Local boundedness of weak solutions to elliptic equations with p,q-growth

http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(3): 1–28.
DOI:10.3934/mine.2023065
Received: 27 September 2022
Revised: 13 November 2022
Accepted: 13 November 2022
Published: 29 November 2022

Research article

Local boundedness of weak solutions to elliptic equations with $p, q-$growth[†]

Giovanni Cupini[1,*], **Paolo Marcellini**[2] **and Elvira Mascolo**[2]

[1] Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 - Bologna, Italy

[2] Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 - Firenze, Italy

[†] **This contribution is part of the Special Issue:** PDEs and Calculus of Variations–Dedicated to Giuseppe Mingione, on the occasion of his 50th birthday
Guest Editors: Giampiero Palatucci; Paolo Baroni
Link: `www.aimspress.com/mine/article/6240/special-articles`

* **Correspondence:** Email: giovanni.cupini@unibo.it.

Abstract: This article is dedicated to Giuseppe Mingione for his 50^{th} birthday, a leading expert in the regularity theory and in particular in the subject of this manuscript. In this paper we give conditions for the *local boundedness* of weak solutions to a class of nonlinear elliptic partial differential equations in divergence form of the type considered below in (1.1), under $p, q-$growth assumptions. The novelties with respect to the mathematical literature on this topic are the general growth conditions and the explicit dependence of the differential equation on u, other than on its gradient Du and on the x variable.

Keywords: regularity; local boundedness; weak solutions; elliptic equations; $p, q-$growth

1. Introduction

We consider the general second order elliptic equation in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i\left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right), \qquad x \in \Omega, \tag{1.1}$$

where Ω is an open set of $\mathbb{R}^n$, $n \geq 2$, the vector field $\left(a^i\left(x, u, \xi\right)\right)_{i=1,\ldots,n}$ and the right hand side $b\left(x, u, \xi\right)$ are Carathéodory applications defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. We study the elliptic equations (1.1) under some *general growth conditions* on the gradient variable $\xi = Du$, named *$p, q-$conditions*, which we are going to state in the next Section 3.2. Under these assumptions we will obtain the *local boundedness of the weak solutions*, as stated in Theorem 3.2.

A strong motivation to study the local boundedness of solutions to (1.1) relies on the recent research in [53], where the *local Lipschitz continuity* of the weak solutions of the Eq (1.1) has been obtained under general growth conditions, precisely some $p, q-$growth assumptions, with the explicit dependence of the differential equation on u, other than on its gradient Du and on the x variable. In [53] the Sobolev class of functions where to start in order to get more regularity of the weak solutions was pointed out, precisely $u \in W^{1,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$. That is, in particular the *local boundedness* $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$ of weak solutions is a starting assumption for more interior regularity; i.e., for obtaining $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and more. When we refer to the classical cases this is a well known aspect which appears in the mathematical literature on a-priori regularity: in fact, for instance, under the so-called *natural growth conditions*, i.e., when $q = p$, then the a-priori boundedness of u often is a natural assumption to obtain the boundedness of its gradient Du too; see for instance the classical reference book by Ladyzhenskaya-Ural'tseva [45, Chapter 4, Section 3] and the $C^{1,\alpha}-$regularity result by Tolksdorf [60].

The aim of this paper is to derive the local boundedness of solutions to (1.1); i.e., to deduce the local boundedness of u only from the growth assumptions on the vector field $\left(a^i(x,u,\xi)\right)_{i=1,\dots,n}$ and the right hand side $b(x,u,\xi)$ in (1.1). The precise conditions and the related results are stated in Section 3.

We start with a relevant aspect to remark in our context, which is *different from what happens in minimization problems* and it is peculiar for equations: although under $p, q-$growth conditions (with $p < q$) the Eq (1.1) is elliptic and coercive in $W^{1,p}_{\mathrm{loc}}(\Omega)$, it is not possible a-priori to look for weak solutions only in the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$, but it is necessary to emphasize that the notion of *weak solution* is consistent if a-priori we assume $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$. This is detailed in Section 2.

Going into more detail, in this article we study the local boundedness of weak solutions to the $p-$elliptic equation (1.1) with $q-$growth, $1 < p \leq q < p+1$, as in (3.2), (3.3) and (3.7)–(3.10). Starting from the integrability condition $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ on the weak solution, under the bound on the ratio $\frac{q}{p}$

$$\frac{q}{p} < 1 + \frac{1}{n-1}$$

we obtain $u \in L^{\infty}_{\mathrm{loc}}(\Omega)$. The proof is based on the powerful De Giorgi technique [29], by showing first a Caccioppoli-type inequality and then applying an iteration procedure. The result is obtained via a *Sobolev embedding theorem on spheres*, a procedure introduced by Bella and Schäffner in [3], that allows a dimensional gain in the gap between p and q. This idea has been later used by the same authors in [4], by Schäffner [58] and, particularly close to the topic of our paper, by Hirsch and Schäffner [43] and De Rosa and Grimaldi [30], where the local boundedness of scalar *minimizers* of a class of *convex energy integrals* with $p, q-$growth was obtained with the bound $\frac{q}{p} < 1 + \frac{q}{n-1}$.

Some references about the *local boundedness* of solutions to elliptic equations and systems, with general and $p, q-$growth conditions, start by Kolodiĭ [44] in 1970 in the specific case of some *anisotropic* elliptic equations. The local boundedness of solution to classes of anisotropic elliptic equations or systems have been investigated by the authors [18–24] and by Di Benedetto, Gianazza and Vespri [31]. Other results on the boundedness of solutions of PDEs or of minimizers of integral functionals can be found in Boccardo, Marcellini and Sbordone [7], Fusco and Sbordone [37, 38], Stroffolini [59], Cianchi [14], Pucci and Servadei [57], Cupini, Leonetti and Mascolo [17], Carozza, Gao, Giova and Leonetti [12], Granucci and Randolfi [42], Biagi, Cupini and Mascolo [5].

Interior $L^\infty-$*gradient bound*, i.e., the *local Lipschitz continuity*, of weak solutions to nonlinear elliptic equations and systems under non standard growth conditions have been obtained since 1989 in [46–50]. See also the following recent references for other Lipschitz regularity results: Colombo and Mingione [16], Baroni, Colombo and Mingione [1], Eleuteri, Marcellini and Mascolo [34, 35], Di Marco and Marcellini [32], Beck and Mingione [2], Bousquet and Brasco [9], De Filippis and Mingione [26, 27], Caselli, Eleuteri and Passarelli di Napoli [13], Gentile [39], the authors and Passarelli di Napoli [25], Eleuteri, Marcellini, Mascolo and Perrotta [36]; see also [53]. For other related results see also Byun and Oh [10] and Mingione and Palatucci [55]. The local boundedness of the solution u can be used to achieve further regularity properties, as the Hölder continuity of u or of its gradient Du; we limit here to cite Bildhauer and Fuchs [6], Düzgun, Marcellini and Vespri [33], Di Benedetto, Gianazza and Vespri [31], Byun and Oh [11] as examples of this approach. For recent boundary regularity results in the context considered in this manuscript we mention Cianchi and Maz'ya [15], Bögelein, Duzaar, Marcellini and Scheven [8], De Filippis and Piccinini [28]. A well known reference about the regularity theory is the article [54] by Giuseppe Mingione. We also refer to [51–53] and to De Filippis and Mingione [27], Mingione and Rădulescu [56], who have outlined the recent trends and advances in the regularity theory for variational problems with non-standard growths and non-uniform ellipticity.

2. On the definition of weak solution

In order to investigate the consistency of the notion of weak solution, we anticipate the ellipticity and growth conditions of Section 3, in particular the growth in (3.3), (3.4),

$$\begin{cases} \left|a^i(x,u,\xi)\right| \le \Lambda \left\{|\xi|^{q-1} + |u|^{\gamma_1} + b_1(x)\right\}, & \forall\, i = 1,\dots,n, \\ |b(x,u,\xi)| \le \Lambda \left\{|\xi|^r + |u|^{\gamma_2} + b_2(x)\right\}. \end{cases} \tag{2.1}$$

As well known the integral form of the equation, for a smooth test function φ with compact support in Ω, is

$$\int_\Omega \sum_{i=1}^n a^i(x,u,Du)\,\varphi_{x_i}dx + \int_\Omega b(x,u,Du)\,\varphi\,dx = 0\,.$$

Let us discuss the summability conditions for the pairings above to be well defined. Since each a^i in the gradient variable ξ grows at most as $|\xi|^{q-1}$, more generally we can consider test functions $\varphi \in W_0^{1,q}(\Omega)$. In fact, starting with the first addendum and applying the Young inequality with conjugate exponents $\frac{q}{q-1}$ and q, we obtain the L^1 local summability

$$\begin{aligned} \left|a^i(x,u,Du)\,\varphi_{x_i}\right| &\le \Lambda\left\{|Du|^{q-1} + |u|^{\gamma_1} + b_1(x)\right\}\left|\varphi_{x_i}\right| \\ &\le \Lambda\tfrac{q-1}{q}\left\{|Du|^{q-1} + |u|^{\gamma_1} + b_1(x)\right\}^{\frac{q}{q-1}} + \tfrac{\Lambda}{q}\left|\varphi_{x_i}\right|^q \in L^1_{\mathrm{loc}}(\Omega) \end{aligned}$$

if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and if $\frac{q}{q-1}\gamma_1 \le q^*$, where q^* is the Sobolev conjugate exponent of q, and $b_1 \in L^{\frac{q}{q-1}}_{\mathrm{loc}}(\Omega)$. On γ_1 equivalently we require (if $q < n$) $\gamma_1 \le q^*\frac{q-1}{q} = \frac{nq}{n-q}\frac{q-1}{q} = \frac{n(q-1)}{n-q}$, which essentially corresponds to our assumption (3.8) below (the difference being the strict sign "$<$" for compactness reasons). We also observe that the summability condition $b_1 \in L^{\frac{q}{q-1}}_{\rm loc}(\Omega)$ is satisfied if $b_1 \in L^{s_1}_{\rm loc}(\Omega)$, with $s_1 > \frac{n}{q-1}$, as in (3.10).

Similar computations apply to $|b(x,u,\xi)\,\varphi|$, again if $q < n$ and with conjugate exponents $\frac{q^*}{q^*-1}$ and q^*,

$$\begin{aligned} |b(x,u,Du)\,\varphi| &\leq \Lambda\{|Du|^r + |u|^{\gamma_2} + b_2(x)\}\,|\varphi| \\ &\leq \Lambda\frac{q^*-1}{q^*}\{|Du|^r + |u|^{\gamma_2} + b_2(x)\}^{\frac{q^*}{q^*-1}} + \frac{\Lambda}{q^*}\,|\varphi|^{q^*} \in L^1_{\rm loc}(\Omega) \end{aligned}$$

and we obtain $b_2 \in L^{\frac{q^*}{q^*-1}}_{\rm loc}(\Omega)$ (compare with (3.10), where $b_2 \in L^{s_2}_{\rm loc}(\Omega)$ with $s_2 > \frac{n}{p}$, since $\frac{q^*}{q^*-1} \leq \frac{p^*}{p^*-1} \leq \frac{p^*}{p^*-p} = \frac{n}{p}$) and the conditions for r and γ_2 expressed by $r\frac{q^*}{q^*-1} \leq q$ and $\gamma_2\frac{q^*}{q^*-1} \leq q^*$; i.e., for the first one,

$$r \leq q\frac{q^*-1}{q^*} = q\frac{\frac{nq}{n-q}-1}{\frac{nq}{n-q}} = q + \frac{q}{n} - 1\,,$$

which correspond to the more strict assumption (3.9), with $r < p + \frac{p}{n} - 1$, with the sign "$<$" and where q is replaced by p. Finally for γ_2 we obtain $\gamma_2 \leq q^* - 1$, which again corresponds to our assumption (3.8) with the strict sign.

