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ORIGINAL ARTICLE

The Critical Space for Orthogonally Invariant Varieties

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Abstract

Let q be a nondegenerate quadratic form on V. Let $X \subset V$ be invariant for the action of a Lie group G contained in $SO(V, q)$. For any $f \in V$ consider the function d_f from X to $\mathbb{C}$ defined by $d_f(x) = q(f - x)$. We show that the critical points of d_f lie in the subspace orthogonal to $\mathfrak{g} \cdot f$, that we call critical space. In particular any closest point to f in X lie in the critical space. This construction applies to singular t-ples for tensors and to flag varieties and generalizes a previous result of Draisma, Tocino and the author. As an application, we compute the Euclidean Distance degree of a complete flag variety.

Keywords Tensors · Singular t-ples · Critical space · EDdegree · Euclidean Distance degree

Mathematics Subject Classification (2010) 14N07 · 14L30 · 14N05 · 15A69 · 15A18

1 Introduction and Main Result

Let V be a complex vector space equipped with a nondegenerate symmetric bilinear form $q \in \mathrm{Sym}^2 V$, identified in this paper with its associated quadratic form. The orthogonal group $SO(V, q) = SO(V)$ consists of linear transformations of V leaving q invariant. Let $X \subset V$ be an algebraic variety defined over $\mathbb{R}$, this includes the case when X is the cone over a projective variety defined over $\mathbb{R}$. We assume that X is G-invariant for the action of a Lie group $G \subset SO(V)$. In many cases of interest X is H-invariant for a larger group H and we can take $G = SO(V) \cap H$, see Section 2 for the case of partially symmetric tensors.

We denote by $\mathfrak{g} = T_e G$ the Lie algebra of G, where e is the identity element, note that $\mathfrak{g} \subset \mathfrak{so}(V)$. The tangent space to the orbit $G \cdot f$ at f is $f + \mathfrak{g} \cdot f$. Denoting by $G_f = \{g \in G \mid g \cdot f = f\}$ the isotropy group of f, we have $\dim \mathfrak{g} \cdot f = \dim \mathfrak{g} - \dim G_f$.

Dedicated to Bernd Sturmfels on the occasion of his 60th birthday.

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We define for $f \in V$ the *critical space* of f as the subspace

$$H_f := (\mathfrak{g} \cdot f)^{\perp} = \{v \in V \mid q(v, w) = 0 \quad \forall w \in \mathfrak{g} \cdot f\}. \tag{1.1}$$

We remark that $\operatorname{codim} H_f = \dim \mathfrak{g} \cdot f$, so we have $\operatorname{codim} H_f \leq \dim \mathfrak{g}$ and the equality holds for general f in many cases, but it cannot hold in the cases when $\dim \mathfrak{g} \geq \dim V$ (this happens when $V = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ and c is large, in the setting of Section 2.2). Consider the function $d_f : X \to \mathbb{C}$, $d_f(x) = q(f - x, f - x)$, which, in the case f is real, extends the squared distance function from f defined over $\mathbb{R}$.

Note that at a critical point x of d_f we have $f - x \in (T_x X)^{\perp}$.

Lemma 1.1 *If $g \in \mathfrak{g}$ then $q(g \cdot x, y) = -q(x, g \cdot y)$, in particular $q(g \cdot x, x) = 0$.*

Proof If $\mathfrak{g} = 0$ then the statement is trivial. If $\mathfrak{g} \neq 0$, let $g(t) \subset G$ be a path such that $g(0) = e$ and $\dot{g}(0) = g$. Taking the derivative at $t = 0$ of the constant function $q(g(t) \cdot x, g(t) \cdot y)$ the thesis follows. □

Our main result is the following theorem. Its proof is quite simple, nevertheless we will see in the rest of the paper it has some nontrivial consequences.

