



Controllability of nonlinear integral equations of Chandrasekhar type

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Abstract. In this paper, we study the controllability of two problems involving the same Chandrasekhar-type integral equation, but under different kinds of controls. A viability condition is imposed as well. We provide existence results of continuous trajectories coupled to continuous controls. Then, in the non-viable case, we investigate the optimal estimates to be taken in view of the existence of solutions for both problems. The last part of the paper deals with the application of the previous results to the classical Chandrasekhar equation, first showing the existence of a viable continuous solution, then providing also uniqueness and approximability. Two examples of controllability problems governed by this equation are given.

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1. Introduction

In the last decades, a great attention has been given to the quadratic integral equation of Chandrasekhar type

$$x(t) = a + x(t) \int_0^T \frac{t}{t+s} w(s)x(s)ds, \quad t \in [0, T], \quad (1)$$

where $a > 0$ and $w : [0, T] \rightarrow \mathbb{R}$ is a continuous function with $w(0) = 0$. This equation is a first generalization of the quadratic integral equation introduced by Chandrasekhar in [15], which plays an important role in the radiative transfer theory, in the neutron transport theory, and in the kinetic theory of gases (see, e.g., [8, 14, 17, 19, 22, 24, 26]). Equations of type (1) and some of their generalizations were studied in several papers (see, e.g., [2, 3, 5, 6, 9–11, 23]). This fact is mainly connected to the applications of the mentioned classes of integral equations to the description of several real-world events which appear in bioengineering, mechanics, physics, mathematical physics,

porous media, viscoelasticity, control theory and other important branches of sciences and applied mathematics.

Our aim in this paper is to study the controllability of two different generalizations of the equation (1).

In the first case, we work considering the nonlinear integral equation with feedback controls

$$x(t) = a + u(t, x(t)) + g(t, x(t)) \int_0^{\varphi(t)} k(t, s) f(s, x(s)) ds, \quad t \in [0, T], \quad (2)$$

$$u(t, x(t)) \in U(t, x(t)), \quad t \in [0, T], \quad (3)$$

where $a \in \mathbb{R}$, $g, f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : [0, T] \rightarrow [0, T]$, $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$, and U is a real multimap defined on $[0, T] \times \mathbb{R}$.

The presence of the controls naturally leads to the use of multivalued analysis tools. In particular, we provide the existence of an admissible pair trajectory-control by means of a selection theorem for multimaps (see [20, Theorem 1.6.21]), and a version of the Daher fixed point theorem involving a measure of noncompactness on the space $C([0, T])$ (see [13, Theorem 3.2]).

The second case is about the integral equation

$$x(t) = a + [v(t) + g(t, x(t))] \int_0^{\varphi(t)} k(t, s) f(s, x(s)) ds, \quad t \in [0, T], \quad (4)$$

$$v(t) \in V(t), \quad t \in [0, T], \quad (5)$$

where V is a real multimap defined on $[0, T]$. In this other setting, we obtain the existence of an admissible pair by applying the classical Michael selection theorem and contraction fixed point theorem.

The two equations differ from the type of the control. Indeed, in equation (2) a feedback control is considered, while in equation (4) the control function v does not depend on the solution trajectory, but it acts on the integral part.

Notice that in both cases, taking $\varphi(t) = t$ or $\varphi(t) = T$, the integral term becomes a Volterra or a Fredholm integral, respectively.

The paper is organized as follows. In Sect. 2, we collect the necessary preliminary definitions and results. Then, in Sects. 3 and 4, we state the controllability results for the problems (2)–(3) and (4)–(5), respectively, under a viability condition. The solutions obtained are continuous trajectories coupled to controls, continuous as well. In Sect. 5, we study optimal estimates for the existence of solutions for both nonlinear integral Eqs. (2) and (4) in the nonviable case, and, finally, Sect. 6 is devoted to the application of our results to the classical Chandrasekhar equation (1). At first, we show the existence of a viable continuous solution. Then, under different assumptions on the characteristic function w , we provide existence, uniqueness and approximability again of a viable continuous solution. Finally, we achieve optimal estimates on w , allowing us to improve known results in [6] and [9], and extend those in [5], as explained in Remark 6.1. The paper is completed by two examples of controllability problems governed by the classical Chandrasekhar integral equation.

2. Notations and preliminary results

Let X, Y be topological spaces. We will use the following notations:

$$\begin{aligned} \mathcal{P}(X) &= \{\Omega \subset X : \Omega \text{ nonempty}\}; \\ \mathcal{P}_b(X) &= \{\Omega \in \mathcal{P}(X) : \Omega \text{ bounded}\}; \\ \mathcal{P}_{fb}(X) &= \{\Omega \in \mathcal{P}(X) : \Omega \text{ closed and bounded}\}. \end{aligned}$$

If X is also a vector space, then we put

$$\mathcal{P}_{kc}(X) = \{\Omega \in \mathcal{P}(X) : \Omega \text{ compact and convex}\}.$$

It is well known that a multimap $F : X \rightarrow \mathcal{P}(Y)$ is *lower semicontinuous* at a point $x_0 \in X$ if for every open set $W \subset Y$ such that $F(x_0) \cap W \neq \emptyset$ there exists a neighborhood $V(x_0)$ of x_0 such that $F(x) \cap W \neq \emptyset$ for all $x \in V(x_0)$. A multimap is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every $x_0 \in X$. For more details, we refer, e.g., to [21].

Let (X, d) be a metric space. If $A \subset X$ and $x \in X$, the standard distance is $\delta(x, A) = \inf_{y \in A} d(x, y)$. Let $A, B \in \mathcal{P}_b(X)$; their Hausdorff distance $d_H(A, B)$ is the number

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \delta(x, B), \sup_{y \in B} \delta(y, A) \right\}.$$

Then, given X, Y two metric spaces, we recall that a multimap $\mathcal{F} : X \rightarrow \mathcal{P}_b(Y)$ is *Hausdorff continuous (shortly, H-continuous)* if it is continuous with respect to the metric d in X and the Hausdorff metric d_H in $\mathcal{P}_b(Y)$.

Let $(X, \|\cdot\|)$ be a Banach space. We recall (see, e.g., [13]) that a function $\beta : \mathcal{P}_b(X) \rightarrow \mathbb{R}_0^+$ is said to be a *measure of noncompactness* (MNC, for short) if

$$\beta(\overline{\text{co}}(\Omega)) = \beta(\Omega), \text{ for every } \Omega \in \mathcal{P}_b(X), \tag{6}$$

where $\overline{\text{co}}(\Omega)$ is the closed convex hull of Ω .

In the sequel, we will consider a measure of noncompactness β verifying the following properties:

- (β_1) *regularity*: $\beta(\Omega) = 0$ if and only if $\overline{\Omega}$ is compact;
- (β_2) *monotonicity*: $\Omega_1 \subset \Omega_2$ implies $\beta(\Omega_1) \leq \beta(\Omega_2)$;
- (β_3) *semiadditivity*: $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;

for any $\Omega, \Omega_1, \Omega_2 \in \mathcal{P}_b(X)$.