Therefore our assumptions for Theorem 3.2 are more strict than that ones considered in this section and they are consistent with a correct definition of weak solution to the elliptic equation (1.1).

3. Statement of the main result

Let $a^i : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., n$, and $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be Carathéodory functions, Ω be an open set in $\mathbb{R}^n$, $n \geq 2$. Consider the nonlinear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x,u,Du) = b(x,u,Du). \tag{3.1}$$

For the sake of simplicity we use the following notation: $a(x,u,\xi) = (a^i(x,u,\xi))_{i=1,\ldots,n}$, for all $i = 1, \ldots, n$.

We assume the following properties:

- *$p-$ellipticity condition at infinity*:
  there exist an exponent $p > 1$ and a positive constant λ such that
  $$\langle a(x,u,\xi), \xi\rangle \geq \lambda\,|\xi|^p\,, \tag{3.2}$$
  for a.e. $x \in \Omega$, for every $u \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^n$ such that $|\xi| \geq 1$.

- *$q-$growth condition*:
  there exist exponents $q \geq p$, $\gamma_1 \geq 0$, $s_1 > 1$, a positive constant Λ and a positive function $b_1 \in L^{s_1}_{\rm loc}(\Omega)$ such that, for a.e. $x \in \Omega$, for every $u \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^n$,
  $$|a(x,u,\xi)| \leq \Lambda\left\{|\xi|^{q-1} + |u|^{\gamma_1} + b_1(x)\right\}; \tag{3.3}$$

- *growth conditions for the right hand side* $b(x,u,\xi)$:
  there exist further exponents $r \geq 0$, $\gamma_2 \geq 0$, $s_2 > 1$ and a positive function $b_2 \in L^{s_2}_{\mathrm{loc}}(\Omega)$ such that

$$|b(x,u,\xi)| \leq \Lambda \left\{ |\xi|^r + |u|^{\gamma_2} + b_2(x) \right\}, \tag{3.4}$$

  for a.e. $x \in \Omega$, for every $u \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^n$.

Without loss of generality we can assume $\Lambda \geq 1$ and $b_1, b_2 \geq 1$ a.e. in Ω. We recall the definition of weak solution to (3.1).

Definition 3.1. A function $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is a weak solution to (3.1) if

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x,u,Du)\varphi_{x_i} + b(x,u,Du)\varphi \right\} dx = 0 \tag{3.5}$$

for all $\varphi \in W^{1,q}(\Omega)$, $\operatorname{supp}\varphi \Subset \Omega$.

3.1. Assumptions on the exponents

Our aim is to study the local boundedness of weak solutions to (3.1). Since this regularity property is trivially satisfied for functions in $W^{1,q}_{\mathrm{loc}}(\Omega)$ with $q > n$, from now on we only consider the case $q \leq n$; more precisely

$$1 < p < n, \qquad p \leq q \leq n, \tag{3.6}$$

since if $q > n$ then weak solutions are Hölder continuous as an application of the Sobolev-Morrey embedding theorem, see Remark 3.3.

Other assumptions on the exponents are

$$\begin{cases} q < 1 + p \\ \dfrac{q}{p} < 1 + \dfrac{1}{n-1} \end{cases} \tag{3.7}$$

$$0 \leq \gamma_1 < \frac{n(q-1)}{n-p}, \qquad 0 \leq \gamma_2 < \frac{n(p-1)+p}{n-p}, \tag{3.8}$$

$$0 \leq r < p + \frac{p}{n} - 1, \tag{3.9}$$

$$s_1 > \frac{n}{q-1}, \qquad s_2 > \frac{n}{p}. \tag{3.10}$$

3.2. The statement of the boundedness result

Under the conditions described above the following local boundedness result holds.

Theorem 3.2 (Boundedness result)**.** *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $1 < q \leq n$, be a weak solution to the elliptic equation (3.1). If (3.2)–(3.4) and (3.6)–(3.10) hold true, then u is locally bounded. Precisely, for every open set $\Omega' \Subset \Omega$ there exist constants $R_0, c > 0$ depending on the data $n, p, q, r, \gamma_1, \gamma_2, s_1, s_2$ and on the norm $\|u\|_{W^{1,q}(\Omega')}$ such that $\|u\|_{L^\infty(B_{R/2}(x_0))} \leq c$ for every $R \leq R_0$, with $B_{R_0}(x_0) \subseteq \Omega'$.*

Remark 3.3. We already observed that if $q > n$ then the weak solutions to (3.1) are locally Hölder continuous. Let us now discuss why in (3.6) we do not consider the case $p = q = n$. If $p = q$ ($\le n$), the same computations in the proof of Theorem 3.2 work with the set of assumptions (3.8)–(3.10). They can be written, coherently with the previous ones, as

$$0 \le \gamma_1 < p^* \frac{p-1}{p}, \qquad 0 \le \gamma_2 < p^* - 1 \tag{3.11}$$

$$0 \le r < p - \frac{p}{p^*}, \tag{3.12}$$

$$s_1 > \frac{p^* p}{(p^* - p)(p-1)}, \qquad s_2 > \frac{p^*}{p^* - p}. \tag{3.13}$$

Here p^* denotes the Sobolev exponent appearing in the Sobolev embedding theorem for functions in $W^{1,p}(\Omega)$ with Ω bounded open set in $\mathbb{R}^n$; i.e.,

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n \\ \text{any real number} > n, & \text{if } p = n. \end{cases} \tag{3.14}$$

Following the computations in [40, Theorem 2.1] and [41, Chapter 6] it can be proved that the weak solutions to (3.1) are quasi-minima of the functional

$$\mathcal{F}(u) := \int_\Omega \left(|Du|^p + |u|^\tau + b_1^{\frac{p}{p-1}} + b_2^{\frac{p^*}{p^*-1}} \right) dx, \tag{3.15}$$

with $\tau := \max\{\gamma_1 \frac{p}{p-1}, \gamma_2 \frac{p^*}{p^*-1}\}$. It is known that if

$$\tau < p^* \quad \text{and} \quad b_1^{\frac{p}{p-1}} + b_2^{\frac{p^*}{p^*-1}} \in L^{1+\delta} \ \text{ with } \delta > 0 \tag{3.16}$$

then the gradient of quasi-minima of the functional (3.15) satisfies a higher integrability property; i.e., they belong to $W^{1,p+\epsilon}$, for some $\epsilon > 0$.

Under our assumptions, (3.16) is satisfied; indeed, taking into account that we are considering $p = q$, by (3.10)

$$s_1 > \frac{n}{p-1} \ge \frac{p}{p-1}$$

and, by (3.13)

$$s_2 > \frac{p^*}{p^* - p} \ge \frac{p^*}{p^* - 1}.$$

Analogously, by (3.11),

$$\gamma_1 \frac{p}{p-1} < p*, \qquad \gamma_2 \frac{p^*}{p^*-1} < (p^* - 1)\frac{p^*}{p^*-1} = p^*.$$

In particular, if $p = q = n$ the quasi-minima of (3.15) are in $W^{1,n+\epsilon}_{\mathrm{loc}}(\Omega)$ for some $\epsilon > 0$, therefore the weak solutions to (3.1) are Hölder continuous. We refer to [41] Chapter 6 for more details.

4. Notation and remarks

If $p \geq 1$ and $d \in \mathbb{N}$, $d \geq 2$, we define

$$(p_d)^* := \begin{cases} \frac{dp}{d-p} & \text{if } p < d \\ \text{any real number} > d, & \text{if } p = d. \end{cases}$$

The Sobolev exponent appearing in the Sobolev embedding theorem for functions in $W^{1,p}(\Omega)$, $p \geq 1$, with Ω bounded open set in $\mathbb{R}^n$, is $(p_n)^*$ and will be denoted, as usual, p^*.

Let $t \in \mathbb{R}$, $t > 0$. We define t_* as follows:

$$\frac{1}{t_*} := \min\left\{\frac{1}{t} + \frac{1}{n-1}, 1\right\}.$$

We have, if $n \geq 3$,

$$t_* = \begin{cases} \frac{t(n-1)}{t+n-1} & \text{if } t > \frac{n-1}{n-2} \\ 1 & \text{if } 1 \leq t \leq \frac{n-1}{n-2}, \end{cases}$$

and, if $n = 2$, $t_* = 1$ for every t.

We notice that, if $n \geq 3$,

$$((t_*)_{n-1})^* = \begin{cases} t & \text{if } t > \frac{n-1}{n-2} \\ \frac{n-1}{n-2} & \text{if } 1 \leq t \leq \frac{n-1}{n-2} \end{cases}$$

and, if $n = 2$, for every t, $((t_*)_{n-1})^*$ stands for any real number greater than 1.

Remark 4.1. Let us consider the exponents p, q satisfying (3.6) and (3.7) in Section 3. We notice that

$$\frac{1}{\left(\frac{p}{p-q+1}\right)_*} = \begin{cases} \frac{1}{\frac{p}{p-q+1}} + \frac{1}{n-1} & \text{if } q > 1 + \frac{p}{n-1} \\ 1 & \text{if } q \leq 1 + \frac{p}{n-1}. \end{cases} \tag{4.1}$$

Due to assumption (3.7), if $n = 2$, then $\left(\frac{p}{p-q+1}\right)_* = 1$.

Moreover, if we denote $t := \left(\frac{p}{p-q+1}\right)_*$ then, if $n \geq 3$,

$$(t_{n-1})^* = \begin{cases} \frac{p}{p-q+1} & \text{if } q > 1 + \frac{p}{n-1} \\ \frac{n-1}{n-2} & \text{if } q \leq 1 + \frac{p}{n-1}, \end{cases} \tag{4.2}$$

if instead $n = 2$ than $(t_{n-1})^*$ is any real number greater than 1.

Let p, q satisfy (3.6) and (3.7). It is easy to prove that

$$\frac{p}{p-q+1} < q^*. \tag{4.3}$$

In the following it will be useful to introduce the following notation:

$$\nu := \frac{1}{\left(\frac{p}{p-q+1}\right)_*} - \frac{1}{p},$$

or, more explicitly,

$$\nu = \begin{cases} \frac{p-1}{p} & \text{if } q \leq 1 + \frac{p}{n-1} \\ 1 - \frac{q}{p} + \frac{1}{n-1} & \text{if } q > 1 + \frac{p}{n-1}. \end{cases} \tag{4.4}$$

Remark 4.2. Assume $1 < p \leq q$. Then easy computations give

$$\nu > 0 \Leftrightarrow q < \frac{pn}{n-1}, \qquad \nu = 0 \Leftrightarrow q = \frac{pn}{n-1}. \tag{4.5}$$

To get the sharp bound for q, we use a result proved in [43], see also [3,4,30,58]. Here we denote $S_\sigma(x_0)$ the boundary of the ball $B_\sigma(x_0)$ in $\mathbb{R}^n$.

