap H$. It follows that O centralizes H and $R = H \times O$. By Dedekind modular law $HK \cap O \leq HK \cap R \leq H(K \cap R) \leq H$ and hence $HK \cap O \leq H \cap O = 1$. This means that $G = (KH) \times O$ and $K \cap H = Z$ that has order 2. If H and K commute, then $KH = K \curlyvee H$. Suppose by contradiction that $[H, K] = Z$, hence there is $aZ \in H/Z$ that acts non-trivially by conjugation on K/Z. But this is impossible since $KH/Z = \bar{K} \times \bar{H}$. We now prove that H is of Chillag-Mann type. Suppose that this is not the case, so there is $\phi \in \mathrm{Irr}_{\mathbb{R}}(H)$ such that $\phi(1) > 1$. Since $\bar{H}$ is of Chillag-Mann type, we have that $Z \not\leq \ker \phi$, so $\phi_Z = \phi(1)\lambda$, with $\lambda \neq 1_Z$. On the other hand, if θ is the unique character of K of degree 6, then $Z \not\leq \ker \theta$ and $\theta_Z = \theta(1)\lambda$. Now, $KH = K \times H/N$ where $N = \{(z, z) \mid z \in Z\}$ (see [31, Satz I9.10]) and $\psi = \theta \times \phi \in \mathrm{Irr}_{\mathbb{R}}(K \times H)$. Moreover $\psi_N = \phi(1)\theta(1)\lambda^2 = \phi(1)\theta(1)1_N$, so it follows that $N \leq \ker \psi$ and $\psi \in \mathrm{Irr}_{\mathbb{R}}(KH)$. If $\chi = \psi \times 1_O$, then $\chi \in \mathrm{Irr}(G)$ takes real values and has composite degree, impossible. Since $\mathrm{SL}_2(5)$ does not satisfy the hypotheses, we have that $Z < H$. Part 2 follows. □

We remark that in Problem 4.4 of [34], we can find a stronger version of the argument used in the proof above.

As a consequence, we get some control on character degrees.

Corollary 3.3.9. *Let G be a non-solvable group and suppose that* $\mathrm{cd}_{\mathbb{R}}(G)$ *consists of prime-power numbers. Then either* $\mathrm{cd}_{\mathbb{R}}(G) = \mathrm{cd}_{\mathbb{R}}(\mathrm{L}_2(8))$ *or* $\mathrm{cd}_{\mathbb{R},2'}(G) = \mathrm{cd}_{\mathbb{R},2'}(A_5)$.

Proof. Apply Theorem 3.3.8. In case 1 there is nothing to prove. Suppose 2, we have that $G = (KH) \times O$ with O of odd order, $K = G^{(\infty)}$ and H is a normal 2-subgroup. Call S the simple section KH/H, hence $S \simeq A_5$. Take $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ a real non-linear character of odd degree. Hence $\chi(1) = p^n$ with p odd and χ is a character of HK since, by part 1 of Lemma 3.2.13, $O \leq \ker(\chi)$. The degree of every irreducible constituent of χ_H divides $(|H|, \chi(1)) = 1$, hence $\chi_H = e \sum_i \lambda_i$ for $\lambda_i \in \mathrm{Lin}(H)$. By hypothesis we have that $\chi(1)$ is a non-trivial p-power for an odd prime p and by 3.2.27 we have that $\chi(1)/\lambda(1)$ divides $[HK : H] = |S|$, where $S \simeq A_5$. Hence $p \leq \chi(1) \leq |S|_p$, the p-part of the number $|S|$, that is equal to p if p is an odd prime. It follows that $\chi(1) = p$. We have proved that $\mathrm{cd}_{\mathbb{R},2'}(G) \subseteq \{3, 5\} = \mathrm{cd}_{\mathbb{R},2'}(A_5)$. The right-to-left inclusion follows observing that A_5 is a quotient of G. □

3.3.1 Remarks

In [42], Manz characterized the groups where all the irreducible characters have prime power degrees. It turns out that if G is a group with this property, then G is isomorphic to $\mathrm{L}_2(8)$ or A_5. Unfortunately, for real characters, our results are not easily improvable, in the sense that the intersection $H \cap K$ in part 2 of Theorem 3.3.8 not need to be trivial and, consequently, H not need to be a group of Chillag-Mann type. Indeed, if G is the `SmallGroup(240,93)`, then $K \simeq \mathrm{SL}_2(5)$, $|H| = 4$ and $K \cap H = \mathbf{Z}(K)$.

One second point worth mentioning, is that the proofs of the previous section would have been much easier if the bijection in Theorem 3.2.26 conserved values of characters. On the other hand, it seems that in general, this is a property that is not affordable with these types of correspondences, at least not for reality. Also Gallagher's correspondence [34, Corollary 6.17] has the same problem. Nevertheless, there are some partial, and deep, results in this context. See for example [52, Theorem 5.3] and [51, Theorem 5.1].

3.4 Real degrees in solvable groups

In this section, we prove that if G is a group such that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of odd prime powers, then only one prime is involved, see Proposition 3.4.3. As a consequence, we get that a solvable group G such that $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime powers involving more than one prime, then the primes involved are p odd and 2; this is the content of Theorem 3.4.4.

Lemma 3.4.1. *Let G be a group and $T \in \mathrm{Syl}_2(G)$. Suppose that every real irreducible character of G has odd degree. Then $\mathbf{O}_{2'}(G)\Phi(T) \leq \ker\chi$ for every $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$.*

Proof. By Theorem 3.2.7, we have that $T \trianglelefteq G$ and T is of Chillag-Mann type, namely every real irreducible character of T is linear and contain $\Phi(T)$ in its kernel by Corollary 3.2.4. Note that $[T, \mathbf{O}_{2'}(G)] = 1$, because T and $\mathbf{O}_{2'}(G)$ have coprime orders and are normal in G. Let $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$, by part 1 of Lemma 3.2.13 we have that $\mathbf{O}_{2'}(G) \leq \ker\chi$. Now, since $\chi(1)$ is odd, χ_T has odd number of irreducible constituents and the complex conjugation acts on them. Thus one is real contains $\Phi(T)$. Since all the irreducible constituents of χ_T are G-conjugated, $\Phi(T)$ is contained in all constituents of χ_T and this means that $\Phi(T) \leq \ker\chi$. $\square$

The following lemma is a very useful tool in the context of character degrees or conjugacy class sizes.

Lemma 3.4.2. *Let A act via automorphisms on G, where A and G are groups, and suppose that $(|A|, |G|) = 1$. Suppose that there exist A-orbits of size m and n in G where $(n, m) = 1$. Then there is also an orbit of size mn.*

Proof. See [35, Theorem 3.34], where either G or A has odd order and is solvable by Feit-Thompson Theorem.

Proposition 3.4.3. *Let G be a group, $T \in \mathrm{Syl}_2(G)$ and $L \in \mathrm{Hall}_{2'}(G)$. The following conditions are equivalent.*

1. *$\mathrm{cd}_{\mathbb{R}}(G)$ consists of odd prime-power numbers;*

2. *T is normal in G and $V = [L, T/\Phi(T)]$ is an L-module and there is an odd prime q such that $[L : \mathbf{C}_L(v)]$ is a non-trivial q-power for all $v \in V^{\#}$; moreover $\mathbf{C}_L(V) = \mathbf{C}_L(T) = \mathbf{O}_{2'}(G)$;*

3. *The degree of every real irreducible character of G is a q-power for a fixed odd prime q.*

Proof. We first prove that 1 implies 2.
Note that $\chi(1)$ is odd for all $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$. So, by Theorem 3.2.7, $T \trianglelefteq G$ and T is of Chillag-Mann type. In particular, G is solvable. Let $L \in \mathrm{Hall}_{2'}(G)$, then $G = T \rtimes L$. Call $\bar{T} = T/\Phi(T)$. Note that L acts coprimely on $\bar{T}$; write $\bar{T} = \mathbf{C}_{\bar{T}}(L) \times [L, \bar{T}]$ the Fitting decomposition. Observe that every non-trivial element $v \in V$ generates a non-trivial L-orbit. Moreover, $\mathbf{C}_L(V) = \mathbf{C}_L(T) = \mathbf{O}_{2'}(G)$. Suppose that there are two L-orbits of V of coprime lengths m and n. By Lemma 3.4.2, there is an L-orbit of V, with representative λ, with length mn. The actions of L on V and $\hat{V} = \mathrm{Irr}(V)$ are permutation isomorphic by [34, Theorem 13.24], so there is $\lambda \in \hat{V}$ such that $[L : I] = mn$, where $I = \mathrm{I}_G(\lambda)$. The character λ is real since it is linear and $o(\lambda) = 2$. If $\hat{\lambda} \in \mathrm{Irr}(I \mid \lambda)$ is the canonical extension of λ to I (see [34, Corollary 8.16]), then $\hat{\lambda}$ is real. Thus, $\chi = \hat{\lambda}^G \in \mathrm{Irr}_{\mathbb{R}}(G)$ and $\chi(1) = mn$. Since $\mathrm{cd}_{\mathbb{R}}(G)$ consists of odd prime powers, it follows that there is an odd prime q such that $[L : \mathbf{C}_L(v)]$ is a q-power for all $v \in V$. Moreover, for all $1 \neq v \in V$, then $[L : \mathbf{C}_L(v)] > 1$ is non-trivial, because $\mathbf{C}_V(L) = 1$.
Assuming now 2, we prove 3 by induction. Call $U = \mathbf{O}_{2'}(G)$; by Lemma 3.4.1, $U\Phi(T)$ is contained in the kernel of every real character, hence assume that $U\Phi(T) = 1$; note that T is an elementary abelian 2-group and that $\mathrm{Irr}(T) = \mathrm{Irr}_{\mathbb{R}}(T)$. Call $G_0 = [T, L]L$ and $Z = \mathbf{C}_T(L)$. The group Z is a direct central factor of G and therefore $G = Z \times G_0$. An irreducible character of G is of the form $\theta\chi$, for $\theta \in \mathrm{Irr}(Z)$ and $\chi \in \mathrm{Irr}(G_0)$, and it is real if and only if χ is. It follows that we can assume $Z = 1$. Note that $T = V$ is a faithful L-module and $[L : \mathbf{C}_L(v)] > 1$ for all $v \in V$. Take χ a non-principal real irreducible character of G and consider λ an irreducible constituent of χ_T. Since T is elementary abelian, $\lambda(1) = 1$ and $o(\lambda) \leq 2$. Moreover, we have that $\lambda \neq 1_T$. Indeed, if λ is principal, $T \leq \ker \chi$ and $\chi \in \mathrm{Irr}_{\mathbb{R}}(G/T) = 1_{G/T}$, that is against the assumption. Since the action of L on T and $\hat{T}$ are permutation isomorphic, we have that $[L : \mathrm{I}_L(\lambda)]$ is a non-trivial q-power. Hence $\chi(1) = [L : \mathrm{I}_L(\lambda)]$ is a q-power.
Finally, part 1 follows easily from 3. □

Recall that, according to Definition 3.2.5, $\mathcal{V}_G$ is the vertex set of $\Delta_{\mathbb{R}}(G)$, the prime graph on the real degrees of the finite group G. The following theorem appears as Theorem B in [14].

Theorem 3.4.4. *Let G be a solvable group. Suppose that $\mathrm{cd}_{\mathbb{R}}(G)$ consists in prime-power numbers. Then $\mathcal{V}_G \subseteq \{2, p\}$, with $p > 2$.*

Proof. By Theorem 3.2.6, we have that $\Delta_{\mathbb{R}}(G)$ has at most 2 connected components. So $\mathcal{V}_G \subseteq \{p, q\}$ for two distinct primes p, q. Suppose that $p, q > 2$. Then $p = q$ by Proposition 3.4.3. □

Solvable groups in which *all* character degrees are prime-power numbers are characterized in [43]. A generalization of this result for real characters appears to be very complicated. This is due to the flexibility of the structure of solvable groups. One natural way to proceed is to factor out $\Phi(G)$ and apply module theoretic methods as well as results in Chapter 2. Unfortunately, factoring over the Frattini subgroup has some unpredictable consequences on the real character degrees. For example, consider G is the `SmallGroup(869,19309)`; then, $\mathrm{cd}_{\mathbb{R}}(G) = \{1, 7, 8\}$, but $\mathrm{cd}_{\mathbb{R}}(G/\Phi(G)) = \{1, 7\}$.

Nevertheless, for solvable groups, this problem appears only for real characters of even degree, as the following application of Gow's theorem shows.

Proposition 3.4.5. *Let G be a solvable group. Let $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ of odd degree. Then $\Phi(G) \leq \ker \chi$.*

Proof. By Lemma 3.2.13, we have that $\mathbf{O}_{2'}(\Phi(G)) \leq \ker \chi$.
Let $N = \mathbf{O}_2(\Phi(G))$. By Gow's Theorem 3.2.11 there is a subgroup $L \leq G$ and $\lambda \in \mathrm{Lin}(L)$ such that $o(\lambda) = 2$ and $\chi = \lambda^G$. Since L has odd index in G, we have that $N \leq L$. If we prove that N is contained in $\ker(\lambda)$, the result follows by the definition of induced characters. Suppose by contradiction that $N \not\leq \ker(\lambda)$, hence $L = N \ker(\lambda)$. Let $H \in \mathrm{Hall}_{2'}(G)$, we have that $G = LH = \ker(\lambda)H$, since $N \leq \Phi(G)$. By Dedekind's modular law, $L = \ker(\lambda)(L \cap H)$, but $L \cap H$ has odd order, hence $L/\ker(\lambda)$ has odd order and this is impossible since λ is real and non-principal. $\square$

3.5 Limits of the study of real characters

In general, when we deal with "reality" of character degrees or class sizes, the prime 2 plays a special role in these contexts. Recall that the real versions of Ito-Micheler and Thompson theorems hold for the prime $p = 2$ (Theorems 3.2.7 and 3.2.9). One may ask if such results hold for p odd. Unfortunately, this is not the case in general. However, a weaker version of Ito-Michler for real characters and p odd has already been encountered and it is Theorem 3.2.8. In the work of Tiep [55], also real Brauers characters have been treated. A real version of Thompson's Theorem for p odd recently appeared, see [53].

A very wide branch of character theory is the McKay conjecture. If G is a group and $P \in \mathrm{Syl}_p(G)$ and $\mathrm{Irr}_{p'}(G)$ is the set of irreducible characters of p'-degree, then the McKay conjecture says that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(p))|.$$

In a stronger stronger version of McKay conjecture, there is a "canonical" bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)).$$

If $p = 2$, there is such a bijection for real characters, as we will see soon.

In [32], Isaacs proved that if A is a group that acts coprimely on a group of odd order G, then there is a choice-free bijection $\mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A))$. If $|G|$ has even order and $(|G|, |A|) = 1$, then $|A|$ has odd order and it is solvable by Feit-Thompson Theorem. In this case, such a bijection exists and it is commonly known as Galuberman correspondence. In the case that both $|A|$ and $|G|$ are odd numbers, Gauberman and Isaacs are both defined and actually they concide. This was proved by Wolf in [59], so nowdays we refer to "Glauberman-Isaacs" correspondence. We note that this correspondence $\mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A))$ is "natural", in the sense that there is algorithm for constructing it and that the result is independent of any choices made in application of the algorithm. So, the Glaubermann-Isaacs correspondence commutes with Galois action and, in particular, sends real characters to real characters.

The techniques introduced by Isaacs lead to the McKay bijection for $p = 2$ (see [32, Theorem 10.9]). This bijection is natural in the meaning as above, so there is a bijection

$$\mathrm{Irr}_{\mathbb{R},2'}(G) \to \mathrm{Irr}_{\mathbb{R},2'}(\mathbf{N}_G(P)),$$

where $P \in \mathrm{Syl}_2(G)$. For p odd, the "real version" of McKay conjecture does not hold. Indeed if $G = \mathrm{GL}_2(3)$ and $P \in \mathrm{Syl}_3(G)$, then $|\mathrm{Irr}_{\mathbb{R},3'}(G)| = 4$ and $|\mathrm{Irr}_{\mathbb{R},3'}(\mathbf{N}_G(P))| = 6$. In literature, it seems that there is any reference in this context. We suspect that McKay conjecture is not a good topic for the theory of real characters.

In general, a good strategy to produce results in the context of real characters is to find "real versions" of known theorems that are true for all characters. However, one must be careful with this strategy.
Suppose we want to find a "real version" of the following Theorem.

Theorem 3.5.1. *[48, Theorem A] Let $\rho \subseteq \pi$ be two sets of primes. Suppose that G is π-separable and ρ-separable. If $\mathrm{Irr}_\pi(G) = \mathrm{Irr}_\rho(G)$ then, called $H \in \mathrm{Hall}_{\pi'}(G)$ e $K \in \mathrm{Hall}_{\sigma'}(G)$, the following holds.*

1. *$\mathbf{N}_G(H) = K'\mathbf{N}_G(K)$;*
2. *$K' \cap \mathbf{N}_G(H) = H'$.*

As the prime 2 must play a role, the minimal choice is $\rho = \{2, p\}$. Therefore, for π there are these two possibilities: $\pi = 2'$ or $\pi = p'$. For $\pi = 2'$, we can see that `SmallGroup(42,2)` is a counterexample.
Consider $\pi = p'$. Hence a good candidate for a "real version" of the Theorem above would be the following statement

- Let G be a group and $p > 2$ a prime. Take $H \in \mathrm{Hall}_{\{2,p\}}(G)$, $P \in \mathrm{Syl}_p(H)$ and $N = \mathbf{N}_G(H)$. If $\mathrm{Irr}_{\mathbb{R},p'}(G) = \mathrm{Irr}_{\mathbb{R},\{2,p\}'}(G)$, then $\mathbf{N}_G(P) \leq N$.

Unfortunatly, also this statement is not true. Indeed `SmallGroup(1500,123)` is a counterexample.

Chapter 4

Prime graphs on real class sizes

In this chapter are discussed the groups G such that $\mathrm{cs}_{\mathbb{R}}(G)$, the set of the sizes of the real classes of G, consists of prime powers. These groups are characterized in the main Theorem 4.2.10 under the assumption $\mathrm{Re}(\mathbf{O}_2(\mathbf{F}(G))) \subseteq \mathbf{Z}(\mathbf{O}_2(\mathbf{F}(G)))$. Sections 4.1 and 4.2 contain some preliminary and technical results, which are well known, except maybe for Lemma 4.1.17.
In Section 4.2 is proved the main theorem; in the first part of this section, it is assumed that $\mathcal{O}(G)$, the largest normal subgroup of odd order that is centralised by a Sylow 2-subgroup of G, is trivial. This assumption is removed later. The proof of Theorem 4.2.10 is divided in the case where $\mathbf{F}(G)$ is a 2-group (Subsection 4.2.1) and in the case where $\mathbf{F}(G)$ is not a 2-group and $\mathcal{O}(G) = 1$ (Subsection 4.2.2).
Section 4.3 investigates the possibility to generalize the main Theorem 4.2.10, removing the hypothesis $\mathrm{Re}(\mathbf{O}_2(\mathbf{F}(G))) \subseteq \mathbf{Z}(\mathbf{O}_2(\mathbf{F}(G)))$. The removal of this additional condition leads to a difficulty that is not new in the study of real classes, which is Conjecture 4.3.1.
Section 4.4 includes some additional material, general remarks and examples.

4.1 Preliminaries

We start with a definition.

Definition 4.1.1. Let G be a group.

1. If $x \in G$ and $H \leq G$, we define $x^H = \{x^h \mid h \in H\}$. So, $x^G = \{x^g \mid g \in G\}$ is a conjugacy class of G.

2. If $\mathcal{C}$ is a conjugacy class of G, then $|\mathcal{C}|$ is the *size* or *length* of $\mathcal{C}$.

3. An element $x \in G$ is said *real* if it is G-conjugated to its inverse, i.e. there exists $g \in G$ such that $x^g = x^{-1}$.

4. if $\mathcal{C}$ is a conjugacy class, we say that $\mathcal{C}$ is *real* if it is generated by a real element or, equivalently, if $x \in \mathcal{C}$, then $x^{-1} \in \mathcal{C}$.

