



On the Existence of Greatest Elements and Maximizers

Federico Quartieri¹

Received: 5 March 2022 / Accepted: 27 June 2022 / Published online: 16 July 2022
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Abstract

We obtain several characterizations of the existence of greatest elements of a total preorder. The characterizations pertain to the existence of unconstrained greatest elements of a total preorder and to the existence of constrained greatest elements of a total preorder on every nonempty compact subset of its ground set. The necessary and sufficient conditions are purely topological and, in the case of constrained greatest elements, are formulated by making use of a preorder relation on the set of all topologies that can be defined on the ground set of the objective relation. Observing that every function into a totally ordered set can be naturally conceived as a total preorder, we then reformulate the mentioned characterizations in the more restrictive case of an objective function with a totally ordered codomain. The reformulations are expressed in terms of upper semi- and pseudo-continuity by showing a topological connection between the two notions of generalized continuity.

Keywords Existence of greatest elements · Existence of maximizers · Characterization · Comparison of topologies

Mathematics Subject Classification 49J27 · 54A10 · 91B02

1 Introduction

A known result of the literature states that a preorder has a maximal element on every nonempty compact subset of its ground set provided that each of its upper sections is closed. This known result is—in a sense that will be clear in the course of the paper—a generalization of the Weierstrass extreme value theorem that must be essentially credited to Wallace [26]: see also [22] for a brief historical account. An immediate

Communicated by Juan-Enrique Martinez Legaz.

✉ Federico Quartieri
federico.quartieri@unifi.it

¹ Department of Economics and Management, University of Florence, Via delle Pandette 9, 50127 Florence, Italy

consequence thereof is that a total preorder has a greatest element on every nonempty compact subset of its ground set provided that each of its upper sections is closed. The present contribution inquires on the necessity of the closedness of the upper sections of a total preorder for it to possess a greatest element on every nonempty compact subset of its ground set. The inquiry yields several characterizations of the existence of greatest elements of a total preorder.

A point made in a quite recent article by Martínez-Legaz [15] is that the thesis of the Weierstrass theorem “does not refer to any specific topology” and hence that “one is free to consider the most convenient topology.” By leaning on this freedom, the sufficient conditions of one of the versions of the Weierstrass theorem are shown to be necessary by Theorem 1 in [15]. A function into a totally ordered set can be viewed as a total preorder and, in this perspective, the previous point can in fact extend to topology-based theorems on the existence of maximals—or of the greatest elements—of a relation. Certainly, the point holds true not only when one considers *unconstrained optimization problems*—as in the just mentioned paper—but also when one considers *families of constrained optimization problems* whose constraint sets are expressed with respect to some starting natural topology. On the importance of the last-mentioned type of problems for Economic Theory, see Walker [25] and the Introduction in [22].

The point raised in [15] was in fact implicit in Alcantud [1, Theorem 4], where the author provided a characterization of the existence of unconstrained maximals of an acyclic relation by showing the necessity of the sufficient topological conditions of a theorem dubbed in [1] as the “Bergstrom–Walker Theorem.” An implicit¹ presence of that same point can be found in the more recent Bosi and Zuanon [8, Corollary 3.2] and Quartieri [22, Theorem 4], where the authors provide, respectively, characterizations of the existence of unconstrained maximals of a preorder and of the existence of maximals of an arbitrary relation on every nonempty compact subset of its ground set. Somehow relatedly, the same point is tacitly present also in Andrikopoulos and Zacharias [5], where Smith and Schwartz sets are considered with respect to fixed choice sets. None of the results mentioned so far provides *purely topological conditions* that characterize the existence of optimal points on every nonempty compact subset of the ground set of an objective relation (or, in the case of functions, of the domain of an objective function).

Theorem 1 in [15] shows that a function has unconstrained maximizers if and only if there exists a topology that makes its domain compact and for which the function is upper semi-continuous. The aim of the present paper is to extend such a result to total preorders and to complement it with a characterization of the conditions for the existence of greatest elements on every nonempty compact subset of the ground set of a total preorder. In the case in which the objective relation to be optimized is a function, attention is posed also on upper pseudo-continuous functions—a generalization of upper semi-continuous functions introduced by Morgan and Scalzo [18] as a variant of [17]’s sequential upper pseudo-continuity—for which [18] formulated a Weierstrass-type extreme value theorem. Whether or not similar purely topological characterizations can be formulated for any (possibly not totally preordered) relation is

¹ Note that [8, Corollary 3.2] and [22, Theorem 4] employ the necessity part of [1, Theorem 4].

an open issue. Actually, total preorders enjoy some particular status within part of the economic literature: the fact that an authoritative economic textbook for graduate students like [16, Definition 1.B.1] dubs totally preordered individual preference relations “rational” bears witness of this. However, the tenet that totally preordered individual preference relations are the only “rational” individual preference relations has been disputed by the literature and some preference relations of economic interest—the usual stochastic order—are intrinsically not total. So, the mentioned open issue remains an important issue.

