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LYAPUNOV STABILITY FOR MEASURE DIFFERENTIAL EQUATIONS

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Abstract. Recently, the concept of measure differential equation was introduced in [17]. Such a concept allows for deterministic modeling of uncertainty, finite-speed diffusion, concentration, and other phenomena. Moreover, it represents a natural generalization of ordinary differential equations to measures.

In this paper, we deal with the stability of fixed points for measure differential equations. In particular, we discuss two concepts related to classical Lyapunov stability in terms of measure support and first moment. The two concepts are not comparable, but the latter implies the former if the measure differential equation is defined by an ordinary one. Finally, we provide results concerning Lyapunov functions.

1. **Introduction.** The concept of measure differential equations (briefly MDEs), introduced in [17], allows for the modeling of various phenomena utilizing the time-evolution of Radon measures with finite mass over Euclidean spaces, or, more generally, topological manifolds. Moreover, this concept is a natural generalization of ordinary differential equations (briefly ODEs) in two ways: It conceptually extends the definition of vector field as section of the tangent bundle to measures over the same spaces, and it is possible to define an MDE associated to an ODE extending the set of solutions. General existence results for weak solutions are achieved using lattice approximate solutions (briefly LAS), which consist of Dirac masses centered at lattice points. However, uniqueness can be obtained only at the semigroup level using Dirac germs, which consist of small-time evolution of finite sums of Dirac masses. We refer the reader to [17] for details. Notice that a direct comparison with stochastic differential equations (SDEs) is not easy to provide. Indeed, the concept of solutions for SDEs is usually based on Brownian motion and Ito integrals, while

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the concept of solution to an MDE is based on the weak integral formulation. Intuitively, the MDE approach is more close to the deterministic representation of uncertanties as in differential inclusions [3].

MDE theory has been analyzed and generalized in various directions. A superposition principle was proved in [7]. An extension to differential inclusions was provided in [16]. A different direction was taken by the authors of [5] using multivalued maps between measures and vector fields, and using Lipschitz-type continuity as in [20]. A variational formulation was given in [10] for λ-dissipative multifunctions for measures. An extension to evolution equations with sources is found in [22], which allows for applications to crowd dynamics [21]. Applications of this approach to biology is discussed in [13]. We refer the reader to [19] for a general perspective on other applications.

More generally, MDEs are evolution equations for measures. The latter attracted a lot of attention in recent years. For instance, the general framework of gradient flow on metric spaces, see [1], provide tools and examples. Both MDEs and gradient flows are strictly related to optimal transport theory [24, 25]. For instance, the theory of MDEs is developed based on continuity and Lipschitz continuity property w.r.t. to the Wasserstein distance on the space of Radon measures and functionals defined in term of such distances.

Finally, let us point out the connections of measure evolutions with control theory. Starting from a classical control system $\dot{x} = f(x, u)$, $u \in U$, one can look for vector fields compatible with the associated multifunction $\dot{x} \in F(x) = \{f(x, u) : u \in U\}$ and measures on the state space, see [8]. Various results in this direction are available, including characterizing the value function of optimal control problems as the viscosity solution of the Hamilton-Jacobi-Bellmann equations [9, 14]. On the other hand, MDEs can be used to study classical control problems such as disturbance rejection [18] or provide generalizations to relaxed controls [2, 12].

In this paper, we introduce a stability theory for MDEs in the sense of Lyapunov. Existence and uniqueness theory for MDEs was developed using the tools provided by the theory of optimal transport and, in particular, on the Wasserstein distance, also known in the operations research community as Earth-mover's distance, see [15].

After recalling the basic definitions and properties of MDEs, we define two concepts of stability and asymptotical stability. Since the evolving object is a measure, one can define stability both in terms of the size of the support and the first moment of the measure. Interestingly, the two concepts are not comparable in the sense that stability in one sense does not imply stability in the other. This is proven by explicit counterexamples and occurs for asymptotic stability as well. To further clarify the picture, we focus on the meaning of convergence to a stable point for a measure. Convergence in integral sense is equivalent to weak convergence of measures, or convergence for the Wasserstein distance. The latter is implied by convergence of the support (in norm of the Hausdorff distance), but still the two concepts of stability remain independent.

Then, we focus on the special case of an MDE defined by an ODE. In such a case, stability of the first moment, called integral stability, implies support stability. The latter happens to be equivalent to the classical Lyapunov stability for the ODE. The same properties hold for asymptotic stability. The key aspect here is the additional regularity inherited by the MDE from the ODE. In particular, since every measure can be approximated by finite sums of Dirac masses for the Wasserstein distance, the proof relies directly on reducing to the ODE solution, or solutions to the corresponding linear transport equation.

Finally, we turn our attention to Lyapunov functions. We provide the definition of the Lyapunov function for the support stability, and prove the sufficiency of the existence of a Lyapunov function for stability. It is interesting to notice that the decrease of the Lyapunov function is necessary only on the part of the measure support maximizing the function. On the other hand, we have to restrict to MDEs which admit a Lipschitz semigroup obtained as a limit of LAS. A similar approach is developed for the case of first moment stability. However, we have to require the Lyapunov function to be estimated from above and below by the Euclidean norm. Moreover, we introduce a stronger concept of Lyapunv function, called the measure Lyapunov function. We prove that the existence of a measure Lyapunov function implies stability in both senses, and that it is equivalent to the existence of a classical Lyapunov function for the case of an MDE defined by an ODE.

2. **Basic definitions.** This section provides notation and basic definitions. Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean space. We indicate by $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ its tangent bundle and $\pi_1 : T\mathbb{R}^n \to \mathbb{R}^n$ the projection map to the base $\mathbb{R}^n$, i.e. $\pi_1(x, v) = x$. Moreover, $\pi_{13} : (T\mathbb{R}^n)^2 \to (\mathbb{R}^n)^2$ indicates the projection on the bases (first and third component), i.e. $\pi_{13}(x, v, y, w) = (x, y)$. Given a subset $A \subset \mathbb{R}^n$, χ_A denotes the characteristic function of A and $\mathcal{C}_c^\infty(A)$, the space of real smooth functions with compact support in A.

We denote by $\mathcal{P}(X)$ the set of probability measures, i.e. Radon measures with total mass equal to one, on the set X (X will usually be $\mathbb{R}^n$ or $T\mathbb{R}^n$). Given $\mu \in \mathcal{P}(X)$, $\text{Supp}(\mu)$ is the support of μ. The symbol $\mathcal{P}_c(X)$ denotes the set of probability measures with compact support. Similarly, $\mathcal{M}(X)$ denotes the set of positive Radon measures with finite mass and $\mathcal{M}_c(X) \subset \mathcal{M}(X)$ is the subset of measures with compact support. Given $\mu \in \mathcal{M}(X)$, we set $|\mu| = \mu(X)$ as its total mass.

