DYNAMIC INTERACTIVE MEDIATORS IN DISCOURSE ON INDETERMINATE QUANTITIES: A CASE STUDY

Samuele Antonini, Chiara Bonadiman and Bernardo Nannini

Department of Mathematics and Computer Science, University of Florence, Italy

This paper investigates, with a commognitive perspective, the role of dynamic interactive mediators (DIMs) in promoting students' discourse on indeterminate quantities. We analyze a case study of two high school students with a history of low achievement in mathematics, focusing on whether their discourse, developed in activities with DIMs, integrate the (meta)-arithmetical discourse. We show how words, narratives and visual mediators produced interacting with DIMs expand and compress the arithmetical discourse, shaping, in this way, the meta-arithmetical discourse.

CONCEPTUAL BACKGROUND

For more than three decades, research has shown how the teaching and learning of school algebra is a challenging issue, often source of difficulties for many students (e.g., Sfard & Linchevski, 1994; Kieran, 1992). More recently, a stream of studies focused on proposing characterizations of algebraic thinking, on studying the process involved in its formation, and on looking for forms of algebraic thinking in activities apparently distant from school algebra or that even precede it (e.g., Radford, 2014; Caspi & Sfard, 2012; Kaput et al., 2008). Radford (2014) characterizes thinking as algebraic when it deals with indeterminate quantities as if they were known. Kieran (2022) frames three dimensions – analytic, structural, functional - of early algebraic thinking, and defines as analytic the thinking dimension related to the dealing with unknows as if they were knowns. In this view, thinking algebraically is not necessarily related to the use of algebraic symbolism: unknows and variables can be represented with symbols, everyday language, gestures and different signs.

In this paper, we focus on the development of algebraic thinking, specifically regarding the use of indeterminate quantities, of low-achieving high school students. This study is part of a funded research project on the learning of high school students with a long history of failure, with support of digital environments.

We adopt the *commognitive framework* (Sfard, 2008) where doing mathematics is seen as to engage in an established discourse, learning mathematics consists in becoming able to participate in this discourse and to study students' mathematical learning means to analyze their mathematical discourse. The term *discourse* applies to a form of communication characterized by *specific words* (e.g. "equation", "variable", "function"), *visual mediators* (perceptually accessible objects pre-existing to the discourse or artefacts produced for communicative purposes), *narratives* (descriptions of objects, of relations between objects and of activities with them) and *routines* (repetitive patterns characteristic of the given discourse).

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Multilevel structure of algebraic thinking

In the framework of Commognition, mathematical objects are neither extra-discursive nor pre-existing entities. Rather, they constitute part of the discourse itself, they are discursive constructs (Sfard, 2008, p. 129). The process of object construction is called *objectification*. It may be achieved in three ways (*saming*, *encapsulating*, *reification*) that develop with the use of a noun which will be employed in depersonalized narratives. In these narratives, the human subject disappears, as if the referent of the noun exists independently of it. In this way, the discursive construct becomes an object of mathematical exploration and then new mathematical narratives can emerge. Therefore, mathematical discourse develops by addition of new discursive layers, any of which subsumes a previous discourse. In this perspective, elementary algebra can be described as "*metaarithmetic*, or more precisely, as the *unification of arithmetic with its own metadiscourse*" (Sfard, 2008, p. 120), that is the mathematical discourse on arithmetical relations and processes (Caspi & Sfard, 2012).

The development of discourse can be described in terms of alternating *expansions* and *compressions* (Sfard, 2008, pp. 119-123). The increase in the amount and complexity of routines and of new discourses leads to a discursive expansion, while the compression reduces the complexity of the discourse through the rise to the metalevel.

Dynamic interactive mediators

Following the distinction between static and dynamic mediators (Ng, 2016), some studies (Baccaglini-Frank, 2021; Antonini et al., 2020) have formulated the notion of *dynamic interactive mediators* (DIMs), mediators that are dynamic and that respond to a person's manipulations. Examples of DIMs are digital manipulable objects constructed within technological environments. In our study, we are interested in studying how the arithmetical and meta-arithmetical discourses are shaped by the discourses emerging from the interaction with DIMs consisting in digital representation of the balance model. The balance model is a common metaphor in teaching linear equations, for conceptualizing the equal sign and promoting strategies to deal with unknowns. Gains and pitfalls of this model are discussed in the literature (for a review, Otten et al., 2019).