Theorem 1.2 *Let X be G-invariant for the action of $G \subset SO(V)$.*

1. *The critical points of d_f on X lie in H_f.*
2. *When f is real, any closest point to f in $X_{\mathbb{R}}$ (with respect to q) belongs to H_f.*
3. *$f \in H_f$.*

Proof Let x be a critical point. We need to prove $q(x - f, g \cdot f) = 0\ \forall g \in \mathfrak{g}$. We have $q(g \cdot f, f) = 0 \quad \forall g \in \mathfrak{g}$ from Lemma 1.1. So it is enough to show that $q(x, g \cdot f) = 0$ $\forall g \in \mathfrak{g}$. The crucial remark is that since X is G-invariant then $\mathfrak{g} \cdot x \subset T_x X$. Since x is critical it follows the chain of equalities (the second and the third one by Lemma 1.1 $0 = q(g \cdot x, x - f) = -q(g \cdot x, f) = q(x, g \cdot f)$, which proves (1).

(2) is an immediate consequence of (1).

(3) follows by $q(f, g \cdot f) = 0$. □

A partial converse to Theorem 1.2 is the following.

Theorem 1.3 *Let X be G-invariant for the action of $G \subset SO(V)$. Let $x \in H_f \cap X$.*

1. *If the orbit $G \cdot x$ is dense in X then x is a critical point of d_f restricted to X.*
2. *If X is a cone, x is not isotropic and the orbit $G \cdot [x]$ is dense in $\mathbb{P}X$ then there is $\lambda \in \mathbb{C}$ such that λx is a critical point of d_f restricted to X.*

Proof We have the equality $\mathfrak{g} \cdot x = T_x X$ by assumption, and with this equality all the steps of the proof of Theorem 1.2 are invertible. This proves (1). The assumption of (2) implies that $\mathfrak{g} \cdot x + \langle x \rangle = T_x X$. Since x is not isotropic there is λ such that since $q(\lambda x, \lambda x - f) = 0$, namely $\lambda = \frac{q(x,f)}{q(x,x)}$, so that orthogonality is guaranteed on the subspace $\langle x \rangle \subset T_x X$. To check orthogonality on the remaining part of $T_x X$ we may replace x with $\frac{q(x,f)}{q(x,x)} x$ and the same argument in (1) works. □

A stronger converse form will be proved for tensors, see Theorems 2.2 and 2.4 and for Grassmann varieties, see Theorem 3.1. In Theorem 3.4 we will compute the Euclidean

Distance degree (EDdegree) of a complete flag variety with respect to the Frobenius product.

We recall that EDdegree(X) (introduced in [4] by following an idea by Bernd Sturmfels) is the number of critical points of d_f restricted to X for general f. In many cases of interest it happens that $H_f \cap X$ is finite and reduced for general f, in these cases its cardinality counts EDdegree(X), see (2.4) and Theorem 3.4.

The critical space was introduced for tensors in [14] and for partially symmetric tensors in [5], see also [15]. In Corollary 2.6 we get an alternative proof of the fact proved in [5] by Draisma, Tocino and the author that any best rank q approximation of a partially symmetric tensor f lies in the critical space.

Our approach is somehow dual to the one in [3, 6], where EDdegree was considered in an orthogonally invariant setting, but certain subvarieties of X were constructed in order to cut transversally the orbits.

2 Symmetric and Partially Symmetric Tensors

2.1 Symmetric Tensors

Let W a space of dimension $n+1$ and $V = \mathrm{Sym}^d W$. We assume that W is equipped with a nondegenerate quadratic form q_W and we choose coordinates in W such that $q_W = \sum_{i=0}^n x_i^2$. There is a unique nondegenerate bilinear form q such that

$$q(x^d, y^d) = q_W(x, y)^d \quad \forall x, y \in W, \tag{2.1}$$

which is called the Frobenius (or Bombieri-Weyl) form. Since every polynomial in $\mathrm{Sym}^d W$ can be written as a sum of powers of linear forms, it is enough to ask (2.1) for any power x^d, y^d. The group $G = SO(W, q_W)$ acts over V by the analogous rule $g \cdot (x^d) = (g \cdot x)^d$. We get the inclusion $G \subset SO(V, q)$, so that we are in the setting of Section 1; our aim is to apply Theorem 1.2. The Frobenius form has the coordinate expression $q\left(\sum_\alpha \binom{d}{\alpha} f_\alpha x^\alpha, \sum_\alpha \binom{d}{\alpha} g_\alpha x^\alpha\right) = \sum_\alpha \binom{d}{\alpha} f_\alpha g_\alpha$ which, up to a scalar factor, has the nice M2 [10] implementation

