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On Geodesic Triangles in Non-Euclidean Geometry

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Abstract: In this paper, we study centroids, orthocenters, circumcenters, and incenters of geodesic triangles in non-Euclidean geometry, and we discuss the existence of the Euler line in this context. Moreover, we give simple proofs of the existence of a totally geodesic 2-dimensional submanifold containing a given geodesic triangle in the hyperbolic or spherical 3-dimensional geometry.

Keywords: non-Euclidean geometry; notable points of hyperbolic and spherical triangles; polar triangle; totally geodesic hypersurfaces

MSC: 53A35; 58A05

1. Introduction

Let T be a triangle in the Euclidean plane, the medians of T intersect in the centroid, the altitudes of T intersect in the orthocenter, and the perpendicular bisectors of the sides of T intersect in the circumcenter; if T is not equilateral, then these points define the Euler line. Also, the bisectors of the interior angles intersect in the incenter. Moreover, any triangle in Euclidean space is contained in a plane, that is, in a totally geodesic 2-dimensional submanifold of the Euclidean space. In the first part of this paper, we investigate previous geometrical properties in the context of non-Euclidean geometry. We first consider the hyperbolic setting. Starting from the results of [1] about the existence of centroid, orthocenter, circumcenter and incenter, in Section 2, we give algebraic conditions under which the three perpendicular bisectors of the sides, or the three altitudes, have a finite common point, or are asymptotically parallel, or are ultra-parallel geodesics. We describe explicit examples in the hyperbolic setting, where the analogue of the Euler line does not exist. In Section 3, we prove that every geodesic triangle in the hyperbolic 3-dimensional space is contained in a totally geodesic hypersurface. Sections 4 and 5 are devoted to the study of geodesic triangles of the sphere. Specifically, in Section 4, we first compute the circumradius and the inradius of a spherical geodesic triangle and describe relationships with the same geometrical quantities of the polar triangle. Then, we prove that the circumcenter, the orthocenter, and the centroid of a spherical geodesic triangle belong to a common geodesic of the 2-dimensional sphere if and only if the triangle is isosceles. Moreover, starting from the well-known property that, in Euclidean geometry, the distance between the orthocenter and the centroid of a triangle is twice the distance between the circumcenter and the centroid, we prove that this is no longer true in non-Euclidean geometry, and we compute formulas for the distance of these points for some special spherical triangle. In Section 5, we prove that, as in the hyperbolic case, every geodesic triangle in the 3-dimensional sphere is contained in a totally geodesic hypersurface. The main purpose of this paper is to provide a simple and organic treatment of some similarities and differences between geometrical properties of geodesic triangles in Euclidean and non-Euclidean geometry. Indeed, to the best of our knowledge, the mathematical literature in this area, in particular regarding notable points of a geodesic triangle in non-Euclidean geometry, is lacking. In this sense, good references are [1–4]. This paper is a sort of extension of the results in [1,3]. Nevertheless,

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we would like to remark that many interesting papers describe the general framework of non-Euclidean geometry and its applications; in particular, for applications to physics, a good reference is [5].

2. On Geodesic Triangles of the Hyperbolic Plane

2.1. Preliminaries

(a) Let (M, g) be a complete, simply connected n-dimensional Riemannian manifold, $n \geq 2$, with constant sectional curvature equal to -1; then, we say that (M, g) is an *n-dimensional hyperbolic space* . It is well known that n-dimensional hyperbolic space is unique up to isometries. Therefore, in statements that are invariant under isometries, we can refer to a generic n-dimensional hyperbolic space $\mathbf{H}_n$, rather than to any Riemannian model (M, g).

When $n = 2$, the 2-dimensional hyperbolic space $\mathbf{H}_2$ is called a *hyperbolic plane*.

As is known, it is possible to intrinsically define the *boundary* $\partial\mathbf{H}_n$ of $\mathbf{H}_n$, whose elements are called *points at infinity* of $\mathbf{H}_n$ (see, for instance, [6], (pp. 29–30)).

The geodesics of $\mathbf{H}_n$ are also more simply called (*hyperbolic*) *lines*. It is possible to associate every (hyperbolic) line of $\mathbf{H}_n$ with exactly two points at infinity in $\partial\mathbf{H}_n$; moreover, given $p, q \in \partial\mathbf{H}_n$, $p \neq q$, there exists one and only one line of $\mathbf{H}_n$ with p and q as points at infinity. Recall that two distinct lines of $\mathbf{H}_n$ are

- ⋄ *Incident* if they intersect in $\mathbf{H}_n$ (necessarily in a single point);
- ⋄ *Asymptotically parallel* if they have one common point at infinity (and therefore do not intersect in $\mathbf{H}_n$);
- ⋄ *Ultra-parallel* if they are neither incident nor asymptotically parallel.

(b) Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the complex unitary disk, and let $h := \dfrac{4\, dzd\bar{z}}{(1 - |z|^2)^2}$ be the Poincaré metric on Δ. It is well known that the Riemannian surface (Δ, h) is a model for the hyperbolic plane $\mathbf{H}_2$. We will often refer to this model during the calculations necessary for the proofs of theorems.

The boundary (i.e., the set of points at infinity) of (Δ, h) can be naturally identified with $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. The geodesics of (Δ, h) are the curves γ of the form $\gamma = \widehat{\gamma} \cap \Delta$, where $\widehat{\gamma}$ is any Euclidean line or any Euclidean circle in both cases perpendicular to S^1. The points at infinity of the geodesic γ are the two points of $\widehat{\gamma} \cap S^1$. The distance d, defined by the Poincaré metric h, is given by

$$d(z, w) = \ln\left(\frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|}\right), \ \text{ for every } z, w \in \Delta, \ \text{ where } \ \varphi_w(z) := \frac{z - w}{1 - \overline{w}z} \,.$$

In particular, for every $z \in \Delta$, we have

$$d(z, 0) = \ln\left(\frac{1 + |z|}{1 - |z|}\right); \text{ from this : } |z| = \frac{e^{d(z,0)} - 1}{e^{d(z,0)} + 1} == \sqrt{\frac{\cosh\big(d(z, 0)\big) - 1}{\cosh\big(d(z, 0)\big) + 1}} \,.$$

It is well known that the *hyperbolic circles* of (Δ, h) are exactly the Euclidean circles of $\mathbb{C}$ contained in Δ, although, obviously, the hyperbolic center and radius do not agree with the Euclidean center and radius.

If we denote by $\mathbf{I}^+(\Delta, h)$ the group of isometries that preserve the orientation of (Δ, h), we have $\mathbf{I}^+(\Delta, h) = \{f : \Delta \to \Delta : f(z) = e^{\theta \mathbf{i}} \cdot \varphi_w(z), \text{ with } \theta \in \mathbb{R},\ w \in \Delta\}$.

It is also well known that, given $z_1, z_2, w_1, w_2 \in \Delta$, with $z_1 \neq z_2$, $w_1 \neq w_2$, and $d(z_1, z_2) = d(w_1, w_2)$, there exists a unique $f \in \mathbf{I}^+(\Delta, h)$ such that $f(z_1) = w_1$ and $f(z_2) = w_2$.

(c) Given a *geodesic triangle* T of the hyperbolic plane $\mathbf{H}_2$ with $A, B, C \in \mathbf{H}_2$ as vertices, in the following we will denote by $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$, the hyperbolic lengths of the sides BC, AC, AB, respectively. Moreover, we will denote by α, β, γ the hyperbolic measures of the interior angles of T in the vertices $A, B, C,$ respectively. We also need to define the following symmetric functions of a, b, c and α, β, γ, respectively, which will appear in some later theorems:

$$\mathbf{\Phi}(a,b,c) := \frac{2(\cosh(a)-1)(\cosh(b)-1)(\cosh(c)-1)}{1-\cosh^2(a)-\cosh^2(b)-\cosh^2(c)+2\cosh(a)\cosh(b)\cosh(c)};$$

$$\mathbf{\Theta}(a,b,c) := 3\cosh^2(a)\cosh^2(b)\cosh^2(c)+\cosh^2(a)\cosh^2(b)+\cosh^2(a)\cosh^2(c)+ \cosh^2(b)\cosh^2(c)-2\cosh(a)\cosh(b)\cosh(c)\big(\cosh^2(a)+\cosh^2(b)+\cosh^2(c)\big);$$

$$\mathbf{\Omega}(\alpha,\beta,\gamma) := \cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)+2\sin(\beta)\sin(\gamma)\cos(\alpha)+ 2\sin(\alpha)\sin(\gamma)\cos(\beta)+2\sin(\alpha)\sin(\beta)\cos(\gamma)-3;$$

$$\mathbf{\Psi}(\alpha,\beta,\gamma) := 3\cos^2(\alpha)\cos^2(\beta)\cos^2(\gamma)+2\cos^3(\alpha)\cos(\beta)\cos(\gamma)+ 2\cos(\alpha)\cos^3(\beta)\cos(\gamma)+2\cos(\alpha)\cos(\beta)\cos^3(\gamma)+\cos^2(\alpha)\cos^2(\beta)+ \cos^2(\alpha)\cos^2(\gamma)+\cos^2(\beta)\cos^2(\gamma);$$

$$\mathbf{Y}(\alpha,\beta,\gamma) := \frac{\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)+2\cos(\alpha)\cos(\beta)\cos(\gamma)-1}{2\big(1+\cos(\alpha)\big)\big(1+\cos(\beta)\big)\big(1+\cos(\gamma)\big)}.$$

(d) As in Euclidean geometry, for each geodesic triangle T of the hyperbolic plane, it is possible to define particular geodesics, which are, respectively, the three bisectors of the interior angles, the three altitudes, the three medians, and the three perpendicular bisectors of the sides of T. Furthermore, in hyperbolic geometry, the following theorem holds.