Lemma 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$. Consider $B_\sigma(x_0)$ ball in $\mathbb{R}^n$ and $u \in L^1(B_\sigma(x_0))$ and $s > 1$. For any $0 < \rho < \sigma < +\infty$, define*

$$I(\rho, \sigma, u) := \inf \left\{ \int_{B_\sigma(x_0)} |u||D\eta|^s \, dx \ : \ \eta \in C_0^1(B_\sigma(x_0)), \ 0 \leq \eta \leq 1, \ \eta = 1 \text{ in } B_\rho(x_0) \right\}.$$

Then for every $\delta \in]0, 1]$,

$$I(\rho, \sigma, v) \leq (\sigma - \rho)^{s-1+\frac{1}{\delta}} \left(\int_\rho^\sigma \left(\int_{S_r(x_0)} |v| \, d\mathcal{H}^{n-1} \right)^\delta dr \right)^{\frac{1}{\delta}}.$$

The following result is the Sobolev inequality on spheres.

Lemma 4.4. *Let $n \in \mathbb{N}$, $n \geq 3$, and $\gamma \in [1, n-1[$. Then there exists c depending on n and γ such that for every $u \in W^{1,p}(S_1(x_0), d\mathcal{H}^{n-1})$*

$$\left(\int_{S_1(x_0)} |u|^{(\gamma_{n-1})^*} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{(\gamma_{n-1})^*}} \leq c \left(\int_{S_1(x_0)} (|Du|^\gamma + |u|^\gamma) \, d\mathcal{H}^{n-1} \right)^{\frac{1}{\gamma}}.$$

Lemma 4.5. *Let $n = 2$. Then there exists c such that for every $u \in W^{1,1}(S_1(x_0), d\mathcal{H}^1)$ and every $r > 1$,*

$$\left(\int_{S_1(x_0)} |u|^r \, d\mathcal{H}^1 \right)^{\frac{1}{r}} \leq c \left(\int_{S_1(x_0)} (|Du| + |u|) \, d\mathcal{H}^1 \right).$$

Proof. By the one-dimensional Sobolev inequality

$$||u||_{L^\infty(S_1(x_0))} \leq c||u||_{W^{1,1}(S_1(x_0))}.$$

Then, for every $r > 1$,

$$\left(\int_{S_1(x_0)} |u|^r \, d\mathcal{H}^{n-1} \right)^{\frac{1}{r}} \leq c||u||_{L^\infty(S_1(x_0))} \leq c||u||_{W^{1,1}(S_1(x_0))}.$$

□

We conclude this section, by stating a classical result; see, e.g., [41]. that will be useful to prove Theorem 3.2.

Lemma 4.6. *Let $\alpha > 0$ and (J_h) a sequence of real positive numbers, such that*

$$J_{h+1} \leq A \, \lambda^h J_h^{1+\alpha},$$

with $A > 0$ and $\lambda > 1$.
If $J_0 \leq A^{-\frac{1}{\alpha}} \lambda^{-\frac{1}{\alpha^2}}$, then $J_h \leq \lambda^{-\frac{h}{\alpha}} J_0$ and $\lim_{h\to\infty} J_h = 0$.

5. Caccioppoli's inequality

Under the assumptions in Section 3 we have the following Caccioppoli-type inequality.

Given a measurable function $u : \Omega \to \mathbb{R}$, with Ω open set in $\mathbb{R}^n$, and fixed $x_0 \in \mathbb{R}^n$, $k \in \mathbb{R}$ and $\tau > 0$, we denote the super-level set of u as follows:

$$A_{k,\tau}(x_0) := \{x \in B_\tau(x_0) \,:\, u(x) > k\};$$

usually dropping the dependence on x_0. We denote $|A_{k,\tau}|$ its Lebesgue measure.

Proposition 5.1 (Caccioppoli's inequality)**.** *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (3.1). If (3.6)–(3.10) hold true, then there exists a constant $c > 0$, such that for any $B_{R_0}(x_0) \Subset \Omega$, $0 < \rho < R \le R_0$*

$$\begin{aligned}
\int_{B_\rho} |D(u-k)_+|^p \, dx \le{}& C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times \\
&\times \|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p}{p-q+1}} |A_{k,R}|^{\frac{p}{p-q+1}\nu} \\
&+ c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p\gamma_1}{q-1}} |A_{k,R}|^{1-\frac{p\gamma_1}{p^*(q-1)}} + c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p}{p-r}} |A_{k,R}|^{1-\frac{1}{p^*\frac{p-r}{p}}} \\
&+ c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2+1} |A_{k,R}|^{1-\frac{\gamma_2+1}{p^*}} + c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2} |A_{k,R}|^{1-\frac{\gamma_2}{p^*}} \\
&+ ck^{\gamma_2}\|(u-k)_+\|_{W^{1,p}(B_R)} |A_{k,R}|^{1-\frac{1}{p^*}} + c\left(k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2}\right)|A_{k,R}| \\
&+ c\|(u-k)_+\|_{W^{1,p}(B_R)} |A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} + c|A_{k,R}|^{1-\frac{p}{s_1(q-1)}}
\end{aligned} \tag{5.1}$$

with ν as in (4.4) and c is a constant depending on n, p, q, r, R_0, the L^{s_1}-norm of b_1 and the L^{s_2}-norm of b_2 in B_{R_0}.

Proof. Without loss of generality we assume that the functions b_1, b_2 in (3.3) are a.e. greater than or equal to 1 in Ω. We split the proof into steps.

Step 1. Consider $B_{R_0}(x_0) \Subset \Omega$, $0 < \frac{R_0}{2} \le \rho < R \le R_0 \le 1$.

We set

$$\mathcal{A}(\rho, R) := \{\eta \in C_0^\infty(B_R(x_0)) \,:\, \eta = 1 \text{ in } B_\rho(x_0),\ 0 \le \eta \le 1\}. \tag{5.2}$$

For every $\eta \in \mathcal{A}(\rho, R)$ and fixed $k > 1$ we define the test function φ_k as follows

$$\varphi_k(x) := (u(x)-k)_+[\eta(x)]^\mu \quad \text{for a.e. } x \in B_{R_0}(x_0),$$

with

$$\mu := \frac{p}{p-q+1} \tag{5.3}$$

that is greater than 1 because $q > 1$.

Notice that $\varphi_k \in W_0^{1,q}(B_{R_0}(x_0))$, supp $\varphi_k \Subset B_R(x_0)$.

Step 2. Let us consider the super-level sets:

$$A_{k,R} := \{x \in B_R(x_0) \,:\, u(x) > k\}.$$

In this step we prove that

$$
\begin{aligned}
&\int_{A_{k,\rho}} |Du|^p\,dx \le c\left\{\int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,dx\right.\\
&+\int_{A_{k,R}}\left((u-k)^{\frac{p\gamma_1}{q-1}}+(u-k)^{\frac{p}{p-r}}+(u-k)^{\gamma_2+1}+(u-k)^{\gamma_2}\right)dx\\
&\left.+\,c\int_{A_{k,R}}\left(k^{\gamma_2}(u-k)+b_2(u-k)+k^{\frac{p\gamma_1}{q-1}}+k^{\gamma_2}+b_1^{\frac{p}{q-1}}\right)dx\right\}
\end{aligned}
\tag{5.4}
$$

for some constant c independent of u and η.

Using φ_k as a test function in (3.5) we get

$$
\begin{aligned}
I_1 &:= \int_{A_{k,R}} \langle a(x,u,Du),Du\rangle\,\eta^\mu\,dx\\
&= -\mu\int_{A_{k,R}} \langle a(x,u,Du),D\eta\rangle\eta^{\mu-1}(u-k)\,dx\\
&\quad -\int_{A_{k,R}} b(x,u,Du)(u-k)\,\eta^\mu\,dx =: I_2+I_3.
\end{aligned}
\tag{5.5}
$$

Now, we separately consider and estimate I_i, $i=1,2,3$.

Estimate of I_3

Using (3.4) we obtain

$$
I_3 \le \Lambda\int_{A_{k,R}} \eta^\mu\left\{|Du|^r(u-k)+|u|^{\gamma_2}(u-k)+b_2(u-k)\right\}\,dx.
$$

We estimate the right-hand side using the Young inequality, with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$, and (3.2). There exists c, depending on λ, Λ, n, p, r, such that

$$
\begin{aligned}
&\Lambda|Du|^r(u-k) \le \frac{\lambda}{4}|Du|^p + c(u-k)^{\frac{p}{p-r}}\\
&\le \frac{1}{4}\langle a(x,u,Du),Du\rangle + c(u-k)^{\frac{p}{p-r}} \qquad \text{a.e. in } \{|Du|\ge 1\}.
\end{aligned}
\tag{5.6}
$$

and, recalling that $b_2 \ge 1$,

$$
\Lambda|Du|^r(u-k) \le \Lambda(u-k) \le \Lambda b_2(u-k) \qquad \text{a.e. in } \{|Du|<1\}.
$$

Therefore,

$$
\begin{aligned}
I_3 &\le \frac{1}{4}\int_{A_{k,R}\cap\{|Du|\ge 1\}} \langle a(x,u,Du),Du\rangle\eta^\mu\,dx\\
&\quad + c\int_{A_{k,R}} \eta^\mu\left\{(u-k)^{\frac{p}{p-r}}+|u|^{\gamma_2}(u-k)+b_2(u-k)\right\}\,dx.
\end{aligned}
\tag{5.7}
$$

Collecting (5.5)–(5.7) we get

$$\frac{3}{4}\int_{A_{k,R}\cap\{|Du|\geq 1\}} \langle a(x,u,Du),Du\rangle \eta^\mu \,dx \leq I_2 - \int_{A_{k,R}\cap\{|Du|\leq 1\}} \langle a(x,u,Du),Du\rangle \eta^\mu \,dx$$
$$+ c\int_{A_{k,R}} \eta^\mu \left\{(u-k)^{\frac{p}{p-r}} + |u|^{\gamma_2}(u-k) + b_2(u-k)\right\} dx.$$

Using (3.2) and (3.3) we get

$$\frac{3\lambda}{4}\int_{A_{k,R}\cap\{|Du|\geq 1\}} |Du|^p\eta^\mu \,dx \leq I_2 + 2\Lambda\int_{A_{k,R}\cap\{|Du|\leq 1\}} (|u|^{\gamma_2} + b_1)\eta^\mu \,dx$$
$$+ c\int_{A_{k,R}} \eta^\mu \left\{(u-k)^{\frac{p}{p-r}} + |u|^{\gamma_2}(u-k) + b_2(u-k)\right\} dx. \tag{5.8}$$

Estimate of I_2. For a.e. $x \in A_{k,R} \cap \{\eta \neq 0\}$ we have

$$\mu|\langle a(x,u,Du),D\eta\rangle|(u-k)\eta^{\mu-1} \leq \mu\Lambda\left\{|Du|^{q-1} + |u|^{\gamma_1} + b_1\right\}|D\eta|(u-k)\eta^{\mu-1}. \tag{5.9}$$

For a.e. $x \in \{|Du| \geq 1\} \cap A_{k,R} \cap \{\eta \neq 0\}$, by $q < p+1$ and the Young inequality with exponents $\frac{p}{q-1}$ and $\frac{p}{p-q+1}$, and noting that $\mu - 1 = \mu\frac{q-1}{p}$, we get

$$\mu\Lambda|Du|^{q-1}|D\eta|(u-k)\eta^{\mu-1}$$
$$\leq \frac{\lambda}{4}|Du|^p\eta^\mu + c(\lambda,\Lambda)\mu^{\frac{p}{p-q+1}}|D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}. \tag{5.10}$$

On the other hand we have

$$\mu\Lambda|Du|^{q-1}|D\eta|(u-k)\eta^{\mu-1} \leq \mu\Lambda|D\eta|(u-k)\eta^{\mu-1} \tag{5.11}$$

a.e. in $\{|Du| < 1\} \cap A_{k,R} \cap \{\eta \neq 0\}$.

