Remarks on Poincaré and interpolation estimates for Truncated Hierarchical B-spline

ANNALISA BUFFA
Institute of Mathematics, Ecole Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland
annalisa.buffa@epfl.ch

CARLOTTA GIANNELLI
Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, viale Morgagni 67a, 50134 Firenze, Italy
carlotta.giannelli@unifi.it

This paper should be considered as an addendum to [Math. Models Methods Appl. Sci. Vol. 26, No. 1 (2016) pp. 1–25] and [Math. Models Methods Appl. Sci. Vol. 27, No. 14 (2017) pp. 2781–2802] where Poincaré and approximation estimates are used as theoretical tools to study properties of adaptive numerical methods based on hierarchical B-splines. After noting that the support of truncated hierarchical B-splines may be disconnected (and thus no Poincaré estimate can hold) we study minimal extensions of their support on suitable mesh configurations such that *i)* Poincaré estimates can be established on them and *ii)* their overlaps stay independent of the number of levels. The Poincaré estimates proposed in this note should replace the ones used in the proofs of Theorem 11 and Lemma 7 in Ref. 1 and 2, respectively, in order to include the most general meshes, i.e., the cases when basis functions support can be disconnected.

1. Introduction

This short note complements the two papers in Refs. 1 and 2. In particular, we examine the validity of Poincaré inequalities on the support (or extensions of it) of truncated hierarchical B-splines. In what follows, we first show a few key properties of hierarchical splines in Section 2, and we remind the problem we are willing to solve in Section 3. In Section 4, we show the validity of suitable Poincaré estimates on well constructed extensions of the truncated hierarchical splines supports. Finally, in Section 5, we apply these estimates to clarify the proofs of global and local upper bounds in Refs. 1, 2.

2. THB-splines on refined strictly admissible meshes

Let $V^\ell \subset V^{\ell+1}$, for $\ell = 0, \ldots, N-1$, be a sequence of tensor-product spline spaces defined on a closed hyper-rectangle $D \in \mathbb{R}^d$. We assume open knot vectors in any direction at level 0 and dyadic mesh refinement between consecutive hierarchical levels. Note that V^0 may contain repeated knots, while any knot at subsequent levels appears with multiplicity one. At any level ℓ, we consider the B-spline basis $\widehat{\mathcal{B}}^\ell$ of fixed degree $\mathbf{p} := (p_1, \ldots, p_d)$ defined on the rectilinear grid $\widehat{G}^\ell$. Any (non-empty) element $\widehat{Q}$ of the grid is the Cartesian product of d open intervals defined by consecutive breakpoints. We consider a sequence of domains $\widehat{\Omega}^\ell$, for $\ell = 0, \ldots, N-1$, with $\widehat{\Omega}^N = \emptyset$, such that $\widehat{\Omega}^{\ell+1} \subseteq \widehat{\Omega}^\ell$ and $\widehat{\Omega}^{\ell+1}$ is the union of elements in $\widehat{\Omega}^\ell$. We also define the set $\widehat{\mathcal{G}}^\ell$ of active elements of level ℓ, and the hierarchical mesh $\widehat{\mathcal{Q}}$ as follows: $\widehat{\mathcal{G}}^\ell := \left\{ \widehat{Q} \in \widehat{G}^\ell : \widehat{Q} \subset \Omega^\ell \wedge \widehat{Q} \not\subset \Omega^{\ell+1} \right\}$, $\widehat{\mathcal{Q}} := \left\{ \widehat{Q} \in \widehat{\mathcal{G}}^\ell, \ell = 0, \ldots, N-1 \right\}$.

The mesh $\widehat{\mathcal{Q}}$ of active elements with respect to the domain hierarchy $\widehat{\Omega}^{\ell-1} \supseteq \widehat{\Omega}^\ell$, for $\ell = 1, \ldots, N$, is *strictly admissible* of class m if $\widehat{\Omega}^\ell \subseteq \widehat{\omega}^{\ell-m+1}$, for $\ell = m, m+1, \ldots, N-1$, where

$$\widehat{\omega}^{\ell-m+1} := \bigcup \left\{ \overline{\widehat{Q}} : \widehat{Q} \in \widehat{G}^{\ell-m+1} \wedge S(\widehat{Q}, \ell - m + 1) \subseteq \widehat{\Omega}^{\ell-m+1} \right\},$$

and

$$S(\widehat{Q}, k) := \left\{ \widehat{Q}' \in \widehat{G}^k : \exists \widehat{\beta} \in \widehat{\mathcal{B}}^k, \ \operatorname{supp} \widehat{\beta} \cap \widehat{Q}' \neq \emptyset \wedge \operatorname{supp} \widehat{\beta} \cap \widehat{Q} \neq \emptyset \right\},$$

with $0 \leq k \leq \ell$. In what follows, we will consider strictly admissible meshes of class m that are generated by the REFINE routine[a] introduced in Ref. 1. For the sake of brevity, we will call these meshes *refined strictly admissible meshes of class* m and denote them RSAm.