5. $\mathrm{cl}(G)$ denotes the set of all conjugacy classes of G and $\mathrm{cl}_{\mathbb{R}}(G) \subseteq \mathrm{cl}(G)$ denotes the set of all real conjugacy classes of G.

6. $\mathrm{cs}(G)$ and $\mathrm{cs}_{\mathbb{R}}(G)$ denote respectively the set of lengths of classes in $\mathrm{cl}(G)$ and $\mathrm{cl}_{\mathbb{R}}(G)$.

7. If p is a prime, we denote with $\mathrm{cl}_{\mathbb{R},p}(G)$ the set of all *non-central* classes in $\mathrm{cl}_{\mathbb{R}}(G)$ whose length is a p-power number.

We recall some notation. If X is a set of positive integers, we denote $\pi(X)$ the set of all the primes that divide some element in X.

Definition 4.1.2. Let G be a group, $\mathrm{cs}_{\mathbb{R}}(G)$ the set of lengths of real conjugacy classes of G and denote $\mathcal{V}_G^* = \pi(\mathrm{cs}_{\mathbb{R}}(G))$. Consider the graph $\Delta_{\mathbb{R}}^*(G)$ whose vertex-set is $\mathcal{V}_G^*$ and two vertices $p, q \in \mathcal{V}_G^*$ are adjacent if and only if there is $s \in \mathrm{cs}_{\mathbb{R}}(G)$ such that pq divides s. We call $\Delta_{\mathbb{R}}^*(G)$ the *prime graph on real conjugacy class sizes* of G.

Our goal is to study the groups G such that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-power numbers. In terms of the prime graph, this is equivalent to requiring that $\Delta_{\mathbb{R}}^*(G)$ consists of isolated vertices. We remark that if there are more than two primes involved, then one of them is the prime 2. A direct proof of this can be obtained using the argument in Theorem 3.4.3. Nevertheless, it is more convenient to use the following general result from [57] and [17].

Theorem 4.1.3. *Let G be a group and suppose that $\Delta_{\mathbb{R}}^*(G)$ is disconnected. Then G is solvable, there are exactly two connected components and one of them contains the prime* 2*.*

Proof. By Theorem A and B of [57], we have that G is solvable and 2 appears as a divisor of some real class size. By Theorem 6.2 of [17] we have that the number of connected components of $\Delta_{\mathbb{R}}^*(G)$ is at most 2. $\square$

The following Theorem is an analogue of Ito-Michler Theorem for real conjugacy classes.

Theorem 4.1.4. *[17, Theorem 6.1] Let G be a group and $T \in \mathrm{Syl}_2(G)$. Then all real classes of G have odd sizes if and only if $T \trianglelefteq G$ and $\mathrm{Re}(T) \subseteq \mathbf{Z}(T)$.*

We point out that a real version of Thompson's Theorem for real conjugacy class sizes does not hold. Indeed, if G is isomorphic to either the solvable group $\mathrm{SL}_2(3)$ or the non-solvable group $\mathrm{SL}_2(5) = 2.A_5$, then $\mathrm{cs}_{\mathbb{R}}(G)$ consists of 1 or even numbers, but G does not possess a normal $2'$-complement.

Now we present a deep result that connects the worlds of real degrees and real class sizes.

Theorem 4.1.5. *[28, Theorem B] Let G be a group and p an odd prime. If $p = 3$, assume in addition that G has no composition factor isomorphic to* $\mathrm{SL}_3(2)$*. If p does not divide $|\mathcal{C}|$ for every real class $\mathcal{C}$ of G, then p does not divide $\chi(1)$ for every real-valued $\chi \in \mathrm{Irr}(G)$.*

If $\mathrm{cs}_{\mathbb{R}}(G)$ consists of 2-powers (or, in terms of prime graphs, $\Delta^*_{\mathbb{R}}(G)$ consists of the isolated vertex 2) then the structure of the group G is characterized.

Theorem 4.1.6. *Let G be a group. Then $\mathcal{V}^*_G = \{2\}$ if and only there is and $K \in \mathrm{Hall}_{2'}(G)$ normal in G, $\mathrm{Re}(G) \subseteq \mathbf{C}_G(K)$ and $[K', T] = 1$ for $T \in \mathrm{Syl}_2(G)$.*

Proof. By Theorem C of [49], we have that $\mathcal{V}^*_G = \{2\}$ if and only if $K \trianglelefteq G$ and $\mathrm{Re}(G) \subseteq \mathbf{C}_G(K)$. Assume that $\mathcal{V}^*_G = \{2\}$. We have that G is solvable and thus G has no composition factor isomorphic to $\mathrm{SL}_3(2)$. By Theorem 4.1.5, the degree of all real characters are 2-powers and by Theorem 3.2.10 we have that $[K', T] = 1$. □

4.1.1 Technical lemmas

The following is a list of useful technical lemmas.

Lemma 4.1.7. *Let G be a group.*

1. *If $x \in \mathrm{Re}(G)$ and $|x^G|$ is odd, then $x^2 = 1$.*

2. *Suppose N a normal subgroup and $xN \in \mathrm{Re}(G/N)$. If either $|N|$ or the order of xN in G/N is odd, then $xN = yN$ for some $y \in \mathrm{Re}(G)$ (of odd order if the order of Nx is odd).*

3. *If $\mathcal{B}, \mathcal{C} \in \mathrm{cl}(G)$ such that $(|\mathcal{B}|, |\mathcal{C}|) = 1$, then $\mathcal{BC} = \mathcal{CB} \in \mathrm{cl}(G)$ and $|\mathcal{BC}|$ divides $|\mathcal{B}||\mathcal{C}|$. If further $\mathcal{B} = x^G$ and $\mathcal{C} = y^G$ with $x, y \in \mathrm{Re}(G)$ such that $xy = yx$, then $xy \in \mathrm{Re}(G)$ and $\mathcal{BC} \in \mathrm{cl}_{\mathbb{R}}(G)$.*

4. *If $xN \in G/N$, then $|(xN)^{G/N}|$ divides $|x^G|$.*

5. *If $N \trianglelefteq G$ is a 2-group and $x \in G$ is an element of odd order such that $xN \in \mathrm{Re}(G/N)$, then x is real in G.*

Proof. Parts 1 and 2 are parts (3) and (6) of [57, Lemma 2.1]. Part 3 is a combination of [3, Lemma 1(b)] and [17, Lemma 6.3b)]. Part 4 is well known and part 5 is [17, Lemma 6.3c)]. □

Lemma 4.1.8. *[17, Proposition 6.4] Let G be a group. The following are equivalent.*

1. *Every non-trivial element in* $\mathrm{Re}(G)$ *has even order;*
2. *Every element in* $\mathrm{Re}(G)$ *is a* 2*-element;*
3. *G is has a normal Sylow* 2*-subgroup.*

Lemma 4.1.9. *Let G be a group and suppose that $\Delta^*_{\mathbb{R}}(G)$ has two connected components with vertex sets π_1, π_2, where $2 \notin \pi_2$. If $x \in \mathrm{Re}(G)$ and $|x^G| > 1$ is a π_2-number, then x is an involution and $\mathbf{C}_G(x)$ is 2-closed.*

Proof. The proof is contained in [57, Lemma 3.1]. □

Lemma 4.1.10. *[57, Lemma 3.2] Let G be a group and N a normal subgroup of odd order. Then $\Delta^*_{\mathbb{R}}(G/N)$ is a subgraph of $\Delta^*_{\mathbb{R}}(G)$.*

Definition 4.1.11. We say that G satisfies the assumption $(*)$ if G is solvable, $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-power numbers and $\mathcal{V}^*_G = \{2, p\}$ for some odd prime p. $(*)$

The condition $(*)$ is introduced for convenience.

Lemma 4.1.12. *Let G be a group. Suppose that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. Then one of the following holds.*

1. $\mathcal{V}^*_G \subseteq \{2\}$.
2. $\mathcal{V}^*_G \subseteq \{p\}$ *for an odd prime p.*
3. *The property $(*)$ holds for G.*

Proof. Suppose that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers involving more than one prime. Then Theorem 4.1.3 applies: G is solvable, $\mathcal{V}^*_G = \{2, p\}$ and 3. follows. It remains the case where $\mathcal{V}^*_G = \{q\}$ for a prime q. □

The previous lemma shows that if we want to describe the structure of G in the case where $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers, we are reduced to assuming $(*)$.

Let G be a group and $N, M \trianglelefteq G$ such that N and M have odd order and commute with $S \in \mathrm{Syl}_2(G)$. Then NM has odd order and commutes with S. Therefore, we give the following definition.

Definition 4.1.13. Let G be a group and $S \in \mathrm{Syl}_2(G)$. Let $\mathcal{A}$ be the collection of normal subgroups of G, of odd order and that commute with S. Then we define

$$\mathcal{O}(G) = \prod_{N \in \mathcal{A}} N,$$

that is the largest normal subgroup of odd order that commutes with S.

It is worth noting that $\mathrm{Irr}_{\mathbb{R}}(G) = \mathrm{Irr}_{\mathbb{R}}(G/\mathcal{O}(G))$ by part 1 of Lemma 3.2.13.

Lemma 4.1.14. *Let G be a group and $M \trianglelefteq G$ a subgroup of odd order.*

1. *Suppose that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. If $M \leq \Phi(G)$ and $p \notin \mathcal{V}^*_{G/M}$ for an odd prime p, then $p \notin \mathcal{V}^*_G$.*

2. *If $M \leq \Phi(G)$ and $2 \notin \mathcal{V}^*_{G/M}$, then $2 \notin \mathcal{V}^*_G$.*

3. *Suppose $\mathrm{cd}_{\mathbb{R}}(G)$ consists of prime powers and $M \leq \mathcal{O}(G)$. Then $\mathcal{V}^*_G = \mathcal{V}^*_{G/M}$ and $\langle x, M\rangle = \langle x\rangle \times M$ for every $x \in \mathrm{Re}(G)$.*

Proof. Let $T \in \mathrm{Syl}_2(G)$ and call $\bar{G} = G/M$.
We prove 1. Suppose that $p \notin \mathcal{V}^*_{\bar{G}}$ and assume working by contradiction that $p \in \mathcal{V}^*_G$. So, there is an element $t \in \mathrm{Re}(G)$ such that $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. By part 1 of Lemma 4.1.7, t is an involution; we can furthermore assume that $T \leq \mathbf{C}_G(t)$. Note that $t \notin M$, hence $\bar{t}$ is an involution in $\bar{G}$. By part 4 of Lemma 4.1.7, $\mathrm{cs}_{\mathbb{R}}(\bar{G})$ consists of prime power numbers and therefore, by Theorem 4.1.3, we have that $\mathcal{V}^*_{\bar{G}} \subseteq \{2\}$. Since $|t^G|$ has odd order, we have that $\bar{t} \in \mathbf{Z}(\bar{G})$. So, $\overline{t^x} = \bar{t}^{\bar{x}} = \bar{t}$ for every $x \in G$. It follows that $\langle t^x, M\rangle = \langle t, M\rangle$ for every $x \in G$. This means that $\langle t, M\rangle \trianglelefteq G$ and $\langle t, M\rangle$ contains $\langle t\rangle$ as a Sylow 2-subgroup. By the Frattini argument $G = \mathbf{C}_G(t)M = \mathbf{C}_G(t)$ because $M \leq \Phi(G)$, and this is impossible.
We now prove 2. Suppose that $2 \notin \mathcal{V}^*_{\bar{G}}$; then, by Theorem 4.1.4, we have that $\bar{T} \trianglelefteq \bar{G}$ and $\mathrm{Re}(\bar{T}) \subseteq \mathbf{Z}(\bar{T})$. Since M has odd order, then $T \simeq \bar{T}$ and therefore $\mathrm{Re}(T) \subseteq \mathbf{Z}(T)$. Now, $\bar{T} \leq \mathbf{F}(\bar{G}) = \mathbf{F}(G)/M$ because $M \leq \Phi(G)$, therefore $\mathbf{F}(G)$ contains T. This means that $TM = T \times M$, since both subgroups are normal in $\mathbf{F}(G)$. It follows that $T \trianglelefteq G$ and $\mathrm{Re}(T) \subseteq \mathbf{Z}(T)$. Hence, again by Theorem 4.1.4, we have that $2 \notin \mathcal{V}^*_G$.
Finally, we prove 3. Since $M \leq \mathcal{O}(G)$, we have that $[G : \mathbf{C}_G(M)]$ is odd. So, $T \leq \mathbf{C}_G(M)$ because $\mathbf{C}_G(M) \trianglelefteq G$. We prove that $\mathcal{V}^*_G = \mathcal{V}^*_{\bar{G}}$. Suppose that $2 \notin \mathcal{V}^*_{\bar{G}}$. By Theorem 4.1.4 we have that $\bar{T} \trianglelefteq \bar{G}$ and $\mathrm{Re}(\bar{T}) \subseteq \mathbf{Z}(\bar{T})$. Hence, as $\bar{T} \simeq T$, we have that $\mathrm{Re}(T) \subseteq \mathbf{Z}(T)$. Since $T \leq \mathbf{C}_G(M)$, we have that $TM = T \times M$ and $T \trianglelefteq G$. This means that $2 \notin \mathcal{V}^*_G$ by Theorem 4.1.4. Suppose that $p \notin \mathcal{V}^*_{\bar{G}}$ for an odd prime p and let $K \in \mathrm{Hall}_{2'}(G)$. As before, we can see that $\mathrm{cs}_{\mathbb{R}}(\bar{G}) \subseteq \{2\}$; so, by Theorem 4.1.6, we have that $\bar{K} \trianglelefteq \bar{G}$ and $[(\bar{K})', \bar{T}] = 1$. Since M has odd order and $\bar{K} = K/M$, it follows that $K \trianglelefteq G$. Moreover, $\overline{[K', T]} \leq [(\bar{K})', \bar{T}] = 1$ and therefore we have that $[K', T] \leq M$. Hence $[K', T] = [K', T, T] \leq [M, T] = 1$ and $p \notin \mathcal{V}^*_G$. So, $\mathcal{V}^*_G = \mathcal{V}^*_{\bar{G}}$.
Let $x \in \mathrm{Re}(G)$. Suppose that $y \in \langle x\rangle \cap M$. Hence y, that is a power of x, is real in G. On the other hand, y commutes with a Sylow 2-subgroup of G. Therefore $|y| \leq 2$. Since M has odd order, we have that $y = 1$. $\square$

Corollary 4.1.15. *Let G be a group and $M \trianglelefteq G$ of odd order. Suppose that property $(*)$ holds for G. If either $M \leq \Phi(G)$ or $M \leq \mathcal{O}(G)$, then property $(*)$ holds for G/M.*

Proof. Observe that $\mathrm{cs}_{\mathbb{R}}(G/M)$ consists of prime power numbers by part 4 of Lemma 4.1.7. By Lemma 4.1.12, it is enough to prove that $|\mathcal{V}^*_{G/M}| > 1$, i.e. if a prime divides some real class size of G, then it divides some real class size of G/M. If $[G : \mathbf{C}_G(M)]$ is odd, then, by part 3 of Lemma 4.1.14, we have that $|\mathcal{V}_{G/M}| = |\mathcal{V}^*_G| = 2$.
Suppose that $M \leq \Phi(G)$. By part 1 and 2 of Lemma 4.1.14, we have that $\mathcal{V}^*_{G/M} = \mathcal{V}^*_G$, that has size 2, because $(*)$ holds for G. □

4.1.2 Ultraspecial groups

Definition 4.1.16. Let T be a group. We say that T is *ultraspecial* if the following hold.

1. $\Phi(T) = T' = \mathbf{Z}(T) = \mathbf{\Omega}_1(T)$,
2. $|T/\Phi(T)| = 2^{2n}$ and $|\Phi(T)| = 2^n$.

Now, following paragraph 2 of [46], we give an explicit construction of groups that are ultraspecial. Let n be a positive integer and $\mathbb{F}$ a field of order 2^n. Let $\mathbb{L}$ be an extension of degree 2 of $\mathbb{F}$ and $\lambda \neq 1$ an element of $\mathbb{L}$ whose multiplicative order is a divisor of $2^n + 1$. Clearly, $\lambda \notin \mathbb{F}$ and $\epsilon = \lambda + \lambda^{-1} \in \mathbb{F}$. We define the group T_n as the set all triads $(\alpha, \beta, \gamma) \in \mathbb{F}^3$ with the multiplication

$$(\alpha, \beta, \gamma)(\alpha_1 \beta_1, \gamma_1) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2 + \alpha_1\alpha_2 + \epsilon\alpha_1\beta_2 + \beta_1\beta_2).$$

The group T_n depends on λ and the identity of T_n is the element $(0, 0, 0)$. Moreover T_n is ultraspecial and $\Phi(T_n) = \{(0, 0, \gamma) \mid \gamma \in \mathbb{F}\}$. Every element in $\Phi(T_n)$ is a square and the mapping $\sigma_\lambda \colon (\alpha, \beta, \gamma) \to (\beta, \alpha + \epsilon\beta, \gamma)$ is an automorphism such that $\sigma_\lambda = |\lambda|$. We observe that T_n has real non-central elements. This can be easily seen viewing T_n as the Sylow 2-subgroup of the simple group $\mathrm{PSU}_3(2^n)$ or by direct check. Indeed $a^b = a^{-1}$ for $a = (\alpha, 0, 0)$ and $b = (0, \alpha\epsilon^{-1}, 0)$, where α is any element of $\mathbb{F}$. We have that $|a| > 2$, because $\alpha \neq 0_{\mathbb{F}}$ and $a^2 = (0, 0, \alpha^2) \neq 1_{T_n}$. It follows that a is real and non-central. Note that $a \notin \Phi(T_n)$.

Lemma 4.1.17. *Let G be a 2-group and $X \leq \mathrm{Aut}(G)$ a subgroup of order p, an odd prime. Suppose that $\mathbf{\Omega}_1(G) \leq \mathbf{C}_G(X)$. Then there is $x \in \mathrm{Re}(G)$ such that $|t| > 2$.*

Proof. The proof relies on results in [46]. Let $G_0 = [G, X]$. Since the action is coprime, we have that $G = G_0\mathbf{C}_G(X)$; therefore $G_0 > 1$, because $X \leq \mathrm{Aut}(G)$ is non-trivial. The action of X on $\bar{G}_0 = G_0/\ \Phi(G_0)$ is completely reducible. Moreover, it is fixed-point-free and, therefore, is faithful. To see this, take $g\ \Phi(G_0)$ such that $g^x\Phi(G_0) = g\Phi(G_0)$, with $X = \langle x \rangle$. Then, using [31, Satz I18.6], there is $t \in \mathbf{C}_{G_0}(X)$ such that $g\Phi(G_0) = t\Phi(G_0)$. By coprimity, we have that $\mathbf{C}_{\bar{G}_0}(X) = \overline{\mathbf{C}_{G_0}(X)}$. Moreover, $[\bar{G}_0, X] = \overline{[G_0, X]}$ and, by Fitting's Theorem, $\mathbf{C}_{G_0}(X) \cap [G_0, X] = 1$. Since $G_0 = [G_0, X]$, we have that $t \in [G_0, X] \cap \mathbf{C}_{G_0}(X) \leq \Phi(G_0)$. By Lemma 3 of [46], there is $T \leq G_0$ that is X-invariant and such that the action of X on T is faithful. By Theorem 1 in [46], we have that T is isomorphic to T_n for some n. Now, T has an element $t \in T \setminus \Phi(T)$ that is real of order > 2 (see the discussion in the paragraph before the Lemma) $\square$

4.2 Groups G such that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers

In this section we prove the main result of this chapter, Theorem 4.2.10, where we characterize the structure of a group G such that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers, assuming $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$. As a consequence, we obtain a bound for $\ell_2(G)$ and $\ell_{2'}(G)$ in Corollary 4.2.11.
In Lemma 4.1.12 we have seen that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers involving more than one prime if and only if property $(*)$ (see Definition 4.1.11) holds for G. In this case, by Corollary 4.1.15, we have that $(*)$ holds for $G/\mathcal{O}(G)$. Therefore, in the two following subsections, we assume that $\mathcal{O}(G) = 1$. Using Lemma 4.2.8, we will prove in Theorem 4.2.10 that $O(G)$ is a direct summand in G. We divide the work in two cases. The first is when $\mathbf{F}(G)$ is a 2-group and it is analysed in Subsection 4.2.1. The second, in Subsection 4.2.2, is when $\mathbf{F}(G)$ is not a 2-group .

