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Research Paper

Element orders and codegrees of characters in non-solvable groups ☆

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ABSTRACT

Given a finite group G and an irreducible complex character χ of G, the *codegree* of χ is defined as the integer $\mathrm{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$. It was conjectured by G. Qian in [16] that, for every element g of G, there exists an irreducible character χ of G such that $\mathrm{cod}(\chi)$ is a multiple of the order of g; the conjecture has been verified under the assumption that G is solvable ([16]) or almost-simple ([13]). In this paper, we prove that Qian's conjecture is true for every finite group whose Fitting subgroup is trivial, and we show that the analysis of

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the full conjecture can be reduced to groups having a solvable socle.

Introduction

Let G be a finite group, and let $\mathrm{Irr}(G)$ denote the set of the irreducible complex characters of G; given a character χ in $\mathrm{Irr}(G)$, the *codegree* of χ is defined (following [18]) as the integer

$$\mathrm{cod}(\chi) = \frac{|G : \ker(\chi)|}{\chi(1)}.$$

An interesting problem related to character codegrees was introduced by G. Qian in [17], and then formulated by the same author as a conjecture in [16]: it is asked whether, for every element g of G, there exists $\chi \in \mathrm{Irr}(G)$ such that $\mathrm{cod}(\chi)$ is divisible by the order of g. This conjecture, which also appears as Problem 20.78 in [12] and Problem 8.3 in [15], is proved to be true in [16] under the assumption that G is solvable.

A "modular version" of the above conjecture is also considered, and proved true for finite solvable groups, by X. Chen and G. Navarro in [4]. As for non-solvable groups, Qian's conjecture has been verified by E. Giannelli for finite symmetric and alternating groups ([7]); furthermore, S.Y. Madanha proves in [13] that the conjecture is true for finite almost-simple groups as well.

The present note is a contribution in this framework. The following main result extends the validity of Qian's conjecture to finite groups whose Fitting subgroup is trivial.

Theorem A. *Let G be a finite group whose Fitting subgroup is trivial, and let g be an element of G. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\mathrm{cod}(\chi)$ is a multiple of the order of g.*

As a further step towards a possible proof of Qian's conjecture in full generality, we will also see that a minimal counterexample would be a (non-solvable) group whose minimal normal subgroups are all abelian (Remark 3.1).

In the following discussion we will freely use basic facts concerning character theory, for which we refer to [11]; also, every group will be tacitly assumed to be a finite group.

1. Preliminaries

We start by recalling some standard facts and establishing some notation in Remark 1.1.

Remark 1.1. Let G be a group, and assume that G has a unique minimal normal subgroup M; assume also that M is non-solvable, thus $M = S_1 \times \cdots \times S_n$ where the S_i are pairwise isomorphic non-abelian simple groups.

Let $\Omega = \{S_1, \ldots, S_n\}$, $N = \mathbf{N}_G(S_1)$, and let $T = \{t_1 = 1, \ldots, t_n\}$ be a right transversal for N in G. The (transitive) action of G by conjugation on Ω, i.e. the action of G by right multiplication on the set $\{Nt_i \mid i \in \{1, \ldots, n\}\}$, defines a homomorphism $g \mapsto \sigma_g$ from G to $\mathrm{Sym}(\Omega) \cong \mathrm{Sym}(n)$; moreover, for $g \in G$ and $i \in \{1, \ldots, n\}$, the element $g_i = t_i g t_{\sigma_g(i)}^{-1}$ lies in N.

Considering the factor group $\overline{N} = N/\mathbf{C}_G(S_1)$ and adopting the bar convention, we see that

$$g \mapsto (\overline{g_1}, \ldots, \overline{g_n})\sigma_g$$

defines an injective homomorphism from G to the wreath product $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$: this homomorphism is in fact the composition map of the injective homomorphism $g \mapsto (g_1, \ldots, g_n)\sigma_g$ (see [6, 13.3]) with the natural homomorphism from $N \wr \mathrm{Sym}(n)$ onto $\overline{N} \wr \mathrm{Sym}(n)$ (of course, here we are regarding $\overline{N}$ as a subgroup of $\mathrm{Aut}(S_1)$), and the injectivity of this composition map is guaranteed by the fact that the normal core of $\mathbf{C}_G(S_1)$ in G is $\mathbf{C}_G(M) = 1$.

Identifying G with a subgroup of Γ, if α_1 is an irreducible character of S_1 (or, more generally, of a subgroup X_1 of $\mathrm{Aut}(S_1)$) we will say that $\alpha_1^{t_i}$ is the character of S_i (of $X_1^{t_i}$) corresponding to α_1.

Our proof of Theorem A relies on Lemma 1.7, concerning monolithic groups (i.e., groups having a unique minimal normal subgroup) with a non-solvable socle; as a relevant preliminary ingredient, we gather some properties of characters of non-abelian simple groups in Lemma 1.2 and Lemma 1.3.

Lemma 1.2. *Let S be a non-abelian simple group, and assume $S \not\cong \mathrm{PSL}_2(3^f)$ for any odd positive integer f. Then there exist two distinct non-principal irreducible characters α and β of S, both having an extension to $\mathrm{Aut}(S)$, such that $\dfrac{|S|}{\alpha(1)} \cdot \dfrac{|S|}{\beta(1)}$ is a multiple of the exponent of S.*

Proof. Suppose that S is an alternating group $\mathrm{Alt}(n)$ where $n \geq 7$, or a sporadic simple group or the Tits group. According to Theorem 3 and Theorem 4 in [3], there exist two non-principal irreducible characters α and β of S whose degrees are coprime, and both characters extend to $\mathrm{Aut}(S)$. Consequently, $\dfrac{|S|}{\alpha(1)} \cdot \dfrac{|S|}{\beta(1)}$ is a multiple of the order of S, thus a multiple of the exponent of S.

Consider next the case when S is a simple group of Lie type (thus including $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$), and denote by p the characteristic of S. If $p > 3$, then Theorem B in [8] guarantees the existence of a non-principal irreducible character α of S whose degree is

Table 1
Classical groups of Lie type in characteristic $p \in \{2, 3\}$.

