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Paths and flows for centrality measures in networks

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Abstract

We consider the number of paths that must pass through a subset X of vertices of a capacitated network N in a maximum sequence of arc-disjoint paths connecting two vertices y and z. We consider then the difference between the maximum flow value from y to z in N and the maximum flow value from y to z in the network obtained from N by setting to zero the capacities of arcs incident to X. When X is a singleton, those quantities are involved in defining and computing the flow betweenness centrality and are commonly identified without any rigorous proof justifying the identification. On the basis of a deep analysis of the interplay between paths and flows, we prove that, when X is a singleton, those quantities coincide. Moreover they are both equal to the global flow that must pass through X in any maximum flow from y to z. On the other hand, we prove that, when X has at least two elements, those quantities and the global flow that must pass through X in any maximum flow from y to z may be different from each other. We next show that, by means of the considered quantities, two conceptually different group centrality measures, based on paths and flows respectively, can be naturally defined. Such group centrality measures both extend the flow betweenness centrality to groups of vertices and are proved to satisfy a desirable form of monotonicity.

KEYWORDS

arc-disjoint paths, flow betweenness, group centrality, network flow

1 | INTRODUCTION

The concept of flow is certainly one of the most fruitful concepts in network theory with a plethora of recent applications varying from transport engineering to social choice theory and financial networks (see, for instance, [6, 9, 22]). The powerful maxflow-mincut theorem by Ford and Fulkerson [12] immensely contributed to the success of that concept giving rise, among other things, to the manageable augmenting path algorithm for computing maximum flows. Flows and paths are also at the core of the well-known centrality measure called *flow betweenness* due to Freeman et al. [14]. That centrality measure, here denoted by Λ_1, is defined, for a capacitated network N with vertex set V and $x \in V$, by

$$\Lambda_1^N(x) = \sum_{\substack{y,z \in V \setminus \{x\} \\ y \neq z}} \lambda_{yz}^N(x),$$

where $\lambda_{yz}^N(x)$ is introduced as "the maximum flow from y to z that passes through the vertex x" in [14, pp. 147-148]. Since a flow is a function on the arcs of the network, the description of $\lambda_{yz}^N(x)$ is not perfectly clear. However, as one can understand from the examples in [14] and the comments to [14] by Borgatti and Everett [4], $\lambda_{yz}^N(x)$ corresponds to the number of paths that

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surely pass through x in any maximum sequence of arc-disjoint paths in N connecting y and z. In fact, the flow betweenness was designed to solve some of the drawbacks of the betweenness centrality measure, proposed by Freeman [13] and based on geodesics, by considering not only the shortest paths but all the possible paths between two vertices. Thus, despite its name, the flow betweenness is conceived as a centrality measure based on paths.

The flow betweenness, and some of its variations, are present in UCINET [3] and in the R sna package [20]. One of those variations, proposed by Borgatti and Everett [4] and here denoted by Λ_2, is defined, for every capacitated network N with vertex set V and $x \in V$, by

$$\Lambda_2^N(x) = \sum_{\substack{y,z \in V \setminus \{x\} \\ y \neq z, \lambda_{yz}^N > 0}} \frac{\lambda_{yz}^N(x)}{\lambda_{yz}^N},$$

where λ_{yz}^N corresponds to the maximum number of arc-disjoint paths from y to z in N.[1]

Borgatti and Everett [4, p. 475] observe, however, that, since there is not in general a unique maximum sequence of arc-disjoint paths between two vertices, "flow betweenness cannot be calculated directly by counting paths." Thus, they explain that in UCINET the number $\lambda_{yz}^N(x)$ is computed as the difference $\varphi_{yz}^N(x) = \varphi_{yz}^N - \varphi_{yz}^{N_x}$, where φ_{yz}^N is the maximum flow value from y to z in N and $\varphi_{yz}^{N_x}$ is the maximum flow value from y to z in the network N_x obtained by N by setting to zero all the capacities on arcs incident to x. In other words, $\lambda_{yz}^N(x)$ is replaced by the amount of flow that gets lost when all the communications through x are interrupted.[2] As a consequence, the centrality measures Λ_1 and Λ_2 are de facto replaced by the centrality measures Φ_1 and Φ_2 based on flows and defined, for every capacitated network N with vertex set V and $x \in V$, by[3]

$$\Phi_1^N(x) = \sum_{\substack{y,z \in V \setminus \{x\} \\ y \neq z}} \varphi_{yz}^N(x), \qquad \Phi_2^N(x) = \sum_{\substack{y,z \in V \setminus \{x\} \\ y \neq z, \varphi_{yz}^N > 0}} \frac{\varphi_{yz}^N(x)}{\varphi_{yz}^N}.$$

Note that in [4] the centrality measure Φ_2 is called flow centrality. Of course, the computation of $\varphi_{yz}^N(x)$ is definitely much less expensive than the one of $\lambda_{yz}^N(x)$, since the two numbers φ_{yz}^N and $\varphi_{yz}^{N_x}$ can be simply computed via the augmenting path algorithm. Thus, $\Phi_1^N(x)$ and $\Phi_2^N(x)$ can be computed much more easily than $\Lambda_1^N(x)$ and $\Lambda_2^N(x)$. On the other hand, $\Lambda_1^N(x)$ and $\Phi_1^N(x)$ (resp. $\Lambda_2^N(x)$ and $\Phi_2^N(x)$) might be in principle different for some networks and some of their vertices. In other words, Λ_1 and Φ_1 (resp. Λ_2 and Φ_2) might be different centrality measures. As a consequence, an analysis of the relation between the two numbers $\lambda_{yz}^N(x)$ and $\varphi_{yz}^N(x)$ is certainly crucial in order to understand the relation among the considered centrality measures. At the best of our knowledge, however, no general result linking $\lambda_{yz}^N(x)$ and $\varphi_{yz}^N(x)$ is available in the literature.

As the main result of the paper, we prove that the equality

$$\lambda_{yz}^N(x) = \varphi_{yz}^N(x), \tag{1}$$

always holds true (Theorem 20). That immediately implies that actually $\Lambda_1 = \Phi_1$ and $\Lambda_2 = \Phi_2$ and shows then that replacing $\lambda_{yz}^N(x)$ by $\varphi_{yz}^N(x)$ is perfectly justified. We stress that, while proving the inequality $\lambda_{yz}^N(x) \geq \varphi_{yz}^N(x)$ is quite simple, the proof of the equality $\lambda_{yz}^N(x) = \varphi_{yz}^N(x)$ requires a quite sophisticated argument involving some delicate aspects of flow theory and, in particular, the Flow Decomposition Theorem (Theorem 12). The heart of the matter is that, as shown in detail in Section 3, it is possible to reconstruct flows from the knowledge of paths and conversely to derive paths from the knowledge of flows. Hence, in many situations, one can conceptually interchange flows and paths but that interchange is not obvious at all, especially when the focus is on paths passing through a fixed vertex. Relying on that delicate interchange we also give a complete justification of the intuitive description of $\lambda_{yz}^N(x)$ appearing in [14]. Indeed, we formally define the concept of global flow that must pass through a fixed vertex x in any maximum flow from y to z in the network N, denoted by $\delta_{yz}^N(x)$, and we prove that $\lambda_{yz}^N(x) = \delta_{yz}^N(x)$ (Proposition 23).

In the paper, we also study the natural extensions of $\lambda_{yz}^N(x)$ and $\varphi_{yz}^N(x)$ to the case where groups of vertices are considered. Given a group X of vertices, we consider the number $\lambda_{yz}^N(X)$ of paths that must pass through X in a maximum sequence of arc-disjoint paths connecting two distinct vertices y and z. Moreover, we consider the number $\varphi_{yz}^N(X) = \varphi_{yz}^N - \varphi_{yz}^{N_X}$, where N_X is the network obtained by N by setting to zero all the capacities related to arcs incident to X. On the basis of (1), one might expect that $\lambda_{yz}^N(X)$ equals $\varphi_{yz}^N(X)$, but that is not true, in general, when X is not a singleton (Proposition 21). Indeed, in general, we only have $\lambda_{yz}^N(X) \geq \varphi_{yz}^N(X)$ (Proposition 24). These facts make clear that $\lambda_{yz}^N(X)$ and $\varphi_{yz}^N(X)$ are in fact different concepts based on diverse rationales. Using those extensions, the centrality measures Λ_1, Λ_2, Φ_1 and Φ_2 can be immediately generalized to groups of vertices. Indeed, for a capacitated network N with vertex set V and $X \subseteq V$, one can set

$$\Lambda_1^N(X) = \sum_{\substack{y,z \in V \setminus X \\ y \neq z}} \lambda_{yz}^N(X), \qquad \Phi_1^N(X) = \sum_{\substack{y,z \in V \setminus X \\ y \neq z}} \varphi_{yz}^N(X),$$

[1] The definition of Λ_2 is inspired to [13].
[2] Also in the R sna package $\lambda_{yz}^N(x)$ is replaced by $\varphi_{yz}^N(x)$.
[3] The equality $\lambda_{yz}^N = \varphi_{yz}^N$ is well known (see, for instance, [2, Lemma 7.1.5]).

and

$$\Lambda_2^N(X) = \sum_{\substack{y,z\in V\setminus X\\ y\neq z,\lambda_{yz}^N>0}} \frac{\lambda_{yz}^N(X)}{\lambda_{yz}^N}, \qquad \Phi_2^N(X) = \sum_{\substack{y,z\in V\setminus X\\ y\neq z,\varphi_{yz}^N>0}} \frac{\varphi_{yz}^N(X)}{\varphi_{yz}^N}.$$

Such extensions of centrality measures to groups of vertices are in line with the ideas by Everett and Borgatti [10], as pointed out by the fact that the sums only involve vertices not belonging to X. Of course, since $\lambda_{yz}^N(X)$ and $\varphi_{yz}^N(X)$ may be different, we have that Λ_1 and Φ_1 (resp. Λ_2 and Φ_2) do not coincide as group centrality measures. Moreover, as noticed in [4] for single vertices, Λ_1 and Λ_2 are hard to compute and thus their concrete application seems quite problematic.

In the last part of the paper, using the quantities $\lambda_{yz}^N(X)$ and $\varphi_{yz}^N(X)$, we propose two new group centrality measures (Section 5). The first one, based on $\lambda_{yz}^N(X)$ and denoted by Λ, is called *full-flow betweenness group centrality measure* and is a variation of Λ_2. The second one, based on $\varphi_{yz}^N(X)$ and denoted by Φ, is called *full-flow vitality group centrality measure* and is a variation of Φ_2. Precisely, for a capacitated network N with vertex set V and $X \subseteq V$, we define

$$\Lambda^N(X) := \sum_{\substack{y,z\in V\\ y\neq z,\lambda_{yz}^N>0}} \frac{\lambda_{yz}^N(X)}{\lambda_{yz}^N}, \qquad \Phi^N(X) := \sum_{\substack{y,z\in V\\ y\neq z,\varphi_{yz}^N>0}} \frac{\varphi_{yz}^N(X)}{\varphi_{yz}^N}.$$

As evident, the new idea is just to take into account all the vertices of the network, including those belonging to X. On the one hand, the computational complexity of Λ (resp. Φ) is the same as the computational complexity of Λ_1 and Λ_2 (resp. Φ_1 and Φ_2). On the other hand, Λ and Φ satisfy a natural form of monotonicity which surely does not hold true for Λ_1, Λ_2, Φ_1, Φ_2. More precisely, we show that if $X \subseteq Y \subseteq V$, then $\Lambda^N(X) \leq \Lambda^N(Y)$ as well as $\Phi^N(X) \leq \Phi^N(Y)$ (Proposition 28). Such a form of monotonicity is invoked as a main property by Everett and Borgatti [10] and constitutes a basic requirement for other desirable properties for group centrality measures. A detailed analysis of the properties of Λ and Φ, especially in view of their applications, is an interesting topic for future research. We are particularly confident in the use of Φ that appears very clear in scope and, differently from Λ, tractable from the computational viewpoint. In fact, one of the objectives of our research is to support and promote the use of (group) centrality measures defined in terms of flows, like Φ. Indeed, it seems that flows had not played yet, in the context of centrality, the deep role that they would deserve. Flows allow us to investigate the characteristics of a network through a very complete and global approach that does not seem to be possible by making use of other concepts. Moreover, flows have a long consolidated mathematical history that could pave the road for a wide investigation of many interesting properties.

