Nonrationality Degree of Toric Quasifolds

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FIAMMETTA BATTAGLIA AND ELISA PRATO

ABSTRACT. Toric quasifolds are nonrational generalizations of toric manifolds and orbifolds. We introduce the notion of *nonrationality degree*, an invariant that measures the extent to which a toric quasifold fails to be Hausdorff. We do so by first defining analogous notions for the underlying quasilattice and the corresponding quasitorus. We conclude by applying the nonrationality degree to reframe Gordan's lemma in the nonrational setting.

Keywords. Toric quasifold, quasilattice, quasitorus, nonrational fan, nonrational polytope.

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INTRODUCTION

Toric quasifolds are a natural generalization of toric manifolds and orbifolds in the nonrational case. They were first introduced by the second author, in the symplectic setting, in [18]. Their complex and algebraic–geometric counterparts were introduced by both authors in [1] and [5], respectively. These spaces are locally modeled by $\mathbb{C}^n \cong \mathbb{R}^{2n}$ modulo the action of countable groups and are generally not Hausdorff.

The aim of this article is to introduce the notion of *nonrationality degree* of toric quasifolds, which measures the extent to which they fail to be Hausdorff.

We recall that the starting data for a toric quasifold is provided by the *fundamental triple*

$$(\Sigma, Q, \{v_1, \ldots, v_d\}),$$

where Σ is a complete simplicial fan in $\mathbb{R}^n$, Q is a quasilattice, namely the $\mathbb{Z}$–span of a set of spanning vectors in $\mathbb{R}^n$, and the vectors $v_1, \ldots, v_d$ are in Q and generate the rays of Σ. The corresponding toric quasifold, which we denote by X_Σ, is acted upon by the real quasitorus $\mathbb{R}^n/Q$ in the symplectic setting, and by the complex quasitorus $\mathbb{C}^n/Q$ in the complex and algebraic–geometric settings.

In order to define the nonrationality degree of X_Σ, we first investigate the structure of the underlying quasilattice Q.

As a consequence of the structure theorem for closed groups of $\mathbb{R}^n$, we prove that Q admits a canonical decomposition

$$Q = (Q \cap V) \oplus (Q \cap W),$$

where V and W are complementary subspaces of $\mathbb{R}^n$, $Q \cap V$ is dense in V and $Q \cap W$ is a lattice in W (see Theorem 1.3). The subspace V is unique and maximal with this property. We define the nonrationality degree of Q to be the dimension of V. This integer will ultimately define the nonrationality degree of the real and complex quasitori and of the toric quasifold.

In fact, the decomposition of Q naturally yields structure theorems for the corresponding quasitori. For example, in the complex case, we get

$$\mathbb{C}^n/Q \cong D^h_{\mathbb{C}} \times (\mathbb{C}^*)^{n-h},$$

where h is the nonrationality degree of Q and $D^h_{\mathbb{C}}$ is a totally nonrational complex quasitorus of dimension h (see Theorem 2.4). Accordingly, we define h to be the nonrationality degree of $\mathbb{C}^n/Q$. The same applies to $\mathbb{R}^n/Q$. Observe that Q is a lattice and both quasitori are tori if, and only if, their nonrationality degree h is zero. Increasing values of h correspond to the presence of higher–dimensional totally nonrational directions.

As in the rational case, the action of the complex quasitorus has a dense open orbit. This naturally leads to defining the notion of nonrationality degree of the toric quasifold to coincide with that of its residing quasitorus.

A given complete simplicial fan may be viewed in the context of infinitely many different triples, and the degree of the corresponding quasifolds will vary accordingly. We take the minimum possible degree of nonrationality of these quasifolds to define the intrinsic degree of nonrationality of the fan. The fan is rational if, and only if, its nonrationality degree is zero. This applies also to simple polytopes, by considering their normal fan.

The motivation for introducing the notion of nonrationality degree came from two different contexts: on the one hand, from the framework of the foliated models of toric quasifolds and their measures of nonrationality (see [7, Section 2.2.3]); on the other hand, from studying nonrational toric geometry in the algebraic–geometric setting (see [5]). In the latter, replacing a lattice with a quasilattice has significant repercussions, such as the different behavior of duals (see Remarks 1.9 and 5.2) and the failure of Gordan's Lemma, a fundamental result in classical toric geometry. In fact, we recast the lemma by proving that it holds if, and only if, the degree of nonrationality of the quasilattice is zero (see Theorem 5.1).

We conclude by remarking that quasilattices are key objects in several different areas of mathematics. For example, they form the basis for the study of nonperiodic tilings that are related to quasicrystals (see, for example, [21, 2, 3]). They also appear, explicitly or implicitly, in most other works addressing nonrational toric geometry (see [7, 8, 20, 14, 13, 17, 16, 9]). Moreover, they arise naturally in the framework of diffeology (see, for example, [12, 15]).

