Journal of Functional Analysis 285 (2023) 110091

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

Wulff shape symmetry of solutions to overdetermined problems for Finsler Monge-Ampère equations

Andrea Cianchi *, Paolo Salani

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy

A R T I C L E   I N F O

Article history:
Received 7 September 2022
Accepted 27 June 2023
Available online 13 July 2023
Communicated by Guido De Philippis

MSC:
35J06
35J96

Keywords:
Anisotropic Monge-Ampère equation
Overdetermined boundary conditions
Symmetry of solutions
Wulff shape

A B S T R A C T

We deal with Monge-Ampère type equations modeled upon general Finsler norms H in $\mathbb{R}^n$. An overdetermined problem for convex solutions to these equations is analyzed. The relevant solutions are subject to both a homogeneous Dirichlet condition and a second boundary condition, designed on H, on the gradient image of the domain. The Wulff shape symmetry associated with H of the solutions is established.

© 2023 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction and main result

The standard Monge-Ampère operator is formally defined, for a real-valued function on an open set $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, as

* Corresponding author.
E-mail addresses: andrea.cianchi@unifi.it (A. Cianchi), paolo.salani@unifi.it (P. Salani).

https://doi.org/10.1016/j.jfa.2023.110091
0022-1236/© 2023 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

$$Mu = \det(\nabla^2 u). \tag{1.1}$$

Here, $\nabla^2 u$ denotes the Hessian $n \times n$–matrix of the second-order derivatives of u and "det" stands for determinant. The operator M lies at one endpoint of a family of so-called Hessian operators, whose opposite terminal is occupied by the Laplacian. Besides the usual divergence form $\Delta u = \operatorname{div}(\nabla u)$, the latter can be expressed as

$$\Delta u = \operatorname{tr}(\nabla^2 u), \tag{1.2}$$

where "tr" denotes trace.

The structure of these classical operators is in a sense related to the use of the Euclidean norm in the ambient space $\mathbb{R}^n$. For instance, the Laplacian emerges from the Euler equation of the Dirichlet energy integral, defined in terms of the Euclidean norm $|\nabla u|$ of the gradient ∇u of u. Replacing this norm with a more general Finsler norm $H : \mathbb{R}^n \to [0, \infty)$ results in the functional

$$\int_\Omega E(\nabla u)\, dx, \tag{1.3}$$

where $E : \mathbb{R}^n \to [0, \infty)$ is the function given by

$$E(\xi) = \tfrac{1}{2} H(\xi)^2 \quad \text{for } \xi \in \mathbb{R}^n. \tag{1.4}$$

Recall that a Finsler norm in $\mathbb{R}^n$ is a nonnegative convex function which vanishes only at 0 and is positively homogeneous of degree 1. Hence, unlike standard norms, it need not be an even function. Of course, any norm in $\mathbb{R}^n$ is, in particular, a Finsler norm.

The Finsler Laplacian built upon H can be defined as the differential operator $\Delta_H u = \operatorname{div}\big(H(\nabla u)\nabla_\xi H(\nabla u)\big)$ appearing in the Euler equation of functional (1.3). Here, and in what follows, the subscript ξ attached to a differential operator denotes differentiation in the "gradient variable". With this regard, observe that

$$-\int_\Omega u\Delta_H u\, dx = 2\int_\Omega E(\nabla u)\, dx \tag{1.5}$$

provided that u vanishes on $\partial\Omega$. In analogy with (1.2), the operator Δ_H admits the alternate form

$$\Delta_H u = \operatorname{tr}\big(\nabla(\nabla_\xi E(\nabla u))\big). \tag{1.6}$$

Plainly, definition (1.2) is recovered from (1.6) when H is the Euclidean norm, since in this case $\nabla_\xi E(\xi) = \xi$ for $\xi \in \mathbb{R}^n$.

The operator Δ_H and its p-generalization, obtained analogously after replacing the exponent 2 by any $p \in (1, \infty)$ in functional (1.3), have been investigated under various aspects. A sample of contributions on this subject is furnished by [1–3,6,23,24,26,27,25,30,41].

The Finsler Monge-Ampère operator M_H is defined as

$$M_H u = \det\big(\nabla(\nabla_\xi E(\nabla u))\big). \tag{1.7}$$

Besides being suggested by the mere replacement of the trace with the determinant on the right-hand side of equation (1.6), definition (1.7) originates from the Euler equation of the functional

$$\int_\Omega E(\nabla u)^{\frac{n+1}{2}} k_H(x)\, dx, \tag{1.8}$$

where $k_H(x)$ denotes the Finsler Gauss curvature of the level set of u at the point x. This assertion can be verified via the identity

$$-\int_\Omega u\, M_H u\, dx = \frac{2^{\frac{n+1}{2}}}{n} \int_\Omega E(\nabla u)^{\frac{n+1}{2}} k_H(x)\, dx, \tag{1.9}$$

which holds for sufficiently smooth convex functions u vanishing on $\partial\Omega$ and sufficiently smooth Finsler norms H. When H is the Euclidean norm, the integral on the left-hand side is the customary energy functional of the Monge-Ampère operator and equation (1.9) is classical. Functional (1.8) thus provides us with a natural generalization to the Finsler realm and stands to the operator M_H as functional (1.3) stands to Δ_H. We refer to [20,21] for the derivation of equation (1.9) and properties of the operator M_H in connection with ad hoc symmetrizations.

Formally, $\det\big(\nabla(\nabla_\xi E(\nabla u))\big) = \det\big(\nabla^2_\xi E(\nabla u)\nabla^2 u\big) = \det\big(\nabla^2_\xi E(\nabla u)\big)\det(\nabla^2 u)$. Thereby, M_H can be regarded as a classical Monge-Ampère operator with a gradient-depending coefficient, whose special structure depends on the Finsler norm H. We prefer to introduce it in the form (1.7), which allows for a definition of generalized solution in the sense of Alexandrov without any a priori regularity assumption on H.