Isomorphism type	order	$\alpha(1)$	Exponent of a Sylow p-subgroup
$A_1(q) \simeq \mathrm{SL}_2(q)$, $q = 2^{2f+1}$	$q(q^2-1)$	$q-1$	2
$A_1(q) \simeq \mathrm{PSL}_2(q)$, $q = p^{2f}$	$\dfrac{q(q^2-1)}{(2,q-1)}$	$q+1$	$2, 3$
$A_2(q) \simeq \mathrm{PSL}_3(q)$	$\dfrac{q^3(q^2-1)(q^3-1)}{(3,q-1)}$	$q(q^2+q+1)$	$2^2, 3$
$A_n(q) \simeq \mathrm{PSL}_{n+1}(q)$, $n \geq 3$	$\dfrac{q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^{i+1}-1)}{(n+1,q-1)}$	$\dfrac{q(q^n-1)}{q-1}$	$\leq np$
${}^2A_2(q^2) \simeq \mathrm{PSU}_3(q)$	$\dfrac{q^3(q^2-1)(q^3+1)}{(3,q+1)}$	$q(q^2-q+1)$	$2^2, 3$
${}^2A_n(q^2) \simeq \mathrm{PSU}_{n+1}(q)$, $n \geq 3$	$\dfrac{q^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^{i+1}-(-1)^{i+1})}{(n+1,q+1)}$	$\dfrac{q(q^n-(-1)^n)}{q+1}$	$\leq np$
$B_2(q) \simeq C_2(q) \simeq \mathrm{PSp}_4(q)$	$\dfrac{q^4(q^2-1)(q^4-1)}{(2,q-1)}$	$\dfrac{q(q^2+1)}{2}$	$2^2, 3^2$
$B_n(q) \simeq \Omega_{2n+1}(q)$, $C_n(q) \simeq \mathrm{PSp}_{2n}(q)$, $n \geq 3$	$\dfrac{q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)}{(2,q-1)}$	$\dfrac{q(q^n+1)(q^{n-1}-1)}{2(q-1)}$	$\leq (2n-1)p$
$D_n(q) \simeq \mathrm{P}\Omega_{2n}^{+}(q)$, $n \geq 4$	$\dfrac{q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)}{(4,q^n-1)}$	$\dfrac{q(q^{n-2}+1)(q^n-1)}{q^2-1}$	$\leq (2n-3)p$
${}^2D_n(q^2) \simeq \mathrm{P}\Omega_{2n}^{-}(q)$, $n \geq 4$	$\dfrac{q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1}(q^{2i}-1)}{(4,q^n+1)}$	$\dfrac{q(q^{n-2}-1)(q^n+1)}{q^2-1}$	$\leq (2n-3)p$

not divisible by p and which extends to $\mathrm{Aut}(S)$; as for β, we can choose the Steinberg character of S (whose degree is the full p-part of the order of S and which, by [19], has as extension to $\mathrm{Aut}(S)$ as well). Also in this situation $\dfrac{|S|}{\alpha(1)} \cdot \dfrac{|S|}{\beta(1)}$ is clearly a multiple of $|S|$.

To complete the proof, it remains to establish our claim for simple groups of Lie type in characteristic $p \in \{2, 3\}$. Assume first that S is a classical group of Lie type, and take β to be the Steinberg character of S: clearly $|S|/\beta(1)$ is a multiple of the p'-part of the order of any element in S. Therefore, our aim is to determine a non-principal $\alpha \in \mathrm{Irr}(S)$ such that α is extendible to $\mathrm{Aut}(S)$ and $|S|/\alpha(1)$ is divisible by the exponent of a Sylow p-subgroup of S. Such a character α is described for each isomorphism type of S in Table 1, which provides the degree of α and a bound for the exponent of a Sylow p-subgroup of S; the data of Table 1 are taken from Section 3 of [13], with the further remark that all characters listed there are unipotent except for the first two rows, hence they have an extension to $\mathrm{Aut}(S)$ by Theorem 2.4 and Theorem 2.5 in [14] (for the first two rows, see Theorem A and Lemma 5.3(ii) of [20]).

Table 2
Exceptional groups of Lie type in characteristic $p \in \{2,3\}$ (part I).

Isomorphism type	Order	Label of α and β	$\alpha(1)$ and $\beta(1)$
$G_2(q)$	$q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\phi_{2,1}$ $G_2[1]$	$\frac{1}{6}q\Phi_2^2\Phi_3$ $\frac{1}{6}q\Phi_1^2\Phi_6$
${}^3D_4(q^3)$	$q^{12}\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$	$\phi_{1,3'}$ $\phi_{2,1}$	$q\Phi_{12}$ $\frac{1}{2}q^3\Phi_2^2\Phi_6^2$
$E_6(q)$	$q^{36}\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	$\phi_{64,4}$ $D_4, 1$	$q^4\Phi_2^3\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$ $\frac{1}{2}q^3\Phi_1^4\Phi_3^2\Phi_5\Phi_9$
${}^2E_6(q^2)$	$q^{36}\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$	$\phi_{9,6}{}'$ ${}^2E_6[\theta]$	$q^6\Phi_3^2\Phi_6^3\Phi_{12}\Phi_{18}$ $\frac{1}{3}q^2\Phi_1^4\Phi_2^6\Phi_4^2\Phi_8\Phi_{10}$
$E_7(q)$	$q^{63}\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\cdot$ $\cdot\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$E_6[\theta], 1$ $\phi_{27,2}$	$\frac{1}{3}q^7\Phi_1^6\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}$ $\Phi_{14}q^2\Phi_3^2\Phi_6^2\Phi_9\Phi_{12}\Phi_{18}$
$E_8(q)$	$q^{120}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\cdot$ $\cdot\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$\phi_{8,1}$ $E_8[i]$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$ $\frac{1}{4}q^{16}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_5^2\Phi_6^4\Phi_7$ $\Phi_9\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}$
${}^2F_4(q^2)$, $q^2 = 2^{2f+1} > 2$	$q^{24}\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8^2\Phi_{12}\Phi_{24}$	ϵ' cuspidal	$q^2\Phi_{12}\Phi_{24}$ $\frac{1}{3}q^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8^2$

Table 3
Exceptional groups of Lie type in characteristic $p \in \{2,3\}$ (part II).

