## Self-Similar Solutions for the Heat Equation with a Signum-Type Source Term

# Self-Similar Solutions for the Heat Equation with a Signum-Type Source Term

Angiolo Farina[*,a], Roberto Gianni[a], Fabio Rosso[a]

[a]*Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Viale Morgagni 67/a, Firenze, 50134, Italy*

**Abstract**

We study the existence and multiplicity of self-similar solutions for the semilinear heat equation

$$u_t = u_{xx} + S(u),$$

where $S$ is a signum-type nonlinearity. Such reaction term, characterized by discontinuous and non-Lipschitz behavior, gives rise to qualitative features that sharply differ from the classical theory, including the loss of uniqueness and the emergence of sign-changing profiles.

By exploiting the invariance under parabolic rescaling, we reduce the problem to the analysis of a nonlinear ODE for the similarity profile and show that the resulting boundary value problem may admit at most two sign-changing solutions. A key point of the analysis is the characterization of the parameter domains in which sign-changing self-similar solutions exist, which is achieved through a detailed study of an algebraic constraint arising from the matching of the similarity solutions with the initial condition.

These results highlight how weak regularity of the source/sink term can induce nonstandard dynamical effects in semilinear parabolic equations, including

*Corresponding author
*Email address:* `angiolo.farina@unifi.it` (Angiolo Farina )

multiplicity of solutions and the onset of sign-changing structures in problems where monotone behavior would normally be expected.

**Keywords**: semilinear heat equation, nonlinear ODE, non-uniqueness, self-similar solutions.

## 1. Introduction

We consider the initial value problem

$$\begin{cases} u_t = u_{xx} + S(u), & x \in \mathbb{R}, \quad 0 < t, \\ \\ u(x,0) = -cx^2, & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $c > 0$, and where $S$ is a signum-type graph

$$\begin{cases} S(\xi) = 1, & \text{if} \quad \xi > 0, \\ S(\xi) \in [\varepsilon, 1], & \text{if} \quad \xi = 0, \\ S(\xi) = \varepsilon, & \text{if} \quad \xi < 0, \end{cases} \tag{2}$$

with $\varepsilon < 0$. It is obvious that the form (2) corresponds to the signum graph for $\varepsilon = -1$ and that any graph for which $S(\xi) = a$ for $\xi < 0$, $S(0) \in [a, b]$ and $S(\xi) = b$ for $\xi > 0$, with $a < 0$ and $b > 0$, can be reduced to the form (2) by a suitable scaling of the function $u$. The case $\varepsilon = 0$, corresponding to $S(\xi)$ equal to the Heaviside graph, has been analyzed in [1] and [2].

Reaction terms of this type arise naturally in a variety of physical and biological models, including threshold-type activation mechanisms, singular reaction kinetics, combustion and effective descriptions of diffusion-driven processes in heterogeneous media (see, e.g., [3–16]). The lack of regularity of the source/sink term plays a central role in producing mathematical behavior that strongly departs from the classical theory of semilinear parabolic equations. The central aim of this study is to establish a detailed existence and multiplicity theory for self-similar solutions of the form

$$u(x,t) = t^{\gamma} f\left(\frac{x}{\sqrt{t}}\right), \tag{3}$$

where $\gamma$ is a suitable scaling exponent determined by the structure of the nonlinearity, and where the similarity profile $f$ solves a nonlinear ordinary differential equation obtained via the invariance of the problem under parabolic rescaling. Self-similar solutions are of particular interest because they often describe intermediate asymptotic of more general solutions and can shed light on the long-time and singularity formation behavior of the system (see, e.g., [17–20]).

Our analysis proves the existence of multiple self-similar profiles. In particular, we establish the presence of one strictly negative self-similar solution, as well as the existence of two sign-changing profiles. These sign-changing solutions highlight a striking contrast with the classical case of smooth, Lipschitz continuous reaction terms, where the regularity of the source/sink term typically precludes oscillations and rules out the existence of sign-changing structures.

A key aspect of the problem is the phenomenon of *non-uniqueness*. The failure of the Lipschitz condition for $S(u)$ enables the emergence of distinct regular solutions from the same initial configuration, a phenomenon that is, here, rigorously demonstrated through the construction of at most three self-similar profiles. This behavior stands in sharp contrast with the standard theory for parabolic equations with smooth nonlinearities, where uniqueness is usually guaranteed by classical Picard-type arguments.

An additional and particularly intriguing consequence of the non-Lipschitz structure being the appearance of *sign-changing self-similar solutions* despite the initial datum is non-positive for $x \in (-\infty, \infty)$, which implies that, except at $t = 0$, $S\left(u\right)$ acts as sink term. Their existence indicates that the interplay between diffusion and a weakly regular reaction term can generate complex dynamics, including the possibility of transient or persistent changes of sign that are not directly dictated by the sign of the source. These complex dynamics are determined not only by the structure of the source/sink term, i.e., by $S\left(u\right)$, but are also strongly dependent on the profile of the initial datum. Indeed, if $u\left(x,0\right)$ is monotone and there exists $x_0$ such that $u\left(x_0,0\right) = 0$, then the solution to problem $(1)_1$ coupled with such an initial datum is unique [21].

Taken together, our findings reveal a rich variety of behaviors for non-

Lipschitz semilinear heat equations. They also provide a framework for the systematic study of self-similar structures in problems where the failure of standard regularity assumptions leads to non-unique evolutions, oscillatory patterns, and unconventional solution profiles. These results contribute to a growing body of literature highlighting the subtle and often counterintuitive effects of low regularity in nonlinear parabolic equations.

The remainder of this work is organized as follows. In Sect. 2 we introduce the self-similar framework, derive the associated similarity ODE, and describe the matching procedure across the curve where the sign of the solution changes. Section 3 contains the core of our analysis: we establish the existence of at most two sign-changing similarity profiles, characterize their qualitative properties, and determine the $\varepsilon$ parameters interval in which such solutions may arise. In the same section we also analyze the algebraic condition linking the similarity profiles to the initial datum and identify the critical thresholds governing existence and multiplicity. Finally, Sect. 4 summarizes the main results and discusses their relevance to the dynamics of reaction-diffusion systems with signum-type source terms.

## 2. Similarity solutions

We seek solutions of problem (1) in a self-similar form and which change sign (i.e. solutions which are positive in a neighborhood of the origin). These are of particular interest from a physical point of view. In fact, if $u$ represents the temperature, $u > 0$ means that the reaction has been activated. In other words, an initial negative value everywhere except at one point is sufficient to trigger a positive reaction, despite the term $S(u)$, which, at least at the initial time, acts as a negative source.

The sign-changing similarity solutions will be added to the "classical" one (see, e.g., [22]), i.e.

$$u(x,t) = -(2c + |\varepsilon|)\, t - cx^2 = t\left[-(2c + |\varepsilon|) - c\left(\frac{x}{\sqrt{t}}\right)^2\right],$$

which is clearly in the form (3) but which never attains positive value (and for which $S(u)$ is identically $\varepsilon$).