Our focus here is on the symmetry of the solution to an overdetermined boundary value problem for the operator M_H. The interest in symmetry properties of solutions to overdetermined boundary value problems for partial differential equations was ignited half a century ago by the seminal paper [38] by Serrin. A special case of his result concerns the Poisson equation, coupled with both a homogeneous Dirichlet condition and a constant Neumann condition at the boundary. It asserts that, if Ω is bounded and sufficiently smooth, and c is any positive constant, then the problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ |\nabla u| = c & \text{on } \partial\Omega \end{cases} \tag{1.10}$$

admits a solution if and only if Ω is (up to translations and dilations) the Euclidean unit ball B centered at 0. Moreover,

$$u(x) = \frac{|x|^2 - 1}{2} \qquad \text{for } x \in B. \tag{1.11}$$

Over the years, Serrin's result has inspired a wealth of investigations on related questions – see e.g. the surveys [31,33] on developments along this line of research. In particular, overdetermined boundary value problems for a family of Hessian-type equations are the subject of [9]. The results of that paper include convex solutions to the Monge-Ampère equation, with a constant right-hand side, on a bounded convex set Ω and subject to the same Dirichlet and Neumann boundary conditions as in (1.10). Observe that, because of the convexity of Ω and u, the latter condition is equivalent to requiring that

$$\nabla u(\Omega) = B(c), \tag{1.12}$$

where $B(c)$ denotes the Euclidean ball, centered at 0, with radius c. This is a special case of the "second boundary condition" in the theory of the Monge-Ampère operator, so called as opposed to the Dirichlet boundary condition. In its general formulation, it amounts to imposing that $\nabla u(\Omega) = \Omega'$ for some bounded convex set Ω'. It is also named "natural boundary condition", inasmuch as it arises naturally in the solution to the Monge-Kantorovich mass transportation problem.

The conclusion of [9] is that, if the problem

$$\begin{cases} Mu = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \nabla u(\Omega) = B(c) \end{cases} \tag{1.13}$$

admits a convex solution in a bounded convex domain Ω, for some positive constant c, then (up to translations and dilations) the domain agrees with the Euclidean unit ball B, and the solution u obeys (1.11). An alternate proof is offered in [10]. An extension of this result to overdetermined problems for a class of fully nonlinear elliptic equations more general than that considered in [10] can be found in [39], and rests upon a different approach.

The analysis of overdetermined problems in the Finsler ambient was initiated in [13], where a version of Serrin's theorem was established for (sufficiently smooth) Finsler norms H. Loosely speaking, it tells us that a symmetry result still holds, provided that the role of Euclidean balls is replaced by balls in the Finsler norm H in the "gradient variable" and balls in the dual Finsler norm H_0 in the "space variable". Balls according to H_0 are usually said to have the Wulff shape associated with H. This terminology comes after G. Wulff, who, at the beginning of the last century, employed anisotropic geometric functionals built upon general Finsler norms H in his mathematical theory of crystals [43]. The functionals in question replace the standard perimeter of sets in $\mathbb{R}^n$. They are defined as the integral, over the boundary of a set, of the function H evaluated at the unit normal vector. Of course, the boundary and the unit normal vector of a set have to be properly defined according to geometric measure theory. Wulff-shaped balls are known to solve the corresponding isoperimetric problem among sets of prescribed Lebesgue measure [40].

Specifically, the paper [13] concerns the problem

$$\begin{cases} \Delta_H u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ H(\nabla u) = c & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

Under suitable regularity assumptions on the bounded domain Ω, it asserts that a solution to problem (1.14) exists if and only if $\Omega = B_{H_0}$ (up to translations and dilations). Moreover, the solution u is obtained on replacing the norm $|x|$ with $H_0(x)$ in equation (1.11). Here, B_{H_0} denotes the unit ball, centered at 0, in the metric of H_0; balls with radius $c > 0$ will be denoted by $B_{H_0}(c)$. Analogous notations are adopted for balls in the Finsler metric of H. To be precise, the result of [13] is stated in the case when H is a classical norm (hence, an even function) and the right-hand side equals -1; however, a close inspection of the proof shows that it also applies to deduce the above conclusion about problem (1.14) for Finsler norms.

In the last decade, further contributions appeared on overdetermined boundary value problems for the Finsler Laplacian and its p-Laplacian version and, more generally, on symmetry properties of solutions to problems involving these operators. A partial list includes [5,7,8,14–16,19,22].

In the present paper, we complement the picture outlined above and show the Wulff shape symmetry of the solution to the Finsler Monge-Ampère equation, when simultaneously subject to the homogeneous Dirichlet condition and a Wulff shape symmetric second boundary condition. This is the content of the following theorem. In its statement, the notation C^2_+ denotes the class of twice continuously differentiable functions whose Hessian matrix is everywhere positive definite.

Theorem 1.1. *Let Ω be a convex bounded open set in $\mathbb{R}^n$. Let H be a Finsler norm in $\mathbb{R}^n$ such that $H^2 \in C^2_+(\mathbb{R}^n \setminus \{0\})$. Assume that there exists an Alexandrov convex solution u to the problem*

$$\begin{cases} M_H u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \nabla u(\Omega) = B_H(c) \end{cases} \tag{1.15}$$

for some constant $c > 0$. Then,

$$\Omega = B_{H_0} \tag{1.16}$$

and

$$u(x) = \frac{H_0(x)^2 - 1}{2} \qquad \textit{for } x \in B_{H_0}, \tag{1.17}$$

up to translations and dilations.

As observed above, owing to the convexity of Ω and u, the last condition in problem (1.15) can be equivalently formulated as the boundary condition

$$H(\nabla u) = c \qquad \text{on } \partial\Omega.$$

Let us mention that overdetermined problems parallel to (1.15), for Hessian type operators modeled upon H which are intermediate between Δ_H and M_H, could be considered. A definition of these operators can be found in [21]. Results along this direction would extend those of [10] for classical Hessian equations. However, this falls beyond the scope of the present work and we leave it for possible future investigations.

Our approach to Theorem 1.1 departs from the original technique employed in [38], which is based on a variant of the method of moving planes. This method was introduced by Alexandrov in his proof of the symmetry of bodies whose boundary has constant mean curvature and, after [38], it was adapted to the proof of a variety of symmetry results for PDEs. We instead resort to arguments ultimately rooted in an alternate proof of Serrin's result by Weinberger [42] and later developed in [9]. They rely upon integral identities and inequalities involving the solution u to problem (1.15). The overall idea is that overdetermination causes all the inequalities to hold as equalities. This piece of information forces ∇u to agree with the gradient of the right-hand side of equation (1.17), whence the conclusion follows.

Besides other ingredients, the derivation of the relevant integral relations makes use of duality arguments pertaining to the theory of convex functions and sets. Specific properties linking the Finsler norm H with its dual H_0, and the solution u with its Young conjugate $\widetilde{u}$ enter the game. The regularity of the solution u is critical in substantiating several steps of the argument. The nowadays classical $C^{1,\alpha}$ regularity theory by Caffarelli, as well as the $W^{2,1}$ regularity theory more recently inaugurated by De Philippis and Figalli, plays a key role in this connection.

The paper is organized as follows. We begin by gathering definitions and some properties of the functions H and H_0. The notion of Legendre conjugate and its basic properties are also recalled. This is the content of Section 2. Section 3 is devoted to a precise formulation of problem (1.15). The definition of its solution in the framework of the theory of Monge-Ampère type equations is discussed in that section, where its regularity properties of use for our analysis are presented as well. The proof of Theorem 1.1 is then accomplished in Section 4.

