Serrin's problem and Alexandrov's Soap Bubble Theorem: enhanced stability via integral identities

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16 July 2024

SERRIN'S PROBLEM AND
ALEXANDROV'S SOAP BUBBLE THEOREM:
ENHANCED STABILITY VIA INTEGRAL IDENTITIES

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ABSTRACT. We consider Serrin's overdetermined problem for the torsional rigidity and Alexandrov's Soap Bubble Theorem. We present new integral identities, that show a strong analogy between the two problems and help to obtain better (in some cases optimal) quantitative estimates for the radially symmetric configuration. The estimates for the Soap Bubble Theorem benefit from those of Serrin's problem.

1. INTRODUCTION

The pioneering symmetry results obtained by A. D. Alexandrov [Al1], [Al2] and J. Serrin [Se] are now classical but still influential. The former — the well-known *Soap Bubble Theorem* — states that a compact hypersurface, embedded in $\mathbb{R}^N$, that has constant mean curvature must be a sphere. The latter — Serrin's symmetry result — has to do with certain overdetermined problems for partial differential equations. In its simplest formulation, it states that the overdetermined boundary value problem

$$(1.1) \qquad \Delta u = N \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \Gamma,$$

$$(1.2) \qquad u_\nu = R \ \text{ on } \ \Gamma,$$

admits a solution for some positive constant R if and only if Ω is a ball of radius R and, up to translations, $u(x) = (|x|^2 - R^2)/2$. Here, Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with sufficiently smooth boundary Γ, say C^2, and u_ν is the outward normal derivative of u on Γ. This result inaugurated a new and fruitful field in mathematical research at the confluence of Analysis and Geometry and has many applications to other areas of mathematics and natural sciences. To be sure, that same result was actually motivated by two concrete problems in mathematical physics regarding the torsion of a straight solid bar and the tangential stress of a fluid on the walls of a rectilinear pipe.

The two problems share several common features. To prove his result, Alexandrov introduced his *reflection principle*, an elegant geometric technique that also works for other symmetry results concerning curvatures. Serrin's proof hinges on his *method of moving planes* — an adaptation and refinement of the reflection principle — that proves to be a very flexible tool, since it allows to prove radial symmetry for *positive* solutions of a far more general class of non-linear equations, that includes the semi-linear equation

$$(1.3) \qquad \Delta u = f(u),$$

where f is a locally Lipschitz continuous non-linearity.

1991 *Mathematics Subject Classification.* Primary 35N25, 53A10, 35B35; Secondary 35A23.

Key words and phrases. Serrin's overdetermined problem, Alexandrov Soap Bubble Theorem, torsional rigidity, constant mean curvature, integral identities, stability, quantitative estimates.

Also, alternative proofs of both symmetry results can be given, based on certain integral identities and inequalities, and the maximum principle. H. F. Weinberger's proof ([We]) of symmetry, even if it is known to work only for problem (1.1)-(1.2) and other few instances, leaves open the option of considering less regular settings. Such a possibility may be extended to the Soap Bubble Theorem, by using R. C. Reilly's argument ([Re]), in which the relevant hypersurface is regarded as a level surface of the solution of (1.1) (Γ itself, indeed).

In this paper, we shall further analyze the analogies in Weinberger's and Reilly's arguments to obtain quantitative estimates of the desired radial symmetries. Roughly speaking, we shall address the problem of estimating how close to a sphere is a hypersurface Γ, if either its mean curvature H is close to be constant or, alternatively, if the normal derivative on Γ of the solution of (1.1) is close to be constant.

In both problems, the radial symmetry of Γ will descend from that of the solution u of (1.1), by showing that in *Newton's inequality*

$$(1.4) \qquad (\Delta u)^2 \le N\, |\nabla^2 u|^2,$$

that, by Cauchy-Schwarz inequality, holds pointwise in Ω, the equality sign is identically attained in Ω. In fact, such an occurrence holds if and only if u is a quadratic polynomial q of the form

$$(1.5) \qquad q(x) = \frac{1}{2}\,(|x - z|^2 - a),$$

for some choice of $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$. The boundary condition in (1.1) will then tells us that Γ must be a sphere centered at z.

The starting points of our analysis are the following two integral identities:

$$(1.6) \qquad \int_\Omega (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx = \frac{1}{2} \int_\Gamma \left(u_\nu^2 - R^2 \right) (u_\nu - q_\nu)\, dS_x$$

and

$$(1.7) \qquad \frac{1}{N-1} \int_\Omega \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx + \frac{1}{R} \int_\Gamma (u_\nu - R)^2 dS_x =$$
$$\int_\Gamma (H_0 - H)\,(u_\nu - q_\nu)\, u_\nu\, dS_x + \int_\Gamma (H_0 - H)\,(u_\nu - R)\, q_\nu\, dS_x.$$

The two identities hold regardless of how the point z or the constant a are chosen in (1.5). In (1.6) and (1.7), R and H_0 are reference constants given by

$$(1.8) \qquad R = \frac{N\,|\Omega|}{|\Gamma|}, \quad H_0 = \frac{1}{R} = \frac{|\Gamma|}{N\,|\Omega|}.$$

If u_ν is constant on Γ, that constant must be equal to R, by the identity

$$(1.9) \qquad \int_\Gamma u_\nu\, dS_x = N\,|\Omega|,$$

and hence we obtain symmetry by using (1.6), since the equality sign must hold in (1.4) (in fact, $-u > 0$ in Ω by the maximum principle). If in turn H is constant on Γ, then the *Minkowski's identity*,

$$(1.10) \qquad \int_\Gamma H\, q_\nu\, dS_x = |\Gamma|,$$

implies that $H \equiv H_0$ on Γ, and hence once again symmetry follows by an inspection of (1.7), instead.

Identity (1.6), which appears without proof in [MP, Remark 2.5], puts together Weinberger's identities and some remarks of L. E. Payne and P. W. Schaefer [PS]; (1.7) is a slight modification of one that was proved in [MP, Theorem 2.2] and

will turn out to be useful to improve our desired quantitative estimates. For the reader's convenience, we will present the proofs of (1.6) and (1.7) in Section 2.

Identity (1.6) certainly holds under Serrin's smoothness assumptions ($\Gamma \in C^2$), but it is clear that it can be easily extended by approximation to the case of $\Gamma \in C^{1,\alpha}$, since in that case u_ν is continuous on Γ. Nevertheless, it should be noticed that, to infer the radial symmetry of $|\Omega|$, it suffices to show that the surface integral on the right-hand side of (1.6) is zero.

The main motivation of this paper is to investigate how the use of (1.6) and (1.7) benefits the study of the stability of the radial configuration in the Soap Bubble Theorem and Serrin's problem. Technically speaking, one may look for two concentric balls B_{ρ_i} and B_{ρ_e}, with radii ρ_i and ρ_e, such that

$$(1.11) \qquad\qquad B_{\rho_i} \subset \Omega \subset B_{\rho_e}$$

and

$$(1.12) \qquad\qquad \rho_e - \rho_i \le \psi(\eta),$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function vanishing at 0 and η is a suitable measure of the deviation of u_ν or H from being a constant.