Isomorphism type	Order	Label of α	$\alpha(1)$	Exponent of a Sylow p-subgroup
$F_4(q)$	$q^{24}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	$\phi_{4,1}$	$\frac{1}{2}q\Phi_2^2\Phi_6^2\Phi_8$	$2^4, 3^3$
${}^2G_2(q^2)$, $q^2 = 3^{2f+1} > 3$	$q^6\Phi_1\Phi_2\Phi_4\Phi_{12}$	cuspidal	$\frac{1}{\sqrt{3}}q\Phi_1\Phi_2\Phi_4$	3^2
${}^2B_2(q^2)$, $q^2 = 2^{2f+1} > 2$	$q^4\Phi_1\Phi_2\Phi_8$	${}^2\mathrm{B}_2[a]$	$\frac{1}{\sqrt{2}}q\Phi_1\Phi_2$	2^2

In Table 2 some exceptional groups of Lie type are considered. Here, for each group S, we list two irreducible characters α and β that satisfy the conclusions of the statement. The data of Table 2 can be found in [5, Section 13.9], and the extendability of the relevant characters (which are all unipotent) is again ensured by Theorem 2.4 and Theorem 2.5 in [14].

Finally, we focus on the exceptional groups of Lie type listed in Table 3. Again we consider β as the Steinberg character of the relevant group S, and α as the character appearing in the table. It is clear that the p'-part of $|S|$ divides $\dfrac{|S|}{\alpha(1)} \cdot \dfrac{|S|}{\beta(1)}$, and we also see that the exponent of a Sylow p-subgroup of S divides the p-part of $\dfrac{|S|}{\alpha(1)} \cdot \dfrac{|S|}{\beta(1)}$. The data of Table 3 are taken from [5, Section 13.9], [13, Section 4] and, for what concerns the exponent of a Sylow p subgroup of $F_4(q)$, from [9, Theorem 3.1]; the extendability of the unipotent character α is ensured, as usual, by Theorem 2.4 and Theorem 2.5 in [14]. $\square$

Note. We are grateful to the referee for pointing out that the statement of [13, Lemma 3.8] is not accurately transcribed from the original text [2, Corollary 9]. Upon her/his recommendation, we take the opportunity to note that the formula $n(t) = \dfrac{t^{n-1}+3}{2}$ has to be corrected in $n(t) = \dfrac{p^{t-1}+3}{2}$, and $n(t) < t$ in (i) should be changed to $n(t) < n$. However, this does not affect our results.

Lemma 1.3. *Let S be a non-abelian simple group such that $S \not\cong \mathrm{PSL}_2(3^f)$ for any odd positive integer f, and let x be an element of S. Then there exists a non-principal irreducible character α of S which has an extension to $\mathrm{Aut}(S)$ and such that $|S|/\alpha(1)$ is a multiple of the order of x.*

Proof. Note first that, by the main theorem of [13], the statement is true whenever $\mathrm{Out}(S)$ is trivial, thus we may assume $\mathrm{Out}(S) \neq 1$.

Let us consider the case when S is a sporadic simple group or the Tits group. According to the isomorphism type of S, in Table 1 of [3] it is possible to find two non-principal irreducible characters of S that both extend to $\mathrm{Aut}(S)$; it can be checked that, unless S is isomorphic to Fi_{22}, one of those is suitable to be taken as a character α such that $|S|/\alpha(1)$ is a multiple of $\mathrm{o}(x)$. As for $S \cong \mathrm{Fi}_{22}$, referring to the notation of [1], an appropriate character α can be found in the set $\{\chi_2, \chi_{56}\}$.

Now, assume that S is isomorphic to an alternating group $\mathrm{Alt}(n)$ for $n \geq 7$. In this case the desired conclusion can be easily deduced from the proof of [7, Theorem A], where Qian's conjecture is established for symmetric and alternating groups; for the convenience of the reader, we sketch next the relevant argument.

Consider the prime factorization

$$\mathrm{o}(x) = 2^k \cdot p_1^{k_1} \cdots p_t^{k_t}$$

of the order of x, where $k \geq 0$ and $k_i > 0$ for every $i \in \{1, \ldots, t\}$ (taking into account that the set of odd primes $\{p_1, \ldots p_t\}$ can be empty). The proof of [7, Theorem A] yields a non-principal irreducible character α of S such that $|S|/\alpha(1)$ is a multiple of $2^{2k-1} \cdot p_1^{2k_1-1} \cdots p_t^{2k_t-1}$ if $k \neq 0$, and of $p_1^{2k_1-1} \cdots p_t^{2k_t-1}$ if $k = 0$ (hence, in any case, a multiple of $\mathrm{o}(x)$): for our purposes, it is then enough to check whether α has an extension to $\mathrm{Aut}(S) \cong \mathrm{Sym}(n)$ and, as we will see, this does happen in most cases.

In fact, depending on the prime decomposition of $\mathrm{o}(x)$, the character α is chosen as an irreducible constituent of χ_S, where $\chi \in \mathrm{Irr}(\mathrm{Sym}(n))$ is the character associated to one of the following partitions: $\lambda = (n-1, 1)$ or $\mu = (n-2, 2)$ if $k = 0$; $\nu = (2^k + 1, 1^{n-2^k-1})$ if $k \neq 0$. Observe that λ, μ and ν are not self-associated, hence χ_S lies in $\mathrm{Irr}(S)$ as we want, except for ν in the case when ($k \neq 0$ and) $n = 2^{k+1} + 1$. But in the latter case, still following the argument in the proof of [7, Theorem A], we get $p_1^{k_1} + \cdots + p_t^{k_t} \leq 2^k - 1$, thus the largest prime power that divides $o(x)$ is 2^k; since 2^k is smaller than $n-1 = 2^{k+1}$, denoting by $\chi^\lambda \in \mathrm{Irr}(\mathrm{Sym}(n))$ the character associated to the partition λ, it turns out that $|\mathrm{Sym}(n)|/\chi^\lambda(1) = 2|S|/\chi^\lambda(1)$ is a multiple of $2^{2k-1} \cdot p_1^{2k_1-1} \cdots p_t^{2k_t-1}$. We deduce that $|S|/\chi^\lambda(1)$ is a multiple of $2^{2k-2} \cdot p_1^{2k_1-1} \cdots p_t^{2k_t-1}$, which is in turn a multiple of $o(x)$ unless $k = 1$: but $k = 1$ yields $n = 5$, not our case, and the desired conclusion follows taking into account that χ_S^λ lies in $\mathrm{Irr}(S)$.