Let us now focus on the similarity solutions to (1), searching among them for those that are positive for $t > 0$ and $x$ within a suitable interval, and negative outside this interval. Considering

$$\eta = \frac{x}{\sqrt{t}},$$

we look for $u$ in the form (3) with $\gamma = 1$, namely

$$u(x,t) = t f\left(\frac{x}{\sqrt{t}}\right). \tag{4}$$

Upon substitution in $(1)_1$, we find

$$f'' + \frac{\eta}{2} f' - f = -S(f), \tag{5}$$

where the prime denotes the derivative with respect to $\eta$. Because of symmetry, we solve (5) for $\eta \geq 0$. As for the "initial" conditions to be coupled to (5), we take

$$f'(0) = 0, \qquad f(0) = \alpha > 0. \tag{6}$$

The first one is due to the symmetry of problem (1): $u$ must be an even function in the space variable $x$ and so $f$ has to be even. The second one is motivated by the fact that we look for a solution $u$ which is positive in a neighborhood of $x = 0$. Hence, recalling (2), equation (5) reduces to

$$f'' + \frac{\eta}{2} f' - f + 1 = 0, \quad \text{if} \quad f > 0, \tag{7}$$

since $S(f) = 1$, because $f$ is positive. If we take

$$f(\eta) = \alpha - \frac{1-\alpha}{2}\eta^2, \tag{8}$$

we have that both (7) and (6) are fulfilled. Moreover $f > 0$ provided

$$0 \leq \eta < \sqrt{\frac{2\alpha}{1-\alpha}}, \quad \text{and} \quad 0 < \alpha < 1. \tag{9}$$

Setting

$$\beta = \sqrt{\frac{2\alpha}{1-\alpha}}, \quad \Leftrightarrow \quad \alpha = \frac{\beta^2}{2+\beta^2}, \quad 0 < \beta < +\infty, \tag{10}$$

(8) rewrites as

$$f(\eta) = \frac{\beta^2}{2+\beta^2}\left(1 - \frac{\eta^2}{\beta^2}\right), \quad 0 \leq \eta \leq \beta, \tag{11}$$

while condition (9), understood on the $(x,t)$ plane, means that $u$ is positive when $0 \leq x < \beta\sqrt{t}$, and $u = 0$ for $\eta = \beta$, i.e. $x = \beta\sqrt{t}$. In particular

$$f'(\eta)|_{\eta=\beta} = -\frac{2\beta}{2+\beta^2} = -\sqrt{2\alpha(1-\alpha)}.$$

If $\eta > \beta$, the Cauchy problem (7), (6) has to be replaced with the following one (which holds if $f < 0$)

$$\begin{cases} f'' + \dfrac{\eta}{2}f' - f + \varepsilon = 0, & \eta > \beta, \\ \\ f(\beta) = 0, \quad f'(\beta) = -\dfrac{2\beta}{2+\beta^2}\ , \end{cases} \tag{12}$$

where the two conditions on $\eta = \beta$ are the matching conditions (i.e. $f|_{\eta=\beta^-} = f|_{\eta=\beta^+}$ and $f'|_{\eta=\beta^-} = f'|_{\eta=\beta^+}$) with (11). The solution to (12) is obtained by setting $\varphi(\eta) = f(\eta) - \varepsilon$, and then proceeding as in [1], Appendix 1. We thus have

$$f(\eta) = -2\beta e^{\beta^2/4}(1-\varepsilon)\left(\eta^2+2\right)\int_\beta^\eta \frac{e^{-s^2/4}}{\left(s^2+2\right)^2}ds - \varepsilon\left(\frac{\eta^2+2}{\beta^2+2} - 1\right), \quad \beta < \eta < +\infty,$$

which, when $\varepsilon = 0$, coincides with formula (20) of [2].

## 3. Existence of similarity solutions

Summarizing, we have found the following self-similar profile

$$f(\eta) = \begin{cases} \dfrac{\beta^2 - \eta^2}{2+\beta^2}, & 0 \le \eta < \beta, \\ \\ -2\beta e^{\beta^2/4}(1-\varepsilon)\left(\eta^2+2\right)\displaystyle\int_\beta^\eta \frac{e^{-s^2/4}}{\left(s^2+2\right)^2}ds & \\ -\varepsilon\left(\dfrac{\eta^2+2}{\beta^2+2} - 1\right), & \beta < \eta < +\infty, \end{cases} \tag{13}$$

with $\beta \in (0,\infty)$ linked to $\alpha \in (0,1)$ via formula (10). However $f$ given by (13) is not the self-similar solution of problem (1), since we still have to impose the initial condition $(1)_2$. To this end, we have to require an additional condition for $f$. Since $(1)_2$ entails

$$\frac{u(x,t)}{t} \Big/ \frac{x^2}{t} \longrightarrow -c, \quad \text{as} \quad t \to 0,$$

then $f$ has to satisfy

$$\lim_{\eta \to +\infty} \frac{f(\eta)}{\eta^2} = -c. \tag{14}$$

Hence, exploiting $(13)_2$ we define

$$F(\beta,\varepsilon) \quad = \quad -\lim\nolimits_{\eta\to\infty} \frac{f(\eta)}{\eta^2},$$

getting

$$F(\beta,\varepsilon) = 2\beta e^{\beta^2/4}(1-\varepsilon)\int\limits_\beta^\infty \frac{e^{-s^2/4}}{\left(s^2+2\right)^2}ds + \frac{\varepsilon}{\beta^2+2}, \tag{15}$$

defined in the domain $\{0<\beta<\infty,\ -\infty<\varepsilon<0\}$.

Condition (14) therefore turns into the following algebraic equation

$$F(\beta,\varepsilon) = c. \tag{16}$$

It is thus a matter of determining for which values of $\varepsilon < 0$ and $c > 0$ the above equation admits solutions $\beta > 0$, and how many such solutions there are.