There is some limited recent literature about new centrality measures based on flows. Remarkably, Gómez et al. [17] introduce the flow-cost closeness centrality measure and the flow-cost betweenness centrality measure. As declared by their names, those measures take into account not only the maximum flows but also the costs that can be associated to the paths of the network. The authors rely on ordered sets of the so-called nondominated vectors and use methods of linear programming. Since the definitions and the methods are very different from ours, a comparison with our centrality measures is not at hand but could certainly be of some interest.

The paper is organized as follows. In Section 2 some well-known concepts of network theory are recalled, among which are those of flow, generalized path, path, cycle, and sequence of arc-disjoint paths. We propose precise and formal definitions in order to fix notation and allow the proofs to run smoothly. We define then the two main concepts of our research, namely the numbers $\varphi_{yz}^N(X)$ and $\lambda_{yz}^N(X)$. In Section 3 we explain how to recover flows from the knowledge of sequences of arc-disjoint paths and conversely. Section 4 is about the analysis of the properties of $\varphi_{yz}^N(X)$ and $\lambda_{yz}^N(X)$. In particular, in Section 4.1 we prove our main result (1); in Section 4.2 we focus on the global flow that must pass through X in any maximum flow, showing that it coincides with $\lambda_{yz}^N(X)$ when X is a singleton. In Section 5 we introduce the two flow group centrality measures Λ and Φ and we comment on them. Moreover, we show that they both satisfy the aforementioned type of monotonicity. The conclusions close the paper. For convenience of the reader, we collect in Table 1 the fundamental notation introduced in the paper.

2 | MAIN DEFINITIONS

2.1 | Notation and preliminary definitions

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $m \in \mathbb{N}_0$ we set $[m] = \{n \in \mathbb{N} : n \leq m\}$. In particular, $[0] = \varnothing$ and $|[m]| = m$ for all $m \in \mathbb{N}_0$. As usual, the sum of real numbers (of real-valued functions) over an empty set of indices is assumed to be the number 0 (the constant function 0).

Let X be a (possibly empty) set and $m \in \mathbb{N}$. A sequence of m elements in X is an element of the Cartesian product X^m. Given $\boldsymbol{x} = (x_j)_{j\in[m]} = (x_1, \dots, x_m) \in X^m$ and $j \in [m]$, we say that $x_j \in X$ is the jth component of $\boldsymbol{x}$. Of course, different components of the same sequence can be equal. Note that $X^m \neq \varnothing$ if and only if $X \neq \varnothing$ so that there are sequences of m elements in X if

TABLE 1 Fundamental notation

Symbol	Description	Definition
N_X	Network obtained by N by deleting the arcs incident to X	Section 2.2
$\mathcal{F}(N,y,z)$	Set of flows	Section 2.2
φ^N_{yz}	Maximum flow value	Section 2.2
$f(X)$	Flow through X	Definition 1
$\varphi^N_{yz}(X)$	Reduction of maximum flow when the arcs incident to X are deleted	Definition 2
$A(\gamma)$	Arcs of the path γ	Section 2.4
$A(\gamma)^+$	Forward arcs of the path γ	Section 2.4
$A(\gamma)^-$	Backward arcs of the path γ	Section 2.4
P^N_{yz}	Set of paths	Section 2.5
C^N	Set of cycles	Section 2.5
$\boldsymbol{\gamma}$	Sequence of paths	Section 2.5
$l(\boldsymbol{\gamma})$	Number of components of the sequence of paths $\boldsymbol{\gamma}$	Section 2.5
$(P^N_{yz})^m$	Set of sequences of m paths	Section 2.5
$\mathcal{S}^{N,m}_{yz}$	Set of sequences of m arc-disjoint paths	Section 2.5
$\mathcal{S}^N_{yz}$	Set of sequences of arc-disjoint paths	Section 2.5
λ^N_{yz}	Maximum of $l(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}$	Section 2.5
$\mathcal{M}^N_{yz}$	Set of the elements of $\mathcal{S}^N_{yz}$ that maximize l	Section 2.5
$l_X(\boldsymbol{\gamma})$	Number of components of $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}$ passing through X	Section 2.5
$\lambda^N_{yz}(X)$	Minimum of $l_X(\boldsymbol{\gamma})$ for $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}$	Definition 5
$\mathcal{M}^N_{yz}(X)$	Set of the elements of $\mathcal{M}^N_{yz}$ that minimize l_X	Section 2.6
χ_a	Arc function associated with the arc a	Definition 8
χ_γ	Path (cycle) function associated with the path (cycle) γ	Definition 8
$f_{\boldsymbol{\gamma}}$	Flow associated with the sequence of paths $\boldsymbol{\gamma}$	Definition 10
$(\boldsymbol{\gamma}, \boldsymbol{w})$	Decomposition of a flow	Theorem 12
$\mathcal{S}^N_{yz}(f)$	Set of sequences of arc-disjoint paths associated with f	Definition 14
$\mathcal{T}^{N,m}_{yz}$	Set of sequences of arc-disjoint paths for flows of value m	Definition 14
$\mathcal{T}^N_{yz}$	Set of sequences of arc-disjoint paths for maximum flows	Definition 14
$\delta^N_{yz}(X)$	Global flow that must pass through X in any maximum flow	Definition 22
$\Phi^N(X)$	The full flow vitality GCM	Definition 26
$\Lambda^N(X)$	The full flow betweenness GCM	Definition 27

and only if $X \neq \varnothing$. We also set $X^0 = \{()\}$ and call the symbol () the sequence of 0 elements of X. In order to have a uniform notation for sequences of 0 elements of X and sequences of $m \geq 1$ elements of X, we will always interpret as () any writing of the type $(x_j)_{j\in[0]}$. Finally, given two sequences of elements of X, we say that they are *equivalent* if they both have 0 elements or if they have the same number of elements and one can be obtained from the other by a permutation of the components.

Let V be a finite set with $|V| \geq 2$. The complete digraph on V is the digraph $K_V = (V, A)$ with vertex set V and arc set $A = \{(x, y) \in V^2 : x \neq y\}$. Note that in a complete digraph the set A of arcs is completely determined by the choice of the set V of vertices. The set of the complete digraphs is denoted by $\mathscr{K}$. A network is a pair $N = (K_V, c)$, where $K_V = (V, A) \in \mathscr{K}$ and c is a function from A to $\mathbb{N}_0$ called capacity. If convenient we will also indicate a network in a more detailed way by $N = (V, A, c)$. The set of networks is denoted by $\mathscr{N}$.[4]

An important family of networks, often considered in network literature, are the 0-1 networks. Recall that a network $N = (V, A, c)$ is called 0-1 if $c(a) \in \{0, 1\}$ for all $a \in A$. Of course, there is a natural bijection between 0-1 networks on V and digraphs on V. Thus, the concept of network can be seen as an extension of the one of digraph.

2.2 | Flows in a network

Let $N = (K_V, c) \in \mathscr{N}$ be fixed in the rest of this section. If $a = (x, y) \in A$ we call x and y the endpoints of a. Moreover, we say that a exits from x and enters in y. Let $X \subseteq V$. An arc $a \in A$ is called incident to X if at least one of its endpoints belongs to X. We define

$$A^+_X := \{(x, u) \in A : x \in X, u \in V \setminus X\}, \quad A^-_X := \{(u, x) \in A : x \in X, u \in V \setminus X\}, \quad A_X := A^+_X \cup A^-_X.$$

[4]To avoid insidious set theory issues one can, of course, assume $V \subseteq \mathbb{N}$.

Note that A_X is the set of arcs in A with a unique endpoint belonging to X.[5] We define by N_X the network $(K_V, c_X) \in \mathscr{N}$ where, for every $a \in A$,

$$c_X(a) := \begin{cases} 0 & \text{if } a \text{ is incident to } X \\ c(a) & \text{otherwise.} \end{cases}$$

Within flow theory, the capacity of X is defined by $c(X) := \sum_{a \in A_X^+} c(a)$. Note that if $x \in V$, then $c(x)$ is the so-called outdegree of x while $c(V \setminus \{x\})$ is the so-called indegree of x.

Let $y, z \in V$ be distinct. Recall that a flow from y to z in N is a function $f : A \to \mathbb{N}_0$ such that, for every $a \in A$,

$$0 \leq f(a) \leq c(a) \quad \text{(compatibility)} \tag{2}$$

and, for every $x \in V \setminus \{y, z\}$,

$$\sum_{a \in A_x^-} f(a) = \sum_{a \in A_x^+} f(a) \quad \text{(conservation law).} \tag{3}$$

The function $f_0 : A \to \mathbb{N}_0$ defined by $f_0(a) = 0$ for all $a \in A$ is a flow, called the null flow. We denote the set of flows from y to z in N by $\mathcal{F}(N, y, z)$.

When we represent networks and flows via a figure, we are going to use some standard conventions: a single number attached to an arc represents the capacity of that arc; two numbers attached to an arc, respectively, represent the flow and the capacity of that arc; if an arc is not drawn, then its capacity is zero.

Recall that, given $f \in \mathcal{F}(N, y, z)$, the value of f is the nonnegative integer

$$v(f) := \sum_{a \in A_y^+} f(a) - \sum_{a \in A_y^-} f(a).$$

The number

$$\varphi_{yz}^N := \max_{f \in \mathcal{F}(N,y,z)} v(f),$$

is called the maximum flow value from y to z in N. If $f \in \mathcal{F}(N, y, z)$ is such that $v(f) = \varphi_{yz}^N$, then f is called a maximum flow from y to z in N. We denote the set of maximum flows from y to z in N by $\mathcal{M}(N, y, z)$.

Given $N' = (K_V, c') \in \mathscr{N}$ with $c' \leq c$, it is immediate to observe that

$$\mathcal{F}(N', y, z) \subseteq \mathcal{F}(N, y, z), \tag{4}$$

and

$$\varphi_{yz}^{N'} \leq \varphi_{yz}^N. \tag{5}$$

Let us introduce now an important definition.

Definition 1. Let $f \in \mathcal{F}(N, y, z)$. For every $x \in V$, we set

$$f(x) := \begin{cases} \sum_{a \in A_x^+} f(a) & \text{if } x \notin \{y, z\} \\ v(f) & \text{if } x \in \{y, z\}. \end{cases}$$

For every $X \subseteq V$, we next set

$$f(X) := \sum_{x \in X} f(x)$$

and we call $f(X)$ the flow that passes through X in the flow f.

Note that $f(X) \geq 0$ and that if $X \cap \{y, z\} \neq \emptyset$, then $f(X) \geq v(f)$.

2.3 | The number $\varphi_{yz}^N(X)$

Let us introduce now the first main concept of our research, namely the number $\varphi_{yz}^N(X)$.

Definition 2. Let $N = (K_V, c) \in \mathscr{N}$, $y, z \in V$ be distinct and $X \subseteq V$. We define

$$\varphi_{yz}^N(X) := \varphi_{yz}^N - \varphi_{yz}^{N_X},$$

[5]Throughout the paper, in all the writings involving a subset X of the set of vertices of a network, we write x instead of X when $X = \{x\}$, for some vertex x. Thus, for instance, we write A_x^+ instead of $A_{\{x\}}^+$.

The number $\varphi^N_{yz}(X)$ represents the reduction of maximum flow value from y to z in N when the capacity of all the arcs incident to X are set to zero. Note that $X \cap \{y,z\} \neq \varnothing$ implies $\varphi^N_{yz}(X) = \varphi^N_{yz}$. The next proposition states a monotonicity property of $\varphi^N_{yz}(X)$.

Proposition 3. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $X \subseteq Y \subseteq V$. Then*

$$0 \leq \varphi^N_{yz}(X) \leq \varphi^N_{yz}(Y).$$

Proof. By Definition 2, we have that $\varphi^N_{yz}(X) = \varphi^N_{yz} - \varphi^{N_X}_{yz}$ and $\varphi^N_{yz}(Y) = \varphi^N_{yz} - \varphi^{N_Y}_{yz}$. Since $X \subseteq Y$ we have that $N_X = (K_V, c_X)$ and $N_Y = (K_V, c_Y)$ are such that $c_Y \leq c_X \leq c$. Thus, by (5), we have that $\varphi^{N_Y}_{yz} \leq \varphi^{N_X}_{yz} \leq \varphi^N_{yz}$ which in turn implies $0 \leq \varphi^N_{yz}(X) \leq \varphi^N_{yz}(Y)$, as desired. ∎

2.4 | Generalized paths and cycles in a complete digraph

Let $K_V \in \mathcal{K}$ and $y, z \in V$ be distinct. Consider a pair $\gamma = ((x_1, \dots, x_m), (a_1, \dots, a_{m-1}))$, where $m \geq 2$, $x_1, \dots, x_m \in V$ are called the vertices of γ, $a_1, \dots, a_{m-1} \in A$ are called the arcs of γ. The set of vertices of γ is denoted by $V(\gamma)$ and the set of arcs by $A(\gamma)$. Given $X \subseteq V$, we say that γ passes through X if $X \cap V(\gamma) \neq \varnothing$. We are interested in the following specifications for γ.