4.2.1 The case where $\mathbf{F}(G/\mathcal{O}(G))$ is a 2-group

Proposition 4.2.1. *Let G be a group. Suppose that $F = \mathbf{F}(G) < G$ is a 2-group and that $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$. If $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers, then the followings hold.*

1. *$\mathcal{V}^*_G \subseteq \{2, p\}$ for an odd prime p and $\mathbf{F}_2(G) = F \rtimes L$, where L is a non-trivial p-group.*

2. *$\mathbf{\Omega}_1(F)$ elementary abelian and central in F.*

3. *L acts faithfully on $\mathbf{\Omega}_1(F)$;*

4. *If* $V = [\mathbf{\Omega}_1(F), \mathbf{F}_2(G)]$ *and* $U = \mathbf{C}_{\mathbf{\Omega}_1(F)}(\mathbf{F}_2(G))$*, then* V *and* U *are* G*-invariant and* $\mathbf{C}_G(V) = F$*.*

5. $|v^G|$ *is a* p*-power for all* $v \in \mathbf{\Omega}_1(F)$*.*

Proof. By Theorem 4.1.3, G is solvable and $\mathcal{V}_G^* \subseteq \{2, p\}$ for an odd prime p. By hypothesis, $\mathbf{F}_2(G)/F > 1$ and has odd order. Write $\mathbf{F}_2(G) = L \ltimes F$ for some nilpotent $2'$-group L. The group L acts faithfully on F. Take $Q \in \mathrm{Syl}_q(L)$, for some prime $q \neq p$. Since $FQ \trianglelefteq G$, $\mathrm{cs}_{\mathbb{R}}(FQ)$ consists of prime powers involving only the primes 2 and p. Therefore, since FQ is a $\{2, q\}$-group, we have that $\mathrm{cs}_{\mathbb{R}}(FQ)$ are 2-powers. By Theorem 4.1.6, we have that $FQ = F \times Q$. This means that Q centralizes F and hence $Q = 1$. So part 1 follows.
Note that $\mathbf{\Omega}_1(F)$ is elementary abelian and central in F, since $\mathbf{\Omega}_1(F) \subseteq \langle \mathrm{Re}(F) \rangle \subseteq \subseteq \mathbf{Z}(F)$. So part 2 follows.
We now prove part 3. Observe that L acts on $\mathbf{\Omega}_1(F)$. Suppose that there is $1 < X < L$ that acts trivially on $\mathbf{\Omega}_1(F)$. Clearly, we can assume that $X \leq \mathbf{Z}(L)$ and has prime order. The group X acts faithfully on F and fixes all the involutions of F. By Lemma 4.1.17, F has a real element t of order > 2, but this is against $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$.
We prove part 4. Call $U = \mathbf{C}_{\mathbf{\Omega}_1(F)}(\mathbf{F}_2(G))$ and $V = [\mathbf{\Omega}_1(F), \mathbf{F}_2(G)]$ and note that V and U are characteristic subgroups of $\mathbf{F}_2(G)$. It follows that V and U are G-invariant. Moreover, observe that $V = [\mathbf{\Omega}_1(F), L]$ and $U = \mathbf{C}_{\mathbf{\Omega}_1(F)}(L)$. Let $C = \mathbf{C}_G(V)$, we prove that $C = F$. Observe that the group C is 2-closed. Indeed, $\mathbf{C}_V(L) = 1$ and if $v \in V^{\#}$, then $v^G \in \mathrm{cl}_{\mathbb{R},p}(G)$, since v, being an involution, is real in G. Therefore, $C \leq \mathbf{C}_G(v)$, that is 2-closed by Lemma 4.1.9. As before, write $\mathbf{F}_2(C) = T \rtimes R$, for a nilpotent group R of odd order. Since $T = \mathbf{O}_2(C) = F$, the group R is contained in some conjugate of L that acts faithfully on V, and this implies that $R = 1$. We have proved that C is a non-trivial 2-group and this implies that $C = F$.
We now prove part 5. Observe that $\mathbf{\Omega}_1(F) = U \times V$ is the Fitting decomposition of the action of L on $\mathbf{\Omega}_1(F)$. Suppose that there is $w \in \mathbf{\Omega}_1(F)$ such that $w^G \in \mathrm{cl}_{\mathbb{R},q}(G)$ for a prime q different from p. Hence, without loss of generality, we can assume that $\mathbf{C}_G(w)$ contains L. So, $w \in U$. On the other hand, as $\mathbf{C}_L(V) = 1$, there is $v \in V$ such that $|v^L|$ is divisible by p. Since $v^L = v^{\mathbf{F}_2(G)}$, it follows by hypothesis $v^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. Note that $U, V \trianglelefteq G$ and $U \cap V = 1$. Therefore $\mathbf{C}_G(wv) = \mathbf{C}_G(w) \cap \mathbf{C}_G(v)$ and it follows that pq divides $|(uv)^G|$, that is impossible. □

Proposition 4.2.2. *In the situation of Proposition 4.2.1, suppose that* $(*)$ *holds for a group* G*. Then the followings hold.*

1. V *is an irreducible* G/F*-module and* $\mathbf{C}_G(V) = 1$*.*

2. *G/F is isomorphic to a subgroup of* $\Gamma(V)$ *and G/F is dihedral of order* 18 *or* $2p$*, with p a Fermat prime.*

3. $\mathbf{\Omega}_1(F) = U \times V$ *where* $U \leq \mathbf{Z}(G)$.

4. *If yF is an involution of G/F for some 2-element $y \in G$, then* $y \notin \mathrm{Re}(G)$.

Proof. Call $H = G/F$. Suppose by contradiction that H has odd order. By hypothesis $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$, and by Theorem 4.1.4 all the real class sizes of G have odd size. On the other hand, by $(*)$, we have that $\mathcal{V}_G^* = \{2, p\}$ and this is impossible. Therefore H has even order. Let V and U be as in the proof of Proposition 4.2.1. For every $v \in V^{\#}$, $v^G \in \mathrm{cl}_{\mathbb{R},p}(G)$ and, by Lemma 4.1.9, $\mathbf{C}_G(v)$ is 2-closed. Hence $\mathbf{C}_H(v)$ contains a Sylow 2-subgroup of H as a normal subgroup. Then, (H, V) satisfies the condition $\mathcal{N}_2$ and by Proposition 2.4.4, we have that V is irreducible and $H \leq \Gamma(V)$. By part 3 of Proposition 2.2.6, H is a dihedral group of order 18 or $2p$ with p a Fermat prime. This means that $\mathbf{F}(H)$ is cyclic and its order is a Fermat prime or 9. Clearly, $\mathbf{F}(H)$ is the image of L in H, where L is the group in part 1 of Proposition 4.2.1. Moreover, L acts trivially on U, that is central in F, and hence $|G/\mathbf{C}_G(U)| \leq 2$. By part 5 of Proposition 4.2.1, we have that $|v^G|$ is a p-power for every $v \in V \times U$, therefore $[G : \mathbf{C}_G(U)] = 1$ and hence $U \leq \mathbf{Z}(G)$.
Let $1 \neq t$ be an involution of H, hence $t = yF$ for some 2-element $y \in G$ such that $y^2 \in F$. Suppose by contradiction that $y \in \mathrm{Re}(G)$. Now, y does not centralize $\mathbf{\Omega}_1(F)$, otherwise we would have $[y, V] = 1$ and $y \in \mathbf{C}_G(V) = F$, but $y \notin F$. Therefore, $|y^{\mathbf{\Omega}_1(F)}| > 1$. This implies that 2 divides $[G : \mathbf{C}_G(y)]$ and, by hypothesis, $y^G \in \mathrm{cl}_{\mathbb{R},2}(G)$. So, a conjugate of L, say L^g for $g \in G$, centralizes y. Therefore $G/F = (L^gF/F) \times \langle t \rangle$ is cyclic and not dihedral, impossible. □

4.2.2 The case where $\mathbf{F}(G/\mathcal{O}(G))$ is not a 2-group

Definition 4.2.3. We write $M \cdot\!\lhd G$ to indicate that M is a minimal normal subgroup of the group G. If M is an elementary abelian p-group, then we call p the *characteristic* of M and we write $p = \mathrm{char}(M)$.

We recall that, according to Definition 2.3.6, diagonal elements of a wreath product are those in the base group whose entries are all equal.

Proposition 4.2.4. *Suppose that $(*)$ holds for G. Let $M \cdot\!\lhd G$ of odd order and not contained in $\mathcal{O}(G)$, and $t \in \mathrm{Re}(G)$ such that $1 < |t^G|$ is odd. Then, $|t| = 2$ and t acts as the inversion on M. Moreover,* $\mathrm{char}(M) \in \mathcal{V}_G^*$ *and, called $\bar{G} = G/\mathbf{C}_G(M)$, one of the following holds.*

1. *$\bar{G}$ is a 2-group and $\bar{t} \in \mathbf{Z}(\bar{G})$.*

2. $\mathrm{char}(M) = 3$ *and* $\bar{G}$ *is isomorphic to a subgroup of* $H \wr K$*, where* K *is a* 2-*group,* $H \simeq \mathrm{SL}_2(3)$ *or* $\mathrm{GL}_2(3)$ *and* $H = \mathbf{N}_{\bar{G}}(M_0)/\mathbf{C}_{\bar{G}}(M_0)$ *for some subspace* M_0 *such that* $M_0^G = M$ *and* $|M_0| = 3$*. In this case,* $\bar{t}$ *the image of* $\bar{t}$ *in* $H \wr K$ *is central and diagonal in* $H \wr K$*. In particular,* $\bar{t}$ *is central in* $\bar{G}$*.*

Proof. By $(*)$, the group G is solvable, $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers and $\mathcal{V}_G^* = \{2, p\}$. Since $M \cdot\!\lhd G$, we can see M as an irreducible faithful $\bar{G}$-module. Suppose that $\mathrm{char}(M) = q$, with q an odd prime, and let $L_0 = \mathbf{O}_{2'}(\bar{G})$. Since M is not contained in $\mathcal{O}(G)$, $\bar{G}$ has even order and hence there is an involution $z \in \bar{G}$. Note that there is $v \in M^{\#}$ such that $v^z = v^{-1}$ (indeed, there is $m \in M$ such that $m^z \neq m$, it is enough to choose $v = m^{-1}m^z$); therefore, v is real and if $\tilde{z} \in G$ is an element in the preimage of z, then $\tilde{z} \in \mathbf{N}_G(\mathbf{C}_G(v)) \setminus \mathbf{C}_G(v)$. Hence 2 divides $[G : \mathbf{C}_G(v)]$. By hypothesis $v^G \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $L_0 \leq \mathbf{C}_{\bar{G}}(v)$. So, $1 \neq v \in \mathbf{C}_M(L_0)$. By the irreducibility of M, we have that $M = \mathbf{C}_M(L_0)$. It follows that $L_0 = 1$ and that $\mathbf{F}(\bar{G})$ is a 2-group. Let $Z = \mathbf{\Omega}_1(\mathbf{Z}(\mathbf{F}(\bar{G})))$. Note Z that is an elementary abelian 2-group. Since Z is normal in $\bar{G}$ and M is irreducible, we have that $M = [Z, M]$ and $\mathbf{C}_M(Z) = 1$. By Maschke's Theorem, M is a completely reducible Z-module. We now prove that Z acts as the inversion on every irreducible Z-constituent of M. Let M_1 an irreducible Z-constituent of M. Since Z is elementary abelian, $Z/\mathbf{C}_Z(M_1)$ has order 2 by [45, Lemma 0.5]. Thus, $Z/\mathbf{C}_Z(M_1)$ acts as the inversion on M_1. In addition, M_1 has order p and every element of M_1 is real.
Since $p \in \mathcal{V}_G^*$, there is $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. By Lemma 4.1.9, the element t is an involution. Call $\bar{t}$ the image of t in $\bar{G}$. Note that $\mathbf{C}_G(t)$ contains a Sylow 2-subgroup of G and therefore $\bar{t}$ centralizes $\mathbf{F}(\bar{G})$, so $\bar{t} \in Z$. The element t acts on M; suppose by contradiction that $\mathbf{C}_M(t) > 1$. Hence, $\mathbf{C}_M(\bar{t}) > 1$. Take $M_1 \leq \mathbf{C}_M(\bar{t})$ and irreducible Z-constituent. Then, as remarked earlier, Z acts on M_1 as the inversion. Thus, if $1 \neq m \in M_1$, the element m is real. Note that m and t have coprime orders, commute and $m^G \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. By part 1 of Lemma 4.1.7, tm is a real element that generates a class of size a multiple of $2p$, against the hypothesis. Therefore, $\mathbf{C}_M(t) = 1$ and t acts as the inversion in M. As a consequence, we have that q divides $|t^G|$, that therefore is a q-power. Thus, $q = p \in \mathcal{V}_G^*$. Moreover, for all $v \in M$, $\mathbf{C}_G(v)$ contains a Hall $2'$-subgroup of G. Observe that t centralizes a Sylow 2-subgroup of G. If $\bar{G}$ is a 2-group, $\bar{t} \in \mathbf{Z}(\bar{G})$. Suppose that $\bar{G}$ is not a 2-group. By Theorem 2.3.7, we have that either $\bar{G}$ is a 2-group or $p = 3$ and there is a monomorphism $i \colon \bar{G} \to H \wr K$ where K is a 2-group, $H \simeq \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ and $H = \mathbf{N}_{\bar{G}}(M_0)/\mathbf{C}_{\bar{G}}(M_0)$ for some $M_0 \leq M$ such that M, as G-module, is induced by M_0. Note that $t \in \mathbf{N}_G(M)$. Therefore, by part 4 of Theorem 2.3.7, the image of t in $H \wr K$ is diagonal and it is centralized by K. Since H is isomorphic to $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$, we have that $\mathbf{F}(H)$ is the quaternion group and $\bar{t}\mathbf{C}_G(M) \in \mathbf{F}(H)$. It follows that $\bar{t}\mathbf{C}_G(M)$ is the unique central involution of H. It follows that the image of t in $H \wr K$ is central. Since $\bar{G}$ is isomorphic to a

subgroup of $H \wr K$, $\bar{t} \in \mathbf{Z}(\bar{G})$. □

Lemma 4.2.5. *Let G be a group, $F = \mathbf{F}(G)$ and $N = \mathbf{O}_{2'}(F)$. Suppose that N is an elementary abelian 3-group, that $\mathbf{C}_G(N) = F$ and that $N = N_1 \times \cdots \times N_k$, where, for every i, $N_i \cdot\!\lhd G$ and N_i is not contained in $\mathcal{O}(G)$ for all i. Assume that, for all i, $G/\mathbf{C}_G(N_i)$ is either a 2-group or isomorphic to $\mathrm{SL}_2(3)$. Let G_0 be a complement for N in G and $x \in \mathrm{Re}(G_0)$ such that 3 divides $|x^{G_0}|$. Then G contains a real class whose length is a composite number.*

Proof. We work by contradiction, hence we assume that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. Since $G_0 \simeq G/N$, we have that $xN \in \mathrm{Re}(G/N)$ and 3 divides $|(xN)^{G/N}|$. By part 4 of Lemma 4.1.7, 3 divides $|x^G|$. Therefore $x^G \in \mathrm{cl}_{\mathbb{R},3}(G)$ and x is an involution by Lemma 4.1.9. In particular, $3 \in \mathcal{V}_G^*$. Moreover, G/F is isomorphic to a subgroup of $\prod_i G/\mathbf{C}_G(N_i)$ and, since 3 divides $|G/N|$, without loss of generality we can assume that $G/\mathbf{C}_G(N_1) \simeq \mathrm{SL}_2(3)$. The central involution of $G/\mathbf{C}_G(N_1)$ acts as the inversion on N_1 and hence every element of N_1 is real. Take $1 \neq n \in N_1$, then n has odd order and, by assumption, $n^G \in \mathrm{cl}_{\mathbb{R},2}(G)$. It follows that $2 \in \mathcal{V}_G^*$ and, by Lemma 4.1.12, property $(*)$ holds for G.
Since $\cap_i \mathbf{C}_G(N_i) = \mathbf{C}_G(N) = F$, the group G/F is isomorphic to a subgroup of $\prod_i G/\mathbf{C}_G(N_i)$, applying Proposition 4.2.4 to the action of x every N_i, we have that x acts as the inversion on N and xF is central in G/F. Since $x^G \in \mathrm{cl}_{\mathbb{R},3}(G)$ and $\mathbf{C}_G(N) = N\mathbf{C}_{G_0}(N)$, there is $S \in \mathrm{Syl}_2(G_0)$ such $x \in S$. Clearly, $x \in \mathbf{\Omega}_1(\mathbf{Z}(S))$. Let $H \in \mathrm{Hall}_{2'}(G)$ and $T = \mathbf{O}_2(G)$. Note that $\mathbf{C}_{G_0}(N) \trianglelefteq G$ and $\mathbf{C}_{G_0}(N) \leq F$, so $\mathbf{O}_{2'}(\mathbf{C}_{G_0}(N)) \leq N \cap G_0 = 1$. So, $\mathbf{C}_{G_0}(N)$ is a 2-group and it follows that $\mathbf{C}_{G_0}(N) = T \leq S$. Note that $G_0/\mathbf{C}_{G_0}(N_i) \simeq G/\mathbf{C}_G(N_i)$ is 2-closed for every i; so, it follows that $\prod_i G/\mathbf{C}_G(N_i)$ is 2-closed. As G_0/T is isomorphic to a subgroup of $\prod_i G/\mathbf{C}_G(N_i)$, we have that G_0 is 2-closed. This means that $S \trianglelefteq G_0$ and $x^{G_0} \subseteq \mathbf{\Omega}_1(\mathbf{Z}(S))$. Since xF is central in G/F, we have that $x^G F = xF$. Therefore, for every $h \in H$, $x^h F = xF$ and $[x,h] = xx^h \in F \cap S = T = \mathbf{O}_2(G)$. Call $V = [H, \langle x^{G_0}\rangle]$. Note that $V \leq \mathbf{\Omega}_1(\mathbf{Z}(S)) \cap T$. Moreover, H acts non-trivially on V, otherwise, using coprime actions, the following

$$1 = [V,H] = [\langle x^{G_0}\rangle, H, H] = [\langle x^{G_0}\rangle, H]$$

implies that H commutes with x and this is impossible since $x \in \mathrm{cl}_{\mathbb{R},3}(G)$. Therefore, there is $v \in V$ such that $|v^H| > 1$. Since $v \in \mathbf{\Omega}_1(\mathbf{Z}(S)) \cap T$, we have that $v^H = v^{G_0}$. Moreover, v commutes with N and thus $v^H = v^G$ and $v^G \in \mathrm{cl}_{\mathbb{R},3}(G)$. Take any non-trivial $n \in N$, since x acts as the inversion on N, n is real and $n^G \in \mathrm{cl}_{\mathbb{R},2}(G)$, because n has odd order and we are assuming that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. By part 3 of Lemma 4.1.7, $nv \in \mathrm{Re}(G)$ and 6 divides $|(nv)^G|$ and this is impossible. □

Compare the method used in Proposition 4.2.5 with Proposition 4.3.5.