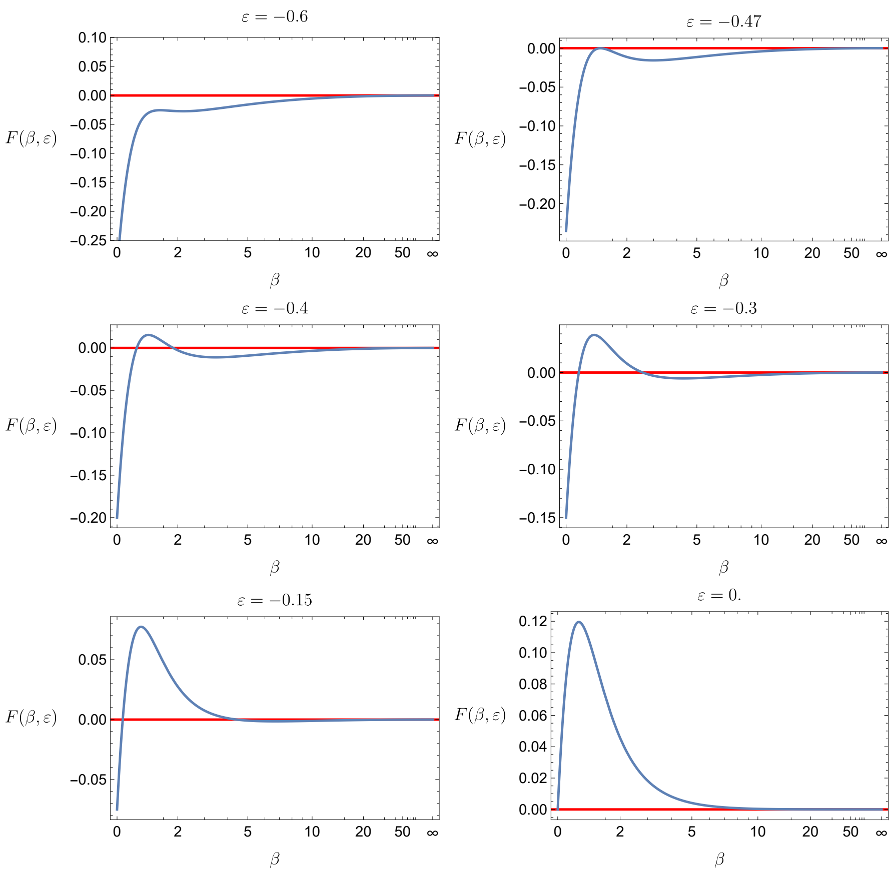


Figure 1: Effective behavior of $F(\beta,\varepsilon)$ for $\varepsilon \leq 0$. We can see that for $\varepsilon$ sufficiently small $F(\beta,\varepsilon)$ is always negative so that equation (16) does not have any solution. We note that the case $\varepsilon = 0$ reproduces exactly Fig. 1 in [2].

Figure 1, showing six possible behaviors of $F$, clearly highlights that there can be values of $\varepsilon$ for which (16) has no solutions.

Introducing

$$D(\beta,\varepsilon) = 2(1-\varepsilon)\int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2+2)^2}ds + \frac{\varepsilon e^{-\beta^2/4}}{\beta(\beta^2+2)}, \tag{17}$$

so that

$$F(\beta,\varepsilon) = \beta e^{\beta^2/4} D(\beta,\varepsilon), \tag{18}$$

equation (16) rewrites as

$$\beta e^{\beta^2/4} D(\beta,\varepsilon) = c. \tag{19}$$

So the existence of self-similar solutions with a region of positivity is linked to the positivity of $D(\beta,\varepsilon)$ in the domain $\{0<\beta<\infty,\ -\infty<\varepsilon<0\}$.

In the sequel we prove that, for suitable $\varepsilon<0$, equation (16) admits at most two solutions, provided the parameter $c$ lies in a suitable interval. This fact, which we emphasize once again, implies that problem (1) admits two sign-changing self-similar solutions. The proof goes through the following steps:

**1.** We first focus on the $(\beta,\varepsilon)$-plane and show that, in the set $\{0<\beta,\ \varepsilon<0\}$, there exists a non-empty domain, denoted by $Q$, with the property that if $(\beta,\varepsilon)\in Q$, then the function $D$ is positive and hence equation (19) admits a solution for some positive value of $c$. Conversely, if $(\beta,\varepsilon)\notin\bar{Q}$, then $D$ is strictly negative and therefore equation (19) has no solutions.

In particular, we prove that the region $Q$ has nonempty intersection with line of constant positive $\beta$, and is bounded above by the line $\varepsilon=0$ and below by a continuous curve $\varepsilon_0(\beta)$. Moreover, $\varepsilon_0(\beta)$ is shown to attain a negative minimum $\varepsilon_{0\,\min}$. This allows us to conclude that if $\varepsilon<\varepsilon_{0\,\min}$, then equation (19) has no solutions and, consequently, the problem admits no changing sign self-similar solutions.

**2.** Still working in the set $\{0 < \beta,\ \varepsilon < 0\}$, we consider the intersection of the domain $Q$ with a horizontal line, that is, a line parallel to the $\beta$-axis, at height $\varepsilon$, with $\varepsilon_{0\,\min} < \varepsilon < 0$. We show that this intersection is an interval.

**3.** We now focus the analysis to this interval (where we know that $F$ is positive) and show (and this is the key point) that $F$ has a unique positive maximum (which we denote by $c_{\text{crit}}(\varepsilon)$) and vanishes at the endpoints, that is $F = 0$, at both interval extremes. This immediately implies that, for each $\varepsilon$, with $\varepsilon_{0\,\min} < \varepsilon \leq 0$, equation (16) admits at most two solutions, provided that the parameter $c$ is smaller than the maximum of $F$, i.e. provided that $0 < c < c_{\text{crit}}(\varepsilon)$.

**4.** The final step consists in showing that $c_{\text{crit}}(\varepsilon)$ is a monotonically increasing function of $\varepsilon$.

**Point 1.**

Let us take an arbitrary positive value of $\beta$. We can state the following proposition.

**Proposition 1.** *Given an arbitrary $\beta > 0$, there exists $\varepsilon_0 < 0$ such that $D(\beta, \varepsilon) > 0$ for $\varepsilon_0 < \varepsilon \leq 0$, $D(\beta, \varepsilon_0) = 0$ and $D(\beta, \varepsilon) < 0$ for $\varepsilon < \varepsilon_0$.*

**Proof.** Given $\beta > 0$, we consider the lines

$$r_{1,\beta}(\varepsilon) = (\varepsilon - 1)\, m_1(\beta), \qquad r_{2,\beta}(\varepsilon) = \varepsilon m_2(\beta),$$

with

$$m_1(\beta) = 2\int_\beta^\infty \frac{e^{-s^2/4}}{(s^2+2)^2} ds, \qquad m_2(\beta) = \frac{e^{-\beta^2/4}}{\beta(\beta^2+2)}, \quad \text{with} \quad \beta > 0,$$

so that we can write definition (17) as

$$D(\beta, \varepsilon) = -r_{1,\beta}(\varepsilon) + r_{2,\beta}(\varepsilon) = -(\varepsilon - 1)\, m_1(\beta) + \varepsilon m_2(\beta).$$

Hence $D(\beta,\varepsilon) > 0$ for $\varepsilon < 0$ entails $r_{2,\beta}(\varepsilon) > r_{1,\beta}(\varepsilon)$. Now the slope of the line $r_{2,\beta}(\varepsilon)$ is greater than the one of $r_{1,\beta}(\varepsilon)$, as we will show shortly. As a consequence there exists $\varepsilon_0 < 0$ such that $r_{1,\beta}(\varepsilon_0) = r_{2,\beta}(\varepsilon_0)$ and $r_{1,\beta}(\varepsilon) < r_{2,\beta}(\varepsilon)$ for $\varepsilon \in (\varepsilon_0, 0]$. The corresponding geometrical interpretation in terms of the lines $r_{1,\beta}(\varepsilon)$ and $r_{2,\beta}(\varepsilon)$ is illustrated in Fig. 2.