That kind of problem has been considered for Serrin's problem for the first time in [ABR]. There, for a $C^{2,\alpha}$-regular domain Ω, it is considered a *positive* solution u of (1.3), such that $u = 0$ on Γ, and it is proved that (1.11) and (1.12) hold with $\psi(\eta) = C \, |\log \eta|^{-1/N}$ and $\eta = \|u_\nu - c\|_{C^1(\Gamma)}$. (Here and in the remaining paragraphs of this introduction, C denotes a positive constant that depends on some geometric and regularity parameters associated to Ω and, when applicable, f.) The proof is based on a quantitative study of the method of moving planes. In [CMV], in the same general framework and with a similar proof, that stability estimate has been improved. There, it is in fact shown that (1.11) and (1.12) hold with $\psi(\eta) = C \, \eta^\tau$, where η is the Lipschitz seminorm $[u_\nu]_\Gamma$ defined by

$$\sup_{\substack{x,y\in\Gamma \\ x\neq y}} \frac{|u_\nu(x) - u_\nu(y)|}{|x - y|}.$$

The exponent $\tau \in (0,1)$ can be computed for a general setting and, if Ω is convex, is arbitrarily close to $1/(N+1)$.

Our stability estimate for the original Serrin's problem (1.1)-(1.2) improves on a different technique, based on Weinberger's integral identities and first employed in [BNST]. There, a quantitative estimate of Hölder type was obtained for the first time, by using the weaker deviation $\eta = \|u_\nu - c\|_\infty$ of u_ν from some reference constant. In [BNST], it is also considered the possibility to measure the deviation of u_ν from a constant by the L^1-norm and it is shown that Ω can be approximated in measure by a finite number of mutually disjoint balls B_i. The error in the approximation is $\psi(\eta) = C \, \eta^\tau$, with $\eta = \|u_\nu - c\|_{1,\Gamma}$ and $\tau = 1/(4N+9)$. Recently, that approach has been greatly improved in [Fe] where, rather than by (1.11) and (1.12), the closeness of Ω to a ball is measured by the following slight modification of the so-called Fraenkel asymmetry:

$$(1.13) \qquad \mathcal{A}(\Omega) = \inf\left\{ \frac{|\Omega \Delta B^x|}{|B^x|} : x \text{ center of a ball } B^x \text{ with radius } R \right\},$$

where $\Omega \Delta B^x$ denotes the symmetric difference of Ω and B^x, and R is the constant defined in (1.8). In [Fe], the deviation of u_ν from a constant is measured in L^2-norm and the estimate obtained is of Lipschitz type: $\mathcal{A}(\Omega) \le C \, \|u_\nu - R\|_{2,\Gamma}$.

In this paper, we turn back to an approximation of type (1.11) and (1.12) and, in Theorems 3.6 and 3.7, we derive two inequalities:

$$(1.14) \qquad \rho_e - \rho_i \leq C \left\| u_\nu - R \right\|_{2,\Gamma}^{2/(N+2)}$$

and

$$(1.15) \qquad \rho_e - \rho_i \leq C \left\| u_\nu - R \right\|_{1,\Gamma}^{\frac{1}{N+2}}.$$

The latter clearly improves that obtained in [BNST], while the former makes better than [CMV], even if we replace the Lipschitz semi-norm by an L^2-deviation.

Another salient result in our paper is a quantitative inequality for simmetry in the Soap Bubble Theorem, that is obtained as a benefit from the analysis employed to derive (1.14). In fact, in Theorem 4.3 , we prove that

$$(1.16) \qquad \rho_e - \rho_i \leq C \left\| H_0 - H \right\|_{2,\Gamma}^{\tau_N},$$

where $\tau_N = 1$ for $N = 2, 3$ and $\tau_N = 2/(N+2)$ for $N \geq 4$. Inequality (1.16) enhances, in all dimensions, estimates obtained in [CM] with the uniform deviation $\|H_0 - H\|_{\infty,\Gamma}$, for strictly mean convex surfaces. Also, it improves by a factor of 2 the results derived by the authors in [MP]. More importantly, for the cases $N = 2, 3$ but with a weaker deviation and for a more general class of hypersurfaces, gains the (optimal) Lipschitz stability proved in [CV] and [KM, Theorem 1.8] for strictly mean convex hypersurfaces, by using the uniform deviation.

As a final important achievement, by arguments similar to those of [Fe], we also get in Theorem 4.8 the (optimal) inequality for the asymmetry (1.13):

$$\mathcal{A}(\Omega) \leq C \left\| H_0 - H \right\|_{2,\Gamma}.$$

In the remaining paragraphs of this introduction, we shall pinpoint the salient remarks that lead us to the proof of (1.14) and (1.16).

We start to simplify matters by noticing that, by (1.6), the *harmonic* function $h = q - u$ satisfies:

$$(1.17) \qquad \int_\Omega (-u) \left| \nabla^2 h \right|^2 dx = \frac{1}{2} \int_\Gamma (R^2 - u_\nu^2) \, h_\nu \, dS_x.$$

Also, notice that $h = q$ on Γ and hence

$$(1.18) \qquad \max_\Gamma h - \min_\Gamma h = \frac{1}{2} \left(\rho_e^2 - \rho_i^2 \right) \geq \frac{1}{2} \left(|\Omega| / |B| \right)^{1/N} (\rho_e - \rho_i).$$

Now, observe that (1.17) holds regardless of the choice of the parameters z and a defining q, so that we will complete the first step of our proof by choosing $z \in \Omega$ in a way that the *oscillation* of h on Γ on the left-hand side of (1.18) (which is indeed the oscillation of h on $\overline{\Omega}$) can be bounded by a suitable function of the left-hand side of (1.17).

To carry out this plan, we use three ingredients. First, we choose $z \in \Omega$ as a minimum (or any critical) point of u and, as done in [MP, Lemma 3.3], we show that

$$\max_\Gamma h - \min_\Gamma h \leq C \left(\int_\Omega h^2 dx \right)^{1/(N+2)}.$$

Secondly, we observe that, depending on the regularity of Γ, we can easily obtain the bound

$$C \, \delta_\Gamma(x)^\alpha \leq -u \quad \text{on} \quad \overline{\Omega},$$

where $\delta_\Gamma(x) = \text{dist}(x, \Gamma)$ for $x \in \overline{\Omega}$ and $\alpha = 1$ or 2. Thirdly, we apply to the first (harmonic) derivatives of h the Hardy-Poincaré-type inequality

$$\int_\Omega v(x)^2 dx \leq C \int_\Omega \delta_\Gamma(x)^\alpha |\nabla v(x)|^2 dx,$$

that holds for fixed $\alpha \in [0, 1]$ and for any harmonic function $v \in W^{1,2}(\Omega)$ that is zero at some given point in Ω (in our case that point will be z) and, to h, the Poincaré-type inequality

$$\int_\Omega v^2 dx \leq C \int_\Omega |\nabla v|^2 dx,$$

that holds for any harmonic function $v \in W^{1,2}(\Omega)$ with zero mean value on Ω. Thus, summing up the last 5 inequalities gives:

$$(1.19) \qquad \max_\Gamma h - \min_\Gamma h \leq C \left(\int_\Omega (-u) \, |\nabla^2 h|^2 \, dx \right)^{1/(N+2)}.$$

Next, we work on the right-hand side of (1.17). An important observation is that, if $u_\nu - R$ tends to 0, also h_ν does. Quantitavely, this fact can be expressed by the inequality

$$\|h_\nu\|_{2,\Gamma} \leq C \|u_\nu - R\|_{2,\Gamma},$$

that can be derived from [Fe]. Thus, (1.14) will follow by using this inequality, after an application of Hölder's inequality to the right-hand side of (1.17).

