Minimizers of the prescribed curvature functional in a Jordan domain with no necks

Original Citation:

Minimizers of the prescribed curvature functional in a Jordan domain with no necks / Leonardi G.P.; Saracco G.. - In: ESAIM. COCV. - ISSN 1292-8119. - ELETTRONICO. - 26:(2020), pp. 76.1-76.20. [10.1051/cocv/2020030]

Availability:

This version is available at: 2158/1343051 since: 2024-04-18T13:36:25Z

Published version:

DOI: 10.1051/cocv/2020030

MINIMIZERS OF THE PRESCRIBED CURVATURE FUNCTIONAL IN A JORDAN DOMAIN WITH NO NECKS

GIAN PAOLO LEONARDI AND GIORGIO SARACCO

Abstract. We provide a geometric characterization of the minimal and maximal minimizer of the prescribed curvature functional $P(E) - \kappa|E|$ among subsets of a Jordan domain Ω with no necks of radius κ^{-1}, for values of κ greater than or equal to the Cheeger constant of Ω. As an application, we describe all minimizers of the isoperimetric profile for volumes greater than the volume of the minimal Cheeger set, relative to a Jordan domain Ω which has no necks of radius r, for all r. Finally, we show that for such sets and volumes the isoperimetric profile is convex.

1. Introduction

The existence and the study of properties of hypersurfaces in $\mathbb{R}^n$, with mean curvature given by some prescribed function $g\colon \mathbb{R}^n \to \mathbb{R}$, are classical problems in geometric analysis and in Calculus of Variations, see e.g. [20–26, 37, 38, 49, 50] and the references therein. In the setting of oriented boundaries, the variational approach to the prescribed mean curvature problem is based on the minimization of the functional

$$\mathcal{F}_g[F] = P(F) - \int_F g \,\mathrm{d}x, \tag{1.1}$$

where $P(F) = P(F; \mathbb{R}^n)$ is the total perimeter, intended in the BV framework (see [5, 36]). The function g that shows up in (1.1) plays the role of a prescribed mean curvature, in the sense that any smooth critical point F for $\mathcal{F}_g$ satisfies $H_F(x) = g(x)$ at any $x \in \partial F$, where $H_F(x)$ is the mean curvature of ∂F at x. A nice introduction to the problem in $\mathbb{R}^2$ and $\mathbb{R}^3$ is available in [7]. When $g \geq 0$, the minimization of the functional (1.1) is tied to the weighted isoperimetric problem with volume density given by g: any minimizer E of (1.1) is as well a perimeter minimizer among all sets F that have the same "weighted volume" of E, i.e. $\int_F g = \int_E g$. Some results in this setting have been obtained for instance in [2, 3, 41, 42] with in mind applications such as Hardy–Sobolev inequalities [9, 13], capillarity [17, 18, 22, 34], and even politics [14, 44].

In this paper we are interested in studying the structure of minimizers of (1.1) when g is a positive constant, among subsets of an open, bounded set $\Omega \subset \mathbb{R}^2$. Specifically, for a given positive constant κ we consider the

---

2010 *Mathematics Subject Classification.* Primary: 49Q10. Secondary: 35J93, 49Q20.

Key words and phrases. perimeter minimizer, prescribed mean curvature, Cheeger constant.

G. P. L. and G. S. have been partially supported by the INdAM–GNAMPA Project 2019 "Problemi isoperimetrici in spazi Euclidei e non" (n. prot. U-UFMBAZ-2019-000473 11-03-2019).

minimization of the functional

$$\mathcal{F}_\kappa[F] = P(F) - \kappa|F| \tag{1.2}$$

among measurable sets $F \subset \Omega$, where $|\cdot|$ denotes the 2-dimensional Lebesgue measure. It is well known that the internal boundary $\partial E_\kappa \cap \Omega$ of any nontrivial minimizer E_κ of (1.2) is smooth and made of an at most countable union of circular arcs with curvature equal to κ. Existence of minimizers of (1.2) follows from the Direct Method of the Calculus of Variations, see [36, Section 12.5], but it may happen that the minimum is achieved by the empty set. A special value of κ is given by the *Cheeger constant* of Ω, defined as

$$h_\Omega = \inf\left\{ \frac{P(F)}{|F|} : |F| > 0\,, F \subseteq \Omega \right\},$$

and any nontrivial set E attaining the infimum is called *Cheeger set* of Ω. The computation of the constant h_Ω and the characterization of the Cheeger sets of Ω are referred to as the *Cheeger problem*. The existence of Cheeger sets is well known, see for instance [30, 39, 40, 45]. Clearly, any Cheeger set E is a nontrivial minimizer of (1.2) for the choice $\kappa = h_\Omega$, i.e. of

$$\mathcal{F}_{h_\Omega}[F] = P(F) - h_\Omega|F|.$$

Notice that $\min \mathcal{F}_{h_\Omega} = 0$ and that $\min \mathcal{F}_\kappa \leq \mathcal{F}_\kappa[\emptyset] = 0$, for all $\kappa > 0$. On the one hand, if $\kappa > h_\Omega$, one has

$$\min \mathcal{F}_\kappa \leq P(E) - \kappa|E| < P(E) - h_\Omega|E| = 0,$$

where E is a Cheeger set of Ω; this shows that $\mathcal{F}_\kappa$ admits nontrivial minimizers. On the other hand, if $\min \mathcal{F}_\kappa \geq 0$, then $P(F)|F|^{-1} \geq \kappa$ for all subset $F \subseteq \Omega$ such that $|F| > 0$, hence by taking the infimum one finds that $\kappa \leq h_\Omega$. Therefore, the unique minimizer of (1.2) whenever the strict inequality $\kappa < h_\Omega$ holds is the empty set. In the equality case, both the empty set and the Cheeger sets of Ω solve (1.2); in this limiting case, we shall always consider the nontrivial minimizers.

The Cheeger problem has been widely studied in the past, due to its deep connections with other problems ranging from eigenvalue estimates to capillarity. Several authors addressed the question about how to characterize and efficiently compute the value of the Cheeger constant h_Ω. The known results in this direction are essentially limited to the planar setting, as they heavily rely on the rigid characterization of curves with constant curvature in the plane. In particular, under the assumption that Ω is convex [27] or a strip [32] it has been proved that the Cheeger set of Ω is unique and precisely characterized from the geometric viewpoint. If we denote by Ω^r the *inner parallel set* at distance r, i.e.

$$\Omega^r = \{\, x \in \Omega : \operatorname{dist}(x;\partial\Omega) \geq r \,\},$$

then the unique Cheeger set E of Ω is given by the Minkowski sum $\Omega^r \oplus B_r$, where $r = h_\Omega^{-1}$. Equivalently, the Cheeger set E agrees with the union of all balls of radius r contained in Ω. Moreover, the *inner Cheeger formula* holds, i.e. the radius r is the unique positive solution of the equation

$$\pi\rho^2 = |\Omega^\rho|.$$

This formula and this kind of structure for planar Cheeger sets have been recently extended in [31] to a class of planar domains that is essentially the largest possible. Before recalling the statement of the general structure theorem, we need to introduce the following definition of *no necks of radius* r for $r \in (0, \mathrm{inr}(\Omega)]$, where $\mathrm{inr}(\Omega)$ stands for the *inradius* of Ω.

Definition 1.1. A set Ω has *no necks of radius* r, with $r \in (0, \mathrm{inr}(\Omega)]$ if the following condition holds. If $B_r(x_0)$ and $B_r(x_1)$ are two balls of radius r contained in Ω, then there exists a continuous curve $\gamma\colon [0,1] \to \Omega$ such that

$$\gamma(0) = x_0, \qquad \gamma(1) = x_1, \qquad B_r(\gamma(t)) \subset \Omega, \quad \forall t \in [0,1].$$

We remark that having no necks of radius r_1 does not imply the same property for any radius $r_2 < r_1$.

Whenever a set has no necks of radius $r = h_\Omega^{-1}$, then its (maximal) Cheeger set agrees with the union of all balls of radius r contained in Ω, analogously to what happens for convex sets and strips. This remarkable fact was proved in [31], and we recall the theorem below.

Theorem 1.2 (Theorem 1.4 and Remark 5.2 of [31])**.** *Let Ω be a Jordan domain such that $|\partial\Omega| = 0$. If Ω has no necks of radius $r = h_\Omega^{-1}$, then the maximal Cheeger set E of Ω is given by*

$$E = \Omega^r \oplus B_r\,,$$

i.e. the Minkowski sum of Ω^r and B_r. Moreover, r is the unique positive solution of

$$\pi\rho^2 = |\Omega^\rho|\,. \tag{1.3}$$

Finally, if $\Omega^r = \overline{\mathrm{int}(\Omega^r)}$, then E is the unique Cheeger set of Ω.

We remark that $|\partial\Omega|$ is the 2-dimensional Lebesgue measure of $\partial\Omega$, thus sets whose boundary is a plane-filling curve à la Knopp–Osgood (see [43]) are not covered by the theorem. While it is unclear whether the hypothesis $|\partial\Omega| = 0$ is necessary, the other hypothesis of topological flavor, i.e. that Ω is a Jordan domain, and the assumption of no necks of radius h_Ω^{-1}, must be required, otherwise one can produce counterexamples (see [31, 35]). While uniqueness is not always granted in this more general setting, one can speak of *the maximal* Cheeger set because the class of Cheeger sets is closed under countable unions: one can define a maximal Cheeger set (see Definition 2.2) and prove its uniqueness (see Proposition 3.2).

In this paper we show that an analogous result to Theorem 1.2 holds for nontrivial minimizers of the prescribed curvature functional $\mathcal{F}_\kappa$. Specifically, in Theorem 2.3 we show that if a Jordan domain Ω with $|\partial\Omega| = 0$ has no necks of radius $r = \kappa^{-1}$, then the maximal minimizer E_κ^M of $\mathcal{F}_\kappa$ is given by $E_\kappa^M = \Omega^r \oplus B_r$. Moreover, thanks to a careful study of the set $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$, see Proposition 2.1, we are able to give a precise geometric description of the unique minimal minimizer E_κ^m of $\mathcal{F}_\kappa$ and therefore to completely characterize the cases when uniqueness is granted (for the definition of minimal minimizer, we refer the reader to Definition 2.2).

Once these characterizations are proved, we are able to describe *all* possible minimizers of $\mathcal{F}_\kappa$ by suitably "interpolating" between E_κ^m and E_κ^M, and consequently we show that there exists a minimizer E_κ of $\mathcal{F}_\kappa$ such that

$|E_\kappa| = V$, for any prescribed volume V between $|E^m_\kappa|$ and $|E^M_\kappa|$. In Theorem 2.4 we apply this fact to the isoperimetric problem in a Jordan domain Ω with $|\partial\Omega| = 0$ that has no necks of radius r, for all $r \le h_\Omega^{-1}$. For such an Ω we can fully describe the isoperimetric sets relative to volumes $V \ge |E^m_{h_\Omega}|$, and we show that the isoperimetric profile is convex in the volume range $|E^m_{h_\Omega}| \le V \le |\Omega|$.

The paper is structured as follows. In Section 2 we state our main results and comment them. In Section 3 we state some properties of minimizers of (1.2) which are well known in the limit case $\kappa = h_\Omega$, and whose extensions to any $\kappa \ge h_\Omega$ are mostly trivial. In Section 4 we give a characterization of the set difference $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$, when Ω has no necks of radius r. In Section 5 we prove the structure of the maximal and minimal minimizers of (1.2) for κ, whenever Ω has no necks of radius κ^{-1}. In Section 6 we address the isoperimetric problem in sets Ω with no necks of radius r for all $r \le h_\Omega^{-1}$, proving the structure of minimizers with volume greater than a certain threshold and the convexity of the isoperimetric profile above such a threshold.