In what follows, given an RSAm mesh $\widehat{\mathcal{Q}}$, we denote by $\widehat{\mathcal{T}}(\widehat{\mathcal{Q}})$ the collection of truncated hierarchical B-splines (THB-splines) constructed following the algorithm described in Ref. 3 (and used in Ref. 1, 2). Let $\operatorname{mot} \widehat{\tau} := \widehat{\beta}$ be the *mother* B-spline that originates the THB-spline $\widehat{\tau}$ via the truncation mechanism. The level of $\widehat{\tau}$ is the level of its mother function. We define the *extended support* of any THB-spline $\widehat{\tau}$ of level ℓ, $\ell = 0, \ldots, N-1$, as the support of $\operatorname{trunc}^{\ell+1}(\operatorname{mot} \widehat{\tau})$ obtained by applying the truncation mechanism only with respect to level $\ell + 1$. The functions $\widehat{\tau}$ and $\operatorname{trunc}^{\ell+1}(\operatorname{mot} \widehat{\tau})$ coincide only when $\widehat{\tau}$ is no further truncated at any successive level $k > \ell + 1$.

Definition 1. For any THB-spline $\widehat{\tau} \in \widehat{\mathcal{T}}(\widehat{\mathcal{Q}})$ of level ℓ, $\ell = 0, \ldots, N-1$, the extended support $\widehat{\omega}_{\widehat{\tau},\ell+1}$ of $\widehat{\tau}$ is defined as the support of $\operatorname{trunc}^{\ell+1} \operatorname{mot} \widehat{\tau}$, namely $\widehat{\omega}_{\widehat{\tau},\ell+1} := \operatorname{supp}(\operatorname{trunc}^{\ell+1} \operatorname{mot} \widehat{\tau})$. Moreover, we denote by $\widehat{\omega}^0_{\widehat{\tau},\ell+1} := \widehat{\omega}_{\widehat{\tau},\ell+1} \setminus \partial \widehat{\omega}_{\widehat{\tau},\ell+1}$.

Proposition 2. *Let $\widehat{\mathcal{Q}}$ be an RSAm mesh. For any $\widehat{\tau} \in \mathcal{T}(\widehat{\mathcal{Q}})$ of level ℓ, $\ell = 0, \ldots, N-1$, the set $\widehat{\omega}^0_{\widehat{\tau},\ell+1}$ of $\widehat{\tau}$ is a connected subset of $\widehat{\Omega}$, and $\widehat{\omega}^0_{\widehat{\tau},\ell+1} \cap \widehat{\omega}^{\ell+1} = \emptyset$.*

Proof. When the extended support of a THB-spline of level ℓ is considered, the truncation is applied only between level ℓ and $\ell + 1$. This means that the domain $\widehat{\omega}_{\widehat{\tau},\ell+1}$ can be fully covered by considering only (active or non-active) elements of

[a]The REFINE routine defined in Ref. 1 is defined on physical domains (see Section 3). Clearly, each mesh in the physical domain has a parametric counterpart. In what follows, we will use the term RSAm both for meshes in the physical space and for their parametric counterpart.

these two successive levels. Let assume by absurd that $\widehat{\omega}^0_{\widehat{\tau},\ell+1}$ is not connected. Since for any THB-spline $\widehat{\tau}$ introduced at level ℓ, $\mathrm{trunc}^{\ell+1}\widehat{\beta}$, with $\widehat{\beta} = \mathrm{mot}\,\widehat{\tau}$, is the linear combination of B-splines of level $\ell+1$ included in $\mathrm{supp}\,\widehat{\beta}$ but not in $\widehat{\Omega}^{\ell+1}$, any element of level $\ell+1$ in $\widehat{\omega}^0_{\widehat{\tau},\ell+1}$ belongs to the support of one of these B-splines which partially overlap at least one element of level ℓ. Consequently, two disconnected components could be generated only around two active elements, $\widehat{Q}'$ and $\widehat{Q}''$, of level ℓ contained in $\mathrm{supp}\,\widehat{\tau}$.

Indeed, the following reasoning proves that these two components cannot have disconnected interiores.

In order for the supports of two B-splines (one intersecting $\widehat{Q}'$ and one intersecting $\widehat{Q}''$) of level $\ell+1$ to have disconnected interiors, there should be at least $2p+1$ knots of level $\ell+1$ between $\widehat{Q}'$ and $\widehat{Q}''$. On the other hand, by dyadically refining the closure of the inner elements in $\mathrm{supp}\,\widehat{\beta}$ between $\widehat{Q}'$ and $\widehat{Q}''$ we obtain at most $2p-1$ knots of level $\ell+1$ in any coordinate direction. The upper bound of $2p-1$ knots is obtained by computing $(p+2)+(p+1)-4$, where $p+2$ is the number of knots of level ℓ in $\mathrm{supp}\,\widehat{\beta}$, $(p+1)$ is the maximal number of newly inserted knots,[b] and 4 are the two boundary knots together with the new knots of level $\ell+1$ in the first and last intervals of $\mathrm{supp}\,\widehat{\beta}$. Consequently $\widehat{\omega}^0_{\widehat{\tau},\ell+1}$ is connected.