The paper is organized as follows. In Section 1, we investigate the structure of quasilattices and define their nonrationality degree. In Section 2, we establish the structure theorems for quasitori and introduce their nonrationality degree. In Section 3, we extend this notion to toric quasifolds.

In Section 4, we discuss the case of fans and polytopes. Finally, in Section 5, we reframe Gordan's lemma in the nonrational setting.

1. Nonrationality degree of a quasilattice

We begin by recalling the following

Definition 1.1. A *quasilattice* Q in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of a set of spanning vectors.

A quasilattice is, in particular, a free $\mathbb{Z}$–module of finite rank and a subgroup of $\mathbb{R}^n$. The *rank* of Q is the number of vectors in a set of minimal generators. We recall that a lattice L in $\mathbb{R}^n$ is the $\mathbb{Z}$–span of a basis of $\mathbb{R}^n$. It is therefore a quasilattice of rank equal to n. On the other hand, the classical definition of lattice [22, Chapter 7, Section 2.2] is that of a subgroup L of $\mathbb{R}^n$ that satisfies the following equivalent conditions:

- L is discrete and $\mathbb{R}^n/L$ is compact;
- L is discrete and spans $\mathbb{R}^n$;
- there exists an $\mathbb{R}$–basis of $\mathbb{R}^n$ which is a $\mathbb{Z}$–basis of L.

Therefore, a quasilattice Q in $\mathbb{R}^n$ is a lattice if, and only if, it is a discrete subset of $\mathbb{R}^n$.

The aim of this section is to introduce the notion of *nonrationality degree* of a quasilattice. We will be making use of the classification theorem of closed subgroups of $\mathbb{R}^n$, which we recall from Bourbaki [10, Chapter VII, Section 1.2, Thm. 2]:

Theorem 1.2. *Let G be a closed subgroup of $\mathbb{R}^n$, and let $k = \dim(\mathrm{span}_{\mathbb{R}}(G))$. Then there exists a maximal vector subspace $V \subset \mathbb{R}^n$, $\dim(V) \leq k$, such that, for every complement W of V, $\Lambda = W \cap G$ is a discrete subgroup of rank $r = k - \dim V$ and $G = V \oplus \Lambda$.*

Given a quasilattice Q in $\mathbb{R}^n$, we apply this classification theorem to the closed subgroup $\overline{Q}$ and obtain the following

Theorem 1.3. *Let Q be a quasilattice of $\mathbb{R}^n$. Then there exists a unique maximal vector subspace $V \subset \mathbb{R}^n$ and a complement W of V such that*

$$Q = (Q \cap V) \oplus (Q \cap W),$$

where $Q \cap V$ is dense in V and $Q \cap W$ is a lattice in W.

Proof: Since Q is a subgroup of $\mathbb{R}^n$, $\overline{Q}$ is a closed subgroup of $\mathbb{R}^n$. Moreover, since Q spans $\mathbb{R}^n$, so does $\overline{Q}$. Applying Theorem 1.2 to $\overline{Q}$, we obtain a maximal vector subspace $V \subset \mathbb{R}^n$ such that, for every complement W of V, we have $\overline{Q} = V \oplus \Lambda$, where $\Lambda = \overline{Q} \cap W$ is a discrete subgroup of rank $r = n - \dim(V)$, thus a lattice in W. Let

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/V$$

be the quotient map. Since Q is dense in $\overline{Q}$ and π is continuous, $\pi(Q)$ is dense in $\pi(\overline{Q})$. Since

$$\pi(\overline{Q}) \cong \overline{Q}/V \cong \Lambda,$$

the group $\pi(\overline{Q})$ is a lattice in $\mathbb{R}^n/V$; in particular, it is discrete.

Let us show that $\overline{\pi(Q)} = \pi(\overline{Q})$. Since $Q \subset \overline{Q}$, we have $\overline{\pi(Q)} \subset \overline{\pi(\overline{Q})}$. However, $\pi(\overline{Q})$ is discrete, hence closed. Therefore $\overline{\pi(Q)} \subset \pi(\overline{Q})$. Conversely, consider an element $\pi(h)$, $h \in \overline{Q}$ and take a sequence $(q_m) \subset Q$ converging to h. By continuity, the sequence $(\pi(q_m)) \subset \pi(Q)$ converges to $\pi(h)$. Therefore $\pi(\overline{Q}) \subset \overline{\pi(Q)}$.

Since $\pi(\overline{Q})$ is discrete and $\pi(Q)$ is dense in $\pi(\overline{Q})$, we obtain $\pi(Q) = \pi(\overline{Q})$.

Now choose a $\mathbb{Z}$–basis of $\pi(Q)$ and write it as $\{\pi(w_1), \ldots, \pi(w_r)\}$, with $w_1, \ldots, w_r \in Q$. Take the subspace

$$W = \operatorname{span}_{\mathbb{R}}(w_1, \ldots, w_r).$$

Since the vectors $\pi(w_1), \ldots, \pi(w_r)$ form a basis of $\mathbb{R}^n/V$, the subspace W is a complement of V.