2. Statement of the main results

Throughout the paper, with a slight abuse of notation, given a curve $\gamma\colon [0,1] \to \mathbb{R}^2$, we shall write γ in place of $\gamma([0,1])$. For the sake of completeness, we recall that a Jordan domain is the region bounded by an injective and continuous map $\Phi\colon \mathbb{S}^1 \to \mathbb{R}^2$, which is well defined thanks to the Jordan–Schoenflies theorem.

The first result we are going to prove is a characterization of the set difference $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$, whenever Ω is a Jordan domain with no necks of radius r. This, roughly speaking, says that such a difference consists of two families of curves Γ^1_r and Γ^2_r: curves in Γ^1_r correspond to the presence of "tendrils" of width r, while curves in Γ^2_r to the presence of "handles" of width r as shown in Figure 1.

Proposition 2.1. *Let Ω be a Jordan domain with no necks of radius r. The following properties hold:*

- (a) *if Ω^r is nonempty but has empty interior, then either it consists of a single point or there exists an embedding $\gamma\colon [0,1] \to \mathbb{R}^2$ of class $C^{1,1}$, with curvature bounded by r^{-1}, such that $\gamma([0,1]) = \Omega^r$;*
- (b) *if $\mathrm{int}(\Omega^r) \neq \emptyset$, then there exist two (possibly empty) families Γ^1_r and Γ^2_r of embedded curves contained in Ω^r with the following properties. For each $i = 1, 2$ and each $\gamma \in \Gamma^i_r$,*
    - (i) *$\gamma\colon [0,1] \to \Omega^r$ is nonconstant and of class $C^{1,1}$, with curvature bounded by r^{-1};*
    - (ii) *if $i = 1$, then $\overline{\mathrm{int}(\Omega^r)} \cap \gamma = \{\gamma(0)\}$;*
    - (iii) *if $i = 2$, then $\overline{\mathrm{int}(\Omega^r)} \cap \gamma = \{\gamma(0), \gamma(1)\}$;*
    - (iv) *Γ^1_r is finite;*
    - (v) *the following set equality holds*

$$\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)} = \bigcup_{\gamma\in\Gamma^1_r} \gamma\left((0,1]\right) \cup \bigcup_{\gamma\in\Gamma^2_r} \gamma\left((0,1)\right).$$

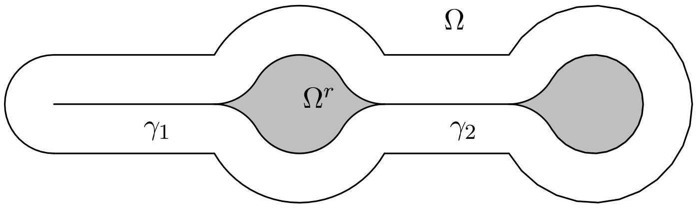


FIGURE 1. Curves γ_2 with both endpoints in $\overline{\mathrm{int}(\Omega^r)}$ correspond to "handles" and connect disjoint connected components of $\mathrm{int}(\Omega^r)$, while curves γ_1 with just one endpoint in $\overline{\mathrm{int}\,(\Omega^r)}$ correspond to "tendrils".

The structure granted by Proposition 2.1 might turn out useful in other contexts. We recall indeed, e.g. the ∞-Laplacian problem [12] and the irrigation problem [8, 48], in which the set Ω^r plays a role.

Definition 2.2. Let $\Omega \subset \mathbb{R}^2$ and $\kappa > 0$ be fixed, and let E_κ be a minimizer of $\mathcal{F}_\kappa$. We say that E_κ is a *maximal* minimizer if for any other minimizer F_κ one has $F_\kappa \subset E_\kappa$; we say that it is a *minimal* minimizer if for any other minimizer F_κ one cannot have the strict inclusion $F_\kappa \subsetneq E_\kappa$.

The existence of maximal and minimal minimizers is proved in Proposition 3.2, along with the uniqueness of the maximal minimizer. Concerning the uniqueness of minimal minimizers, it is verified when $\kappa > h_\Omega$ but may fail in the case $\kappa = h_\Omega$ (see again Proposition 3.2 and Remark 3.3). In what follows we shall denote by E^M_κ the maximal minimizer and by E^m_κ the minimal minimizer in case the latter is unique.

Theorem 2.3. *Let Ω be a Jordan domain with $|\partial\Omega| = 0$ and let $\kappa \geq h_\Omega$ be fixed. Assume Ω has no necks of radius $r = \kappa^{-1}$. Then, both maximal and minimal minimizers E^M_κ and E^m_κ are uniquely characterized as*

$$E^M_\kappa = \Omega^r \oplus B_r, \qquad E^m_\kappa = \Bigg(\overline{\mathrm{int}(\Omega^r)} \cup \bigcup_{\gamma \in \Gamma^2_r} \gamma \Bigg) \oplus B_r.$$

In particular, $\mathcal{F}_\kappa$ has a unique minimizer (i.e., $E^m_\kappa = E^M_\kappa$) as soon as Γ^1_r is empty.

Theorem 2.3 extends Theorem 1.2 on the maximal minimizer for the limit case $\kappa = h_\Omega$, originally proved in [31, Theorem 1.4 and Remark 5.2]. There are two immediate consequences to this theorem. Firstly, we show in Corollary 5.6 the nestedness of minimizers for increasing values $\kappa_2 > \kappa_1$, provided that Ω has no necks of radii κ_1^{-1} and κ_2^{-1}. Secondly, we show that Theorem 1.2 can be "improved", in the following sense. In order to apply it, one needs to know a priori the value of the constant h_Ω, or at least to ensure that Ω has no necks of radius r for a range of values such that h_Ω^{-1} falls within. If this happens, *then* r is the unique positive solution of $\pi\rho^2 = |\Omega^\rho|$. In Corollary 5.5, we prove that one can "reverse" these operations. By this, we mean that one can consider the unique positive solution r to $\pi\rho^2 = |\Omega^\rho|$ and *then* check if the set has no necks of radius r. If it does, then r is the inverse of h_Ω and the maximal Cheeger set is $\Omega^r \oplus B_r$.

We mention that, thanks to the above result, one derives an extension of a result by Chen (see [11, 22], or [19, 27] for convex sets). Chen's theorem provides a criterion for a set Ω to be the unique Cheeger set of itself. This also follows from a more general criterion related to self-minimizers of the prescribed curvature functional $\mathcal{F}_\kappa$, to appear in the forthcoming paper [46].

Finally, notice that any nontrivial minimizer E_κ of $\mathcal{F}_\kappa$ is also a set attaining the minimum of the isoperimetric profile

$$\mathcal{J}(V) = \inf\{\, P(F) : F \subset \Omega, |F| = V\,\},$$

relatively to the volume $V = |E_\kappa|$. Thanks to Theorem 2.3 we are in a position to exhibit the minimizers of $\mathcal{J}(V)$ relatively to volumes $V \geq |E^m_{h_\Omega}|$, provided that Ω has no necks of radius r, for all $r \in (0, h_\Omega^{-1}]$. Specifically, the following result holds.

Theorem 2.4. *Let Ω be a Jordan domain with $|\partial\Omega| = 0$. Assume Ω has no necks of radius $r = \kappa^{-1}$, for all $r \in (0, h_\Omega^{-1}]$. Then, for all volumes $V \geq |E^m_{h_\Omega}|$, there exists $\kappa \in [h_\Omega, +\infty)$ and a minimizer E_κ of $\mathcal{F}_\kappa$ such that*

$$|E_\kappa| = V, \qquad\qquad \mathcal{J}(V) = P(E_\kappa).$$

Under the same hypotheses of Theorem 2.4, we show the convexity of the isoperimetric profile $\mathcal{J}$ for $V \geq |E^m_{h_\Omega}|$ by observing that it coincides with the Legendre transform of the convex function $\mathcal{G}\colon \kappa \mapsto -\min \mathcal{F}_\kappa$, defined on $[h_\Omega, +\infty)$, see Proposition 6.2 and Corollary 6.3. This agrees with the results of [14] relatively to a relaxation of the isoperimetric profile. For the sake of completeness, we recall that the above theorem was known in the convex case, see [47, Theorem 3.32]. In the n-dimensional convex case, existence and uniqueness were discussed in [1, Section 4] (as well as in the Gaussian convex case [10, Theorem 23]).

3. Properties of minimizers

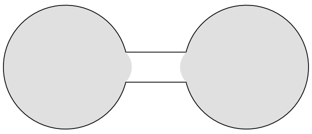
Most of the proofs of the results presented in this section are not given, since they are easy adaptations from the limit case $\kappa = h_\Omega$. The interested reader is referred to the original ones for which we give a precise reference.

We remark that throughout this section the no neck condition is *never* enforced. Same goes for the request that Ω is a Jordan domain but for Section 3.1. The results contained here apply generally to any minimizer in an open, bounded set $\Omega \subset \mathbb{R}^2$.

First of all, notice that any minimizer E_κ of $\mathcal{F}_\kappa$ enjoys many regularity properties which come from the standard regularity theory of perimeter minimizers. Among these, the fact that $\partial E_\kappa \cap \Omega$ has constant (mean) curvature equal to κ, which is the reason why the functional is usually referred to as the *prescribed (mean) curvature functional.* We collect these regularity properties of the boundary in the next proposition.

Proposition 3.1. *Let E_κ be a minimizer of $\mathcal{F}_\kappa$ relatively to $\Omega \subset \mathbb{R}^2$. Then, the following statements hold true:*

- (i) *$\partial E_\kappa \cap \Omega$ is analytic and coincides with a countable union of circular arcs of curvature κ, with endpoints belonging to $\partial\Omega$;*
- (ii) *the length of any arc in $\partial E_\kappa \cap \Omega$ cannot exceed $\pi\kappa^{-1}$;*



(a) The maximal Cheeger set of the balanced dumbell.

(b) A minimal Cheeger set of the balanced dumbell.

(c) A minimal Cheeger set of the balanced dumbell.

FIGURE 2. The above figures show all the nontrivial minimizers of the prescribed curvature functional for $\kappa = h_\Omega$, i.e. the Cheeger sets of Ω, with Ω a balanced dumbell.

(iii) *for Ω with locally finite perimeter, if $x \in \partial E_\kappa \cap \partial^* \Omega$, then $x \in \partial^* E_\kappa$ and $\nu_\Omega(x) = \nu_{E_\kappa}(x)$.*

Point (i) is nowadays standard, and one can refer to [36, Section 17.3]. Point (ii) can be proved as in [32, Lemma 2.11]. Point (iii) is well known for a Lipschitz Ω, see for instance [24]; see also [34, Theorem 3.5] for a proof valid for every Ω with locally finite perimeter.

We recall the notion of P-connectedness which in the theory of sets of finite perimeter replaces the usual notion of connectedness, and from now onwards whenever we write connected it is understood to be P-connected. Given a set A of finite perimeter we say that it is *decomposable* if there exists a partition (E, F) of A such that $P(A) = P(E) + P(F)$ and both $|E|$ and $|F|$ are strictly positive. We say that it is *indecomposable* if it is not decomposable. Given any set of finite perimeter A, there exists a unique finite or countable family $\{E_i\}_i$ of pairwise disjoint indecomposable sets with $|E_i| > 0$ such that $P(A) = \sum_i P(E_i)$, see [4, Theorem 1]. We shall call each of these sets E_i a *P-connected component* of A.

In the next proposition we show that there exist both maximal and minimal minimizers of $\mathcal{F}_\kappa$, which we recall we defined in Definition 2.2.