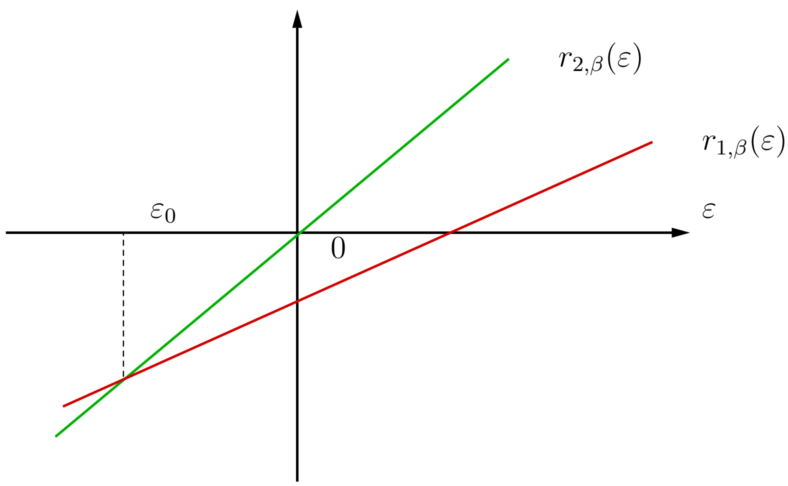


Figure 2: Behavior of the lines $r_{1,\beta}(\varepsilon)$ and $r_{2,\beta}(\varepsilon)$. Since $r_2$ is steeper than $r_1$, relation $r_1 < r_2$ can be satisfied only in a limited portion of $(-\infty, 0)$, i.e. only in the interval $(\varepsilon_0, 0)$.

To show that the slope of $r_{2,\beta}(\varepsilon)$ is greater than the one of $r_{1,\beta}(\varepsilon)$, we consider

$$M(\beta) = m_2(\beta) - m_1(\beta), \tag{20}$$

for $\beta > 0$. Since

$$\lim_{\beta\to 0^+} M(\beta) = +\infty, \qquad \lim_{\beta\to\infty} M(\beta) = 0, \tag{21}$$

and

$$M'(\beta) = -\frac{e^{-\beta^2/4}}{2\beta^2} < 0,$$

we have $M(\beta) > 0$, for any $\beta > 0$.

$\blacksquare$

A direct consequence of this proposition is that we can define a mapping which associates to each $\beta \in (0,\infty)$ a unique $\varepsilon_0 < 0$, such that

$$(\varepsilon_0 - 1)\, m_1(\beta) = \varepsilon_0 m_2(\beta)\,.$$

We therefore define the function, denoted by $\varepsilon_0(\beta)$, as

$$\varepsilon_0(\beta) = -\frac{m_1(\beta)}{m_2(\beta) - m_1(\beta)} \underset{(20)}{=} -\frac{m_1(\beta)}{M(\beta)}. \tag{22}$$

In other words, the curve $\varepsilon_0(\beta)$ is such that $D(\beta, \varepsilon_0(\beta)) = 0$, and i.e. it corresponds to the zero level set of $D$ in the quadrant $\{0 < \beta,\ \varepsilon < 0\}$. In particular, based on the properties of $M(\beta)$ discussed above, it follows that $\varepsilon_0(\beta)$ is a continuous function for all $\beta \in (0,\infty)$, is always negative and

$$\lim_{\beta\to 0^+} \varepsilon_0(\beta) = \lim_{\beta\to\infty} \varepsilon_0(\beta) = 0.$$

As for the computation of the first limit, it suffices to recall (21). Indeed

$$\lim_{\beta\to 0^+} \varepsilon_0(\beta) = -\lim_{\beta\to 0^+}\left(\frac{m_1(\beta)}{M(\beta)}\right) = -\lim_{\beta\to 0^+} \frac{2\displaystyle\int_\beta^\infty \frac{e^{-s^2/4}}{(s^2+2)^2} ds}{M(\beta)}.$$

The numerator clearly converges to a finite value (which can also be easily computed numerically and is approximately 0.44) while, by (21), the denominator tends to $+\infty$. Therefore,

$$\lim_{\beta\to 0^+} \varepsilon_0(\beta) = 0.$$

The second limit, i.e. $\lim_{\beta\to\infty} \varepsilon_0(\beta)$, is computed by evaluating $\lim_{\beta\to\infty} \dfrac{m_2(\beta)}{m_1(\beta)}$, since, by recalling (22), we have

$$\lim_{\beta\to\infty} \varepsilon_0(\beta) = -\lim_{\beta\to\infty} \frac{1}{\dfrac{m_2(\beta)}{m_1(\beta)} - 1}.$$

By applying l'Hôpital's rule, one easily shows that $\lim_{\beta\to\infty} \dfrac{m_2(\beta)}{m_1(\beta)} = \infty$, and thus $\lim_{\beta\to\infty} \varepsilon_0(\beta) = 0$.

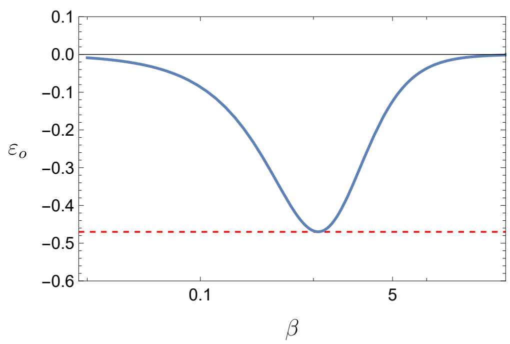


Figure 3: Graph of the function $\varepsilon_0(\beta)$. All values remain between $\approx -0.47$ and 0.

Therefore there exists a finite minimum of $\varepsilon_0(\beta)$, which we denote as $\varepsilon_{0,\min}$. In particular, $\varepsilon_{0,\min} \approx -0.47$ (see Fig. 3).

In the $(\beta, \varepsilon)$ plane we define the domain

$$Q = \left\{ (\beta, \varepsilon) \in \mathbb{R}^2 : 0 < \beta < \infty, \quad \varepsilon_0(\beta) < \varepsilon \leq 0 \right\},$$

which lies within the strip $\{0 < \beta < \infty, \quad \varepsilon_{0\,\min} < \varepsilon \leq 0\}$. We can therefore state the following

**Proposition 2.** *Given $(\beta, \varepsilon) \in Q$, then the algebraic equation (16) is certainly fulfilled for some positive $c$.*

**Proof.** Let $(\beta, \varepsilon) \in Q$, i.e. $\beta > 0$ and $\varepsilon_0(\beta) < \varepsilon \leq 0$, and hence $D(\beta, \varepsilon)$ is positive. Moreover, $D$ is continuous with respect to $\varepsilon$, since it is a linear combination of the continuous functions $m_1$ and $m_2$. Therefore, the intermediate value Theorem implies that, for sufficiently small positive values of $c$, the algebraic equation (16) is certainly fulfilled.