2. Convex functions and Finsler norms

2.1. Legendre conjugate

Let Ω be a convex set and let $u : \Omega \to \mathbb{R}$ be a convex function. Its Legendre conjugate $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\widetilde{u}(\xi) = \sup\{x \cdot \xi - u(x) \, : \, x \in \Omega\} \qquad \text{for } \xi \in \mathbb{R}^n. \tag{2.1}$$

If the function $u \in C^1(\Omega)$ and is strictly convex, then $\widetilde{u} \in C^1(\nabla u(\Omega))$ and is strictly convex (see [34, Theorem 26.5]). Moreover the map $\nabla u : \Omega \to \nabla u(\Omega)$ is invertible and

$$\nabla_\xi \widetilde{u} = (\nabla u)^{-1} . \tag{2.2}$$

The very definition of the function $\widetilde{u}$ entails that

$$x \cdot \xi \leq u(x) + \widetilde{u}(\xi) \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \tag{2.3}$$

Furthermore, if $u \in C^1(\Omega)$ and is strictly convex, then

$$x \cdot \xi = u(x) + \widetilde{u}(\xi) \quad \text{if either } x = \nabla_\xi \widetilde{u}(\xi) \text{ or } \xi = \nabla u(x). \tag{2.4}$$

2.2. Finsler norms in $\mathbb{R}^n$

A function $H : \mathbb{R}^n \to [0, \infty)$ is called a Finsler norm in $\mathbb{R}^n$ if:

$$H \text{ is convex,} \tag{2.5}$$

$$H(\xi) \geq 0 \text{ for } \xi \in \mathbb{R}^n, \text{ and } H(\xi) = 0 \text{ if and only if } \xi = 0\,, \tag{2.6}$$

$$H(t\xi) = tH(\xi) \text{ for } \xi \in \mathbb{R}^n \text{ and } t \geq 0\,. \tag{2.7}$$

Notice that, owing to (2.7), assumption (2.5) can be equivalently replaced by requiring that H be subadditive. The dual Finsler norm H_0 of H is given by

$$H_0(x) = \max_{\xi \neq 0} \frac{x \cdot \xi}{H(\xi)} \qquad \text{for } x \in \mathbb{R}^n, \tag{2.8}$$

where the dot " $\cdot$ " stands for scalar product in $\mathbb{R}^n$. Conversely, H is the dual Finsler norm of H_0, since

$$H(\xi) = \max_{x \neq 0} \frac{x \cdot \xi}{H_0(x)} \qquad \text{for } \xi \in \mathbb{R}^n. \tag{2.9}$$

As mentioned above, given $r > 0$, we denote by $B_H(r)$ the open ball, centered at 0 and with radius r, in the Finsler metric generated by H. Namely,

$$B_H(r) = \{\xi \in \mathbb{R}^n : H(\xi) < r\}.$$

The ball $B_{H_0}(r)$ is defined analogously. If $r = 1$, we shall simply write B_H and B_{H_0}. One has that

$$H_0 \in C^1(\mathbb{R}^n \setminus \{0\}) \text{ if and only if } B_H \text{ is strictly convex,} \tag{2.10}$$

see [36, Corollary 1.7.3]. Of course, an analogous property holds on interchanging the roles of H and H_0.
If $H \in C^1(\mathbb{R}^n \setminus \{0\})$, then

$$\nabla_\xi H(t\xi) = \nabla_\xi H(\xi) \quad \text{for } \xi \neq 0 \text{ and } t > 0\,. \tag{2.11}$$

This is a consequence of property (2.7), Moreover,

$$\xi \cdot \nabla_\xi H(\xi) \, = H(\xi) \quad \text{for } \xi \in \mathbb{R}^n\,, \tag{2.12}$$

where the left-hand side is taken to be 0 when $\xi = 0$.
Assume that $H \in C^1(\mathbb{R}^n \setminus \{0\})$ and B_H is strictly convex. Then, by [13, Lemma 3.1],

$$H_0(\nabla_\xi H(\xi)) = 1 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}\,, \tag{2.13}$$

and

$$H(\nabla H_0(x)) = 1 \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}\,. \tag{2.14}$$

Moreover, under the same assumptions, the map $H\nabla_\xi H : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and

$$H\nabla_\xi H = (H_0\nabla H_0)^{-1}\,. \tag{2.15}$$

Here, and in what follows, $H\nabla_\xi H$ and $H_0\nabla H_0$ are continued by 0 at 0.
If $H \in C^2(\mathbb{R}^n \setminus \{0\})$, then

$$\nabla^2_\xi H(t\xi) = \frac{1}{t}\nabla^2_\xi H(\xi) \quad \text{for } \xi \neq 0 \text{ and } t > 0\,, \tag{2.16}$$

and

$$\nabla^2_\xi H^2(t\xi) = \nabla^2_\xi H^2(\xi) \quad \text{for } \xi \neq 0 \text{ and } t > 0. \tag{2.17}$$

Observe that the matrix-valued function $\nabla^2_\xi H^2$ is discontinuous at 0, unless it is constant. Yet, it is bounded, and hence

$$H\nabla_\xi H \text{ is Lipschitz continuous in } \mathbb{R}^n, \tag{2.18}$$

inasmuch as $\nabla_\xi H^2 = 2H\nabla_\xi H$. Of course, a parallel property holds for H_0, provided that $H_0 \in C^2(\mathbb{R}^n \setminus \{0\})$; namely

$$H_0 \nabla H_0 \text{ is Lipschitz continuous in } \mathbb{R}^n. \tag{2.19}$$

The properties enjoyed by the functions H and H_0 mentioned so far are reflected in properties of the function E defined by equation (1.4) and of the function E_0 defined by replacing H with H_0 in the same equation. In particular, note that, by equations (2.7) and (2.13),

$$E_0(\nabla_\xi E(\xi)) = E(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.20}$$

Thanks to equation (2.12),

$$\xi \cdot \nabla_\xi E(\xi) = 2E(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.21}$$

Furthermore, if $H^2 \in C^2_+(\mathbb{R}^n \setminus \{0\})$, i.e. $E \in C^2_+(\mathbb{R}^n \setminus \{0\})$, then

$$\nabla^2_\xi E(t\xi) = \nabla^2_\xi E(\xi) \quad \text{for } \xi \neq 0 \text{ and } t > 0. \tag{2.22}$$

Hence, there exist constants $\Lambda > \lambda > 0$ such that

$$\lambda \leq |\nabla^2_\xi E(\xi)| \leq \Lambda \quad \text{for } \xi \neq 0, \tag{2.23}$$

and

$$\lambda \leq \det(\nabla^2_\xi E(\xi)) \leq \Lambda \quad \text{for } \xi \neq 0. \tag{2.24}$$