1. γ is called a *generalized path* in K_V if $x_1, \dots, x_m$ are distinct and, for every $i \in [m-1]$, $a_i = (x_i, x_{i+1})$ or $a_i = (x_{i+1}, x_i)$. If $a_i = (x_i, x_{i+1})$, a_i is called a *forward arc*; if $a_i = (x_{i+1}, x_i)$, a_i is called a *backward arc*. Note that, as a consequence, $a_1, \dots, a_{m-1}$ are distinct too. The set of forward arcs is denoted by $A(\gamma)^+$; the set of backward arcs by $A(\gamma)^-$. Clearly, we have $A(\gamma) = A(\gamma)^+ \cup A(\gamma)^-$ and $A(\gamma)^+ \cap A(\gamma)^- = \varnothing$. We say that γ is a generalized path from y to z if $x_1 = y$ and $x_m = z$.
2. γ is called a *path* in K_V if γ is a generalized path and $A(\gamma)^- = \varnothing$ or, equivalently, $A(\gamma) = A(\gamma)^+$.
3. γ is called a *cycle* in K_V if $m \geq 3$, $x_1, \dots, x_{m-1}$ are distinct while $x_m = x_1$ and, for every $i \in \{1, \dots, m-1\}$, $a_i = (x_i, x_{i+1})$.

Let $\gamma = ((x_1, \dots, x_m), (a_1, \dots, a_{m-1}))$ be a path or a cycle in K_V. Then γ is completely determined by its vertices and thus we usually write $\gamma = x_1 \cdots x_m$. Of course, the same simple notation is not possible for generalized paths that are not paths.

2.5 | Sequences of arc-disjoint paths in a network

Let $N = (K_V, c) \in \mathcal{N}$. A path (cycle) γ in K_V is called a path (cycle) in N if, for every arc $a \in A(\gamma)$, we have $c(a) \geq 1$. The set of paths from y to z in N is denoted by P^N_{yz}. The set of cycles in N is denoted by C^N.

Definition 4. Given $m \in \mathbb{N}_0$, $\boldsymbol{\gamma} = (\gamma_j)_{j \in [m]} \in (P^N_{yz})^m$ is called a sequence of m *arc-disjoint paths* if, for every $a \in A$,

$$|\{j \in [m] : a \in A(\gamma_j)\}| \leq c(a). \tag{6}$$

The above definition agrees with the common definition of m arc-disjoint paths used for 0-1 networks. Indeed, if N is a 0-1 network then $\boldsymbol{\gamma} = (\gamma_j)_{j \in [m]} \in (P^N_{yz})^m$ is a sequence of m arc-disjoint paths in N if and only if, for every $j \in [m]$, all the arcs of γ_j have capacity 1 and every arc with capacity 1 in the network is used at most once. In this case, the γ_j are necessarily distinct. On the contrary, for a generic capacitated network, repetitions of paths in a sequence of m arc-disjoint paths are surely possible.

Note that, trivially, if $\boldsymbol{\gamma}$ is a sequence of m arc-disjoint paths and $\boldsymbol{\gamma}'$ is equivalent to $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}'$ is a sequence of m arc-disjoint paths too. We denote the set of sequences of m arc-disjoint paths from y to z in N by $\mathcal{S}^{N,m}_{yz}$. Note that $\mathcal{S}^{N,0}_{yz} = \{(\,)\}$ and that $P^N_{yz} = \varnothing$ implies $\mathcal{S}^{N,m}_{yz} = \varnothing$ for all $m \geq 1$. The set of sequences of arc-disjoint paths from y to z in N is defined by

$$\mathcal{S}^N_{yz} := \bigcup_{m \in \mathbb{N}_0} \mathcal{S}^{N,m}_{y,z}.$$

Note that, since $(\,) \in \mathcal{S}^N_{yz}$, we always have $\mathcal{S}^N_{yz} \neq \varnothing$. Moreover, $P^N_{yz} = \varnothing$ if and only if $\mathcal{S}^N_{yz} = \{(\,)\}$. If $\boldsymbol{\gamma} \in \mathcal{S}^{N,m}_{yz}$, we say that the length of $\boldsymbol{\gamma}$ is m and we write $l(\boldsymbol{\gamma}) = m$. Observe that, $l(\boldsymbol{\gamma}) = 0$ if and only if $\boldsymbol{\gamma} = (\,)$. We also set

$$\lambda^N_{yz} := \max\left\{m \in \mathbb{N}_0 : \ \mathcal{S}^{N,m}_{yz} \neq \varnothing\right\}.$$

Note that λ^N_{yz} is the maximum length of a sequence of arc-disjoint paths from y to z in N and that $\lambda^N_{yz} = 0$ if and only if $P^N_{yz} = \varnothing$. The set of sequences of arc-disjoint paths from y to z in N having maximum length is defined by

$$\mathcal{M}^N_{yz} := \mathcal{S}^{N,\lambda^N_{yz}}_{yz} = \left\{\boldsymbol{\gamma} \in \mathcal{S}^N_{yz} : \ l(\boldsymbol{\gamma}) = \lambda^N_{yz}\right\}.$$

A sequence in $\mathcal{M}^N_{yz}$ is called a maximum sequence of arc-disjoint paths. By Lemma 7.1.5 in [2], we know that

$$\varphi^N_{yz} = \lambda^N_{yz}. \tag{7}$$

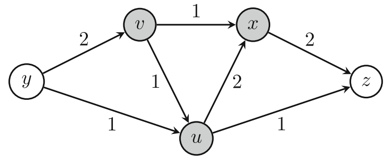

FIGURE 1 The network N in Example 6

Hence, we also have $\mathcal{M}_{yz}^N = \mathcal{S}_{yz}^{N,\varphi_{yz}^N}$ so that if $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N$, then $l(\boldsymbol{\gamma}) = \varphi_{yz}^N$.

We emphasize that, given $\boldsymbol{\gamma} = (\gamma_j)_{j\in[m]} \in \mathcal{S}_{yz}^{N,m}$, where $m \in \mathbb{N}$ and $m < n \leq \varphi_{yz}^N$ (so that $\boldsymbol{\gamma}$ is not of maximum length), it is not generally guaranteed that there exists a sequence $(\gamma_j)_{j\in\{m+1,\dots,n\}}$ of $n-m$ arc-disjoint paths from y to z in N such that $(\gamma_j)_{j\in[n]} \in \mathcal{S}_{yz}^{N,n}$. In other words, one cannot generally add paths to a sequence of arc-disjoint paths to get a new sequence of arc-disjoint paths of higher length. That fact surely introduces an element of complexity in treating the sequences of arc-disjoint paths. We will discuss in more detail that issue after having presented the Flow Decomposition Theorem (Theorem 12) in Section 3.2.

Given $\boldsymbol{\gamma} \in \mathcal{S}_{yz}^N$ and $X \subseteq V$, we denote now by $l_X(\boldsymbol{\gamma})$ the number of components of $\boldsymbol{\gamma}$ passing through X. Formally, if $\boldsymbol{\gamma} = (\gamma_j)_{j\in[m]} \in \mathcal{S}_{yz}^N$, where $m \in \mathbb{N}_0$, we set

$$l_X(\boldsymbol{\gamma}) := |\{j \in [m] : \gamma_j \text{ passes through } X\}|.$$

Note that if $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{S}_{yz}^N$ are equivalent, then $l_X(\boldsymbol{\gamma}) = l_X(\boldsymbol{\gamma}')$.

2.6 | The number $\lambda_{yz}^N(X)$

We now have all the tools for providing the definition of the other main concept of our research, namely the number $\lambda_{yz}^N(X)$.[6]

Definition 5. Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $X \subseteq V$. We define

$$\lambda_{yz}^N(X) := \min_{\boldsymbol{\gamma}\in\mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma}).$$

The number $\lambda_{yz}^N(X)$ represents the number of paths that must pass through X in a maximum sequence of arc-disjoint paths connecting the vertices y and z. Since $\mathcal{M}_{yz}^N \neq \emptyset$, $\lambda_{yz}^N(X)$ is well defined. By (7) we have that

$$0 \leq \lambda_{yz}^N(X) \leq \varphi_{yz}^N. \tag{8}$$

Note that $X \cap \{y, z\} \neq \emptyset$ implies $\lambda_{yz}^N(X) = \varphi_{yz}^N$. Moreover, $\lambda_{yz}^N(X) = 0$ if and only if there exists $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N$ such that $l_X(\boldsymbol{\gamma}) = 0$, that is, none of the paths appearing as components of $\boldsymbol{\gamma}$ passes through X.

We also set

$$\mathcal{M}_{yz}^N(X) := \arg\min_{\boldsymbol{\gamma}\in\mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma}).$$

Note that $\mathcal{M}_{yz}^N(X) \neq \emptyset$ and that if $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N(X)$, then $l_X(\boldsymbol{\gamma}) = \lambda_{yz}^N(X)$ and $l(\boldsymbol{\gamma}) = \varphi_{yz}^N$. In other words, the set $\mathcal{M}_{yz}^N(X)$ collects the maximum sequences of arc-disjoint paths from y to z in N minimally passing through X.

Example 6. In order to clarify Definition 5, let us perform some explicit computations for the network N in Figure 1. First of all, we have that $P_{yz}^N = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$, where

$$\gamma_1 = yvuxz,\ \gamma_2 = yvuz,\ \gamma_3 = yvxz,\ \gamma_4 = yuxz,\ \gamma_5 = yuz.$$

Of course, $\mathcal{S}_{yz}^{N,0} = \{()\}$ and $\mathcal{S}_{yz}^{N,1} = \{(\gamma_1), (\gamma_2), (\gamma_3), (\gamma_4), (\gamma_5)\}$. Up to a reordering of the components, the elements of $\mathcal{S}_{yz}^{N,2}$ are given by

$$(\gamma_1, \gamma_3),\ (\gamma_1, \gamma_5),\ (\gamma_2, \gamma_3),\ (\gamma_2, \gamma_4),\ (\gamma_3, \gamma_4),\ (\gamma_3, \gamma_5),$$

while the elements of $\mathcal{S}_{yz}^{N,3}$ are given by

$$\boldsymbol{\gamma}' = (\gamma_1, \gamma_3, \gamma_5),\ \boldsymbol{\gamma}'' = (\gamma_2, \gamma_3, \gamma_4).$$

[6]When X is a singleton, a definition similar to Definition 5 is proposed in [15].

Moreover, for every $m \geq 4$, $\mathcal{S}_{yz}^{N,m} = \emptyset$. As a consequence, $\lambda_{yz}^N = 3$ and $\mathcal{M}_{yz}^N = \mathcal{S}_{yz}^{N,3}$. Considering now $X = \{x\}$, we have that $l_X(\boldsymbol{\gamma}') = 2$ and $l_X(\boldsymbol{\gamma}'') = 2$. Thus,

$$\lambda_{yz}^N(X) = \min_{\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma}) = \min\{l_X(\boldsymbol{\gamma}'), l_X(\boldsymbol{\gamma}'')\} = 2,$$

and $\mathcal{M}_{yz}^N(X) = \mathcal{M}_{yz}^N$. Considering instead $X = \{x, v\}$, we have that $l_X(\boldsymbol{\gamma}') = 2$ and $l_X(\boldsymbol{\gamma}'') = 3$. Thus,

$$\lambda_{yz}^N(X) = \min_{\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma}) = \min\{l_X(\boldsymbol{\gamma}'), l_X(\boldsymbol{\gamma}'')\} = 2,$$

and the elements of $\mathcal{M}_{yz}^N(X)$ are given by $\boldsymbol{\gamma}''$ and all the sequences equivalent to $\boldsymbol{\gamma}''$.

We now prove a monotonicity property for $\lambda_{yz}^N(X)$ similar to that proved for $\varphi_{yz}^N(X)$ in Proposition 3.