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diff(f,g).
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Note that $SL(W) \cap SO(V) = SO(W)$, but we will not need this fact. The monomials are orthogonal but not orthonormal with respect to q.

Proposition 2.1

$$\mathfrak{so}(W) \cdot f = \left\langle x_j \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial x_j} \right\rangle_{0 \le i < j \le n}. \tag{2.2}$$

Proof It is convenient to denote

$$D_{ij}(f) = x_j \frac{\partial f}{\partial x_i} - x_i \frac{\partial f}{\partial x_j}. \tag{2.3}$$

For any skew-symmetric matrix A we have that e^A is orthogonal. Then $f(e^{tA}x)$ is a path in the SO-orbit of f. By taking the derivative at $t = 0$ we get $\sum_{p=0}^n \frac{\partial f}{\partial x_p}(Ax)_p \in \mathfrak{so}(W) \cdot f$. By taking $A = e_{ij} - e_{ji}$ we get exactly $D_{ij} f$ and these elements span $\mathfrak{so}(W) \cdot f$. □

The rank one tensors in $\mathrm{Sym}^d W$ have the form x^d and make a cone over the Veronese variety $v_d\mathbb{P}W$, where the origin has been removed from the cone. We recall that the eigenvectors of $f \in \mathrm{Sym}^d W$ are the critical points of the function $d_f(x^d) = q(f - x^d)$ restricted to the rank one tensors [11, 12, 16, 17, 19]. In this paper we are interested in the condition $x^d \in H_f$, which does not distinguish between x and its scalar multiples, so by abuse of notation we may shift to projective space $\mathbb{P}W$ and denote by the same symbol the point $x \in \mathbb{P}W$. The eigenvectors correspond to the non isotropic x (i.e. $q(x) \neq 0$) such that $\nabla f(x) = x$ in $\mathbb{P}W$, which means that any representatives of the right and the left hand side differ by a nonzero scalar multiple. The connection with (2.2) and (2.3) is that the eigenvectors of f make the base locus of the linear system $\langle D_{ij} f\rangle$.

It follows from Theorem 1.2 that the eigenvectors of f lie in H_f (which is obvious from the above description since $D_{ij} f$ are the minors of the matrix $\begin{pmatrix}\nabla f\\ x\end{pmatrix}$) and moreover the critical points of d_f on the secant varieties of d-Veronese variety lie in H_f, which is not obvious from the definition and it was proved first in [5, Theorem 1.1]. We will state more precisely this claim in the more general setting of partially symmetric tensors in Corollary 2.6.

We give now a more precise converse to Theorem 1.2 (1) in the case when X is the cone of symmetric tensors of rank one.

Theorem 2.2 *For general $f \in \mathrm{Sym}^d W$, $H_f \cap v_d\mathbb{P}W$ consists exactly of the critical points of d_f restricted to $v_d\mathbb{P}W$, namely of the eigenvectors of f.*

Proof Let $v^d \in H_f \cap v_d\mathbb{P}W$. In particular $q(v^d, g\cdot f) = 0$ for any $g \in \mathfrak{g}$, which implies that $D_{ij} f$ vanishes at v. This is equivalent to the matrix

$$\begin{pmatrix}\nabla f\\ x\end{pmatrix}$$

having rank one at v, which is the condition that v is eigenvector of f, if v is not isotropic. By [6, Lemma 4.2] the critical points of d_f for a general f avoid any proper closed subset of $v_d\mathbb{P}W$, so for general f it is guaranteed that no isotropic v is found. □