Theorem 1 ([1], (Teorema, p. 68))**.** *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$. Then,*

(a) *the three bisectors of the interior angles of T meet at a common point of $\mathbf{H}_2$, called* hyperbolic incenter of *T;*

(b) *for the three perpendicular bisectors of the sides of T, the following events occur: either they meet at a common point of $\mathbf{H}_2$ (that is, T has a finite* hyperbolic circumcenter*), or they are asymptotically parallel with a point at infinity common to the three lines, or they are ultra-parallel with a perpendicular line common to all three;*

(c) *for the three altitudes of T the following events occur: either they meet at a common point of $\mathbf{H}_2$ (that is, T has a finite* hyperbolic orthocenter*), or they are asymptotically parallel with a point at infinity common to the three lines, or they are ultra-parallel with a perpendicular line common to all three;*

(d) *the three medians of T meet at a common point of $\mathbf{H}_2$, called* hyperbolic centroid of *T.*

2.2. On the Circumcenter of a Hyperbolic Triangle

Theorem 2. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let a, b, c be the hyperbolic lengths of its three sides. Let $\mathbf{\Phi}(a,b,c)$ be the function defined in Preliminaries 2.1(c).*

Then,

(i) *the three perpendicular bisectors of the sides of T meet at a common point of $\mathbf{H}_2$ (that is, T has a finite hyperbolic circumcenter) if and only if $\mathbf{\Phi}(a,b,c)) < 1$;*

(ii) *the three perpendicular bisectors of the sides of T are asymptotically parallel with a point at infinity common to the three lines if and only if $\mathbf{\Phi}(a,b,c) = 1$;*

(iii) *the three perpendicular bisectors of the sides of T are ultra-parallel with a perpendicular line common to all three if and only if $\mathbf{\Phi}(a,b,c) > 1$;*

(iv) *if $\mathbf{\Phi}(a,b,c) < 1$, denoted by r_0 the hyperbolic radius of the circle passing through the vertices of T (i.e., r_0 is the* hyperbolic circumradius *of T), we have*

$$r_0 = \frac{1}{2}\ln\left(\frac{1+\sqrt{\mathbf{\Phi}(a,b,c)}}{1-\sqrt{\mathbf{\Phi}(a,b,c)}}\right).$$

Proof. Without loss of generality, we can assume that the hyperbolic plane $\mathbf{H}_2$ is the Poincaré disk (Δ, h). The Poincaré metric h is conformally equivalent to the Euclidean metric on $\Delta \subset \mathbb{C}$; then, the hyperbolic measure of each interior angle of T coincides with its Euclidean measure. Up to hyperbolic isometries of (Δ, h), we can suppose that the vertices A, B, C of T are $A = 0$, $B = t$, $C = \xi + \eta\,\mathbf{i}$, where $t, \xi, \eta \in \mathbb{R}$, with $t \in (0,1)$, $\eta > 0$, $0 < \xi^2 + \eta^2 < 1$. We denote the measures of the sides and interior angles of T as in *Preliminaries* 2.1(c).

We denote the function $\mathbf{\Phi}(a,b,c)$ more simply by $\mathbf{\Phi}$. We recall that

$$\mathbf{\Phi} = \frac{2(\cosh(a)-1)(\cosh(b)-1)(\cosh(c)-1)}{1-\cosh^2(a)-\cosh^2(b)-\cosh^2(c)+2\cosh(a)\cosh(b)\cosh(c)}\,.$$

From the law of cosines for hyperbolic triangles ([7], (Formula (21) p. 85)), we deduce

$$\cos(\alpha) = \frac{\cosh(b)\cosh(c)-\cosh(a)}{\sinh(b)\sinh(c)} = \frac{\cosh(b)\cosh(c)-\cosh(a)}{\sqrt{(\cosh^2(b)-1)(\cosh^2(c)-1)}}\,;$$

hence:

$$\sin^2(\alpha) = \frac{1-\cosh^2(a)-\cosh^2(b)-\cosh^2(c)+2\cosh(a)\cosh(b)\cosh(c)}{(\cosh^2(b)-1)(\cosh^2(c)-1)}\,.$$

Furthermore we have

$$|C| = \sqrt{\frac{\cosh(b)-1}{\cosh(b)+1}},\ t = |B| = \sqrt{\frac{\cosh(c)-1}{\cosh(c)+1}},$$

$$\xi = |C|\cdot\cos(\alpha) = \frac{\cosh(b)\cosh(c)-\cosh(a)}{(\cosh(b)+1)\cdot\sqrt{\cosh^2(c)-1}},$$

$$t^2+|C|^2-2t|C|\cos(\alpha) = \frac{2(\cosh(a)-1)}{(\cosh(b)+1)\cdot(\cosh(c)+1)}\,;$$

so we obtain

$$\frac{t^2+|C|^2-2t|C|\cos(\alpha)}{\sin^2(\alpha)} = \mathbf{\Phi}.$$

The midpoint of the hyperbolic segment of extremes $A = 0$ and an arbitrary $z \in \Delta$ is the point $\hat{z} = \dfrac{1-\sqrt{1-|z|^2}}{|z|^2}\,z$. It follows that the midpoint of AB is the point $\hat{t} = \dfrac{1-\sqrt{1-t^2}}{t}$, while the midpoint of AC is the point

$$\widehat{C} = \frac{1-\sqrt{1-|C|^2}}{|C|^2}\,C = \frac{1-\sqrt{1-|C|^2}}{|C|}\big(\cos(\alpha)+\sin(\alpha)\,\mathbf{i}\,\big)\,.$$

In particular, the perpendicular bisector of the hyperbolic segment AB (that is, the geodesic perpendicular to AB through $\hat{t}$) has the following equation:

$$t(x^2+y^2)-2x+t=0\,;$$

while the equation of the perpendicular bisector of the hyperbolic segment AC is

$$|C|(x^2+y^2)-2\cos(\alpha)\,x-2\sin(\alpha)\,y+|C|=0\,.$$

In these equations, x and y denote the real and imaginary parts of an arbitrary $z \in \mathbb{C}$, respectively. Direct computation shows that the two Euclidean circles corresponding to the perpendicular bisectors of the hyperbolic segments AB and BC intersect in the complex plane $\mathbb{C}$ if and only if

$$\frac{t^2 + |C|^2 - 2t|C|\cos(\alpha)}{\sin^2(\alpha)} = \mathbf{\Phi} \leq 1.$$

When $\mathbf{\Phi} \leq 1$, the real coordinates of intersection points are the following:

$$x_{\pm} = \frac{t\big(1 \pm \sqrt{1-\mathbf{\Phi}}\big)}{\mathbf{\Phi}}, \quad y_{\pm} = \Big(\frac{|C| - t\cos(\alpha)}{t\sin(\alpha)}\Big)\left[\frac{t\big(1 \pm \sqrt{1-\mathbf{\Phi}}\big)}{\mathbf{\Phi}}\right],$$

and so

$$x_{\pm}^2 + y_{\pm}^2 = \frac{\left[1 \pm \sqrt{1-\mathbf{\Phi}}\right]^2}{\mathbf{\Phi}}.$$

From $\mathbf{\Phi} \leq 1$, we obtain

$$x_{+}^2 + y_{+}^2 = \frac{\left[1 + \sqrt{1-\mathbf{\Phi}}\right]^2}{\mathbf{\Phi}} \geq 1, \;\; x_{-}^2 + y_{-}^2 = \frac{\left[1 - \sqrt{1-\mathbf{\Phi}}\right]^2}{\mathbf{\Phi}} \leq 1\,;$$

in both cases equality holds if and only if $\mathbf{\Phi} = 1$.

Then, the perpendicular bisectors of the hyperbolic segments AB and AC are incident lines of (Δ, h) if and only if $\mathbf{\Phi} < 1$, they are asymptotically parallel lines of (Δ, h) if and only if $\mathbf{\Phi} = 1$, and they are ultra-parallel lines of (Δ, h) if and only if $\mathbf{\Phi} > 1$. So, taking Theorem 1 (b) into account, we obtain statements (i), (ii) and (iii).

If $\mathbf{\Phi} < 1$, as seen, the circumcenter is

$$Z_0 = x_- + y_-\,\mathbf{i} = \frac{t\big(1 - \sqrt{1-\mathbf{\Phi}}\big)}{\mathbf{\Phi}} + \Big(\frac{|C| - t\cos(\alpha)}{t\sin(\alpha)}\Big)\left[\frac{t\big(1 - \sqrt{1-\mathbf{\Phi}}\big)}{\mathbf{\Phi}}\right]\mathbf{i},$$

whose modulus is

$$|Z_0| = \frac{1 - \sqrt{1-\mathbf{\Phi}}}{\sqrt{\mathbf{\Phi}}}.$$

Hence, the circumradius of T is

$$r_0 = d(Z_0, A) = \ln\left(\frac{1+|Z_0|}{1-|Z_0|}\right) = \ln\left(\sqrt{\frac{1+\sqrt{\mathbf{\Phi}}}{1-\sqrt{\mathbf{\Phi}}}}\right) = \frac{1}{2}\ln\left(\frac{1+\sqrt{\mathbf{\Phi}}}{1-\sqrt{\mathbf{\Phi}}}\right).$$

This proves (iv). □

The previous theorem can be stated in terms of angles instead of sides as follows.

Theorem 3. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let α, β, γ be the measures of its three interior angles. Let $\mathbf{\Omega}(\alpha, \beta, \gamma)$ be the function defined in Preliminaries 2.1(c). Then,*

(i) the geodesic triangle T has a finite hyperbolic circumcenter if and only if $\mathbf{\Omega}(\alpha, \beta, \gamma) > 0$;
(ii) the three perpendicular bisectors of the sides of T are asymptotically parallel with a point at infinity common to the three lines if and only if $\mathbf{\Omega}(\alpha, \beta, \gamma) = 0$;
(iii) the three perpendicular bisectors of the sides of T are ultra-parallel with a perpendicular line common to all three if and only if $\mathbf{\Omega}(\alpha, \beta, \gamma) < 0$.