The DIMs we have designed and that we analyze in this paper consist in different versions of two-pan balances. A first DIM consists of a two-pan balance with weights that are known and are represented as colored shapes with a number inside, while some weights are indeterminate and are represented by white shapes (Fig. 2,3). The user can insert one value for the white weights in an input field. By the key "Let's try!", one can interact with the balance which will move depending on whether the total weight on the right pan is less than, greater than or equal to the total weight on the left pan (Fig. 1b), where the white shapes are worth the value inserted in the input field. We call DIM_{TB} this DIM, where TB stands for "test-balance". It models linear dependency between quantities, as its position depends on the input.

To model an equation, we design a visual mediator of a two-pan balance with, on its pans, known weights, colored and with reference to their values, and unknown weights (the balance on the left in Fig. 1a, 4a). This is a fixed, not interactive, balance that we denote by VM_{FB}, where VM stands for "visual mediator" and FB for "fixed balance". Finally, we call $DIM_{(FB,TB)}$ the DIM embedding the DIM_{TB} and VM_{FB} (Fig. 1a), where the DIM_{TB} is a version of the VM_{FB} in which the unknown weights are white (Fig. 1a). In $DIM_{(FB,TB)}$, the fixed balance is unmovable since every weight has fixed value, even if unknown, while the test-balance can assume different positions according to the value given to the white weights. They correspond to two different ways of thinking about a relation between two algebraic expressions. For example, one can think about 3+2x=11 as an equality between two quantities where x is *one* unknown number (VM_{FB}), or as a relation that can be true or false depending on x (DIM_{TB}).

Research questions

This study is part of a wider research project investigating the impact of DIM-based teaching interventions with second year high school students with a history of low achievement in mathematics. Under the hypothesis that the discourse about DIM can foster students' participation in mathematical discourse (Baccaglini-Frank, 2021; Antonini et al., 2020), we are interested in investigating the role of the discourse emerging from the interaction with DIMs (hereon DIMs-discourse) in promoting students' mathematical discourse. To guide the study, we ask the following questions: *how do the $DIM_{(FB,TB)}$- and DIM_{TB}–discourses integrate the arithmetical and meta-arithmetical discourses of students with a history of low achievement? How can these DIMs-discourses shape the production of a discourse on indeterminate quantities?*

METHOD

Participants were 12 students of 10th grade from three Italian high schools, selected by their teacher for their history of severe and persistent difficulties in mathematics and on voluntary participation. The sequence consisted in 4 or 5 sessions of two hours each, which took place in an out-of-school center. Students worked in pairs under the guidance of an expert and were provided with touch-screen tablets for the activities. This paper focuses on a pair of students engaged in activities with DIM_{TB} and $DIM_{(FB,TB)}$ during the first two sessions. Data consist of audio-video recordings, students' written productions, and screen recordings of the tablets used for the activities. In tune with the Commognition, the analysis will focus on the use of words, narratives and visual mediators related to dealing with indeterminate quantities.

A CASE STUDY

We present three episodes from the case study of Andrea and Hugo, two students coming from a professional high school. During a preliminary individual interview, both students showed difficulties in dealing with indeterminate quantities. For example, in looking for a solution of $13 - a = 13 + 11$, Andrea said "I don't remember how to do it" and Hugo wrote "$a = 13 + 11 = 24$". His discourse appears purely ritualistic, focused on performing (meaningless) procedures.

Episode 1

Andrea and Hugo are asked to find out the weights of the triangle, knowing that the balance with the colored shapes (Fig. 1a) is balanced off. Hugo immediately answers:

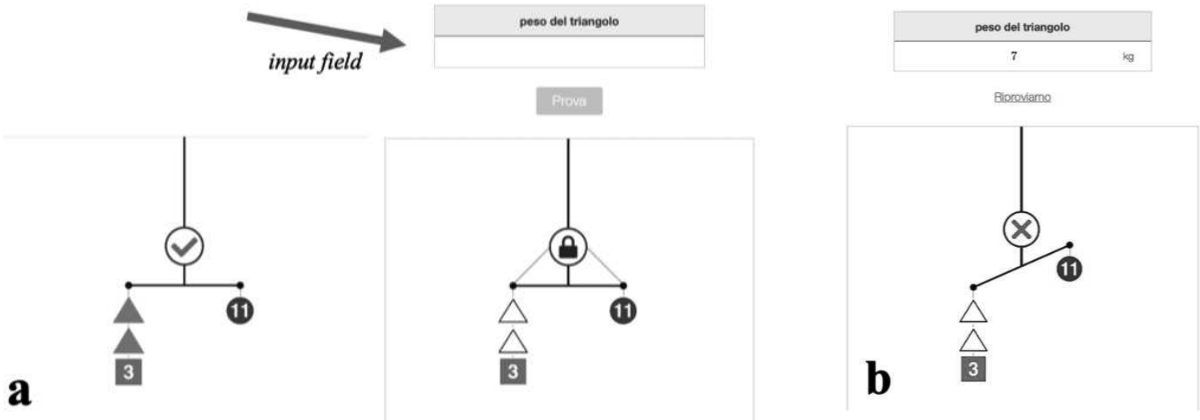

Figure 1: (a) $DIM_{(FB,TB)}$, with VM_{FB} (left) and DIM_{TB} (right); (b) DIM_{TB} with number 7 in the input field.