In order to prove (1.16), we use the new identity (1.7). In fact, discarding the first summand at its left-hand side and applying Hölder's and the last inequality to its right-hand side yield that

$$\|u_\nu - R\|_{2,\Gamma} \leq C \|H_0 - H\|_{2,\Gamma}.$$

Inequality (1.16) then follows again from (1.7) and the estimate

$$\max_\Gamma h - \min_\Gamma h \leq C \left(\int_\Omega |\nabla^2 h|^2 \, dx \right)^{\tau_N/2}$$

already obtained in [MP].

The constants C in our inequalities generally depend on N, the *volume* $|\Omega|$ and the *diameter* d_Ω of Ω, and the radii r_i and r_e of the largest balls osculating Γ and, respectively, contained in Ω or in $\mathbb{R}^N \setminus \overline{\Omega}$. Their dependence on those parameters will be clarified later.

The paper is organized as follows. In Section 2, we collect the relevant identities on which our results are based. Section 3 is dedicated to the stability of Serrin's problem. Subsection 3.2 contains the estimates on harmonic functions that are instrumental to derive (1.15) and (1.14). Then, in Subsection 3.3, we assemble all the relevant identities and inequalities to estabilish (1.15) and (1.14) (respectively Theorems 3.6 and 3.7). In Section 4, we present the new stability result (1.16) for Alexandrov's Soap Bubble Theorem (Theorem 4.3).

2. Identities for Serrin's problem and the Soap Bubble Theorem

We begin by setting some relevant notations. By $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we shall denote a bounded domain and call Γ its boundary. By $|\Omega|$ and $|\Gamma|$, we will denote indifferently the N-dimensional Lebesgue measure of Ω and the surface measure of Γ. When Γ is of class C^1, ν will denote the (exterior) unit normal vector field to Γ and, when Γ is a hypersurface of class C^2, $H(x)$ will denote its mean curvature (with respect to $-\nu(x)$) at $x \in \Gamma$.

We will also use the letter q to denote the quadratic polynomial defined in (1.5), where z is any point in $\mathbb{R}^N$ and a is any real number; furthermore, we will always use the letter h to denote the harmonic function

$$h = q - u.$$

Finally, as already mentioned in the introduction, for a point $z \in \Omega$ to be determined, ρ_i and ρ_e shall denote the radius of the largest ball contained in Ω and that of the smallest ball that contains Ω, both centered at z; in formulas,

$$(2.1) \qquad \rho_i = \min_{x \in \Gamma} |x - z| \quad \text{and} \quad \rho_e = \max_{x \in \Gamma} |x - z|.$$

To start, we will provide the proof of identity (1.6) appearing in [MP, Remark 2.5]. The proof is as minimal as possible. It follows the tracks of and improves on the work of Weinberger [We] and its modification due to Payne and Schaefer [PS].

Identity (1.6) is a consequence of two ingredients: the differential identity for the solution u of (1.1),

$$(2.2) \qquad |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} = \Delta P,$$

that associates the Cauchy-Schwarz deficit on the left-hand side with the *P-function*,

$$P = \frac{1}{2} |\nabla u|^2 - u,$$

and is easily obtained by direct computation; the *Rellich-Pohozaev* identity for the solution u of (1.1) (see [Po]):

$$(2.3) \qquad (N + 2) \int_\Omega |\nabla u|^2 \, dx = \int_\Gamma (u_\nu)^2 \, q_\nu \, dS_x.$$

Notice that (2.2) also implies that P is subharmonic, since the left-hand side is non-negative by Cauchy-Schwarz inequality.

Theorem 2.1 (Fundamental identity for Serrin's problem)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{1,\alpha}$, $0 < \alpha \le 1$, and R be the positive constant defined in (1.8).*

Then the solution u of (1.1) satisfies identity (1.6):

$$\int_\Omega (-u) \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx = \frac{1}{2} \int_\Gamma \left(u_\nu^2 - R^2 \right) (u_\nu - q_\nu) \, dS_x.$$

Therefore, if the right-hand side of (1.6) is non-positive, Γ must be a sphere (and hence Ω a ball) of radius R. The same conclusion holds, in particular, if u_ν is constant on Γ.

Proof. First, suppose that Γ is of class $C^{2,\alpha}$, so that $u \in C^{2,\alpha}(\overline{\Omega})$. Integration by parts then gives:

$$\int_\Omega (u \, \Delta P - P \, \Delta u) \, dx = \int_\Gamma (u \, P_\nu - u_\nu \, P) \, dS_x;$$

thus, since u satisfies (1.1), we have that

$$(2.4) \qquad \int_\Omega (-u) \, \Delta P \, dx = -N \int_\Omega P \, dx + \frac{1}{2} \int_\Gamma u_\nu^3 \, dS_x,$$

being $P = |\nabla u|^2/2 = u_\nu^2/2$ on Γ.

Now, we compute the first summand on the right-hand side of (2.4) by the divergence theorem:

$$N \int_\Omega P \, dx = \frac{N}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega u \, \Delta u \, dx = \left(\frac{N}{2} + 1 \right) \int_\Omega |\nabla u|^2 \, dx.$$

Thus, (2.4), (2.2), and (2.3) give (1.6), since in (2.4) we can replace $(u_\nu)^3$ by $(u_\nu^2 - R^2) \, (u_\nu - q_\nu)$, being

$$\int_\Gamma (u_\nu - q_\nu) \, dS_x = 0,$$

since $u - q$ harmonic in Ω.

If Γ is of class $C^{1,\alpha}$, then $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$. Thus, by a standard approximation argument, we conclude that (1.6) holds also in this case.

Now, if the right-hand side of (1.6) is non-positive, being the left-hand side non-negative by (1.4) and the maximum principle for u, then (1.4) must hold with the equality sign, since $u < 0$ on Ω, by the strong maximum principle. Being $\Delta u = N$, we infer that $\nabla^2 u$ coincides with the identity matrix I. Thus, u must be a quadratic polynomial q of the form (1.5), for some $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$.

Since $u = 0$ on Γ, then $|x - z|^2 = a$ for $x \in \Gamma$, that is a must be positive and

$$\sqrt{a}\,|\Gamma| = \int_\Gamma |x - z|\,dS_x = \int_\Gamma (x - z)\cdot\nu(x)\,dS_x = N\,|\Omega|.$$

In conclusion, Γ must be a sphere centered at z with radius R.

Finally, if $u_\nu \equiv c$ on Γ for some constant c, then

$$c\,|\Gamma| = \int_\Gamma u_\nu\,dS_x = N\,|\Omega|,$$

that is $c = R$, and hence we can apply the previous argument. $\qquad\square$

Corollary 2.2. *Let u be the solution of* (1.1) *and set $h = q - u$. Then, h is harmonic in Ω and* (1.17) *holds true:*

$$\int_\Omega (-u)\,|\nabla^2 h|^2\,dx = \frac{1}{2}\int_\Gamma (R^2 - u_\nu^2)\,h_\nu\,dS_x.$$

Moreover, if the point z in (1.5) *is chosen in Ω, then* (1.18) *holds.*

Proof. Simple computations give that $|\nabla^2 h|^2 = |\nabla^2 u|^2 - (\Delta u)^2/N$ and $h_\nu = q_\nu - u_\nu$, and hence (1.17) easily follows from (1.6).