Lemma 4.2.6. *Let G be a finite group and π a set of primes. Suppose that $\mathbf{O}_\pi(\Phi(G)) = 1$. Then $\mathbf{O}_\pi(\mathbf{F}(G))$ has a complement in G and is completely reducible as G-module.*

Proof. Call $N = \mathbf{O}_\pi(\mathbf{F}(G))$, $F = \mathbf{F}(G)$ and consider $\bar{G} = G/\Phi(G)$. Observe that N has a complement G_0 in G by [31, III Hilfssatz 4.4]. By Gaschütz theorem (see [45, Theorem 1.12]), we have that $\bar{F}$ is a completely reducible $\bar{G}$-module. Since $\bar{N} \simeq N$ as G-module (because $N \cap \Phi(G) = 1$), we have that N is completely reducible under the action of G. □

Proposition 4.2.7. *Suppose that G is a finite group, $\mathcal{O}(G) = 1$ and that $(*)$ holds for G. Suppose that $\mathbf{O}_{2'}(\Phi(G)) = 1$. Call $F = \mathbf{F}(G)$ and $N = \mathbf{O}_{2'}(F)$. Assume that $N > 1$. Then, N is a p-group with $p \in \mathcal{V}_G^*$, $\mathbf{C}_G(N) = F$ and G/F is a 2-group. Moreover, if $t \in \mathrm{Re}(G)$ is such that $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$, then $|t| = 2$ and t acts as the inversion on N.*

Proof. We recall that from $(*)$ it follows that G is solvable, $\mathcal{V}_G^* = \{2, p\}$ for an odd prime p and that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. By Lemma 4.2.6, N has a complement G_0 in G and it is completely reducible as a G-module. If we call $C = \mathbf{C}_G(N)$, then C has a complement T for N and hence $C = T \times N$. Note furthermore that $T \lhd\lhd G$, that $\mathbf{F}(\mathbf{O}_{2'}(T)) \leq T \cap N = 1$ and, therefore, that $\mathbf{O}_{2'}(T) = 1$. Moreover, since $(*)$ holds for G and T is subnormal in G, $\mathrm{cs}_{\mathbb{R}}(T)$ consists of prime-powers, involving at most the primes 2 and p. Suppose $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(T)$. Then, by Lemma 4.1.9, the class $\mathcal{C}$ is generated by an involution x. Since T is subnormal in G, it follows by hypothesis that $x^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. But x acts trivially on N and hence x acts trivially on a minimal normal subgroup of G contained in N. This is against Proposition 4.2.4. Hence the real classes of T have lengths that are 2-powers. By Theorem 4.1.6, T is 2-nilpotent. Since $\mathbf{F}(\mathbf{O}_{2'}(T)) = 1$, it follows that T is a 2-group, $T = \mathbf{O}_2(F)$ and $C = F$.
Since $(*)$ holds for G, there is $t \in \mathrm{Re}(G)$ such that $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$ and t is an involution by Lemma 4.1.9. Write $N = N_1 \times \cdots \times N_n$, where N_i is a minimal normal subgroup of G. Note that $G/F \simeq G_0/T$ and $\mathbf{C}_{G_0}(N) = T$. Applying Proposition 4.2.4 to N_i for every i, it follows that N is a p-group (where $p \in \mathcal{V}_G^*$) and that t acts as the inversion on N. In particular, every $v \in N$ is real. In addition, again by Proposition 4.2.4, $G_0/\mathbf{C}_{G_0}(N_i)$ is either a 2-group or it is isomorphic to a subgroup of $H_i \wr K_i$, where H_i is the quotient of a subgroup of G and it is isomorphic to $\mathrm{GL}_2(3)$ or $\mathrm{SL}_2(3)$.
Assume by contradiction that G/F is not a 2-group. So, G_0/T is not a 2-group. As $T = \bigcap_i \mathbf{C}_{G_0}(N_i)$ and G_0 is isomorphic to a subgroup of $\prod_i G_0/\mathbf{C}_{G_0}(N_i)$, we can assume that N_1 is such that $G_0/\mathbf{C}_{G_0}(N_1)$ is isomorphic to a subgroup of $H_1 \wr K_1$, where H_1 is isomorphic to either $\mathrm{GL}_2(3)$ or $\mathrm{SL}_2(3)$. In this case, we have that $\mathrm{char}(N_i) = 3$ for every i, so N is a 3-group and $3 \in \mathcal{V}_G^*$. Suppose that

$H_1 \simeq \mathrm{GL}_2(3)$; by Proposition 4.2.4, there is $M \leq N_1$ such that $\mathbf{N}_{G_0}(M)/\mathbf{C}_{G_0}(M)$ is isomorphic to $\mathrm{GL}_2(3)$ and $|M| = 9$. Since $\mathrm{GL}_2(3)$ contains a real element of order 3, there is $g\mathbf{C}_{G_0}(M) \in \mathrm{Re}(H_1)$ of order 3. By part 2 of Lemma 4.1.7, we can assume that $g \in \mathrm{Re}(\mathbf{N}_{G_0}(M))$ and that g has odd order. So, 2 divides $|g^G|$. On the other hand, 3 divides $|g^G|$, because g acts non-trivially on N, that is a 3-group. This means that g^G is a real class whose length is a composite number and this is impossible. This argument shows that $H_i \not\simeq \mathrm{GL}_2(3)$ for all i. So, without loss of generality, we can assume the following working situation: N is a 3-group and there is $1 \leq k \leq n$ such that $G_0/\mathbf{C}_{G_0}(N_i) \simeq H_i \wr K_i$ with $H_i \simeq \mathrm{SL}_2(3)$ for $i \leq k$ and $G_0/\mathbf{C}_{G_0}(N_i)$ a 2-group for $i > k$.
If $i \leq k$, write B_i for the base group of $H_i \wr K_i$; if $i > k$, write $B_i = G/\mathbf{C}_{G_0}(N_i)$, that is a 2-group. There is a monomorphism $\sigma \colon G_0/T \to \prod_{i=1}^{n} G_0/\mathbf{C}_{G_0}(N_i)$. Let B be the preimage of $\sigma^{-1}(\prod_{i=1}^{n} B_i)$ in G_0. Observe that $B \trianglelefteq G_0$ and that G_0/B is isomorphic to a subgroup of $\prod_{i=1}^{k} K_i$, that is a 2-group. Since G_0 is not a 2-group, we have that B contains every Hall $2'$-subgroup of G_0. Note that $\Delta^*_{\mathbb{R}}(G_0) = \Delta^*_{\mathbb{R}}(G/N)$ and that $\Delta^*_{\mathbb{R}}(G/N)$ is a subgraph of $\Delta^*_{\mathbb{R}}(G)$, by Lemma 4.1.10. Therefore, $\mathcal{V}^*_{G_0} \subseteq \{2, 3\}$ and $\mathrm{cs}_{\mathbb{R}}(G_0)$ consists of prime power numbers. Suppose that $3 \in \mathcal{V}^*_{G_0}$. Then, there is $x \in G_0$ such that $x^{G_0} \in \mathrm{cl}_{\mathbb{R},3}(G_0)$ and x is an involution, by Lemma 4.1.9. By Lemma 4.1.7, we have that $x^G \in \mathrm{cl}_{\mathbb{R},3}(G)$, so, as remarked earlier, x acts as the inversion on N. If $i > k$, then $x\mathbf{C}_{G_0}(N_i) \in B_i$. If $i \leq k$, by Proposition 4.2.4, we have that $x\mathbf{C}_{G_0}(N_i)$ lies in the base group of $H_i \wr K_i$. So, $\sigma(x) \in \prod_{i=1}^{n} B_i$. This implies that $x \in B$.
We now prove that the hypotheses of Lemma 4.2.5 are satisfied for the group NB. Observe firstly that N an elementary abelian 3-group and that N is completely reducible under the action of B, by Clifford's Theorem. Moreover, B acts on every irreducible B-constituent of N as a $\mathrm{SL}_2(3)$ or a 2-group. Since $B \leq G_0$, that is a complement for N, we have that B is a complement for N in NB. Moreover, $\mathbf{O}_{2'}(NB) \leq \mathbf{O}_{2'}(G) = N \leq \mathbf{O}_{2'}(NB)$, so $\mathbf{O}_{2'}(NB) = N$ and $x^{G_0} = x^B$, as $x \in B$ and every Hall $2'$-subgroup of G_0 is contained in B. This means that $x^B \in \mathrm{cl}_{\mathbb{R},3}(B)$. As remarked earlier, x acts as the inversion on N. So, none of the B-constituent of N are contained in $\mathcal{O}(NB)$. Since $NB \trianglelefteq G$, we have that $\mathbf{F}(NB) = F$ and $\mathbf{C}_{BN}(N) \leq \mathbf{C}_G(F) \cap NB = F \cap NB = \mathbf{F}(NB)$. By Lemma 4.2.5 applied on NB, we have that NB, and hence G, has a real class of composite size, which is impossible.
Hence we can assume that $\mathcal{V}^*_{G_0} \subseteq \{2\}$. By Theorem 4.1.6, we have that G_0 is 2-nilpotent. This means that $\mathbf{C}_G(n)$ is 2-nilpotent for every $n \in N$ and, using the fact that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers and that $N \subseteq \mathrm{Re}(G)$, it follows that $\mathbf{C}_G(n)$ contains a Hall $2'$-subgroup of G for all $n \in N$. Therefore $(G/F, N)$ satisfies the property $\mathcal{N}_{2'}$. By Proposition 2.4.4, N is irreducible. Note that $\mathbf{C}_{G_0}(N) = T$ is a 2-group and $G_0/T \simeq \mathrm{SL}_2(3)$, and this is impossible since G_0 is 2-nilpotent. □

Lemma 4.2.8. *Let G be a group and $H \trianglelefteq G$ a subgroup of odd order. Suppose that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. If there is $M \trianglelefteq G$, $M \leq H$, and $t \in G$ such that t acts as the inversion on H/M, then there is $N \leq \mathbf{Z}(H)$ such that $NM = H$ and t acts as the inversion on N. If furthermore $M \leq \mathcal{O}(G)$, then $N \times M = H$.*

Proof. Call $A = H/M$. The automorphism t acts as the inversion on A that, therefore, is abelian. Moreover, every element of A is real in $\langle t \rangle \ltimes A$. Let $A \setminus 1 = \{a_1 \dots a_n\}$. For every i, $1 \neq a_i \in \mathrm{Re}(\langle t \rangle H/M)$ and by part 2 of Lemma 4.1.7, there is $x_i \in \mathrm{Re}(H\langle t \rangle)$ such that $x_i M = a_i$. Note that x_i is non-trivial and 2 divides $|(x_i)^G|$; so, it follows by hypothesis that $x_i^G \in \mathrm{cl}_{\mathbb{R},2}(G)$. Since $|x_i^H|$ divides $|x_i^G|$ and $\mathrm{cs}_{\mathbb{R}}(G)$ consists in prime powers, it follows that $x_i \in \mathbf{Z}(H)$. Moreover, $x_i \in \mathrm{Re}(H\langle t \rangle)$, so, for every i, a conjugate of t inverts x_i; since $x_i \in \mathbf{Z}(H)$, we have that t inverts x_i. If $N_0 = \langle x_1 \dots x_n \rangle$, then $N_0 \leq \mathbf{Z}(H)$ and $N_0 M = H$. If $N = [N_0, \langle t \rangle]$, then $N_0 = N \times \mathbf{C}_{N_0}(\langle t \rangle)$ by the Fitting's decomposition. Let $1 \neq x \in \mathbf{C}_{N_0}(\langle t \rangle)$; then, $xM \in \mathbf{C}_A(\langle t \rangle) = 1$ and $x \in M$. It follows that $H = NM$ and that t acts as the inversion on N, since $\mathbf{C}_N(\langle t \rangle) = 1$. This means that $N \subseteq \mathrm{Re}(G)$. Suppose that $M \leq \mathcal{O}(G)$, and consider $n \in N \cap M$. Then $n \in \mathcal{O}(G)$ and hence $n = 1$ by part 3 of Lemma 4.1.14. This means that $N \cap M = 1$. Since $H = NM$, we have that $H = N \times M$. $\square$

Proposition 4.2.9. *Suppose that $(*)$ holds for a group G such that $\mathcal{O}(G) = 1$. Call $F = \mathbf{F}(G)$ and $N = \mathbf{O}_{2'}(F)$. If $N > 1$, then N is an abelian p-group for $p \in \mathcal{V}_G^*$, $F = \mathbf{C}_G(N)$ and G/F is a 2-group. Moreover, if $t \in \mathrm{Re}(G)$ and $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$, then $|t| = 2$, t acts as the inversion on N and $tN \in \mathbf{Z}(G/N)$.*

Proof. Let $M = \mathbf{O}_{2'}(\Phi(G))$ and $N = \mathbf{O}_{2'}(G)$. Call $\bar{G} = G/M$. Note that $\bar{F} = \mathbf{F}(\bar{G})$ by [31, III3.5] and $\Phi(\bar{G}) = \overline{\Phi(G)}$, since $\Phi(G)$ is the intersection of all maximal subgroups of G. It follows that $\mathbf{O}_{2'}(\Phi(\bar{G})) = 1$. By Lemma 4.1.15, we have that $(*)$ holds for $\bar{G}$. Hence, by Proposition 4.2.7 applied to $\bar{G}$, we have that $\bar{G}/\mathbf{F}(\bar{G}) = (G/M)/(F/M) \simeq G/F$ is a 2-group, that $\mathbf{C}_{\bar{G}}(\bar{N}) = \bar{F}$ and that $\bar{N}$ is a p-group for $p \in \mathcal{V}_G^*$. Observe that $\overline{\mathbf{C}_G(N)} \leq \mathbf{C}_{\bar{G}}(\bar{N}) \leq \bar{F}$, so $\mathbf{C}_G(N) \leq F$.
Now we prove that N is abelian and if $t \in \mathrm{Re}(G)$ such that $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$, then t is an involution that acts as the inversion on N. Let $t \in \mathrm{Re}(G)$ such that $t^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. By part 1 of Lemma 4.1.7, $|t| = 2$. Moreover, t is centralized by a Sylow 2-subgroup of G and therefore $tN \in \mathbf{Z}(G/N)$, because $N \in \mathrm{Hall}_{2'}(G)$. We now prove that N is abelian. Consider $y = t\Phi(N) \in G/\Phi(N)$, then y is an involution and the conjugacy class in $G/\Phi(N)$ generated by y has length a p-power number. If y is central in $G/\Phi(N)$, then $K = \langle t, \Phi(N) \rangle$ is normal in G and $\langle t \rangle$ is a Sylow 2-subgroup of K. By the Frattini argument, $G = \mathbf{C}_G(t)\Phi(N) = \mathbf{C}_G(t)\Phi(G)$, since $\Phi(N) \leq \Phi(G)$. It follows that $G = \mathbf{C}_G(t)$ and this is impossible. Hence y is non-central in $G/\Phi(N)$ and its conjugacy class in $G/\Phi(G)$ has length a non-trivial p-power number. Note that condition $(*)$ holds for $G/\Phi(N)$ by Corollary 4.1.15.

Note that Sylow 2-subgroup S of G acts coprimely on the quotient $V = N/\Phi(N)$, which is a faithful completely reducible S-module. If there is $1 \neq v \in V$ such that $v \in \mathbf{C}_S(V)$, then by [31, I Satz 18.6], there is $1 \neq n \in \mathbf{C}_N(S)$. In particular, $n \in \mathcal{O}(G) = 1$ and this is impossible. So, $\mathbf{C}_S(V) = 1$ and $\mathcal{O}(G/\Phi(N)) = 1$. By Proposition 4.2.4, y acts as the inversion on every irreducible constituent of V. It follows that t acts as the inversion on V. By Lemma 4.2.8, there is $N_0 \leq \mathbf{Z}(N)$ such that $N = N_0\Phi(N)$, therefore $N = N_0$ is abelian and t acts as the inversion on N. So, $N \leq \mathbf{C}_G(N)$ and $\mathbf{C}_G(N) = F$. □

4.2.3 Main Theorem

We now prove the main theorem of this chapter.

Theorem 4.2.10. *Let G be a group such that $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$. Then $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers and $|\mathcal{V}_G^*| > 1$ if and only if $\mathcal{V}_G^* = \{2, p\}$ for an odd prime p, $G = G_0 \times \mathcal{O}(G)$ and, writing $F = \mathbf{F}(G_0)$, $N = \mathbf{O}_{2'}(F)$, one of the following holds.*

1. *N is a non-trivial, abelian p-group, $\mathbf{C}_{G_0}(N) = F$ and G_0/F is a 2-group. Moreover, $\mathrm{Re}(G_0) \setminus F \neq \emptyset$ and if $x \in \mathrm{Re}(G_0) \setminus F$, then $|x| = 2$, $x^G \in \mathrm{cl}_{\mathbb{R},p}(G)$ and x acts as the inversion on N.*

2. *F is a 2-group, $\mathrm{Re}(F) = \mathbf{\Omega}_1(F) = U \times V$, with U and V are elementary abelian 2-subgroups that satisfy the following conditions:*

   (a) $U = \mathbf{C}_{\mathbf{\Omega}_1(F)}(\mathbf{F}_2(G_0))$ and $U \leq \mathbf{Z}(G_0)$,

   (b) $V = [\mathbf{\Omega}_1(F), \mathbf{F}_2(G_0)]$ and $\mathbf{C}_{G_0}(V) = F$.

   Moreover, V is an irreducible G_0/F-module, $G_0/F \lesssim \Gamma(V)$, G_0/F is dihedral of order 18 or $2p$, where p is a Fermat prime and if tF is an involution of G_0/F for $t \in G_0$, then $t \notin \mathrm{Re}(G_0)$.

Proof. Let G be a group such that $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$ and $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. The group G satisfies the property $(*)$ by Theorem 4.1.3. Let K be a normal subgroup such that $K/\mathcal{O}(G) = \mathcal{O}(G/\mathcal{O}(G))$. Hence T centralizes $K/\mathcal{O}(G)$ and by coprimality $K = \mathcal{O}(G)\mathbf{C}_K(T) = \mathbf{C}_K(T)$. This means that $K = \mathcal{O}(G)$. In particular, if $\bar{G} = G/\mathcal{O}(G)$ then $\mathcal{O}(\bar{G}) = 1$. Note that property $(*)$ holds for $\bar{G}$ by Corollary 4.1.15. In particular, G is solvable and $\mathcal{V}_G^* = \{2, p\}$.
Suppose first that $\mathbf{O}_{2'}(\mathbf{F}(\bar{G})) > 1$, call M the preimage of $\mathbf{O}_{2'}(\mathbf{F}(\bar{G}))$ in G. By Proposition 4.2.9, $\bar{G}/\mathbf{F}(\bar{G})$ is a 2-group and $\bar{M}$ is an abelian p-group. Moreover, there is $y \in \mathrm{Re}(\bar{G})$ such that $y^{\bar{G}} \in \mathrm{cl}_{\mathbb{R},p}(\bar{G})$ and y acts as the inversion on $\bar{M}$. By Lemma 4.1.7, there is $x \in \mathrm{Re}(G)$ such that $\bar{x} = y$ and $x^G \in \mathrm{cl}_{\mathbb{R},p}(G)$; so, x is an involution by part 1 of Lemma 4.1.7. The element y acts as the inversion on

$\bar{M}$ and $y\bar{M} \in \mathbf{Z}(\bar{G}/\bar{M})$. Therefore, by Lemma 4.2.8, there is $N \leq M$ such that $N \times \mathcal{O}(G) = M$. Call $G_0 = TN$ where $T \in \mathrm{Syl}_2(G)$. Note that G_0 is a direct factor for G and $\mathbf{C}_{G_0}(N) = F$.
We now prove the remaining part of 1. Suppose that $\mathrm{Re}(G) \subseteq F$. Then, every real element is centralized by N, which is a Sylow p-subgroup of G_0. Therefore, p doesn't appear as a divisor of any real class size and this is against our assumption. This proves that $\mathrm{Re}(G) \setminus F \neq \emptyset$. Let $x \in \mathrm{Re}(G) \setminus F$. Since $\mathbf{C}_G(N) = F$, the element x acts non-trivially on N, therefore $|x^N| > 1$. So p divides $|x^G|$ therefore $x^G \in \mathrm{cl}_{\mathbb{R},p}(G)$ and, by Lemma 4.1.9, x is an involution. By Proposition 4.2.9, x acts as the inversion on N.