Proposition 3.2. *There exists a unique maximal minimizer of $\mathcal{F}_\kappa$, which is given by the union of all minimizers. There exist minimal minimizers of $\mathcal{F}_\kappa$. Moreover, in the case $k > h_\Omega$ one has the uniqueness of the minimal minimizer.*

Proof. We start noticing the following fact. If E_κ and F_κ are both minimizers, then $E_\kappa \cap F_\kappa$ and $E_\kappa \cup F_\kappa$ are minimizers as well, i.e. the class of minimizers is closed under countable unions and intersections. Indeed, by the well-known inequality (see for instance [36, Lemma 12.22])

$$P(E_\kappa \cup F_\kappa) + P(E_\kappa \cap F_\kappa) \leq P(E_\kappa) + P(F_\kappa)\,,$$

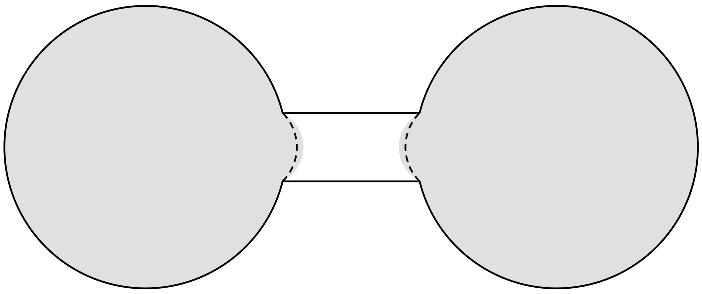


FIGURE 3. The shaded area represents the minimizer of the prescribed curvature functional for κ close to h_Ω, while the dashed curves are the interior boundary of the maximal Cheeger set. Each of the connected components E^i_κ is such that $\mathcal{F}_\kappa[E^i_\kappa] < 0$, hence a component alone is not a minimizer.

we have

$$\begin{aligned} P(E_\kappa) + P(F_\kappa) - 2\min \mathcal{F}_\kappa &= \kappa|E_\kappa| + \kappa|F_\kappa| = \kappa|E_\kappa \cup F_\kappa| + \kappa|E_\kappa \cap F_\kappa| \\ &\le P(E_\kappa \cup F_\kappa) + P(E_\kappa \cap F_\kappa) - 2\min \mathcal{F}_\kappa \le P(E_\kappa) + P(F_\kappa) - 2\min \mathcal{F}_\kappa \end{aligned}$$

thus all inequalities are equalities. Hence, we get

$$P(E_\kappa \cap F_\kappa) - \kappa|E_\kappa \cap F_\kappa| = P(E_\kappa \cup F_\kappa) - \kappa|E_\kappa \cup F_\kappa| = \min \mathcal{F}_\kappa\,. \tag{3.1}$$

Let now $\{F^i_\kappa\}_i$ be a countable family of minimizers. Let $U_\kappa = \cup_i F^i_\kappa$ and $I_\kappa = \cap_i F^i_\kappa$. Then, thanks to (3.1) and the lower semicontinuity of the perimeter, one readily shows that U_κ and I_κ are minimizers too. Notice that in the case $\kappa = h_\Omega$, one can have $I_\kappa = \emptyset$, i.e. the trivial minimizer. However, this can be excluded by requiring $\cap_{i\le j} E^i_\kappa \neq \emptyset$ for all j. Indeed, any nontrivial minimizer satisfies a uniform lower bound on the volume, see Proposition 3.4 below. Finally, observe that if two minimal minimizers E_κ and F_κ have a nonnegligible intersection, then the intersection is also a minimal minimizer and therefore $E_\kappa = F_\kappa$. This also shows that two distinct minimal minimizers must be P-connected components of their union. Hence, if we assume $\kappa > h_\Omega$ we have $\mathcal{F}_\kappa[E_\kappa] < 0$ for every minimal minimizer E_κ. Thus, the existence of another minimal minimizer $F_\kappa \neq E_\kappa$ would lead to $\mathcal{F}_\kappa[E_\kappa \cup F_\kappa] = \mathcal{F}_\kappa[E_\kappa] + \mathcal{F}_\kappa[F_\kappa] < \mathcal{F}_\kappa[E_\kappa]$, against minimality. This shows that when $\kappa > h_\Omega$ the minimal minimizer is unique. □

Remark 3.3. It is rather interesting to notice that there exists a unique, nontrivial minimal minimizer whenever $\kappa > h_\Omega$, given precisely by the intersection of all minimizers. This is in contrast with the limit case $\kappa = h_\Omega$, where one can have multiple minimal minimizers, as the dumbell in Figure 2 shows. The reason is that, for $\kappa = h_\Omega$, any connected component of a minimizer is a minimizer itself (see Figures 2(b) and 2(c)), while this is false for $\kappa > h_\Omega$ (for comparison, see Figure 3).

The following lower bound to the volume of any connected component of a minimizer is readily established.

Proposition 3.4. *Let E_κ be a minimizer of $\mathcal{F}_\kappa$. Then, any of its connected components E^i_κ has volume bounded from below by $4\pi\kappa^{-2}$.*

Proof. If $\kappa = h_\Omega$ this is straightforward from the isoperimetric inequality and the well-known fact that any connected component of a Cheeger set is a Cheeger set itself. Suppose now that $\kappa > h_\Omega$ and without loss of generality that E_κ is decomposable, i.e. there exist $E_\kappa^1, E_\kappa^2 \subset E_\kappa$ with $|E_\kappa^1| \cdot |E_\kappa^2| > 0$ and such that

$$|E_\kappa| = |E_\kappa^1| + |E_\kappa^2|, \qquad\qquad P(E_\kappa) = P(E_\kappa^1) + P(E_\kappa^2).$$

Assume by contradiction that $|E_\kappa^1| < 4\pi\kappa^{-2}$ and denote by $B_{E_\kappa^1}$ the ball with same volume of E_κ^1. Its radius $r_{E_\kappa^1}$ is strictly less than $2\kappa^{-1}$. Thus,

$$\begin{aligned}\mathcal{F}_\kappa[E_\kappa^1] &= P(E_\kappa^1) - \kappa|E_\kappa^1| \geq P(B_{E_\kappa^1}) - \kappa|B_{E_\kappa^1}| \\ &= 2\pi r_{E_\kappa^1} - \kappa\pi r_{E_\kappa^1}^2 = \pi r_{E_\kappa^1}(2 - \kappa r_{E_\kappa^1}) > 0.\end{aligned}$$

Therefore $\mathcal{F}_\kappa[E_\kappa^2] < \mathcal{F}_\kappa[E_\kappa]$, against the minimality of E_κ. □

Finally, we recall the *rolling ball lemma* [32, Lemma 2.12], which was later refined [31, Lemma 1.7]. This still holds for general κ, and the proof is a straightforward adaptation of the original lemma.

Lemma 3.5 (Rolling ball)**.** *Let $\kappa \geq h_\Omega$ be fixed, and let E_κ^M be the maximal minimizer of $\mathcal{F}_\kappa$. If E_κ^M contains a ball $B_r(x_0)$ of radius $r = \kappa^{-1}$, then it contains all balls of same radius that can be reached by rolling $B_r(x_0)$, i.e. it contains any ball $B_r(x_1)$ such that there exists a continuous curve $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$ and $B_r(\gamma(t)) \subset \Omega$ for all $t \in [0,1]$.*

3.1. **Additional properties when Ω is a Jordan domain.**

Here, we state a few additional properties of minimizers when Ω is a Jordan domain. Their proofs are omitted as they closely follow the corresponding ones presented in [31] for the case $\kappa = h_\Omega$.

Proposition 3.6. *Suppose $\Omega \subset \mathbb{R}^2$ is a Jordan domain with $|\partial\Omega| = 0$, and let E_κ be a minimizer of $\mathcal{F}_\kappa$. Then,*

- (i) *the curvature of ∂E_κ is bounded from above by κ in both variational and viscous senses;*
- (ii) *E_κ is Lebesgue-equivalent to a finite union of simply connected open sets, hence its measure-theoretic boundary ∂E_κ is a finite union of pairwise disjoint Jordan curves;*
- (iii) *E_κ contains a ball of radius κ^{-1}.*

The definitions of curvature in variational and in viscous senses, notions that appear in the above proposition, can be found resp. in [6] and [31, Definition 2.3]. The proof of (i) is obtained by mimicking [31, Lemma 2.2 and Lemma 2.4]. The proof of (ii) follows by arguing as in [31, Propositions 2.9 and 2.10]. The proof of claim (iii) follows from (i) and (ii) combined with [31, Theorem 1.6].

4. THE SET DIFFERENCE $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$

Here we prove Proposition 2.1, i.e. the structure of the set difference $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$, under the assumption that Ω has no necks of radius r. According to Definition 1.1, this means that given any two balls $B_r(x_0)$ and $B_r(x_1)$ contained in Ω, there exists a continuous curve $\gamma\colon [0,1] \to \Omega^r$ such that

$\gamma(0) = x_0$ and $\gamma(1) = x_1$. Thanks to [31, Theorem 1.8], we can further assume γ to be of class $\mathrm{C}^{1,1}$ with curvature bounded by $1/r$.

We now lay down some notation we shall use throughout the paper from now onwards. Given a regular curve $\gamma\colon [0,1] \to \mathbb{R}^2$ of class $\mathrm{C}^{1,1}$, we set

$$\gamma'(0) = \lim_{t\to 0^+} \gamma'(t)\,, \qquad\qquad \gamma'(1) = \lim_{t\to 1^-} \gamma'(t).$$

We denote by $\nu(t)$ the renormalization of $\gamma'(t)$, i.e.

$$\nu(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Owing to the regularity of γ, $\nu(t)$ is continuous and defined on the whole interval $[0,1]$. Given $r > 0$, we define the open half-ball

$$B_r^+(\gamma(t)) = \{\, z \in \mathbb{R}^2 : |z-\gamma(t)| < r, (z-\gamma(t))\cdot\nu(t) > 0 \,\}, \tag{4.1}$$

and the relatively open half-circle

$$S_r^+(\gamma(t)) = \{\, z \in \mathbb{R}^2 : |z-\gamma(t)| = r, (z-\gamma(t))\cdot\nu(t) > 0 \,\},$$

that are "oriented in the direction $\nu(t)$" (note that we have dropped the explicit dependence on $\nu(t)$ in the notation). Finally, the endpoints of $S_r^+(\gamma(t))$ are denoted by

$$z_t^+ = \gamma(t) + r\nu(t)^\perp, \qquad\qquad z_t^- = \gamma(t) - r\nu(t)^\perp. \tag{4.2}$$

Proof of Proposition 2.1. If Ω^r and $\overline{\mathrm{int}(\Omega^r)}$ agree, there is nothing to prove. Let us suppose then that there exists $x \in \Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$, and let us denote by Π_x its "projection set", i.e.

$$\Pi_x = \{y \in \partial\Omega \,:\, |y-x| = r\}\,.$$

We split the proof in two steps, following points (a) and (b) of the statement.

(a) The case $\mathrm{int}(\Omega^r) = \emptyset$. We can distinguish three subcases, according to the properties of the projection set Π_x.

(a1) For all directions $\nu \in \mathbb{S}^1$, there exist two points $y_1, y_2 \in \Pi_x$ such that

$$\nu\cdot(y_1 - x) < 0 < \nu\cdot(y_2 - x).$$

(a2) There exist a direction $\nu \in \mathbb{S}^1$ and two distinct points $y_1\,, y_2 \in \Pi_x$ such that

$$\nu\cdot(y_1 - x) = \nu\cdot(y_2 - x) = 0, \qquad \nu\cdot(y - x) \geq 0, \quad \forall y \in \Pi_x.$$

(a3) There exist a direction $\nu \in \mathbb{S}^1$ and $\delta > 0$ such that

$$\nu\cdot(y - x) \geq \delta, \quad \forall y \in \Pi_x.$$

We start noticing that case (a3) can never happen. Indeed, one could easily show that $x - \varepsilon\nu \in \mathrm{int}(\Omega^r)$, for ε sufficiently small which contradicts $\mathrm{int}(\Omega^r) = \emptyset$.