Finally, let S be a simple group of Lie type (thus including $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$). In this case, our claim is ensured by [13, Theorem 5.1] when $S \not\cong \mathrm{PSL}_2(q)$ for any prime power q. If $S \cong \mathrm{PSL}_2(p^f)$ for $p > 3$, taking into account that the order of x is either p or a p'-number, we can define α as the character provided by [8, Theorem B] or the Steinberg character of S, respectively. As for $S \cong \mathrm{SL}_2(2^f)$, or $S \cong \mathrm{PSL}_2(3^f)$ with an even f, the character α (of degree $2^f + (-1)^f$ or $3^f + 1$, respectively) is provided by Theorem A and Lemma 5.3(ii) of [20] if $o(x) = p$, or as the Steinberg character of S otherwise. □

Remark 1.4. Note that any group $S \cong \mathrm{PSL}_2(3^f)$, where $f \geq 3$ is an odd positive integer, is a genuine exception to Lemma 1.2 and Lemma 1.3. In fact, it is well known that the degrees of the irreducible characters of S are the integers in the set $\{1,\ (3^f-1)/2,\ 3^f-1,\ 3^f,\ 3^f+1\}$ (see [20], for instance); recalling that the outer automorphism group of S has order $2f$, and it is generated by a field automorphism ϕ of order f and a diagonal automorphism $\overline{\delta}$ of order 2, by Lemma 4.1, Lemma 4.5 and Lemma 4.6 of [20] the two irreducible characters of degree $(3^f-1)/2$ are both invariant under ϕ (hence they extend to $S\langle\phi\rangle$), but they are interchanged by $\overline{\delta}$. Also, Lemma 5.2(i) and Lemma 5.3(iii) in [20] show that $\langle\phi\rangle$ does not stabilize any irreducible character of S whose degree is either 3^f-1 or 3^f+1; as a consequence, the only non-principal irreducible character of S that has an extension to $\mathrm{Aut}(S)$ is the Steinberg character (of degree 3^f).

Another key ingredient for the proof of Lemma 1.7 will be the information, provided by Lemma 1.5 and Lemma 1.6, on the extendability of certain irreducible characters in a monolithic group G with non-solvable socle $M \cong S_1 \times \cdots \times S_n$. For these lemmas and for Lemma 1.7, we will assume that an injective homomorphism from G to $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$ as described in Remark 1.1 has been preliminary fixed.

Lemma 1.5. *Let G be a group having a unique minimal normal subgroup M, and assume $M = S_1 \times \cdots \times S_n$, where the S_i are pairwise isomorphic non-abelian simple groups. Let α_1 be a non-principal irreducible character of S_1 which has an extension to $\mathrm{Aut}(S_1)$ and, for every $i \in \{1, \ldots, n\}$, let α_i be the corresponding character in $\mathrm{Irr}(S_i)$. Also, for a given $h \in \{1, \ldots, n\}$, set $M_1 = S_1 \times \cdots \times S_h$ and $M_2 = S_{h+1} \times \cdots \times S_n$. Then the irreducible character $\lambda = (\alpha_1 \times \cdots \times \alpha_h) \times 1_{M_2}$ of M has an extension to its inertia subgroup $I_G(\lambda) = \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$.*

Proof. For $i \in \{1, \ldots, n\}$, define $A_i = \mathrm{Aut}(S_i)$ and set $B_1 = A_1 \times \cdots \times A_h$, $B_2 = A_{h+1} \times \cdots \times A_n$, $B = B_1 \times B_2$. Given an extension $\widehat{\alpha_1}$ of α_1 to A_1, let $\widehat{\alpha}_i$ be the corresponding character in $\mathrm{Irr}(A_i)$ and note that $\widehat{\lambda} = (\widehat{\alpha_1} \times \cdots \times \widehat{\alpha_h}) \times 1_{B_2} \in \mathrm{Irr}(B)$ is an extension of λ. Since B is the base group of the wreath product $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$, by Lemma 25.5(b) in [10] there exists an extension θ of $\widehat{\lambda}$ to its inertia subgroup $I_\Gamma(\widehat{\lambda})$. Now, viewing G as a subgroup of Γ, an element $g = (\overline{g_1}, \ldots, \overline{g_n})\sigma_g \in G$ lies in $I_G(\lambda)$ if and only if σ_g lies in $\mathrm{Stab}_{\mathrm{Sym}(n)}(\{1, \ldots, h\}) = \mathrm{Stab}_{\mathrm{Sym}(n)}(\{h+1, \ldots, n\})$, which means that g lies in $\mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$. Since $I_\Gamma(\widehat{\lambda}) = B\,\mathrm{Stab}_{\mathrm{Sym}(n)}(\{1, \ldots, h\})$ contains $I_G(\lambda)$, we get that $\theta_{I_G(\lambda)}$ is an extension of λ, as wanted. □

The following variation will take care of the exceptions to Lemma 1.2. After that, we will be in a position to prove Lemma 1.7.