We claim that

$$Q \cap W = \operatorname{span}_{\mathbb{Z}}(w_1, \ldots, w_r).$$

Let $w \in Q \cap W$. Since $w \in W$, there exist real numbers $a_1, \ldots, a_r$ such that $w = \sum_{i=1}^r a_i w_i$. Thus $\pi(w) = \sum_{i=1}^r a_i \pi(w_i)$. Since $w \in Q$, its image via π belongs to $\pi(Q)$, and therefore $\pi(w) = \sum_{i=1}^r m_i \, \pi(w_i)$, for uniquely determined integers $m_1, \ldots, m_r$. Thus $a_i = m_i$ for all i. Therefore $w \in \operatorname{span}_{\mathbb{Z}}(w_1, \ldots, w_r)$. The reverse inclusion is obvious, proving the claim. It follows that $Q \cap W$ is a discrete subgroup of W of rank $r = n - \dim(V)$, thus a lattice in W.

Now let $u \in Q$. Since the classes $\pi(w_1), \ldots, \pi(w_r)$ generate $\pi(Q)$, there exist integers $h_1, \ldots, h_r$ such that $\pi(u) = \sum_{i=1}^r h_i \, \pi(w_i)$. Therefore

$$u - \sum_{i=1}^r h_i w_i \in V.$$

Since both terms belong to Q, we obtain

$$u - \sum_{i=1}^r h_i w_i \in Q \cap V.$$

Thus every element of Q can be written as the sum of an element of $Q \cap V$ and an element of $Q \cap W$. Therefore $Q = (Q \cap V) + (Q \cap W)$. Since $V \cap W = \{0\}$, we have $(Q \cap V) \cap (Q \cap W) = \{0\}$, and therefore

$$Q = (Q \cap V) \oplus (Q \cap W).$$

Let us now show that $Q \cap V$ is dense in V. Let $v \in V \subset \overline{Q}$, and take a sequence $(q_m) \subset Q$ converging to v. Since $Q = (Q \cap V) \oplus Q(\cap W)$, we can write $q_m = v_m + w_m$, with $v_m \in Q \cap V$ and $w_m \in Q \cap W$. Now, since $\mathbb{R}^n = V \oplus W$, we can consider the linear projection $p_V \colon \mathbb{R}^n \to V$. Since it is continuous, the sequence $(p_V(q_m)) = (v_m)$ converges to $p_V(v) = v$. Therefore $Q \cap V$ is dense in V.

To conclude, we show that V is unique. Suppose there are two maximal subspaces V_1 and V_2 such that $Q \cap V_1$ is dense in V_1, $Q \cap V_2$ is dense in V_2. Notice that $Q \cap (V_1 + V_2)$ is dense in $V_1 + V_2$. Indeed, take $v_1 + v_2 \in V_1 + V_2$. We know that there exists a sequence $(v_m^1) \subset Q \cap V_1$ converging to v_1 and a sequence $(v_m^2) \subset Q \cap V_2$ converging to v_2. Then the sequence $(v_m^1 + v_m^2) \subset Q \cap (V_1 + V_2)$ converges to $v_1 + v_2$. Therefore, by maximality, $\dim(V_1 + V_2) \leq \dim(V_1) = \dim(V_2)$, which implies $V_1 = V_2$. ■

Definition 1.4 (Maximal totally nonrational subspace)**.** Given a quasilattice Q in $\mathbb{R}^n$, we call the unique subspace V of Theorem 1.3 *maximal totally nonrational subspace* with respect to Q.

We are now finally ready for the following

Definition 1.5 (Nonrationality degree of a quasilattice)**.** Let Q be a quasilattice in $\mathbb{R}^n$. The *nonrationality degree* of Q, denoted $\deg_{\mathrm{NR}}(Q)$, is the dimension of the maximal totally nonrational subspace V.

We clearly have
$$0 \leq \deg_{\mathrm{NR}}(Q) \leq n.$$
As a first, straightforward, consequence of Theorem 1.3, we get

Corollary 1.6. *Let Q be a quasilattice in $\mathbb{R}^n$. Then $\deg_{NR}(Q) = 0$ if, and only if, Q is a lattice of rank n.*

One of the crucial differences between lattices and quasilattices is in the notion of dual. Let us first recall from [5] the following

Definition 1.7 (Weak dual of a quasilattice)**.** Given a quasilattice Q in $\mathbb{R}^n$, we define the *weak dual* of Q to be the $\mathbb{Z}$–module
$$Q^\vee = \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(Q) \subset \mathbb{Z}\}.$$

As a consequence of Theorem 1.3, we see that the elements of $Q^\vee$ only keep track of rational directions:

Corollary 1.8. *Let Q be a quasilattice of $\mathbb{R}^n$ and consider the subspaces V, W of Theorem 1.3. Then we have*

(1) *$Q^\vee \cong (Q \cap W)^*$ and is thus a discrete subgroup of $(\mathbb{R}^n)^*$ of rank $r = n - \deg_{NR}(Q)$;*
(2) *$V = \mathrm{Ann}(Q^\vee)$.*

Proof: (1) The direct sum decomposition $\mathbb{R}^n = V \oplus W$ allows us to write $\lambda = \lambda|_V + \lambda|_W$ for every $\lambda \in (\mathbb{R}^n)^*$. Now notice that, if $\lambda \in Q^\vee$, by continuity, $\lambda(\overline{Q}) \subset \mathbb{Z}$. Thus, since $V \subset \overline{Q}$ and V is connected, we have $\lambda|_V = 0$. Therefore, $\lambda = \lambda|_W$ and
$$Q^\vee = \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(Q) \subset \mathbb{Z}\} \cong \{\mu \in W^* \mid \mu(Q \cap W) \subset \mathbb{Z}\} = (Q \cap W)^*.$$

(2) We have already noticed that $\lambda|_V = 0$, for all $\lambda \in Q^\vee$. Thus $V \subset \mathrm{Ann}(Q^\vee)$. However, by (1) we have

$$\dim(\mathrm{Ann}(Q^\vee)) = n - \mathrm{rank}(Q^\vee) = n - (n - \deg_{\mathrm{NR}}(Q)) = \dim(V).$$

So the two subspaces must coincide. ■

Remark 1.9. Consider now the actual dual, $\mathrm{Hom}(Q, \mathbb{Z})$; it is a free $\mathbb{Z}$–module having the same rank as Q. When Q is a lattice, $\mathrm{Hom}(Q, \mathbb{Z})$ is naturally identified with $Q^\vee$ and can be thus thought of as a lattice in $(\mathbb{R}^n)^*$. However, if $\deg_{\mathrm{NR}}(Q) \neq 0$, by Corollary 1.8, this is no longer the case. This fact has some notable consequences in nonrational toric geometry (see [5] and Remark 5.2 below).

It will be useful, for what is to follow, to have Theorem 1.3 reframed as

Corollary 1.10. *Let Q be a quasilattice in $\mathbb{R}^n$ and let $h = \deg_{NR}(Q)$. Then there is an isomorphism*

$$f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n = \mathbb{R}^h \oplus \mathbb{R}^{n-h}$$

such that

$$Q \cong f(Q) = (f(Q) \cap \mathbb{R}^h) \oplus \mathbb{Z}^{n-h},$$

where $f(Q) \cap \mathbb{R}^h$ is a dense quasilattice in $\mathbb{R}^h$.

Proof: Consider the subspaces V and W of Theorem 1.3. Choose a basis $\{v_1, \ldots, v_h, w_1, \ldots, w_{n-h}\}$ of $\mathbb{R}^n$ such that $\{v_1, \ldots, v_h\}$ is a basis of V and $\{w_1, \ldots, w_{n-h}\}$ is a $\mathbb{Z}$–basis of $Q \cap W$. Let f be the isomorphism that sends this basis to the standard basis of $\mathbb{R}^n$. ■

We conclude this section with some examples.

Example 1.11. Let us consider the quasilattice in $\mathbb{R}$ given by $Q = \mathbb{Z} + a\mathbb{Z}$, with a a positive real number. The nonrationality degree obviously depends on a. In fact, if $a \in \mathbb{Z}_{\geq 0}$, we have $Q = \overline{Q} = \mathbb{Z},\ V = \{0\},\ Q^\vee = \mathbb{Z}$. If $a \in \mathbb{Q}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{\geq 0}$ coprime, then $Q = \overline{Q} = \frac{1}{q}\mathbb{Z},\ V = \{0\},\ Q^\vee = q\mathbb{Z}$. Finally, if $a \notin \mathbb{Q}$, then $\overline{Q} = \mathbb{R},\ V = \mathbb{R},\ Q^\vee = \{0\}$. In conclusion,

$$\deg_{\mathrm{NR}}(Q) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 1.12. Consider the quasilattice $Q = \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$ in $\mathbb{R}^2$, where a is a positive real number. Then, if $a \in \mathbb{Z}_{\geq 0}$, we have $Q = \overline{Q} = \mathbb{Z}^2,\ V = \{0\},\ Q^\vee = \mathbb{Z}^2$. If $a \in \mathbb{Q}$, and we write $a = \frac{p}{q}$, with $p, q \in \mathbb{Z}_{\geq 0}$ coprime, then $Q = \overline{Q} = \mathbb{Z} \times \frac{1}{q}\mathbb{Z},\ V = \{0\},\ Q^\vee = \mathbb{Z} \times q\mathbb{Z}$. Finally, if $a \notin \mathbb{Q}$, then $\overline{Q} = \mathbb{Z} \times \mathbb{R},\ V = \{0\} \times \mathbb{R},\ Q^\vee = \mathbb{Z} \times \{0\}$. Here we have, again,