Remark 2.3 Note that for even d, $\mathfrak{g}\cdot(f + cq^{d/2}) = \mathfrak{g}\cdot f + [q^{d/2}]$ for any $c \in \mathbb{C}\setminus\{0\}$. Conversely, if $\mathfrak{g}\cdot f = \mathfrak{g}\cdot h$ for general f, h then we get $H_f = H_h$, so that f, h have the same eigenvectors and Turatti proves in [20] (generalizing previous results from [1, 2]) that there exists $c \in \mathbb{C}$ such that $f + cq^{d/2} = h$.

2.2 Partially Symmetric Tensors

Consider the tensor product $\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k = V$. We assume we have nondegenerate symmetric bilinear forms q_i on V_i. V is equipped with the Frobenius form q such that on decomposable elements

$$q\left(v_1^{d_1}\otimes\cdots\otimes v_k^{d_k}, w_1^{d_1}\otimes\cdots\otimes w_k^{d_k}\right) = \prod_{i=1}^{k} q_i(v_i, w_i)^{d_i}.$$

The decomposable elements make a cone over the Segre–Veronese variety $X \simeq \mathbb{P}V_1 \times\cdots\times \mathbb{P}V_k$ embedded in $\mathbb{P}V$ with the line bundle $\mathcal{O}(d_1,\ldots,d_k)$. The group $G = SO(V_1, q_1)\times\cdots\times SO(V_k, q_k)$ acts on V, we have again the inclusion $G \subset SO(V, q)$ and Theorem 1.2

applies. Denote by $x_{i,0}\dots x_{i,n_i}$ an orthogonal coordinate system on V_i. Analogously to Proposition 2.1 the orbit $\mathfrak{so}(V_1)\times\cdots\times\mathfrak{so}(V_k)\cdot f$ is spanned by $x_{p,j}\frac{\partial f}{\partial x_{p,i}}-x_{p,i}\frac{\partial f}{\partial x_{p,j}}$ for $0\le i<j\le n_p$, $p=1,\dots,k$. It follows that the critical space H_f defined according to (1.1) coincides with the one defined in [5].

The critical points of $d_f(x)=q(f-x)$ restricted to the Segre–Veronese variety are the singular t-ples of f [12], their number is called EDdegree in [4] and it is counted by the formula in [7], see also [4, §8].

The proof of Theorem 2.2 generalizes to this setting and gives

Theorem 2.4 *For general $f\in \mathrm{Sym}^{d_1}V_1\otimes\cdots\otimes\mathrm{Sym}^{d_k}V_k=V$, let $X\subset\mathbb{P}V$ be the Segre–Veronese variety of rank one tensors. $H_f\cap X$ consists exactly of the singular t-ples of f.*

Since general partially symmetric tensors f have trivial isotropic groups, in the binary case $X_{\mathbf{d}}=\mathbb{P}^1\times\cdots\times\mathbb{P}^1$ embedded in $\mathbb{P}(\mathrm{Sym}^{d_1}\mathbb{C}^2\otimes\cdots\otimes\mathrm{Sym}^{d_k}\mathbb{C}^2)$ with the line bundle $\mathcal{O}(d_1,\dots,d_k)$ we have $G=(\mathbb{C}^*)^k$, $\mathfrak{g}=\mathbb{C}^k$ and the nice coincidence $\operatorname{codim}H_f=k=\dim X_{\mathbf{d}}$. Hence the cardinality of the intersection between H_f and $X_{\mathbf{d}}$ can be counted by Bezout Theorem and it follows an alternative proof of the formula

$$\mathrm{EDdegree}(X_{\mathbf{d}})=\deg X_{\mathbf{d}}=k!d_1\dots d_k, \tag{2.4}$$

already known from [7], [18, Eq. (1.6)]. Our approach explains that the resulting equality between EDdegree and deg of $X_{\mathbf{d}}$ is not a coincidence. Note that Bezout Theorem applies when the intersection scheme has the expected codimension, without assuming H_f being general, see (1) in [8, §8.4]. We will apply again this approach to complete flag varieties in Theorem 3.4.