Lemma 1.6. *Let G be a group having a unique minimal normal subgroup M, and assume $M = S_1 \times \cdots \times S_n$, where the S_i are all isomorphic to $\mathrm{PSL}_2(3^f)$ for a suitable odd integer $f \geq 3$. For every $i \in \{1, \ldots, n\}$, let γ_i be an irreducible character of degree $(3^f - 1)/2$ of S_i; also, fixing $h \in \{1, \ldots, n\}$, set $M_1 = S_1 \times \cdots \times S_h$ and $M_2 = S_{h+1} \times \cdots \times S_n$. Then the irreducible character $\lambda = (\gamma_1 \times \cdots \times \gamma_h) \times 1_{M_2}$ of M has an extension to its inertia subgroup $I_G(\lambda) \subseteq \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$.*

Proof. Note that the characters γ_i are not assumed to be necessarily G-conjugate. As above, for $i \in \{1, \ldots, n\}$, define $A_i = \mathrm{Aut}(S_i)$ and set $B_1 = A_1 \times \cdots \times A_h$, $B_2 = A_{h+1} \times \cdots \times A_n$, $B = B_1 \times B_2$.

Recalling that we have preliminary fixed a right transversal $\{t_1 = 1, \ldots, t_n\}$ of $\mathbf{N}_G(S_1)$ in G, for $i \in \{1, \ldots, n\}$ we define $F_i = (S_1\langle\phi_1\rangle)^{t_i}$, where ϕ_1 is a field automorphism of S_1 having order f: by Remark 1.4, we know that each of the γ_i has an extension $\widehat{\gamma_i}$ to F_i. Also, define $U = F_1 \times \cdots \times F_h \times B_2$.

Note that $\widehat{\lambda} = (\widehat{\gamma_1} \times \cdots \times \widehat{\gamma_h}) \times 1_{B_2} \in \mathrm{Irr}(U)$ is an extension of λ and, still taking into account Remark 1.4, we have $I_B(\widehat{\lambda}) = I_B(\lambda) = U$; therefore $\widehat{\lambda}^B$ is an irreducible character of B, and in fact we have $\widehat{\lambda}^B = (\widehat{\gamma_1}^{A_1} \times \cdots \times \widehat{\gamma_h}^{A_h}) \times 1_{B_2}$. As above, B being the base group of the wreath product $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$, [10, Lemma 25.5(b)] ensures that there exists an extension θ of $\widehat{\lambda}^B$ to the inertia subgroup $I_\Gamma(\widehat{\lambda}^B)$. Now, the restriction of θ to M is the sum of all the conjugates λ^b where b runs over a transversal for U in B; in particular, recalling that U coincides with $I_B(\lambda)$, every irreducible constituent of θ_M appears with multiplicity 1.

Observe that if an element $g = (\overline{g_1}, \ldots, \overline{g_n})\sigma_g$ of $G \leq \Gamma$ lies in $I_G(\lambda)$, then necessarily $\sigma_g \in \mathrm{Stab}_{\mathrm{Sym}(n)}(\{1, \ldots, h\})$. Thus, in particular, we have $I_G(\lambda) \subseteq \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$. Since $I_\Gamma(\widehat{\lambda}^B) = B\,\mathrm{Stab}_{\mathrm{Sym}(n)}(\{1, \ldots, h\}) = \mathbf{N}_\Gamma(M_1)$, we see that $I_G(\lambda)$ is contained in $I_\Gamma(\widehat{\lambda}^B)$, hence we can consider an irreducible constituent ψ of $\theta_{I_G(\lambda)}$ lying over λ. Now, ψ_M is a multiple of λ and λ appears as an irreducible constituent of ψ_M with multiplicity 1: as a consequence, $\psi \in \mathrm{Irr}(I_G(\lambda))$ is an extension of λ, and the proof is complete. □

Lemma 1.7. *Let G be a group having a unique minimal normal subgroup M, and assume $M = S_1 \times \cdots \times S_n$, where the S_i are pairwise isomorphic non-abelian simple groups.*

Also, let g be an element of G, and let r denote the order of $gM \in G/M$. Then the following conclusions hold.

(a) *If $S_1 \ncong \mathrm{PSL}_2(3^f)$ for any odd positive integer f, then there exists a non-principal character $\lambda \in \mathrm{Irr}(M)$ such that λ has an extension to $I = I_G(\lambda)$, g lies in I, and $|M|/\lambda(1)$ is a multiple of $\mathrm{o}(g^r)$.*

(b) *If $S_1 \cong \mathrm{PSL}_2(3^f)$ for some odd positive integer f, then there exist a non-principal character $\lambda \in \mathrm{Irr}(M)$ and a suitable $h \leq n$ such that λ has an extension to $I = I_G(\lambda)$ and $g^{2^h} \in I$. Furthermore, $|M|/\lambda(1)$ is a multiple of $2^h\,\mathrm{o}(g^r)$.*

Proof. Set $\Omega = \{S_1, \ldots, S_n\}$ and $K = \bigcap_{i=1}^n \mathbf{N}_G(S_i)$, so that G/K is isomorphic to a transitive subgroup of $\mathrm{Sym}(\Omega) \cong \mathrm{Sym}(n)$: up to renumbering the elements of Ω, there exists a suitable positive integer $h \leq n$ such that the set $\{S_1, S_2, \ldots, S_h\}$ is an orbit for the action of $\langle gK \rangle$ on Ω. As usual, define $M_1 = S_1 \times \cdots \times S_h$ and $M_2 = S_{h+1} \times \cdots \times S_n$ (where M_2 is meant to be trivial if $h = n$).