Proposition 7. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $X \subseteq Y \subseteq V$. Then*

$$\lambda_{yz}^N(X) \leq \lambda_{yz}^N(Y).$$

Proof. Let $\boldsymbol{\gamma}^* \in \mathcal{M}_{yz}^N(Y)$ so that $l_Y(\boldsymbol{\gamma}^*) = \lambda_{yz}^N(Y)$. Since $X \subseteq Y$, the components of $\boldsymbol{\gamma}^*$ passing through Y are at least as many as those passing through X, that is, $l_X(\boldsymbol{\gamma}^*) \leq l_Y(\boldsymbol{\gamma}^*)$. Thus,

$$\lambda_{yz}^N(X) = \min_{\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma}) \leq l_X(\boldsymbol{\gamma}^*) \leq l_Y(\boldsymbol{\gamma}^*) = \lambda_{yz}^N(Y),$$

as desired. ■

We close this section with a final comment about some misunderstandings that appeared in the literature when $X = \{x\}$. Newman in [19, p. 41, note 3], citing Freeman et al. [14] about their description of $\lambda_{yz}^N(x)$, explains that, in order to take into account the fact that there is, in general, more than one maximum sequence of arc-disjoint paths from y to z in N, they consider "the maximum possible flow through x over all possible solutions to the yz maximum flow problem." Within our notation that means to consider the quantity $\max_{\boldsymbol{\gamma} \in \mathcal{M}_{yz}^N} l_X(\boldsymbol{\gamma})$. It describes the flow that *can* pass through x, and not the one that *must* pass through x, in any maximum flow.[7] That object could be of some interest but surely it is not in line with the original idea in [14].

3 | PATHS AND FLOWS

Once the formal definitions of $\varphi_{yz}^N(X)$ and $\lambda_{yz}^N(X)$ are given, our main purpose is to analyze the relation between those numbers. It turns out fundamental to deepen the link between paths and flows. A careful description of that link constitutes the indispensable tool for the proof of our main theorem, namely Theorem 20. First of all, let us introduce the concepts of generalized path function, path function, and cycle function, having in mind the definitions and comments in Section 2.4.

Definition 8. Let $K_V \in \mathcal{K}$. If $a \in A$, the arc function associated with a is the function $\chi_a : A \to \mathbb{N}_0$ defined by $\chi_a(a) = 1$ and $\chi_a(b) = 0$ for all $b \in A \setminus \{a\}$. If γ is a generalized path in K_V, let $\chi_\gamma : A \to \mathbb{Z}$ be defined by

$$\chi_\gamma := \sum_{a \in A(\gamma)^+} \chi_a - \sum_{a \in A(\gamma)^-} \chi_a, \tag{9}$$

so that, if γ is a path, then

$$\chi_\gamma = \sum_{a \in A(\gamma)} \chi_a. \tag{10}$$

If γ is a cycle in K_V, let $\chi_\gamma : A \to \mathbb{Z}$ be defined by

$$\chi_\gamma := \sum_{a \in A(\gamma)} \chi_a. \tag{11}$$

The functions (9), (10), (11) are, respectively, called the generalized path function, the path function and the cycle function associated with γ.

Note that the generalized path functions assume values in $\{-1, 0, 1\}$. In particular, given a generalized path γ, we have that $\chi_\gamma(a) = -1$ if and only if a is a backward arc of γ. Path functions and cycle functions assume instead only values in $\{0, 1\}$.

[7] A similar problem seems to be present in the description of $\lambda_{yz}^N(x)$ in the recent book by Zweig [23, p. 253].

3.1 | From paths to flows

The next result shows how every sequence of arc-disjoint paths can define a flow.

Proposition 9. *Let $N = (V, A, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $\boldsymbol{\gamma} = (\gamma_j)_{j\in[m]} \in \mathcal{S}_{yz}^{N,m}$, for some $m \in \mathbb{N}_0$. Then the function $f_{\boldsymbol{\gamma}} : A \to \mathbb{N}_0$ defined, for every $a \in A$, by*

$$f_{\boldsymbol{\gamma}}(a) := |\{j \in [m] : a \in A(\gamma_j)\}| \tag{12}$$

is a flow from y to z in N with $v(f_{\boldsymbol{\gamma}}) = m$ and $f_{\boldsymbol{\gamma}} = \sum_{j\in[m]} \chi_{\gamma_j}$. Moreover, for every $x \in V$, the flow $f_{\boldsymbol{\gamma}}(x)$ that passes through x in $f_{\boldsymbol{\gamma}}$ satisfies

$$f_{\boldsymbol{\gamma}}(x) = l_x(\boldsymbol{\gamma}). \tag{13}$$

Proof. By (6), we immediately obtain that $f_{\boldsymbol{\gamma}}$ satisfies the compatibility condition (2). For every $a \in A$, define $U_a = \{j \in [m] : a \in A(\gamma_j)\}$. Note that $U_a \subseteq [m]$ and $|U_a| = f_{\boldsymbol{\gamma}}(a)$.

Let $x \in V$ and $a, b \in A$ with $a \neq b$. If $a, b \in A_x^-$ or if $a, b \in A_x^+$, then we have

$$U_a \cap U_b = \emptyset. \tag{14}$$

Indeed, let $a, b \in A_x^-$ and assume by contradiction that there exists $j \in U_a \cap U_b$. Then both a and b are arcs of the path γ_j entering into its vertex x, contrary to the fact that in a path every vertex has at most one arc entering into it. The same argument applies to the case $a, b \in A_x^+$.

Consider now $x \in V \setminus \{y, z\}$. It is immediately checked that

$$\bigcup_{a\in A_x^-} U_a = \{j \in [m] : \gamma_j \text{ passes through } x\} = \bigcup_{a\in A_x^+} U_a. \tag{15}$$

Then, using (14) and (15), we get

$$\sum_{a\in A_x^-} f_{\boldsymbol{\gamma}}(a) = \sum_{a\in A_x^-} |U_a| = \left|\bigcup_{a\in A_x^-} U_a\right| = l_x(\boldsymbol{\gamma}) = \left|\bigcup_{a\in A_x^+} U_a\right| = \sum_{a\in A_x^+} |U_a| = \sum_{a\in A_x^+} f_{\boldsymbol{\gamma}}(a), \tag{16}$$

which says that $f_{\boldsymbol{\gamma}}$ satisfies the conservation law (3). Thus, we have proved that $f_{\boldsymbol{\gamma}}$ is a flow. By (16), we also see that

$$f_{\boldsymbol{\gamma}}(x) = \sum_{a\in A_x^+} f_{\boldsymbol{\gamma}}(a) = l_x(\boldsymbol{\gamma}).$$

We next show that

$$\bigcup_{a\in A_y^+} U_a = [m]. \tag{17}$$

We surely have $\bigcup_{a\in A_y^+} U_a \subseteq [m]$, so that we are left with proving $[m] \subseteq \bigcup_{a\in A_y^+} U_a$. If $m = 0$, then $[m] = [0] = \emptyset$ and the desired inclusion immediately holds. Assume next that $m \geq 1$. Pick $j \in [m]$ and consider γ_j. Since $y \neq z$, there exists $a \in A(\gamma_j) \cap A_y^+$ and therefore $j \in \bigcup_{a\in A_y^+} U_a$.

We now compute the flow value. Since U_a is empty for $a \in A_y^-$, using (14) and (17), we get

$$v(f_{\boldsymbol{\gamma}}) = \sum_{a\in A_y^+} |U_a| - \sum_{a\in A_y^-} |U_a| = \sum_{a\in A_y^+} |U_a| = m. \tag{18}$$

Now the equality $f_{\boldsymbol{\gamma}} = \sum_{j\in[m]} \chi_{\gamma_j}$ is an immediate consequence of (12) and (10). Finally observe that the equality (13) holds also for $x \in \{y, z\}$ because, by Definition 1 and by (18), we have $f_{\boldsymbol{\gamma}}(x) = v(f_{\boldsymbol{\gamma}}) = m = l_x(\boldsymbol{\gamma})$. ■

The above proposition allows us to give an important definition.

Definition 10. Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $\boldsymbol{\gamma} \in \mathcal{S}_{yz}^N$. The flow $f_{\boldsymbol{\gamma}}$ defined in (12) is called the flow associated with $\boldsymbol{\gamma}$.

Note that if $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \mathcal{S}_{yz}^N$ are equivalent, then $f_{\boldsymbol{\gamma}} = f_{\boldsymbol{\gamma}'}$.

3.2 | From flows to paths

In this section we present the well-known Flow Decomposition Theorem in a form that is useful for our purposes and explore its fundamental consequences for our research. We will make large use of generalized path functions and cycle functions (Definition 8).

We start recalling, within our notation, the well-known concept of augmenting path. A generalized path γ from y to z in K_V is called an *augmenting path* for f in N if $c(a)-f(a) \geq 1$ for all $a \in A(\gamma)^+$, and $f(a) \geq 1$ for all $a \in A(\gamma)^-$. We denote by $AP^N_{yz}(f)$ the set of the augmenting paths from y to z for f in N. By the celebrated Ford and Fulkerson Theorem, $f \in \mathcal{M}(N, y, z)$ if and only if $AP^N_{yz}(f) = \emptyset$. The next proposition is a straightforward but useful interpretation of the flow augmenting path algorithm within our notation.

Proposition 11. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $f \in \mathcal{F}(N, y, z) \setminus \mathcal{M}(N, y, z)$. Then $AP^N_{yz}(f) \neq \emptyset$ and, for every $\sigma \in AP^N_{yz}(f)$, we have that $f + \chi_\sigma \in \mathcal{F}(N, y, z)$ and $v(f + \chi_\sigma) = v(f) + 1$.*

The following result is substantially a technical rephrase of the Flow Decomposition Theorem as it appears in [1, Chapter 3].

Theorem 12. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $f \in \mathcal{F}(N, y, z)$ having value $m \in \mathbb{N}_0$. Then there exist a sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\in[m]}$ of m paths from y to z in N, $k \in \mathbb{N}_0$ and a sequence $\boldsymbol{w} = (w_j)_{j\in[k]}$ of k cycles in N such that*

$$f = \sum_{j\in[m]} \chi_{\gamma_j} + \sum_{j\in[k]} \chi_{w_j}. \tag{19}$$

An ordered pair $(\boldsymbol{\gamma}, \boldsymbol{w})$ satisfying (19) is called a decomposition of f. For every decomposition $(\boldsymbol{\gamma}, \boldsymbol{w})$ of f, we have that $\boldsymbol{\gamma}$ is a sequence of m arc-disjoint paths, that is, $\boldsymbol{\gamma} \in \mathcal{S}^{N,m}_{yz}$.

Proof. Except for the final statement, everything comes from [1, Theorem 3.5]. We need only to show that $(\gamma_j)_{j\in[m]} \in \mathcal{S}^{N,m}_{yz}$. Assume then, by contradiction, that there exists $a \in A$ such that $|\{j \in [m] : a \in A(\gamma_j)\}| > c(a)$. Then, by (19) and recalling that the cycle functions are nonnegative, we deduce

$$f(a) = \sum_{j\in[m]} \chi_{\gamma_j}(a) + \sum_{j\in[k]} \chi_{w_j}(a) = |\{j \in [m] : a \in A(\gamma_j)\}| + \sum_{j\in[k]} \chi_{w_j}(a) > c(a),$$

contrary to the compatibility condition (2). ∎

Example 13. As an illustration of Theorem 12, consider the network N and the flow f from y to z in N described in Figure 2. Note that $v(f) = 2$. A simple check shows that we have

$$f = \chi_{yvxz} + \chi_{yuz} + \chi_{vxuv},$$

where $(yvxz, yuz)$ is a sequence of two arc-disjoint paths from y to z in N and $vxuv$ is a cycle in N. Moreover we also have

$$f = \chi_{yvxuz} + \chi_{yuvxz},$$

where $(yvxuz, yuvxz)$ is a sequence of two arc-disjoint paths from y to z in N and no cycle is involved. In other words, $((yvxz, yuz), (vxuv))$ and $((yvxuz, yuvxz), ())$ are two decompositions of f. That confirms the well-known fact that, in general, a flow can admit diverse decompositions.

By Theorem 12 we deduce that if there exists a flow of value m, then there also exists a flow of the same value of the type $f_{\boldsymbol{\gamma}}$, where $\boldsymbol{\gamma} \in \mathcal{S}^{N,m}_{yz}$. Such a $\boldsymbol{\gamma}$ can be obtained by considering any decomposition $(\boldsymbol{\gamma}^*, \boldsymbol{w}^*)$ of an arbitrarily chosen m-valued flow f^* and setting $\boldsymbol{\gamma} := \boldsymbol{\gamma}^*$.