1	Hugo:	... 4. Because here on one side is 3 […] here [*he points to one of the white triangles*] you put a weight is worth 4, this one is 4, 4+4, 8, +3, 11 [4+4=8 and 8+3=11]. If you put 5, the balance tends to dangle that way, […] To make it dangle on the right you have to put a number […] smaller than 4 […] if you put 3, here is 9 while here is 11.
2	Int:	If we wanted to describe all this thing, in general terms, this balance, how could I say it? […] Let's make a summary.
3	Hugo:	To keep the balance… balanced off, the number, here a triangle is worth 4, to make it dangle to the left you have to find a number greater than 4, while to make it dangle to the right you have to find a number smaller than 4.

The discourse on numbers (e.g., "4, 4+4, 8, +3, 11", "here is 9") is intertwined with the DIM_{TB}-discourse, especially with the narrations on the interaction (of the subject) with DIM_{TB} ("here you put a weight", "if you put 5", "you have to put a number", "if you put 3"). Students use the verb "to put" as a signifier of the action of inserting the number inside the white triangle (Hugo points to *one* white triangle but their discourse shows that they consider that the same number *is put* in every white triangle). However, the action of putting numbers into the shapes cannot actually happen. DIM_{TB} allows to put numbers only in the input field and, in fact, Hugo uses the verb "to put" as a metaphor. In [3] we can also observe a depersonalized narrative ("a triangle is worth 4") without a subject who "puts" numbers.

Episode 2

Observing the DIM_{TB} (Fig. 2), Andrea says that on the right there is "more space [where to] put […] the weights". He then justifies the choice of number 3 to balance off the balance summing the numbers in both pans ("I made 5 plus 5, 10, plus 3, 13, plus 3, 16 [5+5+3+3=16]) saying "I made also […] the spaces and it comes out 16, 3 plus 3". The calculation with specific numbers ["3 plus 3"] is an arithmetical discourse while the sentence used to talk about this calculation ("I made also the spaces") can be used for whatever number ("the weights" in the "spaces") and therefore is a meta-arithmetical narrative. In summary, the use of "space" allows the construction of discourses about indeterminate quantities that can be considered as known quantities (even if Andrea produces a discourse on determinate quantities).

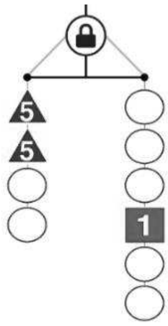

Figure 2: DIM_{TB} of the episode 2.

Episode 3

Andrea and Hugo are asked to identify the weights of the shapes that make the balance hang to the right, to the left or balanced off (Fig. 3). They find the couples of numbers (6,3) and (9,6) to balance off the balance. Then they summarize:

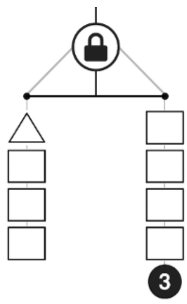

Figure 3: DIM_{TB} with two indeterminate weights and two input fields (episode 3).

4 Andrea: … to every weight [of the square] you add 3 [to get triangle's weight] […]

5 Hugo: It's enough that you take off 9 from the weight of the square with respect to the weight of the triangle […] it's enough that the weight of the square…

The problem is resumed during the next session:

6 Hugo: The triangle should be a square plus 3 […] because it is as if… you take off […] three squares you can tell that a triangle is a square plus 3 […] If you take off both sides three squares [*he makes a sketch*, Fig. 4b] here you are left with a triangle and you know that the triangle is a square plus 3.

The narrative in [4] is a meta-arithmetical summary of how the students have determined the numbers to balance off the DIM_{TB}. The object of the discourse is the human action ("you add 3") to add 3 to "every weight". In [5], Hugo still talks of a human action ("take off") but now the infinite actions, one for "every weight", are expressed as a single action on *one* weight ("the weight of the square"). The numbers 3 and 6 previously identified (as well as any other numbers) are here replaced by the single signifier "the weight", and then, this objectification can be considered as a process of saming. In [6], the human subject disappears, the “weight of the square/triangle” collapses into “the square” and “the triangle”, in the singular form, and the new narrative, completely depersonalized, is about their relation (“the triangle is a square plus 3”). In summary, the transition from the plural to singular form, the saming, and the depersonalization are the indicators of the process of objectification developing from infinite actions to one action, and from one action to the relations between objects. In table 1, from top to bottom, we can read this process.