Notice that $h = q$ on Γ. Thus, the equality in (1.18) follows from (2.1), by choosing z in Ω. The inequality in (1.18) is implied by $\rho_e + \rho_i \geq \rho_e \geq (|\Omega|/|B|)^{1/N}$, since $B_{\rho_e} \supseteq \Omega$. $\qquad\square$

Remark 2.3. The assumptions on the regularity of Γ can further be weakened. For instance, if Ω is a (bounded) convex domain, then inequality (3.3) below, imply that u_ν is essentially bounded on Γ with respect to the $(N-1)$-dimensional Hausdorff measure on Γ. Thus, an approximation argument again gives that (1.6) still holds true.

For the sake of completeness, we present the proof of (1.7), that is a modification of [MP, formula (2.6)].

Theorem 2.4 (Fundamental identity for the Soap Bubble Theorem). *Let Ω be a bounded domain with boundary Γ of class C^2. Then identity* (1.7) *holds true:*

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\}dx + \frac{1}{R}\int_\Gamma (u_\nu - R)^2 dS_x =$$

$$\int_\Gamma (H_0 - H)\,(u_\nu - q_\nu)\,u_\nu\,dS_x + \int_\Gamma (H_0 - H)\,(u_\nu - R)\,q_\nu\,dS_x.$$

Proof. We proceed similarly to the proof of Theorem 2.1, but with two main differences: in place of Pohozaev's identity, we use Minkowski's identity; we use the following well-known formula for the laplacian of u:

$$\Delta u = u_{\nu\nu} + (N-1)H\,u_\nu.$$

This holds pointwise on any regular level surface of u, if we agree to still denote by ν the vector field $\nabla u/|\nabla u|$; it is clear that, on Γ, this coincides with the normal.

We begin with the divergence theorem:

$$\int_{\Omega} \Delta P \, dx = \int_{\Gamma} P_{\nu} \, dS_x.$$

To compute P_{ν}, we observe that ∇u is parallel to ν on Γ, that is $\nabla u = (u_{\nu}) \, \nu$ on Γ. Thus,

$$P_{\nu} = \langle \nabla^2 u \, \nabla u, \nu \rangle - u_{\nu} = u_{\nu} \langle (\nabla^2 u) \, \nu, \nu \rangle - u_{\nu} = u_{\nu\nu} \, u_{\nu} - u_{\nu} =$$
$$u_{\nu} [\Delta u - (N-1) H u_{\nu}] - u_{\nu} = (N-1) \, (1 - H u_{\nu}) \, u_{\nu},$$

where we have used (1.1). Therefore,

$$(2.5) \qquad \frac{1}{N-1} \int_{\Omega} \Delta P \, dx = \int_{\Gamma} (1 - H u_{\nu}) \, u_{\nu} \, dS_x.$$

Now, straightforward calculations that use (1.8), (1.10), and (1.9) tell us that

$$\int_{\Gamma} (H_0 - H) \, (u_{\nu} - q_{\nu}) \, u_{\nu} \, dS_x +$$
$$\int_{\Gamma} (H_0 - H) \, (u_{\nu} - R) \, q_{\nu} \, dS_x = H_0 \int_{\Gamma} u_{\nu}^2 \, dS_x - \int_{\Gamma} H u_{\nu}^2 \, dS_x,$$

while

$$\frac{1}{R} \int_{\Gamma} (u_{\nu} - R)^2 dS_x = H_0 \int_{\Gamma} u_{\nu}^2 \, dS_x - \int_{\Gamma} u_{\nu} \, dS_x.$$

The conclusion then follows by a simple inspection, from these two formulas, (2.5), and (2.2). □

Corollary 2.5. *Let Ω be a bounded domain with boundary Γ of class C^2 and set $h = q - u$.*

Then it holds that

$$(2.6) \quad \frac{1}{N-1} \int_{\Omega} |\nabla^2 h|^2 dx + \frac{1}{R} \int_{\Gamma} (u_{\nu} - R)^2 dS_x =$$
$$- \int_{\Gamma} (H_0 - H) \, h_{\nu} \, u_{\nu} \, dS_x + \int_{\Gamma} (H_0 - H) \, (u_{\nu} - R) \, q_{\nu} \, dS_x.$$

3. Stability for Serrin's overdetermined problem

3.1. **Notations.** The *diameter* of Ω will be indicated by d_{Ω}, while $\delta_{\Gamma}(x)$ denotes the distance of a point x to the boundary Γ.

Even if (1.6) holds for more general domains, in order to consider the stability issue, we shall assume that Ω is a bounded domain with boundary Γ of class C^2. In fact, under this assumption, Ω has the properties of the *uniform interior and exterior sphere condition*, whose respective radii we have designated by r_i and r_e; in other words, there exists $r_e > 0$ (resp. $r_i > 0$) such that for each $p \in \Gamma$ there exists a ball contained in $\mathbb{R}^N \setminus \overline{\Omega}$ (resp. contained in Ω) of radius r_e (resp. r_i) such that its closure intersects Γ only at p. We shall later see that, when Ω is a convex domain, then we can remove the assumption on the regularity of Γ.

The assumed regularity of Ω ensures that the unique solution of (1.1) is of class at least $C^{1,\alpha}(\overline{\Omega})$. Thus, we can define

$$(3.1) \qquad M = \max_{\overline{\Omega}} |\nabla u| = \max_{\Gamma} u_{\nu}.$$

As shown in [MP, Theorem 3.10], the following bound holds for M:

$$(3.2) \qquad M \leq c_N \, \frac{d_{\Omega}(d_{\Omega} + r_e)}{r_e},$$

where $c_N = 3/2$ for $N = 2$ and $c_N = N/2$ for $N \geq 3$. Notice that, when Ω is convex, we can choose $r_e = +\infty$ in (3.2) and obtain

$$(3.3) \qquad\qquad M \leq c_N \, d_\Omega.$$

For other estimates are present in the literature, see [MP, Remark 3.11].

3.2. **Some estimates for harmonic functions.** As already sketched in the introduction, the desired stability estimate for the spherical symmetry of Ω will be obtained, by means of identity (1.17), if we associate the oscillation of h on Γ with

$$\int_\Omega (-u)|\nabla^2 h|^2 dx.$$

To fulfill this agenda, we start by relating the factor $(-u)$ appearing in that quantity to the function $\delta_\Gamma(x)$; we do it in the following lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain such that Γ is made of regular points for the Dirichlet problem, and let u be the solution of (1.1). Then*

$$-u(x) \geq \frac{1}{2}\,\delta_\Gamma(x)^2 \quad \text{for every} \ \ x \in \overline{\Omega}.$$

Moreover, if Γ is of class C^2, then it holds that

$$(3.4) \qquad\qquad -u(x) \geq \frac{r_i}{2}\,\delta_\Gamma(x) \quad \text{for every} \ \ x \in \overline{\Omega}.$$

Proof. If every point of Γ is regular, then a unique solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ exists for (1.1). Now, for $x \in \Omega$, let $r = \delta_\Gamma(x)$ and consider the ball $B = B_r(x)$. If w is the solution of (1.1) in B, that is $w(y) = (|y - x|^2 - r^2)/2$, by comparison we have that $w \geq u$ on $\overline{B}$ and hence, in particular, $w(x) \geq u(x)$, that is we infer the first inequality in the lemma.