Suppose now that $\mathbf{F}(\bar{G})$ is a 2-group. If M is the preimage of $\mathbf{F}(\bar{G})$ in G, then $M = \mathcal{O}(G) \times S$ for a 2-subgroup S. It is easy to see that $S = \mathbf{O}_2(G)$ and that $\mathbf{F}(\bar{G}) \simeq S$. Moreover, there is a bijection between the subgroups of $\mathbf{F}(\bar{G})$ and the subgroups of S. Note that Proposition 4.2.2 applies to $\bar{G}$. Hence, we can write $\mathbf{\Omega}_1(S) = U \times V$ where

$$\bar{V} = [\mathbf{\Omega}_1(\mathbf{F}(\bar{G})), \mathbf{F}_2(\bar{G})]$$

and

$$\bar{U} = \mathbf{C}_{\mathbf{\Omega}_1(\mathbf{F}(\bar{G}))}(\mathbf{F}_2(\bar{G})).$$

Moreover, $\bar{U} \leq \mathbf{Z}(\bar{G})$, $\mathbf{C}_{\bar{G}}(\bar{V}) = \mathbf{F}(\bar{G})$, $\bar{V}$ is an irreducible $\bar{G}$-module and $\bar{G}/\mathbf{F}(\bar{G})$ is dihedral of order 18 or $2p$, with p a Fermat prime. We now prove that $\mathcal{O}(G)$ has a direct complement G_0 in G. Since $\bar{G}/\mathbf{F}(\bar{G})$ is isomorphic to a dihedral group of order 18 or $2p$, a Fermat prime, there is $y \in \bar{G}$ such that $\langle y\mathbf{F}(\bar{G})\rangle$ is a cyclic group of order 9 or p. By part 5 of Lemma 4.1.7, $y \in \mathrm{Re}(\bar{G})$. There is $t \in G$ such that $\bar{t}$ is a 2-element and $\bar{t}$ inverts y. Since $\bar{t} = t\mathcal{O}(G)$, we can assume that t is a 2-element. Let M be the preimage of $\langle y\rangle$ on G. Then t induces an automorphism of M that inverts $M/\mathcal{O}(G)$. By Lemma 4.2.8, $\mathcal{O}(G)$ has a direct factor X in M, therefore $M = X \times \mathcal{O}(G)$ and t acts as the inversion on X. If $T \in \mathrm{Syl}_2(G)$, call $G_0 = XT$. Then G_0 is a complement for $\mathcal{O}(G)$ and $\mathcal{O}(G)$ centralizes G_0. Note that property $(*)$ holds for G_0. Therefore, if tF is an involution of G_0/F for $t \in T$, then $t \notin \mathrm{Re}(G_0)$ by part 4 of Proposition 4.2.2.

Conversely, suppose that G is a group as in the Theorem, we prove that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers.
Assume that G has the form of part 1 of the Theorem. We can assume that $\mathcal{O}(G) = 1$. Let $x \in \mathrm{Re}(G)$. Then $x = x_2 x_{2'}$, where $x_2, x_{2'} \in \mathrm{Re}(G)$ are the 2-part and the $2'$-part of x. If $x_2 \in F$, then N centralizes $x_{2'}$ (because $x_{2'} \in N$ that is abelian $2'$-Hall of G) and x_2, therefore $|x^G|$ is a 2-power. Suppose that $x_2 \notin F$. By hypothesis, we have that x_2 is an involution, $|x_2^G|$ is a non-trivial p-power and x_2 acts as the inversion on N. Since $x_{2'} \in N$, $x_{2'}^{x_2} = x_{2'}^{-1}$. But $[x_2, x_{2'}] = 1$ and,

therefore, $x_{2'} = 1$. So $x = x_2$ and $|x^G|$ is a non-trivial p-power.
Assume now that G is a group as in part 2, we prove that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. We can assume that $\mathcal{O}(G) = 1$. Take $L \in \mathrm{Hall}_{2'}(G)$. Note that L is cyclic of order 9 or p, a Fermat prime. There is a homomorphism $\sigma\colon G \to \Gamma(V)$ such that $\ker\sigma = F$. Since G/F is dihedral, it follows that $\sigma(L) \leq \Gamma_0(V)$ and hence L acts as scalar multiplications on V. Let $x \in \mathrm{Re}(G)$ and write $x = x_2x_{2'}$ as before. By hypothesis, if tF is an involution of G/F then $t \notin \mathrm{Re}(G)$. Therefore, since $|G/F|_2 = 2$ and x_2 is real in G, $x_2 \in F$ and $x_2 \in \mathbf{\Omega}_1(F)$. Write $x_2 = uv$, with $u \in U$ and $v \in V$. Suppose that $v \neq 1$. Without loss of generality, we can assume that $x_{2'} \in L$ and hence $x_{2'}$ acts as scalar multiplication on V. Since v is a non-trivial element that is centralized by $x_{2'}$, we have that $x_{2'} = 1$. Now, $u \in \mathbf{Z}(G)$ and a Sylow 2-subgroup T of G is contained in $\mathbf{C}_G(v)$. This is because F centralizes v and $G/F \lesssim \Gamma(V)$ as a permutation group on V, so every element $v \in V$ is centralized by some involution on G/F (see part 1 of Lemma 2.2.3). So, $|x^G|$ is a p-power. Suppose now that $v = 1$. Since $u \in \mathbf{Z}(G)$, we have that $L \leq \mathbf{C}_G(x)$ and hence $|x^G|$ is a 2-power. □

If all the conjugacy classes of G have prime-power length, then $G/\mathbf{Z}(G)$ is a Frobenius group with an abelian kernel by Theorem A of [3]. In general, we have that $N = N_1 \times \cdots \times N_k$, where every N_i is G-invariant, indecomposable and homocyclic. This can be derived by Remak's Theorem [54, 3.3.2]. It is worth noting that the action of G on $N_i/\Phi(N_i)$is isomorphic to the action of G on $\mathbf{\Omega}_1(N_i)$ by the main Theorem and the Corollary 1) of [29].
As a consequence of Theorem 4.2.10, we get a bound of $\ell_2(G)$ and $\ell_{2'}(G)$ when $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers.

Corollary 4.2.11. *Let G be a group and suppose that $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$. If $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers, then $\ell_{2'}(G) = 1$ and $\ell_2(G) \leq 2$. If $\ell_2(G) = 2$, then $\mathrm{cs}_{\mathbb{R}}(G) = \{2, 9\}$ or $\{2, p\}$ for a Fermat prime p.*

If G is a group and $\mathrm{cs}_{\mathbb{R}}(G)$ consists of p-powers for an odd prime p, then Theorem 4.1.4 characterize the structure of G. Using Proposition 2.3.4 and Proposition 4.2.1 is possible to obtain a better description of such groups.

4.3 Remarks to Theorem 4.2.10

In this section, we discuss the condition $\mathrm{Re}(\mathbf{O}_2(G)) \subseteq \mathbf{Z}(\mathbf{O}_2(G))$ in Theorem 4.2.10. Firstly, we remark that this condition is used only in part 2. Indeed, it is equivalent to the assumption $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$ in Proposition 4.2.1, in the case where $\mathbf{F}(G)$ is a 2-group.

In the discussion after Theorem 3.7 of [57], Tong-Viet states the following.

Conjecture 4.3.1. *Let G be a 2-closed group, then $\Delta^*_{\mathbb{R}}(G)$ is connected.*

If Conjecture 4.3.1 is true, then the assumption $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$ in Proposition 4.2.1 would be a consequence, as the following result shows.

Proposition 4.3.2. *Let G be a group. Suppose that $F = \mathbf{F}(G) < G$ is a 2-group and assume Conjecture 4.3.1. If $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers, then $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$.*

Proof. We can see that part 1 of Proposition 4.2.1 is independent from the assumption $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$. Therefore, $\mathcal{V}^*_G \subseteq \{2, p\}$ for an odd prime p and $\mathbf{F}_2(G) = F \rtimes L$, where L is a non-trivial p-group. Observe that L is not normal in $\mathbf{F}_2(G)$, otherwise $L \leq \mathbf{C}_G(F) = F$, that is a 2-group. So, we have that $\mathcal{V}^*_{\mathbf{F}_2(G)}$ contains an odd prime, by Theorem 4.1.6. Moreover, $\mathcal{V}^*_{\mathbf{F}_2(G)} \subseteq \mathcal{V}^*_G$ as $\mathbf{F}_2(G) \trianglelefteq G$. So, it follows that $p \in \mathcal{V}^*_{\mathbf{F}_2(G)}$. Suppose by contradiction that $2 \in \mathcal{V}^*_{\mathbf{F}_2(G)}$. Since $\Delta^*_{\mathbb{R}}(\mathbf{F}_2(G))$ is connected by Conjecture 4.3.1, there is $\mathcal{C} \in \mathrm{cl}_{\mathbb{R}}(\mathbf{F}_2(G))$ such that $|2p|$ divides $|\mathcal{C}|$. Since $\mathbf{F}_2(G) \trianglelefteq G$, we have that G has a real class of composite length and this is against the assumption. So, it follows that $2 \notin \mathcal{V}^*_{\mathbf{F}_2(G)}$ and therefore $\mathrm{Re}(F) \subseteq \mathbf{Z}(F)$ by Theorem 4.1.4 applied to $\mathbf{F}_2(G)$. $\square$

Assume the notation of the proof above. If $L_0 \leq \mathbf{Z}(L)$ has order p, then FL_0 is subnormal in G and then $\mathrm{cs}_{\mathbb{R}}(FL_0)$ consists of prime powers. We can see that the proof of Proposition 4.3.2 works also if we replace $\mathbf{F}_2(G)$ with FL_0 (because FL_0 is subnormal in G), by applying Conjecture 4.3.1 to FL_0. Therefore, for our purposes, it is still useful to prove the Conjecture 4.3.1 in the particular case where $G = TP$, with $T \in \mathrm{Syl}_2(G)$, $T \trianglelefteq G$ and $|P| = p$, an odd prime.
Even if we do not have a proof of the Conjecture 4.3.1, we present some results that may be useful for this purpose.

Definition 4.3.3. Let G be a group and $\mathcal{B} \in \mathrm{cl}(G)$. Then $\ker \mathcal{B} = \{x \in G \mid x\mathcal{B} = \mathcal{B}\}$ is called the *kernel* of $\mathcal{B}$.

Lemma 4.3.4. *Let G be a group. Suppose that G is 2-closed and that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-power numbers. Consider $\mathcal{B} \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$ for some odd prime p. Then $\mathcal{C}^2 \subseteq \ker \mathcal{B}$.*

Proof. By hypothesis, we have that $T \trianglelefteq G$ for $T \in \mathrm{Syl}_2(G)$. Write $\mathcal{B} = x^G$ and $\mathcal{C} = y^G$ for $x, y \in \mathrm{Re}(T)$. Since $|y^T|$ divides $|\mathcal{C}|$, we have that $y \in \mathbf{Z}(T)$, so y and x commute. Therefore, called $\mathcal{D} = \mathcal{B}\mathcal{C}$, we have that $\mathcal{D} = (xy)^G \in \mathrm{cl}_{\mathbb{R}}(G)$ by part 3 of Lemma 4.1.7. If $|\mathcal{D}|$ is odd, then, as above, T centralizes xy and it follows that $x \in \mathbf{Z}(T)$, since $y \in \mathbf{Z}(T)$. But this is impossible, since $\mathcal{B}$ is non-central in T as $\mathcal{B} = x^G \in \mathrm{cl}_{\mathbb{R},2}(G)$. Hence, $|\mathcal{D}|$ is an even number and, by hypothesis, $\mathcal{D} \in \mathrm{cl}_{\mathbb{R},2}(G)$. So, it follows that $\mathcal{D} = (xy)^G = (xy)^T = x^T y = x^G y$. This means that $|\mathcal{D}| = |\mathcal{B}|$. Using the same argument with $\mathcal{D}$ in place of $\mathcal{B}$, we have that

$\mathcal{C}^2\mathcal{B} = \mathcal{C}\mathcal{D} \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $|\mathcal{C}\mathcal{D}| = |\mathcal{D}| = |\mathcal{B}|$. Since $\mathcal{C}$ is generated by an involution, we have that $1 \in \mathcal{C}\mathcal{C} = \mathcal{C}^2$. Thus, $\mathcal{B} \subseteq \mathcal{C}\mathcal{C}\mathcal{B} = \mathcal{C}\mathcal{D}$ and these two sets have the same cardinality. So, $\mathcal{B} = \mathcal{C}^2\mathcal{B}$ and $\mathcal{C}^2 \subseteq \ker \mathcal{B}$. □

Proposition 4.3.5. *Let G be a group of the form TP where $T = \mathbf{O}_2(G)$ and P is a p-group, for an odd prime p. Assume that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers. Let $\mathcal{B} \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $x \in \mathcal{B}$ such that $P \leq \mathbf{C}_G(x)$. Then, called $M = \langle \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle$ and $N = [M, G]$, the followings hold.*

1. *$M \leq \mathbf{\Omega}_1(\mathbf{Z}(T))$. Moreover, $N = \langle \mathcal{C}^2 \mid \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle$ and $\mathbf{C}_N(P) = 1$.*
2. *$N \leq \ker \mathcal{B}$ and $\mathcal{B} = x_1N \cup \cdots \cup x_kN$, where $x_i \in \mathcal{B}$ is T-conjugated to x and T acts transitively on $\{x_iN\}_{i=1}^k$ by conjugation.*
3. *If $x^t \in xN$ for some $t \in T$, then $t \in \mathbf{N}_G(xN)$ and $t^2 \in \mathbf{C}_G(xN)$.*
4. *For every $n \in N$, there is $t_n \in T$ such that $x^{t_n} = xn$. In this case $xN = x^{t_n}N$, P^{t_n} acts on xN and if $1 \neq g \in P$, then $x^{(g^{t_n})} = xnn^g = x[n, g]$.*

Proof. Observe that $\mathrm{Re}(G) \subseteq T$ part 3 of Lemma 4.1.8. Since $\mathcal{B} \in \mathrm{cl}_{\mathbb{R},2}(G)$, we note that $\mathcal{B} = x^G = x^T$ for $x \in \mathrm{Re}(T)$, namely all the G-conjugates of x are T-conjugates. Observe that if $\mathrm{cl}_{\mathbb{R},p}(G) = \emptyset$, then $1 = M = N$ and $\mathrm{cs}_{\mathbb{R}}(G)$ consists of 2-powers. By Theorem 4.1.6 we have that $P \trianglelefteq G$ and $[T, P] = 1$. So, in this case, Proposition trivially follows. Therefore, without loss of generality, we can assume that $\mathrm{cl}_{\mathbb{R},p}(G)$ is non-empty and $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime-powers involving 2 and p. By part 1 of Lemma 4.1.7 and Lemma 4.3.4, for all $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$ we have that $\mathcal{C}$ is generated by an involution that is central in T and $\mathcal{C}^2 \subseteq \ker \mathcal{B}$. Moreover, it is easy to see that $\mathcal{C} \subseteq \mathbf{\Omega}_1(\mathbf{Z}(T))$.
We prove part 1. Let M and N be as in the hypotheses. Since $\mathcal{C} \subseteq \mathbf{\Omega}_1(\mathbf{Z}(T))$ for all $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$, we have that $N \leq M \leq \mathbf{\Omega}_1(\mathbf{Z}(T))$ and both N and M are normal in G. We now prove that $N = \langle \mathcal{C}^2 \mid \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle$. Consider $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$; we have that $\mathcal{C} = y^G$ for some $y \in M$. Take $y^{g_1}y^{g_2} \in \mathcal{C}^2$ for $g_1, g_2 \in G$. Then

$$y^{g_1}y^{g_2} = y^2y^{g_1}y^{g_2} = yy^{g_1}yy^{g_2} = [y, g_1][y, g_2] \in [M, G] = N.$$

This shows that $\langle \mathcal{C}^2\rangle \leq N$ for all $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$, and hence $\langle \mathcal{C}^2 \mid \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle \leq N$. Now we prove the opposite inclusion. Let $n \in N$, then $n = [y_1, g_1] \ldots [y_s, g_s] = y_1y_1^{g_1} \ldots y_sy_s^{g_s}$, where $y_i \in M$ and $g_i \in G$ for every i. Without loss of generality, we can assume that $y_iy_i^{g_i} \neq 1$ for every i, so $y_i \in \mathbf{\Omega}_1(\mathbf{Z}(T))$ is real and y_i is non-central in G. Hence $y_iy_i^{g_i} \in \mathcal{C}_i^2$, where $\mathcal{C}_i = y_i^G \in \mathrm{cl}_{\mathbb{R},p}(G)$. It follows that $N \leq \langle \mathcal{C}^2 \mid \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle$ and, therefore, equality holds. Using the Fitting's decomposition to the coprime action of P on M, we have that $\mathbf{C}_N(P) = 1$.
We now prove part 2. By Lemma 4.3.4 we have that $\mathcal{C}^2 \subseteq \ker B$ for all $\mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)$.

Therefore, it follows from part 1 that $N \leq \ker \mathcal{B}$ and $\mathcal{B} = x_1N \cup \cdots \cup x_kN$ is a union of N-cosets. Observe that $\mathcal{B}$ consists of T-conjugates, therefore, for all i, x_i is T-conjugated to x. Moreover, T acts transitively on the x_iN's by conjugation. Indeed, for $t_1, t_2 \in T$, we have that $x^{t_1}N = x^{t_2t_2^{-1}t_1}N = (x^{t_2}N)^{t_2^{-1}t_1}$, because $N \leq \mathbf{Z}(T)$. So, T acts transitively by conjugation on $\{x^tN\}_{t\in T}$.
Now we prove part 3. Consider now $t \in T$ such that $x^t \in xN$. Hence $x^t = xn_t$, for a unique $n_t \in N$. Recall that $N \leq \mathbf{\Omega}_1(\mathbf{Z}(T))$ by part 1; therefore

$$(xN)^t = x^tN = xn_tN = xN,$$

and this implies that $t \in \mathbf{N}_G(xN)$. Moreover, for every $n \in N$, we have that

$$(xn)^{(t^2)} = (x)^{(t^2)}n = (xn_t)^tn = x^tn_tn = xn_tn_tn = xn$$

and thus $t^2 \in \mathbf{C}_G(xN)$.
Finally, we prove part 4. Let $n \in N$ and note that $n \in \mathbf{Z}(T)$ by part 1. Since $xn \in xN \subseteq \mathcal{B}$, there is $t_n \in T$ such that $x^{t_n} = xn$; moreover, P^{t_n} centralizes x^{t_n}, as $P \leq \mathbf{C}_G(x)$. Let $1 \neq g \in P$; since t_n centralizes N, we have that $n^{(g^{t_n})} = n^g$. Note that $xN = xnN$; so, we have that $x^{t_n}N = xnN = xN$ and we deduce that both P and P^{t_n} act on xN. It follows that $x^{(g^{t_n})} \in xN$ and hence there is $n_0 \in N$ such that $x^{(g^{t_n})} = xn_0$. Thus,

$$xn = x^{t_n} = (x^{t_n})^{(g^{t_n})} = (xn)^{(g^{t_n})} = x^{(g^{t_n})}n^g = xn_0n^g.$$

Since the action of N on $\mathcal{B}$ is semi-regular, we deduce that $n = n_0n^g$. It follows that $n_0 = n(n^g)^{-1} = [n^{-1}, g] = [n, g]$, because n is an involution. Thus, $x^{(g^{t_n})} = x[n, g]$ and this proves the statement. □

We now include some character-theoretic results that may be useful.

Lemma 4.3.6. *Let G be a finite group, $N \trianglelefteq G$ and p . The followings hold.*

1. *Suppose that $x \in G$ is such that all the elements of xN are G-conjugated. If $N \not\leq \ker \chi$ then $\chi(x) = 0$.*

2. *Suppose that N is a p-group, $x \in N$ and suppose that $p \nmid |x^G|$. Then $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.*

3. *Let χ be a character of G and suppose that $p \nmid \chi(1)$. If g is a p-element, then $\chi(x) \neq 0$.*

Proof. Part 1 is Lemma 3.1 of [27]. Parts 2 and 3 are Theorem 4.18 and Corollary 4.20 in [47]. □

Corollary 4.3.7. *Suppose that $G = TP$, where $T \in \mathrm{Syl}_2(G)$, $T \trianglelefteq G$ and $|P| = p$, an odd prime. Suppose that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime power numbers involving the prime 2 and call $N = [\langle \mathcal{C} \in \mathrm{cl}_{\mathbb{R},p}(G)\rangle, G]$. If $\chi \in \mathrm{Irr}(G)$ and $2 \nmid \chi(1)$, then $N \leq \ker \chi$.*

Proof. It follows from the hypotheses that $\mathrm{cl}_{\mathbb{R},2}(G)$ is non-empty. Let $\mathcal{B} \in \mathrm{cl}_{\mathbb{R},2}(G)$ and $x \in \mathcal{B}$. We apply part 2 of Propostion 4.3.5. Then $N \leq \ker \mathcal{B}$ and all the elements in xN are G-conjugated. Let $\chi \in \mathrm{Irr}(G)$ of odd degree. By part 3 of Lemma 4.3.6, we have that $\chi(g) \neq 0$. By part 1 of the same Lemma, we have that $N \leq \ker \chi$. □

4.3.1 Real q-Baer Groups

Let q be a prime and G be a group. We say that G is a *q-Baer group*, or equivalently G has the *the q-Baer property*, if every q-element generates a conjugacy class that has size a prime-power number.