As examples of measures of noncompactness which satisfy all the previous properties, we recall the Hausdorff and the Kuratowski measures of noncompactness.

To state the fixed point theorem we will use in the following, we also recall (see, e.g., [13]) that a function $F : D \subset X \rightarrow \mathcal{P}(X)$ is *countably condensing* if:

- (I) $F(D)$ is bounded;
- (II) $\beta(F(\Omega)) < \beta(\Omega)$, for all countable $\Omega \in \mathcal{P}_b(D)$ with $\beta(\Omega) > 0$.

In our approach, one of the main tools is the next fixed point theorem, which is a version of the Daher fixed point theorem in [16], holding by virtue of Theorem 3.2 in [13].

Theorem 2.1. *Let M be a closed, convex subset of a Banach space X and $F : M \rightarrow M$ be a continuous map which is countably condensing with respect to a measure of noncompactness.*

Then there exists $x \in M$ with $x = F(x)$.

From now on we will work in the Banach space $X = C([0, T])$ of all the real continuous functions defined on $[0, T]$, endowed with the norm $\|x\| = \max_{t \in [0, T]} |x(t)|$. In X , we will consider the MNC ω_0 defined as follows (cf. [4]).

Let D be a bounded subset of $C([0, T])$. For $x \in D$ and $\varepsilon > 0$, let $\omega(x, \varepsilon)$ be the modulus of continuity of the function x on the interval $[0, T]$, i.e.,

$$\omega(x, \varepsilon) = \max\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}. \tag{7}$$

Put

$$\omega(D, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in D\}, \tag{8}$$

the MNC ω_0 is defined by

$$\omega_0(D) = \lim_{\varepsilon \rightarrow 0^+} \omega(D, \varepsilon). \tag{9}$$

It is well known that ω_0 is equivalent to the Hausdorff MNC of the space $C([0, T])$ (see, e.g., [1]).

3. Controllability of a first type nonlinear integral equation

In this Section, we will study the integral equation

$$x(t) = a + u(t, x(t)) + g(t, x(t)) \int_0^{\varphi(t)} k(t, s) f(s, x(s)) ds, \quad t \in [0, T], \tag{10}$$

subject to feedback controls

$$u(t, x(t)) \in U(t, x(t)), \quad t \in [0, T], \tag{11}$$

and to the viability condition

$$|x(t)| \leq R, \quad t \in [0, T], \tag{12}$$

where $g, f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : [0, T] \rightarrow [0, T]$, $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$, $U : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$, $a \in \mathbb{R}$, and $R > |a|$.

We recall that a problem is said to be *controllable* if there exists an admissible pair $\{x, u\}$, i.e., a continuous function $x : [0, T] \rightarrow \mathbb{R}$ and a measurable function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (10)–(11).

We will suppose the following hypotheses hold on the control multimap U :

- (U1) $U(t, x)$ is closed and convex, for every $(t, x) \in [0, T] \times \mathbb{R}$;
- (U2) there exists $M \in [0, R - |a|]$ such that

$$\|U(t, x)\| \leq M, \quad \text{for every } (t, x) \in [0, T] \times \mathbb{R},$$

where $\|U(t, x)\| := \sup\{|z| : z \in U(t, x)\}$;

- (U3) U is H-continuous and there exists $\alpha \in [0, 1/4]$ such that

$$d_H(U(t, x), U(t, y)) \leq \alpha|x - y|, \quad \text{for every } (t, x), (t, y) \in [0, T] \times \mathbb{R}.$$

For the functions φ, k we assume:

- (φ) φ is continuous and nondecreasing on $[0, T]$;
- (k1) k is continuous on $[0, T] \times [0, T]$.

Finally, for the restrictions of g, f to $[0, T] \times [-R, R]$ we assume the following properties;

- (g) g is continuous on $[0, T] \times [-R, R]$ and there exists $L_g > 0$ such that

$$|g(t, x) - g(t, y)| \leq L_g|x - y|, \text{ for every } t \in [0, T], |x| \leq R, |y| \leq R.$$
- (f1) f is a Carathéodory function on $[0, T] \times [-R, R]$, i.e., $f(\cdot, x)$ is measurable for every $|x| \leq R$ and $f(t, \cdot)$ is continuous for every $t \in [0, T]$;
- (f2) there exist $c \geq 0$ and $h \in L^1_+[0, T]$ such that

$$|f(t, x)| \leq h(t)(c + |x|), \text{ for a.e. } t \in [0, T] \text{ and every } |x| \leq R.$$

We put

$$K := \max_{(t,s) \in [0,T] \times [0,T]} |k(t, s)|, \quad G := \max_{t \in [0,T], |x| \leq R} |g(t, x)|. \tag{13}$$

To achieve the existence of solutions to the integral inclusion (10), we will use the next result on the existence of a selector for a multimap having the properties of our multimap U ; this result is a particular case of [20, Theorem 1.6.21].

Proposition 3.1. *If $U : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}_{kc}(\mathbb{R})$ is a H -continuous multimap satisfying (U2) and (U3) and if $x_*, y_* : [0, T] \rightarrow \mathbb{R}$ are continuous functions satisfying $y_*(t) \in U(t, x_*(t))$ for every $t \in [0, T]$, then there exists a continuous map $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $u(t, x) \in U(t, x)$, $u(t, x_*(t)) = y_*(t)$ and $|u(t, x) - u(t, y)| \leq 4\alpha|x - y|$ for every $t \in [0, T]$, $x, y \in \mathbb{R}$.*

We can now state our controllability result to problem (10)–(12).

Theorem 3.1. *Let (U1)–(U3), (g), (f1), (f2), (φ) (k1) be satisfied, and assume that*

$$K\|h\|_1 < \min \left\{ \frac{R - (M + |a|)}{G(c + R)}, \frac{1 - 4\alpha}{L_g(c + R)} \right\}, \tag{14}$$

where K and G are defined in (13). Then there exists an admissible pair $\{x, u\}$ to problem (10)–(11), with x satisfying the viability condition (12).

