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Periodic solutions in cognitive dynamics with inertia and training

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ABSTRACT

We analyze a class of nonlinear models of cognitive dynamics with inertia-like effects and periodic training. By combining a differential–algebraic formulation with topological degree methods, we prove the existence of periodic solutions whose period matches the external forcing.

1. Introduction

Processes of knowledge involve two distinct dynamics. The first is neural and commonly represented through evolution equations for the neuron firing. These equations can be introduced on empirical basis, as in the classical Hodgkin–Huxley model [1] and its variants (see, e.g., [2–7]). They can be also derived from first principles in terms of representation of interactions and related invariance requirements under changes of observers [8].

The physiological dynamics flows into an interpretative process that depends on the subject's cognitive abilities and cultural background. This results in an epistemic dynamics that can be represented formally in metric spaces.

In modeling this second dynamical process, inertia should play a role: it prevents ultra-fast cognitive response. In general, we cannot merely associate only the standard mechanical momentum with a time-dependent cognitive state $x(t)$, here chosen in $\mathbb{R}^n$, because there is no reason to consider only a 'kinetic energy' with the standard quadratic form for a cognitive process. In principle, we can only say that considering inertia in an epistemic process means referring to a generalized momentum $\phi(x(t), \dot{x}(t)) \in \mathbb{R}^n$, which extends to $\phi(x(t), \dot{x}(t), \lambda h(t))$, with h a C^1 T -periodic function and $\lambda \geq 0$, when training is accounted for (compare with [9,10]).

We also may consider a state-dependent and rate-dependent restoring force g , which represents the effects of cognitive bias. The external input f completes the scenario. For it, we refer to a class of model structures represented by

$$[\phi(x(t), \dot{x}(t), \lambda h(t))] = g(x(t), \dot{x}(t)) + \lambda f(t, x(t), \dot{x}(t), \lambda), \quad (1)$$

where $\lambda \geq 0$, $g : U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous on an open set $U \subset \mathbb{R}^n$, f is continuous and T -periodic in time, and $h : \mathbb{R} \rightarrow \mathbb{R}^s$ is C^1 and T -periodic. The map $\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^n$ is smooth and $\partial_2 \phi(x, y, z)$ is invertible for every triple (x, y, z) .

Our aim is to show that periodic training prevents cognitive inurement under periodic external actions. More precisely, we establish conditions under which Eq. (1) admits T -periodic solutions with period matching the forcing for $\lambda > 0$ (indeed, λ takes small positive values).

Although the case $\lambda = 0$ is of marginal interest for the periodicity problem considered, the branches of initial conditions corresponding to T -periodic solutions extend to zero.

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2. Periodic solutions for small perturbations

A pair $(\lambda, x) \in [0, \infty) \times C_T^1(U)$, such that x is a T -periodic solution of (1), is said to be a T -forced pair for (1). A T -forced pair (λ, x) is called *trivial* if x is constant and $\lambda = 0$. The following result for small positive values of λ provides a quantitative condition ensuring the existence of these solutions.

An initial step is to consider an associated differential–algebraic system (DAE). It is obtained by defining

$$z(t) := \dot{x}(t), \quad y(t) := \phi(x(t), \dot{x}(t)), \quad w(t) = \lambda h(t), \quad (2)$$

and, according to [11], reads as

$$\begin{cases} \dot{x}(t) = z(t), \\ \dot{y}(t) = g(x(t), z(t)) + \lambda f(t, x(t), z(t), \lambda), \\ \dot{w}(t) = -w + \lambda(h(t) + \dot{h}(t)), \\ y(t) - \phi(x(t), z(t), w(t)) = 0. \end{cases} \quad (3)$$

Let $p_0 \in U$ be an isolated zero of $g(\cdot, 0)$ and let U be an isolating neighborhood such that

$$\deg(g(\cdot, 0), U) \neq 0.$$

The existence theory for DAE systems [12, Thm. 5.1], combined with classical topological degree topics, yields the following continuation property.

Proposition 2.1 (DAE Continuation). *Under the above assumptions, there exists a connected branch Γ of nontrivial T -forced pairs whose closure meets $(0, \bar{p}_0)$, where $\bar{p}_0$ is the constant solution $\bar{p}_0 \equiv p_0$, and Γ is not contained in any compact set of $[0, \infty) \times C_T(U)$.*

To obtain solutions for $\lambda > 0$ close to zero, it remains to exclude the possibility that the corresponding branch of DAE-solutions lies entirely in $\{0\} \times C_T(\mathbb{R}^{3n+s})$.