■

**<u>Point 2.</u>**

Now, in order to determine the number of self-similar solutions, we section the domain $Q$ with lines parallel to the $\beta$-axis. Let us indeed fix $\varepsilon$, $\varepsilon_{0,\min} < \varepsilon \leq 0$, and consider the line parallel to the $\beta$-axis at ordinate $\varepsilon$ and, moving along this line, we analyze the behavior of $F$ (that is, we regard $F(\beta,\varepsilon)$ as a function of $\beta$ only, $\beta \in \mathbb{R}^+$). We have

$$F(0,\varepsilon) = -\frac{|\varepsilon|}{2}, \tag{23}$$

and

$$\lim_{\beta\to\infty} F(\beta,\varepsilon) = 0, \tag{24}$$

since, by invoking l'Hôpital's rule, we obtain

$$\lim_{\beta\to\infty} F(\beta,\varepsilon) = (1-\varepsilon)\lim_{\beta\to\infty} \frac{\displaystyle\int_\beta^\infty \frac{e^{-s^2/4}}{\left(s^2+2\right)^2} ds}{\left(2\beta e^{\beta^2/4}\right)^{-1}} = 4(1-\varepsilon)\lim_{\beta\to\infty} \frac{\beta^2}{\left(\beta^2+2\right)^3} = 0.$$

Moreover $F(\beta,\varepsilon) > 0$ only on those portions of the line that lie within the domain $Q$. However, in order to fully analyze the behavior of $F$ along this line, it is necessary to show that the intersection between $Q$ and any line parallel to the $\beta$-axis, at height $\varepsilon$, with $\varepsilon_{0\,\min} < \varepsilon < 0$, is an interval (which becomes the entire half-line $\beta > 0$, if $\varepsilon = 0$). We indeed have the following:

**Proposition 3.** *The intersection between the line parallel to the $\beta$-axis with ordinate $\varepsilon$, $\varepsilon_{0\,\min} < \varepsilon \leq 0$, with the domain $Q$ is a finite interval.*

**Proof.** Every line parallel to the $\beta$-axis at level $\varepsilon$, with $\varepsilon_{0,\min} < \varepsilon \leq 0$, has a non-empty intersection with the domain $Q$. We show that the intersection of this line with the domain $Q$ is actually an interval. Focussing on $D$ we have

$$\lim_{\beta\to 0} D(\beta,\varepsilon) = -\infty, \tag{25}$$

$$\lim_{\beta\to\infty} D(\beta,\varepsilon) = 0, \tag{26}$$

and computing $\dfrac{\partial D}{\partial \beta}$ we get

$$
\begin{aligned}
\frac{\partial D}{\partial \beta} &= \frac{e^{-\beta^2/4}}{2\beta^2\left(\beta^2+2\right)^2}\left[-\varepsilon\beta^4 - 4\beta^2\varepsilon - 4\varepsilon - 4\beta^2\right] \\
&= \frac{e^{-\beta^2/4}}{2\beta^2\left(\beta^2+2\right)^2}\left[|\varepsilon|\left(\beta^2+2\right)^2 - 4\beta^2\right] \\
&= \frac{e^{-\beta^2/4}}{2\beta^2\left(\beta^2+2\right)^2}\left[\sqrt{|\varepsilon|}\left(\beta^2+2\right)+2\beta\right]\left[\sqrt{|\varepsilon|}\left(\beta^2+2\right)-2\beta\right].
\end{aligned}
$$

Since we restrict ourselves to $\beta > 0$, we have that $\dfrac{\partial D}{\partial \beta} \geq 0$ if

$$
\beta^2 - \frac{2\beta}{\sqrt{|\varepsilon|}} + 2 \geq 0, \quad \Rightarrow \quad \beta \leq \beta_1 \;\text{ and }\; \beta_2 \leq \beta,
$$

with

$$
\begin{cases}
\beta_1 = \dfrac{1}{\sqrt{|\varepsilon|}} - \sqrt{\dfrac{1}{|\varepsilon|} - 2}\,, \\[2ex]
\beta_2 = \dfrac{1}{\sqrt{|\varepsilon|}} + \sqrt{\dfrac{1}{|\varepsilon|} - 2}\,.
\end{cases}
$$

Let first $\varepsilon_0^{\min} < \varepsilon < 0$. From the above computation of $\partial D/\partial\beta$, we have

$$
\frac{\partial D}{\partial \beta} > 0, \quad \text{for} \quad 0 < \beta < \beta_1,
$$

$$
\frac{\partial D}{\partial \beta} < 0, \quad \text{for} \quad \beta_1 < \beta < \beta_2,
$$

and

$$
\frac{\partial D}{\partial \beta} > 0, \quad \text{for} \quad \beta > \beta_2.
$$

Since the intersection of the line at height $\varepsilon$ with $Q$ is non-empty, $D$ is positive for some $\beta > 0$, and therefore $D(\beta_1, \varepsilon) > 0$. On the other hand, since $D$ is increasing for $\beta > \beta_2$ and (26) holds true, we necessarily have $D(\beta_2, \varepsilon) < 0$. Recalling also (25), the intermediate value theorem, together with the above monotonicity properties, implies that $D$ has exactly two zeros, one in $(0, \beta_1)$ and the other in $(\beta_1, \beta_2)$. Hence $D$ is positive exactly between these two zeros, and the intersection of the line at height $\varepsilon$ with $Q$ is a finite interval. This behavior is illustrated in Fig. 4.

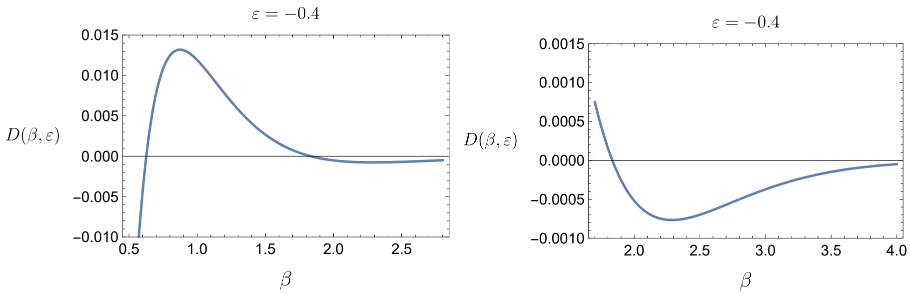


Figure 4: Effective behavior of $D(\beta, \varepsilon)$ for $\varepsilon_{0,\min} < \varepsilon \leq 0$. The function values vary over a very small scale: we have focused over different intervals for visualization purposes. The right panel shows a magnified view of the left panel over the interval $(1.5, 4)$. It is included to emphasize the presence of a negative minimum of $D$.