$$\deg_{\mathrm{NR}}(Q) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 1.13. Consider the fifth roots of unity in $\mathbb{C} \cong \mathbb{R}^2$. Their $\mathbb{Z}$–span, Q_5, is a dense quasilattice in $\mathbb{R}^2$. This quasilattice turns out to be the crucial ingredient in the study of Penrose rhombus tilings and of the Penrose kite from the nonrational symplectic toric viewpoint (see [2, 3] and also Example 3.6 below). Here we have $\overline{Q} = V = \mathbb{R}^2$, $Q^\vee = 0$ and, finally, $\deg_{\mathrm{NR}}(Q) = 2$.

2. Structure theorems for quasitori

We begin by recalling the following

Definition 2.1 (Real and complex quasitori)**.** Let Q be a quasilattice in $\mathbb{R}^n$, the group $\mathbb{R}^n/Q$ is called a *real quasitorus* of dimension n and the group $\mathbb{C}^n/Q$ is called a *complex quasitorus* of dimension n.

Notice that when Q is a lattice, these groups are, respectively, a real torus and a complex torus. At the other extreme we have

Definition 2.2 (Totally nonrational quasitori)**.** Consider a quasilattice Q in $\mathbb{R}^n$. We say that the quasitori $\mathbb{R}^n/Q$ and $\mathbb{C}^n/Q$ are *totally nonrational* if Q is dense in $\mathbb{R}^n$.

The general case is a bit of both, as described in the following structure theorems.

Theorem 2.3 (Structure theorem for a real quasitorus)**.** *Let Q be a quasilattice in $\mathbb{R}^n$, and let $h = \deg_{NR}(Q)$. Then*

$$\mathbb{R}^n/Q \cong D^h \times \mathbb{T}^{n-h},$$

where D^h is a totally nonrational real quasitorus of dimension h and $\mathbb{T}^{n-h}$ is the standard torus of dimension $n-h$.

Proof: By Corollary 1.10, we have

$$Q \cong (f(Q) \cap \mathbb{R}^h) \oplus \mathbb{Z}^{n-h},$$

where $f(Q) \cap \mathbb{R}^h$ is dense in $\mathbb{R}^h$. Thus

$$\mathbb{R}^n/Q \cong \left(\mathbb{R}^h/(f(Q) \cap \mathbb{R}^h)\right) \times \left(\mathbb{R}^{n-h}/\mathbb{Z}^{n-h}\right).$$

■

Similarly, in the complex case, we have

Theorem 2.4 (Structure theorem for a complex quasitorus)**.** *Let Q be a quasilattice in $\mathbb{R}^n$, and let $h = \deg_{NR}(Q)$. Then*

$$\mathbb{C}^n/Q \cong D^h_{\mathbb{C}} \times (\mathbb{C}^*)^{n-h},$$

where $D^h_{\mathbb{C}}$ is a totally nonrational complex quasitorus of dimension h.

Motivated by this, we have the following

Definition 2.5 (Nonrationality degree of a quasitorus)**.** Let Q be a quasilattice in $\mathbb{R}^n$. We call *nonrationality degree* of the quasitorus $\mathbb{R}^n/Q$ (respectively $\mathbb{C}^n/Q$), denoted $\deg_{\mathrm{NR}}(\mathbb{R}^n/Q)$ (respectively $\deg_{\mathrm{NR}}(\mathbb{C}^n/Q)$), the nonrationality degree of Q.

Let us see some examples. To avoid unnecessary repetition, we will be focusing on the real case.

Example 2.6. Let us consider the quasilattice $\mathbb{Z}+a\mathbb{Z}$ in $\mathbb{R}$, where a is a positive real number (see Example 1.11) and take the quasitorus

$$\mathbb{R}/(\mathbb{Z}+a\mathbb{Z}).$$

If $a\in\mathbb{Z}_{\geq 0}$, we obtain $\mathbb{R}/\mathbb{Z}\cong S^1$. If $a\in\mathbb{Q}$, and we write $a=\frac{p}{q}$, with $p,q\in\mathbb{Z}_{\geq 0}$ coprime, we get $\mathbb{R}/\left(\frac{1}{q}\mathbb{Z}\right)\cong S^1$. Finally, if $a\notin\mathbb{Q}$, we get the *tore irrationnel* $\mathbf{T}^a$, which was first introduced within the framework of diffeology by Donato–Iglesias [12]. In conclusion, we have

$$\deg_{\mathrm{NR}}(\mathbb{R}/(\mathbb{Z}+a\mathbb{Z}))=\begin{cases} 0 & a\in\mathbb{Q}\\ 1 & a\notin\mathbb{Q}.\end{cases}$$