Proof. First of all note that U takes values in $\mathcal{P}_{kc}(\mathbb{R})$ (see (U1), (U2)); further, U is H -continuous by (U3). Let us consider the multimap $U_0 : [0, T] \rightarrow \mathcal{P}(\mathbb{R})$ defined by $U_0(t) = U(t, 0)$ for every $t \in [0, T]$. It is easy to see that U_0 satisfies the hypotheses of the Michael selection theorem. Thus, there exists a continuous function $y_* : [0, T] \rightarrow \mathbb{R}$ such that

$$y_*(t) \in U_0(t) \text{ for every } t \in [0, T].$$

Put $x_* : [0, T] \rightarrow \mathbb{R}$, $x_*(t) = 0$ for every $t \in [0, T]$, the previous inclusion can be rewritten as

$$y_*(t) \in U(t, x_*(t)) \text{ for every } t \in [0, T].$$

Hence, we can apply Proposition 3.1 and claim that there exists a continuous map $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$u(t, x) \in U(t, x), \tag{15}$$

$u(t, x_*(t)) = y_*(t)$ and $|u(t, x) - u(t, y)| \leq 4\alpha|x - y|$. Therefore, u is a continuous selection of U such that

$$|u(t, x) - u(t, y)| \leq L_u|x - y|, \text{ for every } t \in [0, T], x, y \in \mathbb{R}, \tag{16}$$

where $L_u = 4\alpha < 1$. Clearly, by (U2) it is also true that

$$|u(t, x)| \leq M, \text{ for every } (t, x) \in [0, T] \times \mathbb{R}. \tag{17}$$

Let us consider now the solution operator $\Phi : \bar{B}(0, R) \rightarrow C([0, T])$,

$$\begin{aligned} \Phi(x)(t) &= a + u(t, x(t)) \\ &+ g(t, x(t)) \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds, \quad t \in [0, T], x \in C([0, T]), \end{aligned}$$

where $\bar{B}(0, R) = \{x \in C([0, T]) : \|x\| \leq R\}$.

The operator Φ is well defined. Indeed, let $x \in \bar{B}(0, R)$ be fixed. From (f1) and (f2), the function $f(\cdot, x(\cdot))$ belongs to $L^1([0, T])$, and $k(t, \cdot)$ is bounded by (k1). Thus, $k(t, \cdot)f(\cdot, x(\cdot)) \in L^1([0, T])$. This, together with the continuity of φ , implies that the map

$$t \mapsto \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds$$

is continuous on $[0, T]$ as well; indeed, for any $t_0 \in [0, T]$, if $(t_n)_n$ is a sequence in $[0, T]$ converging to t_0 , by (f2) we have

$$\begin{aligned} &\left| \int_0^{\varphi(t_n)} k(t_n, s)f(s, x(s))ds - \int_0^{\varphi(t_0)} k(t_0, s)f(s, x(s))ds \right| \\ &\leq \left| \int_0^{\varphi(t_n)} k(t_n, s)f(s, x(s))ds - \int_0^{\varphi(t_0)} k(t_n, s)f(s, x(s))ds \right| \\ &\quad + \left| \int_0^{\varphi(t_0)} k(t_n, s)f(s, x(s))ds - \int_0^{\varphi(t_0)} k(t_0, s)f(s, x(s))ds \right| \\ &\leq K \left| \int_{\varphi(t_0)}^{\varphi(t_n)} f(s, x(s))ds \right| + \int_0^{\varphi(t_0)} |k(t_n, s) - k(t_0, s)||f(s, x(s))|ds \\ &\leq K \left| \int_{\varphi(t_0)}^{\varphi(t_n)} h(s)(c + R)ds \right| + \int_0^{\varphi(t_0)} |k(t_n, s) - k(t_0, s)|h(s)(c + R)ds. \end{aligned}$$

Since φ is continuous, using the absolute continuity of the integral and the Lebesgue convergence theorem, we obtain that the right term in the above inequality tends to 0 as $n \rightarrow +\infty$, that is $\Phi(x(t_n)) \rightarrow \Phi(x(t_0))$ as $n \rightarrow +\infty$, and $\Phi(x)$ is continuous in t_0 .

Now, we show that Φ maps $\bar{B}(0, R)$ in itself. Indeed, for any $t \in [0, T]$ and $x \in \bar{B}(0, R)$ using (17), (g2), (k1), and (f2) we get

$$\begin{aligned} |\Phi(x)(t)| &\leq |a| + M + GK \int_0^{\varphi(t)} h(s)(c + |x(s)|)ds \\ &\leq |a| + M + GK \int_0^T h(s)(c + R)ds \\ &= |a| + M + G(c + R)K\|h\|_1. \end{aligned}$$

From (14) we, therefore, obtain

$$|\Phi(x)(t)| \leq R.$$

Next, we prove that Φ is continuous. Fixed $\bar{x} \in \bar{B}(0, R)$, let $(x_n)_{n \in \mathbb{N}}$, $x_n \in \bar{B}(0, R)$, such that $x_n \rightarrow \bar{x}$ in $C([0, T])$.

Since u, g are uniformly continuous on $[0, T] \times [-R, R]$, we can say that $(u(\cdot, x_n(\cdot)))_{n \in \mathbb{N}}$, $(g(\cdot, x_n(\cdot)))_{n \in \mathbb{N}}$ uniformly converge in $[0, T]$ to $u(\cdot, \bar{x}(\cdot))$ and $g(\cdot, \bar{x}(\cdot))$, respectively.

Further, also the sequence $\left(\int_0^{\varphi(\cdot)} k(\cdot, s)f(s, x_n(s))ds\right)_{n \in \mathbb{N}}$ uniformly converges to $\int_0^{\varphi(\cdot)} k(\cdot, s)f(s, \bar{x}(s))ds$. Indeed, by (13), for every $t \in [0, T]$ we have

$$\left| \int_0^{\varphi(t)} k(t, s) [f(s, x_n(s)) - f(s, \bar{x}(s))] ds \right| \leq K \int_0^T |f(s, x_n(s)) - f(s, \bar{x}(s))| ds$$

and, using (f1) and (f2), the dominated converge theorem leads to the assertion. Therefore, $\|\Phi(x_n) - \Phi(\bar{x})\| \rightarrow 0$ as $n \rightarrow +\infty$, i.e., Φ is continuous in \bar{x} .

We shall prove that Φ is a countably condensing map with respect to the MNC ω_0 in $C([0, T])$ (see (9)). Clearly, $\Phi(\bar{B}(0, R))$ is bounded, so property (I) of the countably condensivity is satisfied. To prove property (II), we consider a countable set $D \subset \bar{B}(0, R)$ with $\omega_0(D) > 0$.

Let $D = \{x_n\}_n$ and fix $n \in \mathbb{N}$. Taking into account the uniform continuity of the maps u, g , and k , and the Lebesgue integrability of h in (f2), we have that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) \leq \varepsilon$ such that for every $t_1, t_2 \in [0, T]$, with $|t_2 - t_1| < \delta(\varepsilon)$, it holds

$$\max_{|x| \leq R} |u(t_2, x) - u(t_1, x)| < \varepsilon, \tag{18}$$

$$\max_{|x| \leq R} |g(t_2, x) - g(t_1, x)| < \varepsilon, \tag{19}$$

$$\max_{s \in [0, T]} |k(t_2, s) - k(t_1, s)| < \varepsilon, \tag{20}$$

$$\left| \int_{\varphi(t_1)}^{\varphi(t_2)} h(s) ds \right| < \varepsilon. \tag{21}$$