In case (a1), it is immediate to see that x is an isolated point in Ω^r. Then, as Ω has no necks of radius r we infer that $\Omega^r = \{x\}$, i.e. it is a constant curve. We are then left with case (a2), which implies that Ω^r satisfies a bilateral ball condition of radius r at x, which means there exist two balls of radius r, B_1 and B_2, such that $\overline{B_1} \cap \overline{B_2} = \{x\}$, and locally at x, $\Omega^r \cap (B_1 \cup B_2) = \emptyset$. As this holds for any choice of x, and as Ω^r is path-connected, this necessarily means that $\Omega^r = \gamma$, with γ a $\mathrm{C}^{1,1}$ curve with curvature bounded by r^{-1}. Further, this curve cannot be a loop: since Ω is a Jordan domain, this would imply that $\mathrm{int}(\Omega^r) \neq \emptyset$. Therefore, Ω^r is diffeomorphic to the closed segment $[0, 1]$.

(b) The case $\mathrm{int}(\Omega^r) \neq \emptyset$*.* We fix $y \in \mathrm{int}(\Omega^r)$, which exists since by hypothesis this set is not empty. The assumption of no necks of radius r paired with [31, Theorem 1.8] yields the existence of a $\mathrm{C}^{1,1}$ curve γ, with curvature bounded by r^{-1}, such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $T > 0$ be the first time for which $\gamma(T) \in \overline{\mathrm{int}(\Omega^r)}$. Thanks to Zorn's lemma, we can extend $\gamma_{|[0,T)}$ to a maximal curve $\widetilde{\gamma}$ in $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$. Moreover, by continuity we can extend $\widetilde{\gamma}$ to a closed interval, and up to a reparametrization we can assume it to be $[0, 1]$. Without loss of generality, suppose that $\widetilde{\gamma}(0) = \gamma(T) \in \partial(\mathrm{int}(\Omega^r))$. Hence, there are two possible cases: either $\widetilde{\gamma}(1)$ belongs as well to $\partial(\mathrm{int}(\Omega^r))$; or $\widetilde{\gamma}(1)$ belongs to $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$.

By reasoning as in the first step, we notice that x has at least two 2 antipodal projections in Π_x. By the bilateral ball condition, which holds at any $x \in \widetilde{\gamma}$, one can show that there exists $\varepsilon = \varepsilon_x > 0$ such that

$$\left(\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}\right) \cap B_\varepsilon(x) = \widetilde{\gamma} \cap B_\varepsilon(x).$$

We define Γ_r^1 as the collection of connected components of $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$ that are diffeomorphic to the half-closed interval $(0, 1]$, and similarly Γ_r^2 as the collection of connected components that are diffeomorphic to the open interval $(0, 1)$.

We are left with showing that $\#\Gamma_r^1 < \infty$. Let us fix any $\gamma \in \Gamma_r^1$ and let $x_\gamma = \gamma(1)$. By reasoning as in the first part of the proof, we have that $z_1^\pm$ as defined in (4.2) belong to Π_{x_γ}. We claim that all $z \in B_r^+(x_\gamma)$ have as unique projection on Ω^r the point x_γ, where $B_r^+(x_\gamma)$ is defined in (4.1). This proves that from any curve $\gamma \in \Gamma_r^1$ stems a contribute to the volume of at least $\frac{\pi}{2}r^2$. The finiteness of $|\Omega|$ implies then the finiteness of the family Γ_r^1.

To show this we argue by contradiction. Let us suppose that some $z \in B_r^+(x_\gamma)$ has as unique projection $y \in \Omega^r$ with $y \neq x_\gamma$. By the no necks assumption there is a $\mathrm{C}^{1,1}$ curve σ from x_γ to y, which lies in Ω^r. We claim that the loop constructed by concatenating σ, the segment $[x_\gamma, z]$ and the segment $[z, y]$ contains either z_1^+ or z_1^- giving a contradiction to the simple connectedness of Ω.

This follows by noticing that both the segment $[z, y]$ and the curve σ cannot pass across the segment $[z_1^-, z_1^+]$. First, assume by contradiction that $[z, y]$ crosses the open segment $[z_1^-, z_1^+]$ in w. Trivially, w cannot coincide with x_γ otherwise this contradicts y being the closest point in Ω^r to z. Moreover, as w is in the open segment $[z_1^-, z_1^+]$ it projects uniquely on x_γ, therefore $|y - w| > |w - x_\gamma|$. By triangular inequality it immediately follows that $|x_\gamma - z| < |y - z|$ which is a contradiction.

Second, on the one hand σ cannot pass through the points lying in the open segments $[x_\gamma, z_1^+]$ and $[x_\gamma, z_1^-]$ as all these have distance from the boundary less than r (since $z_1^-, z_1^+ \in \Pi_{x_\gamma}$). On the other hand, for some $\varepsilon = \varepsilon(x_\gamma) << 1$ we have $\Omega^r \cap B_\varepsilon(x_\gamma) = \gamma$. As $\gamma \in \Gamma_r^1$ and $x_\gamma = \gamma(1)$ one has that $\Omega^r \cap B_\varepsilon^+(x_\gamma) = \emptyset$. Thus, $\sigma \cap B_\varepsilon^+(x_\gamma) = \emptyset$. This establishes that all $z \in B_r^+(x_\gamma)$ have as unique projection on Ω^r the point x_γ. $\square$

Remark 4.1. As can be seen from the proof of the above proposition, we remark that any point x belonging to γ, $x = \gamma(t)$, with γ in either Γ^1_r or Γ^2_r, has two projections on $\partial\Omega$ that are antipodal, given by $z_t^\pm$, defined in (4.2). This in particular implies that any strip

$$\mathcal{S}(\gamma) = \Big\{ \gamma(t) \pm \rho\nu(t)^\perp : t \in [0,1], \rho \in [0,r) \Big\} \tag{4.3}$$

is diffeomorphic to the rectangle $[0,1] \times (-r,r)$. Moreover, given any two curves $\gamma_1, \gamma_2 \in \Gamma^1_r$ the strips $\mathcal{S}(\gamma_1), \mathcal{S}(\gamma_2)$ are pairwise disjoint. Finally, notice that the "lateral boundart" of $\mathcal{S}(\gamma)$

$$\partial_L\mathcal{S}(\gamma) = \Big\{ z_t^\pm = \gamma(t) \pm r\nu(t)^\perp : t \in [0,1] \Big\} \tag{4.4}$$

is contained in $\partial\Omega$.

Remark 4.2. Notice the following: if Ω is a Jordan domain with no necks of radius r for all $r \le R \le \mathrm{inr}(\Omega)$, then for every $r < R$ the set Γ^2_r is empty. Argue by contradiction and suppose $\exists\gamma \in \Gamma^2_r$. The points $\gamma(0)$ and $\gamma(1)$ belong to $\partial(\mathrm{int}\,(\Omega^r))$. Therefore, we can find a point $z_0 \in \mathrm{int}\,(\Omega^r)$ (resp. z_1) arbitrarily close to $\gamma(0)$ (resp. $\gamma(1)$). Clearly one has $r < \bar r$ where $\bar r = \min\{\,\mathrm{dist}(z_0;\partial\Omega);\mathrm{dist}(z_1;\partial\Omega)\,\}$ and without loss of generality we can suppose $\bar r < R$. As Ω has no necks of radius $\bar r$, there exists a curve σ joining these two points contained in $\Omega^{\bar r}$. Being $\bar r > r$, the curves γ and σ cannot meet but in the endpoints. Therefore, by concatenating these two curves, and the segments $[\gamma(i), z_i]$ for $i = 0,1$, one reaches a contradiction as in the proof of Proposition 2.1.

5. Structure of minimizers

In this section we give the proof of Theorem 2.3. The part concerning the structure of the maximal minimizer closely follows the one of [31, Theorem 1.4] for the case $\kappa = h_\Omega$, while the one about the minimal minimizer relies on Proposition 2.1.

We first need to prove that for $\kappa > h_\Omega$, Proposition 2.1 applies, i.e. that Ω^r with $r = \kappa^{-1}$ has nonempty interior. We do so in the next lemma.

Lemma 5.1. *Let Ω be a Jordan domain and let $\kappa \ge h_\Omega$. Assume that Ω has no necks of radius $r = \kappa^{-1}$, then* $\mathrm{int}(\Omega^r)$ *is not empty.*

Proof. Take any $\kappa > h_\Omega$, and let E_{h_Ω} be a Cheeger set of Ω. By Proposition 3.6 (iii) there exists a ball B of radius $1/h_\Omega$ such that $B \subset E_{h_\Omega} \subset \Omega$. Hence, for all $\kappa > h_\Omega$, one has that $\Omega^{1/\kappa}$ contains at least a ball of radius $1/h_\Omega - 1/\kappa$.

We now settle the case $\kappa = h_\Omega$. By the first part we already know that $\Omega^r \neq \emptyset$, because E_{h_Ω} contains at least a ball of radius $r = h_\Omega^{-1}$. Argue by contradiction and suppose that $\mathrm{int}(\Omega^r) = \emptyset$, i.e. Ω^{1/h_Ω} is, by Proposition 2.1 (a), a (possibly constant) $\mathrm{C}^{1,1}$ curve homeomorphic to a closed segment, thus $|\Omega^{1/h_\Omega}| = 0$. This contradicts the inner Cheeger formula (1.3) which states $|\Omega^{1/h_\Omega}| = \pi h_\Omega^{-2}$. ☐

Remark 5.2. Notice that in the proof of the above lemma we use that Ω has no necks of radius κ^{-1} only in the case $\kappa = h_\Omega$. We believe that this is not necessary but we do not have an immediate proof of this fact. In any case, the assumption that Ω is a Jordan domain cannot be avoided: one needs it to apply Proposition 3.6 (iii). Moreover, in the case $\kappa = h_\Omega$ the claim surely fails without such a hypothesis: a counterexample is given by annuli, or more generally by curved annuli [28].

Lemma 5.3. *Let Ω be a Jordan domain with no necks of radius r, and assume that* $\mathrm{int}(\Omega^r) \neq \emptyset$*. Then, the compact sets*

$$C_t = \overline{\mathrm{int}(\Omega^r)} \cup \bigcup_{\gamma\in\Gamma_r^2} \gamma \cup \bigcup_{\gamma\in\Gamma_r^1} \gamma([0,t])\,, \qquad t \in [0,1]\,,$$

are such that $\mathrm{reach}(C_t) \geq r$*. Moreover, they are simply connected.*

For the sake of completeness we recall the definition of reach for a closed set A, which was introduced in the seminal paper [15]. The reach of a closed set A is

$$\mathrm{reach}(A) = \sup\{r : \forall x \in A \oplus B_r\,,\ x \text{ has a unique projection onto } A\}.$$

Proof. Notice that for $t = 1$ the claim corresponds to [31, Lemma 5.1]. To prove the claim we need to show that all the points in

$$A_t = \left(\overline{\mathrm{int}(\Omega^r)} \cup \bigcup_{\gamma\in\Gamma_r^2} \gamma \cup \bigcup_{\gamma\in\Gamma_r^1} \gamma([0,t])\right) \oplus B_r\,, \tag{5.1}$$

have a unique projection on C_t, for all $t \in [0,1]$. Let $\bar t > 0$ and $\gamma \in \Gamma_r^1$ be fixed. First, consider any point x in the strip $\mathcal{S}(\gamma_{|(0,\bar t)})$ defined as in (4.3). We can split its boundary as $\partial_L\mathcal{S}(\gamma_{|(0,\bar t)}) \cup [z_0^+, z_0^-] \cup [z_{\bar t}^+, z_{\bar t}^-]$, where $\partial_L\mathcal{S}(\gamma_{|(0,\bar t)})$ is defined as in (4.4), $z_t^\pm$ as in (4.2), and $[p,q]$ denotes the segment with endpoints p and q. Argue by contradiction and suppose that x has not a unique projection on $C_{\bar t}$. As it has unique projection on C_1, say z, one has $z = \widetilde\gamma(\tau)$ for some $\widetilde\gamma \in \Gamma_r^1$ and $\tau > \bar t$. Clearly all points on the segment $[x,z]$ project on C_1 onto z. If we show that this segment cannot cross $\partial\mathcal{S}(\gamma)$ we get a contradiction. Trivially, the segment cannot cross the lateral boundary $\partial_L\mathcal{S}(\gamma)$, as this is a subset of $\partial A_1 \cap \partial\Omega$ and those points have distance r from C_1. Furthermore, since the balls $B_r(z_t^\pm)$ with $t = 0, \bar t$ are disjoint from C_1, the segment $[x,z]$ cannot cross $[z_0^+, z_0^-]$ but in $\gamma(0)$ (equivalently, $[z_{\bar t}^+, z_{\bar t}^-]$ but in $\gamma(\bar t)$) which gives a contradiction. We remark as well that the points $\gamma(t) + \rho\nu(t)^\perp$ with $\rho < r$ project uniquely onto $\gamma(t)$, thanks to well-known properties of the strip (see [32]).