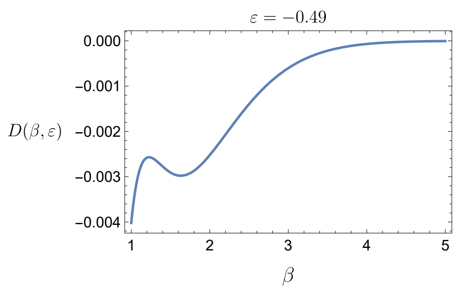


Figure 5: Effective behavior of $D(\beta, \varepsilon)$ for $-1/2 < \varepsilon < \varepsilon_{0,\min}$. In such a case $D$ never attains positive values.

For $\varepsilon = 0$, it follows directly from (17) that $D(\beta, 0) > 0$ for every $\beta > 0$, so that the intersection with $Q$ is the whole half-line $\beta > 0$.

Incidentally, we have also provided a lower bound for $\varepsilon_{0,\min}$. Indeed, $D$ has neither maxima nor minima if $|\varepsilon| \geq 1/2$ (i.e. $\varepsilon \leq -1/2$, since $\varepsilon < 0$). We thus have $\varepsilon_{0,\min} > -1/2$.

$\blacksquare$

Figure 5 illustrates the behavior of $D(\beta, \varepsilon)$, for $-1/2 < \varepsilon < \varepsilon_{0\,\min}$. In this range, the maximum of $D$, still attained at $\beta_1$, is negative, so that $D$ remains negative throughout. In particular, this confirms that the intersection of a line parallel to the $\beta$-axis with $Q$ is a finite interval, on which $D$ attains a unique positive maximum.

**Point 3.**

Now, by analyzing the behavior of $F$ on this segment, we can state the following result:

**Proposition 4.** *For every $\varepsilon \in (\varepsilon_{0,\min}, 0]$, the algebraic equation (16) is fulfilled by at most two values of $\beta$.*

**Proof.** We restrict ourselves to the segment given by the intersection of the line parallel to the $\beta$-axis, at height $\varepsilon \in (\varepsilon_{0,\min}, 0]$, with the domain $Q$. We have that $F$, given by (18), is positive on such an interval. To show that equation (16) has at most two solutions, we need to prove that $F$, viewed as a function of $\beta$, possesses only one maximum in such an interval. We start writing $\frac{\partial F}{\partial \beta}$, i.e.

$$\frac{\partial F}{\partial \beta} = \left(2+\beta^2\right)(1-\varepsilon)\, e^{\beta^2/4} \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{\left(s^2+2\right)^2} ds - \frac{2\beta}{\left(\beta^2+2\right)^2}. \tag{27}$$

It is clear that determining the zeros of $\frac{\partial F}{\partial \beta}$ is equivalent to finding the zeros of

$$G(\varepsilon, \beta) = \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{\left(s^2+2\right)^2} ds - \frac{2\beta e^{-\beta^2/4}}{(1-\varepsilon)\left(\beta^2+2\right)^3}, \tag{28}$$

since, for $\beta > 0$, $\operatorname{sgn} G = \operatorname{sgn}\left(\frac{\partial F}{\partial \beta}\right)$. We first observe that $G(\varepsilon, 0) > 0$ and that $\lim_{\beta \to \infty} G = 0$. Next,

$$\frac{\partial G}{\partial \beta} = \frac{e^{-\beta^2/4}}{\left(\beta^2+2\right)^2} \left[-1 + \frac{1}{(1-\varepsilon)\left(\beta^2+2\right)} \left(-2+\beta^2+\frac{12\beta^2}{\beta^2+2}\right)\right].$$

Hence, setting $s = \beta^2 + 2$, we have that $\frac{\partial G}{\partial \beta} = 0$, provided

$$|\varepsilon|\, s^2 - 8s + 24 = 0,$$

namely

$$s = \begin{cases} \dfrac{4 - \sqrt{16 - 24\,|\varepsilon|}}{|\varepsilon|}, \\ \\ \dfrac{4 + \sqrt{16 - 24\,|\varepsilon|}}{|\varepsilon|}. \end{cases}$$

Before proceeding, it is appropriate to point out a couple of remarks. The first is that $\dfrac{\partial G}{\partial \beta}$ can vanish only if $|\varepsilon| < 2/3$ (which is fully consistent with what has already been noted, namely $|\varepsilon| < |\varepsilon_{0\,\mathrm{min}}| < 1/2$). The second is that

$$s - 2 = \begin{cases} \dfrac{4 - 2\,|\varepsilon| + \sqrt{16 - 24\,|\varepsilon|}}{|\varepsilon|}, \\ \dfrac{4 - 2\,|\varepsilon| - \sqrt{16 - 24\,|\varepsilon|}}{|\varepsilon|}, \end{cases}$$

is certainly non-negative, so that the equation $\beta^2 = s - 2$, does not lead to any contradiction. Thus, considering $\beta > 0$, $\dfrac{\partial G}{\partial \beta}$ vanishes for $\beta = \beta_\pm$, where

$$\beta_\pm = \sqrt{\frac{4 - 2\,|\varepsilon| \pm \sqrt{16 - 24\,|\varepsilon|}}{|\varepsilon|}}.$$

In particular, since $\left.\dfrac{\partial G}{\partial \beta}\right|_{\beta=0} < 0$, we have that $\beta_-$ corresponds to a local minimum while $\beta_+$ a local maximum. The possible behaviors of $G$ are those represented in Fig. 6.

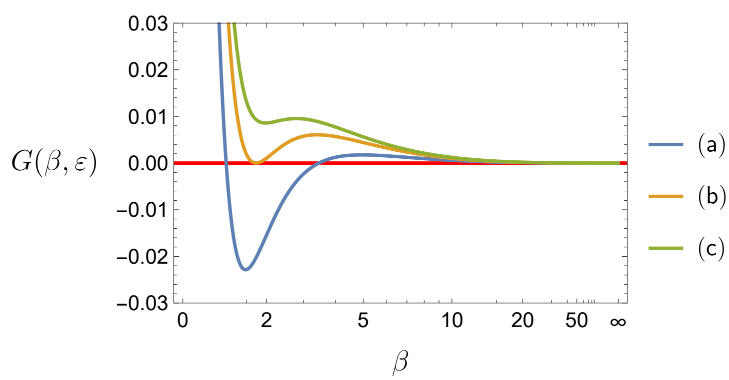


Figure 6: Possible behavior of the function $G$ given by (28).

If $G$ behaved as in cases (b) or (c) of Fig. 6, then $G$ would be non-negative for every $\beta > 0$. Since, for $\beta > 0$, the sign of $\partial F/\partial\beta$ coincides with the sign of $G$, the function $F$ would be non-decreasing. Recalling (23) and (24), this would imply that $F$ is always non-positive, in contradiction with the fact that $F$ is positive for some $\beta$. Therefore, only case (a) can occur.

