An algorithmic approach to the harmonic sets of Just Intonation

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Abstract

This paper investigates the mathematical foundations of Just Intonation through the lens of Renaissance architectural proportions. While Pythagorean tuning and Equal Temperament are governed by transparent, recursive algorithms, the 5-limit system of Just Intonation is often perceived as a non-systematic collection of local optimizations. By shifting the analytical focus from frequency ratios to the classical harmonic mean - a cornerstone of Palladian and Albertian architectural theory - we propose a generative algorithmic approach to pitch selection.

Operating on a foundational nucleus of perfect consonances, the algorithm iteratively "harvests" new pitches only if they establish rigorous harmonic proportions with the existing set. Our results demonstrate that this procedure does not merely replicate the modern diatonic scale; instead, it identifies a finite, algebraically closed set of sounds that mirrors the structural logic of the Renaissance hexachord and the modal finales. This finding suggests that the "natural" scale of the 16th century was not an arbitrary cultural construct, but a mathematically perfectum design - a "rational soul" where internal consistency precludes the infinite drift of Pythagorean fifths. The study concludes by drawing a parallel between this closed harmonic system and the transition from the mystic infinity of Gothic architecture to the unified, proportional space of the Renaissance temple.

1 Introduction

Music theory is a sort of struggle between pure math, which seeks to use every number, and the human ear,which selects a limited subset constrained by historical era and cultural background. While it is an established axiom that the musical scales of any given culture are rooted in arithmetic, a fundamental question remains: does the collection of frequencies defining a specific harmonic system possess an intrinsic mathematical structure, or is it merely an arbitrary list of sounds established by tradition?
This study interrogates whether a musical set is the output of a formal algorithm or simply a culturally curated inventory lacking internal logical mechanisms. To address this, we analyze the structural foundations of sound through two primary lenses:

1. The mathematical foundation of standard tunings: we first review the relationship between mathematics and the two dominant paradigms of Western music history: Pythagorean Tuning and Equal Temperament. In these systems, the connection to mathematical sequences - exponential in the case of temperament and iterative in the case of the the iterative process of the Pythagorean fifths - is a well-documented phenomenon.

2. The formalization of Just Intonation: the focus then shifts to Just Intonation, where the integration of a rigorous mathematical framework proves more elusive. Unlike the symmetry of tempered systems, Just Intonation relies on "pure" rational intervals that do not always align with simple linear or logarithmic models.

The innovative contribution of this study lies within this second focus. This article proposes and discusses mathematical algorithms designed to generate musically coherent harmonic systems within the realm of Just Intonation. By adopting a core set of principles inspired by Renaissance harmonic proportions, we demonstrate how the concept of proportio can formally govern the generation and combination of harmonic elements.

The historical transition from Pythagorean intervals to "just" (natural) intervals is traditionally viewed as a localized, targeted intervention aimed at simplifying specific ratios. This shift was primarily driven by the need to refine major and minor thirds, which became foundational during the Renaissance-era transition from modality to tonality.

This paper establishes that the framework of harmonic proportions provides a rigorous mathematical basis for what has often been dismissed as a purely empirical or practical adjustment. By framing the adoption of natural intervals as a process of "local optimization", we bridge the gap between aesthetic preference and algorithmic necessity, showing that these musical choices are governed by an underlying structural logic.

To conclude this study, we propose an algorithm for constructing a harmonic set based on the principles of harmonic proportion. In developing this model, we adopted the perspective of a Renaissance architect - the era in which the theory of natural sounds was first formalized. Unlike the logic of Equal Temperament, which replicates identical intervals, or Pythagorean Tuning, which prioritizes a singular relationship to a generator, our "architect" adds new elements based on their collective resonance. A new frequency is integrated not because it is equal to the previous ones, but because it achieves a proportional equilibrium within the existing structure, governed by the harmonic proportions it forms with the entire ensemble.

The proposed model employs a fixed-point iterative algorithm. Operating on the reciprocals of frequencies (string lengths) to maintain the arithmetic-harmonic duality, the system calculates the means of its elements and expands the set. This expansion is governed by a 5-limit constraint ($2^n \cdot 3^m \cdot 5^p$), acting as a harmonic filter that reflects the acoustic boundaries of the Renaissance. The process continues until it reaches a state of algebraic closure, where no new harmonic entities can be generated within the defined parameters.

The significance of this iterative process transcends the definition of a single scale. Rather than a linear sequence of notes, the algorithm generates a cohesive harmonic field whose internal topography reveals the foundational structures of Western music. Upon reaching the fixed-point state - resulting in a set of approximately 20 natural frequencies - the system does not merely present a list of sounds; it spontaneously organizes into highly significant historical subsets. Specifically, the algorithm's output reconstructs:

- the Four Finales: the formal kernels of the modal system (Protus, Deuterus, Tritus, and Tetrardus),
- the Hexachordal System: the fundamental scalar unit of Guido d'Arezzo and Renaissance pedagogy.

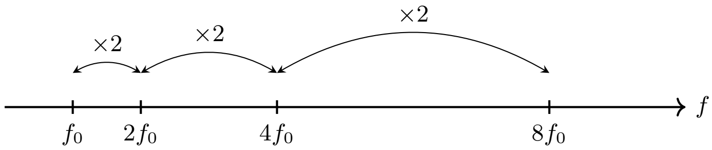
This suggests that the "local optimization" performed by Renaissance theorists was not an isolated adjustment of intervals, but a navigation within a pre-existing mathematical lattice. The modal system, therefore, is revealed as a rigorous sub-structure of a natural harmonic closure, where tradition and logic converge.

2 The mathematics of a tuning system

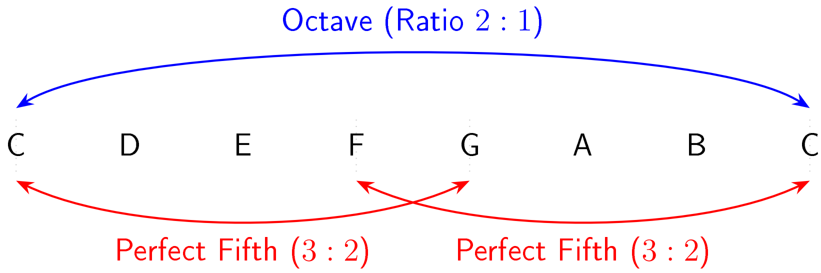
We can represent the range of sound frequencies along the positive semi-axis $\mathbb{R}^+$, disregarding the auditory thresholds of 20 Hz – 20 kHz within this abstract model.

The following prospectus illustrates the fundamental musical concepts arising from the *principle of the octave*, whose theoretical systematization reached its peak in Jean-Philippe Rameau's *Traité de l'harmonie* (1722).

- **The Identity of Octaves:** The octave is defined as a "replica" of a fundamental frequency f, whereby any pitch $2^k f$ (where $k \in \mathbb{Z}$) is perceived as functionally identical. This principle is supported by mathematical simplicity ($1:2$ ratio) and physical resonance, as the first harmonic ($2f$) is naturally embedded within the fundamental sound itself.

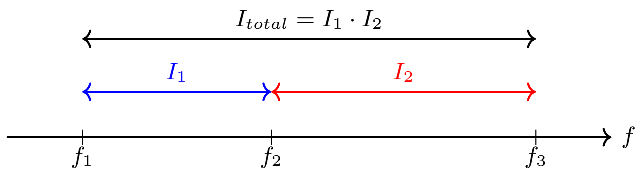


The term "octave" (from Latin *octavus*) refers to the eighth position in the diatonic succession (C–D–E–F–G–A–B–[C]), marking the completion of the cycle at a double frequency ratio (2 : 1). Similarly, the "perfect fifth" is so named because it spans five degrees of the scale (e.g., C–G or F–C), corresponding to the frequency ratio 3 : 2. The following diagram illustrates the justification for the terms "octave" and "fifth", based on the standard sequence of the eight diatonic notes.



- **Ratio-based Measurement of Interval**: musical "distance" is not measured as a linear Euclidean length, but as a ratio between frequencies. This is rooted in how we identify octaves: in the diagram above, the segments covered by each curved double arrow represent the same musical interval, therefore they must be "equal". The interval I between two frequencies f_1 and f_2 is defined as $I = f_2/f_1$. Consequently, the composition of adjacent intervals follows a multiplicative property:

$$I(f_1, f_3) = I(f_1, f_2) \cdot I(f_2, f_3) = \frac{f_2}{f_1} \cdot \frac{f_3}{f_2} = \frac{f_3}{f_1} \tag{1}$$



- **Scale Construction and Tuning:** A tuning system is established by partitioning the octave $[f_0, 2f_0]$ into N steps. Across subsequent octaves, the partition repeats identically, following the principle of octave.

  The identity of a scale is determined by:

  1. **Cultural Context (N):** the number of divisions (e.g., $N = 12$ chromatic, $N = 5$ pentatonic).
  2. **Mathematical Constraints:** the selection of specific ratios. Historically, this involves a choice between rational numbers and irrational numbers.