We use the following classical estimate due to Yorke.

Theorem 2.2 ([13]). *Let ξ be a nonconstant τ -periodic solution of*

$$\dot{x} = F(x),$$

where $F : W \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$ is L -Lipschitz. Then,

$$\tau \geq \frac{2\pi}{L}.$$

Set

$$q_0 = \phi(p_0, 0, 0).$$

For $\lambda = 0$, the T -periodic DAE associated with (1) can be reduced locally near $(p_0, q_0, 0)$ to an ordinary differential equation

$$(\dot{\xi}, \dot{\eta}, \dot{\zeta}) = \mathcal{H}(\xi, \eta, \zeta). \quad (4)$$

Since $\partial_2 \phi(p_0, 0, 0)$ is nonsingular, the implicit function theorem allows one to solve locally the algebraic constraint for the velocity variable. The linearization of the reduced vector field at $(p_0, q_0, 0)$ is

$$\mathcal{H}'(p_0, q_0, 0) = \begin{pmatrix} (\partial_2 \phi)^{-1} \partial_1 \phi & (\partial_2 \phi)^{-1} & 0 \\ \partial_1 g + \partial_2 g (\partial_2 \phi)^{-1} \partial_1 \phi & \partial_2 g (\partial_2 \phi)^{-1} & 0 \\ 0 & 0 & -I \end{pmatrix},$$

where all derivatives are evaluated at $(p_0, 0, 0)$, and I denotes the identity on $\mathbb{R}^s$. Let

$$\sigma := \|\mathcal{H}'(p_0, q_0, 0)\|.$$

Lemma 2.3. *If*

$$T < \frac{2\pi}{\sigma},$$

the connected set of DAE-solutions associated with Γ is not contained in $\{0\} \times C_T(\mathbb{R}^{3n+s})$.

Proof. Choose $\varepsilon > 0$ such that

$$\sigma + \varepsilon < \frac{2\pi}{T}.$$

By continuity of the derivative $\mathcal{H}'$, the reduced vector field $\mathcal{H}$ is Lipschitz with constant smaller than $\sigma + \varepsilon$ in a sufficiently small neighborhood of $(p_0, q_0, 0)$.

By Theorem 2.2, any nonconstant periodic solution of (4) contained in that neighborhood has period

$$\tau \geq \frac{2\pi}{\sigma + \varepsilon} > T.$$

Hence no nonconstant T -periodic solution of (4) can remain in that neighborhood.

Assume by contradiction that the connected branch of DAE-solutions lies entirely in $\{0\} \times C_T(\mathbb{R}^{3n+s})$. Since the branch meets the trivial solution corresponding to $(p_0, q_0, 0)$, connectedness and local continuation would produce nonconstant T -periodic solutions of the reduced equation arbitrarily close to $(p_0, q_0, 0)$. This contradicts the previous estimate. ■

We can now state the local existence result.

Theorem 2.4. *Let p_0 be an isolated zero of $g(\cdot, 0)$ and let U be an isolating neighborhood such that*

$$\deg(g(\cdot, 0), U) \neq 0.$$

Let

$$q_0 = \phi(p_0, 0, 0), \quad \sigma = \|\mathcal{H}'(p_0, q_0, 0)\|.$$

If

$$T < \frac{2\pi}{\sigma},$$

then there exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0]$, Eq. (1) admits a T -periodic solution whose image is contained in U .

Proof. By Proposition 2.1, the nonzero degree assumption yields a connected branch Γ of nontrivial T -forced pairs whose closure meets $(0, p_0)$.

By Lemma 2.3, the corresponding DAE branch is not contained in the slice $\lambda = 0$. Hence Γ contains T -forced pairs with $\lambda > 0$ arbitrarily small.

Since the closure of Γ meets $(0, p_0)$ and U is isolating, standard continuation arguments imply that, for all sufficiently small $\lambda > 0$, the corresponding periodic solutions remain in U (see, e.g., [14]). ■

3. Concluding remarks

We have shown that, within the general model (1), periodic training counteracts cognitive inurement under periodic external actions. More precisely, under a nondegeneracy condition on the equilibrium set and a quantitative bound on the forcing period, one obtains the existence of periodic solutions whose period matches the forcing.

The result highlights the interplay between inertia-like effects and periodic inputs in cognitive dynamics, providing a simple analytical mechanism for the emergence of stable nontrivial responses.

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Data availability

No data was used for the research described in the article.

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