To prove the second property, remember that an element $\widehat{Q}$ of level $\ell+1$ contributes to definition of $\widehat{\omega}^{\ell+1}$ if and only if its support extension of level $\ell+1$ is contained in $\widehat{\Omega}^{\ell+1}$. This means that all the B-splines that are non zero on $\widehat{Q}$ have support in $\widehat{\Omega}^{\ell+1}$. Since the truncation of a B-spline of level ℓ with respect to level $\ell+1$ is a linear combination of B-splines $\widehat{\beta}^{\ell+1}$ of level $\ell+1$ whose support is not fully contained in $\widehat{\Omega}^{\ell+1}$, any element intersecting one of these $\widehat{\beta}^{\ell+1}$ cannot contribute to $\widehat{\omega}^{\ell+1}$. This implies that $\widehat{\omega}_{\widehat{\tau},\ell+1}$ and $\widehat{\omega}^{\ell+1}$ may have a non empty intersection only along their boundaries. □

In what follows $\lesssim$ stands for $\leq C$, where C is a constant independent of all parameters and quantities involved in the inequality.

Corollary 3. *Let $\widehat{\mathcal{Q}}$ be an RSAm mesh of class m, for any $\widehat{\tau} \in \widehat{\mathcal{T}}(\widehat{\mathcal{Q}})$ of level ℓ, $\ell = 0,\dots,N-1$. The number of active elements in $\widehat{\omega}_{\widehat{\tau},\ell+1}$ is bounded independently of ℓ, and*

$$h_{\widehat{\omega}_{\widehat{\tau},\ell+1}} := \mathrm{diam}(\widehat{\omega}_{\widehat{\tau},\ell+1}) \lesssim \min_{\widehat{Q}\subseteq\widehat{\omega}_{\widehat{\tau},\ell+1}} \mathrm{diam}(\widehat{Q}).$$

Proof. Proposition 2, together with the property: $\Omega^{\ell+m} \subseteq \widehat{\omega}^{\ell+1}$ on any RSAm, implies that $\widehat{\omega}^0_{\widehat{\tau},\ell+1} \cap \widehat{\Omega}^{\ell+m} = \emptyset$. Consequently, the active elements in $\widehat{\omega}_{\widehat{\tau},\ell+1}$ can only belong to level ℓ, $\ell+1$, ..., $\ell+m-1$ and are at a bounded distance from each other. □

[b]The maximal number of newly inserted knots when performing dyadic refinement on the support of a B-spline of degree p refers to the case of single knots. Note that higher multiplicities reduce the number of elements in its support.

We define the domain $\bar{S}^*(\widehat{Q})$ associated to an element $\widehat{Q}$ of level $\ell(\widehat{Q})$ of an RSAm mesh as the region covered by the collection of elements in the extended supports of THB-splines $\widehat{\tau}$ so that $\widehat{\omega}_{\widehat{\tau},\ell+1} \cap \widehat{Q} \neq \emptyset$, namely

$$\bar{S}^*(\widehat{Q}) := \bigcup \left\{ \widehat{\omega}_{\widehat{\tau},\ell+1} : \widehat{\tau} \in \widehat{\mathcal{T}} \ \wedge \ \widehat{\omega}_{\widehat{\tau},\ell+1} \cap \widehat{Q} \neq \emptyset \right\},$$

and $S^*(\widehat{Q}) := \mathrm{int}\, \bar{S}^*(\widehat{Q})$ is the set without its boundary. Since $\widehat{\mathcal{Q}}$ is an RSAm mesh, the coarsest level of functions $\widehat{\tau}$ whose extended support intersects the element $\widehat{Q}$ is $\max(0, \ell(\widehat{Q}) - m + 1)$. Consequently, the level of the THB-splines considered in the definition of $\bar{S}^*(\widehat{Q})$ varies between $\max(0, \ell(\widehat{Q}) - m + 1)$ and $\ell(\widehat{Q})$. To highlight this property we denote $S^*(\widehat{Q})$ with $S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)$ in what follows.

Theorem 4. *Let $\widehat{\mathcal{Q}}$ be an RSAm mesh. For all $\widehat{Q} \in \widehat{\mathcal{Q}}$, the set $S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)$ is connected and contains a finite number of active elements $\widehat{Q}' \in \widehat{\mathcal{Q}}$. The number of such elements stays bounded, independently of $\widehat{Q}$ and of its level, but it depends on m. It also holds that:*

$$h_{S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)} := \mathrm{diam}(S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)) \lesssim \mathrm{diam}(\widehat{Q}).$$

Proof. This is implied by Corollary 3, by also observing that $S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)$ is the union of a finite number of extended supports, which all share at least the element $\widehat{Q}$. □

Corollary 5. *Let $\widehat{\mathcal{Q}}$ be an RSAm mesh. Then, there exists a costant $C_{\mathcal{R}}$ such that for all $\widehat{Q} \in \widehat{\mathcal{Q}}$, the number of elements $\widehat{Q}'$ such that $\widehat{Q} \subset S^*(\widehat{Q}', \ell(\widehat{Q}') - m + 1)$ is bounded by $C_{\mathcal{R}}$. The constant $C_{\mathcal{R}}$ depends on m but not on the number of levels of $\widehat{\mathcal{Q}}$.*

Proof. Note that $\widehat{Q} \subset S^*(\widehat{Q}', \ell(\widehat{Q}') - m + 1)$ if and only if $\widehat{Q'} \subset S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)$ since it must exist a truncated basis function whose extended support contains both elements. In view of Theorem 4, the number of active elements $\widehat{Q'}$ in $S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)$ is bounded, thus any element $\widehat{Q}$ belongs to a finite number of $S^*(\widehat{Q}', \ell(\widehat{Q}') - m + 1)$. □

Finally, for further use, we also remark that if $\partial\widehat{Q} \cap \partial\widehat{\Omega} \neq \emptyset$, due to Theorem 4, we have

$$\mu_{d-1}(\partial S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1) \cap \partial\widehat{\Omega}) \leq \mu_{d-1}(\partial S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)) \qquad (2.1)$$

where $\mu_{d-1}(\cdot)$ stands for the $(d-1)-$dimensional Hausdorff measure.