By Theorem 12 we can also better comment upon and comprehend the issue raised in Section 2.5. Let us consider $\boldsymbol{\gamma} = (\gamma_j)_{j\in[m]} \in \mathcal{S}^{N,m}_{yz}$, where $m \in \mathbb{N}$ and $m < n \leq \varphi^N_{yz}$ (so that $\boldsymbol{\gamma}$ has not maximum length). Then, by Proposition 9, $f_{\boldsymbol{\gamma}}$ is not a maximum flow. Applying the flow augmenting path algorithm $n-m$ times using suitable augmenting paths $\sigma_1, \ldots, \sigma_{n-m}$, we find the flow $\hat{f} = \sum_{j\in[m]} \chi_{\gamma_j} + \sum_{j\in[n-m]} \chi_{\sigma_j}$ of value n. Recall that the σ_j are not paths but generalized paths. Now, by Theorem 12, we have that there exist $\boldsymbol{\mu} = (\mu_j)_{j\in[n]} \in \mathcal{S}^{N,n}_{yz}$ and a sequence $\boldsymbol{w} = (w_j)_{j\in[k]}$ of $k \in \mathbb{N}_0$ cycles in N such that $\hat{f} = \sum_{j\in[n]} \chi_{\mu_j} + \sum_{j\in[k]} \chi_{w_j}$. However, the sequence $\boldsymbol{\mu}$ does not contain, in general, the original sequence $\boldsymbol{\gamma}$ as a subsequence and there is no immediate way to get one from the other.

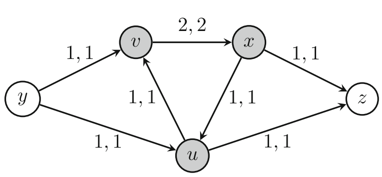

FIGURE 2 The network N in Example 13

Finally, Theorem 12 also allows us to naturally associate with every flow a set of sequences of arc-disjoint paths in the sense of the following definition.

Definition 14. Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $f \in \mathcal{F}(N, y, z)$ with $v(f) = m$. We set

$$\mathcal{S}^N_{yz}(f) := \left\{ \boldsymbol{\gamma} \in \mathcal{S}^N_{yz} \,:\, \exists k \in \mathbb{N}_0 \ \text{ and } \ \boldsymbol{w} \in (C^N)^k \ \text{ such that } \ (\boldsymbol{\gamma}, \boldsymbol{w}) \ \text{ is a decomposition of } f \right\}, \tag{20}$$

and we call $\mathcal{S}^N_{yz}(f)$ the set of sequences of arc-disjoint paths associated with f. We also set

$$\mathcal{T}^{N,m}_{yz} := \bigcup_{\substack{f \in \mathcal{F}(N,y,z)\\ v(f)=m}} \mathcal{S}^N_{yz}(f) \qquad \text{and} \qquad \mathcal{T}^N_{yz} := \mathcal{T}^{N,\varphi^N_{yz}}_{yz}, \tag{21}$$

and we call $\mathcal{T}^{N,m}_{yz}$ the set of sequences of arc-disjoint paths for m-valued flows and $\mathcal{T}^N_{yz}$ the set of sequences of arc-disjoint paths for maximum flows.

Note that, by Theorem 12, if $f \in \mathcal{F}(N, y, z)$ with $v(f) = m$, then $\emptyset \neq \mathcal{S}^N_{yz}(f) \subseteq \mathcal{S}^{N,m}_{yz}$.

We are now in position to clarify the link between sequences of arc-disjoint paths and flows in a network. Proposition 15 significantly extends (7) showing that, whatever is m, the sequences of m arc-disjoint paths are exactly those associated with the flows of value m, through the Flow Decomposition Theorem. Moreover, it shows that the set of sequences of arc-disjoint paths for maximum flows coincides with the set of maximum sequences of arc-disjoint paths.

Proposition 15. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $m \in \mathbb{N}_0$. Then the following facts hold:*

(*i*) $\mathcal{S}^{N,m}_{yz} = \mathcal{T}^{N,m}_{yz}$;
(*ii*) $\mathcal{M}^N_{yz} = \mathcal{T}^N_{yz}$.

Proof.

(*i*) Let $f \in \mathcal{F}(N, y, z)$ with $v(f) = m$. We have already observed that $\mathcal{S}^N_{yz}(f) \subseteq \mathcal{S}^{N,m}_{yz}$. Thus, by (21), we get $\mathcal{T}^{N,m}_{yz} \subseteq \mathcal{S}^{N,m}_{yz}$. Let now $\boldsymbol{\gamma}^* = (\gamma^*_j)_{j\in[m]} \in \mathcal{S}^{N,m}_{yz}$ and consider the flow $f_{\boldsymbol{\gamma}^*}$ associated with $\boldsymbol{\gamma}^*$. By Proposition 9, we have that $v(f_{\boldsymbol{\gamma}^*}) = m$ and $f_{\boldsymbol{\gamma}^*} = \sum_{j\in[m]} \chi_{\gamma^*_j}$, which means that we have a decomposition of $f_{\boldsymbol{\gamma}^*}$ given by $(\boldsymbol{\gamma}^*, ())$ with no cycle involved. Clearly, by (20) and (21), we get $\boldsymbol{\gamma}^* \in \mathcal{S}^N_{yz}(f_{\boldsymbol{\gamma}^*}) \subseteq \mathcal{T}^{N,m}_{yz}$.
(*ii*) Apply (*i*) to $m = \varphi^N_{yz}$. ■

3.3 | Some technical lemmas

In this section we present some technical results to which we will appeal for the proof of the main theorem (Theorem 20). To start with, given a maximum flow f, we show an interesting inequality between $f(x)$ and $\lambda^N_{yz}(x)$.

Lemma 16. *Let $N = (K_V, c) \in \mathcal{N}$, $x, y, z \in V$ with y, z distinct and $f \in \mathcal{F}(N, y, z)$ with $v(f) = m \in \mathbb{N}_0$. Then the following facts hold:*

(*i*) *for every $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}(f)$, we have $f(x) \geq f_{\boldsymbol{\gamma}}(x)$;*
(*ii*) *if $f \in \mathcal{M}(N, y, z)$, then $f(x) \geq \lambda^N_{yz}(x)$;*
(*iii*) *if $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}(x)$, then $f_{\boldsymbol{\gamma}}(x) = \lambda^N_{yz}(x)$.*

Proof.

(*i*) If $x \in \{y, z\}$, then $f(x) = m = f_{\boldsymbol{\gamma}}(x)$. Assume next $x \notin \{y, z\}$. Let $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}(f)$. By Definition 14, there exists a sequence $(w_j)_{j\in[k]}$ of cycles in N such that $f = f_{\boldsymbol{\gamma}} + \sum_{j\in[k]} \chi_{w_j}$. Thus, by Definition 1 and recalling that the cycle functions assume only nonnegative value, we have

$$f(x) = \sum_{a\in A^+_x} f(a) = \sum_{a\in A^+_x} f_{\boldsymbol{\gamma}}(a) + \sum_{a\in A^+_x} \left(\sum_{j\in[k]} \chi_{w_j}(a) \right) \geq f_{\boldsymbol{\gamma}}(x).$$

(*ii*) Assume that $f \in \mathcal{M}(N, y, z)$ and pick $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}(f)$. By (*i*) and by equality (13), we have that $f(x) \geq f_{\boldsymbol{\gamma}}(x) = l_x(\boldsymbol{\gamma})$. Since, by Proposition 15 (*ii*), we have $\mathcal{S}^N_{yz}(f) \subseteq \mathcal{M}^N_{yz}$ then we also have

$$f(x) \geq \min_{\boldsymbol{\gamma}\in\mathcal{M}^N_{yz}} l_x(\boldsymbol{\gamma}) = \lambda^N_{yz}(x).$$

(*iii*) Let $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}(x)$. Then, by (13), we have that $f_{\boldsymbol{\gamma}}(x) = l_x(\boldsymbol{\gamma}) = \lambda^N_{yz}(x)$. ■

The next lemma establishes a natural bound for $\lambda^N_{yz}(x)$ in terms of the outdegree and the indegree of x.

Lemma 17. *Let $N = (K_V, c) \in \mathcal{N}$ and $x, y, z \in V$ be distinct. Then $\lambda^N_{yz}(x) \leq \min\{c(x), c(V \setminus \{x\})\}$.*

Proof. Consider $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}(x)$. By Lemma 16 (*iii*) and Definition 1, we have

$$\lambda^N_{yz}(x) = f_{\boldsymbol{\gamma}}(x) = \sum_{a \in A^+_x} f_{\boldsymbol{\gamma}}(a) \leq \sum_{a \in A^+_x} c(a) = c(x)$$

and also

$$\lambda^N_{yz}(x) = f_{\boldsymbol{\gamma}}(x) = \sum_{a \in A^-_x} f_{\boldsymbol{\gamma}}(a) \leq \sum_{a \in A^-_x} c(a) = c(V \setminus \{x\}).$$

■

In the following two results we explain how some crucial objects of our research behave with respect to a decrease of capacity in the network.

Lemma 18. *Let $N = (K_V, c) \in \mathcal{N}$, $N' = (K_V, c') \in \mathcal{N}$ and $y, z \in V$ be distinct. Assume that $c' \leq c$. Then, the following facts hold:*

(*i*) $\mathcal{S}^{N'}_{yz} \subseteq \mathcal{S}^N_{yz}$. *In particular,* $\mathcal{M}^{N'}_{yz} \subseteq \mathcal{S}^N_{yz}$;
(*ii*) $\varphi^{N'}_{yz}(x)$ *can be greater than* $\varphi^N_{yz}(x)$.

Proof.

(*i*) Let $\boldsymbol{\gamma} = (\gamma_j)_{j \in [m]} \in \mathcal{S}^{N'}_{yz}$, where $m \in \mathbb{N}_0$. Then, for every $a \in A$, we have

$$|\{j \in [m] : a \in A(\gamma_j)\}| \leq c'(a) \leq c(a)$$

and thus $\boldsymbol{\gamma} \in \mathcal{S}^N_{yz}$. Recall now that, by definition, $\mathcal{M}^{N'}_{yz} \subseteq \mathcal{S}^{N'}_{yz}$.
(*ii*) Consider the networks N and N' in Figures 3 and 4 and denote by c and c' their capacities. Of course, we have that $c' \leq c$. It is easily checked that $\varphi^{N'}_{yz}(x) = 1 > \varphi^N_{yz}(x) = 0$. ■

Lemma 19. *Let $N = (K_V, c) \in \mathcal{N}$, $N' = (K_V, c') \in \mathcal{N}$ and $y, z \in V$ be distinct. Assume that $c' \leq c$. Then the following conditions are equivalent:*

(*i*) $\mathcal{M}^{N'}_{yz} \subseteq \mathcal{M}^N_{yz}$;
(*ii*) $\varphi^{N'}_{yz} = \varphi^N_{yz}$;
(*iii*) $\mathcal{M}(N', y, z) \subseteq \mathcal{M}(N, y, z)$.

Proof. (*i*) ⇒ (*ii*) Assume that $\mathcal{M}^{N'}_{yz} \subseteq \mathcal{M}^N_{yz}$. Pick $\boldsymbol{\gamma} \in \mathcal{M}^{N'}_{yz}$. Then $\varphi^{N'}_{yz} = l(\boldsymbol{\gamma})$ and also $\varphi^N_{yz} = l(\boldsymbol{\gamma})$, so that $\varphi^{N'}_{yz} = \varphi^N_{yz}$.
(*ii*) ⇒ (*iii*) Assume that $\varphi^{N'}_{yz} = \varphi^N_{yz}$. Let $f \in \mathcal{M}(N', y, z)$. Then, by (4), $f \in \mathcal{F}(N, y, z)$ and $v(f) = \varphi^{N'}_{yz} = \varphi^N_{yz}$. Thus, $f \in \mathcal{M}(N, y, z)$.