If (X_1, d_1) and (X_2, d_2) are metric spaces, $\mu \in \mathcal{P}(X_1)$, and $\phi : X_1 \to X_2$ is a measurable map, then the push forward measure $\phi\#\mu$ is defined by

$$\phi\#\mu(A) = \mu(\phi^{-1}(A)) = \mu(\{x \in X_1 : \phi(x) \in A\}),$$

for every Borel set A. Given $\mu \in \mathcal{P}(X_1)$ and $\nu_x \in \mathcal{P}(X_2)$, $x \in X_1$, the measure $\mu \otimes \nu_x$ is defined by

$$\int_{X_1 \times X_2} \phi(x, v)\ d(\mu \otimes \nu_x) = \int_{X_1} \int_{X_2} \phi(x, v) d\nu_x(v)\ d\mu(x).$$

Given $\lambda \in \mathcal{P}(X_1 \times X_2)$, by disintegration we can always write $\lambda = \pi_1\#\lambda \otimes \nu_x$, where π_1 is the projection on X_1 and ν_x are probability measures on X_2.

Let (X, d) be a Polish space (metric, separable, and complete) and μ, $\nu \in \mathcal{P}(X)$. The optimal transport problem consists of minimizing the cost of moving the mass of μ to ν defined as follows: A transference plan τ between μ and ν is a probability measure on $X \times X$ with marginals equal to μ and ν, respectively, i.e., such that

$$\tau(A_1 \times \mathbb{R}^n) = \mu(A_1), \quad \tau(\mathbb{R}^n \times A_2) = \nu(A_2),$$

for all Borel sets A_1, $A_2 \subset X$. Let $P(\mu, \nu)$ be the set of transference plans τ from μ to ν, and for every $\tau \in P(\mu, \nu)$ define its transportation cost by

$$J(\tau) = \int_{X^2} d(x, y)\, d\tau(x, y).$$

The Monge-Kantorovich optimal transport problem consists of finding τ that minimizes $J(\tau)$ and the Wasserstein distance is given by

$$W^X(\mu,\nu) = \inf_{\tau \in P(\mu,\nu)} J(\tau).$$

If $X = \mathbb{R}^n$, we will usually drop the superscript to simplify notation. One can prove that an optimal transference plan exists under general conditions, see [24]. Let $P^{opt}(\mu,\nu)$ be the (nonempty) set of optimal transference plans, i.e. minimizing $J(\tau)$, and endow $\mathcal{P}(X)$ with the Wasserstein distance and the relative topology. We also recall the Kantorovich-Rubinstein duality:

$$W^X(\mu,\nu) = \sup \left\{ \int_X f \, d(\mu - \nu) \ : Lip(f) \leq 1 \right\}, \tag{1}$$

with $Lip(f)$ the Lipschitz constant of f. For more details on optimal transport and Wasserstein distance, we refer to [1, 24]. By normalization, the Wasserstein distance can be defined for $\mu, \nu \in \mathcal{M}(X)$ as long as $|\mu| = |\nu|$. Most results will be stated for probability measures, but can be generalized verbatim to measures with finite mass.

3. **Measure differential equations.** Following the approach of [17], we introduce the following concepts.

Definition 3.1. A measure vector field (MVF) on $\mathcal{M}(\mathbb{R}^n)$ is a map $V : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(T\mathbb{R}^n)$ such that $|V[\mu]| = |\mu|$ and $\pi_1 \# V = \mu$. Notice that we use the brackets for the argument of V to underscore the functional dependence on the measure μ.

A measure differential equation (MDE) is an evolution equation given by an MVF V, formally written as

$$\dot{\mu} = V[\mu]. \tag{2}$$

A solution to (2) is interpreted in the weak sense as follows.

Definition 3.2. A map $\mu : [0,T] \to \mathcal{M}(\mathbb{R}^n)$ is a solution to (2) if the following holds. For every $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, the integral $\int_{T\mathbb{R}^n} (\nabla f(x) \cdot v) \, dV(s)(x,v)$ is defined for a.e. s, the map $s \to \int_{T\mathbb{R}^n} (\nabla f(x) \cdot v) \, dV(s)(x,v)$ belongs to $L^1([0,T])$, the map $t \to \int f \, d\mu(t)$ is absolutely continuous, and for a.e. $t \in [0,T]$ it holds that

$$\frac{d}{dt} \int_{\mathbb{R}^n} f(x) \, d\mu(t)(x) = \int_{T\mathbb{R}^n} (\nabla f(x) \cdot v) \, dV(x,v). \tag{3}$$

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$, a Cauchy problem corresponding to (2) is given by

$$\dot{\mu} = V[\mu], \qquad \mu(0) = \mu_0. \tag{4}$$

From now on, for simplicity, we restrict to probability measures, but the theory holds for Radon measures with finite mass. Therefore, we assume $V : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(T\mathbb{R}^n)$. The theory developed in [17] includes results which are the MDE version of the classical Peano Theorem and Cauchy-Lipschitz Theorem for ODEs. More precisely, consider the following assumptions:

(H:bound) V is support sublinear, i.e. there exists $C > 0$ such that for every $\mu \in \mathcal{P}_c(X)$, the following holds: $\sup_{(x,v) \in \mathrm{Supp}(V[\mu])} |v| \leq C \left(1 + \sup_{x \in \mathrm{Supp}(\mu)} |x|\right)$.

(H:cont) Given $R > 0$, denote by $\mathcal{P}_c^R(\mathbb{R}^n)$ the set of probability measures with support contained in $B(0,R)$. For every $R > 0$, the map $V : \mathcal{P}_c^R(\mathbb{R}^n) \to \mathcal{P}_c(T\mathbb{R}^n)$ (restriction of V) is continuous (for the topology given by the Wasserstein distances $W^{\mathbb{R}^n}$ and $W^{T\mathbb{R}^n}$.)

To prove the existence of solutions to a Cauchy problem (4), we construct approximate solutions defined as finite sums of Dirac masses centered at the points of a regular lattice. For $N \in \mathbb{N}$, let $\Delta_N = \frac{1}{N}$ represent a discrete time step, $\Delta_N^v = \frac{1}{N}$ the velocity step, and $\Delta_N^x = \Delta_N^v \Delta_N = \frac{1}{N^2}$ the space step. Let x_i be the $(2N^3+1)^n$ equispaced discretization points of $\mathbb{Z}^n/(N^2) \cap [-N,N]^n$ and v_j to be the $(2N^2+1)^n$ equispaced discretization points of $\mathbb{Z}^n/N \cap [-N,N]^n$. Given $\mu \in \mathcal{P}_c(\mathbb{R}^n)$, define

$$\mathcal{A}_N^x(\mu) = \sum_i m_i^x(\mu)\delta_{x_i}, \quad m_i^x(\mu) = \mu(x_i + Q), \quad Q = ([0, 1/N^2[)^n, \tag{5}$$

and for $\mu \in \mathcal{P}_c(\mathbb{R}^n)$ with $\mathrm{Supp}(\mu) \subset \mathbb{Z}^n/(N^2) \cap [-N,N]^n$, set

$$\mathcal{A}_N^v(V[\mu]) = \sum_i \sum_j m_{ij}^v(V[\mu])\ \delta_{(x_i,v_j)} \tag{6}$$

where $m_{ij}^v(V[\mu]) = V[\mu](\{(x_i,v) : v \in v_j + Q'\})$ and $Q' = ([0, \frac{1}{N}[)^n$. The approximation operators $\mathcal{A}_N^x$ and $\mathcal{A}_N^v$ satisfy the following.