Example 2.7. Consider the quasilattice $Q=\mathbb{Z}\times(\mathbb{Z}+a\mathbb{Z})$ in $\mathbb{R}^2$, where a is a positive real number (see Example 1.12) and take the quasitorus

$$\mathbb{R}^2/(\mathbb{Z}\times(\mathbb{Z}+a\mathbb{Z})).$$

Then, if $a\in\mathbb{Z}_{\geq 0}$, we obtain $\mathbb{R}^2/\mathbb{Z}^2$. If $a\in\mathbb{Q}$, and we write $a=\frac{p}{q}$, with $p,q\in\mathbb{Z}_{\geq 0}$ coprime, then we get $(\mathbb{R}/\mathbb{Z})\times\left(\mathbb{R}/\frac{1}{q}\mathbb{Z}\right)\cong\mathbb{R}^2/\mathbb{Z}^2$. Finally, if $a\notin\mathbb{Q}$ we get $(\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/(\mathbb{Z}+a\mathbb{Z}))\cong S^1\times\mathbf{T}^a$. In particular,

$$\deg_{\mathrm{NR}}(\mathbb{R}^2/(\mathbb{Z}\times(\mathbb{Z}+a\mathbb{Z})))=\begin{cases} 0 & a\in\mathbb{Q}\\ 1 & a\notin\mathbb{Q}.\end{cases}$$

Example 2.8. Take the $\mathbb{Z}$–span, Q_5, of the fifth roots of unity (see Example 1.13). Since Q_5 is a dense quasilattice in $\mathbb{R}^2$, the quasitorus $\mathbb{R}^2/Q_5$ is totally nonrational and its nonrationality degree is 2.

3. Nonrationality degree of a toric quasifold

Let Σ be a complete simplicial fan in $\mathbb{R}^n$ (we refer the reader to [11] for the necessary background on fans).

We recall that a *fundamental triple* for Σ is given by

$$(\Sigma,Q,\{v_1,\ldots,v_d\}),$$

where Q is a quasilattice in $\mathbb{R}^n$, and the vectors $v_1,\ldots,v_d\in Q$ generate the rays of Σ. One way of obtaining a fundamental triple for *any* Σ is to first choose a set of ray generators and then take Q to be their $\mathbb{Z}$–span.

Notice that when the fan is rational with respect to a lattice L, there is an underlying canonical triple, obtained by taking Q to be equal to L and the ray generators to be primitive in L.

To any fundamental triple, there corresponds a toric quasifold, which we will denote by X_Σ. For the symplectic construction of X_Σ, which holds when Σ is also polytopal, see [18]; its complex and algebraic–geometric counterparts are treated in [1] and [5], respectively. The space X_Σ is a toric manifold or orbifold if, and only if, the fan Σ is rational and Q is a lattice, if, and only if, $\deg_{\mathrm{NR}}(Q) = 0$.

Analogously to what happens in the rational case, the quasitorus acts on X_Σ and, in the complex setting, there is a dense open orbit

$$\mathcal{O}_o \cong \mathbb{C}^n/Q$$

which corresponds to the zero cone; all other points of X_Σ belong to the lower dimensional orbits corresponding to the nonzero cones of Σ. It is natural to define the nonrationality degree of X_Σ to be the same as that of the quasitorus, namely

Definition 3.1 (Nonrationality degree of a toric quasifold)**.** We define the *nonrationality degree* of X_Σ to be the nonrationality degree of Q, denoted $\deg_{\mathrm{NR}}(X_\Sigma)$.

Definition 3.2 (Totally nonrational quasifold)**.** We say that the quasifold X_Σ is *totally nonrational* when the corresponding quasilattice Q is dense.

Let us compute the nonrationality degree of some notable examples of toric quasifolds.

Example 3.3 (Quasisphere)**.** Consider the *quasisphere* X_a, namely the toric quasifold corresponding to the triple

$$(\Sigma, \mathbb{Z} + a\mathbb{Z}, \{a, -1\}),$$

where a is a positive real number. Σ is the complete simplicial fan in $\mathbb{R}$ given by the cones $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. We refer the reader to [18, Example 1.13], and [19] for the symplectic construction, to [1, Examples 2.6, 3.8], [4, Example 3.1] for the complex, and to [5] for the algebraic–geometric.

Since the underlying quasilattice is $\mathbb{Z} + a\mathbb{Z}$, we have, by Example 1.11,

$$\deg_{\mathrm{NR}}(X_a) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

The quasisphere is totally nonrational if, and only if, $a \notin \mathbb{Q}$.