The paper is organized as follows. Preliminaries are provided in Sect. 2. Necessary and sufficient conditions for a total preorder to possess unconstrained greatest elements and to possess greatest elements on every nonempty compact subset of its ground set are shown in Sect. 3. Necessary and sufficient conditions for a function to possess unconstrained maximizers and to possess maximizers on every nonempty compact subset of its domain are shown in Sect. 4. Some observations and examples are contained in Sect. 5. The final Sect. 6 concludes.

2 Preliminaries

2.1 Relations

A relation B on a (ground) set X is a subset of the Cartesian product $X \times X$. Here, the second factor is understood as the domain of B and the first as its codomain. Let X be a set and B a relation on X . For all $x \in X$, the upper section of B at x is the set $B(x)$ defined by $B(x) = \{y : (y, x) \in B\}$. We often alternatively denote the membership $(y, x) \in B$ by the membership $y \in B(x)$ or by the juxtaposition yBx . The converse of B is the relation B^c on X defined by $B^c = \{(x, y) : (y, x) \in B\}$. The asymmetric part of B is the relation B^a on X defined by $B^a = B \setminus B^c$. The asymmetric part of the converse of B is denoted by B^{ca} . The relation B is: reflexive iff the membership $x \in X$ implies $(x, x) \in B$; antisymmetric iff the memberships $(y, x) \in B$ and $(x, y) \in B$ imply $y = x$; transitive iff the memberships $(z, y) \in B$ and $(y, x) \in B$ imply $(z, x) \in B$; total iff the membership $(y, x) \in X \times X$ implies $(y, x) \in B \cup B^c$; a preorder iff B is reflexive and transitive; a partial order iff B is reflexive, antisymmetric and transitive; a total preorder iff B is transitive and total; a total order iff B is antisymmetric, transitive and total. When B is a total order, (X, B) is called a totally ordered set. Lemma 2.1—whose proof is omitted—recalls a known fact concerning transitive relations.

Lemma 2.1 *Let X be a set and B a transitive relation on X . If $(z, y) \in B^c$ and $(y, x) \in B^{ca}$, then $(z, x) \in B^{ca}$.*

2.2 Maximals and greatest elements

Let X be a set and B a relation on X . The **set of all** (constrained) **greatest elements of B on** (a constraint set) $Y \subseteq X$ is the set $\mathfrak{M}(B, Y)$ defined by

$$\mathfrak{M}(B, Y) = \{y \in Y : Y \subseteq B^c(y)\}.$$

Therefore, $y \in \mathfrak{M}(B, Y)$ if and only if $y \in Y$ and $[z \in B^c(y)$ for all $z \in Y]$. Equivalently, $y \in \mathfrak{M}(B, Y)$ if and only if $y \in Y$ and $[y \in B(z)$ for all $z \in Y]$. The **set of all (constrained) maximals of B on** (a constraint set) $Y \subseteq X$ is the set $\mathfrak{M}(B, Y)$ defined by

$$\mathcal{M}(B, Y) = \{y \in Y : B^a(y) \cap Y = \emptyset\}.$$

Therefore, $y \in \mathcal{M}(B, Y)$ if and only if $y \in Y$ and $[z \in B^a(y)$ for no $z \in Y]$. Equivalently, $y \in \mathcal{M}(B, Y)$ if and only if $y \in Y$ and $[y \in B(z)$ for all $z \in Y$ such that $z \in B(y)]$. An element of $\mathfrak{M}(B, Y)$ is called a (constrained) greatest element of B on Y and an element of $\mathcal{M}(B, Y)$ is called a (constrained) maximal of B on Y . The inclusion $\mathfrak{M}(B, Y) \subseteq \mathcal{M}(B, Y)$ is always true. Further, the equality $\mathfrak{M}(B, Y) = \mathcal{M}(B, Y)$ holds provided B is total. However, the sets $\mathfrak{M}(B, Y)$ and $\mathcal{M}(B, Y)$ might well differ in general; in particular, $\mathcal{M}(B, Y)$ might be nonempty even when $\mathfrak{M}(B, Y)$ is empty. The **set of all unconstrained greatest elements of B** is the set $\mathfrak{M}(B)$ defined by

$$\mathfrak{M}(B) = \mathfrak{M}(B, X).$$

An element of $\mathfrak{M}(B)$ is called an unconstrained greatest element of B . Analogously, $\mathcal{M}(B, X)$ is called the **set of all unconstrained maximals of B** and an element of $\mathcal{M}(B, X)$ is called an unconstrained maximal of B : unconstrained maximals are discussed only in Sect. 1 and we do not need to fix any special notation. This Sect. 2.2 concludes by briefly recalling that maximals and greatest elements do not exhaust the spectrum of optimality notions. For instance, in Choice Theory and Mathematical Psychology, various alternatives have been proposed and studied in [21] and [2].

