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Cite this article: Mariano PM. 2023 Proof of Straughan's claim on Payne–Song's and modified Guyer–Krumhansl's equations. *Proc. R. Soc. A* **479**: 20230439.
https://doi.org/10.1098/rspa.2023.0439

Received: 18 June 2023
Accepted: 16 October 2023

Subject Areas:
applied mathematics, mathematical modelling, mathematical physics

Keywords:
heat conduction, complex bodies, microstructures, phase fields

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Proof of Straughan's claim on Payne–Song's and modified Guyer–Krumhansl's equations

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We prove a claim by Brian Straughan who conjectured that Payne–Song's and modified Guyer–Krumhansl's equations can be justified in (and derived from) the general model-building framework for the mechanics of complex bodies, so they emerge from the modelling of microstructural effects. The proof is based on taking into account micro-to-macro spatial scaling and a decomposition of the heat flux into a standard Fourier-type component and one measuring microstructural event with associated and balanced actions.

1. Introduction

In an elegant analysis of stationary and oscillatory convection in a Brinkman–Darcy–Kelvin–Voigt fluid, Straughan [1] (see also [2–5]) considered a system of evolution laws including—beyond a regularized balance of momentum—Payne–Song's equation [6] in Eulerian representation, that is an energy balance given by

$$\dot{T} = \kappa \Delta T - \mathrm{div}\mathsf{q}, \tag{1.1}$$

where the superposed dot indicates from now on total derivative with respect to time; T is the absolute temperature, κ the conductivity, taken to be constant, and q the heat flux, which satisfies, *per se*, a version of Guyer–Krumhansl's equation [7,8], given by

$$\ell \frac{\mathcal{D}\mathsf{q}}{\mathcal{D}t} = -\mathsf{q} - \kappa \nabla T + \hat{\varsigma}_1 \Delta \mathsf{q} + \hat{\varsigma}_2 \nabla \mathrm{div}\mathsf{q}, \tag{1.2}$$

with ℓ a time delay and $\mathcal{D}/\mathcal{D}t$ a generic objective derivative, which he considers in the analysis to be the Lie derivative with respect to the macroscopic fluid velocity.

Such equations are in a set of proposals formulated to avoid that, according to Fourier's diffusive law, local perturbations of temperature are instantaneously felt everywhere, which is against causality (see the treatise [9]).

In commenting on equations, in particular Payne–Song's one, Straughan wrote as follows: 'I believe it may be justified from work of Mariano [10, Section 2]'. Here, we prove such a statement.

Indeed, Mariano [10] showed that experimentally recorded finite speed heat propagation can be attributed to effects of temperature-dependent microstructure. The analysis adapts thermodynamic principles to the presence of active material microstructure. The resulting scheme, formulated neglecting macroscopic strain [10], foresees finite-speed heat propagation. Explicit closed-form solutions for pertinent waves with three admissible velocities for the temperature propagation have been derived by Mariano & Spadini [11].

Furthermore, Capriz *et al*. [12] also showed (in one-dimensional setting) that both Maxwell–Cattaneo's and Guyer–Krumhansl's equations are a special offspring of the balance of microstructural interactions, under the presence of appropriate internal constraints linking microstructural descriptors to the heat flux (see also [11,13]). Notice that Guyer–Krumhansl's equation, otherwise largely analysed (see e.g. [14–19] and references therein), involves only the common total time derivative of the heat flux, while the version (1.2) including an objective time derivative has been proposed by Morro [20,21].

Here, in addition to Straughan's claim, we prove that even equation (1.2) emerges from the general model-building framework for the mechanics of complex bodies. The main conceptual ingredients of the analysis developed here go as follows:

— In non-isothermal setting, we consider incompressible viscous complex fluids with microstructure described by a manifold-valued field [22],[1] with values ν referring to a spatial scale λ (here ν is not a scalar).

— *Per se* ν is an observable entity, meaning that it is sensitive to changes of observer along which it varies according to its geometric nature.

— Microstructural actions develop power in the time-variation of ν. They have bulk and contact nature. Their balance emerges from an invariance requirement for the *external power alone* of all standard and microstructural actions over a generic body part with

[1] *Historical note.* The traditional format of continuum mechanics refers to the foundational program developed by C. A. Truesdell's school, starting from W. Noll's work. That format was (in a sense) 'sculptured' in two articles of the Handbuk der Physik [23,24] and motivated further foundational work (see the treatises [25–28]). According to it, a body is considered as an abstract set $\mathfrak{B}$ of the so-called material elements—not otherwise specified—and is presumed to be endowed with (finite-dimensional) manifold structure. Embeddings into (say) $\mathbb{R}^k$ determine its geometric representation: a fit region $\mathcal{B} \subset \mathbb{R}^k$. In this picture, every material element is ideally represented only by a point. Crowding and shearing of neighbouring elements determine interactions described by Cauchy's stress. They are balanced by bulk external actions. The second law of thermodynamics restricts possible constitutive choices. Already before the raising of Truesdell's program, notes by Voigt [29,30] apparently suggested to the brothers E. and F. Cosserat to consider every material element as a 'small' rigid-body, able to rotate independently from its neighbours [31]. Circumstantial reasons left that work aside until in 1958 Ericksen and Truesdell [32] raised attention over it as a setting for building up direct models for shells, rods and their like (notice that even in reference [24, Secs 200, 203, 205], Truesdell and Toupin detach from the path now traditional to discuss Cosserat's viewpoint). Generalizing Cosserat's view, each point of $\mathcal{B}$ acquires additional degrees of freedom with respect to the ones of a mass point in the physical space. The generic material element can be thus portrayed as a system rather than a point. Since 1960, Ericksen adopted this perspective to describe the mechanics of liquid crystals [33–35]. A subsequent rich crop of proposals emerged: different types of additional degrees of freedom were adopted for various physical circumstances. A foundational problem concerning the wide taxonomy of specific models was thus to find whether they can be unified. A proposal in this sense by Germain dates back to 1973 [36]: he considered descriptors of the material morphology in a linear space and the principle of virtual work as a guiding rule. So, he assumed *a priori* the weak form of balance equations, presuming the representation of microstructural interactions. In a 1989 book [22], Capriz summarized and pushed forward aspects of his previous work on this matter; he proposed a unifying framework in which descriptors of material morphology are taken in a generic finite-dimensional differentiable manifold. In fact, he coupled strain with the general view for the representation of microstructures adopted in condensed matter physics [37]. He postulated balance equations in local form, including the balance of microstructural interactions and its link with the one of couples. In question was whether the representation of microstructural actions and their balance could be derived from a more fundamental principle. In 1995, Green and Naghdi suggested to start from the balance of energy, requiring its invariance under changes of observer [38]. The procedure excludes emergence of dissipative stress components. Invariance of the external power alone under $SO(3)$-based changes of observer has been suggested by Mariano [39,40]; such a view is adopted, commented on and refined in the present paper. An approach based on measures over manifolds of maps has been proposed by Segev [41]. Eventually, it is possible to show that requiring invariance of the second law under general diffeomorphism-based changes of observers allows one to derive the representation of contact interactions and the balance laws, in addition to constitutive restriction (proof is in [42]).

non-vanishing volume, under rigid-body-type changes of observer in the physical space and their effects on the space containing ν. The approach follows a procedure developed and progressively refined by Capriz & Mazzini [43] (where the direct action of $SO(3)$ over ν is admitted), [39,40]. It extends to the present multi-field setting Noll's invariance request [44] for the external power of bodies with no direct description of the microstructural morphology—we call them Cauchy's bodies. (Here, the expression of internal—or inner—power of actions is not postulated, as it is done when one starts from the principle of virtual power, that is directly postulating the weak form of balance equations and assuming *a priori* in this way the representation of deformation-conjugated and microstructural interactions in terms of stresses and a microstructural self-action. At variance, we *do not postulate* a form of the internal power, *deriving* it under appropriate conditions.)

— Balances of standard and microstructural actions are supplemented by the first law and the second law of thermodynamics.

— We consider q in equation (1.1) as a portion of total heat flux $\hat{\mathsf{q}}$, which admits the additive decomposition

$$\hat{\mathsf{q}} = -\kappa \nabla T + \mathsf{q}.$$

As mentioned above, we consider every material element at continuum level as a system characterized by a spatial scale λ, rather than an indistinct mass point. The temperature T emerges from averaged evaluations on the ensemble constituting every material element (be it canonical or grand-canonical, the latter feature when transport of microstructure occurs). The vector q represents the fluctuation to $\kappa \nabla T$ determined by microstructural events below λ, a scale that we leave unspecified at the present level of generality.

— It is crucial to consider the vector q as a term of order λ – and we write $\mathsf{q} \sim O(\lambda)$. In this sense, with $f(\mathsf{q}, \nabla \mathsf{q})$ a differentiable function of its arguments, we have

$$\frac{\partial f}{\partial \mathsf{q}} \cdot \mathsf{q} \sim O(\lambda^2) \quad \text{and} \quad \frac{\partial f}{\partial \nabla \mathsf{q}} \cdot \nabla \mathsf{q} \sim O(\lambda^2), \tag{1.3}$$

where the interposed dot indicates duality pairing; it corresponds to the standard scalar product when the metric in space is flat and trivial, meaning that it coincides with the second-rank unit tensor with covariant components.

— Then we impose the internal constraint $\nu = \mathsf{q}$ with additional specific constitutive choices for the interactions considered. We will involve the action of $SO(k)$ in describing changes of observer in the physical space and will consider volume-preserving flows. Under such actions, q behaves as a vector. Under the action of $O(k)$, the full orthogonal group, not involved here, q would display its nature of a pseudo-vector.

To within $O(\lambda^2)$ terms, Payne–Song's equation coincides with the local balance of energy that is appropriate for this setting when there are no heat sources, and viscous components of standard stresses are absent or their power can be negligible with respect to temperature and heat flux contributions.