If Γ is of class C^2, (3.4) centainly holds if $\delta_\Gamma(x) \geq r_i$. If $\delta_\Gamma(x) < r_i$, instead, let z be the closest point in Γ to x and call B the ball of radius r_i touching Γ at z and containing x. Up to a translation, we can always suppose that the center of the ball B is the origin 0. If w is the solution of (1.1) in B, that is $w(y) = (|y|^2 - r_i^2)/2$, by comparison we have that $w \geq u$ in B, and hence

$$-u(x) \geq \frac{1}{2}\left(|x|^2 - r_i^2\right) = \frac{1}{2}\,(r_i + |x|)(r_i - |x|) \geq \frac{1}{2}\,r_i\,(r_i - |x|),$$

which implies (3.4), since $r_i - |x| = \delta_\Gamma(x)$. $\qquad\qquad\square$

By the last lemma, we can estimate the right-hand side of (1.17) from below in terms of the integral

$$\int_\Omega |\nabla h|^2 \delta_\Gamma^{2\alpha} dx,$$

with $\alpha = 1$ or $1/2$. For this kind of integral, useful estimates are present in the literature. We shall briefly report on some of them.

Lemma 3.2 (Hardy-Poincaré-type inequalities). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary Γ of class $C^{0,\alpha}$, and let x_0 be a point in Ω. Then,*
(i) there exists a positive constant $\mu_\alpha(\Omega)$, such that

$$(3.5) \qquad\qquad \int_\Omega v^2 dx \leq \mu_\alpha(\Omega)^{-1} \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha} dx,$$

for every function v which is harmonic in Ω and such that $v(x_0) = 0$;
(ii) there exists a positive constant, $\overline{\mu}_\alpha(\Omega)$ such that

$$(3.6) \qquad\qquad \int_\Omega v^2 dx \leq \overline{\mu}_\alpha(\Omega)^{-1} \int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha} dx,$$

for every function v which is harmonic in Ω and has mean value zero on Ω.

In particular, if Γ is a Lipschitz boundary, the exponent 2α can be replaced by any exponent in $(0,2]$.

Proof. The assertions (i) and (ii) are easy consequences of a general result of Boas and Straube (see [BS]), that improves a work of Ziemer's ([Zi]). In case (i), we apply [BS, Example 2.5]). In case (ii), [BS, Example 2.1] is appropriate.

The variational problems

$$(3.7) \qquad \mu_\alpha(\Omega) = \min\left\{\int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha} dx : \int_\Omega v^2 dx = 1, \, \Delta v = 0 \text{ in } \Omega, \, v(x_0) = 0\right\}$$

and

$$(3.8) \quad \overline{\mu}_\alpha(\Omega) \; = \; \min\left\{\int_\Omega |\nabla v|^2 \delta_\Gamma^{2\alpha} dx : \int_\Omega v^2 \, dx = 1, \, \Delta v = 0 \text{ in } \Omega, \, \int_\Omega v \, dx = 0\right\}$$

then characterize the two constants. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.3. (i) Notice that in the special case that $\alpha = 0$ from (3.5) and (3.6) we recover the Poincaré-type inequality, that we proved and used in [MP]. In the sequel, we shall use the simplified notation $\overline{\mu}(\Omega) = \overline{\mu}_0(\Omega)$. Also, (3.5) in the case $\alpha = 0$ directly follows from the result in [Zi].

(ii) We have that

$$\mu_\alpha(\Omega) \leq \overline{\mu}_\alpha(\Omega),$$

as one can verify by using the function

$$v_0 = \frac{v - v(x_0)}{1 + |\Omega| \, v(x_0)^2},$$

where v is a minimizer for (3.8).

The next lemma, that modifies for our purposes an idea of W. Feldman [Fe], will be useful to bound the right-hand side of (1.17).

Lemma 3.4 (Trace inequality). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary Γ of class C^2 and z be any critical point in Ω of the solution u of (1.1).*

The following inequality holds for $h = q - u$, where q is given by (1.5):

$$(3.9) \qquad \int_\Gamma |\nabla h|^2 dS_x \leq \frac{2}{r_i}\left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)}\right)\int_\Omega (-u)|\nabla^2 h|^2 dx.$$

Proof. It is clear that $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. We begin with the following differential identity:

$$\text{div}\left\{v^2\nabla u - u\,\nabla(v^2)\right\} = v^2\Delta u - u\,\Delta(v^2) = N\,v^2 - 2u\,|\nabla v|^2,$$

that holds for any v harmonic function in Ω, if u is satisfies(1.1). Next, we integrate on Ω and, by the divergence theorem, we get:

$$\int_\Gamma v^2 u_\nu \, dS_x = N\int_\Omega v^2 dx + 2\int_\Omega (-u)|\nabla v|^2 dx.$$

We use this identity with $v = h_i$, and hence we sum up over $i = 1, \dots, N$ to obtain:

$$\int_\Gamma |\nabla h|^2 u_\nu dS_x = N\int_\Omega |\nabla h|^2 dx + 2\int_\Omega (-u)|\nabla^2 h|^2 dx.$$

This formula, together with (3.5) and (3.4), gives us:

$$\int_\Gamma |\nabla h|^2 u_\nu dS_x \leq 2\left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)}\right)\int_\Omega (-u)|\nabla^2 h|^2 dx.$$

The term u_ν at the left-hand side of this last inequality can be bounded from below by r_i, by an adaptation of Hopf's lemma (see also [MP, Theorem 3.10]). Therefore, (3.9) follows at once. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The crucial step in our analysis is Theorem 3.5 below, in which we associate the oscillation of h, and hence $\rho_e - \rho_i$, with a weighted L^2-norm of its Hessian matrix.

Before stating that, we recall from [MP] the following estimate, that links $\rho_e - \rho_i$ with the L^2-norm of h. In fact, if Ω is a bounded domain with boundary of class C^2, we have that

$$(3.10) \qquad \rho_e - \rho_i \leq a_N \, M^{\frac{N}{N+2}} \, |\Omega|^{-\frac{1}{N}} \, \|h\|_2^{2/(N+2)},$$

for

$$(3.11) \qquad \|h\|_2 \leq \alpha_N \, M \, r_i^{\frac{N+2}{2}}.$$

The values of the constants a_N and α_N can be found in [MP, Lemma 3.3].

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary Γ of class C^2 and z be any critical point in Ω of the solution u of (1.1).*

Consider the function $h = q - u$, with q given by (1.5), where the constant a is chosen such that h has mean value zero on Ω.