Now, taking without loss of generality $t_1 < t_2$, we obtain the following inequality (recall the monotonicity of φ)

$$|\Phi(x_n)(t_2) - \Phi(x_n)(t_1)| \leq |u(t_2, x_n(t_2)) - u(t_1, x_n(t_1))|$$

$$\begin{aligned}
 &+|g(t_2, x_n(t_2)) - g(t_1, x_n(t_1))| \int_0^{\varphi(t_2)} |k(t_2, s)f(s, x_n(s))|ds \\
 &+|g(t_1, x_n(t_1))| \left[\int_{\varphi(t_1)}^{\varphi(t_2)} |k(t_2, s)f(s, x_n(s))|ds \right. \\
 &\left. + \int_0^{\varphi(t_1)} |k(t_2, s) - k(t_1, s)||f(s, x_n(s))|ds \right]. \tag{22}
 \end{aligned}$$

For each term of the above equation, we have the next estimates: by (16), (7), and (18)

$$\begin{aligned}
 |u(t_2, x_n(t_2)) - u(t_1, x_n(t_1))| &\leq |u(t_2, x_n(t_2)) - u(t_2, x_n(t_1))| \\
 &\quad + |u(t_2, x_n(t_1)) - u(t_1, x_n(t_1))| \\
 &\leq L_u |x_n(t_2) - x_n(t_1)| + \max_{|x| \leq R} |u(t_2, x) - u(t_1, x)| \\
 &< L_u \omega(x_n, \varepsilon) + \varepsilon; \tag{23}
 \end{aligned}$$

similarly, by (g), (7), and (19)

$$\begin{aligned}
 |g(t_2, x_n(t_2)) - g(t_1, x_n(t_1))| &\leq L_g \omega(x_n, \varepsilon) + \max_{|x| \leq R} |g(t_2, x) - g(t_1, x)| \\
 &< L_g \omega(x_n, \varepsilon) + \varepsilon; \tag{24}
 \end{aligned}$$

by (f2), (k1), (φ)

$$\int_0^{\varphi(t_2)} |k(t_2, s)f(s, x_n(s))|ds \leq K(c + R)\|h\|_1; \tag{25}$$

and, by (21) too,

$$\int_{\varphi(t_1)}^{\varphi(t_2)} |k(t_2, s)f(s, x_n(s))|ds \leq K(c + R) \int_{\varphi(t_1)}^{\varphi(t_2)} h(s)ds < K(c + R)\varepsilon; \tag{26}$$

by (f2) and (20)

$$\begin{aligned}
 \int_0^{\varphi(t_1)} |k(t_2, s) - k(t_1, s)||f(s, x_n(s))|ds &\leq (c + R)\|h\|_1 \max_{s \in [0, T]} |k(t_2, s) - k(t_1, s)| \\
 &< (c + R)\|h\|_1 \varepsilon. \tag{27}
 \end{aligned}$$

Hence, by (22), (23), (24), (13), (25), (26), (27) we obtain

$$\begin{aligned}
 |\Phi(x_n)(t_2) - \Phi(x_n)(t_1)| &\leq L_u \omega(x_n, \varepsilon) + \varepsilon \\
 &\quad + [L_g \omega(x_n, \varepsilon) + \varepsilon] K(c + R)\|h\|_1 \\
 &\quad + G(c + R)\varepsilon [K + \|h\|_1].
 \end{aligned}$$

Therefore (recall (8))

$$\begin{aligned}
 \omega(\Phi(D), \varepsilon) &\leq L_u \omega(D, \varepsilon) + \varepsilon + [L_g \omega(D, \varepsilon) + \varepsilon] K(c + R)\|h\|_1 \\
 &\quad + G(c + R)\varepsilon [K + \|h\|_1].
 \end{aligned}$$

Hence, passing to the limit as $\varepsilon \rightarrow 0$ and using (14), we obtain

$$\omega_0(\Phi(D)) \leq [L_u + L_g K(c + R)\|h\|_1] \omega_0(D) < \omega_0(D),$$

being $L_u = 4\alpha < 1$. By the arbitrariness of D we get that property (II) of the countably condensivity is proved.

Then, we can apply Theorem 2.1 and claim that there exists $x \in \bar{B}(0, R)$ such that $\Phi(x) = x$, i.e., x satisfies equation (10) and the viability condition (12). Further, by (15), the pair $\{x, u\}$ is admissible for problem (10)–(11), with u even continuous. \square

Remark 3.1. From the proof of Theorem 3.1 it can be observed that condition (14) can be slightly improved as follows:

$$K\|h\|_1 \leq \frac{R - (M + |a|)}{G(c + R)}, \quad K\|h\|_1 < \frac{1 - 4\alpha}{L_g(c + R)}.$$

Therefore, in this situation, if

$$\frac{R - (M + |a|)}{G(c + R)} < \frac{1 - 4\alpha}{L_g(c + R)}$$

then (14) can be replaced by

$$K\|h\|_1 \leq \frac{R - (M + |a|)}{G(c + R)}.$$

4. Controllability for a second-type nonlinear integral equation

In this Section, for given $a \in \mathbb{R}$, $f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : [0, T] \rightarrow [0, T]$, $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$, and $V : [0, T] \rightarrow \mathcal{P}(\mathbb{R})$, we consider the integral equation

$$x(t) = a + [v(t) + g(t, x(t))] \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds, \quad t \in [0, T], \quad (28)$$

subject to the controls

$$v(t) \in V(t), \quad t \in [0, T], \quad (29)$$

and to the viability condition (12), with fixed

$$R > |a|. \quad (30)$$

On the control multimap $V : [0, T] \rightarrow \mathcal{P}(\mathbb{R})$ we assume the properties

(V1) $V(t)$ is closed and convex, for every $t \in [0, T]$;

(V2) there exists $N \geq 0$ such that

$$\|V(t)\| \leq N, \quad \text{for every } t \in [0, T],$$

where $\|V(t)\| := \sup\{|z| : z \in V(t)\}$;

(V3) V is lower semicontinuous on $[0, T]$.

Further, suppose that $k : [0, T] \times [0, T] \rightarrow \mathbb{R}$ satisfies

(k2) $k \in L^\infty([0, T] \times [0, T])$ and the map $k(\cdot, s)$ is continuous on $[0, T]$ for a.e. $s \in [0, T]$,

and that for function φ the property (φ) of Sect. 3 holds.

Finally, we assume that the restriction of f to $[0, T] \times [-R, R]$ satisfies the properties

(f4) $f(\cdot, x)$ is measurable, for every $|x| \leq R$;

(f5) there exist $c \geq 0$ and $l \in L^1_+([0, T])$ such that $|f(t, 0)| \leq cl(t)$ for a.e. $t \in [0, T]$ and

$$|f(t, x) - f(t, y)| \leq l(t)|x - y|, \text{ for a.e. } t \in [0, T] \text{ and every } |x| \leq R, |y| \leq R,$$

while the restriction of g to the same set satisfies (g) of Sect. 3.

In this setting, we can apply the classical Michael’s selection theorem joined to the fixed point contraction principle, obtaining the uniqueness of solutions as particular case when the multimap come down to a single-valued function.