The balance of microstructural actions reduces to the modified Guyer–Krumhansl equation (1.2) when

(i) the microstructural self-action includes a dissipative component,
(ii) the free energy depends on q, its gradient, and T in a specific way,
(iii) the microstructural bulk actions have only non-inertial component $\hat{\beta}$, which in an orthonormal frame is such that

$$\int_{\hat{\mathfrak{b}}} \hat{\beta} \cdot \mathsf{q} \, \mathrm{d}y = -\int_{\hat{\mathfrak{b}}} \left(\bar{\mathsf{C}} \nabla v \cdot (\mathsf{q} \otimes \mathsf{q}) + \kappa \nabla T \cdot \mathsf{q} \right) \mathrm{d}y, \tag{1.4}$$

for any choice of q and any connected subset $\hat{\mathfrak{b}}$ of the current configuration ($\bar{\mathsf{C}}$ is a constant adjusting physical dimensions). Equation (1.4) is nothing more than the 'weak version' of

$\hat{\beta}$. However, assuming it, not directly postulating its strong form, allows us to clarify the physics justifying the present choice (exactly as it happens when we identify the inertial component of standard bulk forces by postulating that their power equals the negative of kinetic energy time rate). Precisely, with equation (1.4) we presume that non-uniformity of macroscopic flow and temperature field determines a bulk action over the microstructure, in particular over the heat flux perturbation induced by it (imagine a microstructure that is chemically active).

When we neglect $O(\lambda)$ terms, the theory reduces to the classical Fourier-type conduction in simple fluids.

In summary, we focus on the thermodynamics of fluids with microstructure, the effect of which determines fluctuations $\mathfrak{q}$ with respect to $\kappa\nabla T$. We furnish a general Eulerian description of them, starting from the introduction of manifold-valued phase fields, the time rates of which complement the velocity field. Along this path, we prove Straughan's claim under specific assumptions. We start from a general setting because Straughan's claim refers explicitly to it.

Notations. Consider two finite-dimensional, real, linear spaces $\mathcal{V}$ and $\mathcal{W}$. Let A be an operator mapping linearly $\mathcal{V}$ onto $\mathcal{W}$; in short we write $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$. For $B \in \mathrm{Hom}(\mathcal{W},\mathcal{V})$, we indicate by AB the unique linear operator in $\mathrm{Hom}(\mathcal{V},\mathcal{V})$ with components $A^i_L B^L_j$, where summation over repeated indices is adopted, as we do also below; also, in general, such a product does not commute. We indicate by $\mathcal{V}^*$ and $\mathcal{W}^*$ the dual counterparts of $\mathcal{V}$ and $\mathcal{W}$, namely the spaces of linear maps over $\mathcal{V}$ and $\mathcal{W}$, respectively. For $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$ and $v \in \mathcal{V}$, we indicate by Av the element in $\mathcal{W}$ with components $w^i := A^i_L v^L$; also, if $A \in \mathrm{Hom}(\mathcal{V}^*,\mathcal{W}^*)$ and $a \in \mathcal{V}^*$, Aa will indicate an element $\bar{a} \in \mathcal{W}^*$ with components $\bar{a}_i := A^L_i a_L$. For $P \in \mathrm{Hom}(\mathcal{W}^*,\mathcal{V}^*)$ and $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$, an interposed dot, namely $P \cdot A$, indicates the duality pairing between P and A. The same meaning is attributed to the product $a \cdot v$, with $a \in \mathcal{V}^*$ and $v \in \mathcal{V}$; more precisely, since a is a linear form over $\mathcal{V}$, $a \cdot v$ is defined to be the real value $a(v)$ attained by a over v. We write A^* for the formal adjoint of $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$ and we have $A^* \in \mathrm{Hom}(\mathcal{W}^*,\mathcal{V}^*)$. $A^\top$ is the transpose of A and is an element of $\mathrm{Hom}(\mathcal{W},\mathcal{V})$. For $A \in \mathrm{Hom}(\mathcal{V},\mathcal{V}^*)$, formal adjoint and transpose can be identified; the same property holds when $A \in \mathrm{Hom}(\mathcal{V}^*,\mathcal{V})$. The symbol $\otimes$ indicates as usual the tensor product; for $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$, a natural basis for $\mathrm{Hom}(\mathcal{V},\mathcal{W})$ is given by the tensor products $\hat{\mathbf{e}}_i \otimes \mathbf{e}^L$, where $\hat{\mathbf{e}}_i$ is the ith vector of a basis in $\mathcal{W}$ while $\mathbf{e}^L$ is the Lth element of a basis in $\mathcal{V}^*$, defined to be such that for every element $\mathbf{e}_K$ of a basis in $\mathcal{V}$, $\mathbf{e}^L(\mathbf{e}_K) = \delta^L_K$, where δ^L_K is the 1-contravariant, 1-covariant Kronecker's symbol. For $A \in \mathrm{Hom}(\mathcal{V},\mathcal{W})$, $v \in \mathcal{V}$, and $\bar{a} \in \mathcal{W}^*$, we have $\bar{a} \cdot Av = A^*\bar{a} \cdot v$.

2. Geometry of motion: basic fields

(a) Deformations and gross motions

Consider two isomorphic copies of the three-dimensional real space, namely $\mathbb{R}^k$ and $\tilde{\mathbb{R}}^k$, the isomorphism being simply the identification. Typically, we take $k = 3$.

Take in $\mathbb{R}^k$ an open connected region $\mathcal{B}$. Presume that its boundary has non-zero $\mathcal{H}^{k-1}$ Hausdorff measure (so it is surface-like in the k-dimensional space) and is oriented by the outward unit normal everywhere to within a finite number of corner and edges. Non-singular metrics g and $\tilde{g}$ are assigned to $\mathcal{B}$ and the space $\tilde{\mathbb{R}}^k$, respectively. They are independent of time. $\mathcal{B}$, a fit region in $\mathbb{R}^k$, is a (macroscopic) reference configuration, the result of an embedding in $\mathbb{R}^k$ of a material manifold $\mathfrak{B}$.

Deformations are commonly taken as differentiable, orientation preserving, one-to-one maps $x \longmapsto y := \tilde{y}(x) \in \tilde{\mathbb{R}}^k$. So, $\tilde{y}(\cdot, t)$ is a map from $\mathbb{R}^k$ to $\tilde{\mathbb{R}}^k$ with domain $\mathcal{B}$. The set $\mathcal{B}_c := \tilde{y}(\mathcal{B})$ is the current macroscopic configuration. Time-parametrized families of deformations define gross motions

$$(x, t) \longmapsto y := \tilde{y}(x, t) \in \tilde{\mathbb{R}}^k, \tag{2.1}$$

with the time t ranging in the real line $\mathbb{R}$. The map $\tilde{y}$ is assumed to be twice differentiable with respect to time.

Given (x,t), the time derivative $\partial\tilde{y}(x,t)/\partial t$, indicated in short by $\dot{y}$, defines a vector in the tangent space $T_y\mathcal{B}_c$ at $y\in\mathcal{B}_c$: it is a so-called Lagrangian representation of the velocity. At every t, the map $x\longmapsto\dot{y}$ defines a vector field over $\mathcal{B}_c$ that can be considered as a section of the tangent bundle $T\mathcal{B}_c=\sqcup_{y\in\mathcal{B}}T_y\mathcal{B}_c$ (so, defined to be the disjoint union of all tangent spaces of $\mathcal{B}_c$), a section described by a field $(y,t)\longmapsto v:=\tilde{v}(y,t)$, that is what we call the Eulerian representation of the velocity, so that $\dot{y}=v$ at every y and t.

F indicates the derivative $D_xy=D_xy(x,t)=(\partial y^i/\partial x^A)\tilde{\mathbf{e}}_i\otimes\mathbf{e}^A$, where $\tilde{\mathbf{e}}_i$ is the ith element of a basis in $\tilde{\mathbb{R}}^{k*}$, $\mathbf{e}^A$ is the Ath element of a dual basis in $\mathbb{R}^k$, and the subscript x distinguishes the derivative with respect to x from the analogous operator D computed with respect to y and used below. As it is common, we call F a *deformation gradient*.

Consider the gradient ∇_x of $\tilde{y}(\cdot,t):\mathcal{B}(\subset\mathbb{R}^k)\longrightarrow\tilde{\mathbb{R}}^k$, taking into account that $\mathbb{R}^k$ and $\tilde{\mathbb{R}}^k$ are distinguished only by an isomorphism, that is, specifically, the identification (once again the subscript x distinguishes the gradient computed with respect to x from the analogous operator referring to y, as used below). In defining ∇_x, we assume that there exists a frame of reference in which $\nabla_xy=\nabla_x\tilde{y}(x,t)=(\partial y^i/\partial x)^A\tilde{\mathbf{e}}_i\otimes\mathbf{e}_A$. Then, with respect to that frame, we compute Christoffel's symbols expressing ∇_x in 'curved' frames with respect to the one involving only $(\partial y^i/\partial x)^A$ (e.g. [45]). In that specific frame of reference ∇_xy and D_xy are related by $(\partial y^i/\partial x)^Ag_{AB}=(Dy)^i_B$; so, they can be identified when g_{AB} coincides with δ_{AB}, that is Kronecker's symbol with covariant components.

The requirement that each motion be orientation preserving implies the standard nonlinear constraint

$$\det F>0,\quad \forall x,t. \tag{2.2}$$

We have $\dot{F}=DvF$. Euler's lemma dictates the relation $\dot{\overline{\det F}}=(\det F)\mathrm{div}v$, so that volume-preserving motions are characterized, as it is well known, by the internal constraint $\mathrm{div}v=0$ because the variation of volume, relative to the initial one is measured by the difference $\det F-1$.