Therefore,

$$I_2 \leq \frac{\lambda}{4}\int_{A_{k,R}\cap\{|Du|\geq 1\}} |Du|^p\eta^\mu \,dx + c(\lambda,\Lambda)\mu^{\frac{p}{p-q+1}}\int_{A_{k,R}\cap\{|Du|\geq 1\}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,dx$$
$$+ \int_{A_{k,R}} |D\eta|(u-k)\eta^{\mu-1}\,dx + c\int_{A_{k,R}} |D\eta|\eta^{\mu-1}\{|u|^{\gamma_1} + b_1\}(u-k)\,dx.$$

By (5.8) and the inequality above, we get

$$\frac{\lambda}{2}\int_{A_{k,R}\cap\{|Du|\geq 1\}} |Du|^p\eta^\mu \,dx \leq c(\lambda,\Lambda,p,q)\int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}}(u-k)^{\frac{p}{p-q+1}}\,dx$$
$$+ \int_{A_{k,R}} |D\eta|\eta^{\mu-1}(|u|^{\gamma_1} + b_1)(u-k)\,dx$$
$$+ c\int_{A_{k,R}} \eta^\mu \left((u-k)^{\frac{p}{p-r}} + |u|^{\gamma_2}(u-k) + |u|^{\gamma_2} + b_2(u-k) + b_1\right) dx.$$

Taking into account that $b_1 \geq 1$

$$\int_{A_{k,R}} |Du|^p \eta^\mu \, dx = \int_{A_{k,R} \cap \{|Du| \geq 1\}} |Du|^p \eta^\mu \, dx + \int_{A_{k,R} \cap \{|Du| < 1\}} |Du|^p \eta^\mu \, dx$$
$$\leq \int_{A_{k,R} \cap \{|Du| \geq 1\}} |Du|^p \eta^\mu \, dx + \int_{A_{k,R}} b_1 \eta^\mu \, dx,$$

therefore

$$\int_{A_{k,R}} (|Du|^p - b_1) \eta^\mu \, dx \leq \int_{A_{k,R} \cap \{|Du| \geq 1\}} |Du|^p \eta^\mu \, dx$$

and we obtain

$$\int_{A_{k,\rho}} |Du|^p \, dx \leq c \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx$$
$$+ \int_{A_{k,R}} |D\eta| \eta^{\mu-1} \left(|u|^{\gamma_1} + b_1\right) (u-k) \, dx$$
$$+ c \int_{A_{k,R}} \eta^\mu \left((u-k)^{\frac{p}{p-r}} + |u|^{\gamma_2} (u-k) + |u|^{\gamma_2} + b_2 (u-k) + b_1 \right) dx. \tag{5.12}$$

We have

$$\int_{A_{k,R}} |D\eta| \eta^{\mu-1} |u|^{\gamma_1} (u-k) \, dx \leq c(\gamma_1) \int_{A_{k,R}} |D\eta| \eta^{\mu-1} (u-k)^{\gamma_1+1} \, dx$$
$$+ c(\gamma_1) \int_{A_{k,R}} |D\eta| \eta^{\mu-1} k^{\gamma_1} (u-k) \, dx.$$

By Hölder inequality with exponents $\frac{p}{q-1}$ and $\frac{p}{p-q+1}$, we get

$$\int_{A_{k,R}} |D\eta| \eta^{\mu-1} (u-k)^{\gamma_1+1} \, dx = \int_{A_{k,R}} |D\eta| (u-k) \eta^{\mu-1} (u-k)^{\gamma_1} \, dx$$
$$\leq c \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx + c \int_{A_{k,R}} \eta^{\frac{p(\mu-1)}{q-1}} (u-k)^{\frac{p\gamma_1}{q-1}} \, dx.$$

Analogously,

$$\int_{A_{k,R}} |D\eta| \eta^{\mu-1} k^{\gamma_1} (u-k) \, dx \leq c \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx$$
$$+ c \int_{A_{k,R}} \eta^{\frac{p(\mu-1)}{q-1}} k^{\frac{p\gamma_1}{q-1}} \, dx$$

and

$$\int_{A_{k,R}} |D\eta| \eta^{\mu-1} b_1 (u-k) \, dx \leq c \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx$$
$$+ c \int_{A_{k,R}} \eta^{\frac{p(\mu-1)}{q-1}} b_1^{\frac{p}{q-1}} \, dx,$$

obtaining

$$\int_{A_{k,\rho}} |Du|^p \, dx \le c \Bigg\{ \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx$$
$$+ \int_{A_{k,R}} \Big((u-k)^{\frac{p\gamma_1}{q-1}} + k^{\frac{p\gamma_1}{q-1}} + b_1^{\frac{p}{q-1}} \Big) \, dx$$
$$+ c \int_{A_{k,R}} \Big((u-k)^{\frac{p}{p-r}} + |u|^{\gamma_2}(u-k) + |u|^{\gamma_2} + b_2(u-k) + b_1 \Big) \, dx. \Bigg\}.$$

Therefore,

$$\int_{A_{k,\rho}} |Du|^p \, dx \le c \Bigg\{ \int_{A_{k,R}} |D\eta|^{\frac{p}{p-q+1}} (u-k)^{\frac{p}{p-q+1}} \, dx$$
$$+ \int_{A_{k,R}} \Big((u-k)^{\frac{p\gamma_1}{q-1}} + (u-k)^{\frac{p}{p-r}} + (u-k)^{\gamma_2+1} + (u-k)^{\gamma_2} \Big) \, dx$$
$$+ c \int_{A_{k,R}} \Big(k^{\gamma_2}(u-k) + k^{\gamma_2} + b_2(u-k) + b_1 + k^{\frac{p\gamma_1}{q-1}} + b_1^{\frac{p}{q-1}} \Big) \, dx. \Bigg\}.$$

Since $b_1 \ge 1$ and $q < p+1$, then

$$b_1 + b_1^{\frac{p}{q-1}} \le 2 b_1^{\frac{p}{q-1}},$$

and we get (5.4).

Step 3. In this step we prove that

$$\int_{B_\rho} |D(u-k)_+|^p \, dx \le C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1} - 1 + \frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*} \right)} \times$$
$$\times \|(u-k)_+\|^{\frac{p}{p-q+1}}_{W^{1,p}(B_R(x_0))} |A_{k,R}|^{\frac{p}{p-q+1}\nu}$$
$$+ c \int_{A_{k,R}} \Big((u-k)^{\frac{p\gamma_1}{q-1}} + (u-k)^{\frac{p}{p-r}} + (u-k)^{\gamma_2+1} + (u-k)^{\gamma_2} \Big) \, dx$$
$$+ c \int_{A_{k,R}} \Big(k^{\gamma_2}(u-k) + b_2(u-k) + k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2} + b_1^{\frac{p}{q-1}} \Big) \, dx. \tag{5.13}$$

We obtain this estimate starting by (5.4).

Consider $\tau \in (\rho, R)$ and define the function

$$S_1(0) \ni y \mapsto w(y) := (u-k)_+(x_0 + \tau y)$$

where

$$S_1(0) := \{ y \in \mathbb{R}^n \; : \; |y| = 1 \}.$$

This function w is in $W^{1,\left(\frac{p}{p-q+1}\right)_*}(S_1, d\mathcal{H}^{n-1})$, with

$$\frac{1}{\left(\frac{p}{p-q+1}\right)_*} = \min \left\{ \frac{1}{\frac{p}{p-q+1}} + \frac{1}{n-1}, 1 \right\}. \tag{5.14}$$

Let us consider the case
$$q > 1 + \frac{p}{n-1}.$$
By (4.1) in Remark 4.1, we get
$$\frac{1}{\left(\frac{p}{p-q+1}\right)_*} = \frac{1}{\frac{p}{p-q+1}} + \frac{1}{n-1}. \tag{5.15}$$
By (4.2) and the Sobolev embedding theorem, see Lemma 4.4, we get
$$\left(\int_{S_1} |w|^{\frac{p}{p-q+1}}\, d\mathcal{H}^{n-1}\right)^{\frac{p-q+1}{p}} \leq c(n,p,q)\left(\int_{S_1} (|Dw|^{\left(\frac{p}{p-q+1}\right)_*} + |w|^{\left(\frac{p}{p-q+1}\right)_*})\, d\mathcal{H}^{n-1}\right)^{1/\left(\frac{p}{p-q+1}\right)_*}. \tag{5.16}$$
When
$$q \leq 1 + \frac{p}{n-1},$$
we distinguish among two cases: $n \geq 3$ and $n = 2$. If $n \geq 3$, by using Hölder's inequality, we get
$$\left(\int_{S_1} |w|^{\frac{p}{p-q+1}}\, d\mathcal{H}^{n-1}\right)^{\frac{p-q+1}{p}} \leq c(n,p,q)\left(\int_{S_1} |w|^{\frac{n-1}{n-2}}\, d\mathcal{H}^{n-1}\right)^{\frac{n-2}{n-1}},$$
by (4.2) and the Sobolev embedding theorem, see Lemma 4.4, we obtain the inequality (5.16).

If $n = 2$, then $\left(\frac{p}{p-q+1}\right)_* = 1$, then we obtain the inequality (5.16) by applying Lemma 4.5 with $r = \frac{p}{p-q+1}$.

Let $\mathcal{A}(\rho, R)$ be as in (5.2). We apply Lemma 4.3, with
$$B_R(x_0) \ni y \mapsto v(y) := (u-k)_+^{\frac{p}{p-q+1}}(y),$$
that is a function in $L^1(B_R(x_0))$. Using (5.16) and recalling that $\frac{R_0}{2} \leq \rho < R \leq R_0$, reasoning as in [30], we get
$$\begin{aligned}
&\inf_{\mathcal{A}(\rho,R)} \int_{B_R(x_0)} |D\eta|^{\frac{p}{p-q+1}} (u-k)_+^{\frac{p}{p-q+1}}\, dx \\
&\leq C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1} - 1 + \frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times \\
&\times \left(\int_\rho^R \int_{S_\tau(0)} \left(|D(u-k)_+(x_0+y)|^{\left(\frac{p}{p-q+1}\right)_*}\right.\right. \\
&\qquad \left.\left. + |(u-k)_+(x_0+y)|^{\left(\frac{p}{p-q+1}\right)_*}\right) d\mathcal{H}^{n-1}(y)\, d\tau\right)^{\frac{p}{p-q+1}/\left(\frac{p}{p-q+1}\right)_*}.
\end{aligned} \tag{5.17}$$
By coarea formula, inequality (5.17) implies
$$\begin{aligned}
&\inf_{\mathcal{A}(\rho,R)} \int_{B_R(x_0)} |D\eta|^{\frac{p}{p-q+1}} (u-k)_+^{\frac{p}{p-q+1}}\, dx \\
&\leq C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1} - 1 + \frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times
\end{aligned}$$

$$\times \|(u-k)_+\|^{\frac{p}{p-q+1}}_{W^{1,\left(\frac{p}{p-q+1}\right)_*}(B_R(x_0)\setminus B_\rho(x_0))}$$

and, taking into account (3.7), Remark 4.1 and (4.5)

$$\left(\frac{p}{p-q+1}\right)_* < p \Leftrightarrow \frac{1}{\left(\frac{p}{p-q+1}\right)_*} > \frac{1}{p} \Leftrightarrow \nu > 0 \Leftrightarrow \frac{q}{p} < 1 + \frac{1}{n-1},$$

by Hölder's inequality we get

$$\begin{aligned} &\inf_{\mathcal{A}(\rho,R)} \int_{B_R(x_0)} |D\eta|^{\frac{p}{p-q+1}} (u-k)_+^{\frac{p}{p-q+1}}\, dx \\ &\le C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times \\ &\times \|(u-k)_+\|^{\frac{p}{p-q+1}}_{W^{1,p}(B_R(x_0))} |A_{k,R}|^{\frac{p}{p-q+1}\nu} \end{aligned} \tag{5.18}$$

By (5.4) we get

$$\begin{aligned} &\int_{A_{k,\rho}} |Du|^p\, dx \le C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times \\ &\times \|(u-k)_+\|^{\frac{p}{p-q+1}}_{W^{1,p}(B_R(x_0))} |A_{k,R}|^{\frac{p}{p-q+1}\nu} \\ &+ c \int_{A_{k,R}} \left((u-k)^{\frac{p\gamma_1}{q-1}} + (u-k)^{\frac{p}{p-r}} + (u-k)^{\gamma_2+1} + (u-k)^{\gamma_2}\right) dx \\ &+ c \int_{A_{k,R}} \left(k^{\gamma_2}(u-k) + b_2(u-k) + k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2} + b_1^{\frac{p}{q-1}}\right) dx. \end{aligned}$$

Since

$$\int_{B_\rho} |D(u-k)_+|^p\, dx = \int_{A_{k,\rho}} |D(u-k)_+|^p\, dx = \int_{A_{k,\rho}} |Du|^p\, dx$$

we get (5.13).