Then we have that

$$\rho_e - \rho_i \leq C \left\{ \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx \right\}^{1/(N+2)}$$

if

$$\int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx < \varepsilon^2.$$

Here,

$$C = \frac{a_N \, M^{\frac{N}{N+2}}}{|\Omega|^{\frac{1}{N}} \left[\overline{\mu}(\Omega) \mu_{1/2}(\Omega) \right]^{\frac{1}{N+2}}} \quad and \quad \varepsilon = \alpha_N \, M \, r_i^{\frac{N+2}{2}} \sqrt{\overline{\mu}(\Omega) \mu_{1/2}(\Omega)},$$

and the constants a_N and α_N are those in (3.10) and (3.11).

Proof. We apply (3.5) with $x_0 = z$ and $\alpha = 1/2$ to each first derivative of h (since $\nabla h(z) = 0$) and obtain that

$$\int_\Omega |\nabla h|^2 \, dx \leq \mu_{1/2}(\Omega)^{-1} \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx.$$

Hence, we apply (3.6) with $\alpha = 0$ to h and get

$$\int_\Omega h^2 \, dx \leq \overline{\mu}(\Omega)^{-1} \int_\Omega |\nabla h|^2 dx.$$

Thus,

$$\int_\Omega h^2 dx \leq \left[\overline{\mu}(\Omega) \mu_{1/2}(\Omega) \right]^{-1} \int_\Omega |\nabla^2 h|^2 \delta_\Gamma \, dx,$$

and the conclusion follows from (3.10) and (3.11). $\qquad \square$

3.3. **Stability for Serrin's problem.** We collect here our results on the stability of the spherical configuration by putting together the identities of Section 2 and the estimates in the previous subsection.

Theorem 3.5 above gives an estimate from below of the left-hand side of (1.17). In this subsection, we will take care of its right-hand side an prove our main result for Serrin's problem.

Theorem 3.6 (Stability in L^2-norm)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary Γ of class C^2 and R be the constant defined in (1.8).*

Let u be the solution of problem (1.1) and $z \in \Omega$ be any of its critical points.

Then (1.11) holds, with ρ_i and ρ_e given by (2.1) and such that

$$(3.12) \qquad \rho_e - \rho_i \leq C \, \|u_\nu - R\|_{2,\Gamma}^{2/(N+2)} \quad if \quad \|u_\nu - R\|_{2,\Gamma} < \varepsilon,$$

for some positive constants C and ε.

Proof. We have that

$$\int_\Gamma (R^2 - u_\nu^2)\, h_\nu \, dS_x \le (M+R)\, \|u_\nu - R\|_{2,\Gamma} \|h_\nu\|_{2,\Gamma},$$

after an an application of Hölder's inequality. Thus, by Lemma 3.4, (1.17), and this inequality, we infer that

$$\|h_\nu\|_{2,\Gamma}^2 \le \frac{2}{r_i}\left(1 + \frac{N}{r_i\, \mu_{1/2}(\Omega)}\right) \int_\Omega (-u)|\nabla^2 h|^2 dx \le$$

$$\frac{M+R}{r_i}\left(1 + \frac{N}{r_i\, \mu_{1/2}(\Omega)}\right) \|u_\nu - R\|_{2,\Gamma} \|h_\nu\|_{2,\Gamma},$$

and hence

$$(3.13) \qquad \|h_\nu\|_{2,\Gamma} \le \frac{M+R}{r_i}\left(1 + \frac{N}{r_i\, \mu_{1/2}(\Omega)}\right) \|u_\nu - R\|_{2,\Gamma}.$$

Therefore,

$$\int_\Omega |\nabla^2 h|^2 \delta_\Gamma(x)\, dx \le \frac{2}{r_i}\int_\Omega (-u)|\nabla^2 h|^2 dx \le$$

$$\left(\frac{M+R}{r_i}\right)^2\left(1 + \frac{N}{r_i\, \mu_{1/2}(\Omega)}\right) \|u_\nu - R\|_{2,\Gamma}^2,$$

by Lemma 3.1. These inequalities and Theorem 3.5 then give (3.12). $\qquad\square$

If we want to measure the deviation of u_ν from R in L^1-norm, we get a smaller stability exponent.

Theorem 3.7 (Stability in L^1-norm). *Under the same assumptions of Theorem 3.6, we have that*

$$(3.14) \qquad \rho_e - \rho_i \le C\, \|u_\nu - R\|_{1,\Gamma}^{1/(N+2)} \quad \text{if} \quad \|u_\nu - R\|_{1,\Gamma} < \varepsilon,$$

for some positive constants C and ε.

Proof. Instead of applying Hölder's inequality to the right-hand side of (1.17), we just use the rough bound:

$$\int_\Omega (-u)\, |\nabla^2 h|^2\, dx \le \frac{1}{2}\, (M+R)\, (M+d_\Omega) \int_\Gamma |u_\nu - R|\, dS_x,$$

since $(u_\nu + R)\, |h_\nu| \le (M+R)\, (M+d_\Omega)$ on Γ. The conclusion then follows from similar arguments. $\qquad\square$

Remark 3.8 (On the constants C and ε). For the sake of clarity, we did not display the values of the constants C and ε in Theorems, 3.6, 3.7, and the relevant theorems in the sequel. However, their computation will be clear by following the steps of the proofs.

For instance, an inspection of the proofs of Theorem 3.6 informs us that the constants in (3.12) are:

$$C = a_N \frac{M^{\frac{N}{N+2}}\, (M+R)^{\frac{2}{N+2}}}{|\Omega|^{\frac{1}{N}}\, r_i^{\frac{3}{N+2}}} \left\{\frac{N + r_i\, \mu_{1/2}(\Omega)}{\overline{\mu}(\Omega)\, \mu_{1/2}(\Omega)^2}\right\}^{\frac{1}{N+2}}$$

and

$$\varepsilon = \alpha_N \frac{M}{M+R}\, \frac{\overline{\mu}(\Omega)^{\frac{1}{2}}\, \mu_{1/2}(\Omega)}{\sqrt{N + r_i\, \mu_{1/2}(\Omega)}}\, r_i^{\frac{N+5}{2}}.$$

Here, the constants a_N and α_N are those appearing in (3.10) and (3.11).

The constants C and ε can be shown to depend only on some geometric parameters of Ω. In fact, we can use (3.2) to bound M in terms of d_Ω and r_e. To estimate the ratio R, we can take advantage of the *isoperimetric* inequality, to bound $|\Gamma|$ from below in terms of $|\Omega|^{1-1/N}$, and then the *isodiametric* inequality, to bound $|\Omega|^{1/N}$ from above in terms of d_Ω.

In the next remark, we show how to deal with $\overline{\mu}(\Omega)$ and $\mu_{1/2}(\Omega)$.

Remark 3.9 (Estimating $\overline{\mu}(\Omega)$ and $\mu_{1/2}(\Omega)$)**.** (i) For simplicity, in what follows, k_N denotes a positive number, that only depends on N, whose value may change from line to line.

A lower bound of $\overline{\mu}(\Omega)$ can be obtained as follows. In [HS, Theorem 1.3] a more general form of inequality (3.6) (without the assumption of harmonicity) is proved for the class of the b_0-John domains (see [HS] for the definition) and the constants are explicitly computed; in particular, with the aid of [HS], we can easily deduce that

$$\overline{\mu}(\Omega)^{-1} \le k_N \, |\Omega|^{\frac{2}{N}} \, b_0^{2N},$$

for some constant k_N only depending on N. A domain of class C^2 is obviously a b_0-John domain and it is not difficult to show that

$$b_0 \le d_\Omega/r_i,$$

and hence obtain that

$$(3.15) \qquad \overline{\mu}(\Omega)^{-1} \le k_N \, |\Omega|^{\frac{2}{N}} \left(\frac{d_\Omega}{r_i} \right)^{2N}.$$

An alternative way to estimate $\overline{\mu}(\Omega)$ can be found in [MP, Remark 3.8 (ii)].