Lemma 3.3. *Given $\mu \in \mathcal{P}_c(\mathbb{R}^n)$, for N sufficiently big, it holds that*

$$W(\mathcal{A}_N^x(\mu), \mu) \leq \sqrt{n}\, \Delta_N^x, \qquad W^{T\mathbb{R}^n}(\mathcal{A}_N^v(V[\mu]), V[\mu]) \leq \sqrt{n}\, \Delta_N^v.$$

Using these discretization operators, we can define the approximate solutions.

Definition 3.4. Consider V satisfying (H:bound). Given the Cauchy problem (4), $T > 0$ and $N \in \mathbb{N}$ sufficiently big the lattice approximate solution (LAS) $\mu^N : [0,T] \to \mathcal{P}_c(\mathbb{R}^n)$ is defined as follows.

Initial step. Set $\mu_0^N = \mathcal{A}_N^x(\mu_0)$.

Recursive step. For $\ell \geq 1$, define

$$\mu_{\ell+1}^N = \sum_i \sum_j m_{ij}^v(V[\mu^N(\ell\Delta_N)])\ \delta_{x_i + \Delta_N v_j}. \tag{7}$$

For time-interpolation, we set

$$\mu^N(\ell\Delta_N + t) = \sum_{ij} m_{ij}^v(V[\mu^N(\ell\Delta_N)])\ \delta_{x_i + t\, v_j}. \tag{8}$$

Notice that, by the definition of Δ_N, Δ_N^v, Δ_N^x, and (7), $\mathrm{Supp}(\mu_\ell^N)$ is contained in the set $\mathbb{Z}^n/(N^2) \cap [-N,N]^n$, thus $\mu_\ell^N = \sum_i m_i^{N,\ell}\delta_{x_i}$ for some $m_i^{N,\ell} \geq 0$.

The recursive step is based on first approximating $V[\mu_\ell^N]$ by $\mathcal{A}_N^v(V[\mu_\ell^N])$, and then using the discretized velocities to move the Dirac deltas of μ_ℓ^N. For fixed $T > 0$, it is easy to achieve uniform bounds for the support of LAS (see [17]).

The equivalent of Peano theorem is given by the following.

Theorem 3.5. *Consider an MVF V satisfying (H:bound) and (H:cont). Then, for $T > 0$ and $\mu_0 \in \mathcal{P}_c(\mathbb{R}^n)$ there exists a solution $\mu : [0,T] \to \mathcal{P}_c(\mathbb{R}^n)$ to (4) obtained as the limit of LASs (for $W^{\mathbb{R}^n}$). If $\mathrm{Supp}(\mu_0) \subset B(0,R)$, then*

$$W(\mu(t), \mu(s)) \leq C\, e^{CT}\, (R+1)\ |t-s|. \tag{9}$$

To prove the equivalent of the Cauchy-Lipschitz Theorem, we need some additional notation.

Definition 3.6. Given $V_i \in \mathcal{P}_c(T\mathbb{R}^n)$, $i = 1,2$, set $\mu_i = \pi_1 \# V_i$, and $\mathcal{T}(V_1, V_2) = \{T \in P(V_1, V_2) : \ \pi_{13}\#T \in P^{opt}(\mu_1, \mu_2)\}$. We define

$$\mathcal{W}(V_1, V_2) = \inf \left\{ \int_{(T\mathbb{R}^n)^2} |v - w| \ dT(x, v, y, w) \ : T \in \mathcal{T}(V_1, V_2) \right\}. \quad (10)$$

The functional $\mathcal{W}$ is not a metric since it can vanish for distinct elements of $\mathcal{P}(T\mathbb{R}^N)$. Even adding the term $|x - y|$ would not give a metric, because the triangular inequality fails (see [17].) Our last assumption is the following:

(H:lip) For every $R > 0$, there exists $K = K(R) > 0$ such that if $\mathrm{Supp}(\mu), \mathrm{Supp}(\nu) \subset B(0, R)$, then

$$\mathcal{W}(V[\mu], V[\nu]) \le K \ W(\mu, \nu). \quad (11)$$

We define a Lipschitz semigroup of solutions as follows.

Definition 3.7. For an MVF V satisfying (H:bound) and $T > 0$, a Lipschitz semigroup for (2) is a map $S : [0, T] \times \mathcal{P}_c(\mathbb{R}^n) \to \mathcal{P}_c(\mathbb{R}^n)$ such that for every $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^n)$ and $t, s \in [0, T]$, the following holds:
i) $S_0\mu = \mu$ and $S_t\, S_s\, \mu = S_{t+s}\, \mu$;
ii) the map $t \mapsto S_t\mu$ is a solution to (2);
iii) for every $R > 0$, there exists $C(R) > 0$ such that if $\mathrm{Supp}(\mu), \mathrm{Supp}(\nu) \subset B(0, R)$, then

$$\mathrm{Supp}(S_t\mu) \subset B(0, e^{Ct}(R + 1)), \quad (12)$$

$$W(S_t\mu, S_t\nu) \le e^{C(R)t} W(\mu, \nu), \quad (13)$$

$$W(S_t\mu, S_s\mu) \le C(R)\ |t - s|. \quad (14)$$

Finally, we have the following theorem.

Theorem 3.8. *Given V satisfying (H:bound) and (H:lip), and $T > 0$, there exists a Lipschitz semigroup of solutions to (2) whose trajectories are a limit of LASs for the Wasserstein distance.*

3.1. MDEs defined by ordinary differential equations.

Here we report some results from [17] providing natural connections between ordinary differential equations (ODEs) and MDEs. Let us start by defining the following.

Definition 3.9. Consider $v : \mathbb{R}^n \to \mathbb{R}^n$ and the corresponding ODE $\dot{x} = v(x)$. The MVF V^v, corresponding to v, is defined by $V^v[\mu] = \mu \otimes \delta_{v(x)}$.

Given the usual properties of v, such as sublinear growth, continuity, and Lipschitz continuity, we are interested in understanding if V^v satisfies (H:bound), (H:cont), and (H:lip). (H:bound) holds if v is sublinear: there exists $C > 0$ such that $|v(x)| \le C(1 + |x|)$. Similarly, (H:cont) holds if v is continuous. Finally, we have the following theorem.

Theorem 3.10 ([17])**.** *V^v satisfies (H:lip) for finite sums of Dirac deltas if and only if v is locally Lipschitz continuous.*

Moreover, if v satisfies sublinear growth, then V^v satisfies (H:lip) and there exists a unique Lipschitz semigroup obtained as the limit of LASs.

It is interesting to notice that the relationship between ODEs and MDEs extends to linear transport equations. More precisely, we have the following proposition.

Proposition 3.11 ([17]). *If v is locally Lipschitz continuous with sublinear growth, then the solution to the Cauchy problem $\dot{\mu} = V^v[\mu]$, $\mu(t) = \mu_0 \in \mathcal{P}_c(\mathbb{R}^n)$, is the unique solution to the linear transport equation*

$$\mu_t + \nabla \cdot (v\,\mu) = 0, \quad \mu(0) = \mu_0.$$

3.2. **Further dynamics modeled using MDEs.** The aim of this section is to provide further examples of MDEs. We start with ODEs with one-sided Lipschitz conditions and singular PDEs producing concentration and delta waves. Then, we pass to illustrate examples producing mass splitting and finite-speed diffusion, which are not captured by regular PDEs. Finally, we show an ODE-PDE model for supply chains.