(b) Manifold-valued phase fields describing the microstructural morphology

As already mentioned, in the multi-scale and multi-field view adopted here, a material element is considered as a system characterized by a certain spatial scale λ. Its peculiar features are described by a variable ν, the value of a time-dependent field $(x,t)\longmapsto\nu:=\hat{\nu}(x,t)$, with $x\in\mathcal{B}$ or $(y,t)\longmapsto\nu:=\tilde{\nu}_c(y,t)$, $y\in\mathcal{B}_c:=\tilde{y}(\mathcal{B},t)$, with $\tilde{\nu}_c(\cdot,t)=\hat{\nu}(\cdot,t)\circ[\tilde{y}(\cdot,t)]^{-1}$. The former is a Lagrangian representation of ν, the latter is an Eulerian description.

For example, consider the generic material element at x as a tiny rigid body able to freely rotate with respect to its neighbours. In this case, with $\mathcal{B}\subset\mathbb{R}^k$, the descriptor ν is an element of $SO(k)$ or we can select it in the sphere S^{k-1}. Vectors in $\mathbb{R}^k$ can be chosen when ν represents local polarization or magnetization and ranges into a sphere, but also the choice $\nu\in\mathbb{R}^k$ comes into play to describe linear molecular chains as an end-to-tail stretchable vector.

Several other examples can be added. We can unify them by simply declaring ν to be an element of a finite-dimensional differentiable geodesic-complete Riemannian manifold without boundary—write $\mathcal{M}$ for it—endowed with metric $g_{\mathcal{M}}$. This choice is a starting point for determining a unified framework for the mechanics of complex materials (see [22,39–41,46] and references therein).

By considering $\nu=\tilde{\nu}_c(\tilde{y}(x,t),t)$, we obviously have

$$N:=D_x\nu=D\tilde{\nu}_c(y,t)F. \tag{2.3}$$

The time-rate of ν in Lagrangian representation is $\dot{\nu}:=\dot{\hat{\nu}}(x,t)=\partial\hat{\nu}(x,t)/\partial t$, while we write ξ for the same rate expressed in Eulerian representation, namely

$$\xi:=\tilde{\xi}(y,t)=\frac{\mathrm{d}\tilde{\nu}_c(y,t)}{\mathrm{d}t}=\frac{\partial\tilde{\nu}_c(y,t)}{\partial t}\Big|_{y\,\mathrm{fixed}}+(D\nu)v. \tag{2.4}$$

So, we compute

$$\dot{\overline{Dv}} = \dot{\overline{NF^{-1}}} = D\xi - DvDv, \tag{2.5}$$

where $DvDv$ is the second-rank tensor with components $(Dv^\alpha)_i(Dv^i)_j$, where the index α indicates coordinates over $\mathcal{M}$.

(c) Observers

An observer is the choice of reference frames in *all* spaces that are involved in the description of body morphology and its motion.

We consider rigid-body-type changes of observer in the physical space $\tilde{\mathbb{R}}^k$ and their consequences on $\mathcal{M}$ justified below. We leave invariant the reference space $\mathbb{R}^k$ and the time scale.

In $\tilde{\mathbb{R}}^k$, an observer $\mathcal{O}$ records a place y that is y' for another observer $\mathcal{O}'$ connected with $\mathcal{O}$ by a rigid-body motion so that $y' = w(t) + Q(t)(y - y_0)$, where $t \longmapsto w(t) \in \tilde{\mathbb{R}}^k$ and $t \longmapsto Q(t) \in SO(k)$ are smooth maps, and y_0 is an arbitrary fixed point.

Let $\dot{y}$ be a velocity evaluated by $\mathcal{O}$ in $\tilde{\mathbb{R}}^k$ and $\dot{y}'$ the corresponding value recorded by $\mathcal{O}'$. The pull-back of $\dot{y}'$ in the frame of reference defining $\mathcal{O}'$, namely $\dot{y}^\diamond := Q^\top \dot{y}'$, is given by

$$\dot{y}^\diamond = c + q \times (y - y_0) + \dot{y}, \tag{2.6}$$

where $c := Q^\top \dot{w}$ and $q\times := Q^\top \dot{Q}$. Also, due to the identity $\dot{y} = v$, we have

$$v^\diamond = c + q \times (y - y_0) + v. \tag{2.7}$$

Changing observer in the physical space $\tilde{\mathbb{R}}^k$ possibly implies a change in how the observer perceives (and represents) an observable microstructure, which belongs, in fact, to the physical space. Adopting the representation of microstructure in terms of a field taking values over $\mathcal{M}$ is only a convenient modelling choice. Also, when we discuss in mathematical terms of observers and their changes, more in general of observability, we are essentially formalizing what occurs in a 'laboratory'-type observation of nature and try to reduce at minimum subjective instances, requiring invariance of some entities under changes of observer. Consequently, when we rotate and translate frames of references in the physical space, or alter them in other ways, we need to describe how the observation of microstructure by an observer coinciding with the changed frame is altered: two different observers record two observations that, in principle, can be different. We thus need to represent over $\mathcal{M}$ such a difference because $\mathcal{M}$—we repeat—is the ambient in which we describe peculiar microstructural features. For a formal description of this physical ground, we need to presume a (possibly empty) family of differentiable homeomorphisms

$$\{\phi : SO(k) \longrightarrow \text{Diff}\,(\mathcal{M}, \mathcal{M})\}, \tag{2.8}$$

so that the counterpart of $\dot{y}^\diamond$ for ξ is

$$\xi^\diamond = \xi + \mathcal{A}(\nu)q, \tag{2.9}$$

with $\mathcal{A}(\nu) \in \text{Hom}(\tilde{\mathbb{R}}^k, T_\nu\mathcal{M})$.

When the set $\{\phi\}$ is not empty, with $\nu_{\phi(Q)}$ the value of ν after the action of $\phi(Q) \in \text{Diff}(\mathcal{M}, \mathcal{M})$ (the explicit expression of $\nu_{\phi(Q)}$ depends on the tensor nature of ν and $\phi(Q)$), the linear operator $\mathcal{A}(\nu)$ is given by

$$\mathcal{A}(\nu) = \frac{\mathrm{d}\nu_{\phi(Q)}}{\mathrm{d}\phi} \frac{\mathrm{d}\phi(Q)}{dq}\Big|_{q=0}, \tag{2.10}$$

where we take into account that $Q = \exp(q\times)$. For example, when $\mathcal{M}$ coincides with $\tilde{\mathbb{R}}^k$, we find $\mathcal{A} = -\nu\times$ because in this special case ν becomes $\nu' := Q\nu$ so that $\dot{\nu}' := \dot{Q}\nu + Q\dot{\nu}$ and the pull-back through $Q^\top$ is $\dot{\nu}^\diamond := Q^\top\dot{\nu}' = \dot{\nu} + Q^\top\dot{Q}\nu = \dot{\nu} + q \times \nu$. It does not necessarily mean that $\mathcal{A}$ reduces always to $-\nu\times$ in some limit. For example, when ν is a second-rank tensor, so that we have $\nu' = Q\nu Q^\top$, we can proceed as above computing the time derivative of ν' and pulling it back through $Q^\top$, so that we find $\mathcal{A}(\nu)q = [W, \nu]$, where W is the skew-symmetric second-rank tensor $Q^\top\dot{Q}$ and $[\cdot, \cdot]$ indicates the Lie bracket.

Notice that $\nu^{\diamond}$ is *insensitive* to rigid translations in space of the whole body. In fact, ν at a point describes what is *inside* the material element placed there: it brings information about the inner structure, which translates with the point and changes independently of the translation itself.

Remark 2.1. The introduction of $\{\phi\}$ allows us to include in this treatment the case in which $\{\phi\}$ is empty, a circumstance in which ν would be insensitive to changes of observer; in this case it would behave as an internal variable taking parametric role at equilibrium. Thus, in this case it just describes the removal from equilibrium; differently from what we discuss here, its evolution is ruled by a phenomenological law involving thermodynamic affinities that contribute to the entropy production but not to the mechanical power, so they do not satisfy balances of true interactions, according to the traditional internal variable scheme [47,48]. So, the set $\{\phi\}$ allows us to distinguish between observable microstructures—meaning $\{\phi\}$ is not empty, as we consider here—and non-observable ones—$\{\phi\}$ is empty. The exception seems to be the case in which $\mathcal{M}$ coincides with an open interval of the real line, since $\mathcal{A}$ would vanish because the observation of a scalar (or a pseudo-scalar) does not change under the action of $SO(k)$ in the physical space. However, if we would accept (or consider) more general changes of observers in the physical space, those induced by non-isometric elements of $\mathrm{Diff}(\tilde{\mathbb{R}}^k, \tilde{\mathbb{R}}^k)$, we could develop the previous analysis changing the set (2.8) into $\{\phi : \mathrm{Diff}(\tilde{\mathbb{R}}^k, \tilde{\mathbb{R}}^k) \longrightarrow \mathrm{Diff}(\mathcal{M}, \mathcal{M})\}$; in this case, a scalar (or a pseudo-scalar) could be sensitive to changes of observer depending on whether this last set is non-empty and, when it is not empty, what kind of ϕ we include in it, depending on the phenomenon under analysis.