Proof. We recall that

$$\mathbf{\Omega}(\alpha, \beta, \gamma) = \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + \\ 2\sin(\beta)\sin(\gamma)\cos(\alpha) + 2\sin(\alpha)\sin(\gamma)\cos(\beta) + 2\sin(\alpha)\sin(\beta)\cos(\gamma) - 3.$$

Denoted by a, b, c the hyperbolic lengths of the three sides of T, from the hyperbolic law of cosines for angles ([7], (Formula (22) p. 85)), we obtain

$$\cosh(a) = \frac{\cos(\beta)\cos(\gamma) + \cos(\alpha)}{\sin(\beta)\sin(\gamma)},$$

$$\cosh(b) = \frac{\cos(\alpha)\cos(\gamma) + \cos(\beta)}{\sin(\alpha)\sin(\gamma)},$$

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}.$$

Now, we denote

$$\Xi = 2\cos(\alpha)\cos(\beta)\cos(\gamma) + \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) - 1,$$

and

$$\aleph = 1 - \cosh^2(a) - \cosh^2(b) - \cosh^2(c) + 2\cosh(a)\cosh(b)\cosh(c).$$

From

$$1 < \cosh^2(c) = \frac{\big(\cos(\alpha)\cos(\beta) + \cos(\gamma)\big)^2}{\big(1-\cos^2(\alpha)\big)\big(1-\cos^2(\beta)\big)},$$

we easily obtain $\Xi > 0$, while from $\sin^2(\alpha) = \dfrac{\aleph}{(\cosh^2(b)-1)(\cosh^2(c)-1)}$ (see the proof of Theorem 2), we also obtain $\aleph > 0$.

By means of tedious but elementary calculations, the following equality is proven:

$$\sin^2(\alpha)\sin^2(\beta)\sin^2(\gamma)\cdot\aleph\cdot\big(1-\mathbf{\Phi}(a,b,c)\big) = \Xi\cdot\mathbf{\Omega}(\alpha,\beta,\gamma).$$

Since $\Xi > 0$, $\aleph > 0$, from Theorem 2, we obtain statements (i), (ii), and (iii). □

We remark that, following the suggestion of one referee of this paper, Theorem 2 can be stated in the following form. Before restating the theorem, we take the opportunity to thank the anonymous referee. We did not know this interesting result; however, we have been able to produce a proof that we describe below, improving the structure of our paper.

Theorem 4. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let a, b, c be the hyperbolic lengths of its three sides.*

Let $a \geq b \geq c$ and $\mathbf{F}(a,b,c) := \sinh\big(\frac{b}{2}\big) + \sinh\big(\frac{c}{2}\big) - \sinh\big(\frac{a}{2}\big)$. Then,

(i) the three perpendicular bisectors of the sides of T meet at a common point of $\mathbf{H}_2$ (that is, T has a finite hyperbolic circumcenter) if and only if $\mathbf{F}(a,b,c) > 0$;

(ii) the three perpendicular bisectors of the sides of T are asymptotically parallel with a point at infinity common to the three lines if and only if $\mathbf{F}(a,b,c) = 0$;

(iii) the three perpendicular bisectors of the sides of T are ultra-parallel with a perpendicular line common to all three if and only if $\mathbf{F}(a,b,c) < 0$.

Proof. As before, without loss of generality, we assume that the hyperbolic plane $\mathbf{H}_2$ is the Poincaré disk (Δ, h) and that the vertices A, B, C of T are

$$A = 0,\ B = t,\ C = \xi + \eta\,\mathbf{i},$$

where $t, \xi, \eta \in \mathbb{R}$, with $t \in (0,1), \eta > 0, 0 < \xi^2 + \eta^2 < 1$.

We denote the measures of the sides and interior angles of T as in *Preliminaries* 2.1(c). Condition $b \geq c$ is equivalent to $|C| \geq t$, while, by the law of cosines for hyperbolic triangles, condition $a \geq b$ is equivalent to

$$\cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha) \geq \cosh(b).$$

In particular we obtain

$$\cos(\alpha) \leq \frac{\cosh(b)(\cosh(c)-1)}{\sinh(b)\,\sinh(c)}.$$

From $b = \ln\left(\frac{1+|C|}{1-|C|}\right)$ and $c = \ln\left(\frac{1+t}{1-t}\right)$, we obtain
$\cosh(b) = \frac{1+|C|^2}{1-|C|^2}$, $\sinh(b) = \frac{2|C|}{1-|C|^2}$, and $\cosh(c) = \frac{1+t^2}{1-t^2}$, $\sinh(c) = \frac{2t}{1-t^2}$.
From these equalities, we easily obtain the following expressions:
$\cosh\left(\frac{b}{2}\right) = \frac{1}{\sqrt{1-|C|^2}}$, $\sinh\left(\frac{b}{2}\right) = \frac{|C|}{\sqrt{1-|C|^2}}$, $\cosh\left(\frac{c}{2}\right) = \frac{1}{\sqrt{1-t^2}}$,
$\sinh\left(\frac{c}{2}\right) = \frac{t}{\sqrt{1-t^2}}$.
Moreover, the conditions $a \geq b \geq c$ are equivalent to

$$(\dagger) \quad \cos(\alpha) \leq \frac{t}{2|C|} + \frac{t|C|}{2}, \quad t \leq |C|.$$

By the law of cosines for hyperbolic triangles, we obtain

$$\sinh\left(\frac{a}{2}\right) = \sqrt{\frac{\cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha) - 1}{2}} = \frac{\sqrt{t^2+|C|^2-2t|C|\cos(\alpha)}}{\sqrt{1-|C|^2}\sqrt{1-t^2}}.$$

So, if $\mathbf{F}(a,b,c)$ is the function defined in the statement, we obtain

$$\mathbf{F}(a,b,c) = \frac{|C|\sqrt{1-t^2} + t\sqrt{1-|C|^2} - \sqrt{t^2+|C|^2-2t|C|\cos(\alpha)}}{\sqrt{1-|C|^2}\sqrt{1-t^2}},$$

hence

$$\mathbf{F}(a,b,c)\cdot\left(|C|\sqrt{1-t^2} + t\sqrt{1-|C|^2} + \sqrt{t^2+|C|^2-2t|C|\cos(\alpha)}\right)\sqrt{1-|C|^2}\sqrt{1-t^2}$$
$$= 2t|C|\cdot\left(\cos(\alpha) - t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2}\right).$$

As a consequence, the functions $\mathbf{F}(a,b,c)$ and $\left(\cos(\alpha) - t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2}\right)$ have the same sign.

Remembering the proof of Theorem 2, we have

$$\sin^2(\alpha)\cdot\left(1-\mathbf{\Phi}(a,b,c)\right) =$$
$$\sin^2(\alpha) - t^2 - |C|^2 + 2t|C|\cos(\alpha) = -\left(\cos^2(\alpha) - 2t|C|\cos(\alpha) + t^2 + |C|^2 - 1\right) =$$
$$\left(t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2} - \cos(\alpha)\right)\cdot\left(\cos(\alpha) - t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2}\right).$$

From $(\dagger)$, we easily obtain $\left(t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2} - \cos(\alpha)\right) \gneqq 0$. So the functions $(1-\mathbf{\Phi}(a,b,c))$ and $\left(\cos(\alpha) - t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2}\right)$ also have the same sign.

As an immediate consequence, we obtain the result that the functions $\mathbf{F}(a,b,c)$ and $(1-\mathbf{\Phi}(a,b,c))$ have the same sign; so, from Theorem 2, we obtain the statement. □

Remark 1. *By using previous notations, let us define the function*

$$\mathbf{G}(\alpha,b,c) := \cos(\alpha) - \frac{\sinh\left(\frac{b}{2}\right)\sinh\left(\frac{c}{2}\right) - 1}{\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)}.$$

Direct computation gives $\mathbf{G}(\alpha,b,c) = \cos(\alpha) - t|C| + \sqrt{1-t^2}\sqrt{1-|C|^2}$.

In the proof of Theorem 4 we saw that, if $a \geq b \geq c$, the functions $\mathbf{F}(a,b,c)$, $(1-\mathbf{\Phi}(a,b,c))$ and $\mathbf{G}(\alpha,b,c)$ have the same sign.

This implies that Theorem 4 can be formulated by replacing the function $\mathbf{F}(a,b,c)$ with the function $\mathbf{G}(\alpha,b,c)$ to express the condition on the existence of the hyperbolic circumcenter in terms of the measure of the largest interior angle of the triangle T and the lengths of the sides adjacent to it.

2.3. On the Incenter of a Hyperbolic Triangle

Theorem 5. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let α, β, γ be the measures of its three interior angles. Let $\mathbf{Y}(\alpha, \beta, \gamma)$ be the function defined in Preliminaries* 2.1 (c)*. Then the hyperbolic radius s_0 of the inscribed circle of T (i.e., the* hyperbolic inradius *of T) is given by*

$$s_0 = \frac{1}{2} \ln \left(\frac{1 + \sqrt{\mathbf{Y}(\alpha, \beta, \gamma)}}{1 - \sqrt{\mathbf{Y}(\alpha, \beta, \gamma)}} \right).$$

Proof. Let A, B, C be the three vertices of the triangle T and denote the measures of the sides and interior angles of T as in *Preliminaries* 2.1 (c); also, remember that the function $\mathbf{Y} := \mathbf{Y}(\alpha, \beta, \gamma)$ is given by

$$\mathbf{Y} = \frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2\big(1 + \cos(\alpha)\big)\big(1 + \cos(\beta)\big)\big(1 + \cos(\gamma)\big)}.$$

The existence of the hyperbolic incenter I as the point of intersection of the three bisectors of the interior angles of T is guaranteed by the Theorem 1 (a).