Second, consider x in the open half-ball $B_r^+(\gamma(\bar t))$ defined in (4.1). Arguing as in the last part of the proof of Proposition 2.1 we find that x projects uniquely on $\gamma(\bar t)$. Indeed, suppose that $x \in B_r^+(\gamma(\bar t))$ has a projection $z \in C_{\bar t}$ with $z \neq \gamma(\bar t)$. As Ω has no necks of radius r we find a simple curve σ that runs from $\gamma(\bar t)$ to z. We claim that the loop obtained by concatenating σ and the segments $[\gamma(\bar t), x]$, $[x,z]$ contains either $z_{\bar t}^+$ or $z_{\bar t}^-$ against the simple connectedness of Ω. This follows again by noticing that σ cannot go across the open segments $[z_{\bar t}^\pm, \gamma(\bar t)]$ and cannot intersect any point on $\gamma((\bar t, 1])$ since none of these can be connected to z without passing through $\gamma(\bar t)$.

Third, we are left with showing that the points in

$$D = A_0 \setminus \bigcup_{\gamma\in\Gamma_r^1} B_r^+(\gamma(0)),$$

have unique projection on C_t, for all t. Notice that for all $\gamma \in \Gamma_r^1$ the set D is pairwise disjoint with: (i) the strip $\mathcal{S}(\gamma)$ defined in (4.3); (ii) the open half-ball $B_r^+(\gamma(1))$. Therefore, any $x \in D$ cannot be of the form

$$\begin{aligned} &\gamma(t) \pm \rho\nu(t)^\perp, && t\in(0,1), \rho\in[0,r),\\ &\gamma(1) \pm \rho\nu, && \nu : \nu\cdot\nu(1)\geq 0, \rho\in[0,r). \end{aligned} \tag{5.2}$$

Fix a point $x \in D$. As $D \subset C_1 \oplus B_r$, x has a unique projection z on $(C_1 \oplus B_r)^r = C_1$, and $\mathrm{dist}(x; C_1) < r$. We claim that $z \in C_0 \subset C_1$, which would imply the uniqueness of the projection of x on C_t, for all t. This is equivalent to say that $z \notin \bigcup_{\Gamma_r^1} \gamma((0,1])$. By contradiction suppose that $z = \gamma(t)$ for some $\gamma \in \Gamma_r^1$ and $t \in (0,1]$. Necessarily all points on the segment $[x, \gamma(t)]$ project on $\gamma(t)$. If $t < 1$, this implies that the segment has direction $\nu(t)^\perp$ by orthogonality. If $t = 1$, this implies that the segment has direction ν such that $\nu\cdot\nu(1) \geq 0$. Hence, as $\mathrm{dist}(x; C_1) < r$, x is of the form given in (5.2), against the assumption.

We are left with showing that C_t is simply connected for all $t \in [0,1]$. As Ω has no necks of radius r and by the definition of C_t, we infer the path-connectedness of C_t. The simple connectedness is then a straightforward consequence of Ω being a Jordan domain, thus simply connected. ☐

We are now ready to prove our main theorem.

Proof of Theorem 2.3. As the proof of the structure of the maximal minimizer is substantially the same of the case $\kappa = h_\Omega$ detailed in [31, Theorem 1.4], we here only sketch it. Let E_κ^M be the maximal minimizer. By Proposition 3.6 (iii) E_κ^M contains a ball of radius $r = \kappa^{-1}$. The assumption of no necks of radius r coupled with Lemma 3.5 gives the inclusion $E_\kappa^M \supseteq \Omega^r \oplus B_r$.

To show the opposite inclusion one argues by contradiction. Yet, this part is much more technical and requires using tools such as the structure of the *cut-locus* and the characterization of *focal points*. Since these play no role in this article besides this part of the proof, we do not comment further and we simply refer the interested reader to the original proof for $\kappa = h_\Omega$ available in [31, Theorem 1.4] which can be followed step by step.

Let us now discuss the structure of the minimal minimizer. We split the proof in three steps. By Lemma 5.1 and Proposition 2.1, we know that $\Omega^r \setminus \overline{\mathrm{int}(\Omega^r)}$ consists of the two (possibly empty) families Γ_r^1 and Γ_r^2 satisfying properties (i)–(v) of Proposition 2.1 (b). According to the notation introduced in Lemma 5.3 we denote by A_t the set defined in (5.1). Hence, we aim to prove that A_0 is the unique minimal minimizer of $\mathcal{F}_\kappa$.

Step (i). For each $t \in [0,1]$ the set A_t is a minimizer. Notice that for $t = 1$, $A_1 = E_\kappa^M$, thus the minimality is trivially true. According again to the notation and to the statement of Lemma 5.3, $A_t = C_t \oplus B_r$ and the set C_t is such that $\mathrm{reach}(C_t) \geq r$ and it is simply connected. Hence, by Steiner's formulas (see [15] and [31, Section 2.3]) we have

$$|A_t| = |C_t| + r\mathcal{M}_o(C_t) + \pi r^2\,, \qquad P(A_t) = \mathcal{M}_o(C_t) + 2\pi r\,,$$

where $\mathcal{M}_o(F)$ is the *outer Minkowski content of* F, i.e.

$$\mathcal{M}_o(F) = \lim_{r\to 0} \frac{|F \oplus B_r| - |F|}{r}\,.$$

As $r = \kappa^{-1}$ and $|C_t| = |C_1|$ for all $t \in [0,1]$ it is immediate to check that $\mathcal{F}_\kappa[A_t] = \mathcal{F}_\kappa[E^M_\kappa]$, for all $t \in [0,1]$ which yields the claim.

Step (ii). Let $\kappa > h_\Omega$. By Proposition 3.2 the minimal minimizer is unique, thus we necessarily have $A_0 \supseteq E^m_\kappa$. Let us suppose that the inclusion is strict, and let $p \in A_0 \setminus \overline{E^m_\kappa}$. By definition of A_0 we find y either in $\overline{\mathrm{int}(\Omega^r)}$ or in γ for some $\gamma \in \Gamma^2_r$, such that $p \in B_r(y)$. By Proposition 3.6 (iii), we find $z \in E^m_\kappa$ such that $B_r(z) \subset E^m_\kappa$. By the assumption of no necks of radius r, there is a $\mathrm{C}^{1,1}$ curve σ contained in Ω^r such that $\sigma(0) = z$ and $\sigma(1) = y$.

Let us denote by t^* the last time for which $B_r(\sigma(t)) \subset E^m_\kappa$ for all $t \le t^*$, which by hypothesis satisfies $t^* < 1$. We claim that $\partial E^m_\kappa \cap \Omega$ contains the half-circle $S^+_r(\sigma(t^*))$ of length πr. To show this, let us fix $\varepsilon > 0$ and take $t \in (t^*, 1)$ sufficiently close to t^*, such that $B_r(\sigma(t))$ is not contained in E^m_κ. Therefore, the set $B_r(\sigma(t)) \cap \partial E^m_\kappa$ is nonempty, hence we can select $x_t \in B_r(\sigma(t)) \cap \partial E^m_\kappa$ minimizing the distance from $\sigma(t)$. Let S_t be the connected component of $\partial E^m_\kappa \cap \Omega$ containing x_t, which is actually an arc of circle of radius r with endpoints on $\partial\Omega$. Since the endpoints of S_t lie outside $B_r(\sigma(t)) \cup B_r(\sigma(t^*))$, and since the boundaries of these two balls have the same curvature as S_t, we conclude that the length of S_t must be at least $\pi r - \varepsilon$, provided t and t^* are close enough. Since $\partial E^m_\kappa \cap \Omega$ has finitely many components of length greater than or equal to $\pi r - \varepsilon$, we find a sequence t_n converging to t^* such that $S = S_{t_n}$ is constant and intersects every ball $B_r(\sigma(t_n))$. Since t_n converges to t^*, the distance of S from $\partial B_r(\sigma(t^*))$ is smaller than any positive constant, hence we conclude that S is a half-circle contained in $\partial B_r(\sigma(t^*))$. By construction, we get as well that $S = S^+_r(\sigma(t^*))$.

Consequently, we have $\sigma(t^*) = \gamma(\tau^*)$ for some $\gamma \in \Gamma^2_r$ and $\tau^* \in [0,1]$. It is not restrictive to assume that $\gamma'(\tau^*) = \lambda\sigma'(t^*)$ for some $\lambda > 0$. Pick a point w arbitrarily close to $\gamma(1)$ in the connected component of $\mathrm{int}(\Omega^r)$ whose boundary contains $\gamma(1)$, and let $\widetilde{\gamma} : [0,1] \to \mathbb{R}^2$ be a $\mathrm{C}^{1,1}$ curve contained in Ω^r and connecting $\sigma(t^*)$ to w (its existence is granted by the no necks assumption). We are now ready to define a one-parameter family of minimizers, by rolling balls along $\widetilde{\gamma}$, that is, by applying the same construction as in the proof of Lemma 3.5), and by exploiting the fact that $S^*_r = S^+_r(\widetilde{\gamma}(0)) = S^+_r(\sigma(t^*))$ is contained in $\partial E^m_\kappa \cap \Omega$.

Let us define for $t \in [0,1]$ the sets

$$E_t = \bigcup_{s\in[0,t]} S^+_r(\widetilde{\gamma}(s)),$$

and

$$D_t = E^m_\kappa \cup E_t.$$

If $\overline{E_t}$ and $\overline{E^m_\kappa} \setminus \overline{S^*_r}$ are disjoint, one has (see [32, Section 3])

$$P(D_t) - P(E^m_\kappa) = 2\ell_t, \qquad |D_t \setminus E^m_\kappa| = 2r\ell_t,$$

where ℓ_t is the length of the curve $\widetilde{\gamma}$ restricted to $(0,t)$. Then, from the above formulas, D_t is a minimizer as well. Let $\tilde{t}$ be the supremum of $t \in [0,1]$ such that $\overline{E_t} \cap \overline{E^m_\kappa} \setminus \overline{S^*_r} = \emptyset$. By the lower semicontinuity of the perimeter, the set $D_{\tilde{t}}$ is still a minimizer. If $\tilde{t} = 1$, a contradiction follows immediately. Indeed D_1 would be a minimizer such that $S^+_r(\widetilde{\gamma}(1)) \subset \partial D_1 \cap \Omega$ and, at the same time, $\partial B_r(\widetilde{\gamma}(1)) \cap \partial\Omega = \emptyset$, which would contradict Proposition 3.1(i). If $0 \le \tilde{t} < 1$, there exists another connected component Σ of $\partial E^m_\kappa \cap \Omega$, such that its closure tangentially meets the closure of $S^+_r(\widetilde{\gamma}(\tilde{t}))$ at some point z. Now there are two possibilities: either $z \in \Omega$, or $z \in \partial\Omega$. In the first case we obtain a contradiction with the minimality of $D_{\tilde{t}}$, again by Proposition 3.1(i) (indeed, z would represent a non-admissible singularity for $\partial D_{\tilde{t}} \cap \Omega$). In the second case, we can "cut the cusp" formed by the two connected components at z and obtain a competitor D' of $D_{\tilde{t}}$ such that $\mathcal{F}_\kappa[D'] < \mathcal{F}_\kappa[D_{\tilde{t}}]$, which is again a contradiction. This shows that $E^m_\kappa = A_0$.