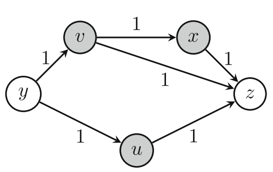

FIGURE 3 The network N used in the proof of Lemma 18

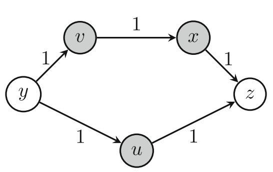

FIGURE 4 The network N' used in the proof of Lemma 18

(*iii*) ⇒ (*i*) Assume that $\mathcal{M}(N', y, z) \subseteq \mathcal{M}(N, y, z)$. Let $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^{N'}$. Then $l(\boldsymbol{\gamma}) = \varphi_{yz}^{N'}$ and, by Lemma 18 (*i*), $\boldsymbol{\gamma} \in \mathcal{S}_{yz}^{N}$. Consider the flow $f_{\boldsymbol{\gamma}}$ associated with $\boldsymbol{\gamma}$, and recall that $v(f_{\boldsymbol{\gamma}}) = l(\boldsymbol{\gamma}) = \varphi_{yz}^{N'}$. Hence $f_{\boldsymbol{\gamma}} \in \mathcal{M}(N', y, z)$ and thus $f_{\boldsymbol{\gamma}} \in \mathcal{M}(N, y, z)$. Thus, $l(\boldsymbol{\gamma}) = \varphi_{yz}^{N}$, which gives $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^{N}$. ■

4 | Properties of $\varphi_{yz}^{N}(X)$ and $\lambda_{yz}^{N}(X)$

4.1 | Main theorem

We are finally ready to prove our main result.[8]

Theorem 20. *Let $N = (K_V, c) \in \mathcal{N}$ and $x, y, z \in V$ with y, z distinct. Then $\lambda_{yz}^{N}(x) = \varphi_{yz}^{N}(x)$.*

Proof. If $x \in \{y, z\}$, then we have $\lambda_{yz}^{N}(x) = \varphi_{yz}^{N}$ and $\varphi_{yz}^{N_x} = 0$. Thus, the equality $\lambda_{yz}^{N}(x) = \varphi_{yz}^{N}(x)$ is certainly true. We complete the proof proving that, if $x \notin \{y, z\}$, then we have $\lambda_{yz}^{N}(x) = \varphi_{yz}^{N}(x)$. Observe first that

$$\lambda_{yz}^{N}(x) = 0 \text{ implies } \varphi_{yz}^{N}(x) = 0. \tag{22}$$

Indeed, if $\lambda_{yz}^{N}(x) = 0$, then there exists $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^{N}$ such that $l_x(\boldsymbol{\gamma}) = 0$. Thus, $\boldsymbol{\gamma} \in \mathcal{S}_{yz}^{N_x}$, which gives $\varphi_{yz}^{N_x} \geq \varphi_{yz}^{N}$. Since by (5) we also have $\varphi_{yz}^{N_x} \leq \varphi_{yz}^{N}$, we deduce that $\varphi_{yz}^{N}(x) = 0$.

Consider now, for $n \in \mathbb{N}_0$, the following statement:

$$\begin{gathered}\text{For every } N = (K_V, c) \in \mathcal{N} \text{ and } x, y, z \in V \text{ distinct with } c(x) + c(V \setminus \{x\}) = n,\\ \text{we have that } \lambda_{yz}^{N}(x) = \varphi_{yz}^{N} - \varphi_{yz}^{N_x}.\end{gathered} \tag{23}$$

We are going to prove the theorem showing, by induction on n, that (23) holds true for all $n \in \mathbb{N}_0$.

Consider first $N = (K_V, c) \in \mathcal{N}$ and $x, y, z \in V$ distinct with $c(x) + c(V \setminus \{x\}) = 0$. Then $c(x) = c(V \setminus \{x\}) = 0$ which, by Lemma 17, implies $\lambda_{yz}^{N}(x) = 0$ and, by (22), the statement holds.

Consider now $N = (K_V, c) \in \mathcal{N}$ and $x, y, z \in V$ distinct with $c(x) + c(V \setminus \{x\}) = n \geq 1$. For brevity, let us set $\lambda_{yz}^{N}(x) = s$ and $\varphi_{yz}^{N} = m$. By (8), we have that $0 \leq s \leq m$. If $s = 0$, then we again conclude by (22). Assume then $s \geq 1$. As a consequence, we also have $m \geq 1$. Choose among the sequences in $\mathcal{M}_{yz}^{N}(x)$ a sequence $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^{N}(x)$ in which the components passing through x are the last s. Let $f_{\boldsymbol{\gamma}}$ be the flow associated with $\boldsymbol{\gamma}$, defined in (12). Recall that $l(\boldsymbol{\gamma}) = m$ and $l_x(\boldsymbol{\gamma}) = s$.

We divide our argument into two cases.

Case (I). Assume that there exists $\tilde{a} \in A_x$ such that $f_{\boldsymbol{\gamma}}(\tilde{a}) < c(\tilde{a})$. Then, obviously, $c(\tilde{a}) \geq 1$.

Consider the network $\tilde{N} = (K_V, \tilde{c})$ where $\tilde{c}$ is defined, for every $a \in A$, as

$$\tilde{c}(a) = \begin{cases} c(a) & \text{if } a \neq \tilde{a} \\ c(\tilde{a}) - 1 & \text{if } a = \tilde{a} \end{cases}$$

and note now that, for every $a \in A$, $\tilde{c}(a) \leq c(a)$. Since $\tilde{c}(x) + \tilde{c}(V \setminus \{x\}) = c(x) + c(V \setminus \{x\}) - 1 = n - 1$, by the inductive assumption we get

$$\lambda_{yz}^{\tilde{N}}(x) = \varphi_{yz}^{\tilde{N}} - \varphi_{yz}^{\tilde{N}_x}.$$

It is immediate to observe that $\tilde{N}_x = N_x$, so that $\varphi_{yz}^{\tilde{N}_x} = \varphi_{yz}^{N_x}$. We also have $\boldsymbol{\gamma} \in \mathcal{S}_{yz}^{\tilde{N}}$ and then $\varphi_{yz}^{\tilde{N}} \geq l(\boldsymbol{\gamma}) = m = \varphi_{yz}^{N}$. Moreover, by (5), we also have $\varphi_{yz}^{\tilde{N}} \leq \varphi_{yz}^{N}$. Thus, $\varphi_{yz}^{\tilde{N}} = \varphi_{yz}^{N}$ and $\boldsymbol{\gamma} \in \mathcal{M}_{yz}^{\tilde{N}}$. As a consequence, $\varphi_{yz}^{N} - \varphi_{yz}^{N_x} = \varphi_{yz}^{\tilde{N}} - \varphi_{yz}^{\tilde{N}_x}$. We are then left with proving that $\lambda_{yz}^{\tilde{N}}(x) = \lambda_{yz}^{N}(x)$. Note that $\lambda_{yz}^{\tilde{N}}(x) \leq l_x(\boldsymbol{\gamma}) = s$. Assume now, by contradiction, that there exists $\tilde{\boldsymbol{\gamma}} \in \mathcal{M}_{yz}^{\tilde{N}}$ such that $l_x(\tilde{\boldsymbol{\gamma}}) < s$. As proved before, $\varphi_{yz}^{\tilde{N}} = \varphi_{yz}^{N}$ and then, by Lemma 19, we have that $\tilde{\boldsymbol{\gamma}} \in \mathcal{M}_{yz}^{N}$ and then $\lambda_{yz}^{N}(x) \leq l_x(\tilde{\boldsymbol{\gamma}}) < s$, a contradiction.

Case (II). Assume now that, for every $a \in A_x$, we have

$$f_{\boldsymbol{\gamma}}(a) = c(a). \tag{24}$$

By Lemma 16(*iii*), we then get

$$s = \sum_{a \in A_x^+} f_{\boldsymbol{\gamma}}(a) = \sum_{a \in A_x^+} c(a) = c(x). \tag{25}$$

[8]Theorem 20 is conjectured in [15].

The component γ_m of $\boldsymbol{\gamma}$ passes through x and reaches $z \neq x$. Thus there exists $\tilde{a} \in A_x^+ \cap A(\gamma_m)$ and, by (24), we have $c(\tilde{a}) \geq 1$.

Define the network $\tilde{N} = (K_V, \tilde{c}) \in \mathcal{N}$ by:

$$\tilde{c}(a) = \begin{cases} c(a) & \text{if } a \neq \tilde{a} \\ c(\tilde{a}) - 1 & \text{if } a = \tilde{a} \end{cases}$$

and note now that, for every $a \in A$, $\tilde{c}(a) \leq c(a)$. Since $\tilde{c}(x) = c(x) - 1$ and $\tilde{c}(V \setminus \{x\}) = c(V \setminus \{x\})$, we have that $\tilde{c}(x) + \tilde{c}(V \setminus \{x\}) = n - 1$. Hence, by the inductive assumption, we get $\lambda_{yz}^{\tilde{N}}(x) = \varphi_{yz}^{\tilde{N}} - \varphi_{yz}^{\tilde{N}_x}$. In order to complete the proof we show the following three equalities:

(a) $\varphi_{yz}^{\tilde{N}} = \varphi_{yz}^{N} - 1$;
(b) $\lambda_{yz}^{\tilde{N}}(x) = \lambda_{yz}^{N}(x) - 1$;
(c) $\varphi_{yz}^{\tilde{N}_x} = \varphi_{yz}^{N_x}$.

Let us start by considering $\tilde{\boldsymbol{\gamma}} \in \mathcal{S}_{yz}^{N}$ obtained by $\boldsymbol{\gamma}$ by deleting the component γ_m. In other words, $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_j)_{j \in [m-1]}$ where, for every $j \in [m-1]$, $\tilde{\gamma}_j = \gamma_j$. By definition of $\tilde{N}$, we surely have $\tilde{\boldsymbol{\gamma}} \in \mathcal{S}_{yz}^{\tilde{N}}$ and thus

$$\varphi_{yz}^{\tilde{N}} \geq l(\tilde{\boldsymbol{\gamma}}) = m - 1 = \varphi_{yz}^{N} - 1. \tag{26}$$

Moreover, by Lemma 17 and (25), we have

$$\lambda_{yz}^{\tilde{N}}(x) \leq \tilde{c}(x) = c(x) - 1 = s - 1. \tag{27}$$

Let us now prove the equalities (a), (b) and (c).

(a) Assume by contradiction that $\varphi_{yz}^{\tilde{N}} > \varphi_{yz}^{N} - 1$, that is, $\varphi_{yz}^{\tilde{N}} \geq \varphi_{yz}^{N}$. By (5), we then obtain $\varphi_{yz}^{N} = \varphi_{yz}^{\tilde{N}}$. By Lemma 19, we also deduce that $\mathcal{M}_{yz}^{\tilde{N}} \subseteq \mathcal{M}_{yz}^{N}$ so that $\lambda_{yz}^{\tilde{N}}(x) \geq s$. On the other hand, by (27), we also have $\lambda_{yz}^{\tilde{N}}(x) \leq s - 1$, a contradiction. As a consequence, $\varphi_{yz}^{\tilde{N}} \leq \varphi_{yz}^{N} - 1$. Using now (26), we conclude $\varphi_{yz}^{\tilde{N}} = \varphi_{yz}^{N} - 1$, as desired.

(b) Let us prove now $\lambda_{yz}^{\tilde{N}}(x) = s - 1$. By (27) it is enough to show

$$\lambda_{yz}^{\tilde{N}}(x) \geq s - 1. \tag{28}$$

Set $\lambda_{yz}^{\tilde{N}}(x) = \tilde{s}$. From (a) we know that $\varphi_{yz}^{\tilde{N}} = m - 1$. Let $\tilde{\mathbf{v}} \in \mathcal{M}_{yz}^{\tilde{N}}(x)$. Thus $l(\tilde{\mathbf{v}}) = m - 1$ and $l_x(\tilde{\mathbf{v}}) = \tilde{s}$. By Proposition 9, we have that $f_{\tilde{\mathbf{v}}} \in \mathcal{F}(\tilde{N}, y, z) \subseteq \mathcal{F}(N, y, z)$ and $v(f_{\tilde{\mathbf{v}}}) = m - 1$. Thus $f_{\tilde{\mathbf{v}}} \in \mathcal{F}(N, y, z) \setminus \mathcal{M}(N, y, z)$ so that $AP_{yz}^{N}(f_{\tilde{\mathbf{v}}}) \neq \emptyset$. Pick then $\sigma \in AP_{yz}^{N}(f_{\tilde{\mathbf{v}}})$. By Proposition 11, we have that $f = f_{\tilde{\mathbf{v}}} + \chi_\sigma \in \mathcal{M}(N, y, z)$. By (13), we then have

$$2f(x) = \sum_{a \in A_x^+} f(a) + \sum_{a \in A_x^-} f(a) = \sum_{a \in A_x} f(a) = \sum_{a \in A_x} f_{\tilde{\mathbf{v}}}(a) + \sum_{a \in A_x} \chi_\sigma(a) = 2f_{\tilde{\mathbf{v}}}(x) + \sum_{a \in A_x} \chi_\sigma(a) \leq 2f_{\tilde{\mathbf{v}}}(x) + 2. \tag{29}$$

The last inequality follows from the fact that

$$\sum_{a \in A_x} \chi_\sigma(a) \leq 2. \tag{30}$$

Indeed, by definition (9), we have

$$\chi_\sigma = \sum_{a \in A(\sigma)^+} \chi_a - \sum_{a \in A(\sigma)^-} \chi_a.$$

In particular, $\chi_\sigma(a) = 0$ for all $a \in A_x \setminus A(\sigma)$ and $\chi_\sigma(a) \leq 1$ for all $a \in A_x \cap A(\sigma)$. Now, by definition of generalized path, we have $|A_x \cap A(\sigma)| \in \{0, 2\}$ and thus (30) holds.