Step 4. In this step we estimate the integrals at the right hand side of (5.13).

Consider

$$J_1 := \int_{A_{k,R}} \left((u-k)^{\frac{p\gamma_1}{q-1}} + (u-k)^{\frac{p}{p-r}} + (u-k)^{\gamma_2+1} + (u-k)^{\gamma_2}\right) dx.$$

Estimate of J_1.

By assumptions (3.8) and (3.9),

$$\max\left\{\frac{p\gamma_1}{q-1}, \gamma_2+1, \frac{p}{p-r}\right\} < p^*.$$

Therefore, by using Hölder inequality with exponent $\frac{p^*(q-1)}{p\gamma_1}$ we get

$$\int_{A_{k,R}} (u-k)^{\frac{p\gamma_1}{q-1}}\, dx \le \left(\int_{A_{k,R}} (u-k)^{p^*}\, dx\right)^{\frac{p\gamma_1}{p^*(q-1)}} |A_{k,R}|^{1-\frac{p\gamma_1}{p^*(q-1)}};$$

Hölder inequality with exponent $p^* \frac{p-r}{p}$ implies

$$\int_{A_{k,R}} (u-k)^{\frac{p}{p-r}}\, dx \leq \left(\int_{A_{k,R}} (u-k)^{p^*}\, dx \right)^{\frac{1}{p^* \frac{p-r}{p}}} |A_{k,R}|^{1-\frac{1}{p^* \frac{p-r}{p}}}.$$

Moreover, by using Hölder inequality with exponent $\frac{p^*}{\gamma_2+1}$ we get

$$\int_{A_{k,R}} (u-k)^{\gamma_2+1}\, dx \leq \left(\int_{A_{k,R}} (u-k)^{p^*}\, dx \right)^{\frac{\gamma_2+1}{p^*}} |A_{k,R}|^{1-\frac{\gamma_2+1}{p^*}};$$

by using Hölder inequality with exponent $\frac{p^*}{\gamma_2}$ we get

$$\int_{A_{k,R}} (u-k)^{\gamma_2}\, dx \leq \left(\int_{A_{k,R}} (u-k)^{p^*}\, dx \right)^{\frac{\gamma_2}{p^*}} |A_{k,R}|^{1-\frac{\gamma_2}{p^*}}.$$

Therefore, by using the Sobolev embedding theorem

$$\begin{aligned} J_1 \leq & \|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p\gamma_1}{q-1}} |A_{k,R}|^{1-\frac{p\gamma_1}{p^*(q-1)}} + \|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p}{p-r}} |A_{k,R}|^{1-\frac{1}{p^* \frac{p-r}{p}}} \\ & + \|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2+1} |A_{k,R}|^{1-\frac{\gamma_2+1}{p^*}} + \|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2} |A_{k,R}|^{1-\frac{\gamma_2}{p^*}}. \end{aligned}$$

Let us consider now the following integral in (5.13):

$$J_2 := \int_{A_{k,R}} \left(k^{\gamma_2}(u-k) + b_2(u-k) + k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2} + b_1^{\frac{p}{q-1}} \right) dx.$$

Trivially,

$$\begin{aligned} \int_{A_{k,R}} k^{\gamma_2}(u-k)\, dx & \leq k^{\gamma_2} \|(u-k)_+\|_{L^{p^*}(A_{k,R})}^{\frac{1}{p^*}} |A_{k,R}|^{1-\frac{1}{p^*}} \\ & \leq k^{\gamma_2} \|(u-k)_+\|_{W^{1,p}(A_{k,R})} |A_{k,R}|^{1-\frac{1}{p^*}}. \end{aligned}$$

By assumption $b_2 \in L^{s_2}$, $s_2 > \frac{n}{p} = \frac{p^*}{p^*-p}$. Since $\frac{p^*}{p^*-p} > \frac{p^*}{p^*-1}$, then $\frac{s_2}{s_2-1} < p^*$. Therefore, by Hölder inequality

$$\begin{aligned} \int_{A_{k,R}} b_2(u-k)\, dx & \leq \|b_2\|_{L^{s_2}(A_{k,R})} \|(u-k)_+\|_{L^{\frac{s_2}{s_2-1}}} \\ & \leq \|b_2\|_{L^{s_2}(B_R)} \|(u-k)_+\|_{L^{p^*}(A_{k,R})} |A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}}, \end{aligned}$$

which implies

$$\int_{A_{k,R}} b_2(u-k)\, dx \leq \|b_2\|_{L^{s_2}(B_R)} \|(u-k)_+\|_{W^{1,p}(B_R)} |A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}}.$$

Now, $b_1 \in L^{s_1}$ with $s_1 > \frac{p}{q-1}$; by using Hölder inequality with exponent $\frac{s_1(q-1)}{p}$ we get

$$\int_{A_{k,R}} b_1^{\frac{p}{q-1}}\, dx \leq \left(\int_{A_{k,R}} b_1^{s_1}\, dx \right)^{\frac{p}{s_1(q-1)}} |A_{k,R}|^{1-\frac{p}{s_1(q-1)}}.$$

We obtain

$$J_2 \le k^{\gamma_2}\|(u-k)_+\|_{W^{1,p}((B_R))}|A_{k,R}|^{1-\frac{1}{p^*}} + \left(k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2}\right)|A_{k,R}|$$
$$+ \|b_2\|_{L^{s_2}(B_R)}\|(u-k)_+\|_{W^{1,p}(B_R)}|A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} + \|b_1\|_{L^{s_1}(B_R)}^{\frac{p}{q-1}}|A_{k,R}|^{1-\frac{p}{s_1(q-1)}}.$$

Step 5. By Steps 3, 4 we get

$$\int_{B_r}|D(u-k)_+|^p\,dx \le C(n,p,q,R_0)(R-\rho)^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)}\times$$
$$\times \|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p}{p-q+1}}|A_{k,R}|^{\frac{p}{p-q+1}\nu}$$
$$+ c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p\gamma_1}{q-1}}|A_{k,R}|^{1-\frac{p\gamma_1}{p^*(q-1)}} + c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\frac{p}{p-r}}|A_{k,R}|^{1-\frac{1}{p^*}\frac{p-r}{p}}$$
$$+ c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2+1}|A_{k,R}|^{1-\frac{\gamma_2+1}{p^*}} + c\|(u-k)_+\|_{W^{1,p}(B_R)}^{\gamma_2}|A_{k,R}|^{1-\frac{\gamma_2}{p^*}}$$
$$+ ck^{\gamma_2}\|(u-k)_+\|_{W^{1,p}(B_R)}|A_{k,R}|^{1-\frac{1}{p^*}} + c\left(k^{\frac{p\gamma_1}{q-1}} + k^{\gamma_2}\right)|A_{k,R}|$$
$$+ c\|b_2\|_{L^{s_2}(B_R)}\|(u-k)_+\|_{W^{1,p}(B_R)}|A_{k,R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} + c\|b_1\|_{L^{s_1}(B_R)}^{\frac{p}{q-1}}|A_{k,R}|^{1-\frac{p}{s_1(q-1)}}$$

and the inequality (5.1) follows.

□

6. Proof of the boundedness result

Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$, $1 < q \le n$, be weak solution to (3.1). Consider $\Omega' \Subset \Omega$ an open set.

I case $q > p$**.** Let $B_{R_0}(x_0) \subseteq \Omega'$.

For every $k \ge 0$

$$\int_{B_{R_0}(x_0)}(u-k)_+^p\,dx + \int_{B_{R_0}(x_0)}|D(u-k)_+|^p\,dx$$
$$\le \int_{B_{R_0}(x_0)}(|u|-k)_+^p\chi_{\{x\in B_{R_0}(x_0)\,:\,|u|>k\}}(x)\,dx + \int_{B_{R_0}(x_0)}|Du|^p\chi_{\{x\in B_{R_0}(x_0)\,:\,|u|>k\}}(x)\,dx$$
$$\le \int_{B_{R_0}(x_0)}(|u|^p + |Du|^p)\chi_{\{x\in B_{R_0}(x_0)\,:\,|u|>k\}}(x)\,dx$$
$$\le \left(\int_{B_{R_0}(x_0)}(|u|^q + |Du|^q)\,dx\right)^{p/q}|\{x\in B_{R_0}(x_0)\,:\,|u|>k\}|^{1-p/q}$$
$$\le \|u\|_{W^{1,q}(B_{R_0}(x_0))}^p|B_{R_0}(x_0)|^{1-p/q}. \tag{6.1}$$

In particular, chosen R_0 such that

$$|B_{R_0}(x_0)| \le \|u\|_{W^{1,q}(\Omega')}^{-\frac{pq}{q-p}}$$

we get

$$\|(u-k)\|_{W^{1,p}(B_{R_0}(x_0))} < 1 \quad \forall k \ge 0. \tag{6.2}$$

II case $q = p$. By a well known result by Giaquinta and Giusti [40], the gradient of the weak solution satisfies a higher integrability property: its gradient is in $L^{p+\varepsilon}(B_{R_0}(x_0))$, for some $\varepsilon > 0$ sufficiently small. Moreover, $u \in L^{p^*}(B_{R_0}(x_0))$; because $p = q$, we can repeat the above argument with q replaced by $p + \varepsilon$ so obtaining (6.1). $R_0 > 0$ depends on the norm $\|u\|_{W^{1,p+\varepsilon}(B_{R_0}(x_0))}$. Again, by the Giaquinta and Giusti result, the norm $\|u\|_{W^{1,p+\varepsilon}(B_{R_0}(x_0))}$ can be estimated in terms of the $\|u\|_{W^{1,p}(\Omega')}$ for $B_{R_0}(x_0) \subseteq \Omega' \Subset \Omega$.

Finally, we can summarize: in both cases, either if $q > p$ or if $q = p$, we can choose R_0 such that (6.2) holds with $R_0 > 0$ depending on the norm $\|u\|_{W^{1,q}(\Omega')}$. We also assume $R_0 < 1$ such that $|B_{R_0}| < 1$, $0 < R \leq R_0$.