More precisely, $\beta_-$ and $\beta_+$ are respectively a local minimum and a local maximum of $G$. Since $G(\varepsilon, 0) > 0$ and $G(\varepsilon, \beta_-) < 0$, while $G(\varepsilon, \beta_+) > 0$, the continuity of $G$ implies the existence of exactly two positive zeros, say $\widehat{\beta}_-$ and $\widehat{\beta}_+$, such that

$$0 < \widehat{\beta}_- < \beta_- < \widehat{\beta}_+ < \beta_+.$$

Consequently,

$$\begin{aligned} \frac{\partial F}{\partial \beta} &> 0, \quad \text{for} \quad 0 < \beta < \widehat{\beta}_-, \\ \frac{\partial F}{\partial \beta} &< 0 \quad \text{for} \quad \widehat{\beta}_- < \beta < \widehat{\beta}_+, \end{aligned}$$

and

$$\frac{\partial F}{\partial \beta} > 0 \quad \text{for} \quad \beta > \widehat{\beta}_+.$$

Hence $F$ attains a unique local maximum at $\widehat{\beta}_-$ and a unique local minimum at $\widehat{\beta}_+$. Therefore, taking (23) into account, the possible behaviors of $F$ are those shown in Fig. 7.

Cases (ii) and (iii) are not possible because $F$ is positive for some $\beta$, provided $\varepsilon \in (\varepsilon_{0,\min}, 0]$. Therefore, the only possible case is (i). Consequently, by the monotonicity properties of $F$ established above, equation (16) admits at most two solutions.

■

So, every $\varepsilon \in (\varepsilon_{0,\min}, 0]$ we denote by $c_{\mathrm{crit}}(\varepsilon)$ the maximum of $F$, i.e.

$$c_{\mathrm{crit}}(\varepsilon) = \max_{\beta > 0} F(\beta, \varepsilon).$$

Therefore $c_{\mathrm{crit}}$ represents the "critical" value of $c$ below which two intersections exist. To determine $c_{\mathrm{crit}}(\varepsilon)$ we use formula (18). Differentiating with respect to $\beta$ we get

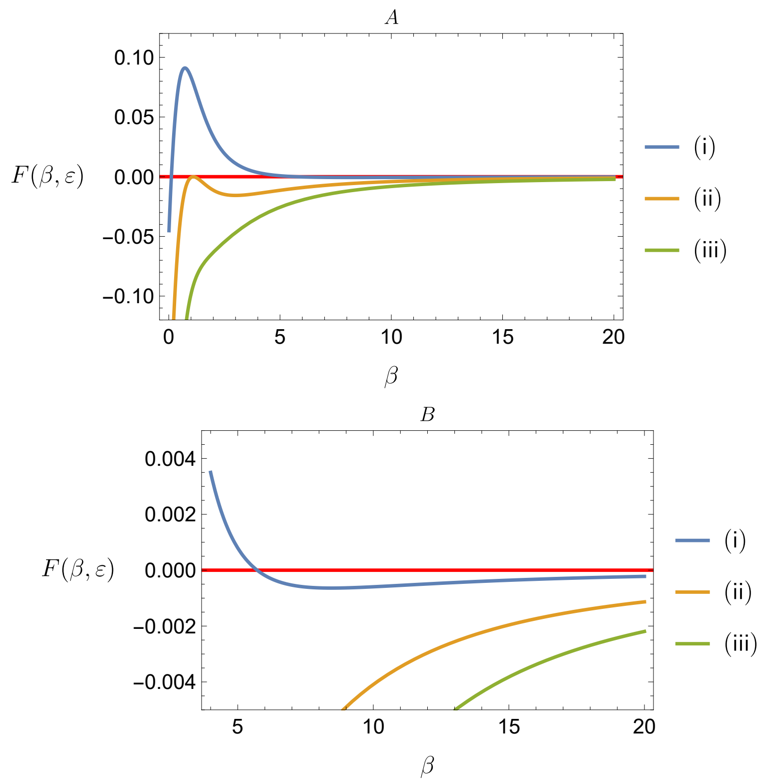


Figure 7: Possible behavior of the function $F$ given by (15). Figure B presents a magnification of Fig. A over the interval $\beta \in (5, 20)$, clearly highlighting that, for $\varepsilon \in (\varepsilon_{0\,\mathrm{min}}, 0)$, i.e. blue line, $F$ indeed attains a negative minimum.

$$D(\beta, \varepsilon) = -\frac{\partial D(\beta, \varepsilon)}{\partial \beta} \frac{\beta}{1 + \beta^2/2}. \tag{29}$$

Then, denoting by $\beta^*(\varepsilon)$ the solution of (29) corresponding to the local maximum, we have

$$c_{\mathrm{crit}}(\varepsilon) = D(\beta^*, \varepsilon)\beta^* e^{\beta^{*2}/4} \underset{(18)}{=} F(\beta^*, \varepsilon). \tag{30}$$

**<u>Point 4.</u>**

Figure 8 shows $c_{\mathrm{crit}}$ in the allowable range of $\varepsilon$ while Tab. 1 provides some of its numerically calculated values.

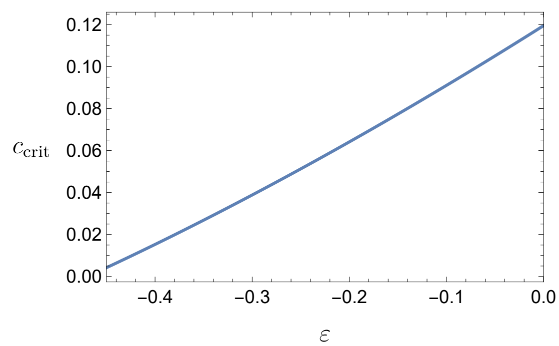


Figure 8: Function $c_{\text{crit}}(\varepsilon)$ in the allowable range of $\varepsilon$

Table 1: Calculated values of $c_{\text{crit}}(\varepsilon)$

| $\varepsilon$ | $c_{\text{crit}}$ |
|---|---|
| -0.45 | 0.0042 |
| -0.4 | 0.0152 |
| -0.35 | 0.0268 |
| -0.3 | 0.0388 |
| -0.25 | 0.0512 |
| -0.2 | 0.0641 |
| -0.15 | 0.0773 |
| -0.1 | 0.0910 |
| -0.05 | 0.1050 |
| 0.0 | 0.1195 |

We actually can show that $c_{\text{crit}}(\varepsilon)$ is a monotonically increasing function. We indeed have the following:

**Proposition 5.** *The function $c_{\text{crit}}(\varepsilon)$ is a monotonic increasing for $\varepsilon \in (\varepsilon_{0,\min}, 0]$.*

**Proof.** We start recalling that $\beta^*(\varepsilon)$ is the solution of (29), namely

$$\frac{\partial}{\partial\beta}\left[\beta e^{\beta^2/4} D(\beta,\varepsilon)\right]\Bigg|_{\beta^*(\varepsilon)} = 0. \tag{31}$$