Theorem 4.3.8. *[9, Theorem A(b)] Let G be a group and q a prime. Assume G is a q-Baer group. Then there is a unique prime p such that each q-element generates a conjugacy class with length a p-power number.*

If π is a set of primes, we say that a group G is a π-Baer group if every π-element generates a conjugacy class that has size a prime power. In [3] appears a generalization Theorem 4.3.8 for π-Baer group.
If π is equal to 2 or $2'$, we can define a real version of π-Baer groups. We say that G is a *real π-Baer group*, or equivalently G has the *real π-Baer property*, if every real π-element generates a class that has size a prime-power number. If a group G is 2-closed and $T \in \mathrm{Syl}_2(G)$, then $\mathrm{Re}(G) \subseteq T$. So, we can see Tong-Viet's Conjecture 4.3.1 as a version of Theorem 4.3.8 for real 2-Baer groups that are 2-closed.

Clearly, if G is a real $2'$-Baer group, every real element of odd order generates a class of 2-power size. So, in this case, a real version of the theorem above is true. Nevertheless, one may ask whether a real $2'$-group is 2-nilpotent. This can be seen as a generalization of Theorem 4.1.6.

4.4 Examples and complements

In conclusion of this chapter, we include some examples of groups G such that $\mathrm{cs}_{\mathbb{R}}(G)$ consists of prime powers.
For example, `Small(486,184)` is a typical group with the structure of part 1 of Theorem 4.2.10. Groups with the structure of part 2 of the same Theorem are a bit harder to find in practice, due to the fact that Fermat primes rapidly

grow in size. Let G be the `SmallGroup(320,1583)`. Then the structure of G is $(C_2 \times C_2 \times C_2 \times C_2) : (C_5 : C_4)$. Note that $\mathbf{F}(G)$ is elementary abelian of order 2^5, $[\mathbf{F}(G), G]$ is elementary abelian of order 2^4 and $G/\mathbf{F}(G)$ is dihedral of order 10, where 5 is a Fermat prime. Moreover, of $t\mathbf{F}(G)$ is an involution of $G/\mathbf{F}(G)$, then $|t| = 4$ is not real and $t^2 \in \mathbf{F}(G)$, according with part 4 of Proposition 4.2.2. The next Fermat prime is 17 and the first n such that 17 divides $2^n - 1$ is $n = 8$. So a group that has the structure of part 2 of Theorem 4.2.10 involving 17 as prime divisor, would have order at least $2^8 \cdot 17 \cdot 4 > 17000$, which is not available in the `SmallGroup` library.

When some theorems on character degrees are proven, to some extent it is natural to consider the corresponding problems for conjugacy class sizes. The analogy between the worlds of class sizes and character degrees have not been explained yet, though there is evidence that these two contexts are related.
It may be interesting to investigate if there is a version for edges of Theorem 4.1.5.

Problem. *Let G be a solvable group, is it true that $\Delta^*_{\mathbb{R}}(G)$ is a subgraph of $\Delta_{\mathbb{R}}(G)$?*

If we do not consider only real characters, this is true. Indeed, it is proved in [15] that $\Delta^*(G)$, the prime graph on all class sizes, is a subgraph of $\Delta(G)$, the prime graph on all character degrees.

Chapter 5

The Gruenberg-Kegel graph

This chapter is dedicated to the study of Grunberg-Kegel graphs. If G is a group, $\Gamma(G)$ is the prime graph on the spectrum of G: the vertex-set of $\Gamma(G)$ is $\pi(G)$, the set of primes that divide $|G|$, and $p, q \in \pi(G)$ are adjacent if there is an element in G with order pq. If σ is a set of primes, we say that σ is a *cut-set* if $\Gamma(G) - \sigma$, the subgraph induced by $\pi(G) \setminus \sigma$, is disconnected. The main Theorem 5.3.7 of the chapter states that if G is a solvable group and σ is a cut-set of $\Gamma(G)$, then there is a normal series whose factors have a controlled structure. As a consequence, in Corollary 5.3.8, we get a bound of the σ-length of G. If σ consists of a single point that is a cut-vertex for $\Gamma(G)$, then the Fitting length of G is bounded. This generalizes [1, Lemma 2.3]. In Section **??** we provide examples where these bounds are attained.

5.1 Definitions

5.1.1 Graph Theory

All the graphs in the following are simple and undirected. In this subsection, Γ is a graph with vertex-set V. We assume the standard definitions of graph theory. So, for example, $n(\Gamma)$ is the number of connected components of Γ, $diam(\Gamma)$ is the diameter of Γ and if $v, w \in V$, we write $v \sim w$ is v and w are adjacent in Γ and $v \nsim w$ otherwise. Moreover, if Σ is a subgraph of Γ, we write $\Sigma \leq \Gamma$.

Definition 5.1.1. Let σ be a set.

1. $\Gamma - \sigma$ is the graph that has vertex-set $V \setminus \sigma$ and such that two vertices of $\Gamma - \sigma$ are adjacent in $\Gamma - \sigma$ if and only if they are adjacent in Γ.

2. σ is a *pseudo cut-set* of Γ if $\Gamma - \sigma$ is disconnected.

3. If Γ is connected and $\sigma \subseteq V$ is a pseudo cut-set of Γ, then we call σ a *cut-set* of Γ.

4. If $\sigma = \{v\}$ is a cut-set for Γ, then v is called a *cut-vertex* of Γ.

The definition of cut-set that we have given is standard in graph theory. The notion of pseudo cut-set is introduced in order to simplify the proofs.
If σ is a pseudo cut-set of Γ, then Γ may not be connected. For example, if Γ is disconnected, then every σ such that $\sigma \cap V = \emptyset$ is a pseudo cut-set for Γ. Note that if σ is a cut-set for Γ, then σ is a pseudo cut-set for Γ.

5.1.2 Grunberg-Kegel graphs

In this subsection, G is a group and τ a set of primes.

Definition 5.1.2. We define the *Grunberg-Kegel graph* $\Gamma(G)$ of G as follows: the vertex-set is $\pi(G)$, the set of the primes that divide $|G|$, and two vertices are joined if and only if there is an element of G with order pq.

The following lemma will be used without explicit reference.

Lemma 5.1.3. *Let $H \leq G$ and $N \trianglelefteq H$. Then $\Gamma(H/N) \leq \Gamma(G)$.*

Proof. The vertex-set of $\Gamma(H/N)$ is contained in $\pi(G)$ and two vertices of $\Gamma(H/N)$ are adjacent only if they are adjacent in $\Gamma(G)$. Indeed, $|tN| = |\langle t\rangle N/N| = |\langle t\rangle/\langle t\rangle \cap N|$. Therefore, $\Gamma(H/N)$ is a subgraph of $\Gamma(G)$. $\square$

5.2 Background

In this section, we include the key tools that are used in this chapter.

Definition 5.2.1. [31, V, Definition 8.1] Let G be a group. We say that G is a *Frobenius group* if G possesses a *Frobenius complement*, namely a non-trivial proper subgroup H of G such that $A \cap A^g = 1$ for all $g \in G \setminus A$. In this case

$$K := (G \setminus \cup_{g \in G} A^g) \cup \{1\}$$

is called the *Frobenius kernel* of G.

Theorem 5.2.2. *[31, V, Hauptsatz 7.6] Let G a Frobenius group with a Frobenius complement A. If K is the Frobenius kernel, then $G = K \rtimes A$.*

See Chapter 7 of [35] for the definition and basic properties of Frobenius groups.

Theorem 5.2.3. *[23, Theorem 10.3.1] If G is a Frobenius group with kernel K and complement A, then the following conditions hold:*

1. *A induces a regular group of automorphism on K.*

2. *$|A|$ divides $|K|-1$.*

3. *K is nilpotent and K is abelian if $|A|$ is even.*

4. *If $P \in \mathrm{Syl}_p(A)$ for some prime p, then P is cyclic for $p \neq 2$ and P is cyclic or a generalized quaternion group for $p = 2$.*

5. *If p and q are primes, any subgroup of A of order pq is cyclic.*

6. *If $|A|$ is odd, then A is metacyclic, while if $|A|$ is even, A possesses a unique involution which necessarily is contained in $\mathbf{Z}(A)$.*

Definition 5.2.4. If G is a group, we say that G is a 2-*Frobenius group* if there is a normal series $1 < H < K < G$ such that K is a Frobenius group with kernel H and G/H is a Frobenius group with kernel K/H. We call H the *lower kernel* and K/H the *upper kernel.*

Lemma 5.2.5. *Let G be a Frobenius group. Then $\mathbf{F}(G)$ is the Frobenius kernel.*

Proof. Let L be the Frobenius kernel of G. Then $L \leq \mathbf{F}(G)$ by part 3 of Theorem 5.2.3., so $\mathbf{Z}(\mathbf{F}(G)) \leq \mathbf{C}_G(L) \leq L$ as L is the Frobenius kernel of G. Then $\mathbf{F}(G) \leq \mathbf{C}_G(\mathbf{Z}(\mathbf{F}(G))) \leq L$, so in fact $\mathbf{F}(G) = L$. $\square$

Proposition 5.2.6. *Let G be a 2-Frobenius group. Then $\mathbf{F}(G)$ is the lower kernel and $\mathbf{F}_2(G)/\mathbf{F}(G)$ is the upper kernel. Moreover, the upper kernel is a cyclic group of odd order, $G/\mathbf{F}_2(G)$ is cyclic and the lower kernel is not cyclic.*

Proof. By definition, there is a normal series $1 < H < K < G$ of G such that H is a Frobenius kernel of K and K/H is a Frobenius kernel of G/H. By Lemma 5.2.5, we have that $\mathbf{F}(G/K) = H/K$, therefore $\mathbf{F}(G) \leq H$. By Lemma 5.2.5 again, we have that $H = \mathbf{F}(K)$, therefore $\mathbf{F}(G) \leq H \leq \mathbf{F}(G)$, so $H = \mathbf{F}(G)$ and $K = \mathbf{F}_2(G)$. The remaining part of the Proposition can be found in [26, Lemma 2.1]. $\square$

The next proposition, that is known as the "Lucido's 3 primes Lemma".

Lemma 5.2.7. *[40, Proposition 1] Let G be a solvable group. If p, q, r are distinct primes dividing $|G|$, then G contains an element of order the product of two of these three primes.*

One of the most important results in the context of Grunberg-Kegel graphs is the Grunberg-Theorem, which appeared in an unpublished manuscript by K. W. Grunberg and O. Kegel in 1975. The following is a version of Grunberg-Kegel Theorem whose statement is suitable for our scopes.

Theorem 5.2.8 (Grunberg-Kegel)**.** *Let G be a solvable group. Suppose that σ is a pseudo cut-set of $\Gamma(G)$ and let $H \in \mathrm{Hall}_{\sigma'}(G)$. Then $\Gamma(H) = \Gamma(G) - \sigma$ and $\Gamma(H)$ consists of two complete connected components. Moreover, one of the following holds.*

1. *H is a Frobenius group and the vertex-set of one component of $\Gamma(H)$ consists of the primes dividing the size of a Frobenius complement of H.*

2. *H is a 2-Frobenius group and the vertex-set of one component of $\Gamma(H)$ consists of the primes dividing the size of a Frobenius complement of $\mathbf{F}_2(H)$.*

Proof. Suppose that σ is a pseudo cut-set of G. Observe that the vertex set of $\Gamma(G) - \sigma$ is equal to the vertex set of $\Gamma(H)$. In addition, two vertices of $\Gamma(H)$ are adjacent in $\Gamma(H)$ if and only if they are adjacent in $\Gamma(G) - \sigma$. This follows from the fact that every σ'-element is contained in some conjugate of H, being G solvable and $H \in \mathrm{Hall}_{\sigma'}(G)$. Therefore, this implies that $\Gamma(G) - \sigma = \Gamma(H)$. By [58, Corollary], $\Gamma(H)$ consrists of two components and either part 1 or part 2 of the theorem holds, where the lower complement of H is the Frobenius complement of H when H is a Frobenius group, and the complement of $\mathbf{F}_2(H)$ when H is a 2-Frobenius group. By Lucido's three Primes Lemma 5.2.7, these connected components are complete subgraphs of $\Gamma(H)$. $\square$

We mention that in [58] are classified also non-solvable groups such that $\Gamma(G)$ is disconnected.

We include the next elementary Lemma for completeness.

Lemma 5.2.9. *Let H be a solvable group. Suppose that $\mathbf{F}(H) = L \times D$ with $(|L|, |D|) = 1$. Then, $H/\mathbf{F}(H)$ is isomorphic to a subgroup of $\mathrm{Out}(L) \times \mathrm{Out}(D)$.*

Proof. Let $F = \mathbf{F}(H)$ and $Z = \mathbf{Z}(F)$. Since $\mathbf{C}_H(F) = Z$ we have that $H/Z \lesssim \mathrm{Aut}(F)$. Now $F/Z \simeq \mathrm{Inn}(F)$ and it follows that

$$\frac{H}{F} \simeq \frac{H/Z}{F/Z} \lesssim \frac{\mathrm{Aut}(F)}{\mathrm{Inn}(F)} = \mathrm{Out}(F).$$

Since $\mathrm{Aut}(F) = \mathrm{Aut}(L) \times \mathrm{Aut}(D)$ and $\mathrm{Inn}(F) = \mathrm{Inn}(L) \times \mathrm{Inn}(D)$ we have $\mathrm{Out}(F) = \mathrm{Out}(L) \times \mathrm{Out}(D)$ and the result follows. $\square$

5.2.1 σ-series

Definition 5.2.10. Let G be a group and σ be a set of primes.

1. A *σ-series* of G is a normal series $1 = G_0 < G_1 < \cdots < G_n = G$ such that G_i/G_{i-1} is a σ-group or a σ'-group for every $i = 1 \ldots n$. If a group G has a σ-series, then G is called σ-separable.

2. If G is a σ-separable group, the *σ-length* $\ell_\sigma(G)$ of G is the minimum possible number of factors that are σ-groups in any σ-series of G.

3. If G is solvable, the smallest n such that $\mathbf{F}_n(G) = G$, where $\mathbf{F}_n(G)$ denotes the n-th term of the Fitting series, is called the *Fitting length* of G and it is denoted with $\ell_F(G)$.

The notion of σ-length is well-behaved with respect to subgroups and quotients.

Lemma 5.2.11. *Let σ be a set of primes and G be a σ-separable group.*

1. *If $N \trianglelefteq G$, then $\ell_\sigma(G) \leq \ell_\sigma(N) + \ell_\sigma(G/N)$.*

2. *If $G = H \times K$, then $\ell_\sigma(G) \leq \max\{\ell_\sigma(H), \ell_\sigma(K)\}$.*

3. *If $H \leq G$, then $\ell_\sigma(H) \leq \ell_\sigma(G)$.*

4. *Suppose that G is solvable. If $\ell_F(G) \geq 2$, then $\ell_\sigma(G) \leq \ell_F(G) - 1$.*

Proof. Parts $1, 2$ and 3 are immediate from the definition. We prove 4 by induction on $\ell_F(G)$. If $\ell_F(G/\mathbf{F}(G)) \geq 2$, then, by induction and part 1, we have that

$$\ell_\sigma(G) \leq \ell_\sigma(\mathbf{F}(G)) + \ell_\sigma(G/\mathbf{F}(G)) \leq 1 + \ell_F(G/\mathbf{F}(G)) - 1 \leq \ell_F(G) - 1.$$

If $\ell_F(G/\mathbf{F}(G)) = 1$, we have that $G = \mathbf{F}_2(G)$. Let $H \in \mathrm{Hall}_{\sigma'}(\mathbf{F}(G))$ and $K \leq G$ such that $K/\mathbf{F}(G) \in \mathrm{Hall}_\sigma(G/\mathbf{F}(G))$. Note that $H, K \trianglelefteq G$. Moreover, $K/\mathbf{F}(G)$ and $\mathbf{F}(G)/H$ are σ-groups, so K/H is a σ-group. Since H and G/K are σ'-groups, we have that $\ell_\sigma(G) = 1$. $\square$

5.3 Main Theorem

We begin with a definition.

Definition 5.3.1. Let G be a solvable group and σ be a set of primes that is a pseudo cut-set of $\Gamma(G)$. Let $H \in \mathrm{Hall}_{\sigma'}(G)$. Then, in view of the Theorem 5.2.8, we adopt the following definitions.

1. If H is a Frobenius group, then $\pi_{2,\sigma}(G)$ consists of primes that divide the order of a Frobenius complement of H.

2. If H is a 2-Frobenius group, then $\pi_{2,\sigma}(G)$ consists of primes that divide the order of a Frobenius complement of $\mathbf{F}_2(H)$.

Moreover, we define, $\pi_{1,\sigma}(G) = \pi(H) \setminus \pi_{2,\sigma}(G)$. If the pseudo cut-set σ is fixed, we write $\pi_1(G) = \pi_{1,\sigma}(G)$ and $\pi_2(G) = \pi_{2,\sigma}(G)$.

Lemma 5.3.2. *Let G be a solvable group and σ be a set of primes that is a pseudo cut-set for $\Gamma(G)$. Then $\pi_1(G)$ and $\pi_2(G)$ are the vertex-sets of the connected components of $\Gamma(G) - \sigma$ and $\mathbf{F}(G/\mathbf{O}_\sigma(G))$ is a $\pi_1(G)$-group.*

Proof. By Theorem 5.2.8, if $H \in \mathrm{Hall}_{\sigma'}(G)$, then $\Gamma(H) = \Gamma(G) - \sigma$ consists of two complete connected components and, by Definition 5.1.2, one of them has $\pi_2(G)$ as vertex-set. So, $\pi_1(G)$ is the vertex set of the other connected component. This is because $\pi(H) = \pi(G) \setminus \sigma = \pi_1(G) \cup \pi_2(G)$. We prove the remaining part of the lemma. Without loss of generality, we can assume that $\mathbf{O}_\sigma(G) = 1$. Then $\mathbf{F}(G)$ is a σ'-group and hence $\mathbf{F}(G) \leq H$. This means that $\mathbf{F}(G) \leq \mathbf{F}(H)$. It follows from Theorem 5.2.8 that $\mathbf{F}(H)$ is a $\pi_1(G)$-group, so $\mathbf{F}(G)$ is a $\pi_1(G)$-group. $\square$

The next result stands at the core of our study on Grunberg-Kegel graphs.