Define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}(1 + \frac{1}{2^h}).$$

Fixed a positive constant $d \geq 2$, to be chosen later, define the increasing sequence of positive real numbers (k_h)

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \ h \in \mathbb{N}.$$

Define the decreasing sequence (J_h),

$$J_h := \|(u - k_h)_+\|^p_{W^{1,p}(B_{\rho_h}(x_0))}.$$

Notice that

$$\rho_0 = R, \qquad \lim_{\rho \to +\infty} \frac{R}{2}(1 + \frac{1}{2^h}) = \frac{R}{2},$$

$$k_0 := \frac{d}{2}, \qquad \lim_{h \to +\infty} k_h = d.$$

Moreover, by (6.2),

$$J_h \leq J_0 = \|(u - \frac{d}{2})_+\|^p_{W^{1,p}(B_R(x_0))} < 1.$$

Let us introduce the following notation:

$$\tau := \max\left\{\frac{pp^*}{p-q+1}\nu + \left(\frac{p}{p-q+1} - 1 + \frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right), p^*\right\}, \tag{6.3}$$

$$\theta := \min\Big\{\frac{pp^*}{p-q+1}\nu, p^* - \frac{p\gamma_1}{q-1}, p^* - \frac{p}{p-r}, p^* - \gamma_2 - 1, p^* - p, \\ p^*\left(1 - \frac{1}{s_2}\right) - 1, p^*\left(1 - \frac{p}{s_1(q-1)}\right)\Big\} \tag{6.4}$$

and

$$\sigma := \min\left\{\frac{1}{p-q+1} + \frac{p^*}{p-q+1}\nu, \frac{p^*}{p} - \frac{p^*}{s_1(q-1)}, \frac{p^*}{p}\left(1 - \frac{1}{s_2}\right)\right\}, \tag{6.5}$$

where ν is defined in (4.4).

Proposition 6.1 (Estimate of J_{h+1}). *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (3.1). Assume (3.2)–(3.4) with the exponents satisfying the inequalities listed in Section 3.1. Then for every $h \in \mathbb{N}$*

$$J_{h+1} \leq c\frac{(2^{\tau})^h}{d^{\theta}} J_h^{\sigma}, \tag{6.6}$$

where c is a constant depending on n, p, q, r, R_0, the L^{s_1}-norm of b_1 and the L^{s_2}-norm of b_2 in B_{R_0}.

We precede the proof with the following remark.

Remark 6.2. We remark that, by assumptions (3.6)–(3.10), then $\tau, \theta > 0$ and $\sigma > 1$. As far as these inequalities are concerned, we remark that

$$p^* > p;$$

$$\nu > 0 \quad \text{(see (4.5))};$$

$$\frac{1}{p-q+1} + \frac{p^*}{p-q+1}\nu > 1 \Leftrightarrow p^*\nu > p-q$$

that is satisfied, because $p \leq q$

$$p^* > \frac{p}{p-r} \Leftrightarrow r < p - \frac{p}{p^*} \Leftrightarrow r < p + \frac{p}{n} - 1;$$

$$p^* > \frac{p\gamma_1}{q-1} \Leftrightarrow \gamma_1 < p^*\frac{q-1}{p} \Leftrightarrow \gamma_1 < \frac{n(q-1)}{n-p};$$

$$\gamma_2 < p^* - 1;$$

$$\frac{p^*}{p} - \frac{p^*}{s_1(q-1)} > 1 \Leftrightarrow \frac{p}{s_1(q-1)} < 1 - \frac{p}{p^*} \Leftrightarrow s_1 > \frac{n}{q-1}$$

that is the first assumption in (3.10); this assumption also implies

$$s_1 > \frac{p}{q-1} > 0$$

that is equivalent to

$$1 - \frac{p}{s_1(q-1)} > \frac{p}{p^*} > 0.$$

By the second assumption in (3.10),

$$s_2 > \frac{n}{p} \Leftrightarrow s_2 > \frac{p^*}{p^*-p} \Leftrightarrow \frac{p^*}{p}\left(1 - \frac{1}{s_2}\right) > 1.$$

Proof of Proposition 6.1. By (5.1), used with $k = k_{h+1}$, $\rho = \rho_{h+1}$, $R = \rho_h$, we have

$$\int_{B_{\rho_{h+1}}} |D(u-k_{h+1})_+|^p\,dx \leq C(n,p,q,R_0)\,(\rho_h - \rho_{h+1})^{-\left(\frac{p}{p-q+1} - 1 + \frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} \times$$
$$\times \|(u-k_{h+1})_+\|^{\frac{p}{p-q+1}}_{W^{1,p}(B_{\rho_h})} |A_{k_{h+1},\rho_h}|^{\frac{p}{p-q+1}\nu}$$

$$
\begin{aligned}
&+ c\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}^{\frac{p\gamma_1}{q-1}}|A_{k_{h+1},R}|^{1-\frac{p\gamma_1}{p^*(q-1)}}\\
&+ c\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}^{\frac{p}{p-r}}|A_{k_{h+1},R}|^{1-\frac{1}{p^*\frac{p-r}{p}}}\\
&+ c\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}^{\gamma_2+1}|A_{k_{h+1},R}|^{1-\frac{\gamma_2+1}{p^*}} + c\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}^{\gamma_2}|A_{k_{h+1},R}|^{1-\frac{\gamma_2}{p^*}}\\
&+ ck_{h+1}^{\gamma_2}\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}|A_{k_{h+1},R}|^{1-\frac{1}{p^*}} + c\left(k_{h+1}^{\frac{p\gamma_1}{q-1}} + k_{h+1}^{\gamma_2}\right)|A_{k_{h+1},R}|\\
&+ c\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_{h+1}})}|A_{k_{h+1},R}|^{1-\frac{1}{s_2}-\frac{1}{p^*}} + c|A_{k_{h+1},R}|^{1-\frac{p}{s_1(q-1)}}. \qquad (6.7)
\end{aligned}
$$

Let us write the estimate above as

$$
\begin{aligned}
\int_{B_{\rho_{h+1}}} |D(u-k_{h+1})_+|^p\,dx \le & c\,(\rho_h-\rho_{h+1})^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)}H_1\\
&+ c\,(H_2+H_3+H_4+H_5+H_6+H_7+H_8+H_9)\,. \qquad (6.8)
\end{aligned}
$$

To estimate the sum at the right-hand side it is useful to remark that, for all h,

$$
k_{h+1}-k_h = \frac{d}{2^{h+2}} \qquad (6.9)
$$

and

$$
k_{h+1}-k_h < u-k_h \quad \text{in } A_{k_{h+1},\rho_h}.
$$

Since

$$
|A_{k_{h+1},\rho_h}| \le \int_{A_{k_{h+1},\rho_h}} \left(\frac{u-k_h}{k_{h+1}-k_h}\right)^{p^*}dx \le \|(u-k)_+\|_{L^{p^*}(B_{\rho_h})}^{p^*}\frac{1}{(k_{h+1}-k_h)^{p^*}},
$$

by the Sobolev inequality we get

$$
|A_{k_{h+1},\rho_h}| \le c(n,p)\frac{J_h^{\frac{p^*}{p}}}{(k_{h+1}-k_h)^{p^*}},
$$

that, together with (6.9), gives

$$
|A_{k_{h+1},\rho_h}| \le c(n,p)J_h^{\frac{p^*}{p}}\left(\frac{2^h}{d}\right)^{p^*}. \qquad (6.10)
$$

Moreover,

$$
\begin{aligned}
\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_h}(x_0))}^p &= \int_{A_{k_{h+1},\rho_h}}(u-k_{h+1})^p\,dx + \int_{A_{k_{h+1},\rho_h}}|D(u-k_{h+1})|^p\,dx\\
&\le \int_{A_{k_h,\rho_h}}(u-k_h)^p\,dx + \int_{A_{k_h,\rho_h}}|D(u-k_h)|^p\,dx\\
&\le J_h. \qquad (6.11)
\end{aligned}
$$

Inequalities (6.10) and (6.11) imply that

$$
\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_h}(x_0))}|A_{k_{h+1},R}|^{-\frac{1}{p^*}} \le c(n,p)J_h^{\frac{1}{p}}\frac{J_h^{\frac{p^*}{p}\left(-\frac{1}{p^*}\right)}}{(k_{h+1}-k_h)^{p^*\left(-\frac{1}{p^*}\right)}}
$$

therefore, by (6.9),

$$\|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_h}(x_0))}|A_{k_{h+1},R}|^{-\frac{1}{p^*}} \le c(n,p)\left(\frac{2^h}{d}\right)^{-1}. \tag{6.12}$$

This estimate, together with (6.10), implies:

$$H_2 \le c(n,p,q,\gamma_1)\left(\frac{2^h}{d}\right)^{-\frac{p\gamma_1}{q-1}}|A_{k_{h+1},R}| \le c(n,p,q,\gamma_1)\left(\frac{2^h}{d}\right)^{p^*-\frac{p\gamma_1}{q-1}} J_h^{\frac{p^*}{p}}, \tag{6.13}$$

and, analogously,

$$H_3 \le c(n,p,r)\left(\frac{2^h}{d}\right)^{p^*-\frac{p}{p-r}} J_h^{\frac{p^*}{p}}, \tag{6.14}$$

$$H_4 \le c(n,p,\gamma_2)\left(\frac{2^h}{d}\right)^{p^*-\gamma_2-1} J_h^{\frac{p^*}{p}}, \tag{6.15}$$

$$H_5 \le c(n,p,\gamma_2)\left(\frac{2^h}{d}\right)^{p^*-\gamma_2} J_h^{\frac{p^*}{p}}, \tag{6.16}$$

$$H_8 \le c(n,p)\left(\frac{2^h}{d}\right)^{-1}|A_{k_{h+1},R}|^{1-\frac{1}{s_2}} \le c(n,p,s_2)\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{1}{s_2}\right)-1} J_h^{\frac{p^*}{p}\left(1-\frac{1}{s_2}\right)}, \tag{6.17}$$

$$H_9 \le c(n,p,q,s_1)\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{p}{s_1(q-1)}\right)} J_h^{\frac{p^*}{p}-\frac{p^*}{s_1(q-1)}}. \tag{6.18}$$

Moreover, taking into account that

$$k_{h+1} = d\left(1-\frac{1}{2^{h+2}}\right) \le d,$$

$$H_6 \le c(n,p)d^{\gamma_2}\left(\frac{2^h}{d}\right)^{p^*-1} J_h^{\frac{p^*}{p}} = c(n,p)\frac{2^{h(p^*-1)}}{d^{p^*-\gamma_2-1}} J_h^{\frac{p^*}{p}} \tag{6.19}$$

$$H_7 \le c\left(\frac{2^{hp^*}}{d^{p^*-\frac{p\gamma_1}{q-1}}}+\frac{2^{hp^*}}{d^{p^*-\gamma_2}}\right) J_h^{\frac{p^*}{p}}. \tag{6.20}$$

Let us now estimate H_1.