Moreover, since

$$\nabla_\xi E = H \nabla_\xi H \quad \text{and} \quad \nabla E_0 = H_0 \nabla H_0,$$

the maps

$$\nabla_\xi E : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad \nabla E_0 : \mathbb{R}^n \to \mathbb{R}^n \quad \text{are Lipschitz continuous,} \tag{2.25}$$

and

$$(\nabla_\xi E)^{-1} = \nabla E_0. \tag{2.26}$$

Especially,

$$\nabla_\xi E : B_H \to B_{H_0} \quad \text{and} \quad \nabla E_0 : B_{H_0} \to B_H. \tag{2.27}$$

One can also verify that E and E_0 are mutual Legendre conjugates.
A property of Legendre conjugation ensures that, if $E \in C^2_+(\mathbb{R}^n \setminus \{0\})$, then

$$E_0 \in C^2_+(\mathbb{R}^n \setminus \{0\}). \tag{2.28}$$

3. Properties of solutions to Finsler Monge-Ampère equations

Let H be any Finsler norm in $\mathbb{R}^n$. A convex function $u : \Omega \to \mathbb{R}$ is a generalized solution in the sense of Alexandrov to the equation

$$M_H = 1 \qquad \text{in } \Omega \tag{3.1}$$

if

$$\mathcal{L}^n\big(\nabla_\xi E(\nabla u(\omega))\big) = \mathcal{L}^n(\omega) \tag{3.2}$$

for every Borel set $\omega \subset \Omega$. Here, $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$, and ∇u and $\nabla_\xi E$ are regarded as multi-valued maps, which are well defined as subgradients, since both u and E are convex functions. We refer to the monographs [4] and [28] for detailed expositions of the classical theory of Alexandrov solutions to Monge-Ampère type equations.
Assume now that H satisfies the additional assumptions of Theorem 1.1. Then u is a solution to equation (3.1) in the sense of Alexandrov if and only if it is an Alexandrov solution to the equation

$$\Phi(\nabla u)\det(\nabla^2 u) = 1 \quad \text{in } \Omega, \tag{3.3}$$

where $\Phi : \mathbb{R}^n \to [0,\infty)$ is the function defined as

$$\Phi(\xi) = \begin{cases} \det(\nabla^2_\xi E(\xi)) & \text{if } \xi \neq 0 \\ \inf_{\{\eta\neq 0\}} \det(\nabla^2_\xi E(\eta)) & \text{if } \xi = 0. \end{cases} \tag{3.4}$$

Recall that a convex function u is an Alexandrov solution to equation (3.3) if

$$\int\limits_{\nabla u(\omega)} \Phi(\xi)\, d\xi = \mathcal{L}^n(\omega) \tag{3.5}$$

for every Borel set $\omega \subset \Omega$. In order to verify this equivalence, notice that, by property (2.25), the map $\nabla_\xi E : \mathbb{R}^n \to \mathbb{R}^n$ is bi-Lipschitz continuous. A change of variables via this map yields

$$\int\limits_{\nabla u(\omega)} \Phi(\xi)\, d\xi = \int\limits_{\nabla u(\omega)} \det(\nabla^2_\xi E(\xi))\, d\xi = \int\limits_{\nabla_\xi E(\nabla u(\omega))} dx = \mathcal{L}^n\big(\nabla_\xi E(\nabla u(\omega))\big) \tag{3.6}$$

for every Borel set $\omega \subset \Omega$. Hence, the claimed equivalence follows.

Next, let Ω be a bounded convex domain and consider the problem obtained by coupling equation (3.1) with the second boundary condition $\nabla u(\Omega) = B_H$. In view of the above remarks, it can be formulated as

$$\begin{cases} \Phi(\nabla u)\det(\nabla^2 u) = 1 & \text{in } \Omega \\ \nabla u(\Omega) = B_H. \end{cases} \tag{3.7}$$

Besides the definition in the Alexandrov sense, another (even weaker) definition of a solution to the equation in (3.7) is available. It was introduced by Brenier in his fundamental work on the Monge-Kantorovich mass transportation problem [11]. A convex function u is a Brenier solution to the equation in (3.7) if

$$\int_{\nabla u(\Omega)} h(\xi)\Phi(\xi)\,d\xi = \int_{\Omega} h(\nabla u(x))\,dx \tag{3.8}$$

for every continuous function $h : \nabla u(\Omega) \to \mathbb{R}$, where ∇u is now regarded as a function defined a.e. in Ω. Since, by equation (2.24), there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda \leq \Phi(\xi) \leq \Lambda \quad \text{for } \xi \in \mathbb{R}^n, \tag{3.9}$$

the result of [11] guarantees that problem (3.7) admits a unique solution in this sense, provided that the compatibility condition

$$\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_{H_0}),$$

dictated by the choice $h = 1$ in (3.8), is fulfilled.

A result from [12] provides us with the following information:

(i) The Brenier solution to problem (3.7) is also a solution in the sense of Alexandrov; hence, we may refer to the (unique) solution u to problem (3.7) without further specification in what follows;

(ii) The Legendre conjugate $\widetilde{u}$ of u is the Brenier and the Alexandrov solution to the problem

$$\begin{cases} \det(\nabla^2 \widetilde{u}) = \Phi(\xi) & \text{in } B_H \\ \nabla \widetilde{u}(B_H) = \Omega. \end{cases} \tag{3.10}$$

Hence,

$$\int_{\Omega} f(x)\,dx = \int_{B_H} f(\nabla \widetilde{u}(\xi))\Phi(\xi)\,d\xi \tag{3.11}$$

for every continuous function $f : \Omega \to \mathbb{R}$, where $\nabla\widetilde{u}$ is regarded as a function defined a.e. in Ω. Moreover,

$$\int_{\varpi} \Phi(\xi)\, d\xi = \mathcal{L}^n\big(\nabla\widetilde{u}(\varpi)\big) \tag{3.12}$$

for every Borel set $\varpi \subset B_H$, where $\nabla\widetilde{u}$ is considered a multi-valued map;
(iii) u and $\widetilde{u}$ are strictly convex;
(iv) $u \in C^{1,\alpha}(\overline{\Omega})$ and $\widetilde{u} \in C^{1,\alpha}(\overline{B_H})$ for some $\alpha > 0$; consequently, ∇u and $\nabla\widetilde{u}$ are, in fact, single-valued maps, and $\nabla u : \Omega \to B_H$ and $\nabla\widetilde{u} : B_H \to \Omega$ are inverses of each other.
In particular, the function u fulfills the equation in (3.7) a.e. in Ω, and $\widetilde{u}$ fulfills the equation in (3.10) a.e. in B_H.
Let us also notice that, thanks to equation (2.26),

$$\Phi(\nabla E_0)\det(\nabla^2 E_0) = \det(\nabla^2_\xi E(\nabla E_0))\det(\nabla^2 E_0) = 1 \quad \text{a.e in } B_{H_0}, \tag{3.13}$$