Step (iii). Let now $\kappa = h_\Omega$; just as before one sees that A_0 is a minimal minimizer, which we shall now denote by E^m_κ. We need to show that it is the unique one. Let F^m_κ be another minimal minimizer, i.e. $F^m_\kappa \cap E^m_\kappa$ is empty. By Proposition 3.6 (iii), there exist two balls of radius $r = \kappa^{-1}$, $B_1 \subset E^m_\kappa$ and $B_2 \subset F^m_\kappa$. The assumption of no necks of radius r grants us the existence of a $\mathrm{C}^{1,1}$ curve $\widetilde{\gamma}$ in Ω^r from the center of B_1 to that of B_2. Arguing as in Step (ii), we could construct a minimizer with a singular point in the interior boundary, which is again a contradiction. □

Remark 5.4. In Step (i) of the above proof, we show that we have a one-parameter family of minimizers $\{A_t\}_t$ which "interpolates" between E^m_κ and E^M_κ. Notice that this is not the only way to "grow" E^m_κ into E^M_κ but there are infinitely many as soon as $\#\Gamma^1_r > 1$. Labelling the curves $\gamma \in \Gamma^1_r$ with indexes $1, \dots, n$, we let $\boldsymbol{\theta}(t) = (\theta_1(t), \dots, \theta_n(t))$ with $\theta_i(t)$ nondecreasing, surjective functions of t from $[0,1]$ in $[0,1]$. Then, one can define the multi-parameter family $\{A_{\boldsymbol{\theta}(t)}\}_t$ for $t \in [0,1]$ as

$$A_{\boldsymbol{\theta}(t)} = \left(\overline{\mathrm{int}(\Omega^r)} \cup \bigcup_{\gamma \in \Gamma^2_r} \gamma \cup \bigcup_{i=1}^{n} \gamma_i([0, \theta_i(t)]) \right) \oplus B_r,$$

and check that these are all minimizers, by reasoning as in the proof of Lemma 5.3 and Step (i) of the proof of Theorem 2.3.

Corollary 5.5. *Let Ω be a Jordan domain with $|\partial\Omega| = 0$, and let r be the unique positive solution of $\pi\rho^2 = |\Omega^\rho|$. If Ω has no necks of radius r, then, $h_\Omega = r^{-1}$.*

Proof. We start noticing that there is a unique positive r such that the equality $\pi r^2 = |\Omega^r|$ holds. This immediately follows from the fact that $\pi\rho^2$ is continuous and strictly increasing, while $|\Omega^\rho|$ is continuous and decreasing. By hypothesis Ω has no necks of radius r, and therefore Ω^r is path-connected. Moreover, as Ω is simply connected, it is easy to see that Ω^r is as well. Recall that by [31, Lemma 5.1] this implies that Ω^r has reach at least r. Therefore, by Steiner's formulas we have

$$P(\Omega^r \oplus B_r) = \mathcal{M}_o(\Omega^r) + 2\pi r, \qquad |\Omega^r \oplus B_r| = |\Omega^r| + r\mathcal{M}_o(\Omega^r) + \pi r^2.$$

The hypothesis $|\Omega^r| = \pi r^2$, paired with the above equalities, implies that

$$P(\Omega^r \oplus B_r) - \frac{1}{r}|\Omega^r \oplus B_r| = 0. \tag{5.3}$$

Therefore, the Cheeger constant of Ω is bounded from above by r^{-1}. By Theorem 2.3, we immediately find that the set $\Omega^r \oplus B_r$ minimizes the prescribed curvature functional $\mathcal{F}_{r^{-1}}$. Then, argue by contradiction and suppose that $h_\Omega < r^{-1}$. As $\min \mathcal{F}_\kappa < 0$ for $\kappa > h_\Omega$, we get

$$P(\Omega^r \oplus B_r) - \frac{1}{r}|\Omega^r \oplus B_r| < 0,$$

against (5.3). □

Corollary 5.6. *Let Ω be a Jordan domain such that $|\partial\Omega| = 0$ and let $\kappa_2 > \kappa_1 \geq h_\Omega$. If Ω has no necks of radius κ_2^{-1} and κ_1^{-1}, then one has*

$$E^M_{\kappa_2} \supseteq E^m_{\kappa_2} \supseteq E^M_{\kappa_1} \supseteq E^m_{\kappa_1}.$$

Proof. Let $r_i = \kappa_i^{-1}$ for $i = 1, 2$. Since $r_2 < r_1$, the set $\overline{\text{int}(\Omega^{r_2})}$ contains Ω^{r_1}, hence $\overline{\text{int}(\Omega^{r_2})} \oplus B_{r_2}$ contains $\Omega^{r_1} \oplus B_{r_1}$. Then the proof directly follows from Theorem 2.3. □

Remark 5.7. Notice that the set inclusion $E^m_{\kappa_2} \supseteq E^M_{\kappa_1}$ is strict as soon as $|\Omega| > |E^M_{\kappa_1}|$. Indeed, this strict volume bound implies that $\partial E^M_{\kappa_1} \cap \Omega$ is not empty. Assume by contradiction that $E^m_{\kappa_2} = E^M_{\kappa_1}$. Then, we infer that the interior boundary $\partial E^M_{\kappa_1} \cap \Omega = \partial E^m_{\kappa_2} \cap \Omega$, which is not empty, must have curvature equal to both κ_1 and κ_2, which is not possible.

Remark 5.8. If one assumes that Ω is a Jordan domain with $|\partial\Omega| = 0$ and that has no necks of radius r for all r, then the solution of (1.2) is unique for almost every $\kappa \geq h_\Omega$, i.e. there are at most countably many κ for which uniqueness does not hold. For the sake of completeness, we remark that this is equivalent to say that there are at most countably many $r \leq \text{inr}(\Omega)$ such that Γ^1_r is not empty. To see this, let us set

$$V_\kappa = |E^M_\kappa| - |E^m_\kappa| = |E^M_\kappa \setminus E^m_\kappa|.$$

For all κ such that uniqueness does not hold, one has $V_\kappa > 0$. At the same time Corollary 5.6 implies that for $\kappa_2 > \kappa_1$ the sets $E^M_{\kappa_1} \setminus E^m_{\kappa_1}$ and $E^M_{\kappa_2} \setminus E^m_{\kappa_2}$ are pairwise disjoint. Therefore, there are at most countably many κ such that $V_\kappa > 0$. An example of set admitting countably many values κ such that uniqueness fails is the “ziggurat” in Figure 4, built as follows. First, let Q be the square $Q = [-2^{-1}, 2^{-1}] \times [0, 1]$ and let $f(n)$ be the sequence

$$f(1) = \frac{1}{2}, \qquad f(n) = \frac{1}{2} + \sum_{i=2}^{n} \frac{1}{2^{i-1}}, \quad \forall n > 1.$$

Let then Q^+_n and Q^-_n be the squares

$$\begin{aligned} Q^+_n &= [f(n), f(n+1)] \times [0, 2^{-n}], \\ Q^-_n &= [-f(n+1), -f(n)] \times [0, 2^{-n}]. \end{aligned}$$

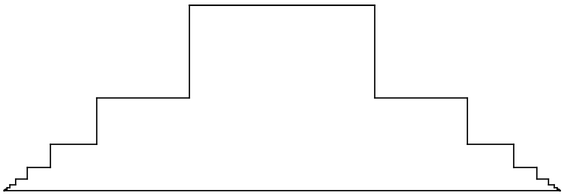


FIGURE 4. A ziggurat: a set with no necks of radius r for all r such that it has infinitely (uncountably) many solutions to the prescribed curvature equation relative to infinitely (countably) many values of κ.

We define the ziggurat as (the interior of)

$$Q \cup \bigcup_{n\geq 1} (Q_n^+ \cup Q_n^-).$$

The resulting set has no necks of radius r for all $r \leq \mathrm{inr}(\Omega)$ and it is such that $\Gamma_r^1 \neq \emptyset$ whenever $r = 2^{-n-1}$ for any $n \in \mathbb{N}$. Therefore, uniqueness of minimizers of $\mathcal{F}_\kappa$ fails whenever $\kappa = 2^{n+1} \geq h_\Omega$.

Similar examples are given by suitable fractals, e.g. by a *square Koch snowflake*, i.e. the set obtained replacing the sides of a unit square with suitable quadratic type 1 Koch curves (e.g. by iteratively replacing each middle n-th part of a segment with a square, with $n > 3$).

6. PROOF OF THEOREM 2.4

In this section we exploit Theorem 2.3 to describe minimizers of the isoperimetric profile $\mathcal{J}$, relatively to a Jordan domain Ω, with $|\partial\Omega| = 0$ and such that it has no necks of radius r for all $r \leq h_\Omega^{-1}$. In particular we shall show that for any volume V greater than or equal to $|E_{h_\Omega}^m|$, there exist a suitable κ and a suitable minimizer E_κ of $\mathcal{F}_\kappa$ such that $|E_\kappa| = V$, hence $\mathcal{J}(V) = P(E_\kappa)$. As a consequence, we are able to prove that for that class of Ω the isoperimetric profile is convex for $V \geq |E_{h_\Omega}^m|$, by showing that it is the Legendre transform of $\mathcal{G}\colon \kappa \mapsto -\min \mathcal{F}_\kappa$, defined for $\kappa \geq h_\Omega$. Trivially, $\mathcal{J}$ (resp. $\mathcal{J}^2$) is concave (resp. convex) for $V \leq |B_R|$, where $R = \mathrm{inr}(\Omega)$; it would be of interest managing to prove that $\mathcal{J}^2$ is convex on the whole range $[0, |\Omega|]$, which up to our knowledge has not been addressed when considering the total perimeter $P(E)$. For the sake of completeness, we recall that the square of the *relative* isoperimetric profile (i.e. of the infimum of the relative perimeter $P(E;\Omega)$ under volume constraint V) is known to be concave in convex bodies (see [29] and [33]).

In order to prove such a theorem we need first the following technical lemma which ensures the semicontinuity of the outer Minkowski content of Ω^r and $\overline{\mathrm{int}(\Omega^r)}$, whenever Ω has no necks of radius r for all $r < R$, for a fixed $R > 0$.