Inequalities (6.10) and (6.11) imply

$$\begin{aligned} H_1 &:= \|(u-k_{h+1})_+\|_{W^{1,p}(B_{\rho_h}(x_0))}^{\frac{p}{p-q+1}}|A_{k_{h+1},\rho_h}|^{\frac{p}{p-q+1}\nu} \\ &\le c(n,p,q) J_h^{\frac{1}{p-q+1}}\left(\frac{J_h^{\frac{p^*}{p}}}{(k_{h+1}-k_h)^{p^*}}\right)^{\frac{p}{p-q+1}\nu} \end{aligned}$$

that gives

$$H_1 \le c(n,p,q)\left(\frac{2^h}{d}\right)^{\frac{pp^*}{p-q+1}\nu} J_h^{\frac{1}{p-q+1}+\frac{p^*}{p-q+1}\nu}.$$

Taking into account that for every h

$$\frac{1}{4}\frac{R_0}{2^{h+1}} \le \rho_h - \rho_{h+1} = \frac{R}{2^{h+2}} \le \frac{1}{4}\frac{R_0}{2^h},$$

we conclude that

$$\begin{aligned}
&(\rho_h - \rho_{h+1})^{-\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)} H_1 \\
&\le c(n,p,q,R_0)\frac{(2^h)^{\frac{pp^*}{p-q+1}\nu+\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)}}{d^{\frac{pp^*}{p-q+1}\nu}} J_h^{\frac{1}{p-q+1}+\frac{p^*}{p-q+1}\nu}.
\end{aligned} \tag{6.21}$$

Collecting (6.13)–(6.21), by (6.8) we get

$$\begin{aligned}
&\int_{B_{\rho_{h+1}}} |D(u-k_{h+1})_+|^p\,dx \le c\frac{(2^h)^{\frac{pp^*}{p-q+1}\nu+\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)}}{d^{\frac{pp^*}{p-q+1}\nu}} J_h^{\frac{1}{p-q+1}+\frac{p^*}{p-q+1}\nu} \\
&+c\left\{\left(\frac{2^h}{d}\right)^{p^*-\frac{p\gamma_1}{q-1}}+\left(\frac{2^h}{d}\right)^{p^*-\frac{p}{p-r}}+\left(\frac{2^h}{d}\right)^{p^*-\gamma_2-1}+\left(\frac{2^h}{d}\right)^{p^*-\gamma_2}\right. \\
&\left.+\frac{2^{h(p^*-1)}}{d^{p^*-\gamma_2-1}}+\frac{2^{hp^*}}{d^{p^*-\frac{p\gamma_1}{q-1}}}+\frac{2^{hp^*}}{d^{p^*-\gamma_2}}\right\} J_h^{\frac{p^*}{p}} \\
&+c\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{1}{s_2}\right)-1} J_h^{\frac{p^*}{p}\left(1-\frac{1}{s_2}\right)}+c\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{p}{s_1(q-1)}\right)} J_h^{\frac{p^*}{p}-\frac{p^*}{s_1(q-1)}}.
\end{aligned} \tag{6.22}$$

Let us now add to both sides of (6.22) the integral $\int_{B_{\rho_{h+1}}} |(u-k_{h+1})_+|^p\,dx$.

By Hölder inequality

$$\int_{B_{\rho_{h+1}}} ((u-k_{h+1})_+)^p\,dx \le \left(\int_{B_{\rho_{h+1}}} ((u-k_{h+1})_+)^{p^*}\,dx\right)^{\frac{p}{p^*}} |A_{k_{h+1},\rho_{h+1}}|^{1-\frac{p}{p^*}}.$$

Since

$$\int_{B_{\rho_{h+1}}} ((u-k_{h+1})_+)^{p^*}\,dx \le \int_{B_{\rho_{h+1}}} ((u-k_h)_+)^{p^*}\,dx \le \int_{B_{\rho_h}} ((u-k_h)_+)^{p^*}\,dx,$$

the Sobolev embedding theorem gives

$$\int_{B_{\rho_{h+1}}} ((u-k_{h+1})_+)^p\,dx \le c\|(u-k_h)_+\|^p_{W^{1,p}(B_{\rho_h})} |A_{k_{h+1},\rho_{h+1}}|^{1-\frac{p}{p^*}}. \tag{6.23}$$

Taking into account (6.10), we obtain

$$|A_{k_{h+1},\rho_{h+1}}|^{1-\frac{p}{p^*}} \le |A_{k_{h+1},\rho_h}|^{1-\frac{p}{p^*}} \le c(n,p)\left(\frac{2^h}{d}\right)^{p^*-p} J_h^{\frac{p^*}{p}-1};$$

therefore, the inequality (6.23) implies

$$\int_{B_{\rho_{h+1}}} ((u-k_{h+1})_+)^p\,dx \le c(n,p)\left(\frac{2^h}{d}\right)^{p^*-p} J_h^{\frac{p^*}{p}}. \tag{6.24}$$

Inequalities (6.22) and (6.24) give

$$\begin{aligned} J_{h+1} \le\ & c\frac{(2^h)^{\frac{pp^*}{p-q+1}\nu+\left(\frac{p}{p-q+1}-1+\frac{\frac{p}{p-q+1}}{\left(\frac{p}{p-q+1}\right)_*}\right)}}{d^{\frac{pp^*}{p-q+1}\nu}} J_h^{\frac{1}{p-q+1}+\frac{p^*}{p-q+1}\nu} \\ & + c\left\{\left(\frac{2^h}{d}\right)^{p^*-\frac{p\gamma_1}{q-1}} + \left(\frac{2^h}{d}\right)^{p^*-\frac{p}{p-r}} + \left(\frac{2^h}{d}\right)^{p^*-\gamma_2-1} + \left(\frac{2^h}{d}\right)^{p^*-\gamma_2} \right. \\ & \left. + \frac{2^{h(p^*-1)}}{d^{p^*-\gamma_2-1}} + \frac{2^{hp^*}}{d^{p^*-\frac{p\gamma_1}{q-1}}} + \frac{2^{hp^*}}{d^{p^*-\gamma_2}} + \left(\frac{2^h}{d}\right)^{p^*-p}\right\} J_h^{\frac{p^*}{p}} \\ & + c\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{1}{s_2}\right)-1} J_h^{\frac{p^*}{p}\left(1-\frac{1}{s_2}\right)} + c\left(\frac{2^h}{d}\right)^{p^*\left(1-\frac{p}{s_1(q-1)}\right)} J_h^{\frac{p^*}{p}-\frac{p^*}{s_1(q-1)}}. \end{aligned} \tag{6.25}$$

where c is a constant depending on n, p, q, r, R_0, the L^{s_1}-norm of b_1 and the L^{s_2}-norm of b_2 in B_{R_0}.

By taking in account the notation in (6.3)–(6.5), we get, by (6.25), the inequality (6.6).

□

We are now ready to prove our regularity result.

Proof of Theorem 3.2. By Proposition 6.1, for every $h \in \mathbb{N}$,

$$J_{h+1} \le c\frac{(2^h)^\tau}{d^\theta} J_h^\sigma,$$

where c is a constant depending on n, p, q, R_0, the L^{s_1}-norm of b_1 and the L^{s_2}-norm of b_2 in B_{R_0} and for every $d \ge 2$. Thus, the following inequality holds:

$$J_{h+1} \le A\lambda^h J_h^{1+\alpha},$$

with

$$A = \frac{c}{d^\theta}, \;\; \lambda = 2^\tau, \;\; \alpha = \sigma - 1,$$

where θ, τ and σ are defined in (6.4), (6.3), (6.5). We recall that $\theta, \tau > 0$, $\sigma - 1 > 0$, see Remark 6.2.

To apply Lemma 4.6, we need

$$\|(u-\frac{d}{2})_+\|^p_{W^{1,p}(B_R(x_0))} = J_0 \le A^{-\frac{1}{\alpha}}\lambda^{-\frac{1}{\alpha^2}} = c^{-\frac{1}{\sigma-1}} 2^{-\frac{\tau}{(\sigma-1)^2}} d^{\frac{\theta}{\sigma-1}}. \tag{6.26}$$

Since

$$\|(u-\frac{d}{2})_+\|^p_{W^{1,p}(B_R(x_0))} \le \|u\|^p_{W^{1,p}(B_R(x_0))},$$

if we choose $d \ge 2$ satisfying

$$d^{\frac{\theta}{\sigma-1}} = 2 + c^{\frac{1}{\sigma-1}} 2^{\frac{\tau}{(\sigma-1)^2}} \|u\|^p_{W^{1,p}(B_R(x_0))}, \tag{6.27}$$

we get $0 = \lim_{h\to+\infty} J_h = \|(u-d)_+\|^p_{W^{1,p}(B_{\frac{R}{2}})}$ and we conclude that

$$u(x) \le d \qquad \text{a.e. in } B_{\frac{R}{2}}(x_0).$$

To prove that u is locally bounded from below, we proceed as follows. The function $-u$ is a weak solution to

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\overline{a}^i(x,u,Du) = \overline{b}(x,u,Du).$$

where

$$\overline{a}(x,u,\xi) := a(x,-u,-\xi) \qquad \text{and} \qquad \overline{b}(x,u,\xi) := b(x,-u,-\xi).$$

Notice that, by (3.2)–(3.4) the following properties hold:

- *$p-$ellipticity condition at infinity*:
  for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}$,

$$\langle \overline{a}(x,u,\xi), -\xi\rangle \ge \lambda|\xi|^p \qquad \forall \xi \in \mathbb{R}^n, |\xi| > 1,$$

- *$q-$growth condition*:
  for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|\overline{a}(x,u,\xi)| \le \Lambda\left\{|\xi|^{q-1} + |u|^{\gamma_1} + b_1(x)\right\},$$

- *growth condition for the right hand side $b(x,u,\xi)$*:

$$|\overline{b}(x,u,\xi)| \le \Lambda\left\{|\xi|^r + |u|^{\gamma_2} + b_2(x)\right\}.$$

To prove the analogue of Proposition 5.1 we now consider the test function $\varphi_k(x) := (k-u(x))_+[\eta(x)]^\mu$ where η is a cut-off function. Let us consider the sub-level sets:

$$B_{k,R} := \{x \in B_R(x_0) \,:\, u(x) < k\}, \qquad k \in \mathbb{R}.$$

Then we obtain, in place of (5.5),

$$\begin{aligned}\int_{B_{k,R}} \langle \overline{a}(x,u,Du), -Du\rangle\, \eta^\mu\, dx = -\mu \int_{B_{k,R}} \langle \overline{a}(x,u,Du), D\eta\rangle \eta^{\mu-1}(k-u)\, dx \\ + \int_{B_{k,R}} \overline{f}(x,u,Du)(k-u)\eta^\mu\, dx.\end{aligned}$$

The proof goes on with no significant changes with respect the previous case, arriving to the conclusion that there exists d' such that we obtain that $B_{\frac{R}{2}} \subseteq \{u \ge d'\}$, and

$$u(x) \ge d' \qquad \text{a.e. in } B_{\frac{R}{2}}(x_0).$$

Collecting the estimates from below and from above for u, we conclude. □

Acknowledgments

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflict of interest

The authors declare no conflict of interest.