2.1 Equal Temperament and Pythagorean Tuning

To understand the challenges of scale construction, this section addresses two foundational historical paradigms. From an axiomatic standpoint, both are governed by transparent arithmetic structures and rigorous algorithmic frameworks; they demonstrate how simple mathematical constraints can be profoundly fruitful, translating iterative logic into two distinct, yet equally coherent, harmonic architectures.

Equal Tempered Scale

From a mathematical perspective, the N-tone Equal Temperament is established by partitioning the octave $[f_0, 2f_0]$ into N equal intervals, where equality is defined by the rule (1). Each frequency is defined by the geometric progression

$$f_{m,k} = 2^{m+\frac{k}{N}} f_0$$

where $m \in \mathbb{Z}$ is the octave index and $k \in \{0, \dots, N-1\}$ is the step index. The system is governed by the constant ratio $r = 2^{1/N}$. The set of equal tempered sounds generated by the fundamental frequency f_0 can be defined as

$$\mathbb{T}_N(f_0) = \Big\{ f \in \mathbb{R} \,|\, f = f_{m,k} := 2^{m+k/N} f_0,\ k \in 0, 1, \dots, N-1,\ m \in \mathbb{Z} \Big\}. \tag{2}$$

Notice that the value $k = N$ is excluded to avoid redundancy, since $f_{m,N} = 2^{m+N/N} f_0 = 2^{m+1} f_0 = f_{m+1,0}$.

All normalized frequencies $2^{m+k/N}$ (for $1 \leq k < N$) are irrational. While they diverge from the simple integer ratios of the harmonic series, they provide a perfectly uniform division of the sonic space. The set $\mathbb{T}_N(f_0)$ is a discrete subset of $\mathbb{R}^+$. The existence of a minimum distance between neighbors ($\Delta f > 0$) ensures that each pitch is an isolated, identifiable point - a mathematical prerequisite for stable musical notation.

The set $\mathbb{T}_N(f_0)$ is an equivalence class of the multiplicative quotient group $\mathbb{R}^+/\langle 2^{1/N} \rangle$. Choosing a reference (e.g., $A4 = 440$ Hz) selects one of infinitely many "parallel and disjoint" tempered systems from the partition of all possible sounds. This is the system's defining feature. Since the interval between any two frequencies depends solely on the difference between their indices, the ratio remains invariant under a shift of τ steps:

$$I(r^\tau f_1, r^\tau f_2) = \frac{r^\tau f_2}{r^\tau f_1} = \frac{f_2}{f_1} = I(f_1, f_2)$$

Musically, this ensures that chords, scales, and melodies retain their internal structural identity regardless of the starting key. In a equal tempered system all N keys are functionally equivalent, allowing for seamless modulation and transposition.

Pythagorean Tuning

Starting from a reference frequency f_0, the admissible sound universe consists of frequencies defined by the formula

$$f = 2^n \cdot 3^m \cdot f_0, \qquad n, m \in \mathbb{Z} \tag{3}$$

which defines the set of Pythagorean pitches generated from the reference frequency f_0:

$$\mathbb{P}(f_0) = \left\{ f \in \mathbb{R} \,|\, f = p_{a,b} = 2^a 3^b f_0,\ a, b \in \mathbb{Z} \right\}. \tag{4}$$

The ancient rationale for this selection lies in the Pythagorean perfection of the *tetraktys*, a sacred emblem represented by the first four integers (1, 2, 3, 4) and the key to understanding the cosmos. Within the Pythagorean worldview, consonances derive only from the ratios between these pure numbers, which were considered both harmonically and cosmologically valid. In musical terms, this led to the belief that only sounds obtained by dividing a string into halves or thirds were harmonious.

According to modern theory, the set defined in (3) represents a 3-limit system [6, 8, 4]. An n-limit system refers to a harmonic space where the prime factorization of any frequency ratio contains no prime number greater than n. Mathematically, any frequency f in such a system can be expressed as:

$$f = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_k^{m_k} \cdot f_0 \tag{5}$$

where $\{p_1, p_2, \dots, p_k\}$ is the set of all prime numbers up to n (such that $p_k \leq n$), and the exponents m_i are integers. Specifically, a 3-limit system derives all its intervals exclusively from the primes 2 and 3 (representing octaves and fifths), excluding higher primes such as 5, which would instead characterize a 5-limit system and enable "pure" major thirds (5/4).

While the 3-limit system defines the boundaries of the available harmonic space, the Pythagorean tuning specifically populates this space through a recursive generator. Instead of choosing values at random from the set in (3), the system "harvests" frequencies by iteratively applying the interval of a perfect fifth (3 : 2) in both ascending and descending directions.Mathematically, this process simplifies the general formula by linking the exponents of the primes 2 and 3. Each step along the chain of fifths corresponds to a frequency $f_k = (3/2)^k \cdot f_0$, where $k \in \mathbb{Z}$ represents the number of generator applications. In the notation of equation (3), this means that for every step $m_2 = k$, there is an inherent $m_1 = -k$. The following diagram illustrates this linear generation process, which extends infinitely in both directions before any octave normalization is applied:

$$(\tfrac{3}{2})^{-k} f_0 \dashleftarrow \cdots \leftarrow (\tfrac{3}{2})^{-2} f_0 \xleftarrow{\times(\frac{3}{2})^{-1}} (\tfrac{3}{2})^{-1} f_0 \xleftarrow{\times(\frac{3}{2})^{-1}} \boxed{f_0} \xrightarrow{\times\frac{3}{2}} \tfrac{3}{2} f_0 \xrightarrow{\times\frac{3}{2}} (\tfrac{3}{2})^2 f_0 \rightarrow \cdots \dashrightarrow (\tfrac{3}{2})^k f_0$$

To map any generated frequency f_k back into the reference octave $[f_0, 2f_0)$, a **normalization** process is required to compensate for the exponential growth:

$$\bar{f}_k = \frac{f_0 \cdot (3/2)^k}{2^{\lfloor \log_2((3/2)^k) \rfloor}} \tag{6}$$

Since no power of 3/2 can ever equal a power of 2, the generation of sounds through the chain of fifths can never close; therefore, a maximum value for k must be chosen to terminate the process. For $k = 12$, the cycle of fifths is forced to a close, leaving a discrepancy known as the Pythagorean comma, which corresponds to the ratio between twelve fifths and seven octaves.

A comparative analysis

A core divergence between the two frameworks lies in their respective frequency ratios: whereas the Pythagorean set $\mathbb{P}(f_0)$ is strictly rational, the Equal-Tempered set $\mathbb{T}_N(f_0)$ is characterized by algebraic irrationality. Consequently, these two infinite systems overlap only at pure octaves:

$$\mathbb{P}(f_0) \cap \mathbb{T}_N(f_0) = \{f_0 \cdot 2^m : m \in \mathbb{Z}\}. \tag{7}$$

The absence of harmonic proportions, due to the geometric progression of the sounds in the tempered scale, prevents the latter from engaging in a “dialogue” with other rational scales, except when comparing irrational values to their nearest rational fractions.

Beyond this numerical divergence, a deeper structural contrast emerges between the two paradigms. While Equal Temperament is intrinsically self-contained - requiring only the Octave Principle and a fixed N to tile the sonic space - the Pythagorean framework generates a set of sounds that is both infinite and dense within the octave. Left to its own mathematical devices, the Pythagorean algorithm never closes, thus failing to define a functional scale without the external imposition of a stopping criterion. This cut represents a moment where mathematical autonomy must be interrupted by a conscious choice to select a finite subset of notes.

Interestingly, the two systems are not entirely disconnected. Fascinating mathematical results suggest that the selection of specific values for N (most notably $N = 12$) is not arbitrary. Instead, these values are privileged because of how successive Pythagorean fifths cluster around the irrational equal tempered points, creating a bridge between the purity of rational ratios and the structural convenience of a closed circle.

The external stopping criterion required by the Pythagorean system is not arbitrary, but is deeply rooted in the theory of continued fractions. The closure of the system depends on solving the proximity between powers of 3 and powers of 2, specifically seeking k and m such that $(3/2)^k \approx 2^m$, which is equivalent to $\frac{k}{m} \approx \log_2(3)$. By expanding the irrational value $\log_2(3)$ into continued fractions, we obtain a sequence of rational convergents that identify $N = 12$ (corresponding to the convergent 7/12) as a highly efficient point of near-closure. This creates what can be described as Pythagorean “sonic zones”: narrow clusters of rational ratios derived from pure fifths that surround and converge upon the irrational tempered pitches. Within these zones, the tempered sound acts as a geometric "centroid," while the Pythagorean values provide the harmonic purity.