Example 3.12. Consider an ODE $\dot{x} = v(x)$ with v satisfying

$$\langle v(x) - v(y), x - y \rangle \leq L\,|x - y|^2, \tag{15}$$

where $\langle \cdot, \cdot \rangle$ indicates the scalar product of $\mathbb{R}^n$, and L is bounded on compact sets. We can associate an MDE as in Section 3.1. Existence and uniqueness theory results are available in such case [4]. This implies that the corresponding MDE has existence properties. Moreover, we have the following proposition.

Proposition 3.13. *Consider a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ with sublinear growth satisfying (15) and let $V[\mu] = \mu \otimes_x \delta_{v(x)}$ be the associated MVF. Then, there exist unique limits of LAS and a Lipschitz semigroup of solutions obtained as the limit of LAS.*

We omit the proof that can be obtained with a proof similar to Theorem 3.10. A general theory including multivalued MVF was developed in [10] using a modification of (H:Lip). Mass concentration can be easily modeled by such MDEs. For instance, if we set $v(x) = \pm 1$ if $\pm x < 0$ and $v(0) = 0$, then any measure with compact support will concentrate the mass at 0 in finite time. Similarly, one can consider the associated semilinear PDE: $u_t + v(t,x) \cdot u_x = 0$, see also [23]. Solutions concentrating masses are obatined as the limit of LAS for the corresponding MDEs.

Example 3.14. We define an MDE splitting mass at the barycenter. For simplicity, consider probability measures μ on $\mathbb{R}$ and define the barycenter as $B(\mu) = \sup\left\{x : \mu(]-\infty, x]) \leq \frac{1}{2}\right\}$.

Set $\eta = \mu(]-\infty, B(\mu)]) - \frac{1}{2}$ so $\mu(\{B(\mu)\}) = \eta + \frac{1}{2} - \mu(]-\infty, B(\mu)[)$. We define $V[\mu] = \mu \otimes \nu_x$, with $\nu_x = \delta_{-1}$ if $x < B(\mu)$, $\nu_x = \delta_1$ if $x > B(\mu)$, and if $\mu(\{B(\mu)\}) > 0$,

$$\nu_{B(\mu)} = \frac{1}{\mu(\{B(\mu)\})} \left(\eta \delta_1 + \left(\frac{1}{2} - \mu(]-\infty, B(\mu)[) \right) \delta_{-1} \right), \tag{16}$$

otherwise $\nu_{B(\mu)} = 0$. The solution to (4) with $\mu_0 = \delta_{x_0}$ is given by $\mu(t) = \frac{1}{2}\delta_{x_0+t} + \frac{1}{2}\delta_{x_0-t}$. Thus, the mass is split in half, traveling in opposite directions.

A generalization of the finite speed diffusion of Example 3.14 is as follows.

Example 3.15. We construct examples of mass diffusion at finite speed. For simplicity, we focus on probability measures. Given a probability measure μ on $\mathbb{R}$ let $F_\mu(x) = \mu(]-\infty, x])$ be its cumulative distribution and let λ indicate the Lebesgue measure. Consider an increasing map $\varphi : [0,1] \to \mathbb{R}$ and define $V_\varphi[\mu] = \mu \otimes_x J_\varphi(x)$, where

$$J_\varphi(x) = \begin{cases} \delta_{\varphi(F_\mu(x))} & \text{if } F_\mu(x^-) = F_\mu(x), \\ \frac{\varphi\#\left(\chi_{[F_\mu(x^-),F_\mu(x)]}\lambda\right)}{F_\mu(x)-F_\mu(x^-)} & \text{otherwise.} \end{cases}$$

Roughly speaking, we order the mass of μ left to right, and then we move at speed $\varphi(a)$ the mass located at position $a \in [0,1]$ according to such order. If $\mu(0) = \delta_0$, then the solution to the Cauchy problem for the corresponding MDE is given by $g(t,x)\lambda$ with

$$g(t,x) = \frac{1}{t\varphi'(\varphi^{-1}(\frac{x}{t}))} = \frac{(\varphi^{-1})'(\frac{x}{t})}{t}.$$

For example, if $\varphi(\alpha) = \alpha - \frac{1}{2}$, then $g(t,x) = \frac{1}{t}\chi_{[-\frac{t}{2},\frac{t}{2}]}$ so we get uniformly distributed mass.

Example 3.16. Supply chains can be modeled by systems of ODE-PDEs as follows:

$$\begin{gathered} \rho^j_t + (\min\{v_j\rho^j, c_j\})_x = 0,\ x \in [a_j, b_j], \\ \dot{q}_j = \min\{v_{j-1}\rho^{j-1}(b_{j-i}-), c_{j-1}\} - f_{j-1}(t), \quad t \geq 0 \\ f_{j-1}(t) = \left\{ \begin{array}{cc} \min\{\min v_{j-1}\rho^{j-1}(b_{j-i}-), c_{j-1}, c_j\} & q_j(t) = 0 \\ c_j & q_j(t) > 0 \end{array} \right. \end{gathered} \tag{17}$$

where ρ^j is the density (of goods) on the j-th processor represented by the interval $[a_j, b_j]$, with $b_{j-1} = a_j$, v_j the processing speed, c_j the maximal capacity, and q_j the buffer queue in front of processor j. In simple words, the queue fills up when the inflow from processor $j-1$ is higher than the outflow to processor j, see [11]. It is easy to prove that if $\rho_j(0) \leq \frac{c_j}{v_j}$, then the same holds for all times, and thus ρ_j solves the semilinear equation $\rho^j_t + v_j\rho^j_x = 0$. We will consider the linear dynamics on processors coupled with the queue dynamics.

To describe such dynamics as an MDE, we follow the ideas of Examples 3.14 and 3.15. Given a measure μ on $\mathbb{R}$ with finite mass, let F_μ be its cumulative distribution and set $\alpha_j = F_\mu(a_j)$, $\beta_j = F_\mu(b_j-)$, and $\gamma_j = \mu(a_j)$. The ordered masses between α_j and β_j move with speed v_j.

If $\mu(a_j) > 0$, then the ordered mass between β_j and $\beta_j + \gamma_j$ move at speed $\frac{c_j}{\gamma_j}$.

4. **Stability for ordinary differential equations.**

Here we recall the basics of Lyapunov stability of ordinary differential equations. Consider the ordinary differential equation

$$\dot{x} = g(x), \tag{18}$$

where $x \in \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and indicate by $t \mapsto x(t, x_0)$ the solution such that $x(0) = x_0$.

Definition 4.1. (Lyapunov stability). Assume $g(\bar{x}) = 0$. Then, $\bar{x}$ is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $|x_0 - \bar{x}| < \delta$, then for every $t \geq 0$ we have $|x(t, x_0) - \bar{x}| < \epsilon$.
The point $\bar{x}$ is asymptotically stable if, in addition, we can choose $\delta > 0$ so that $\lim_{t\to+\infty} x(t, x_0) = \bar{x}$ whenever $|x_0 - \bar{x}| < \delta$.

The stability condition means that, if the initial datum is sufficiently close to $\bar{x}$, then the solution remains close to $\bar{x}$ for all $t > 0$. Asymptotic stability means that $\left|x(t, x_0) - \bar{x}\right|$ converges to zero as $t \to +\infty$.