We start with an observation that will be useful for proving claim (b), so, let us assume $S_1 \cong \mathrm{PSL}_2(3^f)$ for a suitable odd integer $f \geq 3$; in what follows, we will consider the wreath product $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$ and its subgroups U, B as defined in Lemma 1.6, and we recall that an injective homomorphism from G to Γ as in Remark 1.1 is preliminary fixed. Also, we write $\langle g \rangle = X \times Y$, where $|X|$ is an odd number and $|Y|$ is a power of 2. Consider the set

$$\Delta = \{(\gamma_1 \times \cdots \times \gamma_h) \times 1_{M_2} \in \mathrm{Irr}(M) \quad | \quad \gamma_i \in \mathrm{Irr}(S_i) \text{ and } \gamma_i(1) = (3^f - 1)/2\}.$$

We see that both X (which normalizes M_1) and the 2-group B/U act on Δ; moreover, X acts on B/U, the orders of X and B/U are coprime, the action of B/U on Δ is transitive (in fact regular, as $|\Delta| = 2^h = |B/U|$) and we have

$$(\eta^b)^x = (\eta^x)^{b^x}$$

for every $\eta \in \Delta$, $b \in B$ and $x \in X$. Therefore, Glauberman's Lemma 13.8 in [11] yields that there exists an element λ_1 of Δ such that X lies in $I_G(\lambda_1)$; this λ_1 also has an extension to $I_G(\lambda_1)$ by Lemma 1.6. If we choose $\eta = (\gamma_1 \times \cdots \times \gamma_h) \times 1_{M_2}$ in Δ such that the γ_i are all characters corresponding to γ_1, then it is easy to see that $I_\Gamma(\eta)$ lies in $\mathbf{N}_\Gamma(M_1)$ with $|\mathbf{N}_\Gamma(M_1) : I_\Gamma(\eta)| = 2^h$; since there exists $b \in B \subseteq \mathbf{N}_\Gamma(M_1)$ such that $\lambda_1 = \eta^b$, we clearly get $|\mathbf{N}_\Gamma(M_1) : I_\Gamma(\lambda_1)| = 2^h$ as well. But then, as g lies in $\mathbf{N}_\Gamma(M_1)$, we have $|\langle g \rangle : \langle g \rangle \cap I_G(\lambda_1)| \leq |\mathbf{N}_\Gamma(M_1) : I_\Gamma(\lambda_1)| = 2^h$. Taking into account that, as we just proved, the Hall $2'$-subgroup of $\langle g \rangle$ is contained in $I_G(\lambda_1)$, it follows that $|\langle g \rangle : \langle g \rangle \cap I_G(\lambda_1)|$ is in fact a divisor of 2^h and therefore $g^{2^h} \in I_G(\lambda_1)$.

Next, it will also be useful to take into account the following remark, which holds for both (a) and (b) under the assumption that the action of $\langle gK \rangle$ on Ω is transitive (in other words, when $h = n$ and gK is identified with an n-cycle in $\mathrm{Sym}(n)$). Recalling that

r denotes the order of $gM \in G/M$, let us write $g^r = (s_1, ..., s_n) \in M$: we note that the orders of the $s_i \in S_i$ are all the same, for $i \in \{1, \ldots, n\}$. In fact, write $g = (\overline{g_1}, \ldots, \overline{g_n})\sigma_g$ as an element of the wreath product $\Gamma = \mathrm{Aut}(S_1) \wr \mathrm{Sym}(n)$. Conjugating g^r with g, we get $(s_n^{g_n}, s_1^{g_1}, \ldots, s_{n-1}^{g_{n-1}})$. This is clearly the same as g^r, so in particular $s_j = s_{j-1}^{g_{j-1}}$ for every $j \in \{2, \ldots, n\}$ and we get the desired property. As a consequence, the order of g^r is in fact the order of an element of S_1.

We can now work toward a proof of (a) and (b), and we first treat the case when the action of $\langle gK \rangle$ on Ω is *not* transitive, so that we have $1 \leq h < n$.

If S_1 is not isomorphic to $\mathrm{PSL}_2(3^f)$ for any odd positive integer f, then Lemma 1.2 yields the existence of two distinct non-principal characters α_1, $\beta_1 \in \mathrm{Irr}(S_1)$, both having an extension to $\mathrm{Aut}(S_1)$, such that $\dfrac{|S_1|}{\alpha_1(1)} \cdot \dfrac{|S_1|}{\beta_1(1)}$ is a multiple of $\exp(S_1)$. Denoting by α_i and β_i the characters of S_i corresponding to α_1 and β_1 for $i \in \{1, \ldots, n\}$, Lemma 1.5 yields that $\lambda_1 = \alpha_1 \times \cdots \times \alpha_h \times 1_{M_2}$ and $\lambda_2 = 1_{M_1} \times \beta_{h+1} \times \cdots \times \beta_n$ both extend to their inertia subgroup $I = \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$. Define now $\lambda = \lambda_1\lambda_2 \in \mathrm{Irr}(M)$: the inertia subgroup of $\lambda = \alpha_1 \times \cdots \times \alpha_h \times \beta_{h+1} \times \cdots \times \beta_n$ in G is again I (in fact, viewing G as a subgroup of Γ, an element $y = (\overline{y_1}, \ldots, \overline{y_n})\sigma_y \in G$ lies in $I_G(\lambda)$ if and only if σ_y lies in $\mathrm{Stab}_{\mathrm{Sym}(n)}(\{1, \ldots, h\}) = \mathrm{Stab}_{\mathrm{Sym}(n)}(\{h+1, \ldots, n\})$, which means $y \in I$); moreover, I contains the element g and, by [11, Theorem 6.16], λ has an extension to I. Finally, we get $\lambda(1) = \alpha_1(1)^h\beta_1(1)^{n-h}$, therefore $|M|/\lambda(1)$ is certainly a multiple of $\exp(S_1)$ and claim (a) in the non-transitive case immediately follows.