and, as a consequence of equation (3.11) and the change of variable formula for Lipschitz maps,

$$\begin{aligned} \int_{\Omega} f(x)\, dx &= \int_{B_H} f(\nabla\widetilde{u}(\xi))\Phi(\xi)\, d\xi \\ &= \int_{B_{H_0}} f(\nabla\widetilde{u}(\nabla E_0(y))\det(\nabla^2_\xi E(\nabla E_0(y)))\det(\nabla^2 E_0(y))\, dy \\ &= \int_{B_{H_0}} f(\nabla\widetilde{u}(\nabla E_0(y)))\, dy \end{aligned} \tag{3.14}$$

for every continuous function $f : \Omega \to \mathbb{R}$.
Since the functions

$$\nabla_\xi E(\nabla u) : \Omega \to B_{H_0} \quad \text{and} \quad \nabla\widetilde{u}(\nabla E_0) : B_{H_0} \to \Omega$$

are inverses of each other, an application of equation (3.14) with $f = g(\nabla_\xi E(\nabla u))$ yields

$$\int_{B_{H_0}} g(y)\, dy = \int_{\Omega} g(\nabla_\xi E(\nabla u(x)))\, dx \tag{3.15}$$

for every continuous function $g : B_{H_0} \to \mathbb{R}$.
Inasmuch as the function $\widetilde{u}$ solves problem (3.10) and the function Φ fulfills inequalities (3.9), [35, Theorem 1.1] – a global version for second boundary value problems for Monge-Ampère equations of results of [17], [18] and [37] – ensures that

$$\widetilde{u} \in W^{2,1}(B_H). \tag{3.16}$$

Moreover, we claim that

$$u \in W^{2,1}(\Omega). \tag{3.17}$$

This is again a consequence of [35, Theorem 1.1], once one notices that u is also an Alexandrov solution to the problem

$$\begin{cases} \det(\nabla^2 u) = \psi(x) & \text{in } \Omega \\ \nabla u(\Omega) = B_H, \end{cases} \tag{3.18}$$

where the function $\psi : \Omega \to [0, \infty)$, defined as

$$\psi(x) = \frac{1}{\Phi(\nabla u(x))} \quad \text{for } x \in \Omega,$$

satisfies the inequalities $\frac{1}{\Lambda} \leq \psi(x) \leq \frac{1}{\lambda}$ for $x \in \Omega$. The fact that u is an Alexandrov solution to problem (3.18) amounts to the equation

$$\mathcal{L}^n(\nabla u(\omega)) = \int_\omega \psi(x)\, dx \tag{3.19}$$

being fulfilled for every Borel set $\omega \subset \Omega$. Equation (3.19) formally follows from (3.8) with the choice

$$h(\xi) = \frac{\chi_{\nabla u(\omega)}(\xi)}{\Phi(\xi)} \quad \text{for } \xi \in B_H,$$

where $\chi_{\nabla u(\omega)}$ stands for the characteristic function of the set $\nabla u(\omega)$. A rigorous proof can be accomplished through a standard approximation argument for h via convolutions with smooth mollifiers. Note that passage to the limit is justified by the dominated convergence theorem. A role is also played by the fact that, as a consequence of equation (3.5), $\mathcal{L}^n((\nabla u)^{-1}(K)) = 0$ for any set $K \subset B_H$ such that $\mathcal{L}^n(K) = 0$.
Let us mention that [35, Theorem 1.1] ensures that the space $W^{2,1}$ can even be replaced by $W^{2,1+\varepsilon}$ for some $\varepsilon > 0$ in equations (3.16) and (3.17). This additional piece of information will however not be exploited in our proofs.

4. Proof of Theorem 1.1

In this section, we accomplish the proof of Theorem 1.1. A couple of steps rely upon a generalized version of Newton's inequality for matrices. Since we have not been able to locate the relevant inequality in the literature, we state it in a lemma at the end of the section and provide a proof for completeness.

Proof of Theorem 1.1. Up to rescaling in x and u, we may assume that $c = 1$ in problem (1.15). Thus, in the light of the discussion in Section 3, we may assume that u is the Alexandrov solution to the problem

$$\begin{cases} \Phi(\nabla u)\det(\nabla^2 u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ E(\nabla u) = \frac{1}{2} & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Φ is the function defined by (3.4).

The proof is split into several steps.

Step 1. We have that

$$\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_{H_0})\,. \tag{4.2}$$

This is a consequence of equation (3.14) with $f = 1$.

Step2. Owing to property (3.17), we have that $u \in W^{2,1}(\Omega)$. Moreover, by our assumptions on E and property (2.25), the function $\nabla_\xi E \in C^1(\mathbb{R}^n \setminus \{0\}) \cap \mathrm{Lip}(\mathbb{R}^n)$. Hence, the results of [32] on the composition of vector-valued Sobolev functions ensure that

$$\nabla_\xi E(\nabla u) \in W^{1,1}(\Omega) \tag{4.3}$$

and

$$\nabla(\nabla_\xi E(\nabla u)) = \nabla^2_\xi E(\nabla u)\nabla^2 u \quad \text{a.e. in } \Omega. \tag{4.4}$$

Also, inasmuch as the solution u is strictly convex, ∇u vanishes only at the unique minimum point of u, whence

$$\det(\nabla^2_\xi E(\nabla u)\nabla^2 u) = \Phi(\nabla u)\det(\nabla^2 u) \quad \text{a.e. in } \Omega. \tag{4.5}$$

Incidentally, let us notice that equation (4.5) would hold even if u were not convex, since $\nabla^2 u$ vanishes a.e. on a set where ∇u is constant, provided that u is just in $W^{2,1}(\Omega)$.

Recall that the matrix $\nabla^2 u$ is positive semidefinite and the matrix $\nabla^2_\xi E$ is positive definite. Thus, thanks to the generalized version of Newton's inequality of Lemma 4.1 below, to equation (4.5), and to the equation in (4.1), the following chain holds:

$$\Delta_H u = \mathrm{div}(\nabla_\xi E(\nabla u)) = \mathrm{tr}(\nabla^2_\xi E(\nabla u)\nabla^2 u) \geq n\det(\nabla^2_\xi E(\nabla u)\nabla^2 u)^{1/n} = n \quad \text{a.e. in } \Omega. \tag{4.6}$$

Moreover, equality holds in the inequality in (4.6) if and only if

$$\nabla^2_\xi E(\nabla u)\nabla^2 u = I, \tag{4.7}$$

where I denotes the identity matrix in $\mathbb{R}^{n\times n}$.