Lyapunov stability can be proven by constructing a positive function which is decreasing along trajectories of the system.

Definition 4.2. Consider a $\mathcal{C}^1$ vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{x}$ with $g(\bar{x}) = 0$. A $\mathcal{C}^1$ function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is a Lyapunov function for (18) if:
i) V is positive definite, i.e. $V(\bar{x}) = 0$ and $V(x) > 0$ for every $x \neq \bar{x}$;
ii) V decreases along trajectories of (18), i.e. for all x, $\nabla V(x) \cdot g(x) \leq 0$.
We say that V is a strict Lyapunov function, if it is a Lyapunov function and $\nabla V(x) \cdot g(x) < 0$ for every $x \neq \bar{x}$.

We have the following:

Theorem 4.3. *Let g be a $\mathcal{C}^1$ vector field, vanishing at a point $\bar{x} \in \mathbb{R}^n$.*

(i) The equilibrium point $\bar{x}$ is stable if the system (18) admits a Lyapunov function defined on a neighborhood of $\bar{x}$.
(ii) The equilibrium point $\bar{x}$ is asymptotically stable if system (18) admits a strict Lyapunov function defined on a neighborhood of $\bar{x}$.

For more details, we refer the reader to [6].

5. **Stability for measure differential equations.** As shown in Section 4, Lyapunov (asymptotic) stability for ODEs consists of having bounds (asymptotically vanishing) on the norm of solutions depending on the norm of the initial condition. Since solutions to MDEs are measures, there are at least two natural concepts obtained replacing the norm with the size of the support and the first moment of the measure.

From now on, for simplicity we assume that the equilibrium point is the origin ($\bar{x} = 0$ in the notation of Section 4.) The first natural definition of stability for the origin for an MDE is as follows.

Definition 5.1. Consider an MVF V. We say that the origin is support stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(\cdot)$ is a solution to the corresponding MDE with $\mathrm{Supp}(\mu(0)) \subset B(0, \delta)$, then $\mathrm{Supp}(\mu(t)) \subset B(0, \epsilon)$ for every $t \geq 0$.

Similarly, we can define the concept of asymptotic stability.

Definition 5.2. Consider an MVF V for which the origin is support stable. We say that the origin is asymptotically support stable if there exists $\delta > 0$ such that if $\mu(\cdot)$ is a solution to the corresponding MDE with $\mathrm{Supp}(\mu(0)) \subset B(0, \delta)$, then $\sup\{|x| : x \in \mathrm{Supp}(\mu(t))\} \to 0$ as $t \to +\infty$.

The second natural concept is based on the first moment and is stated as follows.

Definition 5.3. Consider an MVF V. We say that the origin is integrally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mu(\cdot)$ is a solution to the corresponding MDE with $\int |x|\, d\mu(0)(x) < \delta$, then $\int |x|\, d\mu(t)(x) < \epsilon$ for every $t \geq 0$.

The concept of asymptotic integral stability is stated as follows.

Definition 5.4. Consider an MVF V for which the origin is integrally stable. We say that the origin is asymptotically integrally stable if there exists $\delta > 0$ such that if $\mu(\cdot)$ is a solution to the corresponding MDE with $\int |x|\, d\mu(0)(x) < \delta$, then $\int |x|\, d\mu(t)(x) \to 0$ as $t \to +\infty$.

5.1. **Comparison of stability concepts for MDEs.** The two concepts of stability for MDEs are not comparable.

Proposition 5.5. *The support stability for an MVF does not imply the integral stability.*

Proof. To see this, first consider an ODE on $\mathbb{R}^2$

$$\dot{x} = f(x) \tag{19}$$

and choose f so that the origin is stable, the unit circle is an unstable periodic orbit, and there are no other equilibria or periodic orbit.

A function satisfying the above properties is

$$f(x,y) = \frac{(-(x(1-\rho^2) - 2y\rho), (-(y(1-\rho^2) + 2x\rho))}{1+\rho^2}$$

with $\rho = \sqrt{x^2+y^2}$. Solutions to (19) can be explicitly computed in polar coordinates and satisfy

$$\rho(t) = e^{\text{arcsinh}(e^t \sinh(\log(\rho_0))}$$

with $\rho_0 = \rho(0)$ the initial radius. The integral curves of f are depicted in Figure 1.

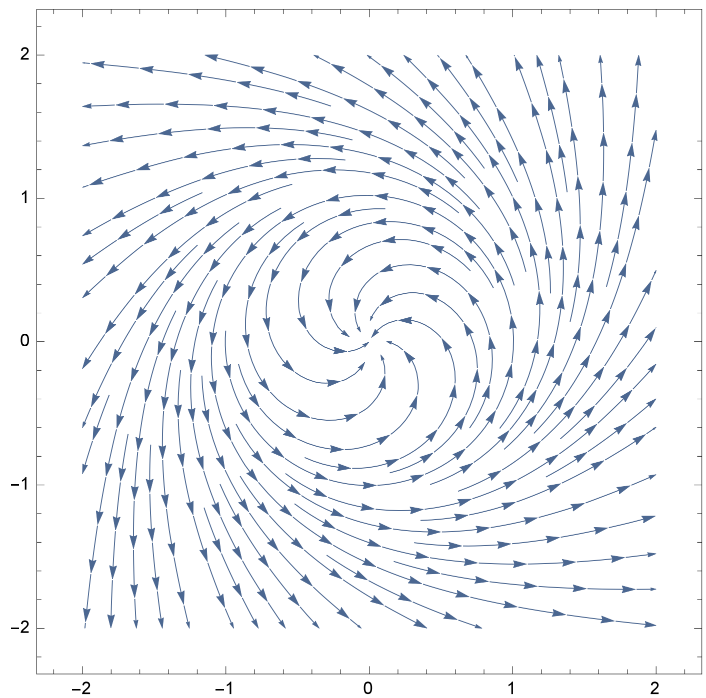

FIGURE 1. Integral curves of f.

Now, define V_f as the corresponding measure vector field. Then, the origin is support stable for V_f. To see this, first let W be a Lyapunov function for f defined on a neighborhood of the origin. For every ϵ, define $W_\epsilon = \inf\{W(x) : x \in \partial B(0,\epsilon)\} > 0$ and set $\delta = \sup\{\eta : B(0,\eta) \subset V^{-1}([0,W_\epsilon])\}$. If $\mu(\cdot)$ is a solution to the MDE corresponding to V_f with $\text{Supp}(\mu(0)) \subset B(0,\eta)$, then $W(x) \leq W_\epsilon$ for every $x \in \text{Supp}(\mu(0))$. This implies $W(x(t,x_0)) \leq W_\epsilon$ for every $x_0 \in \text{Supp}(\mu(0))$, thus $\text{Supp}(\mu(t)) \subset B(0,\epsilon)$.