Remark 2.1. In the statements of Theorem 4 or its Corollary 5 on RSAm meshes, (the unions of the) $\widehat{\omega}_{\widehat{\tau},\ell+1}$ cannot be replaced by (unions of) the full support of the mother B-spline mot τ. Indeed, the number of superpositions of full B-spline supports (instead of extended THB-spline supports) on a single element would depend (and possibly explode) with the level of the element.

3. Mapping, physical domain and differential problems

As in Refs. 1 and 2, the computational domain Ω is the spline mapping of $\widehat{\Omega}$, i.e., $\Omega = \mathbf{F}(\widehat{\Omega})$ and, all geometrical quantities defined in the previous section can be defined on the physical domain by means of the mapping $\mathbf{F}$. In what follows, we will assume that $\mathbf{F} \in C^1(\overline{\Omega})$ and that its inverse is at least Lipschitz regular.

In particular, we set $\mathcal{Q} = \mathbf{F}(\widehat{\mathcal{Q}})$, $\tau := \mathbf{F}(\widehat{\tau})$, $\mathcal{T}$ the collections of all the basis functions τ (and correspondingly $\mathcal{T}_0$ the ones vanishing on $\partial\Omega$), $\omega_{\tau,\ell+1} := \mathbf{F}(\widehat{\omega}_{\widehat{\tau},\ell+1})$. For simplicity, from now on, we assume that all $\tau \in \mathcal{T}$ are C^1 continuous. In addition,

$$S^*(Q, \ell(Q) - m + 1) \;:=\; \mathbf{F}(S^*(\widehat{Q}, \ell(\widehat{Q}) - m + 1)) \tag{3.1}$$

where, clearly $\ell(Q) = \ell(\widehat{Q})$. All geometric properties proved for the sets on the parametric space are valid on the physical space.

As in Refs. 1 and 2, we set the following continuous problem as model problem: for $f \in L^2(\Omega)$, find

$$u \in \mathbb{V} : \quad a(u, v) = (f, v), \quad \forall\, v \in \mathbb{V}, \tag{3.2}$$

where $\mathbb{V} = H^1_0(\Omega)$, $(\cdot, \cdot)$ is the L^2 scalar product, and $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is the bilinear form $a(u, v) := \int_\Omega \mathbf{A}\nabla u \nabla v$. The corresponding Galerkin approximation consists in solving:

$$\text{find } \; U \in \mathbb{S}_D(\mathcal{Q}) : \quad a(U, V) = (f, V), \quad \forall\, V \in \mathbb{S}_D(\mathcal{Q}), \tag{3.3}$$

where $\mathbb{S}_D(\mathcal{Q}) := \operatorname{span} \mathcal{T}(\mathcal{Q}) \cap H^1_0(\Omega)$.

4. Poincaré estimates

In Refs. 1 and 2, we have used Poincaré estimates that need to be revisited. We here present new (valid) Poincaré estimates that will be used in the next section to provide new proofs for the local and global lower bound.

We start with two Poincaré estimates in the reference domain. The following holds:

Theorem 6. *Let $\widehat{\mathcal{Q}}$ be an RSAm mesh. The following holds :*

i) For all $v \in H^1(\widehat{\Omega})$ and $\widehat{\tau} \in \widehat{\mathcal{T}}(\mathcal{Q})$:

$$\inf_{c \in \mathbb{R}} \|v - c\|_{L^2(\widehat{\omega}_{\widehat{\tau},\ell+1})} \leq \widehat{C}_{\mathcal{P}} h_{\widehat{\omega}_{\widehat{\tau},\ell+1}} \|\nabla v\|_{L^2(\widehat{\omega}_{\widehat{\tau},\ell+1})}, \tag{4.1}$$

where $\widehat{C}_{\mathcal{P}}$ is a constant which is independent of $\widehat{\tau}$ and its level, but depends on m and ℓ is the level of $\widehat{\tau}$.

ii) For all $v \in H^1_0(\widehat{\Omega})$, and for all $\widehat{\tau} \in \widehat{\mathcal{T}}(\mathcal{Q})$ such that $\widehat{\omega}_{\widehat{\tau},\ell+1} \cap \partial\widehat{\Omega} \neq \emptyset$, it holds:

$$\|v\|_{L^2(\widehat{\omega}_{\widehat{\tau},\ell+1})} \leq \widehat{C}_{\mathcal{H}} h_{\widehat{\omega}_{\widehat{\tau},\ell+1}} \|\nabla v\|_{L^2(\widehat{\omega}_{\widehat{\tau},\ell+1})}, \tag{4.2}$$

where, as before, $\widehat{C}_{\mathcal{H}}$ is a constant which is independent of $\widehat{\tau}$ and its level ℓ, but depends on m.