3. Eulerian representation of balance of interactions in complex media: consequences of an invariance requirement

Interactions on a body part $\hat{\mathfrak{b}}$ of $\mathcal{B}_c$ are defined by the power that they develop. We define the external power over $\hat{\mathfrak{b}}$ to be the functional

$$\mathcal{P}^{\mathrm{ext}}_{\hat{\mathfrak{b}}}(v, \xi) := \int_{\hat{\mathfrak{b}}} (\hat{b}^{\ddagger} \cdot v + \hat{\beta}^{\ddagger} \cdot \xi)\, \mathrm{d}y + \int_{\partial \hat{\mathfrak{b}}} (\hat{\mathfrak{t}}_{\partial} \cdot v + \hat{\tau}_{\partial} \cdot \xi)\, \mathrm{d}\mathcal{H}^{k-1}(y). \tag{3.1}$$

We impose invariance of the external power under changes of observers defined above, according to a path discussed Mariano [39,40], namely we require

$$\mathcal{P}^{\mathrm{ext}}_{\hat{\mathfrak{b}}}(v, \xi) = \mathcal{P}^{\mathrm{ext}}_{\hat{\mathfrak{b}}}(v^{\diamond}, \xi^{\diamond}) \tag{3.2}$$

for any choice of c and q, which depend only on time, and $\hat{\mathfrak{b}}$. Equation (3.2) implies $\mathcal{P}^{\mathrm{ext}}_{\hat{\mathfrak{b}}}(c + q \times (y - y_0), \mathcal{A}q) = 0$, that is

$$c \cdot \left(\int_{\hat{\mathfrak{b}}} \hat{b}^{\ddagger}\, \mathrm{d}y + \int_{\partial \hat{\mathfrak{b}}} \hat{\mathfrak{t}}_{\partial}\, \mathrm{d}\bar{\mathcal{H}}^2(y) \right) + q \cdot \left(\int_{\hat{\mathfrak{b}}} ((y - y_0) \times \hat{b}^{\ddagger} + \mathcal{A}^* \hat{\beta}^{\ddagger})\, \mathrm{d}y + \int_{\partial \hat{\mathfrak{b}}} ((y - y_0) \times \hat{\mathfrak{t}}_{\partial} + \mathcal{A}^* \hat{\tau}_{\partial})\, \mathrm{d}\bar{\mathcal{H}}^2(y) \right) = 0. \tag{3.3}$$

The arbitrariness of c and q implies the common integral balance of forces and a non-standard integral balance of couples, namely

$$\int_{\hat{\mathfrak{b}}} \hat{b}^{\ddagger}\, \mathrm{d}y + \int_{\partial \hat{\mathfrak{b}}} \hat{\mathfrak{t}}_{\partial}\, \mathrm{d}\bar{\mathcal{H}}^2(y) = 0 \tag{3.4}$$

and

$$\int_{\hat{\mathfrak{b}}} ((y - y_0) \times \hat{b}^{\ddagger} + \mathcal{A}^* \hat{\beta}^{\ddagger})\, \mathrm{d}y + \int_{\partial \hat{\mathfrak{b}}} ((y - y_0) \times \hat{\mathfrak{t}}_{\partial} + \mathcal{A}^* \hat{\tau}_{\partial})\, \mathrm{d}\bar{\mathcal{H}}^2(y) = 0. \tag{3.5}$$

Remark 3.1. Equation (3.4) is a consequence of $\mathcal{P}^{\mathrm{ext}}$-invariance under rigid translations in $\mathbb{R}^k$. We do not have its counterpart involving $\hat{\beta}^{\ddagger}$ and $\hat{\tau}_{\partial}$ because ν is insensitive to rigid translations of observer in $\tilde{\mathbb{R}}^k$. In fact, $\tilde{\nu}_c(y, t)$ describes mechanisms inside a material element at x in the instant t, *relative* to the element itself. Even if ν would indicate a micro-displacement, it would be relative to the material element, so it would not be affected by translations of reference frames in the physical space. Indeed, when $\mathcal{M}$ is a linear space, we could postulate an integral balance of microstructural actions but it would essentially be superfluous as we see below; also, avoiding it would imply an always desirable reduction of axioms. In addition, since the maps $(y,t) \longmapsto \hat{\beta}^{\ddagger}(y,t)$ and $(y,t) \longmapsto \hat{\tau}_{\partial}(y,t)$ take values in the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$, which is the disjoint union of cotangent spaces (each $T^*_\nu\mathcal{M}$ the dual of $T_\nu\mathcal{M}$), in general a nonlinear space, introducing the integrals over $\mathcal{B}_c$ of $\hat{\beta}^{\ddagger}$ and $\hat{\tau}_{\partial}$ would require the embedding of $\mathcal{M}$ in a linear space to allow the integrals themselves to be well defined. Since $\mathcal{M}$ is taken to be finite-dimensional, the embedding is always available; also it can be isometric, according to Nash's theorems; however, it is not unique, while we aim at determining intrinsic balance equations.

Remark 3.2. The occurrence of $\hat{b}^{\ddagger}$ and $\hat{\tau}_{\partial}$ in the integral balance (3.5) does not necessarily mean that such interactions are couples: only their projections induced by $\mathcal{A}^*(\nu) \in \mathrm{Hom}(T^*_\nu\mathcal{M}, \tilde{\mathbb{R}}^{k*})$ into $\tilde{\mathbb{R}}^{k*}$, namely $\mathcal{A}^*\hat{b}^{\ddagger}$ and $\mathcal{A}^*\hat{\tau}_{\partial}$, play the role of couples.

The integral balances (3.4) and (3.5) imply non-trivial consequences; some of them are standard, others are not properly so.

— If $|\hat{b}^{\ddagger}|$ is bounded over $\mathcal{B}_c$ and $\hat{\mathfrak{t}}_{\partial}$ depends continuously on y, we commonly show at first on flat boundaries that the action–reaction principle holds. Then, on this basis we extend the result on not necessarily flat boundaries. This is due to Hamel–Noll's and Cauchy's theorems (e.g. [49, p. 3]). So, $\hat{\mathfrak{t}}_{\partial}$ depends on $\partial\hat{\mathfrak{b}}$ only through the normal n at all points where it is well defined and is such that $\hat{\mathfrak{t}}_{\partial} = \hat{\mathfrak{t}} := \tilde{\mathfrak{t}}(y,t,n) = -\tilde{\mathfrak{t}}(y,t,-n)$. Also, as a function of n, $\tilde{\mathfrak{t}}$ is homogeneous and additive, meaning there exists a second-rank tensor field $(y,t) \longmapsto \sigma(y,t)$, with σ standard Cauchy's stress, such that $\tilde{\mathfrak{t}}(y,t,n) = \sigma(y,t)n(y)$, where n is considered as a co-vector, so that, in components, we have $\sigma_i^j n_j = \hat{\mathfrak{t}}_i$.

— Since $\mathcal{B}$ is bounded, so is $\mathcal{B}_c$. Then, we can choose the arbitrary point y_0 in such a way that the boundedness of $|\hat{b}^{\ddagger}|$ implies that of $|(y-y_0) \times \hat{b}^{\ddagger}|$. Here we also assume that $|\mathcal{A}^*\hat{\beta}^{\ddagger}|$ is bounded over $\mathcal{B}_c$ and $\mathcal{A}^*\hat{\tau}_{\partial}$ depends continuously on y. Previous assumptions on $b^{\ddagger}$ and the boundedness of $|\int_{\hat{\mathfrak{b}}} \hat{\mathfrak{t}}_{\partial}\, \mathrm{d}\mathcal{H}^{k-1}|$ derived from the balance (3.4) imply

$$\left| \int_{\partial\hat{\mathfrak{b}}} \mathcal{A}^*\hat{\tau}_{\partial}\, \mathrm{d}\mathcal{H}^{k-1} \right| \le K \mathrm{vol}(\hat{\mathfrak{b}}), \tag{3.6}$$

for any $\hat{\mathfrak{b}} \subseteq \mathcal{B}$, with K a positive constant. Then, due to the continuity of $\mathcal{A}^*\hat{\tau}_{\partial}$, by means of Cauchy's theorem we realize that $\mathcal{A}^*\hat{\tau}_{\partial}$ depends on $\partial\hat{\mathfrak{b}}$ only through its normal n in all points where it is well defined; moreover, since $\mathcal{A}^*$ does not depend on n, we get $\hat{\tau}_{\partial} := \hat{\tau} = \tilde{\tau}(y,t,n)$ and

$$\mathcal{A}^*(\tilde{\tau}(y,t,n) + \tilde{\tau}(y,t,-n)) = 0. \tag{3.7}$$

Equation (3.7) means that only a projection on the physical space $\tilde{\mathbb{R}}^{k*}$ of the microstructural contact actions satisfies the action–reaction relation. Moreover, Cauchy's theorem implies also that, as a function of n, $\tilde{\tau}$ is homogeneous and additive, meaning there exists a second-rank tensor field $(y,t) \longmapsto \hat{\mathcal{S}}(y,t)$, so-called *microstress*, such that

$$\tilde{\tau}(y,t,n) = \hat{\mathcal{S}}(y,t)n(y). \tag{3.8}$$

We prove the relation (3.8) *without* embedding $\mathcal{M}$ into a linear space. Another proof has been proposed by Capriz & Virga [50], where a microstructural contact interaction is defined as the integral over (say) $\partial\hat{\mathfrak{b}}$ of $\hat{\tau}_{\partial}$; this choice requires the embedding of $\mathcal{M}$ into a linear space, so that, along this path, $\hat{\mathcal{S}}$ depends on the embedding. At variance, here the integral in formula (3.6) is well defined and does not require the embedding of $\mathcal{M}$ in some linear space because $\mathcal{A}^*\hat{\tau}_{\partial} \in \tilde{\mathbb{R}}^{k*}$.

— If $\sigma(\cdot,t)$ is in $C^1(\mathcal{B}_c) \cap C(\bar{\mathcal{B}}_c)$ for every t (we suppress the target space in the functional class for the sake of conciseness because the setting is self-evident) and the bulk actions

$(y,t) \longmapsto \hat{b}^{\ddagger}$, $(y,t) \longmapsto \hat{\beta}^{\ddagger}$ are continuous over $\mathcal{B}_c$, the standard point-wise balance of forces

$$\hat{b}^{\ddagger} + \operatorname{div}\sigma = 0, \tag{3.9}$$

holds.