Example 2.5 If $\dim A=\dim B=\dim C=2$ we denote by Q_A, (resp. Q_B, Q_C) the isotropic quadric consisting of two points on $\mathbb{P}(A)$ (resp. $\mathbb{P}(B)$, $\mathbb{P}(C)$).

We have that $\dim(\mathfrak{so}\cdot f)<3$ if and only if f belongs to one of the following six $\mathbb{P}^3$ linearly embedded in $\mathbb{P}(A\otimes B\otimes C)$ (each item consists of two $\mathbb{P}^3$'s)

$$Q_A\times\mathbb{P}(B\otimes C),\quad Q_B\times\mathbb{P}(A\otimes C),\quad Q_C\times\mathbb{P}(A\otimes B).$$

The following result was proved in [5], joint with J. Draisma and A. Tocino. The proof given here, as a consequence of Theorem 1.2, is maybe simpler.

Corollary 2.6 [5, Theorem 1.1] *Let X_q be the q-secant variety to the Segre–Veronese variety in $\mathbb{P}\left(\mathrm{Sym}^{d_1}V_1\otimes\cdots\otimes\mathrm{Sym}^{d_k}V_k\right)$. Then the critical points of the distance function from a tensor f to X_q lie in H_f. In particular any best rank q approximation of f (when it exists) lie in H_f.*

3 Grassmann and Flag Varieties

3.1 Grassmann Varieties

Let $V=\wedge^k W$, we consider the Grassmann variety $Gr(k,W)$ of k-dimensional subspaces of W, its cone is embedded in V. Again, a nondegenerate quadratic form q_W on W extends to the Frobenius form q on V by requiring $q(v_1\wedge\cdots\wedge v_k, w_1\wedge\cdots\wedge w_k)=\det\left(q_W(v_i,w_j)\right)$

(the Gram determinant). If $v = v_1 \wedge \cdots \wedge v_k \in \wedge^k W$ then the derivative $\frac{\partial v}{\partial x_i} \in \wedge^{k-1} W$ is defined by the Leibniz formula

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^{k} v_1 \wedge \cdots \wedge \frac{\partial v_j}{\partial x_i} \wedge \cdots \wedge v_d$$

and extended by linearity to all $\wedge^k W$. This is compatible with the inclusion $\wedge^k W \subset W^{\otimes k}$ and the form q just defined is the restriction of the Frobenius form on $W^{\otimes k}$ of the previous section. The same formula (2.2) holds formally in case $SO(W)$ acts on $\wedge^k W$.

$$\mathfrak{so}(W) \cdot f = \left\langle \frac{\partial f}{\partial x_i} \wedge x_j - \frac{\partial f}{\partial x_j} \wedge x_i \right\rangle_{0 \le i < j \le n}. \tag{3.1}$$

The EDdegree of Grassmann varieties with respect to the Frobenius form is still unknown in general.

For a general $f \in \wedge^k W$, we have that a non isotropic $v = v_1 \wedge \cdots \wedge v_k$ is a critical point for d_f if $T(v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k) = q(v_i, -)\ \forall i = 1, \ldots k$. Again, the proof of Theorem 2.2 generalizes to this setting and gives

Theorem 3.1 *For general $f \in \wedge^k W$, $H_f \cap Gr(k, W)$ consists exactly of the critical points of d_f restricted to the Grassmann variety $Gr(k, W)$.*

3.2 Flag Varieties

For a flag variety $X = SL(W)/P$, where P is a parabolic subgroup of $SL(W)$ [9, §23.3], embedded by a very ample line bundle L, $H_f \cap X$ consists exactly of the critical points of d_f. The embedding space is a Schur module $S^\alpha W$ where the Frobenius form is defined again by restriction of the one on $W^{\otimes k}$ and again we have $G = SO(W)$.