2.3 Compact Subsets

Let (X, τ) be a topological space. The pair (X, τ) is a compact topological space iff every τ -open cover of X has a finite subcover. Let Y be a subset of X and let $\hat{\tau}$ denote the subspace topology on Y induced by τ . The set Y is a τ -compact subset of X iff $(Y, \hat{\tau})$ is a compact topological space. Equivalently, Y is a τ -compact subset of X if and only if every τ -open cover of Y has a finite subcover. The **set of all nonempty τ -compact subsets of X** is denoted by

$$\mathcal{K}(X, \tau).$$

The **set of all τ -compact subsets of X** is denoted by $\mathcal{K}^\bullet(X, \tau)$. It is readily verified that $\mathcal{K}^\bullet(X, \tau) = \{\emptyset\} \cup \mathcal{K}(X, \tau)$.

2.4 Comparison of Topologies

Let X be a set and let τ_1 and τ_2 be topologies on X . As usual, we say that τ_1 is finer than τ_2 iff $\tau_1 \supseteq \tau_2$ and that τ_2 is coarser than τ_1 iff τ_1 is finer than τ_2 . We also say that τ_1 is **compactly finer** than τ_2 iff $\mathcal{K}^\bullet(X, \tau_1) \supseteq \mathcal{K}^\bullet(X, \tau_2)$ and that τ_2 is **compactly coarser** than τ_1 iff τ_1 is compactly finer than τ_2 . The topologies τ_1 and τ_2 are **compactly equivalent** iff $\mathcal{K}^\bullet(X, \tau_1) = \mathcal{K}^\bullet(X, \tau_2)$. As the empty set is a τ -compact subset of X for any topology τ on X , we have that τ_1 is compactly finer than τ_2 if and only if $\mathcal{K}(X, \tau_1) \supseteq \mathcal{K}(X, \tau_2)$. Likewise, τ_1 is compactly equivalent to τ_2 if and only if $\mathcal{K}(X, \tau_1) = \mathcal{K}(X, \tau_2)$. A set of topologies on X is partially ordered by the *finer than* relation but merely preordered by the *compactly finer than* relation. Indeed, two distinct topologies might well be compactly equivalent. It goes without saying that the coarser a topology, the compactly finer such a topology is: the converse, however, is generally false.

2.5 Topology and Order

Let X be a set, τ a topology on X and B a preorder on X . The relation B is **closed-valued for τ** iff $B(x)$ is τ -closed for all $x \in X$. Assume now that B is also total. The **lower-order topology associated with B** is the topology on X for which $\{\emptyset, X\} \cup \{B(x)\}_{x \in X}$ forms a subbase of closed sets. As B is a total preorder, the lower-order topology associated with B is the topology on X for which $\{\emptyset, X\} \cup \{B^{ca}(x)\}_{x \in X}$ forms a subbase—in fact a base—of open sets. Remark 2.1 is readily verified.

Remark 2.1 Let X be a set and B a total preorder on X . The lower-order topology associated with B is the coarsest topology on X for which B is closed-valued.

2.6 Some Final Considerations on Closed-Valuedness

The 1940s saw the beginning of the systematic study of the interplay of order and topology: much of the early literature—including Nachbin’s monograph [19, 20]—is referenced in Ward’s article on partially ordered topological spaces [27]. The ongoing importance of this subject of study to Mathematical Economics is witnessed, for instance, by the more recent [10] and [6] while that to Domain Theory, for instance, by the more recent [14]. In the already mentioned [26] and [27] of the early literature, closed-valued relations are dubbed “upper semi-continuous.” Part of the recent literature in Mathematical Economics—like [4, p. 44] or [11], [3], [7], [9]—makes use of upper semi-continuity in such an acceptance.² Upper semi-continuous relations are thus named because of their similarity to upper semi-continuous functions, that are historically anterior. Sects. 4.2 and 4.3 will contend that, in point of fact, there does not exist a perfect analogy between upper semi-continuous relations and upper semi-continuous functions.

² For sake of completeness, it is remarked that some authors refer to closed-valuedness as “upper hemi-continuity” (e.g., [13]) while others as “upper semi-closedness” (e.g., [14]) and others provide a definition of upper semi-continuity that—strictly speaking—is in fact different (e.g., [1]).

3 Characterizations for Total Preorders

The existence of constrained greatest elements of a total preorder B implies the compactness of the constraint set for the lower-order topology associated with B .