Step 3. We have that

$$2\int_{B_{H_0}} E_0(y)dy \geq -n\int_\Omega u\,dx\,, \tag{4.8}$$

and equality holds in the inequality if and only if equality (4.7) holds a.e. in Ω. To prove this assertion, one can make use of equation (3.15) to deduce that

$$\int_{B_{H_0}} E_0(y)dy = \int_\Omega E_0(\nabla_\xi E(\nabla u))\,dx.$$

Hence, thanks to equation (2.20),

$$\int_{B_{H_0}} E_0(y)dy = \int_\Omega E_0(\nabla_\xi E(\nabla u))\,dx = \int_\Omega E(\nabla u)\,dx\,. \tag{4.9}$$

The definition of Δ_H, the divergence theorem, equation (2.21) and the Dirichlet boundary condition in (4.1) yield:

$$\begin{aligned} -\int_\Omega u\,\Delta_H u\,dx &= -\int_\Omega \operatorname{div}(u\nabla_\xi E(\nabla u)) - \nabla u\cdot\nabla_\xi E(\nabla u)\,dx \\ &= 2\int_\Omega E(\nabla u)dx - \int_{\partial\Omega} u\nabla_\xi E(\nabla u)\cdot\nu\,d\mathcal{H}^{n-1} = 2\int_\Omega E(\nabla u)dx, \end{aligned} \tag{4.10}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Hence, by equation (4.6),

$$-n\int_\Omega u\,dx \leq 2\int_\Omega E(\nabla u)\,dx. \tag{4.11}$$

Inequality (4.8) follows from (4.11), via equation (4.9). The assertion about the case of equality in (4.8) is a consequence of the case of equality in (4.6).

Step 4. Here, we show that

$$\int_\Omega u\,dx = -\int_{B_{H_0}} \widetilde{u}(\nabla E_0)\,dy + \int_{B_{H_0}} \nabla_\xi\widetilde{u}(\nabla E_0)\cdot\nabla E_0\,dy, \tag{4.12}$$

where $\widetilde{u}$ denotes the Legendre conjugate of u.

Equation (4.12) follows via equation (3.14), which entails that

$$\int_{\Omega} u\,dx = \int_{B_{H_0}} u(\nabla_\xi \widetilde{u}(\nabla E_0))\,dy = \int_{B_{H_0}} \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nabla E_0 - \widetilde{u}(\nabla E_0)\,dy. \tag{4.13}$$

Step 5. We claim that

$$\int_{B_{H_0}} \widetilde{u}(\nabla E_0)\,dy = -(n+1)\int_{\Omega} u\,dx\,. \tag{4.14}$$

Equation (4.14) is a consequence of the following chain:

$$\begin{aligned}\int_{B_{H_0}} \widetilde{u}(\nabla E_0)\,dy &= \int_{\Omega} \widetilde{u}(\nabla u)\,dx = \int_{\Omega} \nabla u\cdot x - u\,dx = \int_{\Omega} \operatorname{div}(ux) - nu - u\,dx \\ &= \int_{\partial\Omega} ux\cdot \nu\,d\mathcal{H}^{n-1} - (n+1)\int_{\Omega} u\,dx = -(n+1)\int_{\Omega} u\,dx.\end{aligned} \tag{4.15}$$

Note that the first equality holds thanks to equations (3.14) and (2.2), the second one by equation (2.4), the fourth one by the divergence theorem, and the last one by the Dirichlet boundary condition in (4.1).

Step 6. The following inequality holds:

$$\int_{B_{H_0}} \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nabla E_0\,dy \geq \frac{n}{2}\left(\mathcal{L}^n(B_{H_0}) - 2\int_{B_{H_0}} E_0\,dy\right). \tag{4.16}$$

By property (3.16), one has that $\widetilde{u} \in W^{2,1}(B_H)$. Also, properties (2.25) and (2.26) ensure that $\nabla E_0 : B_{H_0} \to \Omega$ is a by-Lipschitz map. Hence, the change of variables formula for Sobolev functions tells us that

$$\nabla_\xi \widetilde{u}(\nabla E_0) \in W^{1,1}(B_{H_0}), \tag{4.17}$$

and

$$\nabla(\nabla_\xi \widetilde{u}(\nabla E_0)) = \nabla^2_\xi \widetilde{u}(\nabla E_0)\nabla^2 E_0 \quad \text{a.e. in } B_{H_0}. \tag{4.18}$$

Moreover, by equation (3.13),

$$\det(\nabla^2_\xi \widetilde{u}(\nabla E_0)\nabla^2 E_0) = \frac{\det(\nabla^2 \widetilde{u}(\nabla E_0))}{\Phi(\nabla E_0)} \quad \text{a.e. in } B_{H_0}. \tag{4.19}$$

An application of the divergence theorem yields:

$$\int_{B_{H_0}} \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nabla E_0\, dy = \int_{\partial B_{H_0}} E_0\, \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nu\, d\mathcal{H}^{n-1} - \int_{B_{H_0}} E_0 \operatorname{div}(\nabla_\xi \widetilde{u}(\nabla E_0))\, dy \tag{4.20}$$

$$= \frac{1}{2}\int_{\partial B_{H_0}} \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nu\, d\mathcal{H}^{n-1} - \int_{B_{H_0}} E_0 \operatorname{div}(\nabla_\xi \widetilde{u}(\nabla E_0))\, dy$$

$$= \frac{1}{2}\int_{B_{H_0}} (1-2E_0) \operatorname{div}(\nabla_\xi \widetilde{u}(\nabla E_0))\, dy.$$

Now, observe that $1-2E_0 \geq 0$ in B_{H_0}. Furthermore,

$$\operatorname{div}(\nabla_\xi \widetilde{u}(\nabla E_0)) = \operatorname{tr}(\nabla^2_\xi \widetilde{u}(\nabla E_0)\nabla^2 E_0) \geq n \det(\nabla^2_\xi \widetilde{u}(\nabla E_0)\nabla^2 E_0) \tag{4.21}$$

$$= n\frac{\det(\nabla^2 \widetilde{u}(\nabla E_0))}{\Phi(\nabla E_0)} = n \quad \text{a.e. in } B_{H_0}.$$

Here, we have made use of equation (4.18), of the version of Newton's inequality from Lemma 4.1, of equation (4.19), and of the equation in (3.10). Notice that the application of Lemma 4.1 is legitimate, since the matrix $\nabla^2_\xi \widetilde{u}$ is positive semidefinite and, by property (2.28), the matrix $\nabla^2 E_0$ is positive definite.

From equations (4.20) and (4.21) one deduces that

$$\int_{B_{H_0}} \nabla_\xi \widetilde{u}(\nabla E_0)\cdot \nabla E_0\, dy \geq \frac{n}{2}\int_{B_{H_0}} (1-2E_0)\, dy. \tag{4.22}$$

This establishes inequality (4.16).