On the other hand, fix $\epsilon = 1$. Then, for every $\delta > 0$, there exists μ_0 with $\int |x|\, d\mu_0(x) < \delta$ such that $\mu(\mathbb{R}^2 \setminus B(0,1)) > \frac{\delta}{2} > 0$. All trajectories starting outside

$B(0,1)$ diverge to infinity, and in particular, indicating by $\mu(t)$ the solution to the MDE starting from μ_0, we have $\int |x|\, d\mu(t)(x) \to \infty$. Therefore, the origin is not integrally stable. □

Proposition 5.6. *The integral stability for an MVF does not imply the support stability.*

Proof. Consider now an MVF on $\mathbb{R}$ defined as follows. Given μ with compact support, let $x_- = \inf\{x : x \in \text{Supp}(\mu)\}$, $x_+ = \sup\{x : x \in \text{Supp}(\mu)\}$, and $d = x_+ - x_-$. Consider $d \in [0,1]$ and two functions $D(d)$ and $\eta(d)$ with $D(d) > d$, $1 > \eta(d) > \frac{d}{2}$, and such that $D(d) \to +\infty$ and $\eta(d)\, D(d) \to 0$ as $d \to 0$. Given μ, we can order the mass left to right considering the cumulative distribution function and let the first $1 - \eta(d)$ mass travel towards 0 and the remaining towards $D(d)$. Then, the origin is integrally stable. Indeed fix $\epsilon > 0$, then for every μ_0 with $\int |x|\, d\mu_0(x) < \delta$ we can estimate $\int |x|\, d\mu(t)(x) < \eta(\delta)D(\delta) + \delta(1 - \eta(\delta))$ which is bounded by ϵ for δ sufficiently small. On the other hand, for every δ there exists μ_0 with $\text{Supp}(\mu_0) \subset B(0, \delta)$ such that some mass travel to $D(\delta)$. Since $D(\delta) \to +\infty$ as $\delta \to 0$, we conclude that the origin is not support stable. □

5.2. **Convergence of solutions and asymptotic stability.** The two concepts of asymptotic stability for MDEs are based on the convergence of the support or of the first moment of the solution. Here, we consider other concepts of convergence and compare them. This, in turn, will clarify the relationships between different concepts of asymptotic stability and possible variations.

We first compare different concepts of convergence for a curve with uniformly compact support. To do so, let us recall the definition of Hausdorff distance for closed sets. Given $C_1, C_2 \subset \mathbb{R}^n$ closed, we set

$$d(x, C_1) = \min_{y \in C_1} d(x, y), \; d_H(C_1, C_2) = \max\{\sup_{x_1 \in C_1} d(x_1, C_2), \sup_{x_2 \in C_2} d(x_2, C_1)\}.$$

We have the following.

Proposition 5.7. *Let $\mu : [0, +\infty[\to \mathcal{P}(\mathbb{R}^n)$ be a continuous curve (for the W distance), and assume there exists $r > 0$ such that Supp$(\mu(t)) \subset B(0, r)$ for every $t > 0$. Consider the following concepts of convergence for $t \to +\infty$:*

- *(a)* $\int |x|\, d\mu(t) \to 0$.
- *(b)* $W(\mu(t), \delta_0) \to 0$.
- *(c)* $\mu(t) \rightharpoonup 0$.
- *(d)* $d_H(\text{Supp}(\mu(t)), \{0\}) \to 0$.
- *(e)* $\sup\{|x| : x \in \text{Supp}(\mu(t))\} \to 0$.

Conditions (a), (b), and (c) are equivalent. Conditions (d) and (e) are equivalent. Condition (d) implies (b), but not vice versa.

Proof. (a) $\Longleftrightarrow$ (b). We have $W(\mu(t), \delta_0) = \int |x|\, d\mu(t)$.
(b) $\Longleftrightarrow$ (c). W metrizes the weak convergence for compact supported measures [24].
(d) $\Longleftrightarrow$ (e). We have $d_H(\text{Supp}(\mu(t)), \{0\}) = \sup\{|x| : x \in \text{Supp}(\mu(t))\}$.
(d) $\Longrightarrow$ (b). We have $W(\mu(t), \delta_0) \le d_H(\text{Supp}(\mu(t)), \{0\})$.
Finally, the curve $t \to \frac{t}{t+1}\delta_0 + \frac{1}{t+1}\delta_{re_1}$, where e_1 is the first coordinate vector, satisfies (b) but not (d). □

The meaning of Proposition 5.7 is that, generally speaking, the convergence to zero of the support for the Hausdorff distance is equivalent to the support size going to zero, and is strictly stronger than the convergence to zero of the first moment. Moreover, the latter is equivalent to weak convergence and Wasserstein convergence. Despite this, we cannot compare the two concepts of asymptotic stability as shown by the following proposition.

Proposition 5.8. *Consider an MVF V and assume that the origin is asymptotically support stable. Then, the origin is not necessarily asymptotically integrally stable.*

Vice versa, if the origin is asymptotically integrally stable, then the origin is not necessarily asymptotically support stable.

The proof of the proposition follows from the counterexamples in the proofs of Proposition 5.5 and Proposition 5.6. Indeed, the examples given in the proofs are asymptoticaly stable for one concept but not stable for the other.

5.3. **Stability for an MDE defined by an ODE.** Here we focus on the stability of MDEs corresponding to ODEs as defined in Section 3.1. Our first result shows that, even if the two concepts of stability for MDEs are not comparable, one is stronger than the other for the special case of an MDE defined by an ODE. More precisely, we have the following.

Proposition 5.9. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. If the origin is integrally stable, then it is support stable.*

Proof. Assume by contradiction that the origin is integrally stable, but not support stable. Thus, there exist $\epsilon > 0$ and sequences $\delta_\nu \to 0$, $\mu_\nu(\cdot)$ solution to the MDE, $t_\nu > 0$, such that $\text{Supp}(\mu_\nu(0)) \subset B(0, \delta_\nu)$ and $\text{Supp}(\mu_\nu(t_\nu)) \not\subset B(0, \epsilon)$. Choose $x_\nu \in \text{Supp}(\mu_\nu(t_\nu))$ with $|x_\nu| > \epsilon$. Then, there exists $y_\nu \in \text{Supp}(\mu_v(0)) \subset B(0, \delta_\nu)$ such that $x(t_\nu, y_\nu) = x_\nu$. Consider now the sequence of solutions $\lambda_\nu(\cdot)$ to the MDE with $\lambda_\nu(0) = \delta_{y_\nu}$. Then, $\int |x| d\lambda_\nu(0) = |y_\nu| \to 0$ and $\int |x| d\lambda_\nu(t_\nu) = |y(t_\nu)| = |x_\nu| > \epsilon$ reaching a contradiction with the assumption of integral stability. □

Our second result shows that classical stability of the ODE is equivalent to support stability.

Proposition 5.10. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. The origin is support stable for V^v if and only if it is Lyapunov stable for v.*

Proof. First, assume the origin is support stable for V^v. Fix ϵ and let δ be as in Definition 5.1. Consider $x_0 \in B(0, \delta)$, then the solution to the MDE with initial condition δ_{x_0} satisfies $\text{Supp}(\mu(t)) \subset B(0, \epsilon)$. In turn, $\mu(t) = \delta_{x(t,x_0)}$ by definition of V^v. Therefore the origin is Lyapunov stable for v.

For the converse, assume the origin is Lyapunov stable for v. Fix ϵ and let δ be as in Definition 4.1. Consider μ_0 with $\text{Supp}(\mu(0)) \subset B(0, \delta)$. Then, for every $\eta > 0$ there exists $m > 0$ and $x_1, \ldots, x_m \in B(0, \delta)$ such that $W(\mu_0, \frac{1}{m} \sum_i \delta_{x_i}) < \eta$. By Theorem 3.10, the solution to the MDE with initial datum $\frac{1}{m} \sum_i \delta_{x_i}$ is unique and satisfies $\mu(t) = \frac{1}{m} \sum_i \delta_{x(t,x_i)}$.