Theorem 3.1 *Let X be a set, B a total preorder on X and τ the lower-order topology associated with B . If $Y \subseteq X$ and $\mathfrak{M}(B, Y) \neq \emptyset$, then $Y \in \mathcal{K}(X, \tau)$.*

Proof Suppose $Y \subseteq X$ and $\mathfrak{M}(B, Y) \neq \emptyset$. Then, there exists $y \in Y$ such that

$$z \in B^c(y) \text{ for all } z \in Y. \quad (1)$$

Put $\sigma = \{\emptyset, X\} \cup \{B^{ca}(x)\}_{x \in X}$: the family σ is a subbase of open sets for τ . Let $\hat{\tau}$ denote the subspace topology on Y induced by τ and put $\hat{\sigma} = \{Z \cap Y\}_{Z \in \sigma}$: the family $\hat{\sigma}$ is a subbase of open sets for $\hat{\tau}$. Note that $\{Y\}$ is a cover of Y by members of $\hat{\sigma}$. Noted this, pick an arbitrary cover of Y by members of $\hat{\sigma}$ and denote such a cover by γ . Then, there exists

$$S \in \gamma \quad (2)$$

such that

$$y \in S. \quad (3)$$

Suppose for a moment that $S \neq Y$. As $S \neq Y$, the memberships in (2) and (3) imply the existence of $x \in X$ such that

$$S = B^{ca}(x) \cap Y. \quad (4)$$

An immediate consequence of (3) and (4) is that

$$y \in B^{ca}(x). \quad (5)$$

But the inequality $S \neq Y$ and the equality in (4) entail that $\bar{z} \notin B^{ca}(x)$ for some $\bar{z} \in Y$: an entailment that—by virtue of Lemma 2.1—is in contradiction with (1) and (5). Therefore, $S = Y$ and hence $\{S\}$ is a finite subcover of γ . Said this, Alexander's subbase theorem implies the compactness of the topological space $(Y, \hat{\tau})$. So $Y \in \mathcal{K}(X, \tau)$ in that Y is nonempty. \square

Theorem 3.2—to be credited to Wallace [26]—shows sufficient conditions for a preorder to possess a maximal on every nonempty compact subset of its ground set.

Theorem 3.2 ([26]) *Let (X, τ) be a topological space and B a preorder on X . If B is closed-valued for τ , then $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Theorems 3.1 and 3.2 imply the following characterizations of the existence of greatest elements of a total preorder on every nonempty compact subset of its ground set and of the existence of unconstrained greatest elements of a total preorder.

Corollary 3.1 *Let (X, τ) be a topological space and B a total preorder on X . Assertions I to IV are equivalent.*

- I. $\mathfrak{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.
- II. The lower-order topology associated with B is compactly finer than τ .
- III. The coarsest topology on X for which B is closed-valued is compactly finer than τ .
- IV. There exists a topology on X that is compactly finer than τ and for which B is closed-valued.

Proof The implication $I \Rightarrow II$ is a consequence of Theorem 3.1. The implication $II \Rightarrow III$ follows from Remark 2.1. The implication $III \Rightarrow IV$ is obvious. Now, assume the existence of a topology $\bar{\tau}$ on X that is compactly finer than τ and for which B is closed-valued. Theorem 3.2 ensures that $\mathcal{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \bar{\tau})$. So, $\mathfrak{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \bar{\tau})$ in that B is total. As $\bar{\tau}$ is compactly finer than τ , we have that $\mathfrak{M}(B, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$. Therefore, also the implication $IV \Rightarrow I$ holds true. \square

Corollary 3.2 *Let X be a nonempty set and B a total preorder on X . Assertions I to IV are equivalent.*

- I. $\mathfrak{M}(B) \neq \emptyset$.
- II. X is made compact by the lower-order topology associated with B .
- III. X is made compact by the coarsest topology on X for which B is closed-valued.
- IV. There exists a topology on X that makes X compact and for which B is closed-valued.

Proof The implication $I \Rightarrow II$ is a consequence of Theorem 3.1. The implication $II \Rightarrow III$ follows from Remark 2.1. The implication $III \Rightarrow IV$ is obvious. The totality of B implies $\mathcal{M}(B, X) = \mathfrak{M}(B)$ and hence the implication $IV \Rightarrow I$ is a consequence of Theorem 3.2. \square

4 Characterizations for Functions

We now turn attention to functions and we show characterizations of the existence of their maximizers. In a sense that will be immediately clear, the ongoing examination is a particular case of that concerning total preorders. However, extra investigation is needed to connect two possible analogs of the closed-valuedness of a relation.