Let H be the intersection between side AB of T and the hyperbolic line perpendicular to side AB passing through the incenter I. Therefore, the inradius s_0 is equal to the hyperbolic distance between I and H, and so, from a classical formula of hyperbolic trigonometry applied to both right triangles $\overset{\triangle}{AHI}$ and $\overset{\triangle}{BHI}$ (see [7], (Formula (16) p. 83)), we obtain

$$\tanh(s_0) = \tan\Big(\frac{\alpha}{2}\Big)\sinh(x) = \tan\Big(\frac{\beta}{2}\Big)\sinh(c - x) =$$
$$\tan\Big(\frac{\beta}{2}\Big)\big(\sinh(c)\cosh(x) - \cosh(c)\sinh(x)\big)$$

where x and $c - x = d(A, B) - x$ are the hyperbolic lengths of sides AH and HB, respectively. From the equality $\tan\big(\frac{\alpha}{2}\big)\sinh(x) = \tan\big(\frac{\beta}{2}\big)\big(\sinh(c)\cosh(x) - \cosh(c)\sinh(x)\big)$, we easily obtain

$$\sinh(x) = \frac{\tan\big(\frac{\beta}{2}\big)\sqrt{\cosh^2(c) - 1}}{\sqrt{\tan^2\big(\frac{\alpha}{2}\big) + \tan^2\big(\frac{\beta}{2}\big) + 2\cosh(c)\tan\big(\frac{\alpha}{2}\big)\tan\big(\frac{\beta}{2}\big)}},$$

and so, from this,

$$\tanh(s_0) = \frac{\tan\big(\frac{\alpha}{2}\big)\tan\big(\frac{\beta}{2}\big)\sqrt{\cosh^2(c) - 1}}{\sqrt{\tan^2\big(\frac{\alpha}{2}\big) + \tan^2\big(\frac{\beta}{2}\big) + 2\cosh(c)\tan\big(\frac{\alpha}{2}\big)\tan\big(\frac{\beta}{2}\big)}}.$$

It is well known that

$$\tan\Big(\frac{\alpha}{2}\Big) = \sqrt{\frac{1 - \cos(\alpha)}{1 + \cos(\alpha)}}, \quad \tan\Big(\frac{\beta}{2}\Big) = \sqrt{\frac{1 - \cos(\beta)}{1 + \cos(\beta)}} \quad (\text{being } 0 < \alpha, \beta < \pi).$$

Therefore, substituting these expressions into the formula obtained for $\tanh(s_0)$ and using the hyperbolic law of cosines for angles, through elementary calculations, we obtain

$$\tanh(s_0) = \sqrt{\frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2\big(1 + \cos(\alpha)\big)\big(1 + \cos(\beta)\big)\big(1 + \cos(\gamma)\big)}} = \sqrt{\mathbf{Y}}.$$

From this, the formula $s_0 = \dfrac{1}{2}\ln\left(\dfrac{1 + \sqrt{\mathbf{Y}}}{1 - \sqrt{\mathbf{Y}}}\right)$ is easily deduced. □

2.4. On the Orthocenter of a Hyperbolic Triangle

Theorem 6. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let a, b, c be the hyperbolic lengths of its three sides. Let $\mathbf{\Theta}(a,b,c)$ be the function defined in Preliminaries 2.1 (c). Then,*

(i) the geodesic triangle T has a finite hyperbolic orthocenter if and only if $\mathbf{\Theta}(a,b,c) > 0$;
(ii) the three altitudes of T are asymptotically parallel with a point at infinity common to the three lines if and only if $\mathbf{\Theta}(a,b,c) = 0$;
(iii) the three altitudes of T are ultra-parallel with a perpendicular line common to all three if and only if $\mathbf{\Theta}(a,b,c) < 0$.

Proof. As in the proof of Theorem 2, we can assume that the hyperbolic plane $\mathbf{H}_2$ is the Poincaré disk (Δ, h) and that the vertices A, B, C of the hyperbolic triangle T are $A = 0$, $B = t$, $C = \xi + \eta\,\mathbf{i}$, where $t, \xi, \eta \in \mathbb{R}$, with $t \in (0,1)$, $\eta > 0$, $0 < \xi^2 + \eta^2 < 1$; furthermore, we denote the measures of the sides and interior angles of T as in *Preliminaries* 2.1 (c). We recall that

$$\mathbf{\Theta}(a,b,c) = 3\cosh^2(a)\cosh^2(b)\cosh^2(c) + \cosh^2(a)\cosh^2(b) + \cosh^2(a)\cosh^2(c) + \cosh^2(b)\cosh^2(c) - 2\cosh(a)\cosh(b)\cosh(c)\big(\cosh^2(a) + \cosh^2(b) + \cosh^2(c)\big)\,.$$

First, we note that, in the case $\alpha = \dfrac{\pi}{2}$ (so that $\cosh(b)\cosh(c) = \cosh(a)$), we obtain $\mathbf{\Theta}(a,b,c) = \cosh^2(b)\cosh^2(c)\big(\cosh^2(b) - 1)(\cosh^2(c) - 1) > 0$, and clearly, by Theorem 1 (c), the three altitudes of T intersect in A.

Therefore, when $\alpha = \dfrac{\pi}{2}$, the statement is proved.

Assume now that $\alpha \neq \dfrac{\pi}{2}$. Denoting by x and y, respectively, the real and imaginary parts of any $z \in \mathbb{C}$, the equation of the Euclidean circle that extends the hyperbolic side BC is

$$t|C|\sin(\alpha)(x^2 + y^2) - (1 + t^2)|C|\sin(\alpha)x + \big((1 + t^2)|C|\cos(\alpha) - t(1 + |C|^2)\big)y + t|C|\sin(\alpha) = 0\,;$$

the Euclidean center of this circle is the point

$$P = \frac{1 + t^2}{2t} + \Big(\frac{1 + |C|^2}{2|C|\sin(\alpha)} - \frac{(1 + t^2)\cos(\alpha)}{2t\sin(\alpha)}\Big)\mathbf{i}\,.$$

The altitude h_A relative to side BC passes through A and P; therefore, the Euclidean line $\widehat{h}_A$ which extends h_A has the equation:

$$\big((1 + t^2)|C|\cos(\alpha) - t(1 + |C|^2)\big)x + (1 + t^2)|C|\sin(\alpha)y = 0\,.$$

From $b = \ln\left(\dfrac{1 + |C|}{1 - |C|}\right)$, we easily obtain $2b = \ln\left(\dfrac{1 + \frac{2|C|}{1+|C|^2}}{1 - \frac{2|C|}{1+|C|^2}}\right)$, and so

$$\frac{2|C|}{1 + |C|^2} = \frac{e^{2b} - 1}{e^{2b} + 1} = \tanh(b) \quad \text{and} \quad \coth(b) = \frac{1 + |C|^2}{2|C|}\,.$$

Analogously, we obtain $\tanh(c) = \dfrac{2t}{1 + t^2}$. Therefore the equation of $\widehat{h}_A$ can be rewritten as follows:

$$(\cos(\alpha) - \tanh(c)\coth(b))x + \sin(\alpha)y = 0\,.$$

The altitude h_C relative to side AB is $h_C = \Delta \cap \widehat{h}_C$, where $\widehat{h}_C$ is the Euclidean circle perpendicular to S^1, with the center on the real axis $\{y = 0\}$ and passing through the vertex $C = |C|\cos(\alpha) + |C|\sin(\alpha)\mathbf{i}$. By elementary calculations, it is easy to prove that the Euclidean center of $\widehat{h}_C$ is the point $Q = \dfrac{1 + |C|^2}{2|C|\cos(\alpha)} = \dfrac{\coth(b)}{\cos(\alpha)}$, while its radius is $s = \sqrt{\big(\frac{1+|C|^2}{2|C|\cos(\alpha)}\big)^2 - 1} = \sqrt{\big(\frac{\coth(b)}{\cos(\alpha)}\big)^2 - 1}\,.$

Now, if we denote by $\delta(\widehat{h}_A, Q)$ the Euclidean distance between the Euclidean line $\widehat{h}_A$ and the point Q, we have

$$s^2 - \delta(\widehat{h}_A, Q)^2 = \Big(\frac{\coth(b)}{\cos(\alpha)}\Big)^2 - 1 - \frac{\big(\cos(\alpha) - \tanh(c)\coth(b)\big)^2 \big(\frac{\coth(b)}{\cos(\alpha)}\big)^2}{\big(\cos(\alpha) - \tanh(c)\coth(b)\big)^2 + \sin^2(\alpha)} =$$
$$\frac{\tan^2(\alpha) - \tanh^2(b) - \tanh^2(c) + 2\tanh(b)\tanh(c)\cos(\alpha)}{\tanh^2(b)\big[\big(\cos(\alpha) - \tanh(c)\coth(b)\big)^2 + \sin^2(\alpha)\big]} =$$
$$\frac{\mathbf{\Lambda}}{\tanh^2(b)\big[\big(\cos(\alpha) - \tanh(c)\coth(b)\big)^2 + \sin^2(\alpha)\big]},$$

where we denote

$$\mathbf{\Lambda} := \tan^2(\alpha) - \tanh^2(b) - \tanh^2(c) + 2\tanh(b)\tanh(c)\cos(\alpha) =$$
$$\frac{1}{\cos^2(\alpha)} - 3 + \frac{1}{\cosh^2(b)} + \frac{1}{\cosh^2(c)} + \frac{2\cos(\alpha)\sqrt{(\cosh^2(b)-1)(\cosh^2(c)-1)}}{\cosh(b)\cosh(c)}.$$

Since $\cos(\alpha) = \dfrac{\cosh(b)\cosh(c) - \cosh(a)}{\sqrt{(\cosh^2(b)-1)(\cosh^2(c)-1)}}$, substituting we obtain

$$\mathbf{\Lambda} = \frac{(\cosh^2(b)-1)(\cosh^2(c)-1)}{\big(\cosh(b)\cosh(c) - \cosh(a)\big)^2} - 3 + \frac{1}{\cosh^2(b)} + \frac{1}{\cosh^2(c)} +$$
$$\frac{2\big(\cosh(b)\cosh(c) - \cosh(a)\big)}{\cosh(b)\cosh(c)}.$$

Elementary calculations allow us to obtain:

$$\mathbf{\Theta}(a,b,c) = \big(\cosh(b)\cosh(c) - \cosh(a)\big)^2 \cosh^2(b)\cosh^2(c) \cdot \mathbf{\Lambda}$$

The hypothesis $\alpha \neq \dfrac{\pi}{2}$ implies $\cosh(b)\cosh(c) \neq \cosh(a)$, so we deduce that the three functions $s^2 - \delta(\widehat{h}_A, Q)^2, \mathbf{\Lambda}, \mathbf{\Theta}(a,b,c)$ have the same sign.

Remember that Euclidean circles $\widehat{h}_C$ and S^1 are perpendicular; that Q and s are, respectively, Euclidean center and Euclidean radius of $\widehat{h}_C$; and that $\widehat{h}_A$ passes through the center 0 of S^1. We conclude that the following conditions are equivalent:

(i)′ the altitudes h_A and h_C are incident in the hyperbolic plane Δ;
(ii)′ the Euclidean line $\widehat{h}_A$ intersects the Euclidean circle $\widehat{h}_C$ at two distinct points of the complex plane $\mathbb{C}$;
(iii)′ $s^2 - \delta(\widehat{h}_A, Q)^2 > 0$;
(iv)′ $\mathbf{\Theta}(a,b,c) > 0$.