By assumption, we have $\text{Supp}(\frac{1}{m} \sum_i \delta_{x(t,x_i)}) \subset B(0, \epsilon)$. Again, by Theorem 3.10 and the arbitrariness of η, we conclude that $\text{Supp}(\mu(t)) \subset B(0, \epsilon)$, and thus we are done. □

Let us now pass to consider the concepts of asymptotic stability. We have the following proposition.

Proposition 5.11. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. If the origin is asymptotically integrally stable, then it is asymptotically support stable.*

Proof. From Proposition 5.9 we know that the origin is support stable. Assume by contradiction that the origin is not asymptotically support stable. Thus, there exist sequences $\delta_\nu \to 0$, $\mu_\nu(\cdot)$ solution to the MDE, such that $\mathrm{Supp}(\mu_\nu(0)) \subset B(0, \delta_\nu)$ and $\sup\{|x| : x \in \mathrm{Supp}(\mu_\nu(t))\} \not\to 0$ as $t \to \infty$. Thus, for every ν, there exist $\eta_\nu > 0$ such that $\limsup_t \sup\{|x| : x \in \mathrm{Supp}(\mu_\nu(t))\} \geq \eta_\nu$. We claim that there exists $y^\nu \in \mathrm{Supp}(\mu_\nu(0))$ such that $\limsup_t |x(t, y^\nu)| \geq \frac{1}{2}\eta_\nu$. Otherwise, for every $y \in \mathrm{Supp}(\mu_\nu(0))$, there exists t^y such that $|x(t, y)| \leq \frac{1}{2}\eta_\nu$ for $t > t^y$. By the Lipschitz continuity of v, we have $|x(t, z)| \leq \frac{3}{4}\eta_\nu$ for $t > t^y$ and all z in a neighborhood of y. By compactness, we can take t^y uniformly bounded on $\mathrm{Supp}(\mu_\nu(0))$, reaching a contradiction. Therefore, the trajectory λ^ν of the MDE with initial datum $\lambda_\nu(0) = \delta_{y^\nu}$ satisfies $\int |x| d\lambda_\nu(t) \not\to 0$ as $t \to \infty$. On the other hand, $\int |x| d\lambda_\nu(0) = |y^\nu| \to 0$, and thus we reach a contradiction. □

We also have the following.

Proposition 5.12. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. The origin is asymptotically support stable for V^v if and only if it is asymptotically stable for v.*

Proof. First, assume the origin is asymptotically support stable for V^v. From Proposition 5.10, we know that the origin is Lyapunov stable for v. Fix ϵ and let δ be as in Definition 5.1. Consider $x_0 \in B(0, \delta)$. Then, the solution to the MDE with initial condition δ_{x_0} satisfies $sup\{|x| : x \in \mathrm{Supp}(\mu(t))\} \to 0$. In turn, $\mu(t) = \delta_{x(t)}$, where $x(\cdot)$ is the solution to the ODE with $x(0) = x_0$, by definition of V^v. Therefore, the origin is asymptotically stable for v.

For the converse, assume the origin is asymptotically stable for v. Fix ϵ and let δ be as in Definition 4.1. Consider μ_0 with $\mathrm{Supp}(\mu(0)) \subset B(0, \delta)$. Then, for every $\eta > 0$, there exist $m > 0$ and $x_1, \ldots, x_m \in B(0, \delta)$ such that $W(\mu_0, \frac{1}{m}\sum_i \delta_{x_i}) < \eta$. By Theorem 3.10, the solution to the MDE with initial datum $\frac{1}{m}\sum_i \delta_{x_i}$ is unique and satisfies $\mu(t) = \frac{1}{m}\sum_i \delta_{x(t,x_i)}$. By assumption, we have $sup\{|x| : x \in \mathrm{Supp}(\frac{1}{m}\sum_i \delta_{x(t,x_i)})\} \to 0$. Again, by Theorem 3.10 and the arbitrariness of η, we conclude. □

6. **Lyapunov functions.** In this section, we introduce the definition of the Lyapunov function for an MVF related to the two concepts of stability. We focus on the novelty of the stability concepts, and thus consider smooth Lyapunov functions. Moreover, we need to restrict the MVF so that the corresponding MDE admits a Lipschitz semigroup.

We prove sufficiency results for Lyapunov functions being the necessary results much more difficult to be proven. However, we notice that the concept of support stability is naturally linking ODEs to MDEs defined in terms of the ODEs. Therefore, necessary conditions for support stability are expected to be feasible using standard approaches. On the contrary, necessary conditions results for integral stability are expected to be much harder.

We start with the first concept of stability.

Definition 6.1. Consider an MVF V and the corresponding MDE. We say that $W : \mathbb{R}^n \to \mathbb{R}$ is a support Lyapunov function for the MDE if:

i) W is smooth, $W(0) = 0$, $W(x) > 0$ for $x \neq 0$.
ii) There exists a neighborhood $\mathcal{N}$ of 0 such that the following holds. Given $\mu \in \mathcal{P}$, set $w(\mu) = \inf\{w \in \mathbb{R} : \mu(V^{-1}([0,w])) = |\mu|\}$ (where V^{-1} indicates the preimage via V.) Then, for every $\mu \in \mathcal{P}$ with $\mathrm{Supp}(\mu) \subset \mathcal{N}$ and $(x,v) \in \mathrm{Supp}(V[\mu])$ with $W(x) = w(\mu)$, it holds that

$$\nabla W(x) \cdot v \leq 0. \tag{20}$$

We say that W is strict if the inequality in (20) is replaced by the strict inequality.

We have the following.

Theorem 6.2. *Consider an MVF V satisfying (H:bound) and (H:Lip), and fix a Lipschitz semigroup S obtained as the limit of LAS for the corresponding MDE. If there exists a support Lyapunov function W for the MDE, then the origin is support stable for the semigroup trajectories. Moreover, if W is strict, then the origin is support asymptotically stable for the semigroup trajectories.*

Proof. Fix $\epsilon > 0$ such that $B(0,\epsilon) \subset \mathcal{N}$. Define $w_\epsilon = \min\{W(x) : x \in \partial B(0,\epsilon)\}$. Since $W(0) = 0$ and W is continuous, there exists δ sufficiently small such that $B(0,\delta) \subset \{x : W(x) < w_\epsilon\}$. Fix μ_0 such that $\mathrm{Supp}(\mu_0) \subset B(0,\delta)$, and consider the S-trajectory $t \to S_t\mu_0$. Fix $\eta > 0$. Then, there exist $m > 0$ and $x_1, \ldots, x_m \in B(0,\delta)$ such that $W(\mu_0, \lambda) < \eta$ where $\lambda = \frac{1}{m}\sum_i \delta_{x_i}$. By definition, the S-trajectory from λ is approximated by LAS, and thus

$$W(S_t\lambda, \lambda^N(t)) \leq \eta \tag{21}$$

for N sufficiently big (where λ^N indicates the LAS approximate solution starting from λ). Define $\varphi(t) = \sup\{W(x) : x \in \mathrm{Supp}(\lambda^N(t))\}$. Then, setting $A(t) = \{x \in \mathrm{Supp}(\lambda^N(t)) : W(x) = w(\lambda^N(t))\}$, we get

$$\dot{\varphi}(t) = \int_{A(t)} (\nabla W(x) \cdot v) dV[\lambda^N(t)](x,v) \leq 0$$

by assumption ii) of Definition 6.1 applied to the S trajectory $s \to S_s\lambda^N(t)$. In particular, we get $\mathrm{Supp}(\lambda^N(t)) \subset \{x : W(x) \leq w_\epsilon\}$. From (21) and the arbitrariness of η, we conclude that $\mathrm{Supp}(S_t\mu_0) \subset \{x : W(x) \leq w_\epsilon\} \subset B(0,\epsilon)$, and thus we conclude that the origin is support stable for the semigroup trajectories.