— If also $\hat{\mathcal{S}}$ is in $C^1(\mathcal{B}_c) \cap C(\bar{\mathcal{B}}_c)$ for every t, by exploiting equation (3.9) and Gauss' theorem, the balance (3.5) reduces to

$$\int_{\hat{\mathfrak{b}}} \left(\mathcal{A}^*(\hat{\beta}^{\ddagger} + \operatorname{div}\hat{\mathcal{S}}) + (D\mathcal{A}^*)^t\hat{\mathcal{S}} - \mathbf{e}\sigma\right) \mathrm{d}y = 0, \tag{3.10}$$

where $\mathbf{e}$ is third-rank Ricci's index, and the superscript t means right minor transposition for third-rank tensors. Since at every y and t we have $(\hat{\beta}^{\ddagger} + \operatorname{div}\hat{\mathcal{S}}) \in T^*_{\nu}\mathcal{M}$, equation (3.10), the continuity of integral arguments, and the arbitrariness of $\hat{\mathfrak{b}}$ allow us to read equation (3.10) by saying that there exists a field $(y,t) \longmapsto \hat{z}(y,t) \in T^*_{\nu}\mathcal{M}$ such that

$$\operatorname{skew}\sigma = \frac{1}{2}\mathbf{e}(\mathcal{A}^*\hat{z} + (D\mathcal{A}^*)^t\hat{\mathcal{S}}) \tag{3.11}$$

and

$$\hat{\beta}^{\ddagger} - \hat{z} + \operatorname{div}\hat{\mathcal{S}} = 0. \tag{3.12}$$

Equation (3.12) is the local *balance of microstructural actions*.

— Further on, we find

$$\mathcal{P}^{ext}_{\hat{\mathfrak{b}}}(v,\xi) = \int_{\hat{\mathfrak{b}}} (\sigma \cdot Dv + \hat{z} \cdot \xi + \hat{\mathcal{S}} \cdot D\xi)\, \mathrm{d}y =: \mathcal{P}^{\text{int}}_{\hat{\mathfrak{b}}}(v,\xi). \tag{3.13}$$

The functional $\mathcal{P}^{\text{int}}_{\hat{\mathfrak{b}}}(v,\xi)$ is what we call an *internal* power performed along the velocities v and ξ.

Remark 3.3. Consider two neighbouring material elements. In the viewpoint adopted here each of them is not an indistinct Leibniz's monad as it is in the traditional format of continuum mechanics; rather, it is a more 'articulated' system. The standard stress is associated with bonds solicited in crowding and shearing the two considered material elements with their 'inner systems' presumed to be frozen. Imagine now to fix in space the two material elements, allowing their microstructure to vary. They possibly exchange interactions of first-neighbour type that are represented by $\hat{\mathcal{S}}$ and balanced by inner actions $\hat{z}$ among the elements of each system, together with possible bulk actions $\hat{\beta}^{\ddagger}$ that directly affect their microstructure. This is the essential meaning of equation (3.12).

Remark 3.4. At variance of other works, we do not postulate the balance of microstructural actions in local (e.g. [22,32,35,46,51,52]), integral (e.g. [32, pp. 317–318], [53–55]), or weak form (e.g. [36,56,57]), nor do we obtain it from a variational principle (e.g. [58–60]); rather, we *derive* it from invariance of the external power alone, a way of justifying also its validity even in dissipative setting. Resorting to a variational principle would allow us to consider, beyond the conservative setting, only dissipative effects related with the presence of a dissipation pseudo-potential, which is not always granted *a priori*.

Remark 3.5. If we would adopt the identity (3.13) as a first principle, imposing its validity for any rate field included (principle of virtual power), as done in reference [36] (see also [56,57] and references therein), we were postulating the expression of internal power $\mathcal{P}^{\text{int}}_{\mathfrak{b}}$, so we were postulating both $\hat{z}$ and $\hat{\mathcal{S}}$ instead of deriving them as we do here. In fact, postulating the validity of identity (3.13) for any choice of rate fields would essentially be to prescribe *a priori* the weak form of balance equations imagining in this way all the ingredients that constitute them.

Remark 3.6. In the analysis developed so far, the external power is written in Eulerian form but it implies a reference configuration that is fixed once and for all (although such a configuration does not evidently appear). This is because we are not considering here defects moving relatively to the body. In fact, when we aim at including them, the procedure followed so far can be adopted with the proviso of substituting $\mathcal{P}^{ext}_{\mathfrak{b}}$ with the so-called *relative power* (introduced in [61] and refined in [40]), which is the external power performed relatively to a time-varying reference configuration supplemented by the so-called *power of disarrangements*, associated with the motion of bulk, surface, or line defects; requiring its invariance with respect to changes of observer as above, supplemented by isometric changes of observer in the reference space, allows one to obtain the above results but also to derive in addition the balances of configurational actions (which, at variance, have been otherwise postulated [62]) from a unique source, with a reduction of axioms.

— We assume that $\hat{b}^{\ddagger}$ admits additive decomposition into inertial ($\hat{b}^{\text{in}}$) and non-inertial ($\hat{b}$) terms, namely $\hat{b}^{\ddagger} = \hat{b}^{\text{in}} + \hat{b}$, and an analogous decomposition holds also for $\hat{\beta}^{\ddagger}$, so we have $\hat{\beta}^{\ddagger} = \hat{\beta}^{\text{in}} + \hat{\beta}$. We assume that the inertial components are such that their power on any $\hat{\mathfrak{b}}$ equals the negative time rate of the overall kinetic energy of $\hat{\mathfrak{b}}$ itself minus the flow through $\partial\hat{\mathfrak{b}}$. Write $\hat{\rho}$ for the mass density in the current configuration and presume that the balance of mass is satisfied. Set $\xi_{\text{rel}} := \xi - \mathcal{A}q$ and $\xi^{\flat}_{\text{rel}} = g_{\mathcal{M}}\xi_{\text{rel}}$, which is the microstructural momentum relative to the time rate induced on the microstructure through ϕ by rotations in the physical space ($g_{\mathcal{M}}$—we recall—is the metric over $\mathcal{M}$). We impose that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}_c}\left(\frac{1}{2}\hat{\rho}|v|^2 + \mathfrak{k}(\nu,\xi^{\flat}_{\text{rel}})\right)\mathrm{d}y + \int_{\mathcal{B}_c}\frac{\partial\chi(\nu,\xi_{\text{rel}})}{\partial\nu}\cdot\mathcal{A}q\,\mathrm{d}y \\ - \int_{\partial\mathcal{B}_c}\left(\frac{1}{2}\hat{\rho}|v|^2 + \mathfrak{k}(\nu,\xi^{\flat}_{\text{rel}})\right)(v\cdot n)\,\mathrm{d}\mathcal{H}^{k-1}(y) \\ = -\int_{\mathcal{B}_c}(\hat{b}^{\text{in}}\cdot v + \hat{\beta}^{\text{in}}\cdot\xi_{\text{rel}})\,\mathrm{d}y,\end{aligned} \tag{3.14}$$

for any choice of the velocity fields (v,ξ), where n is the outward unit normal to $\partial\mathcal{B}_c$. In the previous equation, $\mathfrak{k}(\nu,\xi^{\flat}_{\text{rel}})$ is the microstructural kinetic energy; its introduction makes sense when microstructures have a relative motion with respect to the matter around them. Specifically, $\mathfrak{k}$ is assumed to be a twice differentiable function, such that $\mathfrak{k}(\nu,0) = 0$ and $\partial^2_{\xi^{\flat}_{\text{rel}}\xi^{\flat}_{\text{rel}}}\mathfrak{k}\cdot(\xi^{\flat}_{\text{rel}}\otimes\xi^{\flat}_{\text{rel}}) \geq 0$, with the identity holding if and only if $\xi^{\flat}_{\text{rel}} = 0$. Also, $\chi(\nu,\xi_{rel})$ is a function that is convex with respect to ξ_{rel} and is such that its Legendre transform is $\mathfrak{k}$. Recall that, for $f(y,t)$ a differentiable function, Reynolds' transport theorem prescribes that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}_c} f\,\mathrm{d}y &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}} f\det F\,\mathrm{d}x = \int_{\mathcal{B}}(\dot{f}\det F + f\,\overline{\det F}^{\cdot})\,\mathrm{d}x \\ &= \int_{\mathcal{B}_c}\dot{f}\,\mathrm{d}y + \int_{\mathcal{B}} f\,\mathrm{div}v\det F\,\mathrm{d}x = \int_{\mathcal{B}_c}\dot{f}\,\mathrm{d}y + \int_{\mathcal{B}} f\,\mathrm{div}v\,\mathrm{d}y \\ &= \int_{\mathcal{B}_c}\dot{f}\,\mathrm{d}y + \int_{\mathcal{B}} f(v\cdot n)\,\mathrm{d}\mathcal{H}^{k-1}(y),\end{aligned} \tag{3.15}$$

(Euler's identity $\overline{\det F}^{\cdot} = \mathrm{div}v\det F$ plays a role in the proof). When we use the transport theorem in equation (3.14), the arbitrariness of v and ξ_{rel} implies

$$\hat{b}^{\text{in}} = -\hat{\rho}\dot{v}^{\flat}, \tag{3.16}$$

(notice that we assumed validity of the mass balance). What remains is

$$\int_{\mathcal{B}_c}\left(\dot{\mathfrak{k}}(\nu,\xi^{\flat}_{\text{rel}}) + \frac{\partial\chi(\nu,\xi_{\text{rel}})}{\partial\nu}\cdot\mathcal{A}q\right)\mathrm{d}y = -\int_{\mathcal{B}_c}\hat{\beta}^{\text{in}}\cdot\xi_{\text{rel}}\,\mathrm{d}y. \tag{3.17}$$

As anticipated in introducing equation (3.14), due to the properties of $\mathfrak{k}$, we can claim the existence of a function $\chi := \tilde{\chi}(\nu,\xi_{\text{rel}})$ such that

$$\mathfrak{k}(\nu,\xi^{\flat}_{\text{rel}}) = \frac{\partial\chi(\nu,\xi_{\text{rel}})}{\partial\xi_{\text{rel}}}\cdot\xi_{\text{rel}} - \chi(\nu,\xi_{\text{rel}}), \tag{3.18}$$

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so that $\mathfrak{k}$ is the Legendre transform of χ with respect to ξ_{rel}. By substituting into equation (3.17), the arbitrariness of ξ_{rel} implies

$$\hat{\beta}^{\text{in}} = \frac{\partial \chi}{\partial \nu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \chi}{\partial \xi_{\text{rel}}}. \tag{3.19}$$

For detailed analyses on the microstructural inertia, see [63] (and also [22,39,40]).