Likewise, the following conditions are equivalent to each other:

(i)″ the altitudes h_A and h_C are asymptotically parallel in the hyperbolic plane Δ ;
(ii)″ the Euclidean line $\widehat{h}_A$ is tangent to the Euclidean circle $\widehat{h}_C$ at a point of S^1;
(iii)″ $s^2 - \delta(\widehat{h}_A, Q)^2 = 0$;
(iv)″ $\mathbf{\Theta}(a,b,c) = 0$;

just as the following conditions are equivalent to each other:

(i)‴ the altitudes h_A and h_C are ultra-parallel in the hyperbolic plane Δ ;
(ii)‴ the Euclidean line $\widehat{h}_A$ does not intersect the Euclidean circle $\widehat{h}_C$ in $\mathbb{C}$;
(iii)‴ $s^2 - \delta(\widehat{h}_A, Q)^2 < 0$;
(iv)‴ $\mathbf{\Theta}(a,b,c) < 0$.

Taking Theorem 1 (c) into account, we obtain statements (i), (ii) and (iii). □

Theorem 6 can be stated in terms of interior angles as follows:

Theorem 7. *Let T be a geodesic triangle of the hyperbolic plane $\mathbf{H}_2$, and let α, β, γ be the measures of its three interior angles. Let $\mathbf{\Psi}(\alpha, \beta, \gamma)$ be the function defined in Preliminaries 2.1(c). Then,*

(i) the geodesic triangle T has a finite hyperbolic orthocenter if and only if $\mathbf{\Psi}(\alpha, \beta, \gamma) > 0$;
(ii) the three altitudes of T are asymptotically parallel with a point at infinity common to the three lines if and only if $\mathbf{\Psi}(\alpha, \beta, \gamma) = 0$;
(iii) the three altitudes of T are ultra-parallel with a perpendicular line common to all three if and only if $\mathbf{\Psi}(\alpha, \beta, \gamma) < 0$.

Proof. As usual, we denote by a, b, c the hyperbolic lengths of the three sides of T. As in the proof of Theorem 3, we can use the hyperbolic law of cosines for angles to express the function $\mathbf{\Theta}(a, b, c)$ of *Preliminaries* 2.1(c) as a function of the interior angles α, β, γ of T. So we obtain

$$\begin{aligned}
&\sin^4(\alpha)\sin^4(\beta)\sin^4(\gamma)\cdot\mathbf{\Theta}(a,b,c) = \\
&3\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)^2\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)^2\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)^2 + \\
&\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)^2\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)^2\big(1-\cos^2(\alpha)\big)\big(1-\cos^2(\beta)\big)+ \\
&\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)^2\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)^2\big(1-\cos^2(\alpha)\big)\big(1-\cos^2(\gamma)\big)+ \\
&\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)^2\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)^2\big(1-\cos^2(\beta)\big)\big(1-\cos^2(\gamma)\big)+ \\
&-2\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)^3\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)\cdot\big(1-\cos^2(\alpha)\big) + \\
&-2\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)^3\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)\cdot\big(1-\cos^2(\beta)\big) + \\
&-2\big(\cos(\beta)\cos(\gamma)+\cos(\alpha)\big)\big(\cos(\alpha)\cos(\gamma)+\cos(\beta)\big)\big(\cos(\alpha)\cos(\beta)+\cos(\gamma)\big)^3\cdot\big(1-\cos^2(\gamma)\big).
\end{aligned}$$

By means of elementary but very tedious calculations, it is possible to check that the expression on the right side of this equality is equal to

$$\begin{gathered}
\Xi^2\cdot\big(3\cos^2(\alpha)\cos^2(\beta)\cos^2(\gamma)+2\cos^3(\alpha)\cos(\beta)\cos(\gamma)+ \\
2\cos(\alpha)\cos^3(\beta)\cos(\gamma)+2\cos(\alpha)\cos(\beta)\cos^3(\gamma)+\cos^2(\alpha)\cos^2(\beta)+ \\
\cos^2(\alpha)\cos^2(\gamma)+\cos^2(\beta)\cos^2(\gamma)\big) \;=\; \Xi^2\cdot\mathbf{\Psi}(\alpha,\beta,\gamma),
\end{gathered}$$

where $\Xi \;=\; 2\cos(\alpha)\cos(\beta)\cos(\gamma)+\cos^2(\alpha)+\cos^2(\beta)+\cos^2(\gamma)-1,$ so that

$$\sin^4(\alpha)\sin^4(\beta)\sin^4(\gamma)\cdot\mathbf{\Theta}(a,b,c) \;=\; \Xi^2\cdot\mathbf{\Psi}(\alpha,\beta,\gamma).$$

In the proof of Theorem 3, we saw that $\Xi > 0$. Therefore we deduce that the functions $\mathbf{\Theta}(a,b,c)$ and $\mathbf{\Psi}(\alpha,\beta,\gamma)$ have the same sign; hence, by Theorem 6 , we obtain the statements (i), (ii) and (iii). □

Corollary 1. *If all interior angles of a geodesic triangle T of $\mathbf{H}_2$ are acute, then T has a finite hyperbolic orthocenter.*

Proof. If the three angles of T are acute, we have $\cos(\alpha),\ \cos(\beta),\ \cos(\gamma) > 0$; so clearly we have $\mathbf{\Psi}(\alpha,\beta,\gamma) > 0$ and we conclude by means of Theorem 7 (i). □

Remark 2. *Corollary 1 is already known. In fact it can be deduced from a classic Fagnano theorem on the orthic triangle, which is also valid in hyperbolic geometry (see, for instance, [8], (p. 129)).*

2.5. On the Euler Line in Hyperbolic Geometry

In Euclidean geometry, for every given triangle T, there exists a line that contains the circumcenter, the orthocenter, and the centroid of T. This line is called *Euler line of T*. It is easy to verify that the Euler line does not exist for every geodesic triangle of $\mathbf{H}_2$ with finite circumcenter, orthocenter, and centroid. In this sense, we describe the following:

Example 1. *Let us consider the triangle T of the Poincaré disk (Δ, h) with vertices $A = 0$, $B = t$, $C = \eta\mathbf{i}$, where $0 < t, \eta < 1$ and $t^2 + \eta^2 < 1$. Since $\alpha = \frac{\pi}{2}$, the (finite) orthocenter of T exists and is the vertex $A = 0$. On the other hand, since $t^2 + \eta^2 < 1$, the (finite) circumcenter of T also exists and is the point $W_0 = \dfrac{1 - \sqrt{1 - t^2 - \eta^2}}{t^2 + \eta^2}(t + \eta\mathbf{i})$. We deduce that, if the Euler line of T exists, it is the geodesic passing trough A and W_0; hence its equation is $\eta x - ty = 0$. Now, we compute the centroid.*

Let $\hat{t} = \dfrac{1 - \sqrt{1 - t^2}}{t}$ be the midpoint of the side AB and let $\widehat{C} = \dfrac{1 - \sqrt{1 - \eta^2}}{\eta}\mathbf{i}$ be the midpoint of the side AC. The median of the side AB is the geodesic passing through $C = \eta\mathbf{i}$ and $\hat{t}$, so its equation is as follows: $x^2 + y^2 - \frac{2}{t}x - \frac{(1+\eta^2)}{\eta}y + 1 = 0$. The median of the side AC is the geodesic passing through $B = t$ and $\widehat{C}$, and its equation is as follows: $x^2 + y^2 - \frac{(1+t^2)}{t}x - \frac{2}{\eta}y + 1 = 0$. From Theorem 1 (d), we obtain the result that the centroid G_0 of T is obtained by intersecting the two medians. In particular, it belongs to the geodesic of equation $\eta(t^2 - 1)x - t(\eta^2 - 1)y = 0$. Let $G_0 = x_0 + y_0\mathbf{i}$; so, if the Euler line of T exists, we necessarily have

$$\eta x_0 - ty_0 = 0 \text{ and } \eta(t^2 - 1)x_0 - t(\eta^2 - 1)y_0 = 0;$$

this implies the condition $\eta = t$; that is, the triangle T must be isosceles.

We can easily verify that, in this case, the Euler line of T exists, and it is the perpendicular bisector (also altitude and median) of the side BC of T.

Remark 3. *We remark that in [9], it is proved that the orthocenter, the circumcenter, and the centroid of a geodesic triangle T of $\mathbf{H}_2$ are collinear (i.e., the Euler line of T exists) if and only if the triangle T is isosceles.*

3. On Geodesic Triangles of the Hyperbolic 3-Dimensional Space

Let us consider the hyperbolic 3-dimensional space, $\mathbf{H}_3$, given by the upper half space with the Poincaré metric, h: $\mathbf{H}_3 = \left(\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 | z > 0 \right\}, h = \frac{1}{z^2} I \right)$. It is well known that the geodesics of $\mathbf{H}_3$ are the half lines and half circles perpendicular to the Euclidean plane $\{z = 0\}$. Therefore, all Euclidean half planes perpendicular to the Euclidean plane $\{z = 0\}$ and all Euclidean half spheres with their center on $\{z = 0\}$ are totally geodesic hypersurfaces of $\mathbf{H}_3$. Moreover, each of these hypersurfaces, equipped with the restriction of the metric h, is a hyperbolic plane.