Several foundational works have explored this structural resonance between the infinite Pythagorean set and the discrete tempered grid. Notable references include [3] for the psychoacoustic basis of these convergents, [5] which treats the mathematical morphology of these pitch classes [2], specifically for the link between continued fractions and scale construction, [1] for a modern algebraic perspective on how these discrete "zones" stabilize musical structures.

In simple terms, it is compelling to observe how the unmanageable, infinite set of rational Pythagorean pitches - achievable through elementary physical operations such as string divisions - practical equal-tempered sounds.

3 Just Intonation

3.1 Historical and Theoretical Framework

The term Just Intonation refers to the general theoretical principle according to which consonant intervals are expressed by ratios of small integers[1]. This preference for small-integer ratios is not merely an aesthetic convention but is rooted in the physics of wave interference. When two frequencies form a simple ratio (such as $3:2$ or $5:4$), their sound waves achieve a high degree of periodic alignment. Mathematically, the shorter the period of the combined waveform, the more "stable" the interval is perceived by the human ear.

Conversely, complex ratios—such as the Pythagorean major third $(81:64)$—result in wave peaks that rarely coincide. This lack of alignment produces acoustic beats (periodic fluctuations in volume), which the ear interprets as roughness or tension. By "sweetening" these intervals, Renaissance theorists were effectively reducing these micro-fluctuations, allowing the harmonic partials of different notes to lock into a single, cohesive resonance.

Our mathematical inquiry focuses on the specific formulation of Just Intonation developed around the mid-16th century by Gioseffo Zarlino [10]. His objective was to resolve the acoustic friction of Pythagorean intervals by grounding his theory in the Senary Number (*Numero Senario*). Zarlino posited that the first six natural numbers—which he also termed the Sonorous Numbers—contained the totality of musical perfection. In his view, only the integers within this "senarius" possessed the inherent capacity to be "sonorous," or capable of producing harmony; any ratio involving prime numbers beyond this limit, such as 7 or 11, was considered "deaf" (*sordo*, [10]) or dissonant to the ear.

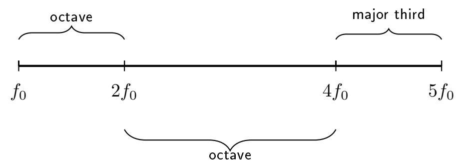
By extracting the superparticular ratios $(n+1:n)$ from this set, Zarlino identified the primary building blocks of harmony: the octave $(2:1)$, the perfect fifth $(3:2)$, the perfect fourth $(4:3)$, the major third $(5:4)$, and the minor third $(6:5)$. This conceptual boundary effectively established what modern mathematics defines as a 5-limit system, as the number 6 is the highest integer that does not introduce a new prime factor beyond 2, 3, and 5.

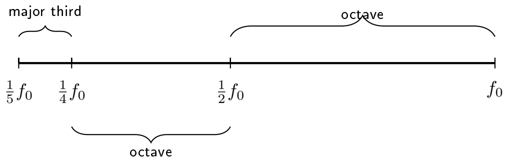
It is precisely the simplification of the 81/64 ratio in favor of 5/4 for the interval of a third - a shift of fundamental importance in music theory from the 16th century onward - that led to the innovations over the Pythagorean scale, summarized by the introduction of the prime factor 5. The term *third* thus refers to the three diatonic degrees spanned by the interval (e.g., C–D–E)[2].

The physical justification is found in the natural vibration of a string divided into five equal parts—the logical progression following the divisions into three and four (the latter being inherent to the Principle of the Octave). Every multiplication of the frequency by 5 raises the pitch by two octaves and a major third; a positive exponent p indicates p such operations performed "forward" (higher in pitch). Conversely, every division by 5 lowers the pitch by the same interval; a negative exponent p repeats this operation $-p$ times "backward" (lower in pitch). These procedures are schematized below: on the left, the transition from f_0 to $5f_0$; on the right, from f_0 to $\frac{1}{5}f_0$.

[1] In musicological literature, the term "natural sounds" is often used as a synonym for Just Intonation: strictly speaking, it refers to the physical correspondence between these ratios and the lower partials of the natural harmonic series. While common, this usage can be imprecise: "natural sounds" refers to the physical phenomenon of the harmonic series (the spontaneous resonance of a vibrating body), Just Intonation represents a human choice, the intentional act of selecting and organizing those specific natural ratios into a coherent tuning system. In this sense, a ratio is deemed "natural" because it coincides with the physical partials of a fundamental frequency, but its integration into a scale remains an algorithmic and cultural construct.

[2] More precisely, this refers to a major third (5:4 ratio); the minor third, which is narrower, corresponds to the 6:5 ratio, as seen in the interval E–F–G.





The following table reports the relative frequencies f/f_0 (setting f_0 as the reference note C) for the diatonic scale in both Pythagorean tuning and Zarlino's Just Intonation[3]:

Table 1: Diatonic Scale Ratios: Pythagorean vs. Zarlino (Just Intonation)

Note	**C**	**D**	**E**	**F**	**G**	**A**	**B**	**C'**
Pythagorean	1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1
Zarlino (JI)	1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1

The genesis of Zarlino's Just Intonation scale—as presented in standard musical theory—consists simply of a local replacement of complex Pythagorean ratios with simpler fractions derived from the Senario. It is, therefore, an empirical method, far removed from a formal algorithmic procedure or the preservation of interval symmetries, both of which characterize the Pythagorean and Equal-Tempered systems.

Rather than following a simple recursive algorithm, JI functions as a system of local optimization. As the ear naturally favors simple integer ratios for consonance—preferring the pure major third (5/4) over the complex Pythagorean third (81/64)—the system adopts a pragmatic approach, prioritizing local purity over global consistency. By favoring the smallest possible integers to maximize acoustic clarity, the global coherence of the algorithm is inevitably broken.

3.2 The modern approach to Just intonation: from algorithms to lattices

While Equal Temperament and Pythagorean tuning rely on transparent, one-dimensional recursive algorithms, the JI Scale formalized by Gioseffo Zarlino represents a fundamental shift toward a system of local optimization.

The mathematical "unmanageability" of Just Intonation stems from its increase in dimensionality. While Pythagorean tuning is a 1-D sequence generated by the prime factor 3, 5-limit Just Intonation inhabits a 2-D discrete lattice, often visualized as a Tonnetz. Any frequency f in this space is defined by the prime factorization:

$$f = f_0 \cdot 2^n \cdot 3^m \cdot 5^r \quad (n, m, r \in \mathbb{Z}) \tag{8}$$

and these frequencies give rise to the sonic universe

$$\mathbb{J}(f_0) = \{f \in \mathbb{R} \,|\, f = 2^n 3^m 5^r f_0,\ n, m, r \in \mathbb{Z}\}\,. \tag{9}$$

The numbers (9) are rational, infinite, and distinct if the exponents are distinct. In particular, if $n \neq 0$ and $p \neq 0$, it is impossible to obtain a power of 2; thus, starting from f_0, no sound in (9) will ever coincide exactly with an octave of f_0.

By normalizing the octave (2^n), we transition from a simple line of fifths (m-axis) to a plane where the introduction of the major third (r-axis) creates a Diophantine challenge. Because the primes 2, 3, and 5 are incommensurable, the axes of this lattice never meet, precluding a closed-loop algorithm.

[3] Henceforth, frequencies are considered normalized, since the results depend only on ratios and are independent of the specific value of f_0.

In this multidimensional framework, selecting a scale (such as the Zarlinian scale) is no longer a matter of following a path, but of performing a constrained optimization. The theorist must select a finite set of points that:

- minimize the distance from the origin, favoring the smallest possible prime exponents to ensure consonance,
- maximize harmonic density, clustering notes that form fundamental musical structures like the major triad ($4:5:6$).

This process is inherently "opportunistic" and non-recursive. The clash between different paths toward the same note—for instance, reaching E via four perfect fifths $(4,0)$ versus one major third $(0,1)$—generates the Syntonic Comma ($\Delta_S = 81/80$).

As modern theorists like J. Tenney [8] and B. Johnston [4] have explored, increasing the limit to higher primes (7, 11, etc.) transforms the musical space into a k-dimensional lattice. In such a space, the "mathematics" remains elegant but loses its predictive simplicity: Just Intonation emerges not as a closed algorithm, but as a complex system of harmonic vectors where the pursuit of local purity precludes global algebraic closure.

The structural "irregularity" of Just Intonation is most evident when we look at the steps of the scale. In the Pythagorean system, the distance between two adjacent notes (the whole tone) is always constant: a uniform ratio of 9/8. Like a staircase with identical steps, the Pythagorean algorithm repeats the same size over and over.