References

1. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var.*, **57** (2018), 62. https://doi.org/10.1007/s00526-018-1332-z
2. L. Beck, G. Mingione, Lipschitz bounds and non-uniform ellipticity, *Commun. Pure Appl. Math.*, **73** (2020), 944–1034. https://doi.org/10.1002/cpa.21880
3. P. Bella, M. Schäffner, On the regularity of minimizers for scalar integral functionals with $(p, q)-$growth, *Anal. PDE*, **13** (2020), 2241–2257. https://doi.org/10.2140/apde.2020.13.2241
4. P. Bella, M. Schäffner, Local boundedness and Harnack inequality for solutions of linear nonuniformly elliptic equations, *Commun. Pure Appl. Math.*, **74** (2021), 453–477. https://doi.org/10.1002/cpa.21876
5. S. Biagi, G. Cupini, E. Mascolo, Regularity of quasi-minimizers for non-uniformly elliptic integrals, *J. Math. Anal. Appl.*, **485** (2020), 123838. https://doi.org/10.1016/j.jmaa.2019.123838
6. M. Bildhauer, M. Fuchs, $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems, *Calc. Var.*, **24** (2005), 309–340. https://doi.org/10.1007/s00526-005-0327-8
7. L. Boccardo, P. Marcellini, C. Sbordone, L^{∞}- regularity for variational problems with sharp nonstandard growth conditions, *Boll. Un. Mat. Ital. A (7)*, **4** (1990), 219–225.
8. V. Bögelein, F. Duzaar, P. Marcellini, C. Scheven, Boundary regularity for elliptic systems with $p, q-$growth, *J. Math. Pure. Appl.*, **159** (2022), 250–293. https://doi.org/10.1016/j.matpur.2021.12.004
9. P. Bousquet, L. Brasco, Lipschitz regularity for orthotropic functionals with nonstandard growth conditions, *Rev. Mat. Iberoam.*, **36** (2020), 1989–2032. https://doi.org/10.4171/RMI/1189
10. S. S. Byun, J. Oh, Global gradient estimates for non-uniformly elliptic equations, *Calc. Var.*, **56** (2017), 46. https://doi.org/10.1007/s00526-017-1148-2
11. S. S. Byun, J. Oh, Regularity results for generalized double phase functionals, *Anal. PDE*, **13** (2020), 1269–1300. https://doi.org/10.2140/apde.2020.13.1269
12. M. Carozza, H. Gao, R. Giova, F. Leonetti, A boundedness result for minimizers of some polyconvex integrals, *J. Optim. Theory Appl.*, **178** (2018), 699–725. https://doi.org/10.1007/s10957-018-1335-0
13. M. Caselli, M. Eleuteri, A. Passarelli di Napoli, Regularity results for a class of obstacle problems with $p, q-$growth conditions, *ESAIM: COCV*, **27** (2021), 19. https://doi.org/10.1051/cocv/2021017

14. A. Cianchi, Local boundedness of minimizers of anisotropic functionals, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **17** (2000), 147–168. https://doi.org/10.1016/S0294-1449(99)00107-9

15. A. Cianchi, V. G. Maz'ya, Global boundedness of the gradient for a class of nonlinear elliptic systems, *Arch. Rational Mech. Anal.*, **212** (2014), 129–177. https://doi.org/10.1007/s00205-013-0705-x

16. M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Rational Mech. Anal.*, **218** (2015), 219–273. https://doi.org/10.1007/s00205-015-0859-9

17. G. Cupini, F. Leonetti, E. Mascolo, Local boundedness for minimizers of some polyconvex integrals, *Arch. Rational Mech. Anal.*, **224** (2017), 269–289. https://doi.org/10.1007/s00205-017-1074-7

18. G. Cupini, P. Marcellini, E. Mascolo, Regularity under sharp anisotropic general growth conditions, *Discrete Contin. Dyn. Syst. B*, **11** (2009), 67–86. https://doi.org/10.3934/dcdsb.2009.11.67

19. G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to quasilinear elliptic systems, *Manuscripta Math.*, **137** (2012), 287–315. https://doi.org/10.1007/s00229-011-0464-7

20. G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to some anisotropic elliptic systems, In: *Recent trends in nonlinear partial differential equations. II. Stationary problems*, Providence, RI: Amer. Math. Soc., 2013, 169–186. http://doi.org/10.1090/conm/595/11803

21. G. Cupini, P. Marcellini, E. Mascolo, Existence and regularity for elliptic equations under $p, q-$growth, *Adv. Differential Equations*, **19** (2014), 693–724.

22. G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of minimizers with limit growth condition, *J. Optim. Theory Appl.*, **166** (2015), 1–22. https://doi.org/10.1007/s10957-015-0722-z

23. G. Cupini, P. Marcellini, E. Mascolo, Regularity of minimizers under limit growth conditions, *Nonlinear Anal.*, **153** (2017), 294–310. https://doi.org/10.1016/j.na.2016.06.002

24. G. Cupini, P. Marcellini, E. Mascolo, Nonuniformly elliptic energy integrals with $p, q-$growth, *Nonlinear Anal.*, **177, Part A** (2018), 312–324. https://doi.org/10.1016/j.na.2018.03.018

25. G. Cupini, P. Marcellini, E. Mascolo, A. Passarelli di Napoli, Lipschitz regularity for degenerate elliptic integrals with $p, q-$growth, *Adv. Calc. Var.*, in press. https://doi.org/10.1515/acv-2020-0120

26. C. De Filippis, G. Mingione, On the regularity of minima of non-autonomous functionals, *J. Geom. Anal.*, **30** (2020), 1584–1626. https://doi.org/10.1007/s12220-019-00225-z

27. C. De Filippis, G. Mingione, Lipschitz bounds and nonautonomous integrals, *Arch. Rational Mech. Anal.*, **242** (2021), 973–1057. https://doi.org/10.1007/s00205-021-01698-5

28. C. De Filippis, M. Piccinini, Borderline global regularity for nonuniformly elliptic systems, *Int. Math. Res. Notices*, in press. https://doi.org/10.1093/imrn/rnac283

29. E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.*, **3** (1957), 25–43.

30. M. De Rosa, A. G. Grimaldi, A local boundedness result for a class of obstacle problems with non-standard growth conditions, *J. Optim. Theory Appl.*, **195** (2022), 282–296. https://doi.org/10.1007/s10957-022-02084-1

31. E. Di Benedetto, U. Gianazza, V. Vespri, Remarks on local boundedness and local Hölder continuity of local weak solutions to anisotropic $p-$Laplacian type equations, *J. Elliptic Parabol. Equ.*, **2** (2016), 157–169. https://doi.org/10.1007/BF03377399

32. T. Di Marco, P. Marcellini, A-priori gradient bound for elliptic systems under either slow or fast growth conditions, *Calc. Var.*, **59** (2020), 120. https://doi.org/10.1007/s00526-020-01769-7

33. F. G. Düzgun, P. Marcellini, V. Vespri, An alternative approach to the Hölder continuity of solutions to some elliptic equations, *Nonlinear Anal. Theor.*, **94** (2014), 133–141. https://doi.org/10.1016/j.na.2013.08.018

34. M. Eleuteri, P. Marcellini, E. Mascolo, Local Lipschitz continuity of minimizers with mild assumptions on the x-dependence, *Discrete Contin. Dyn. Syst. S*, **12** (2019), 251–265. https://doi.org/10.3934/dcdss.2019018

35. M. Eleuteri, P. Marcellini, E. Mascolo, Regularity for scalar integrals without structure conditions, *Adv. Calc. Var.*, **13** (2020), 279–300. https://doi.org/10.1515/acv-2017-0037

36. M. Eleuteri, P. Marcellini, E. Mascolo, S. Perrotta, Local Lipschitz continuity for energy integrals with slow growth, *Annali di Matematica*, **201** (2022), 1005–1032. https://doi.org/10.1007/s10231-021-01147-w

37. N. Fusco, C. Sbordone, Local boundedness of minimizers in a limit case, *Manuscripta Math.*, **69** (1990), 19–25. https://doi.org/10.1007/BF02567909

38. N. Fusco, C. Sbordone, Some remarks on the regularity of minima of anisotropic integrals, *Commun. Part. Diff. Eq.*, **18** (1993), 153–167. https://doi.org/10.1080/03605309308820924

39. A. Gentile, Regularity for minimizers of a class of non-autonomous functionals with sub-quadratic growth, *Adv. Calc. Var.*, **15** (2022), 385–399. https://doi.org/10.1515/acv-2019-0092

40. M. Giaquinta, E. Giusti, Quasi-minima, *Ann. Inst. H. Poincaré C Analyse non-linéaire*, **1** (1984), 79–107. https://doi.org/10.1016/S0294-1449(16)30429-2

41. E. Giusti, *Direct methods in the calculus of variations*, River Edge, NJ: World Scientific Publishing Co. Inc., 2003. https://doi.org/10.1142/5002

42. T. Granucci, M. Randolfi, Local boundedness of Quasi-minimizers of fully anisotropic scalar variational problems, *Manuscripta Math.*, **160** (2019), 99–152. https://doi.org/10.1007/s00229-018-1055-7

43. J. Hirsch, M. Schäffner, Growth conditions and regularity, an optimal local boundedness result, *Commun. Contemp. Math.*, **23** (2021), 2050029. https://doi.org/10.1142/S0219199720500297

44. Ī. M. Kolodīĭ, The boundedness of generalized solutions of elliptic differential equations, *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, **25** (1970), 44–52.

45. O. Ladyzhenskaya, N. Ural'tseva, *Linear and quasilinear elliptic equations*, New York-London: Academic Press, 1968.

46. P. Marcellini, Regularity of minimizers of integrals in the calculus of variations with non standard growth conditions, *Arch. Rational Mech. Anal.*, **105** (1989), 267–284. https://doi.org/10.1007/BF00251503
47. P. Marcellini, Regularity and existence of solutions of elliptic equations with $p, q-$growth conditions, *J. Differ. Equations*, **90** (1991), 1–30. https://doi.org/10.1016/0022-0396(91)90158-6
48. P. Marcellini, Regularity for elliptic equations with general growth conditions, *J. Differ. Equations*, **105** (1993), 296–333. https://doi.org/10.1006/jdeq.1993.1091
49. P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, **23** (1996), 1–25.
50. P. Marcellini, Regularity for some scalar variational problems under general growth conditions, *J. Optim. Theory Appl.*, **90** (1996), 161–181. https://doi.org/10.1007/BF02192251
51. P. Marcellini, Regularity under general and $p, q-$growth conditions, *Discrete Contin. Dyn. Syst. S*, **13** (2020), 2009–2031. https://doi.org/10.3934/dcdss.2020155
52. P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, *J. Math. Anal. Appl.*, **501** (2021), 124408. https://doi.org/10.1016/j.jmaa.2020.124408
53. P. Marcellini, Local Lipschitz continuity for $p, q-$PDEs with explicit $u-$dependence, *Nonlinear Anal.*, **226** (2023), 113066. https://doi.org/10.1016/j.na.2022.113066
54. G. Mingione, Regularity of minima: an invitation to the dark side of the Calculus of Variations, *Appl. Math.*, **51** (2006), 355–426. https://doi.org/10.1007/s10778-006-0110-3
55. G. Mingione, G. Palatucci, Developments and perspectives in nonlinear potential theory, *Nonlinear Anal.*, **194** (2020), 111452. https://doi.org/10.1016/j.na.2019.02.006
56. G. Mingione, V. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, *J. Math. Anal. Appl.*, **501** (2021), 125197. https://doi.org/10.1016/j.jmaa.2021.125197
57. P. Pucci, R. Servadei, Regularity of weak solutions of homogeneous or inhomogeneous quasilinear elliptic equations, *Indiana Univ. Math. J.*, **57** (2008), 3329–3363. https://doi.org/10.1512/iumj.2008.57.3525
58. M. Schäffner, Higher integrability for variational integrals with non-standard growth, *Calc. Var.*, **60** (2021), 77. https://doi.org/10.1007/s00526-020-01907-1
59. B. Stroffolini, Global boundedness of solutions of anisotropic variational problems, *Boll. Un. Mat. Ital. A (7)*, **5** (1991), 345–352.
60. P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differ. Equations*, **51** (1984), 126–150. https://doi.org/10.1016/0022-0396(84)90105-0