Hence $\beta^*(\varepsilon)$ gives the value $\beta$ at which $F$ attains its maximum, i.e. $c_{\text{crit}}(\varepsilon)$. We therefore exploit equation (30) to differentiate with respect to $\varepsilon$ the function $c_{\text{crit}}(\varepsilon)$, namely

$$\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} = \frac{\partial}{\partial\varepsilon}\left[D(\beta^*,\varepsilon)\right]\beta^* e^{\beta^{*2}/4} + \frac{\partial}{\partial\beta^*}\left[D(\beta^*,\varepsilon)\beta^* e^{\beta^{*2}/4}\right]\frac{\partial\beta^*}{\partial\varepsilon}.$$

The second term vanishes because of (31) and, exploiting (17), we obtain

$$\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} = H\left(\beta^*\right)\beta^* e^{\beta^{*2}/4},$$

where

$$H\left(\beta\right) = \frac{e^{-\beta^2/4}}{\beta\left(\beta^2+2\right)} - 2\int_{\beta}^{\infty} \frac{e^{-s^2/4}}{\left(s^2+2\right)^2} ds.$$

We thus have $\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} \geq 0$ provided $H\left(\beta\right) \geq 0$ for $0 < \beta < \infty$. We now remark that

$$\lim_{\beta\to 0} H\left(\beta\right) = +\infty, \qquad \lim_{\beta\to+\infty} H\left(\beta\right) = 0,$$

and that

$$\frac{dH\left(\beta\right)}{d\beta} = -\frac{e^{-\beta^2/4}}{2\left(\beta^2+2\right)} - \frac{e^{-\beta^2/4}}{\beta^2\left(\beta^2+2\right)} < 0, \quad \forall\, \beta > 0.$$

Hence $H\left(\beta\right)$ is positive for $0 < \beta < \infty$, and consequently $\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} > 0$.

■

## 4. Conclusions

In the context of physical models, the present study is motivated by the fundamental question of whether a reaction diffusion process governed by equation (1), with $S(u)$ specified by (2), can be initiated under the condition that the initial datum $u(x,0)$ is negative everywhere (thus rendering the source term inactive) except at a single point $x = 0$, where $u = 0$. A key characteristic of model $(1)_1$ is that the positive source term becomes active exclusively for $u > 0$. From this standpoint, sign-changing solutions of (1) are particularly relevant to elucidating the underlying dynamics of the system.

The findings of this study reveal an unexpected outcome. The source/sink term $S(u)$, which may assume both positive and negative values, can effectively act as a source thereby initiating, for instance, a chemical reaction, provided that the initial condition exhibits small curvature (i.e., limited values of the constant $c$) and that the sink effect is sufficiently moderate, specifically for $|\varepsilon| < |\varepsilon_{0,\min}|$. It has been demonstrated that problem (1) admits at most two self-similar, sign-changing solutions for $\varepsilon \in (\varepsilon_{0,\min}, 0]$, and only for adequately small values of the parameter $c$. Moreover, for a given $\varepsilon \in (\varepsilon_{0,\min}, 0]$, the corresponding critical value of $c$, that is, the threshold beyond which problem (1) no longer admits self-similar sign-changing solutions, has been determined numerically. The behavior of $c_{\mathrm{crit}}(\varepsilon)$ is shown in Fig. 8.


## Acknowledgments

This research was partially supported by GNFM of Italian INDAM.

## References


[1] R. Gianni and J. Hulshof. The semilinear heat equation with a heaviside source term. *European Journal of Applied Mathematics*, 3(4):367–379, 1992.

[2] A. Farina and R. Gianni. Self-similar solutions for the heat equation with a positive non-lipschitz continuous, semilinear source term. *Nonlinear Analysis: Real World Applications*, 79:104121, 2024.

[3] D. G. Aronson. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. *Partial Differential Equations and Related Topics*, pages 1–23, 1975.

[4] J. Keener. Propagation and its failure in coupled systems of discrete excitable cells. *SIAM Journal on Applied Mathematics*, 47(3):556–572, 1987.

[5] M. A. Herrero and J. J. L. Velázquez. Blow-up in parabolic equations with nonlinearities of arbitrary growth. *Journal of Differential Equations*, 70(2):356–374, 1987.

[6] F. Merle and H. Zaag. Blow-up solutions of semilinear heat equations with optimal rates. *Communications on Pure and Applied Mathematics*, 58(11):1597–1654, 2005.

[7] V. A. Galaktionov and J. L. Vázquez. A unified approach to blow-up analysis for nonlinear heat equations. *Journal of Differential Equations*, 127(1):1–40, 1995.

[8] M. Chipot and B. Lovat. On the analysis of some reaction–diffusion systems with singular reaction terms. *Nonlinear Analysis*, 30(6):3801–3816, 1997.

[9] G. A. Pavliotis and A. M. Stuart. Multiscale methods: averaging and homogenization. *Springer*, 2008.

[10] L. A. Brown and D. R. Smith. Thermal ignition in heterogeneous media. *Proceedings of the Royal Society A*, 451(1941):579–607, 1995.

[11] V. Volpert and S. Petrovskii. Traveling waves for reaction–diffusion systems: A review. *Physics of Life Reviews*, 2(1):1–67, 2005.

[12] J. D. Murray. *Mathematical Biology I: An Introduction*. Springer, 3rd edition, 2002.

[13] A. Volpert, Vi. Volpert, and Vl. Volpert. *Traveling Wave Solutions of Parabolic Systems*. American Mathematical Society, 1994.

[14] J. Norbury and A. M. Stuart. Parabolic free boundary problems arising in porous medium combustion. *IMA journal of applied mathematics*, 39(3):241–257, 1987.

[15] J. Norbury and A. M. Stuart. A model for porous-medium combustion. *The Quarterly Journal of Mechanics and Applied Mathematics*, 42(1):159–178, 1989.

[16] A. Friedman and A. E. Tzavaras. Combustion in a porous medium. *SIAM Journal on Mathematical Analysis*, 19(3):509–519, 1988.

[17] G. I. Barenblatt. *Scaling, Self-Similarity, and Intermediate Asymptotics*. Cambridge University Press, 1996.

[18] Ya. B. Zel'dovich and Yu. P. Raizer. *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*. Dover, 2002.

[19] S. Kamin and L. A. Peletier. Large time behaviour of solutions of the porous medium equation. *Annales de l'IHP Analyse Non Linéaire*, 2(3):299–308, 1985.

[20] A. Friedman and S. Kamin. The asymptotic behavior of gas in an expanding universe. *Transactions of the AMS*, 262(2):551–563, 1980.

[21] R. Gianni and R. Ricci. Classical solvability of some free boundary problems through the geometry of the level lines. *Advances in Mathematical Sciences and Applications*, 5:557–567, 1995.

[22] J. R. Cannon. *The One-Dimensional Heat Equation.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1984.