In Zarlino's Just Intonation, however, this uniformity is lost. To ensure that the major third remains "pure" (5/4), the system is forced to alternate between two different sizes of whole tones: the Major Tone (9/8) and the Minor Tone (10/9), the difference between them is the so called Syntonic Comma (81/80). Consequently, a standard major scale is no longer a sequence of identical blocks, but a specific, asymmetrical pattern (e.g., $9/8, 10/9, 16/15, \dots$). This asymmetrical arrangement is what prevents a simple recursive algorithm. In a tempered or Pythagorean system, you can start a melody on any note and it will "fit" the same way because the intervals are identical. In Just Intonation, because the steps have different sizes, moving a melody to a different starting pitch (transposition) would change its internal proportions. This lack of symmetry is the price paid for local purity: by optimizing the third to make it sound sweeter, we break the regular "grid" of the scale, making it impossible to define the system through a single, repeating mathematical rule.

4 Harmonic proportion

In the remainder of the work, we move into an entirely different territory and propose an alternative method for selecting a finite set of Zarlinian pitches from the sonic universe defined in (9), and we shall investigate its outcome. As expected, this approach does not yield the standard diatonic scale shown in Table 1; however, the resulting selection invites compelling insights into the relationships between these pitches and their interpretation according to the musical theory contemporary to the formulation of Just Intonation.

The mathematical tool at the heart of this method is the harmonic proportion; let us therefore begin by recalling some definitions and properties.

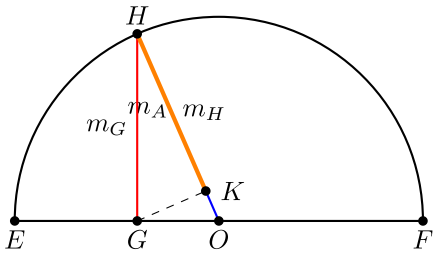
4.1 Harmonic mean and harmonic division

Classical means

The three classical means of two positive numbers a and b are:

- aritmetic mean: $m_A(a,b) = \frac{a+b}{2}$,
- geometric mean: $m_G(a,b) = \sqrt{ab}$,
- harmonic mean: $m_H(a,b) = \frac{2ab}{a+b}$.

They can be visualized through the following geometric construction using a semicircle of diameter $EF = a + b$, with the internal point G such that $EG = a$ and $GF = b$. The corresponding segments are $OH = m_A$ (radius), $GH = m_G$ (altitude), $KH = m_H$ (projection of altitude onto radius). The diagram confirms the known property $m_H \leq m_G \leq m_A$.

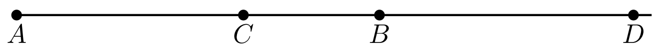


A useful and easily verified property is the following: the harmonic mean is the reciprocal of the arithmetic mean of the reciprocals $1/a$ and $1/b$:

$$m_H(a, b) = \frac{1}{m_A\left(\frac{1}{a}, \frac{1}{b}\right)}. \tag{10}$$

Harmonic division

Let AB be a segment, C a point within it, and D an external point on the same line:



The quadruple (A, B, C, D) forms a *harmonic division* if the following ratio equality holds:

$$\frac{AC}{CB} = \frac{AD}{BD} \tag{11}$$

The relationship between the harmonic mean and the harmonic division is as follows: if the points (A, B, C, D) form a harmonic division, then AB is the harmonic mean of AC and AD, that is

$$AB = m_H(AC, AD). \tag{12}$$

According to property (10), we can also write

$$\frac{1}{AB} = \frac{1}{m_H(AC, AD)} = m_A\left(\frac{1}{AC}, \frac{1}{AD}\right) = \frac{1}{2}\left(\frac{1}{AC} + \frac{1}{AD}\right) \tag{13}$$

Musical counterpart

Let us consider the segment in (11) as a string AD. Given a subdivision at C, we seek the point B that establishes a harmonic division. Since frequencies are inversely proportional to string lengths ($f = \kappa/L$), substituting this into (13) shows that the harmonic division (A, B, C, D) of the string corresponds to the following relationship between frequencies:

$$f_{AB} = \frac{1}{2}(f_{AC} + f_{AD}) \tag{14}$$

where f_{AB} denotes the frequency produced by the string AB, and similarly for the other symbols. Note that this relationship is independent of the proportionality factor κ. We highlight this useful result, which allows us to shift the mathematical-musical analysis from length to frequency:

$$AB = m_H(AC, AD) \qquad \Longleftrightarrow \qquad f_{AB} = m_A(f_{AC}, f_{AD}) \tag{15}$$

Example 1 *If C bisects the string AD, the position of B is determined using (12) as$AB = m_H(AC, 2AC) = \frac{2}{3}AC$. Thus, B must be positioned at $2/3$ of the length to achieve a harmonic division.If we work with frequencies, assigning the frequency f_0 to the full string AD (meaning AC corresponds to double the frequency), according to the second formula in (15), we calculate$f_{AB} = m_A(2f_0, f_0) = \frac{3}{2}f_0$, which is consistent with our previous result.It follows that the full string and the string bisected (which produces the octave sound) have as their harmonic mean the string that produces the sound of the perfect fifth.*

4.2 A Digression on Architecture: The Harmonic System

In the Renaissance, the architect was first and foremost a scientist. Their primary task was to decipher from the Universe the harmonic system of mathematical relationships and to build according to its laws. While no architecture is possible without metric proportions - a fact recognized in both the Middle Ages and the modern era - the explicit reliance on musical theory is a distinctive trait of the Renaissance.

This is where music intervenes: universally valid ratios are revealed through sounds, which result in harmony only when arranged in the correct proportions. In music, nature distinctly exposes ratios that are absolute and objective.

From Alberti to Palladio

The conviction of a universal order of Nature, filtered through music, permeates the architecture of the 15th and 16th centuries, from Leon Battista Alberti (1404–1472)[4] to Andrea Palladio (1508–1580)[5].

The harmony of the Universe reveals itself in music through well-proportioned relationships between sounds. Incorporating these musical ratios into architecture through the universal language of numbers gives the project its proper fulfillment. The goal was not only to find the numerical origin of a single sound but to connect sounds through intermediate elements, finding the reasons for harmony in the presence of proportional means.

Palladian Ratios and the Musical Scale

If in Alberti there is a conscious identification between spatial and musical ratios, it is with the Palladian ratios[6] that architecture and musical theory find an unprecedented accord. Architectural modules are linked like chords; the proportions of sound and space obey a single harmonic system. Palladio selects dimensions as if they were notes, organizing and connecting rooms as if composing a musical scale.

This "symphonic" construction principle—in the etymological sense of "sounding together"—is masterfully narrated by Rudolf Wittkower[7] in his celebrated text [9]. In Part IV, "The Problem of Harmonic Proportion in Architecture," Wittkower guides the reader through Palladian villas, deciphering their harmonic relationships with scientific rigor.

The Three Means for Calculating Height

For Palladio, the height (H) of a room was determined by the relationship between its length (L) and width (W). He employed three mathematical means to ensure the volume of the room "sounded" as well as its floor plan:

1. **The Arithmetic Mean** (for spacious rooms):

$$H = \frac{L + W}{2}$$

   Example: In a 12×6 room, $H = 9$. This creates the intervals of a Fifth ($9 : 6 = 3 : 2$) and a Fourth ($12 : 9 = 4 : 3$).

2. **The Geometric Mean** (for elegant, intermediate spaces):

$$H = \sqrt{L \cdot W}$$

   Example: In a 9×4 room, $H = 6$, maintaining a constant proportion ($9 : 6 = 6 : 4$).

[4] To condense this vast subject into a brief anecdote, we may cite Alberti's famous warning during the construction of the Tempio Malatestiano: one must not alter the proportions of the pillars, otherwise "*si discorda tutta quella musica*" (all that music is put out of tune).

[5] In the same spirit, we recall a memorandum requested from Palladio in 1567 regarding new constructions in Brescia: "...as the proportions of voices are harmony to the ears, so those of measurements are harmony to our eyes..."

[6] The treatise *I Quattro Libri dell'Architettura* (1570) is considered the highest reference point for a correct proportional construction system.

[7] Berlin 1906 – New York 1971. The renowned historian clarified the evidence of the Renaissance model of proportions, countering a purely aesthetic interpretation.

3. **The Harmonic Mean** (for intimate, mathematically dense spaces):

$$H = \frac{2 \cdot L \cdot W}{L + W}$$

Example: In a 12×6 room, $H = 8$. This yields the perfect progression $6 : 8 : 12$, containing the Octave $(12 : 6)$, the Fourth $(8 : 6)$, and the Fifth $(12 : 8)$.

Walking through a Palladian villa provides a sense of visual "peace" because the eye perceives the same frequencies that the ear recognizes in a perfect chord.

