Research Article

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On the local behavior of local weak solutions to some singular anisotropic elliptic equations

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Abstract: We study the local behavior of bounded local weak solutions to a class of anisotropic singular equations of the kind

$$\sum_{i=1}^{s}\partial_{ii}u+\sum_{i=s+1}^{N}\partial_i(A_i(x,u,\nabla u))=0,\quad x\in\Omega\subset\subset\mathbb{R}^N\quad\text{for } 1\le s\le(N-1),$$

where each operator A_i behaves directionally as the singular p-Laplacian, $1<p<2$. Throughout a parabolic approach to expansion of positivity we obtain the interior Hölder continuity and some integral and pointwise Harnack inequalities.

Keywords: anisotropic p-Laplacian, singular parabolic equations, Hölder continuity, intrinsic scaling, expansion of positivity, intrinsic Harnack inequality

MSC 2020: 35J75, 35K92, 35B65

1 Introduction

In this article, we study local regularity properties for bounded weak local solutions to operators whose prototype is

$$\sum_{i=1}^{s}\partial_{ii}u+\sum_{i=s+1}^{N}\partial_i(|\partial_i u|^{p-2}\partial_i u)=0,\quad \text{weakly in } \Omega\subset\mathbb{R}^N,\quad 1<p<2, \tag{1.1}$$

having a nondegenerate behavior along the first s- variables and a singular behavior on the last ones. This kind of operator is useful to describe the steady states of non-Newtonian fluids that have different directional diffusions (see for instance [1]), besides their pure mathematical interest, which still is a challenge after more than 50 years. Precise hypothesis will be given later (in Section 1.2), leaving here the space to describe what are the novelties and significance of the present work in the context of this kind of operator.

Until this moment it is not known whether solutions to equations as (1.1) enjoys the usual local properties as p-Laplacian ones. This is because equation (1.1) is part of a more general group of operators, whose regularity theory is still fragmented and largely incomplete. It is clear that new techniques are needed for a correct

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interpretation of the problem and its resolution. The present work is conceived to introduce a new method, adapted from the theory of singular parabolic equations. In next section, we explain this simple but effective idea that we will apply to a class of equations as (1.1) and that have no homogeneity on the differential operator (hence the epithet *anisotropic*) because they combine both nondegenerate and singular properties.

1.1 Parabolic approach

To introduce our approach, we present an alternative proof of the mean value theorem for solutions to Laplace equation. This brief and modest scheme will highlight the essence of our method, that is, conceived to obtain classical properties of some elliptic equations through a parabolic approach. Let us consider the Laplace equation in an open bounded set $\Omega \subset\subset \mathbb{R}^N$,

$$\sum_{i=1}^{N} \int_{\Omega} (\partial_i u)(\partial_i \phi) \mathrm{d}x = 0, \quad \phi \in C_o^\infty(\Omega).$$

Let $x_o \in \Omega$ be a Lebesgue point for u, and let $B_{2r}(x_o)$ be the ball of radius $2r$ and center x_o. Now for $0 < t < r$ such that $B_{2r}(x_o) \subset \Omega$, consider the test function $\phi(t, x) = (t^2 - |x - x_o|^2)_+$, to obtain the integral equality

$$\sum_{i=1}^{N} \int_{B_t(x_o)} (\partial_i u(x))(x_i - x_{o,i}) \mathrm{d}x = 0.$$

By Green's formula, this is equivalent to

$$\sum_{i=1}^{N} \int_{B_t(x_o)} \partial_i (u(x)\ (x_i - x_{o,i})) \mathrm{d}x - N \int_{B_t(x_o)} u \mathrm{d}x = t \int_{\partial B_t(x_o)} u \mathrm{d}\mathcal{H}^{N-1} - N \int_{B_t(x_o)} u \mathrm{d}x = 0,$$

having used that $\mathbf{n} = (x - x_o)/|x - x_o|$ is the normal unit vector to $\partial B_t(x_o)$ and being $d\mathcal{H}^{N-1}$ the Hausdorff $(N-1)$-dimensional measure. Now, last display can be rewritten as

$$t^{N+1} \frac{\mathrm{d}}{\mathrm{d}t} \left(t^{-N} \int_{B_t(x_o)} u \mathrm{d}x \right) = 0.$$

Finally, we integrate along $t \in (0, r)$ and we use Lebesgue's theorem to obtain

$$t^{-N} \int_{B_t(x_o)} u \mathrm{d}x \Bigg|_0^r = r^{-N} \int_{B_r(x_o)} u \mathrm{d}x - \omega_N u(x_o) = 0,$$

with $\omega_N = |B_1|$ and $|B_r| = \omega_N r^{-N}$. This implies the mean value property

$$u(x_o) = \fint_{B_r(x_o)} u \mathrm{d}x.$$

This point-wise control given in integral average can be used in turn to derive very strong regularity properties of the solutions. We will undergo a similar strategy for solutions to (1.1), by taking into account the degeneracies and singularities that are typical of anisotropic equations.