4.1 Maximizers

Let X be a set, (T, \succeq) a totally ordered set and $f : X \rightarrow T$ a function. The **set of all (constrained) maximizers of f on** (a constraint set) $Y \subseteq X$ is the set $\mathfrak{M}(f, Y)$ defined by

$$\mathfrak{M}(f, Y) = \{y \in Y : f(y) \succeq f(z) \text{ for all } z \in Y\}.$$

The set of all unconstrained maximizers of f is the set $\mathfrak{M}(f)$ defined by

$$\mathfrak{M}(f) = \mathfrak{M}(f, X).$$

An element of $\mathfrak{M}(f, Y)$ is called a (constrained) maximizer of f on Y and an element of $\mathfrak{M}(f)$ is called an unconstrained maximizer of f .

4.2 Upper Pseudo- and Semi-continuity

Let (X, τ) be a topological space, (T, \succeq) a totally ordered set and $f : X \rightarrow T$ a function with image F . The function f is: **upper pseudo-continuous for τ** iff

$$\{x \in X : f(x) \succeq t\} \text{ is } \tau\text{-closed for all } t \in F;$$

upper semi-continuous for τ iff

$$\{x \in X : f(x) \succeq t\} \text{ is } \tau\text{-closed for all } t \in T.$$

Clearly, upper pseudo-continuity is weaker than upper semi-continuity. However, the two notions coincide when F is finite. The definition of upper semi-continuity adopted here is in fact standard (see, e.g., Gaal [12]). The definition of upper pseudo-continuity is slightly more general than that introduced in [18]: the slight generalization—which pertains to the codomain of f —allows for an immediate comparison of the notions of continuity just specified. It has already been observed that a quite popular nomenclature in Mathematics and Mathematical Economics dubs closed-valued relations “upper semi-continuous relations”: Proposition 2.2 in [18] and Fact 2 of the following Sect. 4.3 seem to suggest that the closed-valuedness of a relation bears an analogy with upper pseudo-continuity that is stronger than that with upper semi-continuity.

4.3 Relation Implicit in a Function

Let (X, τ) be a topological space, (T, \succeq) a totally ordered set and $f : X \rightarrow T$ a function with image F . The **relation implicit in f** is the relation B_f on X defined for all $x \in X$ by

$$B_f(x) = \{z \in X : f(z) \succeq f(x)\}. \quad (6)$$

From (6) we infer **Fact 1**: B_f is a total preorder. From (6) we also infer that

$$\{B_f(x)\}_{x \in X} = \{z \in X : f(z) \succeq t\}_{t \in F}. \quad (7)$$

By making use of the last equality we readily verify **Fact 2**: B_f is closed-valued for τ if and only if f is upper pseudo-continuous for τ . For every $(y, x) \in X \times X$, we have that $y \in B_f(x)$ if and only if $f(y) \succeq f(x)$: as B_f is a total preorder, this observation

implies **Fact 3**: $\mathfrak{M}(f, Y) = \mathfrak{M}(B_f, Y)$ for every $Y \subseteq X$. Put $\prec = \succeq^{ca}$. The equality in (7) is equivalent to

$$\{B_f^{ca}(x)\}_{x \in X} = \{z \in X : f(z) \prec t\}_{t \in F} \tag{8}$$

and from (8) we infer **Fact 4**: *The coarsest topology on X for which f is upper pseudo-continuous coincides with the lower-order topology associated with B_f .* For completeness, it is remarked that the coarsest topology on X for which f is upper pseudo-continuous is the topology on X for which the family

$$\sigma_1 = \{\emptyset, X\} \cup \{x \in X : f(x) \prec t\}_{t \in F} \tag{9}$$

forms a subbase—in fact even a base—of open sets and that the coarsest topology on X for which f is upper semi-continuous is the topology on X for which the family

$$\sigma_2 = \{\emptyset, X\} \cup \{x \in X : f(x) \prec t\}_{t \in T} \tag{10}$$

forms a subbase—in fact even a base—of open sets. As $\sigma_1 \subseteq \sigma_2$, we immediately infer **Fact 5**: *The coarsest topology on X for which f is upper semi-continuous is finer than the coarsest topology on X for which f is upper pseudo-continuous.* Clarified all these points, we can reformulate Theorem 3.1 in terms of functions.

Corollary 4.1 (Theorem 3.1 reformulated) *Let X be a set, (T, \succeq) a totally ordered set and $f : X \rightarrow T$. Denote the coarsest topology on X for which f is upper pseudo-continuous by τ . If $Y \subseteq X$ and $\mathfrak{M}(f, Y) \neq \emptyset$, then $Y \in \mathcal{K}(X, \tau)$.*

Proof In view of Facts 1 and 3–4, this is just a consequence of Theorem 3.1. □

Corollary 4.2 reformulates Theorem 3.2 in terms of functions. Actually, the same statement had been obtained in [18, Proposition 2.1] by relying on an existence theorem by [24, Theorem 2]: the proof of Corollary 4.2 shows that such a conclusion follows directly from the much older Theorem 3.2.