When the spatial metric $\tilde{g}$ is the identity, $\dot{v}^\flat$ can be identified with $\dot{v}$, so is for $\xi^\flat_{\text{rel}}$ and ξ_{rel} when $g_{\mathcal{M}}$ is also flat and trivial.

4. First law and second law of thermodynamics in Eulerian representation

(a) The balance of energy

For a generic part $\hat{\mathfrak{b}}$ of $\mathcal{B}_c$, we write the *first law of thermodynamics* in Eulerian representation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\mathfrak{b}}} e\,\mathrm{d}y - \int_{\partial\hat{\mathfrak{b}}} e(v\cdot n)\,\mathrm{d}\mathcal{H}^{k-1}(y) - \mathcal{P}^{\text{ext}}_{\hat{\mathfrak{b}}}(v,\xi) - \int_{\hat{\mathfrak{b}}} \hat{r}\,\mathrm{d}y + \int_{\partial\hat{\mathfrak{b}}} \hat{a}_\partial\,\mathrm{d}\mathcal{H}^{k-1}(y) = 0, \tag{4.1}$$

and presume that it holds for any choice of $\hat{\mathfrak{b}}$ and the velocity fields involved. In the integral balance (4.1), e is the density of internal energy, a state function, $\hat{r}$ the density of heat source, and $\hat{a}_\partial$ a (scalar) heat flux through the boundary $\hat{\mathfrak{b}}$.

Assume that we are under conditions granting the boundedness of $\mathcal{P}^{\text{ext}}_{\hat{\mathfrak{b}}}(v,\xi)$. If, in addition, $\dot{e}$ and $\hat{\mathfrak{r}}$ are bounded, and $\hat{a}_\partial$ is continuous with respect to y, Cauchy's theorem implies that $\hat{a}_\partial$ depends on $\partial\hat{\mathfrak{b}}$ only through the normal n to $\partial\hat{\mathfrak{b}}$ in all points where n is well defined, so that we have $\hat{a}_\partial = \mathfrak{a}(y,t,n)$. We also find

$$\mathfrak{a}(y,t,n) = -\mathfrak{a}(y,t,-n), \tag{4.2}$$

and the existence of a vector $\hat{\mathfrak{q}}$ depending on y and t (not on n) such that

$$\mathfrak{a}(y,t,n) = \hat{\mathfrak{q}}(y,t)\cdot n. \tag{4.3}$$

Then, thanks to the identity $\mathcal{P}^{\text{ext}}_{\hat{\mathfrak{b}}}(v,\xi) = \mathcal{P}^{\text{int}}_{\hat{\mathfrak{b}}}(v,\xi)$, the first law of thermodynamics becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\mathfrak{b}}} e\,\mathrm{d}y - \int_{\partial\hat{\mathfrak{b}}} e(v\cdot n)\,\mathrm{d}\mathcal{H}^{k-1}(y) - \mathcal{P}^{\text{int}}_{\hat{\mathfrak{b}}}(v,\xi) - \int_{\hat{\mathfrak{b}}} \hat{r}\,\mathrm{d}y + \int_{\partial\hat{\mathfrak{b}}} \hat{\mathfrak{q}}\cdot n\,\mathrm{d}\mathcal{H}^{k-1}(y) = 0. \tag{4.4}$$

When $\dot{e}$, $\hat{r}$, and the density of internal power are continuous with respect to y, and $\hat{\mathfrak{q}}(\cdot,t)$ is $C^1(\mathcal{B}_c)\cap C(\bar{\mathcal{B}}_c)$, use of Gauss' and Reynolds' theorems and the arbitrariness of $\hat{\mathfrak{b}}$ imply the local energy balance

$$\dot{e} - \sigma\cdot Dv - \hat{z}\cdot\xi - \hat{\mathcal{S}}\cdot D\xi - \hat{r} + \operatorname{div}\hat{\mathfrak{q}} = 0. \tag{4.5}$$

(b) The entropy inequality

With η the entropy density, $\hat{s}$ the entropy source, and $\hat{h}_\partial$ a (scalar) flux depending on $\partial\hat{\mathfrak{b}}$ besides y and t, we write the *second law of thermodynamics in Eulerian representation* as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\mathfrak{b}}} \eta\,\mathrm{d}y - \int_{\partial\hat{\mathfrak{b}}} \eta(v\cdot n)\,\mathrm{d}\mathcal{H}^{k-1}(y) \geq \int_{\hat{\mathfrak{b}}} \hat{s}\,\mathrm{d}y - \int_{\partial\hat{\mathfrak{b}}} \hat{h}_\partial\,\mathrm{d}\mathcal{H}^{k-1}(y), \tag{4.6}$$

presuming that it holds for any choice of $\hat{\mathfrak{b}}$ and the rate fields involved.

If $\dot{\eta}$ and $\hat{s}$ are bounded, we can apply even in this case Cauchy's theorem: the key aspect in its proof is, in fact, an estimate on the boundary integral, so that we do not necessarily need a balance, rather only an inequality. So, if $\hat{h}_\partial$ is continuous with respect to y, we find that it depends on $\partial\hat{\mathfrak{b}}$ only through its normal n at all points where n is well defined. Then, we have $\hat{h}_\partial(y,t) = \tilde{h}(y,t,n)$ with

$$\tilde{h}(y,t,n) = -\tilde{h}(y,t,-n), \tag{4.7}$$

and there is a vector field with values $\hat{\mathfrak{h}}$ depending only on y and t such that

$$\tilde{h}(y,t,n) = \hat{\mathfrak{h}}(y,t) \cdot n. \tag{4.8}$$

If $\dot{\eta}$ and $\hat{s}$ are continuous with respect to y, while $\hat{\mathfrak{h}}$ is C^1, the arbitrariness of $\hat{\mathfrak{b}}$ implies

$$\dot{\eta} \geq \hat{s} - \mathrm{div}\hat{\mathfrak{h}}. \tag{4.9}$$

With T—we recall—the absolute temperature, we accept the relations

$$\hat{s} = \frac{\hat{r}}{T} \quad \text{and} \quad \hat{\mathfrak{h}} = \frac{\hat{\mathfrak{q}}}{T} + \varpi, \tag{4.10}$$

where ϖ is a *residual entropy flux* due to microstructural effects. Precisely, we presume that

$$\varpi = \tilde{\varpi}(\nu, \dot{\nu}, D\nu, \ldots, T), \tag{4.11}$$

where the dots indicate from now on that the state variables considered can be present with all their derivatives, according to Truesdell's principle of equipresence. The explicit choice of ϖ depends on specific circumstances; here we just look at a general theory, a metamodel from a philosophical viewpoint.

Equation $(4.10)_2$ appeared first in a work by Müller [64].

Taking into account relations (4.10), we can merge in the usual way the local balance (4.5) with the inequality (4.9), so that, with ψ the Helmoltz free energy density defined by $\psi = e - \theta\eta$, we get

$$\dot{\psi} + \dot{T}\eta - \sigma \cdot Dv - \hat{z} \cdot \xi - \hat{\mathcal{S}} \cdot D\xi + \frac{1}{T}\hat{\mathfrak{q}} \cdot DT - T\mathrm{div}\varpi \leq 0, \tag{4.12}$$

which is what we commonly call *the Clausius–Duhem inequality*. We consider it as a source of restrictions to constitutive structures, in accord with the standard interpretation proposed in 1959 by Coleman & Noll [65].

Remark 4.1. When ν is a scalar (or a pseudo-scalar), an appropriate procedure to derive the pertinent balance of microstructural interactions, instead of postulating it (in this specific case, a first postulate of such a scalar balance seems to have been suggested by Nunziato & Cowin [51]), is based on a requirement of *invariance in structure* for the second law, written in terms of Clausius–Duhem's inequality, under general diffeomorphism-based changes of observer, those described in Remark (2.1); this covariance principle for the second law of thermodynamics [42] requires, in short, that if an observer records a process as a dissipative one, any other observer related with it by diffeomorphisms must record the same dissipative character. To obtain a scalar balance of microstructural interactions, without postulating it, we could also adopt covariance of the first principle of thermodynamics (balance of energy), asking its covariance with respect to general diffeomorphism-based changes of observers, and adapting to the present case Marsden–Hughes' theorem [25]; however, we would not be able to obtain dissipative (macro and micro) stress components. For them we need to refer to the covariance principle for the second law [42].

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5. Consequences of the Clausius–Duhem inequality and a first internal constraint

Having in mind *incompressible conducting viscous complex fluids*, we impose first the standard volume-preserving constraint:

$$\mathrm{div}\, v = 0. \tag{5.1}$$

The choice implies considering a reactive component σ^r of the stress σ. More precisely, we adopt the standard decomposition

$$\sigma = \sigma^r + \sigma^e + \sigma^d, \tag{5.2}$$

where σ^e is the *energetic* component, meaning it is determined by the free energy, while σ^d is a dissipative stress component.

We presume, as usual, that the pertinent reactive stress is *powerless*, namely $\sigma^r \cdot Dv = 0$, for every choice of Dv. The arbitrariness of Dv implies

$$\sigma^r = -\pi I^*, \tag{5.3}$$

with I^* the unit $(1, 1)$ tensor with components δ_i^j (for a different and more detailed derivation of σ^r, see [66]).