By (29) and (13), we then obtain $f(x) \leq f_{\tilde{\mathbf{v}}}(x) + 1 = \tilde{s} + 1$. On the other hand, by Lemma 16 (ii), we also have $s \leq f(x)$ and thus $s \leq \tilde{s} + 1$, which is (28).

(c) Clearly we have that $N_x = \tilde{N}_x$ and thus $\varphi_{yz}^{\tilde{N}_x} = \varphi_{yz}^{N_x}$. ■

The next proposition shows that the equality $\lambda_{yz}^{N}(X) = \varphi_{yz}^{N}(X)$ does not hold true in general when X is not a singleton. Proposition 21 follows by an example due to Bang-Jensen (private communication).

Proposition 21. *There exist $N = (K_V, c) \in \mathcal{N}$, $X \subseteq V$ and $y, z \in V$ distinct such that $\lambda_{yz}^{N}(X) > \varphi_{yz}^{N}(X)$.*

Proof. Consider the network N in Figure 5 and $X = \{x_1, x_2\}$. It is immediately checked that $\varphi_{yz}^{N} = 3$ and $\varphi_{yz}^{N_X} = 2$, so that $\varphi_{yz}^{N}(X) = 1$. We show that $\lambda_{yz}^{N}(X) > \varphi_{yz}^{N}(X)$ proving that $\lambda_{yz}^{N}(X) = 2$. Consider

$$\boldsymbol{\gamma} = (yv_2z,\ yu_2x_2v_1z,\ yu_1x_1z) \in \mathcal{M}_{yz}^{N}.$$

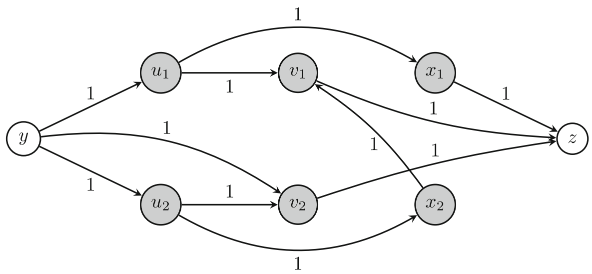

FIGURE 5 $\lambda_{yz}^N(X) > \varphi_{yz}^N(X)$ for $X = \{x_1, x_2\}$

Since $l_X(\boldsymbol{\gamma}) = 2$, we have that $\lambda_{yz}^N(X) \le 2$. Moreover, by (33), we know that $\lambda_{yz}^N(X) \ge \varphi_{yz}^N(X) = 1$. Thus $\lambda_{yz}^N(X) \in \{1, 2\}$.

Assume, by contradiction, that $\lambda_{yz}^N(X) = 1$. Then there exists $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{M}_{yz}^N(X)$ such that only μ_3 passes through X. Thus, $\tilde{\boldsymbol{\mu}} = (\mu_1, \mu_2) \in \mathcal{S}_{yz}^{N_X}$ and, since $\varphi_{yz}^{N_X} = 2$, we deduce that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_{yz}^{N_X}$. Hence, it is immediately observed that there are only two possibilities for a sequence of two arc-disjoint paths from y to z in N_X (up to reordering of the components). More precisely, we have

$$\tilde{\boldsymbol{\mu}} = (yu_1v_1z, yu_2v_2z) \quad \text{or} \quad \tilde{\boldsymbol{\mu}} = (yu_1v_1z, yv_2z).$$

If $\tilde{\boldsymbol{\mu}} = (yu_1v_1z, yu_2v_2z)$, then $\boldsymbol{\mu} = (yu_1v_1z, yu_2v_2z, \mu_3)$. Assume first that μ_3 passes through x_1. Then the only arc entering into x_1 and having capacity 1, that is (u_1, x_1), must be an arc of μ_3. That forces $A(\mu_3)$ to contain also the arc (y, u_1). On the other hand, that arc is also an arc of yu_1v_1z and we contradict the independence requirement. Similarly, if μ_3 passes through x_2, then the only arc entering into x_2 and having capacity 1, that is (u_2, x_2), must be an arc of μ_3. That forces $A(\mu_3)$ to contain also the arc (y, u_2), which is an arc of the path yu_2v_2z, again contrary to the independence requirement.

If now $\tilde{\boldsymbol{\mu}} = (yu_1v_1z, yv_2z)$, then $\boldsymbol{\mu} = (yu_1v_1z, yv_2z, \mu_3)$. As in the previous case, there is no way to include x_1 as a vertex of μ_3. Moreover, if μ_3 passes through x_2, then necessarily $\mu_3 = yu_2x_2v_1z$. Hence, $A(\mu_3)$ must contain the arc (v_1, z), which is an arc of yu_1v_1z contrary to the independence requirement. ■

Proposition 21 ultimately clarifies that the numbers $\lambda_{yz}^N(X)$ and $\varphi_{yz}^N(X)$ stem from different ideas and that Theorem 20 is a pure miracle happening when X is a singleton.[9]

4.2 | Global flow through a set of vertices

Let us introduce a new concept based on Definition 1.

Definition 22. Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $X \subseteq V$. We define

$$\delta_{yz}^N(X) := \min_{f \in \mathcal{M}(N,y,z)} f(X).$$

The number $\delta_{yz}^N(X)$ is called the *global flow that must pass through X in any maximum flow from y to z.*

Since $\mathcal{M}(N, y, z) \neq \varnothing$, $\delta_{yz}^N(X)$ is well defined. Note that $X \cap \{y, z\} \neq \varnothing$ implies $\delta_{yz}^N(X) \ge \varphi_{yz}^N$. Moreover, given $X \subseteq Y \subseteq V$, it is immediately observed that $0 \le \delta_{yz}^N(X) \le \delta_{yz}^N(Y)$.

Propositions 23 and 24 state some interesting links among $\varphi_{yz}^N(X)$, $\lambda_{yz}^N(X)$, and $\delta_{yz}^N(X)$. In particular, we show that when $X = \{x\}$ is a singleton the global flow that must pass through x in any maximum flow coincides with $\lambda_{yz}^N(x)$. That fact clarifies that the original intuitive definition of $\lambda_{yz}^N(x)$ given by Freeman et al. [14, pp. 147-148] is completely sensible.

Proposition 23. *Let $N = (K_V, c) \in \mathcal{N}$, $y, z \in V$ be distinct and $X \subseteq V$. Then $\delta_{yz}^N(X) \ge \lambda_{yz}^N(X)$ and equality holds when X is a singleton.*

Proof. Assume first that $X \cap \{y, z\} \neq \varnothing$. Then we have $\lambda_{yz}^N(X) = \varphi_{yz}^N \le \delta_{yz}^N(X)$. If $X = \{x\}$, so that $x = y$ or $x = z$, recalling Definition 1, we have that, for every $f \in \mathcal{M}(N, y, z)$, $f(x) = \varphi_{yz}^N$ and so also $\delta_{yz}^N(x) = \varphi_{yz}^N = \lambda_{yz}^N(x)$.

Assume next that $X \cap \{y, z\} = \varnothing$. Let $f \in \mathcal{M}(N, y, z)$ and $(\boldsymbol{\gamma}, \boldsymbol{w})$ be a decomposition of f. By Lemma 16 (i), we have

$$f(X) = \sum_{x \in X} f(x) \ge \sum_{x \in X} f_{\boldsymbol{\gamma}}(x) = f_{\boldsymbol{\gamma}}(X), \tag{31}$$

[9]Note that the equality $\lambda_{yz}^N(X) = \varphi_{yz}^N(X)$ trivially holds true also when $X = \varnothing$ or $|V \setminus X| \le 1$.

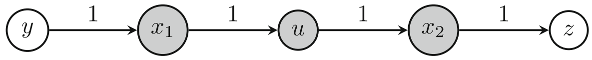

FIGURE 6 $\delta^N_{yz}(X) > \lambda^N_{yz}(X)$ for $X = \{x_1, x_2\}$

where, by Proposition 9, $f_{\boldsymbol{\gamma}} \in \mathcal{M}(N, y, z)$ and $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}$. We now observe that, by (13), we have

$$f_{\boldsymbol{\gamma}}(X) = \sum_{x\in X} f_{\boldsymbol{\gamma}}(x) = \sum_{x\in X} l_x(\boldsymbol{\gamma}) \geq l_X(\boldsymbol{\gamma}) \tag{32}$$

where the inequality in the chain is an equality when X is a singleton. By (31) and (32) it then follows that

$$\delta^N_{yz}(X) = \min_{f\in\mathcal{M}(N,y,z)} f(X) = \min_{\boldsymbol{\gamma}\in\mathcal{M}^N_{yz}} f_{\boldsymbol{\gamma}}(X) \geq \min_{\boldsymbol{\gamma}\in\mathcal{M}^N_{yz}} l_X(\boldsymbol{\gamma}) = \lambda^N_{yz}(X)$$

and the inequality in the chain is an equality when X is a singleton. ■

Proposition 24. *Let $N = (K_V, c) \in \mathscr{N}$, $y, z \in V$ be distinct and $X \subseteq V$. Then*

$$0 \leq \varphi^N_{yz}(X) \leq \lambda^N_{yz}(X) \leq \min\{\delta^N_{yz}(X), \varphi^N_{yz}\}. \tag{33}$$

Proof. Let us prove first that $\varphi^N_{yz}(X) \geq 0$. Consider the network N_X and observe that, for every $a \in A$, $c_X(a) \leq c(a)$. Thus, by (5), $\varphi^N_{yz} \geq \varphi^{N_X}_{yz}$. We deduce then that $\varphi^N_{yz}(X) = \varphi^N_{yz} - \varphi^{N_X}_{yz} \geq 0$, as desired.

Let us prove now that $\varphi^N_{yz}(X) \leq \lambda^N_{yz}(X) \leq \varphi^N_{yz}$. Let $\boldsymbol{\gamma} \in \mathcal{M}^N_{yz}(X)$. Then $l(\boldsymbol{\gamma}) = \varphi^N_{yz}$ and $l_X(\boldsymbol{\gamma}) = \lambda^N_{yz}(X)$. Let $\boldsymbol{\gamma}'$ be a sequence of arc-disjoint paths having as components those components of $\boldsymbol{\gamma}$ not passing through X. Thus, $\boldsymbol{\gamma}' \in \mathcal{S}^{N_X}_{yz}$ and $l(\boldsymbol{\gamma}') = \varphi^N_{yz} - \lambda^N_{yz}(X) \leq \lambda^{N_X}_{yz} = \varphi^{N_X}_{yz}$, where the last equality follows from (7). As a consequence, $\varphi^N_{yz}(X) = \varphi^N_{yz} - \varphi^{N_X}_{yz} \leq \lambda^N_{yz}(X)$. Next note that $\lambda^N_{yz}(X) \leq l_X(\boldsymbol{\gamma}) \leq l(\boldsymbol{\gamma}) = \varphi^N_{yz}$.

Finally the fact that $\lambda^N_{yz}(X) \leq \delta^N_{yz}(X)$ follows from Proposition 23. ■

By Proposition 21 we already know that it may happen that $\lambda^N_{yz}(X) > \varphi^N_{yz}(X)$. The next proposition shows that also the equality $\delta^N_{yz}(X) = \lambda^N_{yz}(X)$ does not hold true in general. That fact, taking into account Proposition 23, reinforces the idea that it is risky to immediately extend any intuition about paths and flows from single vertices to sets of vertices.

Proposition 25. *There exist $N = (K_V, c) \in \mathscr{N}$, $X \subseteq V$ and $y, z \in V$ distinct such that $\delta^N_{yz}(X) > \lambda^N_{yz}(X)$.*

Proof. Consider the network N in Figure 6 and $X = \{x_1, x_2\}$. Then $\delta^N_{yz}(X) = 2 > \lambda^N_{yz}(X) = 1$. ■

5 | TWO FLOW GROUP CENTRALITY MEASURES

Consider the ordered pairs of the type $((K_V, c), X)$, where $(K_V, c) \in \mathscr{N}$ and $X \subseteq V$ and denote the set of such pairs by $\mathscr{U}$. A *group centrality measure* (GCM) is a function from $\mathscr{U}$ to $\mathbb{R}$. If μ is a group centrality measure, we denote the value of μ at $(N, X) \in \mathscr{U}$ by $\mu^N(X)$ and we interpret it as a measure of the importance of the set of vertices X in N. A variety of group centrality measures has been obtained by generalizing classic centrality measures [10, 11].