Step 7. We have that

$$2\int_{B_{H_0}} E_0\, dy \leq -n\int_\Omega u\, dx\,. \tag{4.23}$$

To begin with, observe that merging (4.14) and (4.16) into (4.12) implies that

$$-n\int_\Omega u\, dx \geq \frac{n}{2}\left(\mathcal{L}^n(B_{H_0}) - 2\int_{B_{H_0}} E_0\, dy\right). \tag{4.24}$$

On the other hand,

$$\mathcal{L}^n(B_{H_0}) = \frac{2(n+2)}{n}\int_{B_{H_0}} E_0\, dy. \tag{4.25}$$

To verify this equality, notice that

$$\mathcal{L}^n(B_{H_0}) = \int_{\{E_0(x)\leq\frac{1}{2}\}} dx = \int_0^{\frac{1}{2}} \int_{\{E_0(x)=t\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla E_0(x)|}\, dt \tag{4.26}$$
$$= \int_0^{\frac{1}{2}} \int_{\{E_0(x/\sqrt{t})=1\}} \frac{d\mathcal{H}^{n-1}(x)}{|\nabla E_0(x)|}\, dt = \int_0^{\frac{1}{2}} t^{\frac{n-1}{2}} \int_{\{E_0(y)=1\}} \frac{d\mathcal{H}^{n-1}(y)}{|\nabla E_0(y\sqrt{t})|}\, dt$$
$$= \int_0^{\frac{1}{2}} t^{\frac{n}{2}-1} \int_{\{E_0(y)=1\}} \frac{d\mathcal{H}^{n-1}(y)}{|\nabla E_0(y)|}\, dt = \frac{1}{n2^{\frac{n}{2}-1}} \int_{\{E_0=1\}} \frac{d\mathcal{H}^{n-1}}{|\nabla E_0|}\, dt,$$

where the second equality holds by the coarea formula, the third one since E_0 is positively homogeneous of degree 2, the fourth one by the area formula, and the fifth one since the function $|\nabla E_0|$ is positively homogeneous of degree 1. Analogously,

$$\int_{B_{H_0}} E_0\, dy = \int_0^{\frac{1}{2}} t \int_{\{E_0=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla E_0|}\, dt = \int_0^{\frac{1}{2}} t^{\frac{n}{2}} \int_{\{E_0=1\}} \frac{d\mathcal{H}^{n-1}}{|\nabla E_0|}\, dt$$
$$= \frac{1}{(n+2)2^{\frac{n}{2}}} \int_{\{E_0=1\}} \frac{d\mathcal{H}^{n-1}}{|\nabla E_0|}\, dt. \tag{4.27}$$

Combining equations (4.26) and (4.27) yields (4.25). Inequality (4.23) follows from equations (4.24) and (4.25).

Step 8. Conclusion.

Coupling inequality (4.23) with the reverse inequality (4.8) implies that

$$2\int_{B_{H_0}} E_0\, dy = -n \int_\Omega u\, dx\,. \tag{4.28}$$

This forces all the inequalities derived above to hold as equalities. Especially, equation (4.7) holds a.e. in Ω. Namely,

$$\nabla(\nabla_\xi E(\nabla u)) = I \quad \text{a.e. in } \Omega.$$

Hence, there exists $\overline{x} \in \mathbb{R}^n$ such that

$$\nabla_\xi E(\nabla u) = x - \overline{x} \qquad \text{for } x \in \Omega\,, \tag{4.29}$$

and, by equation (2.26),

$$\nabla u = \nabla E_0(x - \overline{x}) \qquad \text{for } x \in \Omega. \tag{4.30}$$

Equation (1.17) follows form (4.30) and the Dirichlet boundary condition in (4.1). □

Lemma 4.1. *Assume that the matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, and the matrix $B \in \mathbb{R}^{n\times n}$ is symmetric and positive semidefinite. Then,*

$$(\det AB)^{\frac{1}{n}} \le \frac{\operatorname{tr}(AB)}{n}. \tag{4.31}$$

Moreover, equality holds in (4.31) if and only if $AB = \lambda I$ for some constant $\lambda \ge 0$.

Proof. Our assumptions on B ensure that there exist an orthogonal matrix U and a diagonal matrix $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, where $\lambda_1, \dots, \lambda_n \ge 0$ are the eigenvalues of B, such that $\Lambda = U^T BU$. Hence $B = U\Lambda U^T$.
Set $Q = U^T AU$ and $Q = (q_{ij})$. Then

$$\begin{aligned}\frac{\operatorname{tr}(AB)}{n} &= \frac{\operatorname{tr}(AU\Lambda U^T)}{n} = \frac{\operatorname{tr}(Q\Lambda)}{n} = \frac{\sum_{i=1}^n \lambda_i q_{ii}}{n} \ge \left(\Pi_{i=1}^n \lambda_i q_{ii}\right)^{\frac{1}{n}} \\ &= \left(\Pi_{i=1}^n q_{ii}\right)^{\frac{1}{n}} \left(\Pi_{i=1}^n \lambda_i\right)^{\frac{1}{n}} \\ &= \left(\Pi_{i=1}^n q_{ii}\right)^{\frac{1}{n}} (\det B)^{\frac{1}{n}} \ge (\det Q)^{\frac{1}{n}} (\det B)^{\frac{1}{n}} \\ &= (\det A)^{\frac{1}{n}} (\det B)^{\frac{1}{n}} = (\det AB)^{\frac{1}{n}}.\end{aligned} \tag{4.32}$$

Note that the first inequality in (4.32) holds by the arithmetic-geometric mean inequality, with equality if and only if

$$\lambda_1 q_{11} = \cdots = \lambda_{nn} q_{nn}. \tag{4.33}$$

Moreover, the second inequality holds since Q is a symmetric positive definite matrix [29, Theorem 7.8.1], with equality if and only if

$$q_{ij} = 0 \quad \text{if } i \ne j. \tag{4.34}$$

Inequality (4.31) agrees with (4.32).
In particular, if equality holds in (4.31), then both equations (4.33) and (4.34) hold. The latter implies that the matrix $Q\Lambda$ is diagonal. By coupling this piece of information with the former, we deduce that $Q\Lambda$ is a multiple of I. Therefore

$$U^T ABU = U^T AUU^T BU = Q\Lambda = \frac{\operatorname{tr}(Q\Lambda)}{n} I = \frac{\operatorname{tr}(AB)}{n} I. \tag{4.35}$$

Hence, $AB = \lambda I$, with $\lambda = \frac{\operatorname{tr}(AB)}{n}$. Conversely, if $AB = \lambda I$ for some $\lambda \ge 0$, then equality trivially holds in (4.31). □

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments

We wish to thank the referee for carefully reading the paper and for her/his valuable comments.

Funding. This research was partly funded by:

(i) Research Project 201758MTR2 of the Italian Ministry of Education, University and Research (MIUR) Prin 2017 "Direct and inverse problems for partial differential equations: theoretical aspects and applications";

(ii) GNAMPA of the Italian INdAM - National Institute of High Mathematics (grant number not available).