A lower bound for $\mu_{1/2}(\Omega)$ can also be obtained, but it may depend on choice of the particular critical point of u. In fact, by following the argument of [Fe, Lemma 8], one can adapt the result contained in [HS, Theorem 1.3] to the case of harmonic functions vanishing at a given point x_0 (i.e the case of (3.5)). To explicitly perform the computations, the definition of bounded L_0-John domain with base point x_0, given in [Fe], is appropriate. In fact, for that class of domains, by means of [Fe, Lemma 8], we can deduce that

$$\mu_{1/2}(\Omega)^{-1} \le k_N \, |\Omega|^{\frac{1}{N}} \, L_0^{2N}$$

and, by the definition, it is not difficult to prove the following bound:

$$L_0 \le \frac{d_\Omega}{\min[r_i, \delta_\Gamma(x_0)]}.$$

Thus, we obtain:

$$(3.16) \qquad \mu_{1/2}(\Omega)^{-1} \le k_N \, |\Omega|^{\frac{1}{N}} \left(\frac{d_\Omega}{\min[r_i, \delta_\Gamma(x_0)]} \right)^{2N}.$$

Finally, it is clear that we can eliminate the dependence on $|\Omega|$ in (3.15) and (3.16) in favour of d_Ω again by using the isodiametric inequality.

(ii) When Ω is convex, we can further eliminate the dependence on $\delta_\Gamma(x_0)$ appearing in (3.16). In this case, u has a unique minimum point z in Ω, since u is analytic and the level sets of u are convex by a result in [Ko] (see [MS], for a similar argument). Thereby, we have only one choice for the point x_0, that is $x_0 = z$.

Thus, $\delta_\Omega(z)$ may be estimated from below by following arguments established in [BMS]). First, by putting toghether the arguments in [BMS, Theorem 2.7] and [BMS, Remark 2.5], we get that

$$\delta_\Gamma(z) \ge \frac{k_N}{|\Omega| \, d_\Omega^{N-1}} \, \max_{\overline{\Omega}}(-u)$$

and, by a simple comparison argument, the maximum can be bounded from below by $r_i^2/2$. Thus, we have that

$$\delta_\Gamma(z) \geq k_N \, \frac{r_i^{2N}}{|\Omega| \, d_\Omega^{N-1}}$$

and, again, we can replace $|\Omega|$ by d_Ω, owing to the isodiametric inequality. Moreover, in this case, the dependence on r_e for M can be removed, thanks to (3.3).

By similar arguments, we can take care of ε. All in all, we can affirm that C and ε are controlled by r_i and d_Ω, only.

Since the estimates in Theorems 3.6 and 3.7 do not depend on the particular critical point chosen, as a corollary, we obtain results of closeness to a union of balls, similar to [MP, Corollary 4.3]: here, we just illustrate the instance of Theorem 3.6.

Corollary 3.10 (Closeness to an aggregate of balls). *Let Γ, R and u be as in Theorem 3.6.*

Then, there exist points $z_1, \ldots, z_n$ in Ω, $n \geq 1$, and corresponding numbers

$$\rho_i^j = \min_{x \in \Gamma} |x - z_j| \quad and \quad \rho_e^j = \min_{x \in \Gamma} |x - z_j|, \quad j = 1, \ldots, n,$$

such that

$$\bigcup_{j=1}^n B_{\rho_i^j}(z_j) \subset \Omega \subset \bigcap_{j=1}^n B_{\rho_e^j}(z_j)$$

and

$$\max_{1 \leq j \leq n} (\rho_e^j - \rho_i^j) \leq C \, \|u_\nu - R\|_{2,\Gamma}^{2/(N+2)} \quad if \quad \|u_\nu - R\|_{2,\Gamma} < \varepsilon,$$

for some positive constants C and ε.

Proof. We pick one point z_j from each connected component of the set of local minimum points of u. By applying Theorem 3.6 to each z_j, the conclusion is then evident. $\square$

4. Enhanced stability for Alexandrov's Soap Bubble Theorem

In this section, we collect those benefits of the new estimate (3.12) that affect the stability issue for the Soap Bubble Theorem.

We begin by recalling a couple of inequalities concerning the harmonic function h, that we obtained in [MP, Theorem 3.4] and, in the sequel, will play the role of those of Theorem 3.5:

$$(4.1) \qquad \rho_e - \rho_i \leq C_0 \left(\int_\Omega |\nabla^2 h|^2 dx \right)^{1/2},$$

if $N = 2, 3$, and

$$(4.2) \qquad \rho_e - \rho_i \leq C_0 \left(\int_\Omega |\nabla^2 h|^2 dx \right)^{1/(N+2)} \quad \text{for} \quad \int_\Omega |\nabla^2 h|^2 dx < \varepsilon_0^2,$$

that holds for $N \geq 4$. The point z chosen in the definition (2.1) of ρ_i and ρ_e is any critical point of u, as usual. The positive constants C_0 and ε_0 depend on N, $|\Omega|$, d_Ω, r_i r_e, and $\mu_0(\Omega)$. For their values, we refer to [MP, Theorem 3.4].

Remark 4.1. The parameter $\mu_0(\Omega)$ can be estimated by following the arguments used in item (i) of Remark 3.9 to estimate $\mu_{1/2}(\Omega)$. In fact, we deduce that

$$\mu_0(\Omega)^{-1} \leq k_N \, |\Omega|^{\frac{2}{N}} \, L_0^{2N}.$$

Another way to estimate $\mu_0(\Omega)$ can be found in [MP, Remark 3.8 (iii)].

Next, we derive the following lemma, that parallels and is a useful consequence of Lemma 3.4.

Lemma 4.2. *Let $N \geq 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature function for Γ and let H_0 be the constant defined in* (1.8).

Then, the following inequality holds:

$$(4.3) \qquad \|u_\nu - R\|_{2,\Gamma} \leq R \left\{ d_\Omega + \frac{M(M+R)}{r_i} \left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)} \right) \right\} \|H_0 - H\|_{2,\Gamma}.$$

Proof. Discarding the first summand on the left-hand side of (2.6) and applying Hölder's inequality on its right-hand side gives that

$$\frac{1}{R} \|u_\nu - R\|_{2,\Gamma}^2 \leq \|H_0 - H\|_{2,\Gamma} \left(M \|h_\nu\|_{2,\Gamma} + d_\Omega \|u_\nu - R\|_{2,\Gamma} \right),$$

since $u_\nu \leq M$ and $|q_\nu| \leq d_\Omega$ on Γ. Thus, inequality (3.13) implies that

$$\|u_\nu - R\|_{2,\Gamma}^2 \leq$$
$$R \left\{ d_\Omega + \frac{M(M+R)}{r_i} \left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)} \right) \right\} \|H_0 - H\|_{2,\Gamma} \|u_\nu - R\|_{2,\Gamma},$$

from which (4.3) follows at once. $\qquad\square$

Theorem 4.3 (Stability for the Soap Bubble Theorem). *Let $N \geq 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature function for Γ and let H_0 be the constant defined in* (1.8).