By means of the numbers $\varphi^N_{yz}(X)$ and $\lambda^N_{y,z}(X)$, we define in this section two new group centrality measures. Recall that, by Propositions 3 and 7, given $N = (K_V, c) \in \mathscr{N}$, $y, z \in V$ distinct and $X \subseteq V$, we have that $0 \leq \varphi^N_{yz}(X) \leq \varphi^N_{yz}(V)$ and $0 \leq \lambda^N_{yz}(X) \leq \lambda^N_{yz}(V)$. Moreover, $\varphi^N_{yz}(V) = \lambda^N_{yz}(V) = \varphi^N_{yz} = \lambda^N_{yz}$.

Definition 26. The *full flow vitality* GCM, denoted by Φ, is defined, for every $(N, X) \in \mathscr{U}$ with $N = (V, A, c)$, by

$$\Phi^N(X) := \sum_{\substack{(y,z)\in A \\ \varphi^N_{yz}>0}} \frac{\varphi^N_{yz}(X)}{\varphi^N_{yz}}. \tag{34}$$

Note that Φ is a vitality measure in the sense of Koschützki et al. [18]. Indeed, its value at a given set of vertices X takes into consideration how much eliminating the set X from the network impacts the flow of the network.

We emphasize that in (34) we are summing over the maximum set of arcs which makes the definition meaningful, that is over the set $\{(y, z) \in A : \varphi^N_{yz} > 0\}$. That corresponds to the idea of a uniform treatment for the vertices in the network and it is what the adjective *full* in the name of Φ refers to.[10]

[10]In [18] it is called the max-flow betweenness vitality of the vertex x of an undirected connected network N, the number $\sum_{\substack{y,z\in V\setminus\{x\} \\ y\neq z}} \frac{\varphi^N_{yz}(x)}{\varphi^N_{yz}}$.

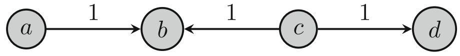

FIGURE 7 A network illustrating the failure of monotonicity of Φ_2 and Λ_2

Definition 27. The *full flow betweenness* GCM, denoted by Λ, is defined, for every $(N, X) \in \mathscr{U}$ with $N = (V, A, c)$, by

$$\Lambda^N(X) := \sum_{\substack{(y,z)\in A \\ \lambda^N_{yz}>0}} \frac{\lambda^N_{yz}(X)}{\lambda^N_{yz}}. \tag{35}$$

Note that Λ is a typical betweenness measure because it takes into considerations to what extent the paths in the network are forced to pass through a set of vertices. In line with (34), also in (35) we are summing over the maximum set of arcs which makes the definition meaningful, that is over the set $\{(y, z) \in A : \lambda^N_{yz} > 0\}$ and that explains the adjective *full* in the name of Λ.

The fact that Λ is a betweenness measure makes Λ conceptually different from Φ, which is instead a typical vitality measure. Anyway some comparison is surely possible. First of all, it is immediate to check that Λ and Φ coincide when $X = \{x\}$. Indeed, by Theorem 20 we know that, for every $y, z \in V$ distinct, we have $\lambda^N_{yz}(x) = \varphi^N_{yz}(x)$. Thus, we also have $\Phi^N(x) = \Lambda^N(x)$. Moreover, by Proposition 24, it is immediately deduced that $\Phi^N(X) \leq \Lambda^N(X)$. Finally, by Propositions 21 and 24 it immediately follows that, when $|X| \geq 2$, we may have $\Phi^N(X) \neq \Lambda^N(X)$.

Observe that, from the computational point of view, there is an important difference between Φ and Λ. Indeed, in order to compute the term $\frac{\varphi^N_{yz}(X)}{\varphi^N_{yz}} = \frac{\varphi^N_{yz} - \varphi^{N_X}_{yz}}{\varphi^N_{yz}}$, it is enough to have just a single maximum flow from y to z in N and a single maximum flow from y to z in N_X. On the other hand, in order to compute the term $\frac{\lambda^N_{yz}(X)}{\lambda^N_{yz}}$, one needs, in principle, to know the decompositions of all the possible maximum flows from y to z in N in the sense of Theorem 12.

We close this section showing that both Φ and Λ satisfy a relevant monotonicity property. It guarantees that enlarging the subset of vertices cannot determine a decrease in the centrality level and appears very desirable for the usual phenomena described through networks. That seems to be at least the opinion expressed by Everett and Borgatti [10] and we certainly agree.

Proposition 28. *Let $N = (K_V, c) \in \mathscr{N}$ and $X \subseteq Y \subseteq V$. Then the following facts hold true:*

(i) $\Phi^N(X) \leq \Phi^N(Y)$*;*
(ii) $\Lambda^N(X) \leq \Lambda^N(Y)$.

Proof. Let $(y, z) \in A$ with $\varphi^N_{yz} > 0$. By Propositions 3 and 7, we have $\varphi^N_{yz}(X) \leq \varphi^N_{yz}(Y)$ and $\lambda^N_{yz}(X) \leq \lambda^N_{yz}(Y)$. Thus summing up over all the arcs $(y, z) \in A$ such that $\varphi_{yz} > 0$, we immediately get $\Phi^N(X) \leq \Phi^N(Y)$ and $\Lambda^N(X) \leq \Lambda^N(Y)$. ■

There is certainly a simple link between Φ and Φ_2 as well as between Λ and Λ_2, where Φ_2 and Λ_2 are considered in the introduction. Indeed, once we have defined the set

$$K^N(X) = \left\{(y, z) \in A : (y, z) \text{ is incident to } X \text{ and } \varphi^N_{yz} > 0\right\},$$

we have

$$\Phi^N(X) = \Phi^N_2(X) + |K^N(X)| \quad \text{and} \quad \Lambda^N(X) = \Lambda^N_2(X) + |K^N(X)|.$$

In particular, the difference between $\Phi^N(X)$ and $\Phi^N_2(X)$ and between $\Lambda^N(X)$ and $\Lambda^N_2(X)$ is the same and is related to connectivity properties of the network N.

However, despite that link, Φ_2 and Λ_2 do not satisfy the monotonicity property considered in Proposition 28. Indeed, consider the 0-1 network N in Figure 7 and note that $\Phi^N_2(b) = \Lambda^N_2(b) = 1$ and $\Phi^N_2(\{a, b, c, d\}) = \Lambda^N_2(\{a, b, c, d\}) = 0$. This failure is surely a consequence of the fact that only vertices not belonging to X are considered in the sums defining Φ_2 and Λ_2 so that, for every network, those measures assume value 0 when X is the whole set of vertices.

The lack of monotonicity of Φ_2 and Λ_2 is far from being their only deficiency. Indeed, consider again the 0-1 network N in Figure 7. A computation shows that

$$\Phi^N_2(b) = \Lambda^N_2(b) = 1 \quad \text{and} \quad \Phi^N_2(x) = \Lambda^N_2(x) = 0, \text{ for } x \in \{a, c, d\}.$$

Moreover

$$\Phi^N_2(\{b, d\}) = \Lambda^N_2(\{b, d\}) = 1 \quad \text{and} \quad \Phi^N_2(X) = \Lambda^N_2(X) = 0, \text{ for every } X \subseteq V \text{ with } |X| = 2 \text{ and } X \neq \{b, d\}.$$

On the other hand, we have

$$\Phi^N(b) = \Lambda^N(b) = \Phi^N(c) = \Lambda^N(c) = 3, \quad \Phi^N(a) = \Lambda^N(a) = 2, \quad \Phi^N(d) = \Lambda^N(d) = 1,$$

and

$$\Phi^N(\{a,c\}) = \Lambda^N(\{a,c\}) = \Phi^N(\{b,c\}) = \Lambda^N(\{b,c\}) = \Phi^N(\{b,d\}) = \Lambda^N(\{b,d\}) = 4,$$
$$\Phi^N(\{a,b\}) = \Lambda^N(\{a,b\}) = \Phi^N(\{a,d\}) = \Lambda^N(\{a,d\}) = \Phi^N(\{c,d\}) = \Lambda^N(\{c,d\}) = 3.$$

The centrality values determined via Φ and Λ seem to better take into account the nature of the network under consideration, allowing a higher level of diversification among its vertices and groups of vertices. In particular, it is reasonable to think that vertex c has a higher level of centrality than vertices a and d in N. That fact is recognized by Φ and Λ but it is not recognized by Φ_2 and Λ_2. Similarly, it is reasonable to think that $\{b,c\}$ has a higher level of centrality than $\{a,b\}$, $\{a,d\}$ and $\{c,d\}$ in N. Again, that fact is recognized by Φ and Λ but it is not recognized by Φ_2 and Λ_2.

6 | CONCLUSIONS AND FURTHER RESEARCH

We have clarified the equality between two fundamental numbers used in network theory that surprisingly were identified without a solid formal argument: the minimum number $\lambda^N_{yz}(x)$ of paths passing through the vertex x in a maximum sequence of arc-disjoint paths from y to z in the network N and the number $\varphi^N_{yz}(x)$ expressing the reduction of maximum flow value from y to z in N when the capacity of all the arcs incident to x is reduced to zero. The proof of that fact has involved a tricky analysis of the relationship between paths and flows which goes beyond the original goal and is interesting in itself. We also proved that the natural generalizations of $\lambda^N_{yz}(x)$ and $\varphi^N_{yz}(x)$ to groups of vertices are not, in general, equal.

Our analysis has led to the definition of two conceptually different centrality measures Φ and Λ both satisfying an important monotonicity property. The contexts in which they could be fruitfully applied are in principle many. Just to give an example, consider a scientific community V of scholars and, for every $x, y \in V$, the number $c(x, y)$ of times that, within a certain fixed period of time, the researcher $x \in V$ cited the researcher $y \in V$. Construct then the corresponding citation network $N = (K_V, c) \in \mathcal{N}$. It is reasonable to think that, for $x, y, z \in V$, if x cited y and y cited z, then indirectly x cited z so that z gains prestige not only from y but also from x and thus, more generally, z gains prestige from any path in N having z as endpoint.[11] Consider, in particular, a situation in which two groups of researchers, say X_1 and X_2, received the same total number of citations from the scholars outside the groups, that is, $\sum_{a\in A^-_{X_1}} c(a) = \sum_{a\in A^-_{X_2}} c(a)$. Suppose that you want to diversify X_1 and X_2 putting in evidence the quality of those citations. For X_i, where $i \in \{1, 2\}$, that quality can be evaluated by looking at the set Y_i of scholars who cited the scholars in X_i and taking into account the number of citations that the scholars in Y_i themselves received. On the other hand, also the quality of the citations received by the scholars in Y_i is important and that can be in turn evaluated by looking at the set Z_i of the scholars who cited the scholars in Y_i. That reasoning can be continued so that the quality of the citations received by the scholars in X_i can effectively emerge only by a global approach which takes into consideration the whole configuration and complexity of the citation network (see, for instance, [5]). That can be obtained by computing which between $\Lambda^N(X_1)$ and $\Lambda^N(X_2)$ is larger. Another reasonable idea could be instead to look at the impact on the amount of direct and indirect citations in the network caused by a hypothetical absence from the scientific scenario of the scholars in X_i, that is, comparing $\Phi^N(X_1)$ and $\Phi^N(X_2)$. Indeed, if $\Phi^N(X_1) > \Phi^N(X_2)$, then the absence of X_1 mostly damages the scientific community in its dynamic exchange of contacts.

We also emphasize that the formal approach we used seems very promising for dealing with the properties of our centrality measures, in particular, with those invoked by Sabidussi [21] as the main desirable. As is well known, an axiomatic satisfactory definition of centrality is missing in the literature. The presence or absence of certain properties can though help in deciding which measure better fits in a certain application, as largely recognized in the bibliometric literature which is recently oriented in using methods from social choice theory (see, for instance, [7, 8]) as well as in the game theoretic approach to centrality (see [16]). Thus, exploring the properties of Φ and Λ is surely an interesting research topic. However, from the practical point of view we already know that the computation of Λ appears very hard and thus, reasonably, the main focus in the future will be on Φ. Of course, a further crucial project is the implementation of Φ on concrete networks in order to discover empirically in which sense, and in which kind of networks, Φ behaves better than other classic centrality measures.

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[11] A similar approach by path consideration is used for sport competitions in [6] with the scope to obtain a ranking of teams.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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