Theorem 4.1. *Let (g), (φ), (V1)–(V3), (f4), (f5), (k2), (30) be satisfied, and assume that*

$$K_\infty \|l\|_1 < \min \left\{ \frac{R - |a|}{(N + G)(c + R)}, \frac{1}{L_g(c + R) + N + G} \right\}, \tag{31}$$

where K_∞ is the ess sup of k (which is finite by (k2)) and G is defined in (13).

Then, there exists an admissible pair $\{x, v\}$ to problem (28)–(29), with x satisfying the viability condition (12).

Proof. First of all, notice that by (V1) and (V3) we can apply the Michael’s selection theorem, so that there exists a continuous function $v : [0, T] \rightarrow \mathbb{R}$ such that

$$v(t) \in V(t), \text{ for every } t \in [0, T]. \tag{32}$$

Let us define the function $p : [0, T] \times [-R, R] \rightarrow \mathbb{R}$ as

$$p(t, x) = v(t) + g(t, x), \text{ } (t, x) \in [0, T] \times [-R, R]$$

and consider the integral equation

$$x(t) = a + p(t, x(t)) \int_0^{\varphi(t)} k(t, s) f(s, x(s)) ds, \text{ } t \in [0, T]. \tag{33}$$

Of course, a solution of (33) will be also a solution of (28).

We define now the solution operator for equation (33), i.e., $\Phi : \bar{B}(0, R) \rightarrow C([0, T])$,

$$\Phi(x)(t) = a + p(t, x(t)) \int_0^{\varphi(t)} k(t, s) f(s, x(s)) ds, \text{ } t \in [0, T].$$

Note that from (f5) the function $f(t, \cdot)$ is continuous on $[-R, R]$ for a.e. $t \in [0, T]$; so, taking into consideration also (f4), we have that for every $x \in \bar{B}(0, R)$ the function $f(\cdot, x(\cdot))$ is measurable on $[0, T]$; moreover, again by (f5), for a.e. $t \in [0, T]$ and every $|x| \leq R$ we get

$$|f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq l(t)(c + |x|) \leq l(t)(c + R). \tag{34}$$

In particular, by (k2) we get $k(t, \cdot) f(\cdot, x(\cdot)) \in L^1([0, T])$.

Fixed $x \in \bar{B}(0, R)$, by virtue of (34) and just taking K_∞, l instead of K, h , respectively, we can retrace the analogous part of the proof of Theorem 3.1 and claim that $\Phi(x)$ is continuous on $[0, T]$.

Moreover, we have $\Phi(\bar{B}(0, R)) \subset \bar{B}(0, R)$. Indeed, from (V2) and (13)

$$|p(t, x)| \leq N + G, \text{ for every } t \in [0, T] \text{ and every } |x| \leq R, \tag{35}$$

so that, using (φ) , (34), and (31), we have

$$\begin{aligned} |\Phi(x)(t)| &\leq |a| + (N + G)K_\infty \int_0^{\varphi(t)} |f(s, x(s))| ds \leq |a| \\ &\quad + (N + G)K_\infty \int_0^T l(s)(c + R) ds \\ &\leq |a| + (N + G)K_\infty \|l\|_1 (c + R) \\ &\leq R, \text{ for every } t \in [0, T] \text{ and every } |x| \leq R. \end{aligned}$$

Now, let us show that Φ is a contraction in $\bar{B}(0, R)$. First of all, observe that condition (g) yields that

$$|p(t, x) - p(t, y)| \leq L_g|x - y|, \text{ for every } t \in [0, T], |x| \leq R, |y| \leq R. \tag{36}$$

Hence, for every $x, y \in \bar{B}(0, R)$ and $t \in [0, T]$, one has

$$\begin{aligned} |\Phi(x)(t) - \Phi(y)(t)| &\leq |p(t, x(t)) - p(t, y(t))| \int_0^{\varphi(t)} |k(t, s)f(s, x(s))| ds \\ &\quad + |p(t, y(t))| \int_0^{\varphi(t)} |k(t, s)||f(s, x(s)) - f(s, y(s))| ds, \end{aligned}$$

so, by (34), (36), (f5), (31), and (35) we obtain

$$\begin{aligned} |\Phi(x)(t) - \Phi(y)(t)| &\leq L_g|x(t) - y(t)|K_\infty \int_0^{\varphi(t)} l(s)(c + R) ds \\ &\quad + (N + G)K_\infty \int_0^{\varphi(t)} l(s)|x(s) - y(s)| ds \\ &\leq \|x - y\|K_\infty \|l\|_1 [L_g(c + R) + N + G], \end{aligned}$$

where $K_\infty \|l\|_1 [L_g(c + R) + N + G] < 1$. Thus the solution operator Φ is a contraction.

We can, therefore, apply the Banach-Caccioppoli fixed point theorem, leading to the existence of a solution to (33), which is unique in $\bar{B}(0, R)$. Hence, the integral inclusion (28) has at least one solution, lying in $\bar{B}(0, R)$.

Finally, noticing that the continuous function v satisfies (32), which is exactly the control condition (29), we can say that the couple $\{x, v\}$ is an admissible pair to problem (28)–(29) with x satisfying the viability property (12). □

Remark 4.1. Notice that, analogously to what observed at the end of Theorem 3.1, see Remark 3.1, condition (31) can be slightly improved by the following:

$$K_\infty \|l\|_1 \leq \frac{R - |a|}{(N + G)(c + R)}, \quad K_\infty \|l\|_1 < \frac{1}{L_g(c + R) + N + G}.$$

5. Optimal estimates for the existence of solutions in the nonviable case

Here we first provide two results on the existence of viable solutions for the two types of nonlinear integral equations, naturally suggested by the study developed in Sects. 3 and 4.

Namely, from the proof of Theorem 3.1 we immediately have the following result.

Theorem 5.1. *Let (g) , (φ) , $(k1)$, $(f1)$, $(f2)$, be satisfied, and assume that $u : [0, T] \times [-R, R] \rightarrow \mathbb{R}$ is a continuous function with the properties*

(u1) *there exists $M \in [0, R - |a|[$ such that*

$$|u(t, x)| \leq M, \text{ for every } t \in [0, T], |x| \leq R.$$

(u2) *there exists $\alpha \in [0, 1/4[$ such that*

$$|u(t, x) - u(t, y)| \leq 4\alpha|x - y|, \text{ for every } t \in [0, T], |x|, |y| \leq R.$$

If (14) is fulfilled, then there exists at least one continuous map $x : [0, T] \rightarrow \mathbb{R}$ satisfying the integral equation

$$x(t) = a + u(t, x(t)) + g(t, x(t)) \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds, \quad t \in [0, T], \quad (37)$$

and the viability condition (12).

Similarly, Theorem 4.1 leads us to the next result.