On the other hand, if $S_1 \cong \mathrm{PSL}_2(3^f)$ for a suitable odd integer $f \geq 3$, then we consider a character $\lambda_1 \in \mathrm{Irr}(M)$ as in the second paragraph of this proof: so, λ_1 has an extension to $I_G(\lambda_1)$ and $g^{2^h} \in I_G(\lambda_1)$. Also, define β_i as the Steinberg character of S_i, set $\lambda_2 = 1_{M_1} \times \beta_{h+1} \times \cdots \times \beta_n$ and observe that λ_2, whose degree is $3^{f(n-h)}$, extends to $I_G(\lambda_2) = \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$ by Lemma 1.5. Set now $\lambda = \lambda_1\lambda_2$; the inertia subgroup of λ turns out to be $I = I_G(\lambda_1)$, and λ extends to I again by Theorem 6.16 of [11]. Recalling that $|S_i| = \dfrac{(3^f-1)\cdot 3^f \cdot (3^f+1)}{2}$, we have

$$\frac{|M|}{\lambda(1)} = 2^h \cdot \frac{|S_1|}{3^f-1} \cdots \frac{|S_h|}{3^f-1} \cdot \frac{|S_{h+1}|}{3^f} \cdots \frac{|S_n|}{3^f},$$

which is certainly a multiple of $2^h \cdot \exp(S_1) = 2^h \cdot \dfrac{(3^f-1)\cdot 3 \cdot (3^f+1)}{4}$ and, in particular, of $2^h\,\mathrm{o}(g^r)$. Claim (b) is thus proved in the non-transitive case.

We move next to the case when the action of $\langle gK \rangle$ on Ω is transitive; as previously observed, in this case the order of g^r is in fact the order of an element of S_1.

If $S_1 \not\cong \mathrm{PSL}_2(3^f)$ for any odd integer $f \geq 3$ then, by Lemma 1.3, there exists an irreducible character $\alpha_1 \in \mathrm{Irr}(S_1)$ such that $|S_1|/\alpha_1(1)$ is a multiple of $\mathrm{o}(g^r)$ and α_1 extends to $\mathrm{Aut}(S_1)$; therefore, by Lemma 1.5, the character $\lambda = \alpha_1 \times \cdots \times \alpha_n$ extends to $I_G(\lambda) = G$ and clearly satisfies the conclusions of claim (a).

It remains to consider the case when $S \cong \mathrm{PSL}_2(3^f)$ for an odd $f \geq 3$ and the action of $\langle gK \rangle$ on Ω is transitive. Since $\mathrm{o}(g^r)$ is the order of an element of S_1, then it is either 3 or a number coprime to 3. For the former case we can consider a character λ_1 as in the second paragraph of this proof (here $h = n$), whereas in the latter case we define λ_1 as the direct product of the Steinberg characters of the S_i, for $i \in \{1, \ldots, n\}$, which extends to $I = I_G(\lambda_1)$ by Lemma 1.6 or Lemma 1.5. It can be easily checked that the conclusions of claim (b) are satisfied by this λ_1, so the proof is complete. □

2. Proof of Theorem A

Note that the conclusion of claim (a) in Lemma 1.7 is stronger than that of claim (b); in fact, the former is just the latter with the additional property that $h = 0$. In other words, claim (b) holds for any isomorphism type of S_1, and this is what will be relevant henceforth.

We are ready to prove Theorem A, that we state again.

Theorem A. *Let G be a group whose Fitting subgroup is trivial, and let g be an element of G. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\mathrm{cod}(\chi)$ is a multiple of the order of g.*

Proof. We can clearly assume $G \neq 1$. Since the group G has a trivial Fitting subgroup, the generalized Fitting subgroup E of G is the socle of G, thus $E = M_1 \times \cdots \times M_k$ where the M_j are non-solvable minimal normal subgroups of G. For every j in $\{1, \ldots, k\}$, M_j is in turn the direct product of pairwise isomorphic non-abelian simple groups, and we denote by n_j the composition length of M_j (i.e. the number of simple direct factors appearing in this direct decomposition of M_j).

Now, set $C_j = \mathbf{C}_G(M_j)$ and denote by V_j the product of all the M_ℓ for $\ell \in \{1, \ldots, k\} - \{j\}$ (in particular, $V_j \subseteq C_j$); the factor group $\overline{G}_j = G/C_j$ has $\overline{M_j}$ as its unique minimal normal subgroup, thus we can apply Lemma 1.7 to $\overline{G}_j$ with respect to the element gC_j, and choose a character $\overline{\lambda}_j \in \mathrm{Irr}(\overline{M_j})$ with a corresponding non-negative integer $h_j \leq n_j$ as in Lemma 1.7(b). Note that each $\overline{\lambda}_j$ can be regarded by inflation as a character of $M_j \times C_j$ whose kernel contains C_j, hence there exists a unique $\lambda_j \in \mathrm{Irr}(M_j)$ such that $\overline{\lambda}_j = \lambda_j \times 1_{C_j}$; given that, we define $\lambda = \lambda_1 \times \cdots \times \lambda_k \in \mathrm{Irr}(E)$.

We know that the character $\overline{\lambda}_j$ extends to $I_G(\overline{\lambda}_j) = I_G(\lambda_j)$, therefore $\lambda_j \times 1_{V_j} \in \mathrm{Irr}(E)$ extends to $I_G(\lambda_j)$ as well. In particular, each $\lambda_j \times 1_{V_j}$ has an extension $\widehat{\lambda}_j$ to $I = I_G(\lambda) = \bigcap_{s=1}^{k} I_G(\lambda_s)$, and it is easy to check that the product $\psi = \prod_{s=1}^{k} \widehat{\lambda}_s$ is an extension of λ to I. Furthermore, defining $h = h_1 + \cdots + h_k$ and recalling that we have $g^{2^{h_j}} \in I_G(\lambda_j)$ for every $j \in \{1, \ldots, k\}$, we get $g^{2^h} \in I$.

Finally, set $\chi = \psi^G \in \mathrm{Irr}(G)$ and note that χ is a faithful character of G, because

$$\ker(\chi) \cap E \leq \ker(\psi) \cap E = \ker(\psi_E) = \ker(\lambda) = 1,$$

and a normal subgroup of G which intersects E trivially is necessarily trivial.