Proposition 1. *Every geodesic triangle T of $\mathbf{H}_3$ is contained in a totally geodesic hypersurface of $\mathbf{H}_3$, and so T is contained in a hyperbolic plane $\mathbf{H}_2 \subset \mathbf{H}_3$.*

Proof. Let T be a geodesic triangle of $\mathbf{H}_3$ with vertices A, B, C. It is sufficient to prove that there exists at least one totally geodesic hypersurface of $\mathbf{H}_3$ passing through the three distinct points $A, B, C \in \mathbf{H}_3$. Let $\wedge$ be the standard vector product of $\mathbb{R}^3$. If $(B - A) \wedge (C - A) = 0$, then the three points are aligned in the Euclidean sense and, obviously, there exists a half plane perpendicular to the plane $\{z = 0\}$ passing trough A, B, and C. If the vector $(B - A) \wedge (C - A)$ is different from 0 and parallel to $\{z = 0\}$, then the half plane passing through A and perpendicular to $(B - A) \wedge (C - A)$ is the required hypersurface. Finally, let us now consider the case where the vector $(B - A) \wedge (C - A) \neq 0$ is not parallel to the plane $\{z = 0\}$. Let π_C be the Euclidean plane passing through $\frac{A+B}{2}$ and perpendicular to $(B - A)$, and let π_B be the Euclidean plane passing through $\frac{A+C}{2}$ and perpendicular to $(C - A)$. Then $\pi_C \cap \pi_B = r$ is a Euclidean line parallel to $(B - A) \wedge (C - A)$. Let $r \cap \{z = 0\} = \{P\}$; we have that P is equidistant (in the Euclidean sense) from A, B and C. Then, the half sphere with its center in P and radius the Euclidean distance between A and P contains A, B and C; so this half sphere is a totally geodesic hypersurface of $\mathbf{H}_3$ containing T. Then, the proof is complete. □

4. On Geodesic Triangles of the Sphere

4.1. Preliminaries

(a) Let (M, g) be a complete and simply connected n-dimensional Riemannian manifold, $n \geq 2$, with constant sectional curvature equal to $+1$; then, as is known, (M, g) is isometric to the standard sphere S^n of $\mathbb{R}^{n+1}$.

Consider first the case $n = 2$. Denote by $\langle\,,\,\rangle$ the Euclidean scalar product and by $\wedge$ the vector (or cross) product of the 3-dimensional Euclidean space $\mathbb{R}^3$. Let $S^2 = \{X \in \mathbb{R}^3 \,:\, ||X||^2 = \langle X, X\rangle = 1\}$ be the unitary sphere and let g be the Riemannian metric on S^2 induced by $\langle\,,\,\rangle$. As is well known, the geodesics of (S^2, g) are the great circles of S^2, while the *spherical distance*, $d = d_{S^2}$, induced by the metric g , is given by $d(P, Q) = \arccos\langle P, Q\rangle$, for every $P, Q \in S^2$.

If P, Q are two distinct non-antipodal points of S^2, there exists a unique geodesic arc joining P and Q, whose length is equal to the spherical distance $d(P, Q)$. We will then say that this geodesic arc is the *spherical segment with endpoints P and Q*, and we will denote it simply by PQ.

Note also that, given three points A, B, C of S^2, there are no geodesics of (S^2, g) passing through them if and only if A, B, C are linearly independent as vectors of $\mathbb{R}^3$, i.e., if and only if $\langle A \wedge B, C\rangle \neq 0$.

Let us now consider three points A, B, C of S^2 through which no geodesic of (S^2, g) passes. Therefore, in the set $\{A, B, C\}$, there are no pairs of antipodal points, and thus, the three spherical segments AB, AC, BC are uniquely defined. Then, the union T of these three spherical segments will be called *the spherical triangle of* S^2 *with vertices* A, B, C *and sides* AB, AC, BC. As in the hyperbolic case, in the following, we will denote by $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$, the spherical lengths of the sides BC, AC, AB of T, respectively, and by α, β, γ the spherical measures of the interior angles of T at the vertices A, B, C, respectively. Clearly, we have $0 < a,\ b,\ c,\ \alpha,\ \beta,\ \gamma < \pi$. Of course, the number $a + b + c$ is the *perimeter* of the triangle T, while, by Girard's well-known theorem, the *area* of T is the number $\alpha + \beta + \gamma - \pi$. We also remember the well-known *spherical law of cosines*:

$$\cos(\gamma) = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}$$

(and similar formulas for the other interior angles α, β).

Finally, note that we can assume that the three vertices A, B, C of the spherical triangle T are ordered in such a way that $\langle A \wedge B, C\rangle > 0$. From now on, we will always assume that this last condition is satisfied.

(b) For the convenience of the reader, we recall some elementary algebraic properties regarding the Euclidean scalar product and the vector product of $\mathbb{R}^3$.

(b.1) The map $(P, Q, R) \mapsto \langle P \wedge Q, R\rangle$ defines an alternating 3-linear form on $\mathbb{R}^3$.

Moreover, if $P, Q, R, S \in \mathbb{R}^3$, the following identities hold:

(b.2) $\langle P \wedge Q, R \wedge S\rangle = \langle P, R\rangle\langle Q, S\rangle - \langle P, S\rangle\langle Q, R\rangle$ (Binet–Cauchy identity);

(b.3) $P \wedge (Q \wedge R) = \langle P, R\rangle\, Q - \langle P, Q\rangle\, R$ (vector triple product formula).

Let T be a spherical triangle of S^2 with vertices A, B, C (which satisfy the condition $\langle A \wedge B, C\rangle > 0$). We define the following points of S^2:

$$A' = \frac{B \wedge C}{||B \wedge C||}, \quad B' = \frac{C \wedge A}{||C \wedge A||}, \quad C' = \frac{A \wedge B}{||A \wedge B||}\ .$$

Using (b.1) and (b.3), it is easy to check that we have

(b.4) $\langle A' \wedge B', C'\rangle = \dfrac{\langle A \wedge B, C\rangle^2}{||B \wedge C|| \cdot ||C \wedge A|| \cdot ||A \wedge B||} > 0$.

Therefore the three points A', B', C' are the vertices of a spherical triangle T' of S^2. Keeping (b.1) in mind, it is easy to realize that the triangle T' is uniquely determined; i.e., it is independent of the order of the vertices A, B, C of T as long as $\langle A \wedge B, C \rangle > 0$. Then T' is called *polar triangle of* T.

Furthermore, from (b.4), (b.1) and (b.3), vice versa we easily obtain that the polar triangle of T' is the triangle T (see, for instance, [3], (Proposizione 6.3.4 p. 80)).

In spherical geometry, the following theorem holds.

Theorem 8 ([3], (Teorema, p. 92))**.** *Let T be a spherical triangle of S^2 with vertices A, B, C. As in Preliminaries 4.1 (b), denote by A', B', C' the vertices of the polar triangle T' of T. Then,*

(a) the three bisectors of the interior angles of T pass through a common point $Q \in S^2$, called spherical incenter of T*;*
(b) the three perpendicular bisectors of the sides of T pass through a common point $Z \in S^2$, called spherical circumcenter of T*;*
(c) if $A \neq A'$, $B \neq B'$, $C \neq C'$, the three altitudes of T pass through a common point $H \in S^2$, called spherical orthocenter of T*;*
(d) the three medians of T pass through a common point $G \in S^2$, called spherical centroid of T*.*

Remark 4. *We remark that two geodesics of S^2 intersect in two antipodal points; then, a priori, incenter, circumcenter, orthocenter (when $A \neq A'$, $B \neq B'$, $C \neq C'$), and centroid, defined in the previous theorem, are a couple of antipodal points. However, we will select and consider the following points, respectively ([3]):*

$$Q = \frac{\sin(a)A + \sin(b)B + \sin(c)C}{||\sin(a)A + \sin(b)B + \sin(c)C||},$$

$$Z = \frac{B \wedge C + C \wedge A + A \wedge B}{||B \wedge C + C \wedge A + A \wedge B||},$$

$$H = \frac{\cos(b)\cos(c)\, B \wedge C + \cos(a)\cos(c)\, C \wedge A + \cos(a)\cos(b)\, A \wedge B}{||\cos(b)\cos(c)\, B \wedge C + \cos(a)\cos(c)\, C \wedge A + \cos(a)\cos(b)\, A \wedge B||},$$

$$G = \frac{A + B + C}{||A + B + C||}.$$

(c) We also need to define the following symmetric functions of a, b, c, which will appear in the next theorems:

$\tilde{\mathbf{\Lambda}}(a,b,c) := 1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\cos(b)\cos(c);$
$\tilde{\mathbf{\Gamma}}(a,b,c) := \sin(a)(\sin(a) + 2\sin(b)\cos(c)) + \sin(b)(\sin(b) + 2\sin(c)\cos(a)) +$
$+ \sin(c)(\sin(c) + 2\sin(a)\cos(b));$
$\tilde{\mathbf{\Xi}}(a,b,c) := 4 - (\cos(a) + \cos(b) + \cos(c) - 1)^2.$

4.2. On the Circumscribed Circle

The following will be useful.

Lemma 1. *Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c. Then, $\langle A \wedge B, C \rangle = \sqrt{\tilde{\mathbf{\Lambda}}(a,b,c)}$.*

Proof. By (b.3), we have

$$A \wedge (B \wedge C) = \cos(b)\, B - \cos(c)\, C,$$

hence

$$||A \wedge (B \wedge C)|| = \sqrt{\cos^2(b) + \cos^2(c) - 2\cos(a)\cos(b)\cos(c)}\,.$$

On the other hand,

$$||A \wedge (B \wedge C)|| = \sin(a)\ \sin(\widehat{A(B \wedge C)})\ ;$$

then

$$\sin(\widehat{A(B \wedge C)}) = \frac{\sqrt{\cos^2(b) + \cos^2(c) - 2\cos(a)\ \cos(b)\cos(c)}}{\sin(a)},$$

and therefore

$$\cos(\widehat{A(B \wedge C)}) = \pm \frac{\sqrt{1 - \cos^2(a) - \cos^2(b) - \cos^2(c) + 2\cos(a)\ \cos(b)\ \cos(c)}}{\sin(a)}\ .$$

Consequently, taking into account (b.1) and the assumption $\langle A \wedge B, C \rangle > 0$, we obtain $\langle A \wedge B, C \rangle = \sqrt{\tilde{\mathbf{\Lambda}}(a,b,c)}$. □

Proposition 2. *Let T be a spherical triangle of S^2 with a, b, c sides; then, the radius, R, of the circumscribed circle to T is defined by* $\cos(R) = \dfrac{\sqrt{\tilde{\mathbf{\Lambda}}(a,b,c)}}{\sqrt{\tilde{\mathbf{\Xi}}(a,b,c)}}$.