Corollary 4.2 (Theorem 3.2 reformulated, [18]) *Let (X, τ) be a topological space, (T, \succeq) a totally ordered set and $f : X \rightarrow T$. If the function f is upper pseudo-continuous for τ , then $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.*

Proof In view of Facts 1–3, this is just a consequence of Theorem 3.2. □

4.4 On Compact Equivalence

The coarsest topologies for which a function is upper semi- and pseudo-continuous are compactly equivalent.

Theorem 4.1 *Let X be a set, (T, \succeq) a totally ordered set and $f : X \rightarrow T$. The coarsest topology on X for which f is upper semi-continuous is compactly equivalent to the coarsest topology on X for which f is upper pseudo-continuous.*

Proof Let τ_1 denote the coarsest topology on X for which f is upper pseudo-continuous and let τ_2 denote the coarsest topology on X for which f is upper semi-continuous. Denoting the image of f by F , let σ_1 and σ_2 be defined as in (9) and (10). Fact 5 has observed that τ_2 is finer than τ_1 . Therefore, τ_1 is compactly finer than τ_2 and to conclude the proof it suffices to show that τ_2 is compactly finer than τ_1 . Suppose $Y \in \mathcal{K}(X, \tau_1)$ and put $\leq = \succeq^c$. Corollary 4.2 ensures the existence of $y \in Y$ such that

$$f(z) \leq f(y) \text{ for all } z \in Y. \tag{11}$$

Let $\hat{\tau}_2$ be the subspace topology on Y induced by τ_2 . The family $\hat{\sigma}_2 = \{Z \cap Y\}_{Z \in \sigma_2}$ is a subbase of open sets for the topology $\hat{\tau}_2$. Note that $\{Y\}$ is a cover of Y by members of $\hat{\sigma}_2$. Noted this, pick an arbitrary cover of Y by members of $\hat{\sigma}_2$ and denote such a cover by γ . Then, there exists

$$S \in \gamma \tag{12}$$

such that

$$y \in S. \tag{13}$$

Suppose for a moment that $S \neq Y$ and put $\prec = (X \times X) \setminus \prec$. As $S \neq Y$, the memberships in (12) and (13) imply the existence of $t \in T$ such that

$$S = \{x \in X : f(x) \prec t\} \cap Y. \tag{14}$$

An immediate consequence of (13) and (14) is that

$$f(y) \prec t. \tag{15}$$

But the inequality $S \neq Y$ and the equality in (14) entail that $f(\bar{z}) \not\prec t$ for some $\bar{z} \in Y$: an entailment that—by virtue of Lemma 2.1—is in contradiction with (11) and (15). Therefore, $S = Y$ and hence $\{S\}$ is a finite subcover of γ . Said this, Alexander’s subbase theorem implies the compactness of the topological space $(Y, \hat{\tau}_2)$. So $Y \in \mathcal{K}(X, \tau_2)$ in that Y is nonempty. □

4.5 Characterizations

Theorem 4.1 and Corollaries 4.1–4.2 imply the characterizations of the existence of constrained maximizers in terms of upper semi- and pseudo-continuity stated below.

Corollary 4.3 *Let (X, τ) be a topological space, (T, \succeq) a totally ordered set and $f : X \rightarrow T$. Assertions I to V are equivalent.*

- I. $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$.
- II. The coarsest topology on X for which f is upper semi-continuous is compactly finer than τ .

- III. The coarsest topology on X for which f is upper pseudo-continuous is compactly finer than τ .
- IV. There exists a topology on X that is compactly finer than τ and for which f is upper semi-continuous.
- V. There exists a topology on X that is compactly finer than τ and for which f is upper pseudo-continuous.

Proof The implication $I \Rightarrow III$ is a consequence of Corollary 4.1. The implication $III \Rightarrow II$ is a consequence of Theorem 4.1. The implications $II \Rightarrow IV$ and $IV \Rightarrow V$ are obvious. Now, to conclude, assume the existence of a topology $\bar{\tau}$ on X that is compactly finer than τ and for which f is upper pseudo-continuous. Corollary 4.2 ensures that $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \bar{\tau})$: as $\bar{\tau}$ is compactly finer than τ , we conclude that $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$. This proves the validity of the implication $V \Rightarrow I$. \square

Theorem 4.1 and Corollaries 4.1–4.2 imply also the characterizations of the existence of unconstrained maximizers in terms of upper semi- and pseudo-continuity stated below. It is noted that the equivalence of assertions I , II and IV of Corollary 4.4 is the conclusion of Theorem 1 in [15].