Assume now that W is strict and fix δ as above. Then, given μ_0 with $\mathrm{Supp}(\mu_0) \subset B(0,\delta)$, define λ, λ^N, and φ as above. Then, we get

$$\dot{\varphi}(t) < 0$$

for every time t. Assume, by contradiction, that $\lim_{t\to\infty} \varphi(t) = l > 0$. Define the function

$$\alpha(\theta) = \sup\{\nabla W(x) \cdot v : \frac{\theta}{2} \leq |x| \leq \theta, (x,v) \in V[\mu], x \in \mathrm{Supp}(\mu) \subset B(0,\epsilon)\}$$

Then, by smoothness of W and (H:cont), we obtain that $\alpha(\theta) < 0$ for every $\theta > 0$. Then, we get $\dot{\varphi}(t) < \alpha(2l) < 0$ for t sufficiently big, reaching a contradiction, and thus $\lim_{t\to\infty} \varphi(t) = 0$. Since $W(x) > 0$ for $x \neq 0$, we conclude. □

For the second concept, we have the following.

Definition 6.3. Consider an MVF V and the corresponding MDE. We say that $W : \mathbb{R}^n \to \mathbb{R}$ is an integral Lyapunov function for the MDE if:

i) W is smooth, $W(0) = 0$, $W(x) > 0$ for $x \neq 0$. Moreover, $C_1|x| \leq W(x) \leq C_2|x|$ for some $C_1, C_2 > 0$.

ii) for every $\mu \in \mathcal{P}$ it holds that

$$\int \nabla W(x) \cdot v \, dV[\mu](x,v) \leq 0. \tag{22}$$

We say that W is strict if the inequality is replaced by the strict inequality in (22)

We have the following.

Theorem 6.4. *Consider an MVF V satisfying (H:bound) and (H:Lip), and fix a Lipschitz semigroup S obtained as the limit of LAS for the corresponding MDE. If there exists an integral Lyapunov function, then the origin is integrally stable for the semigroup trajectories. Moreover, if W is strict, then the origin is integrally asymptotically stable for the semigroup trajectories.*

Proof. Fix ϵ and set $\delta = \frac{C_2 \epsilon}{C_1}$. Consider μ_0 with $\int |x| d\mu_0 \leq \delta$. Then, $\int W(x) d\mu_0 \leq C_2 \delta$. Let $\mu(\cdot) = S_t \mu_0$. Then, using ii) of Definition 6.3, we get

$$\frac{d}{dt} \int W(x) d\mu(t) = \int \nabla W(x) \cdot v \, dV[\mu](x,v) \leq 0,$$

and thus

$$\int W(x) d\mu(t) \leq \int W(x) d\mu(0) \leq C_2 \delta.$$

Therefore,

$$\int |x| d\mu(t) \leq \frac{1}{C_1} \int W(x) d\mu(t) \leq \frac{C_2}{C_1} \delta = \epsilon$$

and we are done.

Assume now that W is a strict Lyapunov function. Fix μ_0, and set $\mu(\cdot) = S_t \mu_0$ and $\varphi = \int W(x) d\mu(t)$. Then, we have

$$\frac{d}{dt} \varphi(t) = \int \nabla W(x) \cdot v \, dV[\mu](x,v) < 0,$$

and thus φ is strictly decreasing. As in the proof of Theorem (6.2), we have that $\lim_{t \to +\infty} \varphi(t) = 0$, and thus

$$\lim_{t \to +\infty} \int |x| d\mu(t) \leq \frac{1}{C_1} \lim_{t \to +\infty} \int W(x) d\mu(t) = 0.$$

□

We now introduce a stronger concept of Lyapunov function, which is global in nature and implies stability for both concepts.

Definition 6.5. Consider an MVF V and the corresponding MDE. We say that $W : \mathbb{R}^n \to \mathbb{R}$ is a measure Lyapunov function for the MDE if:

i) W is smooth, $W(0) = 0$, $W(x) > 0$ for $x \neq 0$.
ii) For every $\mu \in \mathcal{P}$ and $(x,v) \in \text{Supp}(V[\mu])$, it holds that

$$\nabla W(x) \cdot v \leq 0. \tag{23}$$

We say that W is strict if the inequality is replaced by the strict inequality in (23).

One can easily prove the following.

Theorem 6.6. *Consider an MVF V satisfying (H:bound) and (H:Lip), and fix a Lipschitz semigroup S obtained as the limit of LAS for the corresponding MDE. If the MDE admits a measure Lyapunov function, then the origin is support stable for the semigroup trajectories. If, moreover, $C_1|x| \leq W(x) \leq C_2|x|$ for some $C_1, C_2 > 0$, then the origin is integrally stable for the semigroup trajectories.*

Proof. We notice that (23) coincides with (20). Moreover, condition ii) of Definition 6.5 is valid without restrictions on the measure or the point, and thus we conclude that the origin is support stable.

Similarly, (23) implies condition (22). Therefore we conclude that the origin is integrally stable if condition i) of Definition 6.3 also holds. □

6.1. **Lyapunov functions for MDEs defined by ODEs.** In this section, we focus on MDEs defined by ODEs. In this case, the existence of a Lyapunov function for the ODE is equivalent to the existence of a measure Lyapunov function for the MDE. More precisely, we have the following.

Theorem 6.7. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. Then, v admits a Lyapunov function (defined on $\mathbb{R}^n$) if and only if V^v admits a measure Lyapunov function.*

Proof. Assume first that v admits a Lyapunov function W. Then, $\nabla W(x) \cdot v(x) \leq 0$ for every $x \in \mathbb{R}^n$. Since $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$, (23) holds true.

Assume now that V^v admits a measure Lyapunov function. If $\mu = \delta_x$, then from (23) we get $\nabla W(x) \cdot v(x) \leq 0$, and thus we conclude. □

From Theorem 6.6, we immediately get the following.

Corollary 6.8. *Consider an MVF V such that $V^v[\mu] = \mu \otimes_x \delta_{v(x)}$ with v locally Lipschitz with sublinear growth. If v admits a Lyapunov function, then the origin is support stable. If, moreover $C_1|x| \leq W(x) \leq C_2|x|$ for some $C_1, C_2 > 0$, then the origin is integrally stable.*

Proof. Since v is locally Lipschitz with sublinear growth, from Theorem 3.10 solutions are unique, form a semigroup, and are limit of LAS. Thus, we can apply Theorem 6.6. □

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