References

[1] A. Alvino, V. Ferone, P.-L. Lions, G. Trombetti, Convex symmetrization and applications, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 14 (1997) 275–293.

[2] M. Amar, G. Bellettini, A notion of total variation depending on a metric with discontinuous coefficients, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 11 (1994) 91–133.

[3] C.A. Antonini, G. Ciraolo, A. Farina, Interior regularity results for inhomogeneous anisotropic quasilinear equations, Math. Ann., to appear.

[4] I.J. Bakelman, Convex Analysis and Nonlinear Geometric Elliptic Equations, Springer, Berlin, 1994.

[5] L. Barbu, C. Enache, On a free boundary problem for a class of anisotropic equations, Math. Methods Appl. Sci. 40 (2017) 2005–2012.

[6] M. Belloni, V. Ferone, B. Kawohl, Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators, Z. Angew. Math. Phys. 41 (2004) 439–505.

[7] C. Bianchini, G. Ciraolo, Wulff shape characterizations in overdetermined anisotropic elliptic problems, Commun. Partial Differ. Equ. 43 (2018) 790–820.

[8] C. Bianchini, G. Ciraolo, P. Salani, An overdetermined problem for the anisotropic capacity, Calc. Var. Partial Differ. Equ. 55 (2016) 84.

[9] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin type overdetermined problems: an alternative proof, Arch. Ration. Mech. Anal. 190 (2008) 267–280.

[10] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Stability of radial symmetry for a Monge-Ampère overdetermined problem, Ann. Mat. Pura Appl. 188 (2009) 445–453.

[11] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Commun. Pure Appl. Math. 44 (1991) 375–417.

[12] L. Caffarelli, The regularity of mappings with a convex potential, J. Am. Math. Soc. 5 (1992) 99–104.

[13] A. Cianchi, P. Salani, Overdetermined anisotropic elliptic problems, Math. Ann. 345 (2009) 859–881.

[14] G. Ciraolo, A. Figalli, A. Roncoroni, Symmetry results for critical anisotropic p-Laplacian equations in convex cones, Geom. Funct. Anal. 30 (2020) 770–803.

[15] M. Cozzi, A. Farina, E. Valdinoci, Gradient bounds and rigidity results for singular, degenerate, anisotropic partial differential equations, Commun. Math. Phys. 331 (2014) 189–214.

[16] M. Cozzi, A. Farina, E. Valdinoci, Monotonicity formulae and classification results for singular, degenerate, anisotropic PDEs, Adv. Math. 293 (2016) 343–381.
[17] G. De Philippis, A. Figalli, $W^{2,1}$ regularity for solutions of the Monge-Ampère equation, Invent. Math. 192 (2013) 55–69.
[18] G. De Philippis, A. Figalli, O. Savin, A note on interior $W^{2,1+\varepsilon}$ estimates for the Monge-Ampère equation, Math. Ann. 357 (2013) 11–22.
[19] M.G. Delgadino, F. Maggi, C. Mihaila, R. Neumayer, Bubbling with L^2-almost constant mean curvature and an Alexandrov-type theorem for crystals, Arch. Ration. Mech. Anal. 230 (2018) 1131–1177.
[20] F. Della Pietra, N. Gavitone, Symmetrization with respect to the anisotropic perimeter and applications, Math. Ann. 363 (2015) 953–971.
[21] F. Della Pietra, N. Gavitone, C. Xia, Symmetrization with respect to mixed volumes, Adv. Math. 388 (2021) 107887.
[22] S. Dipierro, G. Poggesi, E. Valdinoci, Radial symmetry of solutions to anisotropic and weighted diffusion equations with discontinuous nonlinearities, Calc. Var. Partial Differ. Equ. 61 (2022) 72.
[23] F. Esposito, G. Riey, B. Sciunzi, D. Vuono, Anisotropic Kelvin transform and the Sobolev anisotropic critical equation, arXiv:2203.14903v1.
[24] F. Esposito, G. Riey, B. Sciunzi, D. Vuono, On a Kelvin-type transform for a class of anisotropic elliptic operators, arXiv:2203.14903v3.
[25] V. Ferone, B. Kawohl, Remarks on a Finsler-Laplacian, Proc. Am. Math. Soc. 137 (2009) 247–253.
[26] M. Fazly, Y. Li, Partial regularity and Liouville theorems for stable solutions of anisotropic elliptic equations, Discrete Contin. Dyn. Syst. 41 (2021) 4185–4206.
[27] M. Fazly, Y. Li, Partial regularity for weak solutions of anisotropic Lane-Emden equation, Proc. Am. Math. Soc. 150 (2022) 179–190.
[28] C. Gutierrez, The Monge-Ampère Equation, Birkhäuser, Boston, 2001.
[29] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[30] B. Kawohl, M. Novaga, The p-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric, J. Convex Anal. 15 (2008) 623–634.
[31] R. Magnanini, Alexandrov, Serrin, Weinberger, Reilly: symmetry and stability by integral identities, Bruno Pini Math. Anal. Semin. 8 (2017) 121–141, Univ. Bologna, Alma Mater Stud., Bologna.
[32] M. Marcus, V.J. Mizel, Absolute continuity of tracks and mappings of Sobolev spaces, Arch. Ration. Mech. Anal. 45 (1972) 294–320.
[33] C. Nitsch, C. Trombetti, The classical overdetermined Serrin problem, Complex Var. Elliptic Equ. 63 (2018) 1107–1122.
[34] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N.J., 1970.
[35] O. Savin, H. Yu, Global $W^{2,1+\varepsilon}$ estimates for Monge-Ampère equation with natural boundary condition, J. Math. Pures Appl. 137 (2020) 275–289.
[36] R. Schneider, Convex Bodies: the Brunn-Minkowski Theory, Cambridge University Press, Cambridge, 1993.
[37] T. Schmidt, $W^{2,1+\varepsilon}$ estimates for the Monge-Ampère equation, Adv. Math. 240 (2013) 672–689.
[38] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (1971) 304–318.
[39] L. Silvestre, B. Sirakov, Overdetermined problems for fully nonlinear elliptic equations, Calc. Var. Partial Differ. Equ. 54 (2015) 989–1007.
[40] J. Taylor, Crystalline variational problems, Bull. Am. Math. Soc. 84 (1978) 568–588.
[41] G. Wang, C. Xia, Blow-up analysis of a Finsler-Liouville equation in two dimensions, J. Differ. Equ. 252 (2012) 1668–1700.
[42] H.F. Weinberger, Remark on the preceding paper of Serrin, Arch. Ration. Mech. Anal. 43 (1971) 319–320.
[43] G. Wulff, Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen, Z. Kristallogr. 34 (1901) 449–530.