Proposition 5.3.3. *Let r be a prime, H a solvable group and V a faithful $\mathrm{GF}(r)H$-module. Suppose that σ is a pseudo cut-set for $\Gamma(HV)$ and $r \notin \sigma$. Then, $r \in \pi_1(HV)$ and there is $K \leq H$ such that the following holds.*

1. *K is nilpotent and $\pi(K) \subseteq \pi_2(HV)$.*
2. *$\mathbf{F}(H/\mathbf{O}_\sigma(H)) = K\mathbf{O}_\sigma(H)/\mathbf{O}_\sigma(H)$.*

Proof. Let $\pi_i = \pi_i(HV)$ for $i = 1, 2$. Since $V = \mathbf{F}(HV)$ is an r-group, we have that $\mathbf{O}_\sigma(HV) = 1$ and $r \in \pi_1$ by Lemma 5.3.2. By Theorem 5.2.8, if $U \in \mathrm{Hall}_{\sigma'}(HV)$, we have that $\Gamma(HV) - \sigma = \Gamma(U)$ consists of two complete connected components, having vertex-sets π_1 and π_2 by Lemma 5.3.2. Moreover, U is either a Frobenius group or a 2-Frobenius group. Write $F = \mathbf{F}(U)$ and note that $V = F$. Indeed, we have that $V \leq F$; since $V = \mathbf{F}(HV)$, the opposite inclusion also follows:

$$F = \mathbf{C}_U(F) \leq \mathbf{C}_U(V) \leq V.$$

Consider $\bar{H} = H/\mathbf{O}_\sigma(H)$. Then $\mathbf{O}_\sigma(\bar{H}) = 1$ and $\mathbf{F}(\bar{H})$ is a σ'-group. Let N the preimage in H of $\mathbf{F}(\bar{H})$. Then $N/\mathbf{O}_\sigma(H) = \mathbf{F}(\bar{H})$. Let $U_0 \in \mathrm{Hall}_{\sigma'}(H)$ such that $U_0 \leq U$. Since V is an r group with $r \notin \sigma$ and $U \in \mathrm{Hall}_{\sigma'}(HV)$, we have that $U = VU_0$. Write $K = U_0 \cap N$, so that $KV \trianglelefteq U$ and $N = \mathbf{O}_\sigma(H) \rtimes K$. Observe that K is nilpotent because $K \simeq \mathbf{F}(\bar{H})$. Since $V = \mathbf{F}(U)$, it follows that $K \leq \mathbf{F}_2(U)$. If U is a 2-Frobenius group, we have that V is the Frobenius kernel for $\mathbf{F}_2(U)$ by Proposition 5.2.6, so K is contained in a Frobenius complement of $\mathbf{F}_2(U)$. If U is a Frobenius group, then V is the Frobenius kernel of U by Lemma 5.2.5 and therefore K is contained in a Frobenius complement of U. In any case, K is a π_2-group by Definition 5.3.1. Note that $K\mathbf{O}_\sigma(H)/\mathbf{O}_\sigma(H) = \mathbf{F}(H/\mathbf{O}_\sigma(H))$. This concludes the proof. $\square$

Definition 5.3.4. Let G be a group. We denote with $\mathbf{O}(G)$ the largest normal subgroup of G that has odd order.

Lemma 5.3.5. *Let G be a solvable group. Suppose that $G/\mathbf{O}(G)$ is isomorphic to the extension of* $\mathrm{SL}_2(3)$ *by a cyclic group of order $2q$ with q odd and that a Sylow 2-subgroup of $G/\mathbf{O}(G)$ is a generalized quaternion group. Then $G/\mathbf{O}(G)$ is isomorphic to the* `SmallGroup(48,28)`.

Proof. Call $H = G/\mathbf{O}(G)$ and observe that $\mathbf{O}(H) = 1$. Suppose that H is the extension of $\mathrm{SL}_2(3)$ by a cyclic group of order $2q$, with q odd. Let $N \trianglelefteq G$ such that H/N is cyclic of order q. Then N contains a subgroup of index 2 that is normal in G and that is isomorphic to $\mathrm{SL}_2(3)$. Moreover, a Sylow 2-subgroup of N is a generalized quaternion group. Therefore, by direct check with the software `GAP`, up to isomorphism there is only one such a group, namely the `SmallGroup(48,28)`. Note that $\mathrm{Aut}(N) = C_2 \times S_4$. Let $R \in \mathrm{Hall}_{2'}(H)$, then R acts on N and $R/R_0 \lesssim \mathrm{Aut}(N)$, where $R_0 = \mathbf{C}_N(R)$. So, $[R : R_0] \leq 3$. Take $x \in N$ of order 3. therefore x acts non trivially on N and hence $\langle x \rangle \cap R_0 = 1$. It follows that $R = \langle x \rangle \times R_0$. Since R_0 is cyclic, we have that $NR \leq \mathbf{C}_H(R_0)$ and $R_0 \leq \mathbf{Z}(H)$. Thus, $R_0 \leq \mathbf{O}(H) = 1$. ☐

Definition 5.3.6. We denote by $(2.S_4)^-$ the group `SmallGroup(48,28)`.

Following [1], we call $(2.S_4)^-$ the *binary octaedral group*.
We now prove our main Theorem. The structure of the proof is natural. Firstly, we reduce to the case where $\mathbf{O}_\sigma(G) = 1$ and $\Phi(G) = 1$, so that $\mathbf{F}(G)$ is a completely reducible and faithful $G/\mathbf{F}(G)$-module. Therefore, we proceed using the Proposition 5.3.3.

Theorem 5.3.7. *Let G be a solvable group. Suppose that σ is a set of primes that is a pseudo cut-set for $\Gamma(G)$. Then there is a normal series*

$$1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$$

such that G_0 and G_2/G_1 are σ-groups, G_1/G_0 is a nilpotent $\pi_1(G)$-group, G_3/G_2 is a nilpotent $\pi_2(G)$-group and G/G_3 is not nilpotent only if $2 \in \pi_2(G)$ and $G/\mathbf{O}(G) \simeq (2.S_4)^-$; in this case, $\ell_F(G/G_3) = 2$.

Proof. Let $\pi_i = \pi_i(G)$ for $i = 1, 2$. Call $G_0 = \mathbf{O}_\sigma(G)$ and $\tilde{G} = (G/G_0)/\Phi(G/G_0)$. Suppose that there is a normal series $1 \leq \tilde{G}_1 \leq \tilde{G}_2 \leq \tilde{G}_3 \leq \tilde{G}$ such that $\tilde{G}_1 = \mathbf{F}(\tilde{G})$ is a π_1-group, $\tilde{G}_1/\tilde{G}_2$ is a σ-group, $\tilde{G}_3/\tilde{G}_2$ is a nilpotent π_2-group and $\tilde{G}/\tilde{G}_3$ is not nilpotent if and only if $2 \in \pi_2$ and $\tilde{G}/\mathbf{O}(\tilde{G})$ is isomorphic to $(2.S_4)^-$. Consider G_i the preimage of $\tilde{G}_i$ in G. Then, $(G_1/G_0)/\Phi(G/G_0) = \mathbf{F}(G/G_0)/\Phi(G/G_0)$ by [31, III Satz 3.5] and it follows that $G_1/G_0 = \mathbf{F}(G/G_0)$, that is a nilpotent π_1-group. For $i \geq 2$, it is easy to see that $G_i/G_{i-1} \simeq \tilde{G}_i/\tilde{G}_{i-1}$ and $1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$ satisfies the thesis of the theorem; in particular, G/G_3 is not nilpotent if and only if $\tilde{G}/\tilde{G}_3$ is not nilpotent. This happens if and only if

and $2 \in \pi_2$ and $\tilde{G}/\mathbf{O}(\tilde{G}) \simeq (2.S_4)^-$. Since $\tilde{G} = (G/\mathbf{O}_\sigma(G))/\Phi(G/\mathbf{O}_\sigma(G))$ and $\pi_2 = \pi(G) \setminus (\sigma \cup \pi_1)$, we have that $\tilde{G}$ is the quotient of G by a normal $\sigma \cup \pi_1$-group. Observe that $2 \notin \sigma \cup \pi_1$; so, we deduce that $\mathbf{O}(\tilde{G})$ is a quotient of $\mathbf{O}(G)$ and $G/\mathbf{O}(G) \simeq (2.S_4)^-$. We proved that if $\tilde{G}$ possesses a series that satisfies the thesis of the Theorem, then the same is true for G. Hence, it is no loss to assume $\mathbf{O}_\sigma(G) = 1$, $\Phi(G) = 1$ and $\mathbf{F}(G)$ is a π_1-group.
By Gaschütz's Theorem [45, Theorem 1.12], $\mathbf{F}(G)$ has a complement H in G and $\mathbf{F}(G)$ is a faithful completely reducible H-module, possibly of mixed characteristic. Write $\mathbf{F}(G) = M_1 \times \cdots \times M_n$ as the product of irreducible H-modules, so that M_i is an elementary abelian r_i-group for $r_i \in \pi_1$. Call $H_i = H/\mathbf{C}_H(M_i)$ and $\bar{H} = \prod H_i$. Note that $H \lesssim \bar{H}$, since $\bigcap_i \mathbf{C}_H(M_i) = \mathbf{C}_H(\mathbf{F}(G)) = 1$. The group M_i is a faithful irreducible H_i-module. Note that $M_iH_i = G/\mathbf{C}_H(M_i)\prod_{j\neq i} M_j$, so $\Gamma(M_iH_i)$ is a subgraph of $\Gamma(G)$. Let $L \in \mathrm{Hall}_{\pi_2}(H)$; since no vertex in π_2 is adjacent in $\Gamma(G)$ to any vertex in π_1, we have that $L \cap \mathbf{C}_L(M_i) = 1$. Therefore, for every i, H_i contains a subgroup that is isomorphic to L. In particular, $\pi_2 \subseteq \pi(H_i)$. Since $r_i \in \pi_1$, $r_i \in \pi(H_iM_i)$ and $\Gamma(H_iM_i) \leq \Gamma(G)$, it follows that $\Gamma(M_iH_i) - \sigma$ is disconnected, for every i. This means that σ is a pseudo cut-set for $\Gamma(M_iH_i)$. Note furthermore that $\pi_j(M_iH_i) \subseteq \pi_j$, for $j = 1, 2$ and for all i. By Proposition 5.3.3, for every i, there are $H_{i,2}, H_{i,3} \trianglelefteq H_i$ such that $H_{i,2} = \mathbf{O}_\sigma(H_i)$ and $\mathbf{F}(H_i/H_{i,2}) = H_{i,3}/H_{i,2}$. Moreover, we have that $H_{i,3} = K_iH_{i,2}$, where $K_i \in \mathrm{Hall}_{\pi_2}(H_{i,3})$ and K_i is nilpotent. Suppose that there is i such that $H_i/H_{i,3}$ is not nilpotent. Note that K_i is nilpotent and acts regularly on M_i; so, by Theorem 5.2.3, we have that $K_i = C_i \times D_i$ where C_i is a cyclic group of odd order and D_i is a 2-group that is cyclic, quaternion or generalized quaternion. Moreover, by Lemma 5.2.9 $H_i/H_{i,3} \lesssim \mathrm{Out}(K_i) = \mathrm{Out}(C_i) \times \mathrm{Out}(D_i)$, where $\mathrm{Out}(C_i)$ is abelian. Hence, since $H_i/H_{i,3}$ is not nilpotent, D_i is the quaternion group of order 8 and $H_i/H_{i,3}$ acts on D_i as the symmetric group S_3 (and hence 2 divides $[H_i : H_{i,3}]$). In this case, $\ell_F(H/H_{i,3}) = 2$. In particular, $2 \in \pi_2$ and therefore that $\mathbf{C}_G(M_i)$ has odd order. Let $T \in \mathrm{Syl}_2(H)$; observe that $T \in \mathrm{Syl}_2(G)$, that $T \lesssim H_i$ and that the image of T in H_i acts regularly on M_i. It follows that T is cyclic or a generalized quaternion group by Theorem 5.2.3; since D_i is the quaternion group of order 8 and 2 divides $[H_i : H_{i,3}]$, we have that T is the generalized quaternion of order ≥ 16. Since $G/\mathbf{O}(G)$ is solvable, by [1, Proposition 2] we have that either $G/\mathbf{O}(G) \simeq T$ or $G/\mathbf{O}(G)$ is isomorphic to an extension of the group $\mathrm{SL}_2(3)$ by a cyclic group of order q or $2q$ with q odd. Since $H/H_{i,3}$ acts on D_i as the symmetric group S_3, the group G is not 2-nilpotent. If $G/\mathbf{O}(G)$ is isomorphic to an extension of the group $\mathrm{SL}_2(3)$ by a cyclic group whose order q with q odd, then $|T| = 8$, $T \simeq D_i$ and H_i/G_2^i acts on D_i as a cyclic group of order 3, in this case $\ell_F(H/H_{i,3}) = 1$, against our assumptions. Therefore, by Lemma 5.3.5, $G/\mathbf{O}(G) \simeq (2.S_4)^-$.
Call $\bar{H}_2 = \prod_i H_{i,2}$, $\bar{H}_3 = \prod_i H_{i,3}$, so $1 \leq \bar{H}_2 \leq \bar{H}_3 \leq \bar{H}$ is a normal series such

that $\bar{H}_2$ is a σ-group and $\bar{H}_3/\bar{H}_2$ is a nilpotent π_2-group. With abuse of notation, we write $H \leq \bar{H}$. Call $G_3 = \mathbf{F}(G)(H \cap \bar{H}_3), G_2 = \mathbf{F}(G)(H \cap \bar{H}_2)$ and $G_1 = \mathbf{F}(G)$. Note that G_1 is a π_1-group, G_2/G_1 is a σ-group and G_3/G_2 nilpotent π_2-group. Moreover $G/G_3 \simeq H/H \cap \bar{H}_3 \simeq H\bar{H}_3/\bar{H}_3$ is not nilpotent if and only if there is one i such that $H_i/H_{i,3}$ is not nilpotent and $2 \in \pi_2$. This happens if and only if $G/\mathbf{O}(G) \simeq (2.S_4)^-$ and in this case, $\ell_F(G/G_3) = 2$. □

Corollary 5.3.8. *Let G be a solvable group. Suppose that σ is a cut-set for $\Gamma(G)$, then $\ell_\sigma(G) \leq 3$. Moreover, if σ consists of a cut-vertex p of $\Gamma(G)$, then $\ell_F(G) \leq 6$ and if $G/\mathbf{O}(G) \not\simeq (2.S_4)^-$, then $\ell_F(G) \leq 5$. The bounds are the best possible.*

Proof. By Theorem 5.3.7, there is a series $1 \leq G_0 \leq G_1 \leq G_2 \leq G_3 \leq G$ such that G_0 and G_2/G_1 are σ-groups, G_1/G_0 is a $\pi_1(G)$-group and G_3/G_2 is a $\pi_2(G)$-group. Now, G/G_3 is not nilpotent if and only if $G/\mathbf{O}(G)$ is isomorphic to $(2.S_4)^-$. By Lemma 5.2.11, we have that $\ell_\sigma(G/G_3) \leq 1$. Thus, $\ell_\sigma(G) \leq 3$.
Suppose now that $\sigma = \{p\}$. Clearly G_0 and G_2/G_1 are nilpotent because they are p-groups. The factor G/G_3 is not nilpotent if and only if $G/\mathbf{O}(G) \simeq (2.S_4)^-$. In this case, $\ell_F(G/G_3) = 2$. So $\ell_F(G) \leq 5$ except when $\ell_F(G/G_3) = 2$. In this case, $G/\mathbf{O}(G) \simeq (2.S_4)^-$ and $\ell_F(G) \leq 6$.
For proving that the bounds obtained are the best possible, it suffices to assume that $\sigma = \{p\}$, where p is a cut-vertex. If G is the group constructed in [1, Remark 2], then G is solvable, 3 is a cut-vertex of $\Gamma(G)$ and $\ell_\sigma(G) = 3$, where $\sigma = \{3\}$. Moreover, $\ell_F(G) = 6$. Observe that $G/\mathbf{O}(G) \simeq (2.S_4)^-$. Suppose that $G/\mathbf{O}(G) \not\simeq (2.S_4)^-$, then Theorem 5.4.5 below provides an example of a group G of odd order such that $\ell_F(G) = 5$ and $\Gamma(G)$ has a cut-vertex. □

If $\Gamma(G)$ is a 3-chain, then part of Corollary 5.3.8 appeared in [1, Lemma 2.3], where the authors used tools that are more technical (like [2, (36.4)] or [1, Proposition 4]). For completeness, we include a bound for $\ell_F(G)$ in the case where $\Gamma(G)$ is disconnected.

Lemma 5.3.9. *Suppose that G is a solvable group and that $\Gamma(G)$ is disconnected. Then $\ell_F(G) \leq 4$.*

Proof. By Theorem 5.2.8, we have that G is either a Frobenius group or a 2-Frobenius group. If G is a 2-Frobenius group, then $\ell_F(G) = 3$, by Proposition 5.2.6. Suppose that G is a Frobenius group with kernel N and complement K. By Theorem 5.2.3, we can write $\mathbf{F}(K) = L_0 \times D$, where L_0 is a cyclic group of odd order and D is a 2-group that is cyclic, quaternion of generalized quaternion group. Now, by Lemma 5.2.9

$$K/\mathbf{F}(K) \lesssim \mathrm{Out}(L_0) \times \mathrm{Out}(D),$$

where $\mathrm{Out}(L_0) = \mathrm{Aut}(L_0)$ is abelian and if D is either cyclic or a generalized quaternion group, $\mathrm{Out}(D)$ abelian, a 2-group or the symmetric group S_3. Hence, $\ell_F(\mathrm{Out}(\mathbf{F}(K))) \leq 2$ $\square$

Using the same arguments as in the proof of Theorem 5.3.7, it is not difficult to show that if G solvable and G is disconnected, then $\ell_F(G) = 4$ only if G is a Frobenius group kernel $N = \mathbf{O}(G)$ and complement K such that $\mathbf{F}(K) = L_0 \times D$ where L_0 cyclic, $D \simeq Q_8$, $K/\mathbf{F}(K)$ acts on D as S_3 and on L_0 as a group of odd order. In this case $G/\mathbf{O}(G) \simeq (2.S_4)^-$.

5.4 Examples

ł<abelsection-examples

Example 5.4.1. Let G be a solvable group and σ a cut-set for G. If $|\sigma| = 1$, then there is a bound for $\ell_F(G)$ by Corollary 5.3.3. If $|\sigma| \geq 2$, then there is no such bound for $\ell_F(G)$. In fact, let $n \geq 2$ be a large integer. It is not difficult to find a group G_1 of order $p^\alpha q^\beta$ with Fitting length n, where $p, q \geq 5$ are two primes. Consider $G_2 = S_3$, so $\Gamma(G_2)$ is the union of two connected components that consist of the prime 2 and the prime 3. Let $G = G_1 \times G_2$, then $\{p, q\}$ is a cut-set for $\Gamma(G)$ and $\ell_F(G) = n$.

In the hypotheses of Corollary 5.3.8, if $G/\mathbf{O}(G) \not\simeq (2.S_4)^-$, then the bound obtained is the best possible. In fact, in Theorem 5.4.5 we construct a group G, of odd order and arbitrarily large derived length, such that $\Gamma(G)$ has a cut-vertex and $\ell_F(G) = 5$.
Some concepts of representation theory are involved; we adopt the notation in [34, Ch. 9]. Let r be a prime, $\tilde{R}$ be the ring of local integers for the prime r (see [34, pag. 265]) and $* : \tilde{R} \to \tilde{R}/M$ be the projection map of $\tilde{R}$ on the quotient $\tilde{R}/M$, where M is a maximal ideal of $\tilde{R}$ containing r. In this section, $\mathbb{F}$ denotes the field $\tilde{R}/M$.

Lemma 5.4.2. *Let G be a group and $x \in G$. Let χ be the ordinary character of G afforded by the representation $\mathfrak{X}\colon G \to \mathrm{GL}(V)$. Then $[\chi_{\langle x \rangle}, 1_{\langle x \rangle}] = 0$ if and only if $\mathfrak{X}(x)$ acts fixed-point-freely on V.*

Proof. The principal character $1_{\langle x \rangle}$ appears among the irreducible constituents of $\chi_{\langle x \rangle}$ if and only if $\mathfrak{X}(x)$ has one eigenvector with eigenvalue 1, namely there is a fixed point. $\square$

Let $\mathfrak{X}$ a complex representation of a group G and suppose that, for every $g \in G$, $\mathfrak{X}(g)$ has entries in $\tilde{R}$. Then, following [34, pag. 266], if $\mathbb{F} = \tilde{R}/M$, we can construct an $\mathbb{F}$-representation $\mathfrak{X}^*$ of G by setting $\mathfrak{X}^*(g) = \mathfrak{X}(g)^*$, where $\mathfrak{X}(g)^*$ is

the matrix obtained by applying $*$ to every entry of $\mathfrak{X}(g)$.
If $\mathbb{E} \subseteq \mathbb{F}$ is a subfield and $\mathfrak{Z}$ is an $\mathbb{E}$-representation of G, then $\mathfrak{Z}$ maps G into a group of non-singular matrices over $\mathbb{E}$. We may, therefore, view $\mathfrak{Z}$ as an $\mathbb{F}$-representation of G. As such we denote it by $\mathfrak{Z}^{\mathbb{F}}$ (see [34, pag. 144]).