We presume that ψ, σ^e and $\hat{\mathcal{S}}$ depend *all* on the list of state variables

$$(\nu, D\nu, T) := (\tilde{\nu}_c(y,t), D\tilde{\nu}_c(y,t), T(y,t)), \tag{5.4}$$

while the stress component σ^d depends on $(\nu, D\nu, T, Dv, \ldots)$, where dots indicate possible further state variables and (potentially) all admissible gradients of the whole list, and is *intrinsically dissipative*, meaning it is such that

$$\sigma^d \cdot Dv \geq 0, \tag{5.5}$$

where the identity holds only when $Dv = 0$. This condition is compatible with a structure of the type

$$\sigma^d = \tilde{a}(\cdots)(Dv)^*, \tag{5.6}$$

where $\tilde{a}(\cdots)$ is a (possibly constant) positive definite (scalar) state function.

We also assume that the *microstructural self-action* $\hat{z}$ admits *energetic* ($\hat{z}^e$) and *dissipative* ($\hat{z}^d$) components. They are such that

— $\hat{z}^e$ depends on the list $(\nu, D\nu, T)$ of state variables as ψ, σ^e, and $\hat{\mathcal{S}}$, while
— $\hat{z}^d$ depends on $(\nu, D\nu, T, \xi, \ldots)$ and is *intrinsically dissipative*, meaning

$$\hat{z}^d \cdot \xi \geq 0, \tag{5.7}$$

for every choice of ξ, with the identity that holds only when ξ vanishes. Such a requirement is compatible with the structure

$$\hat{z}^d = \ell(\cdots)\xi^\flat, \tag{5.8}$$

where ℓ is a (possibly constant) positive definite scalar state function.

The coefficients in expressions (5.6) and (5.8) could be second-rank tensor-valued maps. However, we choose the scalar option for the coefficients to avoid possible problems as those discussed by Antman [67].

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Previous assumptions reduce the Clausius–Duhem inequality to

$$\left(\frac{\partial\psi}{\partial\nu} - \hat{z}^e\right)\cdot\xi + \left(\frac{\partial\psi}{\partial D\nu} - \hat{\mathcal{S}}\right)\cdot D\xi - \left((D\nu)^*\frac{\partial\psi}{\partial D\nu} + \sigma^e\right)\cdot Dv + \left(\frac{\partial\psi}{\partial T} + \eta\right)\cdot\dot{T}$$
$$- \hat{z}^d\cdot\xi - \sigma^d\cdot Dv + \frac{1}{T}\hat{\mathfrak{q}}\cdot DT - T\mathrm{div}\varpi \le 0. \tag{5.9}$$

Since ξ is not constrained, we can choose independently ξ and its gradient. The independence of the rate fields involved in the previous inequality implies the identities

$$\eta = -\frac{\partial\psi}{\partial T},\quad \hat{z}^e = \frac{\partial\psi}{\partial\nu},\quad \hat{\mathcal{S}} = \frac{\partial\psi}{\partial D\nu},\ \sigma^e = -(D\nu)^*\frac{\partial\psi}{\partial D\nu} = -(D\nu)^*\hat{\mathcal{S}}, \tag{5.10}$$

and the reduced dissipation inequality

$$\hat{z}^d\cdot\xi + \sigma^d\cdot Dv + \frac{1}{T}\hat{\mathfrak{q}}\cdot DT \ge T\mathrm{div}\varpi. \tag{5.11}$$

It is compatible with the functional structures (5.6), (5.8) and the decomposition

$$\boxed{\hat{\mathfrak{q}} = -\kappa\nabla T + \mathfrak{q}}, \tag{5.12}$$

with $\mathfrak{q}$ *the heat flux component due to microstructural events*, and κ the standard thermal conductivity.

(a) Scaling

As declared in the introduction, ν refers to a spatial scale λ; precisely, ν brings at continuum scale information on microstructure at the considered spatial scale. So, previous results imply

$$\sigma^e = -(\nabla\nu)^*\hat{\mathcal{S}} \sim O(\lambda^2) \quad \text{and} \quad \hat{z}\cdot\xi + \hat{\mathcal{S}}\cdot D\xi \sim O(\lambda^2). \tag{5.13}$$

We have also

$$\mathcal{A}^*\hat{z} + (D\mathcal{A}^*)^t\hat{\mathcal{S}} \sim O(\lambda^2), \tag{5.14}$$

and $\mathrm{skew}\sigma = \mathrm{skew}\sigma^e + \mathrm{skew}\sigma^d$ because σ^r is symmetric. Consequently, by considering the scaling $(5.13)_1$ and (5.14), from the local balance of couples (3.11) we get

$$\mathrm{skew}\sigma^d = O(\lambda^2). \tag{5.15}$$

6. Emergence of Straughan's claim

From now on we will refer to orthonormal frames; thus, we will identify covariant components with contravariant ones. For this reason we will write ∇ instead of D.

(a) From the local balance of microstructural interactions to the modified Guyer–Krumhansl equation

For the proof we deal upon, we may have two options:

(a) We can identify ν with the non-standard heat flux component $\mathfrak{q}$.
(b) We can consider ν as a differentiable function of $\mathfrak{q}$.

Nothing changes in the proof this paper is devoted to: the difference between these two choices rests only on the presence of additional formal complications. In order to reduce the setting to the skeletal basic format, we then opt for the first choice and assume the identification

$$\nu = \mathfrak{q}. \tag{6.1}$$

From a physical viewpoint, it means that we record microstructural events only in terms of the heat perturbation with respect to the macroscopic flow $\kappa\nabla T$ that they determine. From a formal viewpoint, we are identifying $\mathcal{M}$ with $\mathbb{R}^k$, so that, as already recalled in §2c, in this special case we have

$$\mathcal{A}(\nu) = -\nu \times = -\mathfrak{q}\times, \tag{6.2}$$

which renders explicit the above declarations of scaling about the skew component of the stress tensor (see relation (5.15)).

We also presume the following:

H1. $\hat{\mathcal{S}}$ is symmetric.

H2. The free energy density ψ is of the form

$$\psi := \tilde{\psi}(\mathfrak{q}, \mathrm{sym}\nabla\mathfrak{q}, \theta) = \frac{1}{2}\varsigma_0|\mathfrak{q}|^2 + \hat{\varsigma}_1|\mathrm{sym}\nabla\mathfrak{q}|^2 + \frac{\bar{\omega}}{2}(\mathrm{tr}(\mathrm{sym}\nabla\mathfrak{q}))^2 + \mathrm{f}(T), \tag{6.3}$$

where sym(·) extracts the symmetric part of its argument and the factors ς_0, ω, $\bar{\omega}$ are positive constants, while f(·) is a differentiable function.

H3. There is no microstructural inertia:

$$\hat{\beta}^{\mathrm{in}} = 0. \tag{6.4}$$

H4. The non-inertial microstructural bulk action satisfies the identity (1.4), namely

$$\int_{\hat{\mathfrak{b}}} \hat{\beta}\cdot\mathfrak{q}\,\mathrm{d}y = -\int_{\hat{\mathfrak{b}}} \left(\bar{\mathrm{C}}\nabla v\cdot(\mathfrak{q}\otimes\mathfrak{q}) + \kappa\nabla T\cdot\mathfrak{q}\right)\mathrm{d}y$$
$$= -\int_{\hat{\mathfrak{b}}} \left(\bar{\mathrm{C}}\mathrm{sym}\nabla v\cdot(\mathfrak{q}\otimes\mathfrak{q}) + \kappa\nabla T\cdot\mathfrak{q}\right)\mathrm{d}y, \tag{6.5}$$

presumed to hold for any choice of $\mathfrak{q}$ and $\hat{\mathfrak{b}}$, with $\bar{\mathrm{C}}$ a constant to adjust physical dimensions. In other words, the previous identity states that the drag induced on the microstructure by a spatial variability of velocity and temperature fields determines a bulk action on the microstructure itself.

H6. In the expressions of $\hat{z}^d$, the factor $\ell(\cdots)$ is a positive constant ℓ.

H1 implies $\hat{\mathcal{S}}\cdot\nabla\xi = \hat{\mathcal{S}}\cdot\mathrm{sym}\nabla\xi$, so that by using once again the Clausius–Duhem inequality, from assumptions H2 and H5 we get

$$\hat{\mathcal{S}} = \frac{\partial\psi}{\partial\mathrm{sym}\nabla\mathfrak{q}} = 2\hat{\varsigma}_1\mathrm{sym}\nabla\mathfrak{q} + \bar{\omega}(\mathrm{div}\mathfrak{q})I \tag{6.6}$$

and

$$\hat{z}^e = \frac{\partial\psi}{\partial\mathfrak{q}} = \varsigma\mathfrak{q}, \tag{6.7}$$

so that

$$\hat{z} = \varsigma\mathfrak{q} + \ell\dot{\mathfrak{q}} = \varsigma\mathfrak{q} + \ell\left(\frac{\partial\mathfrak{q}}{\partial t} + (\nabla\mathfrak{q})v\right). \tag{6.8}$$