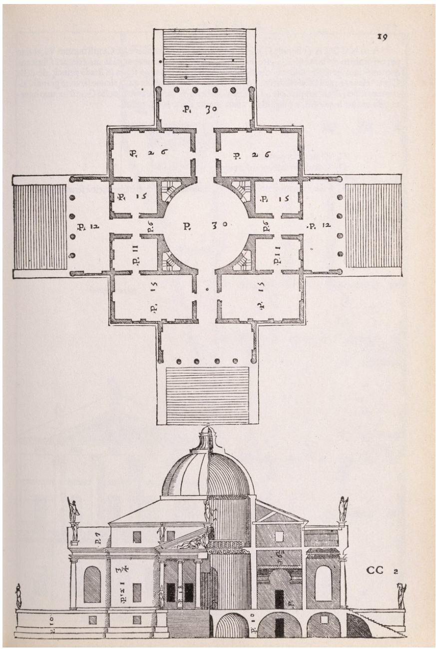


Figure 1: Plan and cross section of Villa Rotonda, from a facsimile of Palladio's *I Quattro Libri dell'Architettura* (1570). Note the numerical annotations provided by Palladio to ensure precise harmonic proportions throughout the design.

5 Searching for Proportions

The lesson to be drawn from Renaissance architectural canons is a preference for the harmonic proportion of spatial dimensions, which function much like musical strings. Our first exploration aims to identify triples of strings forming harmonic proportions within the two sets of diatonic sounds from Table 1 (Section 3.1): the Pythagorean scale (S_P) and the Just Intonation scale (S_J):

$$S_P = \left\{1, \frac{9}{8}, \frac{81}{64}, \frac{4}{3}, \frac{3}{2}, \frac{27}{16}, \frac{243}{128}, 2\right\} \qquad S_J = \left\{1, \frac{9}{8}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{15}{8}, 2\right\} \tag{16}$$

Property (15) allows us to shift the investigation from harmonic proportions of lengths to arithmetic proportions of frequencies (or equivalent quantities inversely proportional to string lengths[8]).

The following tables show the arithmetic means between the elements of S_P and S_J. Shaded cells indicate means that remain within the scale (S_P or S_J), while lighter shading denotes means that fall outside the scales but still belong to the same sound universe (the Pythagorean $P(1)$ or the Natural $N(1)$).

[8]This concept appears as early as Zarlino's treatise [10], where he demonstrates that consonances are determined "by both the arithmetic and harmonic means".

S_P	1	9/8	81/64	4/3	3/2	27/16	243/128	2
1		17/16	145/128	7/6	5/4	43/32	371/256	3/2
9/8			153/128	59/48	21/16	45/32	387/256	25/16
81/64				499/384	177/128	189/128	405/256	209/128
4/3					17/12	145/96	1241/768	5/3
3/2						51/32	435/256	7/4
27/16							459/256	59/32
243/128								499/256

S_J	1	9/8	5/4	4/3	3/2	5/3	15/8	2
1		17/16	9/8	7/6	5/4	4/3	23/16	3/2
9/8			19/16	59/48	21/16	67/48	3/2	25/16
5/4				31/24	11/8	35/24	25/16	13/8
4/3					17/12	3/2	77/48	5/3
3/2						19/12	27/16	7/4
5/3							85/48	11/6
15/8								31/16

The results are revealing; they highlight the "poverty" of internal proportions within the Pythagorean scale as compared to the Natural scale. In the first table (S_P), arithmetic proportions are strictly limited to the triad of unison, fifth, and octave discussed in Example 1, Paragraph 4.1. The fourth (4/3), while a fundamental pillar of the system, fails to emerge as an internal mean, appearing only as an external result of the circle of fifths. All other cells yield complex ratios that do not align with the search for constructive "modules" based on string sections.

The situation in the second table (S_J) is markedly different: every value in the scale is involved in at least one proportion. Furthermore, two additional "natural sounds" (lightly shaded) are generated.

5.1 A First Attempt at a Harmonic Construction of the Just Intonation Scale

The predisposition of natural sounds toward harmonic proportions is further supported by a constructive method that reaches S_J through successive proportions: starting from the fundamental (1) and its octave (2), we sequentially add the following means:

$$m_A(1,2) = \frac{3}{2} \longrightarrow m_A\left(1, \frac{3}{2}\right) = \frac{5}{4} \longrightarrow m_A\left(1, \frac{5}{4}\right) = \frac{9}{8} \tag{17}$$

all of which belong to the natural sound universe $\mathbb{J}(1)$ (see (9)). Conversely, the next step $m_A(1, \frac{9}{8}) = \frac{17}{16}$ does exit the set $\mathbb{J}(1)$.

In musical terms, the calculations (17) show that the perfect fifth is the harmonic mean between a tone and its octave, the major third is the harmonic mean between the tone and the fifth, and the major tone is the harmonic mean between the tone and the major third.

The gap existing between this temporay set[9]

$$\left\{1, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}, 2\right\} \tag{18}$$

and S_J can be compensated once we take into account two remarkable facts:

(i) The only sound in $\mathbb{J}(1)$ lying between $\frac{3}{2}$ and 2 and capable of forming an arithmetic proportion with a pair of sounds belonging to (18) is $\frac{15}{8}$, via the relation $m_A(\frac{9}{8}, \frac{15}{8}) = \frac{3}{2}$.

(ii) The only sounds in $\mathbb{J}(1)$ that can form arithmetic means with the octave boundaries - in the sense that $\sigma_1 = m_A(1, \sigma_2)$ and $\sigma_2 = m_A(\sigma_1, 2)$ - are $\frac{5}{4}$ and $\frac{5}{3}$.+2

[9]We incidentally note that the fractions involved are all superparticular ratios (see Section 3.1).

By incorporating the new values $\frac{15}{8}$ and $\frac{5}{3}$ into the partial set (18), we exactly obtain the scale S_J; indeed, the two aforementioned properties further support the uniqueness of this construction.

However, this mathematical satisfaction is tempered by two points of criticism. First, the decision to stop at seven notes is motivated by musical expectation rather than mathematical necessity; “neutral” mathematics would see no reason not to calculate further existing natural means. Second, the method is not strictly systematic, as its progression relies on "external" musical considerations rather than a purely internal mathematical logic.

However, this mathematical satisfaction is tempered by two critical observations. First, the decision to stop at seven notes is motivated by a priori musical expectations rather than inherent mathematical necessity; “neutral” mathematics would see no reason not to proceed with the calculation of further existing natural means. Second, the method lacks a strictly systematic framework, as its progression relies on “external” musical choices rather than a self-contained recursive logic.

This suggests the need for a generative algorithm capable of deriving the scale’s structure through a purely formal recursive process, independent of historical musical dogmas.

5.2 A harmonic-mean generating method

Just as an architect, guided by the principles of symmetry, introduces structural elements that integrate harmoniously with a growing edifice, we begin with a foundational nucleus of sounds, admitting new ones only if they establish harmonic proportions with the existing set.

To remain strictly within the mathematical framework, we must address the following objective:

> To formulate a generative procedure that organizes sounds based on harmonic proportions - one that operates without “external” interference - to yield an ideal scale defined solely by the internal consistency of its ratios.

5.2.1 The mathematical procedure

In order to formalize this concept mathematically and provide a tool for tracking the occurrence and generation of arithmetic means, we define the mean generator $\mathbb{M}$ as the operator on a set $\mathcal{I}$ acting as follows:

$$\mathbb{M}\langle\mathcal{I}\rangle_{\mathfrak{R}} = \{m_A(x,y) \mid x,y \in \mathcal{I}, m_A(x,y) \in \mathfrak{R}\} \tag{19}$$

where m_A is the arithmetic mean and $\mathfrak{R}$ represents a restriction, according to which the value $m_A(x,y)$ is admitted to the set $\mathbb{M}\langle\mathcal{I}\rangle_{\mathfrak{R}}$ if and only if it belongs to $\mathfrak{R}$. In musical terms, this operator encapsulates all possible harmonic means calculated from pairs of sounds with frequencies in $\mathcal{I}$, where a new sound is accepted only if it belongs to a specific tuning system: essentially, $\mathfrak{R}$ corresponds to Pythagorean Tuning $\mathbb{P}(1)$, Just Intonation $\mathbb{J}(1)$, or Equal Temperament $\mathbb{T}(1)$. Note that we allow $x = y$ in (19): since $m_A(x,x) = x$, it follows that $\mathcal{I} \subseteq \mathbb{M}\langle\mathcal{I}\rangle_{\mathfrak{R}}$. In other words, the operator expands (or leaves fixed) the set $\mathcal{I}$. Musically, the case $x = y$ corresponds to unison, which is the obvious consonance of a sound with itself.