Lemma 5.4.3. *Let $\mathfrak{X}$ be an irreducible $\mathbb{C}$-representation of a group G. Suppose that r is a prime and $r \nmid |H|$. Then, there is a finite field $\mathbb{E} \subseteq \mathbb{F}$, a $\mathbb{C}$-representation $\mathfrak{Y}$ similar to $\mathfrak{X}$ that takes values in $\tilde{R}$ and an absolutely irreducible $\mathbb{E}$-representation $\mathfrak{Z}$ such that $\mathfrak{Y}^* \simeq \mathfrak{Z}^{\mathbb{F}}$.*

Proof. Let $\mathfrak{X}$ be a $\mathbb{C}$-representation of a group H and χ be its complex character. By [34, Theorem 15.8], there exists a $\mathbb{C}$-representation $\mathfrak{Y}$, similar to $\mathfrak{X}$, that takes values in $\tilde{R}$, namely $\mathfrak{Y}(g) \in M_{\chi(1)}(\tilde{R})$ for all $g \in G$. Moreover $\mathfrak{Y}^*$ is an $\mathbb{F}$-representation of G and $\hat{\chi}$ is its Brauer character. Since $r \nmid |G|$, by [34, Theorem 15.13] we have $\hat{\chi} = \chi$ and $\hat{\chi}$ is irreducible. Hence $\mathfrak{Y}^*$ is irreducible. The field $\mathbb{F}$ is algebraically closed over its prime field $\mathbb{F}_p$ by [34, Lemma 15.1c)], so $\mathfrak{Y}^*$ is absolutely irreducible by [34, Corollary 9.4]. Let $\mathbb{E} \subset \mathbb{F}$ a splitting field for the polynomial $x^{|G|} - 1 \in \mathbb{F}_p[x]$. Note that $\mathbb{E}$ is a finite-degree extension of the prime field of $\mathbb{F}$, therefore $\mathbb{E}$ is finite. For every $g \in G$, $\chi^*(g)$ is a sum of $|G|$-roots of unity, so $\chi^*(g) \in \mathbb{E}$ for every $g \in G$. By [34, Theorem 9.14] there exists an absolutely irreducible $\mathbb{E}$-representation $\mathfrak{Z}$ of G such that $\mathfrak{Z}^{\mathbb{F}} \simeq \mathfrak{Y}^*$. □

Proposition 5.4.4. *Let $\mathfrak{X}$ be a $\mathbb{C}$-representation for a group G and W the associated $\mathbb{C}[G]$-module. Then, there is a finite splitting field $\mathbb{E}$ of G and a $\mathbb{E}[G]$-module V, such that $(|G|, |V|) = 1$ and*

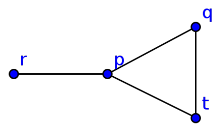
$$\dim_{\mathbb{C}} \mathbf{C}_W(x) = \dim_{\mathbb{E}} \mathbf{C}_V(x)$$

for every $x \in G$.

Proof. Let r be a prime that does not divide $|G|$ and denote $\tilde{R}$ the ring of local integers at the prime r. By Lemma 5.4.3, there is a $\mathbb{C}$-representation $\mathfrak{Y}$, similar to $\mathfrak{X}$, that take values in $\tilde{R}$, a finite field $\mathbb{E} \subseteq \mathbb{F} = \tilde{R}/M$ (where M is the unique maximal ideal of $\tilde{R}$) and absolutely irreducible $\mathbb{E}$-representation $\mathfrak{Z}$ such that $\mathfrak{Y}^* \simeq \mathfrak{Z}^{\mathbb{F}}$. Call W the $\mathbb{C}[G]$-module associated to $\mathfrak{X}$ and V the $\mathbb{E}[G]$-module associated to $\mathfrak{Z}$. Note that V is finite since $\mathbb{E}$ is finite, moreover $(|G|, |V|) = 1$. Let $x \in G$, the number $m = \dim_{\mathbb{C}} \mathbf{C}_W(x)$ is the geometric multiplicity of the eigenvalue 1 of the matrix $\mathfrak{Y}(x)$, that is equal to the algebraic multiplicity of 1 of the matrix $\mathfrak{Y}(x)$. This is because the characteristic of W is 0 and thus the action of $\langle x \rangle$ on W is completely reducible by Maschke's Theorem. Hence, following the proof of [34, Lemma 2.15], $\mathfrak{Y}(x)$ is diagonalizable and, therefore, algebraic and geometric multiplicities of $\mathfrak{Y}(x)$ coincide. Note that m, as the algebraic multiplicity of 1 in $\mathfrak{Y}(x)$, is equal to the algebraic multiplicity of the 1 in $\mathfrak{Y}^*(x)$. Since $\mathfrak{Y}^*(x)$ and

$\mathfrak{Z}^{\mathbb{F}}(x)$ are similar, m is the algebraic multiplicity of 1 for the matrix $\mathfrak{Z}^{\mathbb{F}}(x)$, that is the algebraic multiplicity of 1 in $\mathfrak{Z}(x)$. Using the same argument as above, since the characteristic of V does not divide $|G|$, m is the geometric multiplicity of 1 for $\mathfrak{Z}(x)$, that is equal to $\dim_{\mathbb{E}} \mathbf{C}_V(x)$, □

Figure 5.1:



Theorem 5.4.5. *Let $n \geq 5$ an integer. Then, there is a solvable group G of odd order, with derived length greater than n, such that $\ell_F(G) = 5$ and $\Gamma(G)$ is the graph in Figure 5.1.*

Proof. Let t, q be two odd primes and T, Q two cyclic groups of order respectively t^2 and q^2. Let $T_0 \leq T$ the subgroup of order t and $Q_0 \leq Q$ the subgroup of order q. Choosing t to be a prime divisor of $q^2 - 1$, it is no loss to assume that there is an action of T on Q that has kernel T_0. Consider $L = Q \rtimes T$, $L_0 = Q_0 T_0$ and $\bar{L} = L/L_0$. Let p be an odd prime. By [4, Theorem 22.25], there is a p-group P of derived length n that has a faithful irreducible character θ. If P_0 is the base group of $P \wr \bar{L}$, there is an action of L on P_0 and the kernel of such an action is L_0. Call $H = P_0 \rtimes L$, note that $\mathbf{F}(H) = P_0 L_0$, $\mathbf{F}_2(H) = P_0 F(L)$ and $\mathbf{F}_3(H) = H$. Moreover, the derived length of H is greater than n. Now, $P_0 = \prod_{j \in \bar{L}} P_j$ with $P_j \simeq P$ for every j and there is a character θ^j of P_j that is isomorphic to θ. Consider $\psi = \prod_j \theta^j$. By construction, ψ is a faithful irreducible character of P_0. Consider now two non-trivial characters $\lambda \in \mathrm{Irr}(Q_0)$ and $\mu \in \mathrm{Irr}(T_0)$ such that $\lambda\mu$ is a faithful irreducible character of L_0. Note that $L_0 P_0 = P_0 \times Q_0 \times T_0$, hence $\psi\lambda\mu \in \mathrm{Irr}(P_0 L_0)$. Let $\chi \in \mathrm{Irr}(H \mid \psi\lambda\mu)$, it is easy to see that χ is faithful. Indeed $\ker \chi \cap P_0 = 1$ since $[\chi_{P_0}, \psi] \neq 0$ and ψ is faithful. Moreover, if q divides $|\ker \chi|$, we have that $Q_0 \leq \ker \chi$, but $[\chi_{Q_0}, \lambda] \neq 0$ and this is impossible. Replacing q by t, we have that t does not divide $|\ker \chi|$. So, we have that $\ker(\chi) = 1$ and χ is faithful. The same argument also implies that $[\chi_{Q_0}, 1_{Q_0}] = [\chi_{T_0}, 1_{T_0}] = 0$. Therefore, by Lemma 5.4.2, if $\mathfrak{X}$ is a representation fo H that affords χ and x is an element of order t or q, then x is contained in of either T_0 or Q_0 and $\mathfrak{X}(x)$ acts fixed point freely on W, the $\mathbb{C}[H]$-module associated to χ. Let r be an odd prime such that $r \nmid |H|$. By Proposition 5.4.4 there is a finite field $\mathbb{E}$ of characteristic r and a finite $\mathbb{E}[H]$-module V such that, for every $x \in H$

$$\dim_{\mathbb{C}} \mathbf{C}_W(x) = \dim_{\mathbb{E}} \mathbf{C}_V(x).$$

Note that, since W is faithful, V is faithful. Moreover, an element $x \in H$ acts fixed-point-freely on V whenever it does on W. So, every element in H of order t or q acts fixed-point-freely on V and HV does not contain any element of order tr or qr.
On the other hand, the subgroup P_0 has an elementary abelian subgroup of order p^2. Hence the action of P_0 on V is not regular by Theorem 5.2.3. So, there is one element of order rp. In addition, in HV there are elements of order tp, qp and tq since $\mathbf{F}(H) = P_0L_0$. This means that p is a cut-vertex for the connected graph $\Gamma(HV)$. Note that $\mathbf{F}(HV) = V$ and HV has Fitting length 4. Now consider C_p a group of order p and let $G = C_p \wr (HV)$. Note that G has odd order, $\ell_F(G) = 5$ and that G has derived length greater than n. □

The above theorem shows that bound for $\ell_F(G)$ obtained in Corollary 5.3.8 is the best possible and that is independent from the derived length of the group.

Acknowledgements

I would like to thank my tutor, Prof. Silvio Dolfi, for all the advice and for helping me to understand how research works in mathematics. Without him, this work wouldn't have been possible. I am grateful also to Prof. Lucía Sanus for the collaboration during my visiting in València. I am also thankful to Miquel Martinez for being my first friend there.
I would also like to thank all my colleagues in Florence. The feedback I received from them helped me to organize my work. I also would like to thank Francesca Chiappini for helping me during all my missions abroad. She always has been present.

Two last special thanks. The first is to my mother, that has been always comprehensive and supportive in all my projects. The second is to my partner Núria, that has always been present at my side in all difficult moments of my Phd.

Bibliography

[1] O. A. Alekseeva, A.S. Kondrat'ev, *Finite groups whose prime graphs are triangle-free. I*, (Russian) Tr. Inst. Mat. Mekh. 21 (2015), no. 3, 3–12; translation in Proc. Steklov Inst. Math. 295 (2016), suppl. 1, S11–S20.

[2] M. Aschbacher, *Finite Group Theory*, Corrected reprint of the 1986 original. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 1993.

[3] A. Beltran, M. J. Felipe, *Prime powers as conjugacy class lengths of π-elements*, Bull. Austral. Math. Soc. 69 (2004), no. 2, 317–325.

[4] Y. Berkovich, *Groups of Prime Power Order, Vol 1*, De Gruyter Expositions in Mathematics, 46. Walter de Gruyter GmbH & Co. KG, Berlin, 2008. Cambridge University Press, Cambridge, 1986.

[5] Y. Berkovich, L. Kazarin, *Indices of elements and normal structure of finite groups*, J. Algebra 283 (2005), no. 2, 564–583.

[6] T. R. Berger, *Primitive solvable groups*, J. Algebra 33 (1975), 9–21.

[7] M. Bianchi, D. Chillag, M. L. Lewis, E. Pacifici, *Character Degree Graphs that are Complete Graphs*, Proc. Amer. Math. Soc. 135 (2007), no. 3, 671–676.

[8] L. Bonazzi, *Finite non-solvable groups whose real degrees are prime-powers*, J. Group Theory 25 (2022), no. 4, 741–751.

[9] A. R. Camina, R. D. Camina, *Implications of conjugacy class size*, J. Group Theory 1 (1998), no. 3, 257–269.

[10] A. R. Camina, R. D. Camina, *The influence of conjugacy class sizes on the structure of finite groups: a survey*, Asian-Eur. J. Math. 4 (2011), no. 4, 559–588.

[11] C. Casolo, *Some linear actions of finite groups with q'-orbits*, J. Group Theory 13 (2010), no. 4, 503–534.

[12] C. Casolo, S. Dolfi, E. Pacifici, L. Sanus, *Groups whose character degree graph has diameter three*, Israel J. Math. 215 (2016), no. 2, 523–558.

[13] D. Chillag, A. Mann, *Nearly odd-order and nearly real finite groups*, Comm. Algebra 26 (1998), no. 7, 2041–2064.

[14] Z. Costanzo, H. P. Tong-Viet, *Finite groups whose real-valued irreducible characters have prime power degrees*, J. Algebra 597 (2022), 190–204.

[15] S. Dolfi, *Prime factors of conjugacy class lengths and irreducible character degrees in finite soluble groups*,J. Algebra 174 (1995), no. 3, 749–752.

[16] S. Dolfi, *On independent sets in the class graph of a finite group*, J. Algebra 303 (2006), no. 1, 216–224.

[17] S. Dolfi, G. Navarro, P. H. Tiep, *Primes dividing the degrees of the real characters*, Math. Z. 259 (2008), no. 4, 755–774.

[18] S. Dolfi, E. Pacifici, *Zeros of Brauer characters and linear actions of finite groups*,J. Algebra 340 (2011), 104–113.

[19] S. Dolfi, E. Pacifici, *Zeros of Brauer characters and linear actions of finite groups: small primes*, J. Algebra 399 (2014), 343–357.

[20] S. Dolfi, E. Pacifici, L. Sanus, *Finite groups with real-valued irreducible character of prime degree*, J. Algebra 320 (2008), no. 5, 2181–2195.

[21] S. Dolfi, E. Pacifici L. Sanus, *On zeros of characters of finite groups*, Group theory and computation, 41–58, Indian Stat. Inst. Ser., Springer, Singapore, 2018.

[22] B. Fein, *A note on Brauer-Speiser Theorem*, Proc. Amer. Math. Soc. 25 (1970), 620–621.

[23] D. Gorenstein, *Finite groups*, Second edition. Chelsea Publishing Co., New York, 1980.

[24] R. Gow, *Real-valued characters of solvable groups*, Bull. London Math. Soc. 7 (1975), 132.

[25] N. Grittini, M. A. Pellegrini, *Sylow normalizers and irreducible characters with small cyclotomic field of values*, J. Algebra 608 (2022), 445–464.

[26] A. Gruber, T. M. Keller, M. L. Lewis, K. Naughton, B. Strasser, *A characterization of the prime graph of solvable groups*, J. Algebra 442 (2015), 397–422.

[27] R. Guralnick, G. Navarro, *Squaring a conjugacy class and cosets of normal subgroups*, (English summary) Proc. Amer. Math. Soc. 144 (2016), no. 5, 1939–1945.

[28] R. M. Guralnick, G. Navarro, P. H. Tiep, *Real Class Sizes and Real Character Degrees*, Math. Proc. Cambridge Philos. Soc. 150 (2011), no. 1, 47–71.

[29] M. E. Harris, *On p'-automorphism of abelian p-groups*, Rocky Mountain J. Math. 7 (1977), no. 4, 751–752.

[30] G. Higman, *Finite groups in which every element has prime power order*, J. London Math. Soc. 32 (1957), 335–342.

[31] B. Huppert, *Endliche Gruppen I*, Die Grundlehren der Mathematischen Wissenschaften, Vol. 134, Springer, Berlin–New York, 1967.

[32] I. M. Isaacs, *Characters of solvable and symplectic groups*, Amer. J. Math. 95 (1973), 594–635.

[33] I. M. Isaacs, *Character correspondences in solvable groups*, Adv. in Math. 43 (1982), no. 3, 284–306.

[34] I.M. Isaacs, *Character Theory of Finite groups*, Corrected reprint of the 1976 original [Academic Press, New York; MR0460423]. Dover Publications, Inc., New York, 1994. xii+303 pp.

[35] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, 92. American Mathematical Society, Providence, RI, 2008.

[36] G. D. James,*The Representation theory of Symmetric Groups*, Springer-Verlag Berlin Heidelberg New York 1978.

[37] A. S. Kondratev, *On prime graph components of finite simple groups*, (Russian) Mat. Sb. 180 (1989), no. 6, 787–797, 864; translation in Math. USSR-Sb. 67 (1990), no. 1, 235–247.

[38] H. Kurzweil, B. Stellmacher, *The Theory of Finite Groups*, An introduction. Translated from the 1998 German original. Universitext. Springer-Verlag, New York, 2004.

[39] M. L. Lewis, *An overview of graphs associated with character degrees and conjugacy class sizes in finite groups*, Rocky Mountain J. Math. 38 (2008), no. 1, 175–211.

[40] M. S. Lucido, *The diameter of a prime graph of a finite group*, J. Group Theory 2 (1999), no. 2, 157–172.

[41] M. S. Lucido, *Groups in which the prime graph is a tree*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 5 (2002), no. 1, 131–148.

[42] O. Manz, *Endliche nicht-auflösbare Gruppen, deren sämtliche Charaktergrade Primzahlpotenzen sind*, J. Algebra 96 (1985), no. 1, 114–119.

[43] O. Manz, *Endliche auflösbare Gruppen, deren sämtliche Charaktergrade Primzahlpotenzen sind*, J. Algebra 94 (1985), no. 1, 211–255.

[44] O. Manz, R. Staszewski, W. Willems, *On the number of components of a graph related to character degrees*, Proc. Amer. Math. Soc. 103 (1988), no. 1, 31–37.

[45] O. Manz T. R. Wolf, *Representations of Solvable Groups*, London Mathematical Society Lecture Notes, 185, Cambrige University Press (1993).

[46] V.D. Mazurov, *2-groups possessing an automorphism of odd order that is the identity on involutions*, (Russian) Algebra i Logika 8 1969 674–685.

[47] G. Navarro, *Character theory and the McKay conjecture*, Cambridge Studies in Advanced Mathematics, 175. Cambridge University Press, Cambridge, 2018.

[48] G. Navarro, N. Rizo, L. Sanus, *Character degrees in separable groups*, Proc. Amer. Math. Soc. 150 (2022), no. 6, 2323–2329.

[49] G. Navarro, L. Sanus, P. H. Tiep, *Real characters and degrees*, Israel J. Math. 171 (2009), 157–173.

[50] G. Navarro, P. H. Tiep, A. Turull, *p-rational characters and self-normalizing Sylow p-subgroups*, Represent. Theory 11 (2007), 84–94.

[51] G. Navarro, P.H. Tiep, *Rational irreducible characters and rational conjugacy classes in finite groups*, Trans. Amer. Math. Soc. 360 (2008), no. 5, 2443–2465.

[52] G. Navarro, P. H. Tiep, *Sylow subgroups, exponents, and character values*, Trans. Amer. Math. Soc. 372 (2019), no. 6, 4263–4291.

[53] N. Rizo, L. Sanus, *A real version of Thompson's theorem on degrees*, Ann. Mat. Pura Appl. (4) 200 (2021), no. 3, 1297–1300.

[54] D. J. S. Robinson, *A course in the theory of groups*, Second edition. Graduate Texts in Mathematics, 80. Springer-Verlag, New York, 1996.

[55] P.H. Tiep, *Real Ordinary Character and Real Brauer Characters*, Trans. Amer. Math. Soc. 367 (2015), no. 2, 1273–1312.

[56] P. H. Tiep, H. P Tong-Viet, *Odd-degree rational irreducible characters*, Acta Math. Vietnam. 47 (2022), no. 1, 293–304.

[57] H. P. Tong-Viet, *Real Class Sizes*, Israel J. Math. 228 (2018), no. 2, 753–769.

[58] J. S. Williams, *Prime graph components of finite groups*, J. Algebra 69 (1981), no. 2, 487–513.

[59] T. R. Wolf, *Character correspondences in solvable groups*, Illinois J. Math. 22 (1978), no. 2, 327–340.

[60] J. Zhang, *A Note on Character Degrees of Solvable Groups*, Comm. Algebra 28 (2000), no. 9, 4249–4258.

[61] L. Zang, G. Chen, S. Chen, X. Liu, *Note on finite simple groups whose order has three or four prime divisors*, J. Algebra Appl. 8 (2009), no. 3, 389–399.