Example 3.4 (Generalized weighted projective space)**.** We consider now *generalized weighted projective space* $\mathbb{CP}^2_{(1,1,a)}$. It is the toric quasifold corresponding to the triple

$$(\Sigma, \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z}), \{(1, 0), (-1, -a), (0, 1)\}),$$

where a is any positive real number. Σ is the complete simplicial fan in $\mathbb{R}^2$ whose rays are generated by the vectors $(1, 0)$, $(-1, -a)$, and $(0, 1)$. See [18, Example 3.6], [1, Example 2.8], [4, Example 3.2], and [5] for descriptions of this toric quasifold in the symplectic, complex, and algebraic–geometric settings, respectively.

Since the underlying quasilattice here is $\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$, we have, by Example 1.12,

$$\deg_{\mathrm{NR}}(\mathbb{CP}^2_{(1,1,a)}) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 3.5 (Generalized Hirzebruch surface)**.** Take now the generalized Hirzebruch surface $\mathbb{H}_a$. This is the toric quasifold corresponding to the triple

$$(\Sigma, \mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z}), \{(1,0), (-1,-a), (0,1), (0,-1)\}),$$

where a is again any positive real number. Σ is the complete simplicial fan in $\mathbb{R}^2$ whose rays are generated by the vectors $(1,0)$, $(-1,-a)$, $(0,1)$, and $(0,-1)$. The toric quasifold $\mathbb{H}_a$ was first introduced in both the symplectic and complex setting in [6]; see [5] for the algebraic–geometric version. The corresponding quasilattice is again $\mathbb{Z} \times (\mathbb{Z} + a\mathbb{Z})$, so we have

$$\deg_{\mathrm{NR}}(\mathbb{H}_a) = \begin{cases} 0 & a \in \mathbb{Q} \\ 1 & a \notin \mathbb{Q}. \end{cases}$$

Example 3.6 (The Penrose kite quasifold)**.** Let us consider the toric quasifold X_5 corresponding to the triple

$$(\Sigma, Q_5, \{v_1, v_2, v_3, v_4\}),$$

where $v_1 = -(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5})$, $v_2 = -(\cos\frac{6\pi}{5}, \sin\frac{6\pi}{5})$, $v_3 = (\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5})$, $v_4 = (\cos\frac{8\pi}{5}, \sin\frac{8\pi}{5})$. Σ is the complete simplicial fan whose rays are generated by these vectors. It is the normal fan of the Penrose kite. This toric quasifold was introduced and studied from the symplectic viewpoint in [3]; see [4, Example 3.2] for the complex case and [5] for the algebraic–geometric. Since the underlying quasilattice is the pentagonal quasilattice Q_5, by Example 1.13, we have that $\deg_{\mathrm{NR}}(X_5) = 2$. The toric quasifold X_5 is totally nonrational.

4. Nonrationality degree of fans and polytopes

Let Σ be a complete simplicial fan in $\mathbb{R}^n$. Take the set $\mathcal{Q}_\Sigma$ of all quasilattices $Q \subset \mathbb{R}^n$ that have nonzero intersection with all rays of Σ.

Definition 4.1 (Nonrationality degree of a fan)**.** *We define the nonrationality degree of Σ to be the nonnegative integer*

$$\deg_{NR}(\Sigma) = \min_{Q \in \mathcal{Q}_\Sigma} \deg_{NR}(Q).$$

Let us consider now an n–dimensional simple polytope $\Delta \subset \mathbb{R}^n$ (a standard reference for polytopes is [23]). Let Σ_Δ be its normal fan; it is a complete simplicial fan.

Definition 4.2 (Nonrationality degree of a polytope)**.** We define the *nonrationality degree* of Δ to be the nonrationality degree of Σ_Δ.

Clearly, a fan or polytope is rational if, and only if, its nonrationality degree is 0.

Let us now compute the nonrationality degree of the fans that we introduced in the examples of Section 3.

Example 4.3. The fan of Example 3.3 is rational with respect to $\mathbb{Z}$, so its nonrationality degree is 0.

Example 4.4. The fans of Examples 3.4 and 3.5 are rational with respect to the lattice $\mathbb{Z} \times a\mathbb{Z}$, so the nonrationality degree of both is also 0.

Example 4.5. The Penrose kite (see Example 3.6), and its normal fan, are not rational. Their nonrationality degree is actually 1: normalize the fifth roots of unity so that each has x–coordinate equal to either 0 or 1, and then consider their $\mathbb{Z}$–span, $\widetilde{Q}_5$. By construction, $\deg_{\mathrm{NR}}(\widetilde{Q}_5) = 1$.