For complete flag varieties $\mathbb{F}_n$, which parametrize complete flags $(L_1 \subset \cdots \subset L_n) \subset W$ with $\dim L_i = i$ (partial flags may miss some L_i's), the above principle becomes effective in computing the number of critical points. We recall that $\dim \mathbb{F}_n = n(n+1)/2$ and that $\mathbb{F}_n = SL(n+1)/B$ where B is the Borel subgroup of upper triangular matrices. The following two lemmas are well known, we include the proofs for the convenience of the reader.

Lemma 3.2 $\chi(\mathbb{F}_n, \mathbf{Z}) = (n+1)!$.

Proof A general section of the tangent bundle $T\mathbb{F}_n$ is given by a matrix $A \in SL(n+1) = SL(W)$ with distinct eigenvalues and corresponding eigenvectors $v_1, \ldots v_{n+1}$. The zero locus of this section consists of A-invariant complete flags $(L_1 \subset \ldots \subset L_n)$ with $\dim L_i = i$. There are $(n+1)$ choices for L_n, obtained by the span of n among the v_i. For each L_n there are correspondingly n choices for L_{n-1}, and so on there are $(n+1)!$ choices for each A-invariant complete flag. The thesis follows from Gauss–Bonnet theorem. □

Lemma 3.3 (i) *Let $\mathbb{F}_n$ be embedded with the line bundle $\mathcal{O}(a_1, \ldots, a_n)$ in the projective space over $S^{a_1,\ldots,a_n}\mathbb{C}^{n+1}$, the module with Young diagram having $\sum_{i=j}^{n} a_i$ boxes in the jth row. The degree of the embedded variety is*

$$\binom{n+1}{2}! \prod_{1 \le i < j \le n+1} \frac{a_i + \cdots + a_{j-1}}{j - i}. \tag{3.2}$$

(ii) *When $a_i = 1$ we get* $\deg \mathbb{F}_n = \binom{n+1}{2}!$.

Proof We have $H^0(\mathbb{F}_n, \mathcal{O}(a_1, \ldots, a_n)) = \prod_{1\le i<j\le n+1} \frac{a_i+\cdots+a_{j-1}+j-i}{j-i}$ by Weyl character formula (see [9, equation (15.17)]). Then the Hilbert polynomial is $H^0(\mathbb{F}_n, \mathcal{O}(ta_1, \ldots, ta_n)) = \prod_{1\le i<j\le n+1} \frac{t(a_i+\cdots+a_{j-1})+j-i}{j-i}$ and computing the leading term we get the thesis. In case (ii) the Hilbert polynomial simplifies to $\chi(\mathbb{F}_n, \mathcal{O}(t, \ldots, t)) = (t+1)^{\binom{n+1}{2}}$. □

Theorem 3.4 *Let $B \subset SL(n+1)$ be the Borel subgroup of upper triangular matrices. For a complete flag variety $\mathbb{F}_n = SL(n+1)/B$, embedded by a very ample line bundle $L = \mathcal{O}(a_1, \ldots, a_n)$, with respect to the Frobenius form we have that* $\text{EDdegree}\,\mathbb{F}_n = \deg \mathbb{F}_n$ *is given by* (3.2).

Proof For general $f \in H^0(SL(n+1)/B)$, we have again the nice coincidence that the codimension of H_f is equal to the dimension of $\mathbb{F}_n$, which is $\binom{n+1}{2} = \dim SO(n+1)$, so that the critical points are cut by a linear space of complementary dimension. □

Example 3.5 For $n = 2$, the flag variety $SL(3)/B$ (see I § 3.1 of [13]) embedded with $\mathcal{O}(a, b)$ has $\text{EDdegree}\,\mathbb{F}_2 = \deg \mathbb{F}_2 = 3ab(a+b)$.

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