Proof. We are going to prove both inequalities by using the results of Section 2.3 in Ref. 5. First, we remark that, by Proposition 2, $\forall \widehat{\tau} \in \widehat{\mathcal{T}}(\widehat{\mathcal{Q}})$ the set $\widehat{\omega}_{\widehat{\tau},\ell+1}$ is connected. Second, we consider the mesh obtained by refining all elements in $\widehat{\omega}_{\widehat{\tau},\ell+1}$ to level $\ell+m$. This mesh is a conforming and structured mesh whose elements belong to $\widehat{G}^{\ell+m}$, and, as a consequence of Corollary 3, it contains at most n_τ elements, with n_τ independent of ℓ. Moreover, all elements have comparable sizes. We denote the running element of this refined mesh $\widehat{Q}_i$, $i = 1, \ldots, n_\tau$ and we assume $n_\tau > 2$. Of course, if $n_\tau \leq 2$ the problem is trivial. Moreover, we denote by $\mu(\widehat{Q}_i)$ the d-dimensional Hausdorff measure of the element $\widehat{Q}_i$, for all i.

We cannot apply directly Proposition 2.10 to $\widehat{\omega}_{\widehat{\tau},\ell+1}$ as, in general, this is the union of several hyper-rectangles, and may not be Lipschitz for $d \geq 3$. Instead, we will follow the steps of Section 2.3 in Ref. 5 and show that both (4.1) and (4.2) hold true.

For any $v \in H^1(\widehat{\Omega})$ and $\widehat{\tau} \in \widehat{\mathcal{T}}$, by using the same strategy as in Lemma 2.9 of in Ref. 5,[c] we obtain: for a given index k, $k = 1, \ldots, n_\tau$, for all constants $v_i \in \mathbb{R}$, $i = 1, \ldots, n_\tau$

$$\|v - v_k\|^2_{L^2(\widehat{\omega}_{\widehat{\tau},\ell+1})} \leq \|v - v_k\|^2_{L^2(\widehat{Q}_k)} + \sum_{i=1}^{n_\tau} 2\|v - v_i\|^2_{L^2(\widehat{Q}_i)} + \sum_{i\neq k, i=1}^{n_\tau} 2(n_\tau - 1) A_i W_i^2(v) \tag{4.3}$$

where

$$W_i(v) = \frac{2}{\mu(\widehat{Q}_i)} \|v - v_i\|^2_{L^2(\widehat{Q}_i)} + 4\frac{\operatorname{diam}(\widehat{Q}_i)}{\mu(\widehat{Q}_i)} \|v - v_i\|_{L^2(\widehat{Q}_i)} \|\nabla v\|_{L^2(\widehat{Q}_i)}$$

and $A_k = \mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - \mu(\widehat{Q}_k)$ and $A_i = 2\mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - 2\mu(\widehat{Q}_k) - \mu(\widehat{Q}_i)$. Now, we have the following main facts: for the choice $v_i = \dfrac{\int_{\widehat{Q}_i} v_i}{\mu(\widehat{Q}_i)}$ it holds

$$\|v - v_i\|_{L^2(\widehat{Q}_i)} \leq C \operatorname{diam}(\widehat{Q}_i) \|\nabla v\|_{L^2(\widehat{Q}_i)} \qquad \forall i \quad \forall v \in H^1(\widehat{\Omega}) \tag{4.4}$$

$$\|v\|_{L^2(\widehat{Q}_k)} \leq C \operatorname{diam}(\widehat{Q}_k) \|\nabla v\|_{L^2(\widehat{Q}_k)} \quad \forall v \in H^1_0(\widehat{\Omega}) \text{ and } \overline{\widehat{Q}_k} \cap \partial\widehat{\Omega} \neq \emptyset. \tag{4.5}$$

We use now the estimate (4.3) with the following choices:

(1) If $\widehat{\omega}_{\widehat{\tau},\ell+1}$ does not share any edge with $\partial\widehat{\Omega}$, then $v_i = \dfrac{\int_{\widehat{Q}_i} v_i}{\mu(\widehat{Q}_i)}$, $i = 1, \ldots, n_\tau$.
(2) If $\widehat{\omega}_{\widehat{\tau},\ell+1}$ does share an edge with $\partial\widehat{\Omega}$, i.e., $\overline{\widehat{\omega}_{\widehat{\tau},\ell+1}} \cap \partial\Omega \neq \emptyset$, then: we select k such that $\overline{\widehat{Q}_k} \cap \partial\widehat{\Omega} \neq \emptyset$, we set $v_k = 0$ and $v_i = \dfrac{\int_{\widehat{Q}_i} v_i}{\mu(\widehat{Q}_i)}$, for all $i = 1, \ldots, n_\tau$, $i \neq k$.

[c] This result is based purely on a trace inequality for each single $\widehat{Q}_i$ and the conformity of the partition. This holds without any Lipschitz assumption.