Corollary 4.4 (Partly [15]) *Let X be a nonempty set, (T, \succeq) a totally ordered set and $f : X \rightarrow T$. Assertions I to V are equivalent.*

- I. $\mathfrak{M}(f) \neq \emptyset$.
- II. X is made compact by the coarsest topology on X for which f is upper semi-continuous.
- III. X is made compact by the coarsest topology on X for which f is upper pseudo-continuous.
- IV. There exists a topology on X that makes X compact and for which f is upper semi-continuous.
- V. There exists a topology on X that makes X compact and for which f is upper pseudo-continuous.

Proof The implication $I \Rightarrow III$ is a consequence of Corollary 4.1. The implication $III \Rightarrow II$ is a consequence of Theorem 4.1. The implications $II \Rightarrow IV$ and $IV \Rightarrow V$ are obvious. The implication $V \Rightarrow I$ is a consequence of Corollary 4.2. \square

Remark 4.1 A characterization of the existence of unconstrained maximizers of a function with a *nonempty compact* domain has been provided in [24, Theorem 1]. The characterizing condition is called transfer weak upper continuity. As observed in [24, Remark 6], the preliminary assumption about the compactness of the domain of the objective function is crucial for the validity of that result. It is remarked that no preliminary topological assumption is needed for the validity of Corollary 4.4 of this work.

5 Final Remarks

While applying Theorem 4.1 and Corollaries 3.1 and 4.3 to specific examples, in Sects. 5.1 and 5.2 we provide some final remarks on the tightness of their conclusions.

5.1 Increasing Functions

Endow \mathbb{R} with its natural topology τ and the usual total order relation \geq . Denoting the asymmetric part of \geq by $>$, we say that $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing (strictly increasing) iff $(y, x) \in \mathbb{R} \times \mathbb{R}$ and $y > x$ imply $f(y) \geq f(x)$ (imply $f(y) > f(x)$). Strictly increasing real-valued functions on \mathbb{R} are instances of increasing functions that are pseudo-continuous for τ and that might not be upper semi-continuous for τ : see Remark 2.1 in [18]. An increasing real-valued functions on \mathbb{R} might or might not be upper pseudo-continuous for τ (e.g., the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x - 1$ for all $x \in (-\infty, 0]$ and by $g(x) = 1$ for all $x \in (0, +\infty)$ is not upper pseudo-continuous for τ). Said this, assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing. It is readily seen that $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$, even though $\mathfrak{M}(f)$ can well be empty. So, Theorem 4.1 and Corollary 4.3 guarantee that the coarsest topology on \mathbb{R} for which f is upper semi-continuous is compactly equivalent to the coarsest topology on \mathbb{R} for which f is upper pseudo-continuous and that both topologies retain at least as many compact subsets as those of the natural topology τ . Proposition 5.1 summarizes these conclusions and Proposition 5.2 clarifies that the two compactly equivalent topologies might well differ from each other and need not be compactly equivalent to the natural topology τ .

Proposition 5.1 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. The coarsest topology on \mathbb{R} for which f is upper semi-continuous is compactly equivalent to the coarsest topology on \mathbb{R} for which f is upper pseudo-continuous and both topologies are compactly finer than the natural topology of \mathbb{R} .*

Proof As $\mathfrak{M}(f, Y) \neq \emptyset$ for every $Y \in \mathcal{K}(X, \tau)$, Proposition 5.1 is just a consequence of Theorem 4.1 and Corollary 4.3. \square

Proposition 5.2 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the increasing function defined by $f(x) = x - 1$ for all $x \in (-\infty, 0]$ and by $f(x) = x + 1$ for all $x \in (0, +\infty)$. The coarsest topology on \mathbb{R} for which f is upper semi-continuous differs from the coarsest topology on \mathbb{R} for which f is upper pseudo-continuous and is not compactly equivalent to the natural topology of \mathbb{R} .*

Proof The real interval $(-\infty, 0]$ is an open compact subset of \mathbb{R} for the coarsest topology on \mathbb{R} that makes f upper semi-continuous. However, $(-\infty, 0]$ is not an open subset of \mathbb{R} for the coarsest topology on \mathbb{R} that makes f upper pseudo-continuous and is not a compact subset of \mathbb{R} for the natural topology of \mathbb{R} . \square