Proof. Let A, B, C be the vertices of T. We have $\cos(R) = d(Z, A) = d(Z, B) = d(Z, C)$, where Z is the spherical circumcenter of T. From (b.1) and from the expression of Z we obtain $\cos(d(Z, A)) = \langle Z, A \rangle = \dfrac{\langle A \wedge B, C \rangle}{||B \wedge C + C \wedge A + A \wedge B||}$. Keeping (b.2) in mind, a direct computation gives $||B \wedge C + C \wedge A + A \wedge B||^2 =$ $4 - (\cos(a) + \cos(b) + \cos(c) - 1)^2 = \tilde{\mathbf{\Xi}}(a,b,c)$. Hence, by using Lemma 1, we obtain the statement. □

4.3. On the Inscribed Circle

Proposition 3. *Let T be a spherical triangle of S^2 with a, b, c sides; then, the radius, r, of the inscribed circle of T is defined by* $\sin(r) = \dfrac{\sqrt{\tilde{\mathbf{\Lambda}}(a,b,c)}}{\sqrt{\tilde{\mathbf{\Gamma}}(a,b,c)}}$.

Proof. Let A, B, C be the vertices of T. We have $r = d(Q, sideAB) = d(Q, sideAC) = d(Q, sideBC)$, where Q is the spherical incenter of T. From the expression of Q we obtain (see [3]): $d(Q, sideAB) = \arcsin \dfrac{\langle A \wedge B,\ C \rangle}{||\sin(a)\ A + \sin(b)\ B + \sin(c)\ C||}$. Direct computation gives $||\sin(a)\ A + \sin(b)\ B + \sin(c)\ C||^2 = \sin^2(a) + \sin^2(b) + \sin^2(c) + 2\sin(a)\sin(b)\cos(c) + 2\sin(a)\sin(c)\cos(b) + 2\sin(b)\sin(c)\cos(a)$. Then, by using Lemma 1, we obtain the statement. □

4.4. On Geometrical Properties of the Polar Triangle

The polar triangle T' of a spherical triangle T of S^2 is a very important tool in spherical geometry. We recall the main properties of polar triangles, and we refer to [3] for details. Let T be a spherical triangle of S^2 with vertices A, B, C, whose lengths of the sides and measures of the interior angles are denoted, as in Preliminaries 4.1 (a), by a, b, c and α, β, γ, respectively. Denote with the same letters with apex $'$ the corresponding vertices, lengths of the sides, and measures of interior angles of the polar triangle T' of T. With these agreements, the following theorem holds:

Theorem 9 ([3], (Teorema 6.3.5, pp. 80–81))**.** *The following equalities hold:*
$a' = \pi - \alpha,\ \ b' = \pi - \beta,\ \ c' = \pi - \gamma,\ \ \alpha' = \pi - a,\ \ \beta' = \pi - b,\ \ \gamma' = \pi - c.$

Furthermore, the following laws of cosines for angles are true::

$$\cos(c) = \frac{\cos(\gamma) + \cos(\alpha)\cos(\beta)}{\sin(\alpha)\sin(\beta)},$$

$$\cos(b) = \frac{\cos(\beta) + \cos(\alpha)\cos(\gamma)}{\sin(\alpha)\sin(\gamma)},$$

$$\cos(a) = \frac{\cos(\alpha) + \cos(\beta)\cos(\gamma)}{\sin(\beta)\sin(\gamma)}.$$

Proposition 4. *Let T be a spherical triangle of S^2, and let T' be its polar triangle. We have perimeter (T) + area (T') = perimeter (T') + area $(T) = 2\pi$.*

Proof. From Theorem 9, we have perimeter (T) + area $(T') = a + b + c + \alpha' + \beta' + \gamma' - \pi = a + b + c + \pi - a + \pi - b + \pi - c - \pi = 2\pi$. Analogously, we obtain perimeter (T') + area $(T) = 2\pi$. □

Proposition 5 ([3], (Proposizione 7.3.1, p. 96))**.** *Let T be a spherical triangle of S^2, and let T' be its polar triangle. Let Q and Z be, respectively, the incenter and the circumcenter of T, and denote by Q' and Z', respectively, the incenter and the circumcenter of T'; then, we have*
(a) $Q' = Z$;
(b) $Z' = Q$.

By using this previous result, we can prove the following.

Proposition 6. *Let T be a spherical triangle of S^2, and let T' be its polar triangle. Let r and R be, respectively, the radius of the inscribed circle (inradius) and the radius of the circumscribed circle (circumradius) of T, and denote by r' and R', respectively, the inradius and the circumradius of T'. Then,* $r + R' = r' + R = \dfrac{\pi}{2}$.

Proof. We have $R' = d(Z', A')$; then,

$$\cos(R') = \langle Z', A'\rangle = \langle Q, A'\rangle = \left\langle \frac{\sin(a)A + \sin(b)B + \sin(c)C}{||\sin(a)A + \sin(b)B + \sin(c)C||}, \frac{B \wedge C}{||B \wedge C||}\right\rangle = \frac{\sin(a)\langle A \wedge B, C\rangle}{\sin(a) \cdot ||\sin(a)A + \sin(b)B + \sin(c)C||} = \sin(d(Q, sideAB)) = \sin(r).$$

As the polar triangle of T' is T, we immediately obtain $\cos(R) = \sin(r')$.

Hence, from the identity $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$, $x \in \mathbb{R}$, we have
$\dfrac{\pi}{2} = \arcsin\big(\sin(r)\big) + \arccos\big(\sin(r)\big) = r + \arccos\big(\cos(R')\big) = r + R'$,
$\dfrac{\pi}{2} = \arcsin\big(\sin(r')\big) + \arccos\big(\sin(r')\big) = r' + \arccos\big(\cos(R)\big) = r' + R$. Then the proof is complete. □

4.5. On the Euler Line in Spherical Geometry

As in the hyperbolic setting, we can prove the following.

Theorem 10. *The circumcenter, the orthocenter and the centroid of a spherical triangle T of S^2 belong to a geodesic of S^2 if and only if the triangle T is isosceles. This geodesic will be called the Euler geodesic of T.*

Proof. As noted in Preliminaries 4.1 (a), the points $Z, H, G \in S^2$ belong to the same geodesic of S^2 if and only if $\langle G, Z \wedge H\rangle = 0$ or, equivalently, if and only if
$\langle A + B + C, (B \wedge C + C \wedge A + A \wedge B) \wedge (\cos(b)\cos(c)B \wedge C + \cos(a)\cos(c)C \wedge A +$

$\cos(a)\cos(b)A \wedge B)\rangle = 0$. Direct computation gives the result that the above condition is satisfied if and only if $\langle A + B + C, \cos(a)(\cos(b) - \cos(c))A + \cos(b)(\cos(c) - \cos(a))B + \cos(c)(\cos(a) - \cos(c))C\rangle = 0$ or $\cos^2(a)(\cos(c) - \cos(b)) + \cos^2(b)(\cos(a) - \cos(c)) + \cos^2(c)(\cos(b) - \cos(a)) = (\cos(c) - \cos(b))(\cos(a) - \cos(b))(\cos(a) - \cos(c)) = 0$. As $0 < a, b, c < \pi$, we obtain the statement. □

Proposition 7. *If T is isosceles, its incenter belongs to its Euler geodesic.*

Proof. If we suppose $a = b$, then, by direct computation, we obtain $\langle G, Z \wedge Q\rangle = 0$; from this, the statement follows. □

Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c as before. Let T' be the polar triangle of T with vertices A', B', C' and sides a', b', c'. We can prove the following.

Proposition 8. *T is isosceles if and only if T' is isosceles. In particular, the Euler geodesic exists on T if and only if it exists on T'.*

Proof. We have

$$\cos(a) = \langle B, C\rangle,\ \cos(b) = \langle A, C\rangle,\ \cos(c) = \langle A, B\rangle,$$
$$\cos(a') = \langle B', C'\rangle,\ \cos(b') = \langle A', C'\rangle\ \cos(c') = \langle A', B'\rangle.$$

Direct computation gives $\cos(a') = \dfrac{\cos(b)\ \cos(c) - \cos(a)}{\sin(b)\ \sin(c)}$,
$\cos(b') = \dfrac{\cos(a)\ \cos(c) - \cos(b)}{\sin(a)\ \sin(c)}$, $\cos(c') = \dfrac{\cos(b)\ \cos(a) - \cos(c)}{\sin(b)\ \sin(a)}$.
If we suppose $a = b$, then we immediately obtain the result that $a' = b'$.

Since the polar triangle of T' is T, the proof is complete. □

It is well known that in Euclidean geometry, the distance between the orthocenter and the centroid of a triangle is twice the distance between the circumcenter and the centroid. This is no longer true in non-Euclidean geometry; however, in some special cases, we can compute a relation between the distance of the points H, G, Z, Q.

Lemma 2. *Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c as before. Let us suppose $A \neq A'$, $B \neq B'$, $C \neq C'$, $a = b$. Let H, G, Z, Q be, respectively, the orthocenter, the centroid, the circumcenter, and the incenter of T; then,*

$$\cos(d(Z,G)) = 3\sqrt{\frac{1-2\cos^2(a)-\cos^2(c)+2\cos^2(a)\ \cos(c)}{(3+4\cos(a)+2\cos(c))(3-\cos^2(c)-4\cos(a)+4\cos(a)\ \cos(c))}},$$

$$\cos(d(H,G)) = (2\cos(a)\ \cos(c) + \cos^2(a))\cdot$$
$$\sqrt{\frac{1-2\cos^2(a)-\cos^2(c)+2\cos^2(a)\cos(c)}{(3+4\cos(a)+2\cos(c))\cos^2(a)(1-\cos(c))(2\cos^2(c)+6\cos^2(a)\cos(c)+\cos^2(a))}},$$

$$\cos(d(Q,G)) = \frac{2\sin(a)(1+\cos(c)+\cos(a))+2\cos(a)\ \sin(c)+\sin(c)}{\sqrt{3+4\cos(a)+2\cos(c)}\sqrt{2\sin(a)(2\cos(a)\ \sin(c)+\sin(a)+\sin(a)\ \cos(c))+\sin^2(c)}},$$

$$\cos(d(H,Q)) = \frac{\cos(a)(2\sin(a)\ \cos(c)+\cos(a)\ \sin(c))}{\sqrt{2\sin(a)(\sin(a)+\sin(a)\ \cos(c)+2\cos(a)\ \sin(c))+\sin^2(c)}}\cdot$$
$$\sqrt{\frac{1-2\cos^2(a)-\cos^2(c)+2\cos^2(a)\ \cos(c)}{\cos^2(a)(2\cos^2(c)-2\cos^3(c)+3\cos^2(a)\ \cos^2(c)-4\cos^2(a)\ \cos(c)+\cos^2(a))}},$$

$$\cos(d(H,Z)) = \frac{\cos(a)}{\sqrt{\cos^2(a)(1-\cos(c))(2\cos^2(c)+6\cos(c)\cos^2(a)+\cos^2(a))}}\cdot$$
$$\frac{2\cos(c)(1-\cos(a)-\cos(c)+\cos(a)\ \cos(c))+\cos(a)(2\cos(a)\ \cos(c)-2\cos(a)+1-\cos^2(c))}{\sqrt{3-4\cos(a)+4\cos(a)\ \cos(c)-2\cos(c)-\cos^2(c)}},$$

$$\cos(d(Z,Q)) = (\sin(c) + 2\sin(a))\sqrt{\frac{1-2\cos^2(a)-\cos^2(c)+2\cos^2(a)\cos(c)}{3-4\cos(a)-\cos^2(c)-2\cos(c)+4\cos(a)\ \cos(c)}}\cdot$$
$$\frac{1}{\sqrt{2\sin(a)(\sin(a)+\sin(a)\ \cos(c)+2\cos(a)\ \sin(c))+\sin^2(c)}}.$$