In order to accommodate new elements that harmonize well with existing ones, the procedure operates iteratively by calculating

$$\mathcal{I} \longrightarrow \mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathfrak{R}} \longrightarrow \mathcal{I}_2 = \mathbb{M}\langle\mathcal{I}_1\rangle_{\mathfrak{R}} \rightarrow \mathcal{I}_3 = \mathbb{M}\langle\mathcal{I}_2\rangle_{\mathfrak{R}} \rightarrow \cdots\cdots \tag{20}$$

The process terminates at step k if no new means satisfying the restriction $\mathfrak{R}$ are produced:

$$\mathbb{M}\langle\mathcal{I}_k\rangle_{\mathfrak{R}} = \mathcal{I}_k \tag{21}$$

Remark 1 *It is worth noting that, in the absence of a restriction $\mathfrak{R}$, the iterative application of the operator $\mathbb{M}$ starting from a set such as $\mathcal{I} = \{1,2\}$ does not terminate. Instead, it generates the set of dyadic rationals within the interval $[1,2]$, defined as $\left\{\frac{k}{2^n} \mid n \in \mathbb{N}_0,\, 2^n \leq k \leq 2^{n+1},\, k \in \mathbb{Z}\right\}$ which is an infinite and dense set in the interval $[1,2]$. Musically, this highlights the necessity of the restriction $\mathfrak{R}$: without the constraints of a tuning system like Just Intonation or Equal Temperament, the “mean generation” process would lead to an infinite proliferation of frequencies, eventually escaping any practical tonal organization.*

5.2.2 Results for $\mathcal{I} = \{1, 2\}$

The first investigation takes $\mathcal{I} = \{1, 2\}$ as the initial set, which musically corresponds to a sound and its replica at the octave—the maximum possible consonance. We first examine two predictable cases:

- $\mathfrak{R} = \mathbb{P}(1)$ Pythagorean Tuning: the iterative calculation (20) consists of:

$$\mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{P}(1)} = \{1, 3/2, 2\},$$
$$\mathcal{I}_2 = \mathbb{M}\langle\mathcal{I}_1\rangle_{\mathbb{P}(1)} = \{1, 3/2, 2\} = \mathcal{I}_1.$$

  The stopping condition (21) is therefore satisfied already at the second step. We find a result aligned with what emerged from the table in Paragraph 4.3, which shows that the only Pythagorean harmonic proportion is the triad sound-fifth-octave.

- $\mathfrak{R} = \mathbb{T}_N(1)$ Equal Temperament: the situation is even more drastic, given that (denoting the set again as $\mathcal{I}_1$ to simplify notation):

$$\mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{T}_N(1)} = \{1, 2\} = \mathcal{I}$$

  (indeed, every frequency in (2) is irrational). The practicality and portability of the equal temperament system come at the cost of a total absence of harmonic arrangements among the sounds.

A more compelling theoretical framework emerges when the restriction is

- $\mathfrak{R} = \mathbb{J}(1)$ Just Intonation: the sequence (20) consists of (the calculations are elementary):

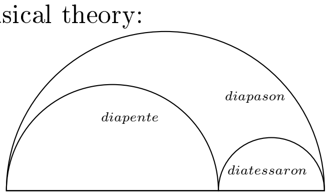
$$\mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{J}(1)} = \{1, 3/2, 2\},$$
$$\mathcal{I}_2 = \mathbb{M}\langle\mathcal{I}_1\rangle_{\mathbb{J}(1)} = \{1, 5/4, 3/2, 2\},$$
$$\mathcal{I}_3 = \mathbb{M}\langle\mathcal{I}_2\rangle_{\mathbb{J}(1)} = \{1, 9/8, 5/4, 3/2, 2\},$$
$$\mathcal{I}_4 = \mathbb{M}\langle\mathcal{I}_3\rangle_{\mathbb{J}(1)} = \{1, 9/8, 5/4, 3/2, 2\} = \mathcal{I}_3.$$

  In this instance, the expansion is at its broadest, returning us to the set (18) derived from the subsequent computation (17) of averages to fill the octave; this confirms Just Intonation's inherent predisposition toward harmonic means.

$$\mathbb{M}\left\langle\mathcal{T}\right\rangle_{(2,3,5)} = \left\{ m_A(1, \frac{3}{2}) = \frac{5}{4},\ m_A(1,2) = \frac{3}{2},\ m_A(\frac{4}{3}, 2) = \frac{5}{3} \right\} \tag{22}$$

5.2.3 Results for $\mathcal{I} = \{1, 4/3, 3/2, 2\}$

The new choice for $\mathcal{I}$ is by no means coincidental; rather, it encapsulates the division of the octave (diapason) into the intervals of the fifth (diapente) and the fourth (diatessaron) according to Pythagorean musical theory:



In terms of formula (1), this partition can be formulated as $\frac{3}{2} \cdot \frac{4}{3} = 2$. This established the system of the three perfect consonances that dominated musical culture for centuries, until the Renaissance[10].

Discarding the Equal Temperament restriction $\mathfrak{R} = \mathbb{T}_N(1)$, which yields the same stasis seen in the previous case, we shall examine:

- $\mathfrak{R} = \mathbb{P}(1)$ Pythagorean Tuning: procedure (20) is even shorter than in the previous case:

$$\mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{P}(1)} = \{1, 4/3, 3/2, 2\} = \mathcal{I}$$

  (once again, the results are essentially contained in the table in Section 4.3). The triad of perfect consonances is unable to generate further Pythagorean sounds in harmonic proportion with one another.

[10] A famous and artistic representation of this division is contained in the tablet at Pythagoras' feet in Raphael's fresco The School of Athens (Stanza della Segnatura, Vatican Stanze).

- $\mathfrak{R} = \mathbb{J}(1)$ Just Intonation: the computation (20) terminates after 5 steps, namely

$$\begin{array}{rcllllllllll}
\mathcal{I} & = & \{1, & & & & 4/3, & & 3/2, & & & 2\} \\
\mathcal{I}_1 = \mathbb{M}\langle \mathcal{I} \rangle_{\mathbb{J}(1)} & = & \{1, & & 5/4, & & 4/3, & & 3/2, & & 5/3, & 2\} \\
\mathcal{I}_2 = \mathbb{M}\langle \mathcal{I}_1 \rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & & 4/3, & & 3/2, & & 5/3, & 2\} \\
\mathcal{I}_3 = \mathbb{M}\langle \mathcal{I}_2 \rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & & 4/3, & & 3/2, & 25/16, & 5/3, & 2\} \\
\mathcal{I}_4 = \mathbb{M}\langle \mathcal{I}_3 \rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & & 4/3, & 45/32, & 3/2, & 25/16, & 5/3, & 2\} \\
\mathcal{I}_5 = \mathbb{M}\langle \mathcal{I}_4 \rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & 81/64, & 4/3, & 45/32, & 3/2, & 25/16, & 5/3, & 2\}
\end{array}$$

(the spacing adopted in the layout above facilitates the understanding of the new elements as they are gradually added) after which $\mathbb{M}\langle \mathcal{I}_5 \rangle_{\mathbb{J}(1)} = \mathcal{I}_5$.

It is undoubtedly significant to examine and comment on the expansion - via the algorithm (20) - from the consonances $\mathcal{I} = \{1, 4/3, 3/2, 2\}$ to the set $\mathcal{I}_5$, which we rename as:

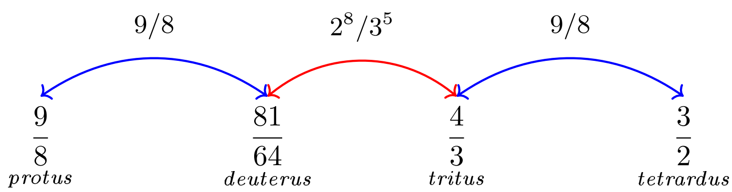
$$\sigma_J = \{1, 9/8, 5/4, 81/64, 4/3, 45/32, 3/2, 25/16, 5/3, 2\} \tag{23}$$

and which features the addition of six tones (in ascending order): 9/8, 5/4, 81/64, 45/32, 25/16, and 5/3. This stage represents the most compelling moment of the procedure we have devised, owing to the historical-musical interpretation that can be associated with it.

Properly speaking, the collection (23) should not be understood as a sequence of sounds to be articulated in succession - an ascending scale - but rather as a repertoire of constructive modules in harmony with one another, arranged for combination according to the elegance of proportion.

The first apparently inconvenient piece of evidence is the existence of two very close sounds: 5/4 (the justly intonated Major Third) and 81/64 (the Pythagorean Major Third). In reality, when placed in the correct context, these two presences trace the two major milestones of the long phase of musical theory and practice preceding the Renaissance.

- On one hand, the four Pythagorean sounds



  form the *diatonic tetrachord*[11], the fundamental particle of ancient Greek harmonic theory. This flows into the musical theory of modes[12] as a central element for the methodical classification of the repertoire through the four *finales* - notes upon which the melody definitively rests and around which the chant develops—named *protus* (D, 9/8), *deuterus* (E, 81/64), *tritus* (F, 4/3), and *tetrardus* (G, 3/2), here legitimately present in their isolated and ancient Pythagorean intonation. The graph highlights the distances between the pitches, which consist exclusively of noteworthy intervals in Pythagorean music theory: the *epogdoon* (9/8) and the *Pythagorean limma* ($2^8/3^5$).