We are ready to conclude the proof. We get

$$\mathrm{cod}(\chi) = \frac{|G|}{\chi(1)} = \frac{|G|}{|G:I|\psi(1)} = \frac{|I|}{|E|} \cdot \frac{|E|}{\psi(1)}$$

and, denoting by $r = |\langle g\rangle E/E|$ the order of $gE \in G/E$,

$$\frac{|I|}{|E|} = \frac{|I|}{|\langle g\rangle E|} \cdot r = \frac{|I : \langle g\rangle E \cap I|}{|\langle g\rangle E : \langle g\rangle E \cap I|} \cdot r = \frac{|I : \langle g\rangle E \cap I|}{|\langle g\rangle : \langle g\rangle \cap I|} \cdot r\,.$$

In order to prove that $\mathrm{cod}(\chi)$ is a multiple of $\mathrm{o}(g)$, taking into account that $|\langle g\rangle : \langle g\rangle \cap I|$ is a divisor of 2^h, it will then suffice to show that $|E|/\psi(1)$ is a multiple of $2^h\,\mathrm{o}(g^r)$.

In fact, for $j \in \{1, \ldots, k\}$, consider $\overline{G}_j = G/C_j$, and denote by r_j the order of $\overline{g}\overline{M_j}$ in $\overline{G}_j/\overline{M_j}$. Clearly all the r_j are divisors of r; since, for every $j \in \{1, \ldots, t\}$, $|M_j|/\lambda_j(1)$ is a multiple of $2^{h_j}\,\mathrm{o}(\overline{g}^{r_j})$ by Lemma 1.7, we see that $|M_j|/\lambda_j(1)$ is a multiple of $2^{h_j}\,\mathrm{o}(\overline{g}^{r})$ as well. Now,

$$\frac{|E|}{\psi(1)} = \frac{|E|}{\lambda_1(1)\cdots\lambda_k(1)} = \frac{|M_1|}{\lambda_1(1)} \cdots \frac{|M_k|}{\lambda_k(1)}$$

is a multiple of $2^h\,\mathrm{o}(g^r C_1)\cdots\mathrm{o}(g^r C_k)$. Recalling that the map $x \mapsto (xC_1, \ldots, xC_k)$ is an injective homomorphism from G to $G/C_1 \times \cdots \times G/C_k$, it follows that the least common multiple of $\mathrm{o}(g^r C_1), \ldots, \mathrm{o}(g^r C_k)$ equals $\mathrm{o}(g^r)$, and the desired conclusion follows. □

3. A reduction

We conclude this note observing that Qian's conjecture can be reduced to groups with a solvable socle.

Remark 3.1. Assume that the group G is a minimal counterexample to the conjecture stated in the Introduction; then we claim that G does not have any non-solvable minimal normal subgroup.

For a proof by contradiction, denote by M a non-abelian minimal normal subgroup of G, set $C = \mathbf{C}_G(M)$, and observe that the factor group $\overline{G} = G/C$ is a monolithic group whose socle is $\overline{M} \cong M$. Therefore, for a fixed element g of G, we can apply Lemma 1.7 with respect to $\overline{g} = gC$ and obtain what follows: there exists a non-principal character $\overline{\lambda} \in \mathrm{Irr}(\overline{M})$ and a non-negative integer h (not exceeding the composition length of M) such that $\overline{\lambda}$ has an extension to $\overline{I} = I_{\overline{G}}(\overline{\lambda})$, $\overline{g}^{2^h} = g^{2^h}C$ lies in $\overline{I}$, and $|\overline{M}|/\overline{\lambda}(1)$ is a multiple of $2^h\,\mathrm{o}(\overline{g}^{\overline{r}})$ where $\overline{r}$ is the order of $\overline{g}\overline{M}$ in $\overline{G}/\overline{M}$. By inflation, $\overline{\lambda}$ can be viewed as a character of $M \times C$ and, as such, it is of the form $\lambda \times 1_C$ for a suitable $\lambda \in \mathrm{Irr}(M)$; clearly, we have $I_G(\lambda) = I_G(\overline{\lambda}) = I$ (hence $g^{2^h} \in I$) and $|\overline{M}|/\overline{\lambda}(1) = M/\lambda(1)$. Observe also that, if r denotes the order of gM in G/M, then r is a multiple of $\overline{r}$ and therefore $\mathrm{o}(\overline{g}^{\overline{r}})$ is a multiple of $\mathrm{o}(\overline{g}^r)$; as the map $x \mapsto \overline{x}$ is an isomorphism of M to $\overline{M}$, we get $\mathrm{o}(\overline{g}^r) = \mathrm{o}(\overline{g^r}) = \mathrm{o}(g^r)$.

Now, we know that λ has an extension $\widehat{\lambda}$ to I such that $\ker(\widehat{\lambda})$ contains C; moreover, the minimality of G yields that there exists $\xi \in \mathrm{Irr}(I/M)$ such that $|I/M : \ker(\xi)|/\xi(1)$ is a multiple of $\mathrm{o}(g^{2^h}M) = r/\gcd(2^h, r)$. Define ψ as $\widehat{\lambda}\xi$, which is in $\mathrm{Irr}(I)$ by Gallagher's Theorem, and $\chi = \psi^G$: by Clifford Correspondence we have $\chi \in \mathrm{Irr}(G)$, and we claim that $\mathrm{cod}(\chi)$ is a multiple of the order of g. It will follow that G is not a counterexample to Qian's conjecture, so we have a contradiction.

In fact,

$$\mathrm{cod}(\chi) = \frac{|G : \ker(\chi)|}{\chi(1)} = \frac{1}{|\ker(\chi)|} \cdot \frac{|I|}{\psi(1)} = \frac{1}{|\ker(\chi)|} \cdot \frac{|I/M|}{\xi(1)} \cdot \frac{|M|}{\lambda(1)}.$$

Since $|I/M : \ker(\xi)|/\xi(1)$ is a multiple of $r/\gcd(2^h, r)$ and $|M|/\lambda(1)$ is a multiple of $2^h\,\mathrm{o}(g^r)$, it will be enough to show that $\ker(\chi)$ is contained in $\ker(\xi)$: this can be deduced by the fact that $\ker(\chi)$ is a normal subgroup of G intersecting M trivially, hence $\ker(\chi) \subseteq C \cap \ker(\psi) = \ker((\widehat{\lambda}\xi)_C) = \ker(\xi_C)$ (recall that $\ker(\widehat{\lambda})$ contains C) and the argument is complete.

Data availability

No data was used for the research described in the article.

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