Table 1: The development of the students’ discourse on indeterminate quantities.

Narrative	Signifier (Math object)	Human action
“To every weight you add 3” [4]	Every weight (Every specific number)	Infinite actions, one for every weight of the square. Every action is on a specific number
“It’s enough that you take off 9 from the weight of the square…” [5]	The weight of the square (An indeterminate quantity)	An action on one indeterminate quantity
“it’s enough that the weight of the square…” [5]	The weight of the square (An indeterminate quantity)	No human subjects
“The triangle should be a square plus three” [6]	The square, the triangle (Two indeterminate quantities)	No human subjects

The process described before is also visible in the drawing (Fig. 4b) that the students use to endorse their narrative “the triangle is a square plus 3”. This drawing is similar to the one made in a previous task (Fig. 4a) with the digital pen on VM_{FB} where the weight of the squares and triangles are 2 and 3 respectively and the circle has a (one), now unknow, *specific* weight. In the drawing produced in episode 3 (Fig. 4b), the signs are the same, but this time on white weights, where it is possible to “put” any number. Therefore, the process of saming described before, which allowed the compression of infinite discourses about numbers through the words “triangle” and “square”, expanded the discourse too.

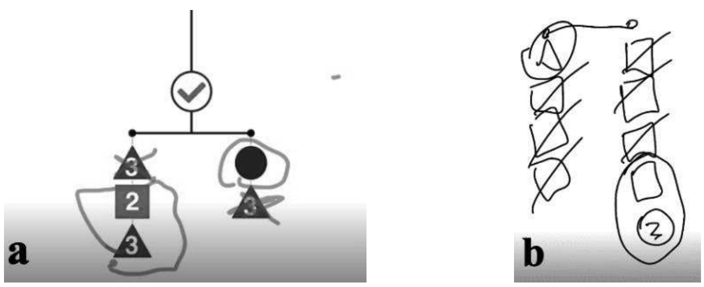

Figure 4: Hugo’s drawings in solving tasks with (a) $DIM_{(FB,TB)}$; (b) DIM_{TB}.

DISCUSSION AND CONCLUSIONS

In response to the research questions, the conducted analysis sheds some light on how the DIMs-discourse ($DIM_{(FB,TB)}$ and DIM_{TB}) promoted the students' discourse on indeterminate quantities. The two students have generated words, narratives, and visual mediators linked to the DIMs and to their interaction with them. The continuous interaction with DIMs modifies the students' discourse. It intertwines with, and then expand, the arithmetical discourse, fostering the production of words, narratives and visual mediators that compress arithmetical discourses, shaping, in this way, the meta-arithmetical discourse. Several elements of the DIMs-discourse expanded. The use of the verb "to put" extends metaphorically to the assignment of numerical values to the "white shapes". The names of the shapes, firstly used for given or unknown but specific weights, extend to a variable value, and finally are used at singular as signifiers of objects that can be manipulated and studied (Table 1). The use of an encapsulated couple, where the objects were numbers (known or unknown), expands into a discourse on indeterminate quantities (Fig. 4a-b). The narrative "the triangle is a square plus 3" arises from a discourse compression accompanied by an objectification. This narrative, in which different numbers could substitute for "triangle" and "square", compresses (potentially infinite) narratives on numbers, reducing the complexity of the discourse. The words "triangle" and "square" allow to make a discourse on indeterminate quantities and to deal with them as they were known numbers. In this way the discourse moves on a meta-arithmetical (algebraic) level.

The case analysis shows how Andrea and Hugo, two high school students with school experience with algebra and a history of low achievement in mathematics, were able to participate to an algebraic discourse. Andrea and Hugo's discourse can be regarded still as *informal*. However, according to Kieran (2022), Radford (2014) and Caspi & Sfard (2012), thinking algebraically is not necessarily related to the use of algebraic symbolism and this can be considered a step towards the formal algebraic discourse. This is in line with Nachlieli & Tabach (2012, p. 24), who, in a study on functions, state that an informal mathematical discourse can be considered as a "solid foundation" for the students' "future discourse".

Furthermore, the analysis presented in this paper confirms previous studies involving low-achieving students that show, in different mathematical fields, how DIM can foster students' participation in mathematical discourse (e.g., Baccaglini-Frank, 2021).

Finally, this study has reported an example of development of informal algebraic discourse after the introduction of school algebra for low-achievers. Similar results are presented by Caspi & Sfard (2012) in a study about the emergence of informal algebraic discourses before the teaching of school algebra. These similarities underline the importance of investigating the relationship between informal and formal algebraic discourses during the long process of learning algebra.

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