With these choices, and using (4.4)-(4.5), we estimate the various terms on the right hand side of (4.3) as follows:

$$\begin{aligned} W_i^2(v) &\lesssim C\frac{\mathrm{diam}^2(\widehat{Q}_i)}{\mu(\widehat{Q}_i)} \qquad \forall\, i = 1,\dots,n_\tau, \\ A_k W_k^2(v) &\lesssim \frac{\mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - \mu(\widehat{Q}_k)}{\mu(\widehat{Q}_k)}\,\mathrm{diam}(\widehat{Q}_k)^2\|\nabla v\|^2_{L^2(\widehat{Q}_k)}, \\ \sum_{i\neq k, i=1}^{n_\tau} A_i W_i^2(v) &\lesssim \mathrm{diam}(\widehat{\omega}_{\widehat{\tau},\ell+1})^2 \sum_{i\neq k,i=1}^{n_\tau} \frac{2\mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - \mu(\widehat{Q}_k) - \mu(\widehat{Q}_i)}{\mu(\widehat{Q}_i)}\|\nabla v\|^2_{L^2(\widehat{Q}_i)}, \end{aligned}$$

where in the last inequality we have used that $\max_{i=1,\dots,n_\tau} \mu(\widehat{Q}_i) \le \mathrm{diam}(\widehat{\omega}_{\widehat{\tau},\ell+1})$. Using that, for $n_\tau > 2$:

$$\frac{\mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - \mu(\widehat{Q}_k)}{\mu(\widehat{Q}_k)} \lesssim 1, \qquad \frac{2\mu(\widehat{\omega}_{\widehat{\tau},\ell+1}) - \mu(\widehat{Q}_k) - \mu(\widehat{Q}_i)}{\mu(\widehat{Q}_i)} \lesssim 1,$$

and that $\mathrm{diam}(\widehat{Q}_k) \lesssim \mathrm{diam}(\widehat{\omega}_{\widehat{\tau},\ell+1})$, the estimates (4.1) and (4.2) follow. □

The inequalities of Theorem 6 remain valid when we move to the physical domain, as it is stated in Ref. 5, Section 2.1. Let us formulate a Theorem in the physical space as follows:

Theorem 7. *Let $\mathcal{Q}$ be an RSAm mesh. The following holds for all $v \in H^1(\Omega)$ and $\tau \in \mathcal{T}$:*

$$\inf_{c\in\mathbb{R}} \|v - c\|_{L^2(\omega_{\tau,\ell+1})} \le C_{\mathcal{P}} h_{\omega_{\tau,\ell+1}} \|\nabla v\|_{L^2(\omega_{\tau,\ell+1})}. \tag{4.6}$$

where $h_{\omega_{\tau,\ell+1}}$ is the diameter of $\omega_{\tau,\ell+1}$ and $C_{\mathcal{P}}$ is a constant which is independent of τ and its level ℓ,but depends on m. Moreover, for all $v \in H^1_0(\Omega)$, and, for all τ such that $\omega_{\tau,\ell+1} \cap \partial\Omega \neq \emptyset$

$$\|v\|_{L^2(\omega_{\tau,\ell+1})} \le C_{\mathcal{H}} h_{\omega_{\tau,\ell+1}} \|\nabla v\|_{L^2(\omega_{\tau,\ell+1})} \tag{4.7}$$

where, as before, $C_{\mathcal{H}}$ is a constant which is independent of τ and its level, but depends on m.

We have also the following result, which can be proved exactly following the same steps as in Theorems 6 and 7:

Theorem 8. *Let $\mathcal{Q}$ be an RSAm mesh. For every $Q \in \mathcal{Q}$, the following holds for all $v \in H^1(\Omega)$:*

$$\inf_{c\in\mathbb{R}} \|v - c\|_{L^2(S^*(Q,\ell(Q)-m+1))} \le C'_{\mathcal{P}}\, h_{S^*(Q,\ell(Q)-m+1)} \|\nabla v\|_{L^2(S^*(Q,\ell(Q)-m+1))}, \tag{4.8}$$

where $h_{S^*(Q,\ell(Q)-m+1)}$ is the diameter of $S^*(Q,\ell(Q)-m+1)$ and $C'_{\mathcal{P}}$ is a constant which is independent of Q and its level, but depends on m. Moreover, for all $v \in H^1_0(\Omega)$, and for all Q that has at least a side on the boundary $\partial\Omega$, it holds:

$$\|v\|_{L^2(S^*(Q,\ell(Q)-m+1))} \le C'_{\mathcal{H}} h_{S^*(Q,\ell(Q)-m+1)} \|\nabla v\|_{L^2(S^*(Q,\ell(Q)-m+1))} \tag{4.9}$$

where, as before, $C'_{\mathcal{H}}$ is a constant which is independent of Q and its level, but depends on m.

5. Global and local upper bound

Let

$$\varepsilon^2_{\mathcal{Q}}(U,\mathcal{Q}) = \sum_{Q\in\mathcal{Q}} \varepsilon^2_{\mathcal{Q}}(U,Q) \qquad \text{with} \qquad \varepsilon^2_{\mathcal{Q}}(U,Q) = h^2_Q ||r||^2_{L^2(Q)}. \tag{5.1}$$

where the residual r is defined by: $\langle r, v\rangle = \int_\Omega \big(f + \operatorname{div}(\mathbf{A}\nabla U)\big)\, v$.