H4 allows the identification of $\hat{\beta}$ to within a term depending on $(\mathrm{skw}\nabla v)\mathfrak{q}$. Then, together with H3, H4 is compatible with

$$\hat{\beta}^{\ddagger} = \hat{\beta} = -\bar{\mathrm{C}}(\mathrm{sym}\nabla v)\mathfrak{q} - \kappa\nabla T + \hat{\ell}(\mathrm{skw}\nabla v)\mathfrak{q}, \tag{6.9}$$

where a constant multiplies the skew-symmetric part of the velocity gradient; in fact,

$$\hat{\beta}\cdot\mathfrak{q} = -\bar{\mathrm{C}}(\mathrm{sym}\nabla v)\cdot(\mathfrak{q}\otimes\mathfrak{q}) - \kappa\nabla T + \hat{\ell}(\mathrm{skw}\nabla v)\cdot(\mathfrak{q}\otimes\mathfrak{q}) = -\bar{\mathrm{C}}(\mathrm{sym}\nabla v)\cdot(\mathfrak{q}\otimes\mathfrak{q}) - \kappa\nabla T, \tag{6.10}$$

because $\mathfrak{q}\otimes\mathfrak{q}$ is symmetric

By introducing in equation (3.12) relations (6.6), (6.8) and (6.9), we get

$$\ell\dot{\mathsf{q}} + \bar{\mathsf{C}}(\mathrm{sym}\nabla v)\mathsf{q} - \hat{\ell}(\mathrm{skw}\nabla v)\mathsf{q} = -\varsigma\mathsf{q} - \kappa\nabla T + \hat{\varsigma}_1\Delta\mathsf{q} + (\hat{\varsigma}_1 + \bar{\omega})\nabla\mathrm{div}\mathsf{q}, \tag{6.11}$$

which we rewrite as

$$\ell\left(\frac{\partial\mathsf{q}}{\partial t} + (\nabla\mathsf{q})v + \frac{\bar{\mathsf{C}}}{\ell}(\mathrm{sym}\nabla v)\mathsf{q} - \frac{\hat{\ell}}{\ell}(\mathrm{skw}\nabla v)\mathsf{q}\right) = -\varsigma\mathsf{q} - \kappa\nabla T + \hat{\varsigma}_1\Delta\mathsf{q} + (\hat{\varsigma}_1 + \bar{\omega})\nabla\mathrm{div}\mathsf{q}. \tag{6.12}$$

By setting

$$\gamma := \frac{\bar{\mathsf{C}}}{\ell}, \quad \frac{\hat{\ell}}{\ell} = 1, \quad \varsigma = 1, \quad \hat{\varsigma}_2 = \hat{\varsigma}_1 + \bar{\omega}, \tag{6.13}$$

equation (6.12) reduces to

$$\ell\left(\frac{\partial\mathsf{q}}{\partial t} + (\nabla\mathsf{q})v + \gamma(\mathrm{sym}\nabla v)\mathsf{q} - (\mathrm{skw}\nabla v)\mathsf{q}\right) = -\mathsf{q} - \kappa\nabla T + \hat{\varsigma}_1\Delta\mathsf{q} + \hat{\varsigma}_2\nabla\mathrm{div}\mathsf{q}. \tag{6.14}$$

The balance (6.14) is nothing more than the modified Guyer–Krumhansl equation (1.2), namely

$$\ell\frac{\mathcal{D}\mathsf{q}}{\mathcal{D}t} = -\mathsf{q} - \kappa\nabla T + \hat{\varsigma}_1\Delta\mathsf{q} + \hat{\varsigma}_2\nabla\mathrm{div}\mathsf{q}, \tag{6.15}$$

with $\mathcal{D}\mathsf{q}/\mathcal{D}t$ the objective derivative

$$\frac{\mathcal{D}\mathsf{q}}{\mathcal{D}t} = \frac{\partial\mathsf{q}}{\partial t} + (\nabla\mathsf{q})v + \gamma(\mathrm{sym}\nabla v)\mathsf{q} - (\mathrm{skw}\nabla v)\mathsf{q}, \tag{6.16}$$

as introduced by Morro [20,21]. When

$$\gamma = -1, \tag{6.17}$$

$\mathcal{D}\mathsf{q}/\mathcal{D}t$ simply reduces to the (total) Lie derivative $L_v\mathsf{q}$ of q along the vector field v, namely

$$L_v\mathsf{q} = \frac{\partial\mathsf{q}}{\partial t} + (\nabla\mathsf{q})v - (\nabla v)\mathsf{q}, \tag{6.18}$$

so that the balance (6.14) reduces to

$$\ell L_v\mathsf{q} = -\mathsf{q} - \kappa\nabla T + \hat{\varsigma}_1\Delta\mathsf{q} + \hat{\varsigma}_2\nabla\mathrm{div}\mathsf{q}, \tag{6.19}$$

which is the version of Guyer–Krumhansl's equation Straughan eventually refers to in his analysis.

(b) From the local balance of energy to the Payne–Song equation

From the Clausius–Duhem inequality, assumption H2 above implies once again the identity

$$\sigma^e = -(\nabla\mathsf{q})^\top\frac{\partial\psi}{\partial\mathrm{sym}\nabla\mathsf{q}} = -(\nabla\mathsf{q})^\top\hat{\mathcal{S}} \sim O(\lambda^2). \tag{6.20}$$

H2 is also compatible with a structure for the internal energy e of the type

$$e := \tilde{e}(\mathsf{q}, \mathrm{sym}\nabla\mathsf{q}, T). \tag{6.21}$$

Thus, by setting

$$c_v := \frac{\partial e}{\partial T}, \tag{6.22}$$

computed by fixing the other state variables, and considering the decomposition (5.2), the energy balance (4.5) becomes

$$c_v\dot{T} = \kappa\Delta T - \mathrm{div}\mathsf{q} + \hat{r} + \sigma^d\cdot\nabla v + \sigma^e\cdot\nabla v + \hat{z}\cdot\xi + \hat{\mathcal{S}}\cdot\nabla\xi - \frac{\partial e}{\partial\mathsf{q}}\cdot\dot{\mathsf{q}} - \frac{\partial e}{\partial\mathrm{sym}\nabla\mathsf{q}}\cdot\overline{\mathrm{sym}\nabla\mathsf{q}}^{\cdot}. \tag{6.23}$$

Thus, by taking into account the scaling (5.13), which applies also to the derivatives of e with respect to q and its gradient, the energy balance (6.23) can be rewritten as

$$c_v\dot{T} = \kappa\Delta T - \mathrm{div}\mathsf{q} + \hat{r} + \sigma^d\cdot\nabla v + O(\lambda^2). \tag{6.24}$$

When

— $c_v = 1$, $\hat{r} = 0$, and
— $\sigma^d \cdot \nabla v$ is negligible,

to within $O(\lambda^2)$ *terms*, the energy balance (6.24) reduces to the Payne–Song equation

$$\dot{T} = \kappa \Delta T - \operatorname{div}\mathfrak{q}, \tag{6.25}$$

as considered by Straughan.

(c) The balance of standard forces to within $O(\lambda^2)$ terms

The scaling (5.15) implies that σ^d is symmetric *to within* $O(\lambda^2)$ *terms*. So, in the same approximation,

$$\sigma^d \cdot \nabla v = \sigma^d \cdot \operatorname{sym}\nabla v \geq 0, \tag{6.26}$$

an inequality that is compatible with the structure

$$\sigma^d = \tilde{a}(\cdots)\operatorname{sym}\nabla v. \tag{6.27}$$

Specifically, if we choose here $\tilde{a}(\cdots) = \tilde{a}(T)$. Then, neglecting the skew-symmetric part of the dissipative stress component, the stress σ turns out to be

$$\sigma = -\pi I - (\nabla\mathfrak{q})^\top \hat{\mathcal{S}} + \tilde{a}(T)\operatorname{sym}\nabla v, \tag{6.28}$$

so that, by taking into account the internal constraint $\operatorname{div} v = 0$, indicating by $\tilde{a}'(T)$ the derivative of $\tilde{a}$ with respect to T, and setting $\hat{\rho} = 1$ for the sake of simplicity, the balance of forces (3.9) reduces to the following equation:

$$\frac{\partial v}{\partial t} + (\nabla v)v = -\nabla\pi + \tilde{a}(T)\Delta v + \tilde{a}'(T)\nabla v \nabla T - \operatorname{div}((\nabla\mathfrak{q})^\top \hat{\mathcal{S}}) + \hat{b}. \tag{6.29}$$

Then, to within $O(\lambda^2)$ terms, equation (6.29) reduces to

$$\frac{\partial v}{\partial t} + (\nabla v)v = -\nabla\pi + \tilde{a}(T)\Delta v + \tilde{a}'(T)\nabla v \nabla T + \hat{b}. \tag{6.30}$$

If the viscosity $\tilde{a}(\cdots)$ is constant, say $\tilde{a}$, equation (6.30) obviously becomes

$$\frac{\partial v}{\partial t} + (\nabla v)v = -\nabla\pi + \tilde{a}\Delta v + \hat{b}, \tag{6.31}$$

which is the Navier–Stokes system with external bulk force $\hat{b}$.

7. Additional remarks

When we describe at macroscopic scale the effects of microstructural events by using manifold-valued observer-sensitive descriptors (say phase fields), we are introducing—even if only implicitly—a spatial scale to which we pay attention, the one defining what we consider a *microstructural level* in each specific case. Referring explicitly to such a scale may allow us to make appropriate approximations as the one showing in which sense Payne–Song's equation emerges from the local energy balance of a complex body when the variable ν depends on the microstructure-induced heat flux fluctuation $\mathfrak{q}$ with respect to $\kappa\nabla T$, heat sources are absent, and power of the dissipative stress component is negligible.

Under the same constraint between ν and $\mathfrak{q}$ (see also [12]), with additional constitutive assumptions the balance of microstructural actions reduces to Guyer–Krumhansl's equation even in the form modified by the introduction of an objective time derivative. The present proof inserts Payne–Song's and Guyer–Krumhansl's equations in a setting that allows us to generalize them further, pushing the scheme beyond boundaries of its descriptive ability.

In this way, we 'acquire universality', meaning that the scheme adopted by Straughan [1] for non-Fourier heat propagation turns out to be justified by the consideration of microstructural effects *independently* of the type of microstructure.

Ethics. No ethic problems are directly related with this text.
Data accessibility. This article has no additional data.
Declaration of AI use. We have not used AI-assisted technologies in creating this article.
Authors' contributions. P.M.M.: conceptualization, formal analysis, investigation, methodology, validation, writing—original draft, writing—review and editing.
Conflict of interest declaration. I declare I have no competing interests.
Funding. No funding has been received for this article.
Acknowledgements. This work belongs to activities of the research group 'Theoretical Mechanics' in the 'Centro di Ricerca Matematica Ennio De Giorgi' of the Scuola Normale Superiore in Pisa. The support of GNFM-INDAM is acknowledged.

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