Lemma 6.1. *Let* $R > 0$ *be fixed and let* Ω *be a Jordan domain with no necks of radius* r *for all* $r < R$*. Then the functions*

$$m(r) = \mathcal{M}_o(\Omega^r), \qquad \mu(r) = \mathcal{M}_o(\overline{\mathrm{int}(\Omega^r)})$$

are, respectively, upper semicontinuous and lower semicontinuous on $(0, R)$*.*

Proof. We assume without loss of generality that Ω^r is not empty for all $0 < r < R$. By [31, Lemma 5.1] we know that $\mathrm{reach}(\Omega^r) \geq r$. Moreover, Ω^r is simply connected, hence by Steiner's formulas we have for all $0 < \varepsilon < r$

$$|\Omega^r \oplus B_\varepsilon| = |\Omega^r| + \varepsilon m(r) + \pi\varepsilon^2,$$

whence

$$m(r) = \frac{|\Omega^r \oplus B_\varepsilon| - |\Omega^r|}{\varepsilon} + \pi\varepsilon. \tag{6.1}$$

Fix now $r_0 \in (0, R)$ and $0 < \varepsilon < r_0$. Thanks to (6.1), the upper semicontinuity of $m(r)$ at r_0 follows as soon as we prove that

$$\limsup_{r\to r_0} \alpha_\varepsilon(r, r_0) + \beta(r, r_0) \leq 0,$$

where we have set

$$\alpha_\varepsilon(r, r_0) = |\Omega^r \oplus B_\varepsilon| - |\Omega^{r_0} \oplus B_\varepsilon|, \qquad \beta(r, r_0) = \alpha_0(r, r_0) = |\Omega^r| - |\Omega^{r_0}|.$$

The fact that $\limsup_{r\to r_0} \beta(r, r_0) \leq 0$ immediately follows from the following simple observations. First, as $r \to r_0^-$, we have $\Omega^r \supset \Omega^{r_0}$ and $|\Omega^r \setminus \Omega^{r_0}| \to 0$, so that in particular $|\Omega^r| \to |\Omega^{r_0}|$. Second, as $r \to r_0^+$, we have $\Omega^r \subset \Omega^{r_0}$ and thus $|\Omega^r| \leq |\Omega^{r_0}|$. We are left with showing $\limsup_{r\to r_0} \alpha_\varepsilon(r, r_0) \leq 0$. We first consider the case of the left upper limit, i.e. when $r \to r_0^-$. In this case we have $\Omega^{r_0} \oplus B_\varepsilon \subset \Omega^r \oplus B_\varepsilon$ by monotonicity. Moreover,

$$\lim_{r\uparrow r_0} (\Omega^r \oplus B_\varepsilon) \setminus (\Omega^{r_0} \oplus B_\varepsilon) = \bigcap_{r<r_0} (\Omega^r \oplus B_\varepsilon) \setminus (\Omega^{r_0} \oplus B_\varepsilon) \subset \partial(\Omega^{r_0} \oplus B_\varepsilon), \tag{6.2}$$

where to prove the last inclusion one can rely on the fact that Ω^r converges to Ω^{r_0} w.r.t. the Hausdorff distance as $r \to r_0^-$. Since $\varepsilon < r_0$ the set $\partial(\Omega^{r_0} \oplus B_\varepsilon)$ is Lipschitz, thus it has zero Lebesgue measure. Hence by (6.2), we find

$$\limsup_{r\to r_0^-} \alpha_\varepsilon(r, r_0) \leq |\partial(\Omega^{r_0} \oplus B_\varepsilon)| = 0.$$

Concerning the right upper limit, i.e. when $r \to r_0^+$, we simply observe that $\alpha_\varepsilon(r, r_0) \leq 0$ whenever $r > r_0$ by monotonicity, hence a fortiori we obtain the desired lim sup inequality. This completes the proof of the upper semicontinuity of $m(r)$.

We now set $W^r = \overline{\mathrm{int}(\Omega^r)}$. By Remark 4.2 one has $\Gamma_r^2 = \emptyset$, thus by Lemma 5.3 we know that W^r is simply connected and $\mathrm{reach}(W^r) \geq r$. If we denote by $\xi : \overline{W^r \oplus B_{r/2}} \to W^r$ the unique projection map onto W^r (which is well defined by the reach property of W^r), we infer by [15, Theorem 4.8] that the restriction of ξ to the boundary of $W^r \oplus B_{r/2}$ is 2-Lipschitz and its image is the boundary of W^r. This shows that ∂W^r is the Lipschitz image of a smooth curve with finite length (indeed, the boundary of $W^r \oplus B_{r/2}$ is of class $\mathrm{C}^{1,1}$ with bounded curvature). Consequently, we have that $\mu(r) = \mathcal{H}^1(\partial W^r) = P(W^r)$, where the first equality follows from [16, Section 3.2.39] and the second from the fact that ∂W^r is a continuous curve.

At the same time, the mapping $r \mapsto W^r$ is continuous w. r. t. the L^1-topology, which can be proved by observing that W^r is Lebesgue equivalent to both $\mathrm{int}(\Omega^r)$ (a consequence of $\mathcal{H}^1(\partial W^r) < +\infty$) and Ω^r (a consequence of Proposition 2.1). We thus conclude that $\mu(r)$ is lower semicontinuous on $(0, R)$. ☐

We are now ready to prove the result concerning the minimizers of the isoperimetric profile. Since we know by Theorem 2.3 the structure of E^m_κ and of E^M_κ, it is easy to show that there is a family of minimizers with volumes spanning the range from $|E^m_\kappa|$ to $|E^M_\kappa|$. Actually, this has already been done in Step (i) of Theorem 2.3, where we proved that the sets

$$A_t = \left(\overline{\mathrm{int}(\Omega^r)} \cup \bigcup_{\gamma \in \Gamma^2_r} \gamma \cup \bigcup_{\gamma \in \Gamma^1_r} \gamma([0,t]) \right) \oplus B_r \,,$$

with $r = \kappa^{-1}$, are minimizers. It is straightforward that $|A_t|$ is a continuous function in t from $[0, 1]$ to $[|E^m_\kappa|, |E^M_\kappa|]$. Thus, the proof boils down to show that there are no volume gaps when increasing the value of κ, and this is exactly what the previous lemma is needed for.

Proof of Theorem 2.4. We can directly assume that $|\Omega| > |E^m_{h_\Omega}|$, otherwise there is nothing to prove. If V equals either $|\Omega|$ or $|E^m_{h_\Omega}|$, there is nothing to prove. Then, let $|\Omega| > V > |E^m_{h_\Omega}|$ and set κ^* and κ_* as follows

$$\kappa^* = \inf\{\, \kappa : |E^M_\kappa| > V \,\}, \qquad \kappa_* = \sup\{\, \kappa : |E^m_\kappa| < V \,\}.$$

Since the functions $\kappa \mapsto |E^M_\kappa|$ and $\kappa \mapsto |E^m_\kappa|$ are nondecreasing, and for all κ one has $|E^M_\kappa| \geq |E^m_\kappa|$, the inequality $\kappa^* \geq \kappa_*$ obviously holds. We first show that these two values agree. Suppose that $\kappa^* > \kappa_*$. Then, for any $\kappa \in (\kappa_*, \kappa^*)$ one can consider the minimizers of $\mathcal{F}_\kappa$. On the one hand, the lower bound $\kappa > \kappa_*$ implies $|E^m_\kappa| \geq V$, while the upper bound $\kappa < \kappa^*$ implies $|E^M_\kappa| \leq V$. As $|E^M_\kappa| \leq |E^m_\kappa|$, we immediately find that $|E^M_\kappa| = |E^m_\kappa| = V$ for all $\kappa \in (\kappa_*, \kappa^*)$. The strict nestedness granted by Corollary 5.6 and Remark 5.7 immediately yields a contradiction.

Hence, $\kappa^* = \kappa_* = \hat\kappa$. Setting $\hat r = 1/\hat\kappa$, by Steiner's formulas we have that

$$|E^M_{\hat\kappa}| = \pi \hat r^2 + \hat r \mathcal{M}_o(\Omega^{\hat r}) + |\Omega^{\hat r}|\,, \quad |E^m_{\hat\kappa}| = \pi \hat r^2 + \hat r \mathcal{M}_o(\overline{\mathrm{int}(\Omega^{\hat r})}) + |\overline{\mathrm{int}(\Omega^{\hat r})}|\,.$$

Therefore, by Lemma 6.1 we get that

$$|E^M_{\hat\kappa}| \geq V \geq |E^m_{\hat\kappa}|\,.$$

If either one of the two inequalities is not strict, we are done. Hence, suppose that both are strict. This implies that $\Gamma^1_{\hat r}$ is not empty. The sets A_t in (5.1) give a family of minimizers with volume increasing continuously from $|E^m_{\hat\kappa}|$ up to $|E^M_{\hat\kappa}|$, thus one finds a suitable t such that $|A_t| = V$, as required. ☐

Proposition 6.2. *Let Ω be a Jordan domain such that $|\partial\Omega| = 0$. Assume that Ω has no necks of radius r, for all $r \leq h_\Omega^{-1}$. Then, the isoperimetric profile $\mathcal{J}$ for $V \geq |E^m_{h_\Omega}|$ is the Legendre transform of $\mathcal{G}\colon \kappa \mapsto -\min \mathcal{F}_\kappa$ defined for $\kappa \geq h_\Omega$.*

Proof. First, we prove that the map $\mathcal{G}$ is convex. Set $\kappa_1, \kappa_2 \geq h_\Omega$ and consider the convex combination $\hat{\kappa} = t\kappa_1 + (1-t)\kappa_2$. As usual we denote by E_κ a minimizer of $\mathcal{F}_\kappa$. Then, one has

$$\begin{aligned}\mathcal{G}(t\kappa_1 + (1-t)\kappa_2) &= \mathcal{G}(\hat{\kappa}) = -\min \mathcal{F}_{\hat{\kappa}} = -P(E_{\hat{\kappa}}) + \hat{\kappa}|E_{\hat{\kappa}}| \\ &= -t(P(E_{\hat{\kappa}}) - \kappa_1|E_{\hat{\kappa}}|) - (1-t)(P(E_{\hat{\kappa}}) - \kappa_2|E_{\hat{\kappa}}|) \\ &\leq -t \min \mathcal{F}_{\kappa_1} - (1-t)\min \mathcal{F}_{\kappa_2} \\ &= t\mathcal{G}(\kappa_1) + (1-t)\mathcal{G}(\kappa_2).\end{aligned}$$

Therefore, one can consider the Legendre transform of $\mathcal{G}$. By definition we have

$$\mathcal{G}^*(V) = \sup_{\kappa \geq h_\Omega} \{\, \kappa V - \mathcal{G}(\kappa)\,\} = \sup_{\kappa \geq h_\Omega} \left\{\, \kappa V + \min\{P(E) - \kappa|E|\,\}\right\}.$$

By Theorem 2.4 for all $V \geq |E^m_{h_\Omega}|$ there exist $\bar{\kappa} \geq h_\Omega$ and $E_{\bar{\kappa}}$ minimizer of $\mathcal{F}_{\bar{\kappa}}$ with $|E_{\bar{\kappa}}| = V$ and such that $\mathcal{J}(V) = P(E_{\bar{\kappa}})$. Hence, on the one hand

$$\begin{aligned}\mathcal{G}^*(V) &\geq \bar{\kappa}V + \min\{\, P(E) - \bar{\kappa}|E|\,\} \\ &= \bar{\kappa}V + P(E_{\bar{\kappa}}) - \bar{\kappa}|E_{\bar{\kappa}}| = P(E_{\bar{\kappa}}) = \mathcal{J}(V).\end{aligned}$$

On the other hand, for all κ we have

$$\kappa V - \mathcal{G}(\kappa) = \kappa V + \min \mathcal{F}_\kappa \leq \kappa V + P(E_{\bar{\kappa}}) - \kappa|E_{\bar{\kappa}}| = P(E_{\bar{\kappa}}).$$

Thus,

$$\begin{aligned}\mathcal{G}^*(V) &\leq \sup_{\kappa \geq h_\Omega} \{\, \kappa V + P(E_{\bar{\kappa}}) - \kappa|E_{\bar{\kappa}}|\,\} \\ &= \sup_{\kappa \geq h_\Omega} \{\, P(E_{\bar{\kappa}})\,\} = P(E_{\bar{\kappa}}) = \mathcal{J}(V),\end{aligned}$$

and the claim follows at once. ☐

Since the Legendre transform maps convex functions in convex functions, one has the following corollary.