The Poincaré estimates (4.6) and (4.7) can be used to prove that $\varepsilon_{\mathcal{Q}}(U,\mathcal{Q})$ is an upper bound for the discretisation error.

Theorem 9 (Global upper bound). *Let $\mathcal{Q}$ be an RSAm mesh. Let u be the exact weak solution of the model problem (3.2). The error of the Galerkin approximation $U \in \mathbb{S}(\mathcal{Q})$ in (3.3) is bounded in terms of the error indicator $\varepsilon_{\mathcal{Q}}(U)$ introduced in (5.1) as follows:*

$$||u - U||_{\mathbb{V}} \leq C_{\rm up}\varepsilon_{\mathcal{Q}}(U,\mathcal{Q}), \tag{5.2}$$

where the constant $C_{\rm up}$ is independent on the mesh size, on the level of hierarchy but depends upon m.

Proof. This proof follows exactly the lines of the classical proof of upper bound in residual based error estimators. For completeness we repeat here the steps that can be found in, e.g., Theorem 6 in Ref. 4.

Using (17) of Ref. 1, we have $\|u - U\|_{\mathbb{V}} \lesssim \dfrac{1}{\alpha_1}\|r\|_{\mathbb{V}^\star}$, and we will prove that $\|r\|_{\mathbb{V}^\star} \lesssim \varepsilon_{\mathcal{Q}}(U,\mathcal{Q})$. We recall that $\mathcal{T}$ forms a partition of unity and we denote by $\mathcal{T}_0$ the collection of all basis functions that vanish at the boundary $\partial\Omega$. It holds:

$$\langle r, v\rangle = \sum_{\tau\in\mathcal{T}} \langle r, \tau\, v\rangle = \sum_{\tau\in\mathcal{T}_0} \inf_{c_\tau\in\mathbb{R}} \langle r, \tau\,(v - c_\tau)\rangle + \sum_{\tau\in\mathcal{T}\setminus\mathcal{T}_0} \langle r, \tau\, v\rangle.$$

By standard Cauchy-Schwarz inequality, we estimate the terms in the right hand side as follows

$$\langle r, \tau\,(v - c_\tau)\rangle = \int_\Omega r\,\tau(v - c_\tau) \leq \|r\,\tau^{1/2}\|_{L^2(\Omega)} \|\tau^{1/2}(v - c_\tau)\|_{L^2(\Omega)},$$

and the same holds for $c_\tau = 0$. We denote by $\omega_\tau = \operatorname{supp}\tau$. We can deduce by the Poincaré inequalities in Theorem 7 that for all $v \in \mathbb{V}$:

$$\begin{aligned}
\|\tau^{1/2}(v - c_\tau)\|_{L^2(\omega_\tau)} &\lesssim \|(v - c_\tau)\|_{L^2(\omega_{\tau,\ell+1})} \lesssim C_{\mathcal{P}} h_{\omega_{\tau,\ell+1}} \|\nabla v\|_{L^2(\omega_{\tau,\ell+1})^d} && \forall\, \tau \in \mathcal{T}_0,\\
\|\tau^{1/2} v\|_{L^2(\omega_\tau)} &\lesssim \|v\|_{L^2(\omega_{\tau,\ell+1})} \lesssim C_{\mathcal{H}} h_{\omega_{\tau,\ell+1}} \|\nabla v\|_{L^2(\omega_{\tau,\ell+1})^d} && \forall\, \tau \in \mathcal{T}\setminus\mathcal{T}_0.
\end{aligned}$$

By taking into account Corollary 5, we know that each element Q of $\mathcal{Q}$ is contained in at most $C_{\mathcal{R}}$ neighbourhoods $\omega_{\tau,\ell+1}$. Thus:

$$\sum_{\tau\in\mathcal{T}} \|\nabla v\|^2_{L^2(\omega_{\tau,\ell+1})^d} \lesssim \|\nabla v\|^2_{L^2(\Omega)^d}.$$

Similarly, we let h be the piecewise constant function which takes values $h(\mathbf{x}) = |Q|^{1/d}$, $\mathbf{x} \in Q$ for all $Q \in \mathcal{Q}$. Using Corollary 3, we know that $h_{\omega_{\tau,\ell+1}} \lesssim h(\mathbf{x})$, for all $x \in \omega_{\tau,\ell+1}$, thus it holds

$$\sum_{\tau\in\mathcal{T}} h^2_{\omega_{\tau,\ell+1}} \|r\,\tau^{1/2}\|^2_{L^2(\omega_{\tau,\ell+1})} \lesssim \sum_{\tau\in\mathcal{T}} \int_{\omega_{\tau,\ell+1}} h^2\, r^2 \tau = \int_\Omega h^2\, r^2 = \varepsilon^2_{\mathcal{Q}}(U,\mathcal{Q}).$$

The estimate (5.2) follows. □

The following stability property is also valid on RSAm meshes.