Theorem 5.2. *Let $v : [0, T] \rightarrow \mathbb{R}$ be a continuous function and let (g) , (φ) , $(k2)$, $(f4)$, $(f5)$, (30) be satisfied. If*

$$K_\infty \|l\|_1 < \min \left\{ \frac{R - |a|}{(\bar{V} + G)(c + R)}, \frac{1}{L_g(c + R) + \bar{V} + G} \right\}, \quad (38)$$

where $\bar{V} = \max_{t \in [0, T]} |v(t)|$, then there exists a unique continuous map $x : [0, T] \rightarrow \mathbb{R}$ satisfying the integral equation

$$x(t) = a + [v(t) + g(t, x(t))] \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds, \quad t \in [0, T], \quad (39)$$

and the viability condition (12).

Moreover, for any $x_0 \in \bar{B}(0, R)$, the sequence $(x_n)_n$,

$$x_n(t) = a + [v(t) + g(t, x_{n-1}(t))] \int_0^{\varphi(t)} k(t, s)f(s, x_{n-1}(s))ds, \quad t \in [0, T], n \in \mathbb{N},$$

uniformly converges to the solution.

Proof. The assertion is a direct consequence of the application of the classical Banach-Caccioppoli Theorem to the solution operator $\Phi : \bar{B}(0, R) \rightarrow \bar{B}(0, R)$ defined by

$$\Phi(x)(t) = a + [v(t) + g(t, x(t))] \int_0^{\varphi(t)} k(t, s)f(s, x(s))ds, \quad t \in [0, T], x \in \bar{B}(0, R).$$

□

In case the viability condition is not assigned, then we can ask ourselves if we can choose the constant R in Theorem 5.1 and in Theorem 5.2 so as to optimize the conditions (14) and (38), respectively.

Let us first consider the result stated in Theorem 5.1. Assume that $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying (u2) but in $[0, T] \times \mathbb{R}$ and

(u1') there exists $M > 0$ such that

$$|u(t, x)| \leq M, \text{ for every } t \in [0, T], x \in \mathbb{R}.$$

Further, let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy (f1) and (f2) but on the whole $[0, T] \times \mathbb{R}$, and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function continuous on $[0, T] \times \mathbb{R}$ and such that

$$(g)_s \text{ for every } R > 0 \text{ there exists } L(R) > 0 \text{ such that} \\ |g(t, x) - g(t, y)| \leq L(R)|x - y|, \text{ for every } t \in [0, T], |x|, |y| \leq R. \quad (40)$$

Let us put $L_g(R) := \inf\{L(R) : L(R) \text{ satisfies (40)}\}$.

Notice that the constant $L_g(R)$ just defined and the constant $G = G(R)$ defined in (13) depend on R . Thus, in the present setting, they have to be considered as functions.

Let $P_1 : [M + |a|, +\infty) \rightarrow \mathbb{R}^+$ be the function

$$P_1(R) = \min \left\{ \frac{R - (M + |a|)}{G(R)(c + R)}, \frac{1 - 4\alpha}{L_g(R)(c + R)} \right\}, \quad R \geq M + |a|. \quad (41)$$

Since L_g, G are continuous, positive and nondecreasing, then P_1 is continuous and $\lim_{R \rightarrow \infty} P_1(R) = 0$. Then, as $P_1(M + |a|) = 0$, the function P_1 has maximum on $[M + |a|, +\infty)$. Let $\bar{R} > M + |a|$ be a point of maximum for P_1 , i.e.,

$$\bar{P}_1 = \max_{R \geq M + |a|} P_1(R) = P_1(\bar{R}). \quad (42)$$

Clearly, the choice of \bar{R} in (14) gives the optimal condition for $K\|h\|_1$. This result can be stated as follows.

Theorem 5.3. *Let (u1'), (g)_s, (φ), (k1) be satisfied and (u2), (f1), (f2) be fulfilled on $[0, T] \times \mathbb{R}$. If*

$$K\|h\|_1 < \bar{P}_1,$$

where \bar{P}_1 is defined in (42), then there exists at least one continuous map $x : [0, T] \rightarrow \mathbb{R}$ satisfying the integral equation (37).

Reasoning similarly as for the result stated in Theorem 5.2, if $v : [0, T] \rightarrow \mathbb{R}$ is continuous, (φ), (k2), (g)_s are fulfilled and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (f4) and (f5) on the whole $[0, T] \times \mathbb{R}$, then the function $P_2 : [|a|, +\infty) \rightarrow \mathbb{R}^+$

$$P_2(R) = \min \left\{ \frac{R - |a|}{(\bar{V} + G(R))(c + R)}, \frac{1}{L_g(R)(c + R) + \bar{V} + G(R)} \right\}, R \geq |a|, \quad (43)$$

has maximum on $[|a|, +\infty)$, and this maximum gives the optimal condition in (38). Put

$$\bar{P}_2 = \max_{R \geq |a|} P_2(R), \tag{44}$$

the following holds.

Theorem 5.4. *Let $v : [0, T] \rightarrow \mathbb{R}$ be a continuous function, let (f4), (f5) be satisfied on $[0, T] \times \mathbb{R}$ and (φ) , (k2), $(g)_s$ be fulfilled. If*

$$K_\infty \|l\|_1 < \bar{P}_2,$$

where \bar{P}_2 is defined in (44), then there exists at least one continuous map $x : [0, T] \rightarrow \mathbb{R}$ satisfying the integral equation (39).

We point out that, finding the optimal constants \bar{P}_1, \bar{P}_2 , it is possible, in some cases, to compare the existence results stated in Theorems 5.3 and 5.4. An example is given in the next section.

Remark 5.1. Let us note that all the existence theorems proved in Sects. 3, 4, 5 extend in a broad sense analogous results proved in [5, 6, 9] (see for instance next Remark 6.1).

6. Applications to the classical Chandrasekhar’s quadratic integral equation

6.1. Existence results for the viability problem

Consider the following Chandrasekhar viability problem

$$\begin{cases} x(t) = a + x(t) \int_0^T \frac{t}{t+s} w(s)x(s) ds, & t \in [0, T], \\ |x(t)| \leq R, & t \in [0, T]. \end{cases} \tag{45}$$

We are in position to state the following result.

Theorem 6.1. *Let $a, R \in \mathbb{R}$, $R > |a|$, and let $w : [0, T] \rightarrow \mathbb{R}$ be a continuous function such that $w(0) = 0$ and*

$$\max_{t \in [0, T]} |w(t)| < \frac{R - |a|}{TR^2}. \tag{46}$$

Then there exists a continuous solution $x : [0, T] \rightarrow \mathbb{R}$ for the Chandrasekhar viability problem (45).

Proof. Consider the setting

$$\begin{aligned} u(t, x) &= 0, & g(t, x) &= x, & f(t, x) &= x, & (t, x) &\in [0, T] \times [-R, R] \\ \varphi(t) &= T, & t \in [0, T], & & k(t, s) &= \begin{cases} 0, & t \in [0, T], s = 0 \\ \frac{t}{t+s} w(s), & t \in [0, T], s \in]0, T]. \end{cases} \end{aligned}$$

Then the Chandrasekhar integral equation in (45) can be seen as a particular case of (37).