Proof. Direct computation gives

$$||A+B+C||^2 = 3+4\cos(a)+2\cos(c)$$
$$||A\wedge B + B\wedge C + C\wedge A||^2 = 3-\cos^2(c) - 2\cos(c) - 4\cos(a) + 4\cos(a)\ \cos(c),$$
$$||\sin(a)(A+B)+\sin(c)\,C||^2 = 2\sin(a)(\sin(a)(1+\cos(c)) + 2\cos(a)\sin(c)),$$
$$||\cos(a)\cos(c)\,B\wedge C + \cos(a)\cos(c)\,C\wedge A + \cos^2(a)\,A\wedge B||^2 =$$
$$\cos^2(a)(2\cos^2(c) - 2\cos^3(c) + 3\cos(a)\cos^2(c) - 4\cos^2(a)\cos(c) + \cos^2(a)),$$
$$\langle G,Z\rangle = \frac{3\cdot\langle A\wedge B, C\rangle}{||A+B+C||\,||A\wedge B+B\wedge C+C\wedge A||},$$
$$\langle H,G\rangle = \frac{\cos(a)(\cos(a)+2\cos(c))\langle A\wedge B, C\rangle}{||A+B+C||\ \ ||\cos(a)\cos(c)\,B\wedge C+\cos(a)\cos(c)\,C\wedge A+\cos^2(a)\,A\wedge B||},$$
$$\langle Q,G\rangle = \frac{2\sin(a)(1+\cos(a)+\cos(c))+2\sin(c)\cos(a)+\sin(c)}{||A+B+C||\,||\sin(a)(A+B)+\sin(c)\,C||},$$
$$\langle H,Q\rangle =$$
$$\frac{\cos(a)(2\cos(c)\sin(a)+\cos(a)\sin(c))\langle A\wedge B, C\rangle}{||\sin(a)(A+B)+\sin(c)\,C||\,||\cos(a)\cos(c)B\wedge C+\cos(a)\cos(c)C\wedge A+\cos^2(a)A\wedge B||},$$
$$\langle H,Z\rangle =$$
$$\frac{\langle\cos(a)\cos(c)(C\wedge A+B\wedge C)+\cos^2(a)\,A\wedge B, B\wedge C+C\wedge A+A\wedge B\rangle}{||\cos(a)\cos(c)B\wedge C+\cos(a)\cos(c)C\wedge A+\cos^2(a)A\wedge B||\,||A\wedge B+B\wedge C+C\wedge A||},$$

$$\langle Z,Q\rangle = \frac{(2\sin(a)+\sin(c))\langle A\wedge B, C\rangle}{||A\wedge B+B\wedge C+C\wedge A||\,||\sin(a)(A+B)+\sin(c)\,C||}.$$

By substituting, we obtain the statement. □

Corollary 2. *Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c as before. Let us suppose $A \neq A'$, $B \neq B'$, $C \neq C'$, $a = b$ and $c = \frac{\pi}{2}$. Let H, G, Z, Q be, respectively, the orthocenter, the centroid, the circumcenter, and the incenter of T. Then,*

$$\cos(d(Z,G)) = 3\sqrt{\frac{1-2\cos^2(a)}{9-16\cos^2(a)}},$$
$$\cos(d(H,G)) = \sqrt{\frac{1-2\cos^2(a)}{3+4\cos(a)}},$$
$$\cos(d(Q,G)) = \frac{1+2\sin(a)+2\cos(a)+2\sin(a)\cos(a)}{\sqrt{3+4\cos(a)}\sqrt{2\sin(a)(2\cos(a)+\sin(a))+1}},$$
$$\cos(d(H,Q)) = \sqrt{\frac{1-2\cos^2(a)}{2\sin(a)(\sin(a)+2\cos(a))+1}},$$
$$\cos(d(H,Z)) = \frac{1-2\cos(a)}{\sqrt{3-4\cos(a)}},$$
$$\cos(d(Z,Q)) = (1+2\sin(a))\sqrt{\frac{1-2\cos^2(a)}{(3-4\cos(a))(2\sin(a)(\sin(a)+2\cos(a))+1)}}\,.$$

Proof. The formulas are obtained by substituting into previous formulas $\cos(c) = 0$ and $\sin(c) = 1$. □

Proposition 9. *Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c as before. Let us suppose $A \neq A'$, $B \neq B'$, $C \neq C'$, $a = b$ and $c = \frac{\pi}{2}$. Let H, G, Z, Q be, respectively, the orthocenter, the centroid, the circumcenter and the incenter of T. Then,*

$$(1+2\sin(a))(\cos(d(G,Z))\cos(d(H,Q))) = 3\cos(d(H,G))\cos(d(Z,Q)),$$
$$(1-2\cos(a))\cos(d(G,Z)) = 3\cos(d(Z,H))\cos(d(G,H)).$$

Proof. The formulas are obtained directly by the previous corollary. □

Corollary 3. *Let T be a spherical triangle of S^2 with vertices A, B, C and sides a, b, c as before. Let us suppose $A \neq A'$, $B \neq B'$, $C \neq C'$, $a = b$, and $c = \frac{\pi}{2}$. Let H, G, Z, Q be, respectively, the orthocenter, the centroid, the circumcenter, and the incenter of T. Then,*

$$\left(\frac{3}{2}\frac{\cos(d(H,G))\cos(d(Z,Q))}{\cos(d(G,Z))\cos(d(H,Q))} - \frac{1}{2}\right)^2 + \left(-\frac{3}{2}\frac{\cos(d(H,G))\cos(d(Z,H))}{\cos(d(G,Z))} + \frac{1}{2}\right)^2 = 1.$$

Proof. From Proposition 9, we obtain $\sin(a) = \frac{3}{2}\frac{\cos(d(H,G))\cos(d(Z,Q))}{\cos(d(G,Z))\cos(d(H,Q))} - \frac{1}{2}$ and $\cos(a) = -\frac{3}{2}\frac{\cos(d(H,G))\cos(d(Z,H))}{\cos(d(G,Z))} + \frac{1}{2}$, and then we obtain the statement. □

5. On Geodesic Triangles of the 3-Dimensional Sphere

Let us consider $\mathbb{R}^4$ with the standard scalar product, $\langle , \rangle$.

Let $S^3 = \{X \in \mathbb{R}^4 \mid ||X||^2 = \langle X, X \rangle = 1\}$ be the unitary sphere, and let g be the Riemannian metric on S^3 induced by $\langle , \rangle$. In coordinates, we have

$$S^3 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1 \right\}, \quad g = \langle , \rangle_{|S^3}.$$

It is well known that the totally geodesic submanifolds of (S^3, g) are the greatest spheres ([10]).

Proposition 10. *Every geodesic triangle in (S^3, g) is contained in a totally geodesic hypersurface.*

Proof. Let T be a geodesic triangle in (S^3, g) with vertices A, B, C. As A, B, C are linearly independent, then there exists a Euclidean hyperplane, π, passing through A, B, C and the center of the 3-sphere. Specifically, if $A = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, B = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix}, C = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ t_3 \end{pmatrix}$, then π is given by $ax + by + cz + dt = 0$, where $a, b, c, d \in \mathbb{R}$ are non-trivial solutions of the following linear system: $\begin{cases} ax_1 + by_1 + cz_1 + dt_1 = 0 \\ ax_2 + by_2 + cz_2 + dt_2 = 0 \\ ax_3 + by_3 + cz_3 + dt_3 = 0. \end{cases}$

Then, $S^3 \cap \pi$ is the totally geodesic surface containing the given geodesic triangle, and the proof is complete. □

We close this section with an example where geodesic triangles are not contained in a totally geodesic surface.

Example 2. *The 3-dimensional Heisenberg group, H, with a suitable left-invariant Riemannian metric has no totally geodesic surfaces ([11]). Then, geodesic triangles in H are not contained in a totally geodesic surface.*

6. Discussion and Conclusions

We study notable points for a geodesic triangle in non-Euclidean geometry, and we discuss the existence of the Euler line in this context. Moreover, we give simple proofs of the existence of a totally geodesic 2-dimensional submanifold containing a given geodesic triangle in the hyperbolic or spherical 3-dimensional geometry.

We give algebraic conditions under which the three perpendicular bisectors of the sides, or the three altitudes, have a finite common point, or are asymptotically parallel, or are ultra-parallel geodesics. We describe explicit examples, in the hyperbolic setting, where the analogue of the Euler line does not exist. We prove that every geodesic triangle in the hyperbolic 3-dimensional space is contained in a totally geodesic hypersurface. We compute the circumradius and the inradius of a spherical triangle, and we describe relationships with the same geometrical quantities of the polar triangle. We prove that the circumcenter, the orthocenter, and the centroid of a spherical triangle belong to a common geodesic of the 2-dimensional sphere if and only if the triangle is isosceles. We prove that, as in the hyperbolic case, every geodesic triangle in the 3-dimensional sphere is contained in a totally geodesic hypersurface. The main purpose of this paper is to provide a simple and organic treatment of some similarities and differences between geometrical properties of geodesic triangles in Euclidean and non-Euclidean geometry. This paper extends the results of [1,3].

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