Proposition 10. *Let $\mathcal{Q}$ be an RSAm mesh, and $\mathcal{I}_{\mathcal{Q}}$ the operator defined by equation (3.6) in Ref. 2. We have*

$$||\mathcal{I}_{\mathcal{Q}} v||_{L^2(Q)} \lesssim ||v||_{L^2(S^*(Q,\ell(Q)-m+1))}, \qquad \forall v \in L^2(\Omega), \tag{5.3}$$

$$||v - \mathcal{I}_{\mathcal{Q}} v||_{L^2(Q)} \lesssim h_Q ||v||_{H^1(S^*(Q,\ell(Q)-m+1))}, \qquad \forall v \in H^1_0(\Omega), \tag{5.4}$$

where $S^*(Q,\ell(Q)-m+1)$ is defined in (3.1).

The proof follows verbatim the proof of Proposition 5 of Ref. 2.

Lemma 11 (Localized upper bound). *Let $\mathcal{Q}$ and $\mathcal{Q}^*$ be two RSAm meshes so that $\mathcal{Q}^* \succeq \mathcal{Q}$, and $\mathcal{T}$ and $\mathcal{T}^\star$ the corresponding truncated hierarchical B-spline functions. We denote by $\mathcal{R}^\star$ the set of elements in the supports of the newly introduced basis functions in $\mathcal{T}^\star$. The corresponding Galerkin solutions $U \in \mathbb{S}_D(\mathcal{Q})$ and $U^* \in \mathbb{S}_D(\mathcal{Q}^*)$ of problem* (3.3) *satisfy*

$$|||U - U^*|||^2_\Omega \lesssim \varepsilon^2_{\mathcal{Q}}(U,\mathcal{R}^\star). \tag{5.5}$$

Proof. Let $\mathcal{I}_{\mathcal{Q}}$ be the operator defined in Section 3.2 in Ref. 2, and $E^* = U - U^*$. Let $\Omega_{\mathcal{R}^\star}$ be the union of the support of the newly introduced basis functions in $\mathcal{T}^\star$, and $\Omega_{\mathcal{Q}} = \Omega \setminus \Omega_{\mathcal{R}^\star}$.

In view of (3.11) in Ref. 2, we can consider the approximation $V \in \mathbb{S}_D(\mathcal{Q})$ defined as

$$V = \begin{cases} \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}^\star}, \\ E^* & \text{in } \Omega_{\mathcal{Q}}, \end{cases} \quad \text{so that} \quad E^* - V = \begin{cases} E^* - \mathcal{I}_{\mathcal{Q}} E^* & \text{in } \Omega_{\mathcal{R}^\star}, \\ 0 & \text{in } \Omega_{\mathcal{Q}}. \end{cases} \tag{5.6}$$

By combining

$$a(E^*,E^*) = a(U,E^*) - a(U^*,E^*)$$

with $a(E^*,E^*) = a(E^*,E^*-V)$ and taking into account (5.6), we have

$$a(E^*,E^*) \le \sum_{Q\in\mathcal{R}^\star} ||r(U)||_{L^2(Q)} ||E^* - \mathcal{I}_{\mathcal{Q}} E^*||_{L^2(Q)},$$

which in turn, due to (5.1) and (5.4), reduces to

$$|||E^*|||_\Omega^2 = a(E^*, E^*) \lesssim \sum_{Q\in\mathcal{R}^\star} \varepsilon_{\mathcal{Q}}(U,Q)||E^*||_{H^1(S^*(Q,\ell(Q)-m+1))}$$
$$\lesssim \left(\sum_{Q\in\mathcal{R}^\star} \varepsilon_{\mathcal{Q}}^2(U,Q)\right)^{1/2} \left(\sum_{Q\in\mathcal{R}^\star} ||E^*||^2_{H^1(S^*(Q,\ell(Q)-m+1))}\right)^{1/2}.$$

Now, by using Corollary 5, it holds:

$$\left(\sum_{Q\in\mathcal{R}^\star} ||E^*||^2_{H^1(S^*(Q,\ell(Q)-m+1))}\right) \leq C_{\mathcal{R}}|||E|||_\Omega^2,$$

which directly implies (5.5). □

Acknowledgments

The authors would like to thank Gregor Gantner, Rafael Vázquez and Andreas Veeser for interesting discussions on this topic. The first author kindly acknowledges the support of the European Research Council through the ERC Advanced Grant n. 654515, project CHANGE. The second author is member of the INdAM Research group GNCS. The INdAM support through GNCS and Finanziamenti Premiali SUNRISE is gratefully acknowledged.

References

1. A. Buffa and C. Giannelli. Adaptive isogeometric methods with hierarchical splines: Error estimator and convergence. *Math. Models Methods Appl. Sci.*, 26:1–25, 2016.
2. A. Buffa and C. Giannelli. Adaptive isogeometric methods with hierarchical splines: Optimality and convergence rates. *Math. Models Methods Appl. Sci.*, pages 2781–2802, 2017.
3. C. Giannelli, B. Jüttler, and H. Speleers. THB–splines: the truncated basis for hierarchical splines. *Comput. Aided Geom. Design*, 29:485–498, 2012.
4. R. H. Nochetto and A. Veeser. Primer of adaptive finite element methods. In *Multiscale and adaptivity: modeling, numerics and applications*, volume 2040 of *Lecture Notes in Math.*, pages 125–225. Springer, Heidelberg, 2012.
5. A. Veeser and R. Verfürth. Poincaré constants for finite element stars. *IMA J. Numer. Anal.*, 32:30–47, 2012.