5.2 Lexicographic Orders

Let α denote a nonzero ordinal. When an ordinal δ is a member of α we write $\delta \sqsubset \alpha$. When δ is an ordinal such that either $\delta \sqsubset \alpha$ or $\delta = \alpha$, we write $\delta \sqsubseteq \alpha$. The product of α copies of \mathbb{R} is denoted by \mathbb{R}^α and is understood as a real vector space. The

lexicographic order on \mathbb{R}^α is the total order relation Λ_α on \mathbb{R}^α defined by

$$\Lambda_\alpha(x) = \left\{ \begin{array}{l} y \in \mathbb{R}^\alpha : \text{either } y = x \text{ or there exists a nonzero} \\ \text{ordinal } \beta \sqsubseteq \alpha \text{ such that } x_\beta < y_\beta \text{ and } x_\gamma = y_\gamma \\ \text{for every nonzero ordinal } \gamma \sqsubset \beta. \end{array} \right\}.$$

The **lexicographic lower-order topology on \mathbb{R}^α** is the lower-order topology associated with Λ_α : such a topology is the coarsest topology on \mathbb{R}^α for which Λ_α is closed-valued. The **natural topology of \mathbb{R}^α** is the product topology that arises by endowing each copy of \mathbb{R} with its natural topology. The two topologies are distinct for every nonzero ordinal α . When $\alpha = 1$, the natural topology of \mathbb{R}^α is finer than the lexicographic lower-order topology on \mathbb{R}^α . When $\alpha \neq 1$, the natural topology of \mathbb{R}^α is neither finer nor coarser than the lexicographic lower-order topology on \mathbb{R}^α . Lemma 2.1 in Schouten [23, Lemma 2.1] ensures the existence of a greatest element of Λ_α on every nonempty subset of \mathbb{R}^α that is compact for the natural topology of \mathbb{R}^α . So, Corollary 3.1 guarantees that the lexicographic lower-order topology on \mathbb{R}^α is compactly finer than the natural topology of \mathbb{R}^α whatever nonzero ordinal is α . Proposition 5.3 summarizes this conclusion and Proposition 5.4 clarifies that the two topologies are not compactly equivalent whatever nonzero ordinal is α .

Proposition 5.3 *The lexicographic lower-order topology on \mathbb{R}^α is compactly finer than the natural topology of \mathbb{R}^α .*

Proof A consequence of Lemma 2.1 in [23] and Corollary 3.1. □

Proposition 5.4 *The lexicographic lower-order topology on \mathbb{R}^α is not compactly equivalent to the natural topology of \mathbb{R}^α .*

Proof Let τ_1 denote the natural topology of \mathbb{R}^α , τ_2 the lexicographic lower-order topology on \mathbb{R}^α and X_1 the set of all nonpositive integers. Put $X_\beta = \{0\}$ for every nonzero ordinal $\beta \sqsubseteq \alpha$ other than 1 and put

$$Y = \prod_{\beta=1}^\alpha X_\beta.$$

The zero vector of \mathbb{R}^α belongs to $\mathfrak{M}(\Lambda_\alpha, Y)$. Therefore, $Y \in \mathcal{K}(X, \tau_2)$ by Theorem 3.1. Let I_z denote the real interval $(z-1/2, z+1/2)$ for all $z \in X_1$ and let $\pi_1 : \mathbb{R}^\alpha \rightarrow \mathbb{R}$ be projection map defined by $\pi_1(x) = x_1$. The family $\{\pi_1^{-1}(I_z)\}_{z \in X_1}$ is a τ_1 -open cover of Y that does not possess any finite subcover. So, $Y \notin \mathcal{K}(X, \tau_1)$. □

6 Conclusions

This first part of the present work has characterized the existence of the greatest elements of a total preorder on every nonempty compact subset of its domain as well as the existence of its unconstrained greatest elements. These characterizations have been provided by Corollaries 3.1 and 3.2, respectively. The two mentioned Corollaries follow from Theorems 3.1 and 3.2: the latter is a known result on the existence of

maximals due to Alexander Doniphan Wallace while the former—the first main contribution of the paper—proves the compactness for the lower-order topology of any subset of the domain of a total preorder on which a greatest element of that relation exists. By viewing a function into a totally ordered set as a total preorder, the second part of the present work has then characterized the existence of the maximizers of such a type of function on every nonempty compact subset of its domain as well as the existence of its unconstrained maximizers. These characterizations have been provided by Corollaries 4.3 and 4.4, respectively. The last-mentioned Corollary extends a known result of the literature. The characterizations of the second part of this work follow almost directly from those of the first after having proved in Theorem 4.1—the second main contribution of the paper—that the coarsest topology on the domain of a function into a totally ordered set that makes it upper semi-continuous admits exactly the same compact subsets as those admitted by the coarsest topology on the domain of such a function that makes it upper pseudo-continuous.

Acknowledgements The author gratefully acknowledges the helpful comments and suggestions received during the review process.

Funding Open access funding provided by Università degli Studi di Firenze within the CRUI-CARE Agreement.

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