Note that k is continuous on $[0, T] \times [0, T]$, i.e., (k1) holds. Moreover f satisfies (f1) and (f2) with $h(t) \equiv 1$ and $c = 0$. Further, g and u satisfy,

respectively (g) [with $L_g = 1$], and (u1) [with $M = 0$], (u2) [with $\alpha = 0$]. Finally, property (46) assures that (14) holds. Indeed, we have

$$\begin{aligned} K\|h\|_1 &\leq \max_{t \in [0, T]} |w(t)| T < \frac{R - |a|}{R^2} = \min \left\{ \frac{R - |a|}{R^2}, \frac{1}{R} \right\} \\ &= \min \left\{ \frac{R - (M + |a|)}{GR}, \frac{1 - 4\alpha}{L_g R} \right\}, \end{aligned}$$

being $G = R$ and $K = \max_{t, s \in [0, T]} |k(t, s)|$.

Then Theorem 5.1 can be applied, and (45) has a continuous solution. □

On the other hand, from Theorem 5.2, also the following result holds.

Theorem 6.2. *Let $a, R \in \mathbb{R}$, $R > |a|$, and let $w : [0, T] \rightarrow \mathbb{R}$ be a L^1 -function such that*

$$\|w\|_1 < \min \left\{ \frac{R - |a|}{R^2}, \frac{1}{2R} \right\}. \tag{47}$$

Then the Chandrasekhar viability problem (45) has a unique continuous solution $\bar{x} : [0, T] \rightarrow \mathbb{R}$ such that for every $x_0 \in \mathbb{R}$, $|x_0| \leq R$, the sequence $(x_n)_n$ defined by

$$\begin{aligned} x_0(t) &= x_0, \quad \forall t \in [0, T] \\ x_n(t) &= a + x_{n-1}(t) \int_0^T \frac{t}{t+s} w(s)x_{n-1}(s) ds, \quad \forall t \in [0, T], n \in \mathbb{N} \end{aligned}$$

uniformly converges to \bar{x} in $[0, T]$.

Proof. First of all, notice that, by considering the maps

$$\begin{aligned} v(t) &= 0 \text{ and } \varphi(t) = T, \quad t \in [0, T], \\ k(t, s) &= \begin{cases} 1, & t \in [0, T], s = 0 \\ \frac{t}{t+s}, & t \in [0, T], s \in]0, T]. \end{cases} \\ g(t, x) &= x \text{ and } f(t, x) = w(t)x, \quad (t, x) \in [0, T] \times [-R, R], \end{aligned}$$

the Chandrasekhar equation in (45) can be seen as a particular case of the integral equation (39). Further, f satisfies (f4) and (f5) [taking $l(\cdot) = |w(\cdot)|$ and $c = 0$], g satisfies (g) [with $L_g = 1$], φ and k trivially have properties (φ) and (k2), respectively. Finally, since $G = R$, $\bar{V} = 0$, $K_\infty = 1$ and $\|l\|_1 = \|w\|_1$, condition (38) reads as (47). Therefore, by applying Theorem 5.2, we have the assertions. □

6.2. Optimal estimates for the existence of solutions of (1)

Consider now the problem of the existence of a continuous solution for the Chandrasekhar equation (1).

If $w : [0, T] \rightarrow \mathbb{R}$ is a continuous function such that $w(0) = 0$, we can apply Theorem 5.3. Indeed, the analogous setting of the proof of Theorem 6.1 assures that equation (1) can be seen as a particular case of (37). Further, (u1'), (u2), (f1), (f2), (g)_s are satisfied on $[0, T] \times \mathbb{R}$, with $M = 0$, $\alpha = 0$, $h(t) =$

1, $c = 0$ and $L_g = 1$, and (φ) and (k1) hold. Since $G(R) = R$, then we have (see (41))

$$P_1(R) = \min \left\{ \frac{R - |a|}{R^2}, \frac{1}{R} \right\} = \frac{R - |a|}{R^2}, \quad R \geq |a|.$$

It is easy to verify that the function P_1 has on $[|a|, +\infty)$ a unique point of maximum \bar{R} , and

$$\bar{R} = 2|a|, \quad \bar{P}_1 = P_1(\bar{R}) = \frac{1}{4|a|}, \tag{48}$$

thus condition $K\|h\|_1 < \bar{P}_1$ in Theorem 5.3 becomes in this setting

$$\max_{t \in [0, T]} w(t)T < \frac{1}{4|a|}.$$

On the other hand, if $w : [0, T] \rightarrow \mathbb{R}$ is a L^1 -function, we can apply Theorem 5.4. The same setting of the proof of Theorem 6.2 assures that equation (1) can be seen as a particular case of (39), and (f4), (f5), (φ) , (k2) are satisfied with $l(\cdot) = |w(\cdot)|$ and $c = 0$, $(g)_s$ is fulfilled with $L_g = 1$. Since $G(R) = R$ and $\bar{V} = 0$ then we have (see (43))

$$P_2(R) = \min \left\{ \frac{R - |a|}{R^2}, \frac{1}{2R} \right\} = \begin{cases} \frac{R - |a|}{R^2}, & |a| \leq R \leq 2|a| \\ \frac{1}{2R}, & R > 2|a| \end{cases},$$

where $\bar{R} = 2|a|$ is the unique point of maximum of P_2 on $[|a|, +\infty)$, with

$$\bar{P}_2 = P_2(2|a|) = P_1(2|a|) = \frac{1}{4|a|}.$$

Condition $K_\infty\|l\|_1 < \bar{P}_2$ in Theorem 5.4 becomes in this setting (here $K_\infty = 1$)

$$\|w\|_1 < \frac{1}{4|a|}.$$

Since $\|w\|_1 \leq T \max_{t \in [0, T]} w(t)$, then for equation (1), Theorem 5.4 gives a more general condition for the existence of a continuous solution, i.e., the following holds.

Corollary 6.1. *Let $w : [0, T] \rightarrow \mathbb{R}$ be a L^1 -function such that*

$$\|w\|_1 < \frac{1}{4|a|}. \tag{49}$$

Then the Chandrasekhar equation (1) has a continuous solution $x : [0, T] \rightarrow \mathbb{R}$, satisfying $|x(t)| \leq 2|a|$, $t \in [0, T]$.