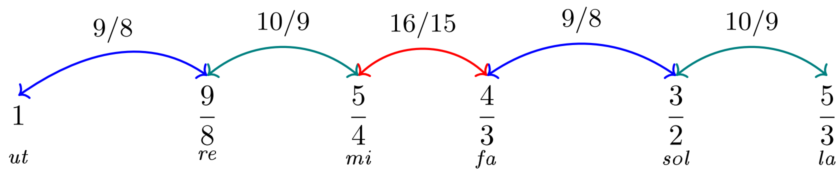
  The four modes induced by the *finales* represent a stable point of reference not only for the universal organization of Gregorian chant carried out in the 8th century[13], but even for the late Renaissance repertoire of madrigals, motets, toccatas, etc., which remains systematically difficult to categorize ([7]).

- The co-presence in (23) of the other E, at the pitch 5/4, well documents the dispersion of Pythagorean sounds in favor of other solutions and fits perfectly within the available selection of the *natural hexachord*

[11] The other two genera, chromatic and enharmonic tetrachords, were abandoned during the transition from the classical to the medieval world.

[12] The meaning of the term *modality* is so extensive that much of what does not fall within the scope of our tonal system is qualified as modal, with greater or lesser propriety.

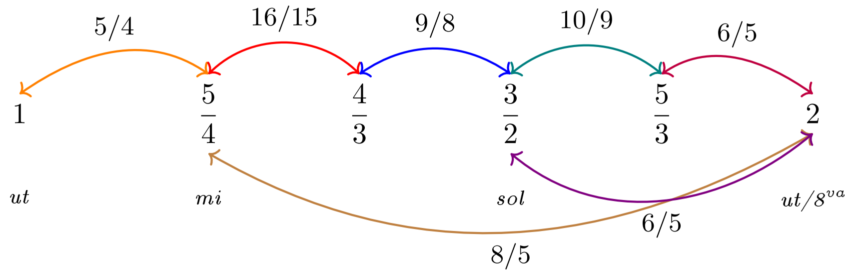
[13] the Byzantine *octoechos*



which, from the explicit formulation of Guido d'Arezzo, represented for several centuries the basic reference system for musical theory, practice, and pedagogy[14]. While settling within the grid of modality, the hexachord leaves a fundamental mark in directing the many ferments of a continuously evolving era - the late Medieval and Renaissance periods - dense with heterogeneous sonorous materials. It stands out as a system projected toward the future, for instance, in the practice of passing from one hexachord to another, lower or higher (the so-called *mutatio*), something that today is comparable to modulation from one key to another. The interplay between the modal *finales* and the hexachord - pillars of the fluid musical doctrine of many centuries - constitutes the most convincing theoretical framework for reviewing at least a millennium of multifaceted musical theory and practice. As illustrated in the diagram, the hexachord is structured through three fundamental scalar steps: the major tone (9/8), the minor tone (10/9), and the major semitone (16/15).

In (23), other interesting presences besides the four *finales* and the *hexachord* can be identified, contributing this time in terms of consonances:

- In examining the prospect



  one finds the imprint of the principle proclaimed by Zarlino in [11] regarding musical consonances based on the *Senario*—that is, the succession of the first six integers which extend the Pythagorean concept of consonance based on the first four integers. For Zarlino, consonance arises from "superparticular" ratios (fractions where the numerator exceeds the denominator by one unit) 2/1, 3/2, 4/3, 5/4, 6/5 and from the ratios 5/3 and 8/5 (the latter still connectable to the Senario). These were of crucial importance for the rise of polyphonic music and the birth of instrumental music of the time[15]. Despite the selection of distances appearing aimed at highlighting the conclusion, it is easy to see that all other distances - with the exception of the major tone, minor tone, and major semitone, which are not considered consonances - always reproduce the same values.

- The remaining sounds 45/32 and 25/16 in (23) offer a proposal regarding the complex issue of the division of the whole tone. Starting from the Pythagorean limma[16] marked by the arduous ratio 256/243, the problem of further thickening the diapason has over the centuries given life to a passionate search that stimulated theorists also from the perspective of arithmetic calculation and geometry applied to acoustics[17]. For example, Zarlino himself provided the demonstration that the

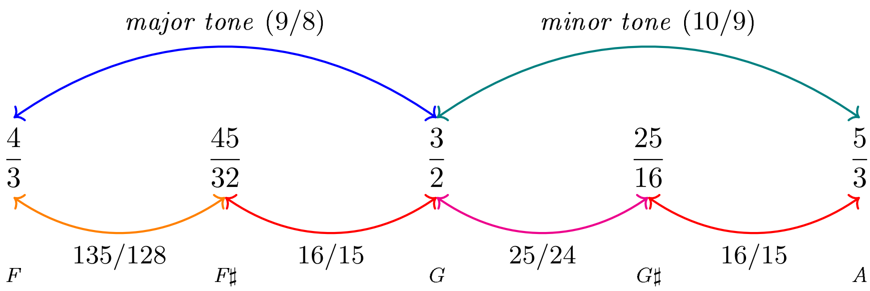
[14] The so-called method of *solmization*.

[15] In modern terminology, 6/5, 5/3, and 8/5 give rise to the intervals of a minor third, major sixth, and minor sixth, respectively; a melodic line superimposed on another largely utilizes intervals of thirds and sixths to harmonically contrast with the existing one.

[16] Listing the other Pythagorean sub-units, calculable based on the continuation of the cycle of fifths, would shift the focus to other topics.

[17] Among the numerous cases, we mention the treatise *Alia Musica*, of French or Flemish origin, in which the geometric solution for calculating the harmonic mean proportional appears.

major tone cannot be divided into two equal intervals; that is, in mathematical terms, the fraction 9/8 is not the product of equal rational numbers. Regarding the search for smaller steps that are in harmony with the rest, the resolution (23) points to indications in the sounds of the interval between 4/3 and 5/3, represented here[18]



Indeed, in the subdivision of both the major and minor tone, the role of the major semitone measuring 16/15 is evident. On the other hand, the 25/24 fraction, the so-called *minor semitone*, another central element in Zarlino's harmonic subdivisions, is important for managing and enriching the repertoire of sounds. The inevitable 135/128 gap required to cover the major tone is not without meaning: we take the opportunity to write $\frac{135}{128} = \frac{5}{4} \times (\frac{3}{2})^3 \times \frac{1}{2^2}$ and translate the operation as "the addition of three diapente (or fifths) to the sound 5/4 (E) and a downward transposition of two diapasons (or octaves)", with the simple intention of offering a single glimpse into the way sonorous material has been reasoned out (in all ages) through fractions - embracing, however, contents and methods that go far beyond our current aims[19].

5.2.4 Results for $\mathcal{I} = S_p$, $\mathcal{I} = S_J$

As a final experiment, we evaluate the degree to which the diatonic scales (16) are "closed" within their respective sound universes (3) and (9). In the case of Pythagorean tones, by setting $\mathcal{I} = S_P = \{1, 9/8, 81/64, 4/3, 3/2, 27/16, 243/128, 2\}$ in (20) and $\mathfrak{R} = \mathbb{P}(1)$, we find - consistently with the values calculated in the table for S_P in Section 4.3:

$$\mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{P}(1)} = \mathcal{I}$$

meaning that no new elements of the same sound type are produced.

The situation is entirely different for the tones of the S_J scale (see 16): the algorithm (20) with $\mathcal{I} = S_J$ and $\mathfrak{R} = \mathbb{J}(1)$ produces

$$\begin{array}{rcllllllllllll}
\mathcal{I} & = & \{1, & 9/8, & & 5/4, & 4/3, & & 3/2, & & 5/3, & & 15/8, & 2\} \\
\mathcal{I}_1 = \mathbb{M}\langle\mathcal{I}\rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & & 4/3, & & 3/2, & 25/16, & 5/3, & 27/16, & 15/8, & 2\} \\
\mathcal{I}_2 = \mathbb{M}\langle\mathcal{I}_1\rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & & 4/3, & 45/32, & 3/2, & 25/16, & 5/3, & 27/16, & 15/8, & 2\} \\
\mathcal{I}_3 = \mathbb{M}\langle\mathcal{I}_2\rangle_{\mathbb{J}(1)} & = & \{1, & 9/8, & 5/4, & 81/64, & 4/3, & 45/32, & 3/2, & 25/16, & 5/3, & 27/16, & 15/8, & 2\}
\end{array}$$

terminating at $k = 3$, since $\mathcal{I}_4 = \mathcal{I}_3$.

Consequently, following the addition of 81/64, 45/32, and 25/16, the set

$$\Sigma_J = \{1, 9/8, 5/4, 81/64, 4/3, 45/32, 3/2, 25/16, 5/3, 27/16, 15/8, 2\} \tag{24}$$

At this stage, the algorithm finds no further "compatible rungs", as any subsequent averaging attempt systematically yields numbers containing higher prime factors. These results are thus rejected by the $2^n 3^m 5^p$ filter - much like non-standard components failing to fit a modular blueprint - ensuring the stability of the final scale Σ_J.