5. Gordan's Lemma revisited

Let Q be a quasilattice in $\mathbb{R}^n$ and consider a maximal simplicial cone σ in $\mathbb{R}^n$ whose rays have nonzero intersection with Q. Gordan's Lemma (see, for example, [11, Proposition 1.2.17]) fails in this setting. As it turns out, this failure is measured precisely by the nonrationality degree of Q:

Theorem 5.1 (Gordan's Lemma revisited)**.** *Let Q be a quasilattice in $\mathbb{R}^n$ and let σ be a maximal simplicial cone whose rays all have nonzero intersection with Q. Then the semigroup $\sigma \cap Q$ is finitely generated if, and only if, $\deg_{NR}(Q) = 0$.*

Proof: If $\deg_{\mathrm{NR}}(Q) = 0$, then Q is a lattice, and the semigroup $\sigma \cap Q$ is finitely generated by Gordan's Lemma. Conversely, suppose that the semigroup $\sigma \cap Q$ is finitely generated. We first prove that it is discrete. First, consider a finite set of generators of $\sigma \cap Q$. Since each ray of σ has nonzero intersection with Q, the above set of generators must contain at least one generator lying on each ray of σ. Indeed, in a simplicial cone, a ray cannot be generated unless one of the generators lies on that ray. Denote these n vectors by $\{v_1, \dots, v_n\}$ and let $\{v_1, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$ be the whole set of generators of the semigroup $\sigma \cap Q$. Then, for each $j = n+1, \dots, n+k$ we have $v_j = \sum_{l=1}^{n} a_{lj} v_l$, with a_{lj} nonnegative real numbers. Write each $a_{lj} = m_{lj} + b_{lj}$, with m_{lj} a nonnegative integer and $0 \leq b_{lj} < 1$. Let $u_j = \sum_{l=1}^{n} b_{lj} v_l$ and discard the vectors u_j that are equal to 0. After renumbering, we are left with $u_{n+1}, \dots, u_{n+h}$, with $h \leq k$. Then it is straightforward to verify that the set $\{v_1, \dots, v_n, u_{n+1}, \dots, u_{n+h}\}$ still generates the semigroup $\sigma \cap Q$. Moreover, the vectors of this set all lie in the zonotope

$$\Delta = \left\{ \sum_{i=1}^{n} t_i v_i \mid 0 \leq t_i \leq 1 \right\}$$

defined by $v_1, \ldots, v_n$. We now prove that $\Delta \cap Q = \Delta \cap (Q \cap \sigma)$ is finite. Indeed, let $u \in Q \cap \Delta$. Then, since $u \in Q \cap \sigma$, we have

$$u = \sum_{i=1}^{n} m_i v_i + \sum_{j=n+1}^{n+h} n_j u_j$$

with m_i, n_j nonnegative integers. Hence

$$u = \sum_{i=1}^{n} \left(m_i + \sum_{j=n+1}^{n+h} n_j b_{ij} \right) v_i.$$

The matrix (b_{ij}) has entries $0 \leq b_{lj} < 1$ and all of its columns are nonzero, since the u_j's are nonzero and the v_i's are linearly independent. On the other hand, since $u \in Q \cap \Delta$ we must have

$$0 \leq m_i + \sum_{j=n+1}^{n+h} n_j b_{ij} \leq 1.$$

Each m_i can only be 0 or 1. Moreover, since every column of the matrix (b_{ij}) is nonzero, for every $j = n+1, \ldots, n+h$ there exists an index i such that $b_{ij} > 0$. Therefore, $n_j b_{ij} \leq \sum_{l=n+1}^{n+h} n_l b_{il} \leq 1$, which implies that $n_j \leq \frac{1}{b_{ij}}$. Hence each n_j is bounded above and there are only finitely many possible tuples $(m_1, \ldots, m_n, n_{n+1}, \ldots, n_{n+h})$. Therefore $\Delta \cap (Q \cap \sigma)$ is finite. Moreover, the semigroup is the union of the translates $\Delta + \sum_{i=1}^{n} m_i v_i$ with m_i nonnegative integers. It is therefore discrete.

Finally, we observe that this implies that Q is in fact a lattice. Indeed, if it were a quasilattice of rank greater than n, then there would be a sequence $(q_m) \subset Q \setminus \{0\}$ converging to 0. Now, consider the point $v = v_1 + \ldots + v_n$ in the interior of σ. Then, the sequence $(q_m + v)$ is contained in the interior of σ for all sufficiently large m. Hence $(q_m + v)$ is a sequence in $Q \cap \sigma$ converging to v, contradicting the discreteness of $Q \cap \sigma$. ■

Remark 5.2. We have already seen in Remark 1.9 that, if $\deg_{\mathrm{NR}}(Q) \neq 0$, the $\mathbb{Z}$–module $\mathrm{Hom}(Q, \mathbb{Z})$ can no longer be thought of as a quasilattice in $(\mathbb{R}^n)^*$. Even if that were the case, by Theorem 5.1, the semigroup algebra associated with the cone σ would not be finitely generated. Consequently, it cannot, in general, serve as the coordinate ring of an affine toric quasifold.

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Dipartimento di Matematica e Informatica "U. Dini"
Università degli Studi di Firenze
Viale Morgagni 67/A, 50134 Firenze, ITALY
fiammetta.battaglia@unifi.it, elisa.prato@unifi.it