Remark 6.1. Usually in the literature, the existence of solutions to the Chandrasekhar equation is derived under the assumption that the characteristic function w appearing inside the integral is an even polynomial in s (cf. [7, Chapter 5]). Extensions are considered, for instance, in [5, 6, 9]. Corollary 6.1 extends the results for the Chandrasekhar equation in [5] and improves the (non-viable) existence results in [6] and [9]. Actually, in [6] the existence of a solution is achieved in the space $L^1(I)$, whereas here we prove the existence of

a (viable) continuous solution. Moreover, to prove the existence of a continuous solution, we require w to be a L^1 -function, instead of the more restrictive condition $w \in L^\infty(I)$ used in [5] and [6], or $w \in C(I)$ assumed in [9]. Finally, condition $\sup \text{ess}_{[0,1]} |w(t)| < 1/4$ required in [6] (or $\max_{[0,1]} |w(t)| < 1/4$ in [9]) implies (49). As a significant example, let us consider the characteristic function

$$w_\alpha(t) = t^\alpha, \quad \alpha > 0, t \in [0, 1].$$

Clearly, $w_\alpha \in C(I) \cap L^\infty(I) \cap L^1(I)$ and

$$\max_{t \in [0,1]} |w_\alpha(t)| = \sup \text{ess}_{t \in [0,1]} |w_\alpha(t)| = 1 > \frac{1}{4},$$

hence, whatever $\alpha > 0$, the existence theorems proved in [6] and [9] do not apply to the Chandrasekar equation governed by w_α . Analogously for the result in [5], since

$$\max_{t \in [0,1]} |w_\alpha(t)| = 1 > \frac{1}{4 \ln 2}.$$

On the other hand,

$$\|w_\alpha\|_1 = \frac{1}{\alpha + 1} < \frac{1}{4}, \quad \forall \alpha > 3.$$

Thus, for every $\alpha > 3$, the characteristic function w_α satisfies all the hypotheses required in Corollary 6.1.

6.3. Examples of controllability results

Here we consider the following controllability problem with viable condition, monitored by the Chandrasekar integral equation:

$$\begin{cases} x(t) = a + [v(t) + x(t)] \int_0^T \frac{t}{t+s} w(s)x(s) ds, & t \in [0, T], \\ v(t) \in V(t), & t \in [0, T], \\ |x(t)| \leq R, & t \in [0, T], \end{cases} \tag{50}$$

where $a \in \mathbb{R}$, and the control multimap $V : [0, T] \rightarrow \mathcal{P}(\mathbb{R})$ is defined by

$$V(t) = \begin{cases} \{0\}, & t = 0 \\ [-t, \beta(t)], & t \in]0, T], \end{cases}, \quad \text{where } \beta : [0, T] \rightarrow \mathbb{R}_0^+ \text{ is fixed.}$$

Our controllability result for the above mentioned problem reads as follows.

Theorem 6.3. *If $w : [0, T] \rightarrow \mathbb{R}$ is a given L^1 -function having the property (47), $\beta : [0, T] \rightarrow \mathbb{R}_0^+$ is a continuous map and $R > |a|$, then there exists an admissible pair $\{x, v\}$ to the viable problem (50).*

Proof. To apply Theorem 4.1, we consider the functions φ, k, g, f defined as in the proof of Theorem 6.2. Then, using the same arguments of the proof of Theorem 6.2 we can say that (f4), (f5), (φ), (k2), (g) and (38) hold. Moreover, the control-multimap V is H-lower semicontinuous in $[0, T]$ (see [25, 2.Remark] and [12, Corollario 3.2]), i.e., V satisfies property (V3) of Theorem 4.1, and V obviously has the properties (V1) and (V2). Therefore, using

Theorem 4.1 we can deduce that exists an admissible pair $\{x, v\}$, where x and v are both continuous on $[0, T]$, for the viable problem (50). \square

Finally, we obtain a controllability proposition for the following viable problem, governed again by a Chandrasekhar integral equation, but controlled by a feedback multimap, i.e.,

$$\begin{cases} x(t) = a + q(t, x(t)) + x(t) \int_0^T \frac{t}{t+s} w(s)x(s) ds, & t \in [0, T], \\ q(t, x(t)) \in [-\gamma_1(t, x(t)), \gamma_2(t, x(t))], & t \in [0, T], \\ |x(t)| \leq R, & t \in [0, T]. \end{cases} \tag{51}$$

Theorem 6.4. *If $a \in \mathbb{R}, R > |a|, w : [0, T] \rightarrow \mathbb{R}$ is a continuous function such that $w(0) = 0$ and (46) is fulfilled, $\gamma_1, \gamma_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are nonnegative continuous functions satisfying the conditions*

- (γ_1) *there exist $\beta_1, \beta_2 > 0$ with $\max\{\beta_1, \beta_2\} \leq R - |a|$, such that*

$$-\beta_1 \leq -\gamma_1(t, x) \leq 0 \leq \gamma_2(t, x) \leq \beta_2, \text{ for every } (t, x) \in [0, T] \times \mathbb{R};$$
- (γ_2) *there exists $\alpha \in [0, \frac{1}{4}[$ such that*

$$|\gamma_i(t, x) - \gamma_i(t, y)| \leq \alpha|x - y|, \text{ for every } (t, x), (t, y) \in [0, T] \times \mathbb{R}, i = 1, 2;$$

then there exists an admissible pair $\{x, u\}$ to the viable problem (51).

Proof. Define the multimap $U : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$U(t, x) = [-\gamma_1(t, x), \gamma_2(t, x)], \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Using (γ_1), for every $(t, x) \in [0, T] \times \mathbb{R}$ we have

$$0 \in U(t, x), \text{ and } \|U(t, x)\| \leq \max\{\beta_1, \beta_2\} := M \leq R - |a|.$$

Therefore, the multimap U has nonempty values and satisfies (U2). Note that (U1) is trivially satisfied. Moreover, due to the continuity of γ_1, γ_2 , it is easy to prove that U is H-continuous in $[0, T] \times \mathbb{R}$. To show that also hypothesis (U3) is fulfilled, it remains to show that

$$d_H(U(t, x), U(t, y)) \leq \alpha|x - y|, \text{ for every } (t, x), (t, y) \in [0, T] \times \mathbb{R}, \tag{52}$$

where α is the constant in (γ_2). Let $(t, x), (t, y) \in [0, T] \times \mathbb{R}$ be fixed. Then, for every $z \in U(t, x)$ we have (see (γ_2))

$$\begin{aligned} \delta(z, U(t, y)) &\leq \max\{-\gamma_1(t, y) - z, z - \gamma_2(t, y)\} \\ &\leq \max\{-\gamma_1(t, y) + \gamma_1(t, x), \gamma_2(t, x) - \gamma_2(t, y)\} \\ &\leq \alpha|x - y|, \end{aligned}$$

and then

$$\sup_{z \in U(t, x)} \delta(z, U(t, y)) \leq \alpha|x - y|.$$

Analogously, we have

$$\sup_{q \in U(t, y)} \delta(q, U(t, x)) \leq \alpha|x - y|,$$

and (52) holds.

Now, with the setting

$$g(t, x) = x, f(t, x) = x, (t, x) \in [0, T] \times \mathbb{R}$$

$$\varphi(t) = T, t \in [0, T], \quad k(t, s) = \begin{cases} 0, & t \in [0, T], s = 0 \\ \frac{t}{t+s}w(s), & t \in [0, T], s \in]0, T], \end{cases}$$

using the same arguments as in the proof of Theorem 6.1, assumptions (k1), (f1), (f2), (g) are satisfied. Moreover, (46) assures that (14) holds. Therefore, from Theorem 3.1 it follows that an admissible pair $\{x, u\}$ exists for the viable problem (51). \square

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