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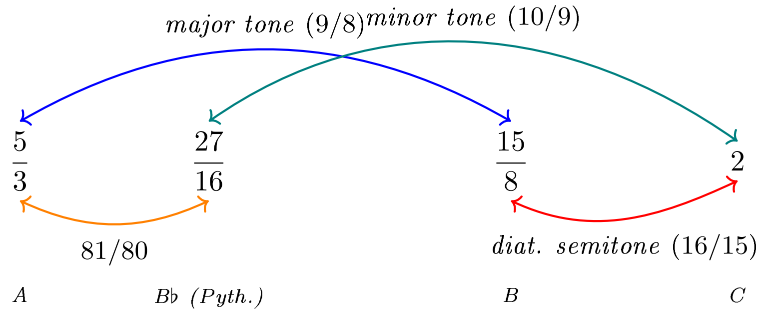
[18] If a modern general naming can help, setting aside the question of diatonic or chromatic semitones, 45/32 between 4/3 (F) and 3/2 (G) corresponds to F-sharp, while 25/16 between 3/2 (G) and 5/3 (A) corresponds to G-sharp.

[19] For those with the patience to follow the solution, simplifying into equal-tempered sounds: three fifths from E leads through E $, \rightarrow, B, \rightarrow, F\#, \rightarrow, C\#$; the latter, transposed two octaves down, finds a sound between the C and D of the original scale.

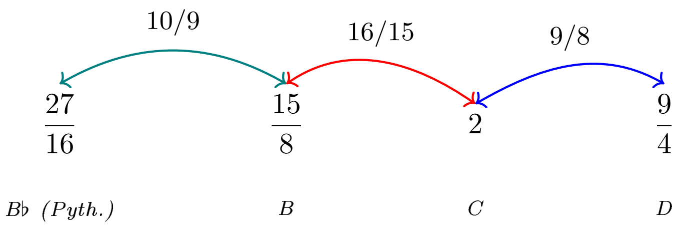
Let us now turn our attention to the musical significance of the second list (24). While Σ_J extends σ_J, the expansion is subtle: the only new additions are $\frac{27}{16}$ and $\frac{15}{8}$.These two pitches[20] fill the widest gap in $\mathfrak{S}_{\mathcal{N}}^{(1)}$, specifically the interval between 5/3 and 2:



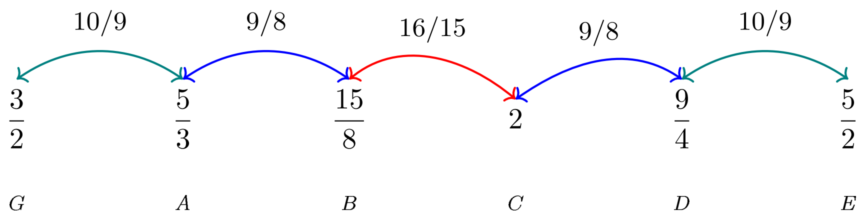
This occurs without introducing any new interval sizes. The 81/80 ratio is the smallest "building block" in the collection; it belongs to the category of commas - a term literally translating to "fragment". This specific ratio is so prevalent in musical theory that it has earned various names[21] depending on its function. Here, it clearly acts as the discrepancy between the major and minor tones; similarly, it can be derived from the difference between the Pythagorean and Just Major Thirds: $\frac{81}{64} : \frac{5}{4} = \frac{81}{80}$.

The appearance of the Pythagorean pitch $\frac{27}{16}$ offers at least two compelling interpretations:

- On one hand, it establishes a new structural framework—specifically, a new set of finales



  and a new hexachord



  (considering the homologous pitches in the upper octave). This configuration retains the dualism of the two adjacent sounds - the Pythagorean $\frac{27}{16}$ and the Just $\frac{5}{3}$ - but with a different internal distribution of intervals compared to the previous presentation[22], resulting in a distinct harmonic "color".

- On the other hand, $\frac{27}{16}$ completes the set of diapente (fifths) within the natural hexachord. Specifically, every pitch in the natural hexachord, when transposed up by a fifth (multiplied by 3/2), maps back onto a pitch within the same hexachord (octave equivalents notwithstanding), with the sole exception of $\frac{9}{8}$, which maps precisely to $\frac{9}{8} \times \frac{3}{2} = \frac{27}{16}$.

[20]They correspond respectively to the Pythagorean Major Sixth (A) and the Just Major Seventh (B).
[21]Primarily the syntonic, Didymian, or Ptolemaic comma.
[22]This refers to the so-called "Hard Hexachord" (hexachordum durum).

Despite the modest difference between σ_J and Σ_J, one crucial point must be noted: starting from the tetrachord $\mathcal{T}$ of essential consonances, it is impossible to derive the seven diatonic notes that form the "C Major scale" in common-practice tonality. The seventh pitch, B (15/8), must already be present - as it is in σ_J - to be included. The "scale" most naturally aligned with this harmonious arrangement of pitches appears to be the hexachordal one, which establishes itself in both σ_J and Σ_J alongside the previously examined subdivisions.

6 Conclusion

The mathematical investigations presented in this study lead to several fundamental realizations regarding the transition from Pythagorean to Natural sonorities. While Renaissance architectural treatises, from Alberti to Palladio, catalog various planimetric proportions derived from classical geometry, the true convergence with Just Intonation lies in the generative method used to determine room heights. The application of the harmonic mean to define the vertical dimension is not merely a construction expedient; rather, it represents a formal application of the principle of proportio. In our proposed algorithm, this same principle allows for the derivation of 5-limit ratios (such as $5:3$ or $6:5$) that characterize Renaissance harmony. This approach transcends the rigidity of the Pythagorean system by utilizing a process of internal mediation rather than external iteration.

The Limits of Pythagorean Harmony

Pythagorean sonorities are inherently incapable of expanding the presence of harmonic proportions within consonance. When the sonic gamut is restricted to values generated solely by the bipartition or tripartition of a string ($2^n/3^m$), the formation of harmonic proportions becomes impossible. Consequently, adding Pythagorean pitches S_P to the four fundamental consonances - the unison, octave, fifth, and fourth - does not intensify the harmonic depth of the diapente and diatessaron. This "hostility" toward mutual harmony reflects the musical reality of the early Middle Ages. The monodic nature of Gregorian chant and the early medieval organum—characterized by the superposition of sounds strictly at the diapason, diapente, and diatessaron[23] - reflect a mathematical framework that is refractory to a broader, integrated harmonic arrangement.

The Emergence of Natural Sonorities

The shift toward Just Intonation - incorporating "imperfect" consonances such as the major third (5/4) and major sixth (5/3) - arose from the structural demands of polyphony as it rose to prominence from the late Middle Ages onward. If harmonic proportion is the criterion that enhances and validates consonance, its pervasive presence in the natural set S_J provides a qualitative confirmation of Zarlino's theoretical intuition.By extending the base of perfect consonances through an algorithm that selects only natural sounds in harmonic proportion with existing ones, we arrive at the set (23). Notably, this process does not inherently yield the modern C major scale. This absence is significant: the seven-note diatonic scale is a later construct, and forcing it upon Renaissance theory is a common historical anachronism in modern literature. Our methodology, by rejecting "non-standard blocks" that do not conform to the established architectural module, identifies the hexachord as the authentic structural unit of the period.The Architectural Parallel: From Infinite to PerfectumThis mathematical evolution finds a profound resonance in the history of architecture. The construction of the Pythagorean scale, based on an infinite succession of fifths, mirrors the "mystic infinity" of Gothic architecture. In the Gothic spire, each element relates primarily to the one immediately below it, lacking a unified, closed proportional memory of the whole. In contrast, the Renaissance model - inspired by the Albertain concept of perfectum (that which is complete or "finished") - seeks a totalizing harmony between the parts and the whole. Our procedural approach is "closed"; it reaches a definitive mathematical end. Like a Renaissance temple, the system of natural sounds explored in (23) and (24) represents a completed design. Here, the elements do not merely follow one another; they recognize and complete each other within a finite, rationalized space.

[23] In music theory, these are termed perfect consonances, a nomenclature that reflects their perceived stability and mathematical purity in the Pythagorean tradition.

Final Reflections: Mathematics as Harmony

The relationship between mathematics and music is often fraught with the risk of finding patterns where none were intended. However, the Renaissance principle of metric proportion stands as a rare example of a scientific lever that triggers a genuine aesthetic perception of the absolute. While the Pythagorean system possesses the sacred, linear perfection of a Greek temple, and equal temperament reflects the functional, repetitive regularity of a modern skyscraper, natural sonorities embody the Renaissance masterpiece. They offer an unity where every detail is a function of the whole. This study has sought to demonstrate that an interlocking web of harmonic proportions grants natural sounds a "rational soul", allowing us to see mathematics not just as an explanation of harmony, but as the very masonry from which harmony is built.

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