Corollary 6.3. *Let Ω be a Jordan domain such that $|\partial\Omega| = 0$. If Ω has no necks of radius r for all $r \in (0, h_\Omega^{-1}]$, then the isoperimetric profile $\mathcal{J}$ is convex in $[|E^m_{h_\Omega}|, |\Omega|]$.*

Remark 6.4. Notice that whenever $\Gamma^1_r \neq \emptyset$, $\mathcal{J}$ is linear in the interval of volumes delimited by $|E^m_{r^{-1}}|$ and $|E^M_{r^{-1}}|$. There are sets with no necks of radius r for all r that display such linear growth on countably many intervals of smaller and smaller size. Think of the ziggurat described in Remark 5.8 and shown in Figure 4. Moreover, Remark 5.4 shows that nestedness of minimizers of the isoperimetric profile is not ensured — even though a nested family can always be chosen. This is achieved, for instance, by interpolating between E^m_κ and E^M_κ, always through the family $\{A_t\}_t$ defined in (5.1).

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

Acknowledgements

The authors would like to thank Hugo Lavenant, Manuel Ritoré, and Aldo Pratelli for fruitful conversations and for their comments on a preliminary version of the paper.

References

[1] F. Alter, V. Caselles, and A. Chambolle. A characterization of convex calibrable sets in $\mathbb{R}^n$. *Math. Ann.*, 332(2):329–366, 2005. `doi:10.1007/s00208-004-0628-9`.

[2] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, and M. R. Posteraro. Some isoperimetric inequalities on $\mathbb{R}^n$ with respect to weights $|x|^\alpha$. *J. Math. Anal. Appl.*, 451(1):280–318, 2017. `doi:10.1016/j.jmaa.2017.01.085`.

[3] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, and M. R. Posteraro. On weighted isoperimetric inequalities with non-radial densities. *Appl. Anal.*, 98(10):1935–1945, 2019. `doi:10.1080/00036811.2018.1506106`.

[4] L. Ambrosio, V. Caselles, S. Masnou, and J.-M. Morel. Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc. (JEMS)*, 3(1):39–92, 2001. `doi:10.1007/PL00011302`.

[5] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discountinuity Problems*. Oxford University Press, 2000.

[6] E. Barozzi, E. Gonzalez, and U. Massari. The mean curvature of a Lipschitz continuous manifold. *Rend. Mat. Acc. Lincei*, 14(4):257–277, 2003. URL: `http://www.bdim.eu/item?id=RLIN_2003_9_14_4_257_0`.

[7] F. Bethuel, P. Caldiroli, and M. Guida. Parametric surfaces with prescribed mean curvature. *Rend. Sem. Mat. Univ. Torino*, 60(4):175–231, 2002. URL: `http://www.seminariomatematico.unito.it/rendiconti/60-4.html`.

[8] G. Buttazzo and E. Stepanov. Optimal transportation networks as free Dirichlet regions for the Monge–Kantorovich problem. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 2(4):631–678, 2003.

[9] X. Cabré and X. Ros-Oton. Sobolev and isoperimetric inequalities with monomial weights. *J. Differential Equations*, 255:4312–4336, 2013. `doi:10.1016/j.jde.2013.08.010`.

[10] V. Caselles, M. Jr. Miranda, and M. Novaga. Total variation and Cheeger sets in Gauss space. *J. Funct. Anal.*, 259(6):1491–1516, 2010. `doi:10.1016/j.jfa.2010.05.007`.

[11] J. T. Chen. On the existence of capillary free surfaces in the absence of gravity. *Pacific J. Math.*, 88(2):323–361, 1980. `doi:10.2140/pjm.1980.88.323`.

[12] G. Crasta and I. Fragalà. On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: regularity and geometric results. *Arch. Rational Mech. Anal.*, 218(3):1577–1607, 2015. `doi:10.1007/s00205-015-0888-4`.

[13] G. Csató. An isoperimetric problem with density and the Hardy Sobolev inequality in $\mathbb{R}^2$. *Diff. Int. Equations*, 28(9–10):971–988, 2015.

[14] D. De Ford, H. Lavenant, Z. Schutzman, and J. Solomon. Total variation isoperimetric profiles. *SIAM J. Appl. Algebra Geom.*, 3(4):585–613, 2019. `doi:10.1137/18M1215943`.

[15] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93(3):418–491, 1959. `doi:10.2307/1993504`.

[16] H. Federer. *Geometric Measure Theory*, volume 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York Inc., New York, 1969.

[17] R. Finn and E. Giusti. Existence and non existence of capillary surfaces. *Manuscripta Math.*, 28(1–3):1–11, 1979. `doi:10.1007/BF01647961`.

[18] R. Finn and E. Giusti. Nonexistence and existence of capillary surfaces. *Manuscripta Math.*, 28(1–3):13–20, 1979. `doi:10.1007/BF01647962`.

[19] R. Finn and A. A. Jr. Kosmodem'yanskii. Some unusual comparison properties of capillary surfaces. *Pacific J. Math.*, 205(1):119–137, 2002. `doi:10.2140/pjm.2002.205.119`.

[20] M. Giaquinta. Regolarità delle superfici $BV(\Omega)$ con curvatura media assegnata. *Boll. Un. Mat. Ital. (4)*, 8:567–578, 1973.

[21] M. Giaquinta. On the Dirichlet problem for surfaces of prescribed mean curvature. *Manuscripta Math.*, 12:73–86, 1974. `doi:10.1007/BF01166235`.

[22] E. Giusti. On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions. *Invent. Math.*, 46(2):111–137, 1978. `doi:10.1007/BF01393250`.

[23] M. Goldman and M. Novaga. Volume-constrained minimizers for the prescribed curvature problem in periodic media. *Calc. Var. Partial Differential Equations*, 44(3–4):297–318, 2012. `doi:10.1007/s00526-011-0435-6`.

[24] E. Gonzalez, U. Massari, and I. Tamanini. Minimal boundaries enclosing a given volume. *Manuscripta Math.*, 34(2–3):381–395, 1981. `doi:10.1007/BF01165546`.

[25] M. Guida and S. Rolando. Symmetric κ-loops. *Diff. Int. Equations*, 23:861–898, 2010.

[26] X. Huang. Closed surface with prescribed mean curvature in $\mathbb{R}^3$. *Science in China*, 34(10), 1991. `doi:10.1360/ya1991-34-10-1162`.

[27] B. Kawohl and T. Lachand-Robert. Characterization of Cheeger sets for convex subsets of the plane. *Pacific J. Math.*, 225(1):103–118, 2006. `doi:10.2140/pjm.2006.225.103`.

[28] D. Krejčiřík and A. Pratelli. The Cheeger constant of curved strips. *Pacific J. Math.*, 254(2):309–333, 2011. `doi:10.2140/pjm.2011.254.309`.

[29] E. Kuwert. *Geometric Analysis and Nonlinear Partial Differential Equations*, chapter Note on the Isoperimetric Profile of a Convex Body, pages 195–200. Springer, Berlin, Heidelberg, 2003. `doi:10.1007/978-3-642-55627-2_12`.

[30] G. P. Leonardi. An overview on the Cheeger problem. In *New Trends in Shape Optimization*, volume 166 of *Internat. Ser. Numer. Math.*, pages 117–139. Springer Int. Publ., 2015. `doi:10.1007/978-3-319-17563-8_6`.

[31] G. P. Leonardi, R. Neumayer, and G. Saracco. The Cheeger constant of a Jordan domain without necks. *Calc. Var. Partial Differential Equations*, 56:164, 2017. `doi:10.1007/s00526-017-1263-0`.

[32] G. P. Leonardi and A. Pratelli. On the Cheeger sets in strips and non-convex domains. *Calc. Var. Partial Differential Equations*, 55(1):15, 2016. `doi:10.1007/s00526-016-0953-3`.

[33] G. P. Leonardi, M. Ritoré, and E. Vernadakis. Isoperimetric inequalities in unbounded convex bodies. To appear in *Mem. Amer. Math. Soc.*

[34] G. P. Leonardi and G. Saracco. The prescribed mean curvature equation in weakly regular domains. *NoDEA Nonlinear Differ. Equ. Appl.*, 25(2):9, 2018. `doi:10.1007/s00030-018-0500-3`.

[35] G. P. Leonardi and G. Saracco. Two examples of minimal Cheeger sets in the plane. *Ann. Mat. Pura Appl. (4)*, 197(5):1511–1531, 2018. `doi:10.1007/s10231-018-0735-y`.

[36] F. Maggi. *Sets of Finite Perimeter and Geometric Variational Problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. `doi:10.1017/CBO9781139108133`.

[37] U. Massari. Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.*, 55:357–382, 1974. `doi:10.1007/BF00250439`.

[38] F. Morgan and A. Ros. Stable constant-mean-curvature hypersurfaces are area minimizing in small L^1 neighborhoods. *Interfaces Free Bound.*, 12(2):151–155, 2010. `doi:10.4171/IFB/230`.

[39] E. Parini. An introduction to the Cheeger problem. *Surv. Math. Appl.*, 6:9–21, 2011. URL: `http://www.utgjiu.ro/math/sma/v06/v06.html`.

[40] A. Pratelli and G. Saracco. On the generalized Cheeger problem and an application to 2d strips. *Rev. Mat. Iberoam.*, 33(1):219–237, 2017. `doi:10.4171/RMI/934`.

[41] A. Pratelli and G. Saracco. On the isoperimetric problem with double density. *Nonlinear Anal.*, 177(Part B):733–752, 2018. `doi:10.1016/j.na.2018.04.009`.

[42] A. Pratelli and G. Saracco. The $\varepsilon - \varepsilon^\beta$ property in the isoperimetric problem with double density, and the regularity of isoperimetric sets. *Adv. Nonlinear Stud.*, Ahead of publication. `doi:10.1515/ans-2020-2074`.

[43] H. Sagan. *Space-Filling Curves*. Springer New York, 1994. `doi:10.1007/978-1-4612-0871-6`.

[44] A. Saracco and G. Saracco. A discrete districting plan. *Netw. Heterog. Media*, 14(4):771–788, 2019. `doi:10.3934/nhm.2019031`.

[45] G. Saracco. Weighted Cheeger sets are domains of isoperimetry. *Manuscripta Math.*, 156(3–4):371–381, 2018. `doi:10.1007/s00229-017-0974-z`.

[46] G. Saracco. A sufficient criterion to determine planar self-Cheeger sets, 2019. Preprint. `arXiv:1906.12101`.

[47] E. Stredulinsky and W. P. Ziemer. Area minimizing sets subject to a volume constraint in a convex set. *J. Geom. Anal.*, 7(4):653–677, 1997. `doi:10.1007/BF02921639`.

[48] P. Tilli. Some explicit examples of minimizers for the irrigation problem. *J. Convex Anal.*, 17(2):583–595, 2010. URL: `http://www.heldermann.de/JCA/JCA17/JCA172/jca17039.htm`.

[49] A. Treibergs and S. W. Wei. Embedded hyperspheres with prescribed mean curvature. *J. Differential Geom.*, 18(3):513–521, 1983. `doi:10.4310/jdg/1214437786`.

[50] S. T. Yau. Problem section. In *Seminar on Differential Geometry*, volume 102, pages 669–706. Princeton Univ. Press, Princeton, N.J., 1982.

Dipartimento di Matematica, via Sommarive 14, IT-38123 Povo - Trento (Italy)
Email address: `gianpaolo.leonardi@unitn.it`

Dipartimento di Matematica, via Ferrata 5, IT-27100 Pavia - Pavia (Italy)
Email address: `giorgio.saracco@unipv.it`