There is a point $z \in \Omega$ such that

(i) if $N = 2$ or $N = 3$, then

$$(4.4) \qquad\qquad \rho_e - \rho_i \leq C \, \|H_0 - H\|_{2,\Gamma},$$

for some positive constant C;

(ii) If $N \geq 4$, then

$$(4.5) \qquad \rho_e - \rho_i \leq C \, \|H_0 - H\|_{2,\Gamma}^{2/(N+2)} \quad if \quad \|H_0 - H\|_{2,\Gamma} < \varepsilon,$$

for some positive constants C and ε.

Proof. Discarding the second summand on the left-hand side of (2.6) and applying Hölder's inequality on its right-hand side, as in the previous proof, gives that

$$\frac{1}{N-1} \int_\Omega |\nabla^2 h|^2 dx \leq$$
$$R \left\{ d_\Omega + \frac{M(M+R)}{r_i} \left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)} \right) \right\} \|H_0 - H\|_{2,\Gamma} \|u_\nu - R\|_{2,\Gamma}, \leq$$
$$R^2 \left\{ d_\Omega + \frac{M(M+R)}{r_i} \left(1 + \frac{N}{r_i \, \mu_{1/2}(\Omega)} \right) \right\}^2 \|H_0 - H\|_{2,\Gamma}^2,$$

where the second inequality follows from Lemma 4.2.

Inequalities (4.4) and (4.5) then result from (4.1) and (4.2). $\qquad\square$

Remark 4.4 (On the constants C and ε). Needless to say, the proof of Theorem 4.3 tells us that, by proceeding as in Remarks 3.8, 3.9, and 4.1, we can reduce the dependence of C and ε to the parameters N, r_i, r_e, d_Ω, and $\delta_\Gamma(z)$, if Γ is of class C^2, and to N, r_i, and d_Ω, if Ω is also convex.

Remark 4.5. Assertion (ii) of Theorem 4.3 can also be proved as a direct corollary of Theorem 3.6, by noting that relation (4.3) together with (3.12) gives (4.5).

Remark 4.6. Analogs of Theorem 3.7 and Corollary 3.10 can be easily derived by following the steps of their proofs.

Remark 4.7. The assumption of smallness of the relevant deviation η required in Theorems 3.6, 3.7, Corollary 3.10 and in (ii) of Theorem 4.3 is only apparent, because if $\eta \geq \varepsilon$, then it is a trivial matter to obtain an upper bound for $\rho_e - \rho_i$ in terms of η. Thus, all the stability estimates that we presented are *global*.

Another consequence of Lemma 4.2 is the following inequality that shows an *optimal* stability exponent for any $N \geq 2$. The number $\mathcal{A}(\Omega)$, defined in (1.13), is some sort of asymmetry similar to the so-called *Fraenkel asymmetry* (see [Fr]).

Theorem 4.8 (Stability by asymmetry)**.** *Let $N \geq 2$ and let Γ be the connected boundary of class C^2 of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by H the mean curvature function for Γ and let H_0 be the constant defined in (1.8).*
    Then it holds that

$$(4.6) \qquad \mathcal{A}(\Omega) \leq C \, \|H_0 - H\|_{2,\Gamma},$$

for some positive constant C.

Proof. We use [Fe, inequality (2.14)]: we have that

$$\frac{|\Omega \Delta B_R^z|}{|B_R^z|} \leq C \, \|u_\nu - R\|_{2,\Gamma},$$

where B_R^z is a ball of radius R (as defined in (1.8)) and centered at a minimum point z of u. Hence, we obtain that

$$\mathcal{A}(\Omega) \leq C \, \|u_\nu - R\|_{2,\Gamma},$$

by the definition (1.13). Thus, thanks to (4.3), we obtain (4.6).

Here, if we estimate L_0 as done in Remark 3.9, we can see that C depends on $N, d_\Omega, r_i, |\Gamma|/|\Omega|$, and $\delta_\Gamma(z)$ if Γ is of class C^2; the dependence on the last parameter can be removed when Ω is also convex. $\qquad\square$

Remark 4.9 (On the asymmetry $\mathcal{A}(\Omega)$)**.** Notice that, for any $x \in \Omega$, we have that

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \leq \frac{|B_{\rho_e}^x \setminus B_{\rho_i}^x|}{|B_R^x|} = \frac{\rho_e^N - \rho_i^N}{R^N} \leq \frac{N \, \rho_e^{N-1}}{R^N} \, (\rho_e - \rho_i),$$

and $\rho_e \leq d_\Omega$. Thus, if d_Ω/R remains bounded and $(\rho_e - \rho_i)/R$ tends to 0, then the ratio $|\Omega \Delta B_R^x|/|B_R^x|$ does it too.

The converse is not true in general. For example, consider a lollipop made by a ball and a stick with fixed length L and vanishing width; as that width vanishes, the ratio d_Ω/R remains bounded, while $|\Omega \Delta B_R^x|/|B_R^x|$ tends to zero and $\rho_e - \rho_i \geq L > 0$.

If we fix r_i and r_e, we have the following result.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain satisfying the uniform interior and exterior sphere conditions with radii r_i and r_e.*
    Then, we have that

$$\rho_e - \rho_i \leq 4 \, R \, \mathcal{A}(\Omega)^{\frac{1}{N}} \quad \text{if} \quad \mathcal{A}(\Omega) \leq \left(\frac{r_i}{R}\right)^N.$$

Proof. Let x be any point in Ω. It is clear that

$$(4.7) \qquad \max(\rho_e - R, R - \rho_i) \geq \frac{\rho_e - \rho_i}{2}.$$

If that maximum is $\rho_e - R$, at a point y where the ball centered at x with radius ρ_e touches Γ, we consider the interior touching ball B_{r_i} .

If $2r_i < (\rho_e - \rho_i)/2$ then $B_{r_i} \subset \Omega \setminus B_R^x$ and hence

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \left(\frac{r_i}{R}\right)^N.$$

If, else, $2r_i \geq (\rho_e - \rho_i)/2$, B_{r_i} contains a ball of radius $(\rho_e - \rho_i)/4$ still touching Γ at y. Such a ball is contained in $\Omega \setminus B_R^x$, and hence

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \left(\frac{\rho_e - \rho_i}{4\,R}\right)^N.$$

Thus, we proved that

$$\frac{|\Omega \Delta B_R^x|}{|B_R^x|} \geq \min\left\{\left(\frac{r_i}{R}\right)^N, \left(\frac{\rho_e - \rho_i}{4\,R}\right)^N\right\},$$

and the conclusion easily follows, since x was arbitrarily chosen in Ω.

If, else, the maximum in (4.7) is $R - \rho_i$, we proceed similarly, by reasoning on the exterior ball B_{r_e} and $\mathbb{R}^N \setminus \overline{\Omega}$, instead. $\square$

Remark 4.11. Theorem 4.10 and (4.6) give the inequality

$$\rho_e - \rho_i \leq C\,\|H_0 - H\|_{2,\Gamma}^{1/N}$$

that, for any $N \geq 2$, is poorer than what obtained in 4.3.

Acknowledgements

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