1.2 Definitions and main results

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $N \geq 2$, and let us denote with ∂_i the ith partial weak derivative. For $1 < p < 2$ and $1 \leq s \leq N-1$, we consider the elliptic partial differential equation

$$\sum_{i=1}^{s}\partial_{ii}u + \sum_{i=s+1}^{N}\partial_i A_i(x,u,\nabla u) = 0, \quad \text{weakly in } \Omega, \tag{1.2}$$

where the Caratheodory[1] functions $A_i(x,u,\xi): \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are subject to the following structure conditions for almost every $x\in\Omega$,

$$\begin{cases} \sum_{i=s+1}^{N} A_i(x,u,\xi)\cdot\xi_i \geq C_1 \sum_{i=s+1}^{N} |\xi_i|^p - C, & \text{for } \xi\in\mathbb{R}^N,\\ |A_i(x,u,\xi)| \leq C_2|\xi_i|^{p-1} + C, & \text{for } i\in\{s+1,\dots,N\}, \end{cases} \tag{1.3}$$

where $C_1, C_2 > 0$, $C \geq 0$ are given constants that we will always refer to as *the data*. A function $u \in L^\infty_{\text{loc}}(\Omega)\cap W^{1,[2,p]}_{\text{loc}}(\Omega)$, where

$$W^{1,[2,p]}_{\text{loc}}(\Omega) := \{u\in L^1_{\text{loc}}(\Omega) | \partial_i u \in L^2_{\text{loc}}(\Omega) \quad \forall i=1,\dots,s, \quad \partial_i u\in L^p_{\text{loc}}(\Omega) \quad \forall i = s+1,\dots,N\},$$
$$W^{1,[2,p]}_o(\Omega) := W^{1,1}_o(\Omega)\cap W^{1,[2,p]}_{\text{loc}}(\Omega),$$

is called a local weak solution to (1.2)–(1.3) if for each compact set $K\subset\subset\Omega$ it satisfies

$$\iint_K \sum_{i=1}^{s}\partial_i u\, \partial_i\varphi \mathrm{d}x + \iint_K \sum_{s+1}^{N} A_i(x,u,\nabla u)\partial_i\varphi \mathrm{d}x = 0, \quad \forall\varphi\in W^{1,[2,p]}_o(K). \tag{1.4}$$

Throughout this study, we will suppose that truncations $\pm(u-k)_\pm$ of local weak solutions to (1.2)–(1.3) preserve the property of being sub-solutions: for any $k\in\mathbb{R}$, every compact subset $K\subset\Omega$, and $\psi\in W^{1,[2,p]}_o(K)$ we have

$$\iint_K \left\{\sum_{i=1}^{s}\partial_i(u-k)_\pm\partial_i\psi + \sum_{i=s+1}^{N} A_i(x,(u-k)_\pm,\partial_i(u-k)_\pm)\partial_i\psi\right\}\mathrm{d}x \leq 0. \tag{1.5}$$

Remark 1.1. Previous assumption (1.5) is very natural. In case of homogeneous coercivity, that is, if in the first formula of (1.3) we have just

$$\sum_{i=s+1}^{N} A_i(x,u,\xi)\cdot\xi_i \geq C_1\sum_{i=s+1}^{N}|\xi_i|^p, \quad \text{for } \xi\in\mathbb{R}^N,$$

then by a simple limit argument it can be shown that (1.5) is always in force (see for example [10], Lemma 1.1 Chap. II).

Properties of anisotropic Sobolev spaces have first been investigated in [18,22,29], and boundedness of local weak solutions has been first considered in [19] and refined in [14]. Limit growth conditions have been investigated in [16] and then refined in [8,9] in great generality. Henceforth, it is a well-known fact in the literature that local weak solutions to our equation (1.2) are bounded, provided $p_{\max} \leq N\bar{p}/(N-\bar{p})$, being $\bar{p} = N(\sum_{i=1}^{N}(p_i)^{-1})^{-1}$ the harmonic mean. We consider the prototype equation to (1.2) as a special case of the full anisotropic analogue

$$-\sum_{i=1}^{N}\partial_i(|\partial_i u|^{p_i-2}\partial_i u) = 0, \tag{1.6}$$

with $p_i = 2$ for $i = 1,\dots,s$ and $1 < p_i = p < 2$ on the remaining components. This last equation suffers heavily from the combined effect of singular and degenerate behavior, even when for instance all p_is are greater than two. This is because the natural intrinsically scaled geometry of the equation that maintains invariant the volume $|\mathcal{K}| = \rho^N$ can be shaped on anisotropic cubes as

1 Measurable in (u,ξ) for all $x\in\Omega$ and continuous in x for a.e. $(u,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

$$\mathcal{K} = \prod_{i=1}^{N} \left\{ |x_i| < M^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\},$$

where M is a number depending on the solution u itself (indeed the epithet *intrinsic*) that vanishes as soon as u vanishes. Therefore, when M approaches zero, for those directions whose index satisfies $p_i > \bar{p}$ the anisotropic cube $\mathcal{K}$ shrinks to a vanishing measure, while for the remaining ones it stretches to infinity. For a detailed description of this geometry and its derivation through self-similarity we refer to [5], where the evolutionary, fully anisotropic prototype equation is considered (see also [15] for the singular case).

We state our two main results hereafter. The first one is a result of the local Hölder continuity.

Theorem 1.1. *Let u be a bounded local weak solution to* (1.2), (1.3), *and* (1.5). *Then there exists $\alpha \in (0, 1)$ depending only on the data such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.*

Next we fix some geometrical notations and conventions. For a point $x_o \in \Omega$, let us denote it by $x_o = (x_o', x_o'')$ where $x_o' \in \mathbb{R}^s$ and $x_o'' \in \mathbb{R}^{N-s}$. Let $\theta, \rho > 0$ be two parameters and define the polydisc

$$Q_{\theta,\rho}(x_o) := B_\theta(x_o') \times B_\rho(x_o''). \tag{1.7}$$

We will say $Q_{\theta,\rho}$ is an *intrinsic* polydisc when θ depends on the solution u itself. We will call first s variables the *nondegenerate* variables and last $(N - s)$ ones *singular* variables. Using this geometry we state our main result, an intrinsic form of Harnack inequality.

Theorem 1.2. *Let u be a nonnegative, bounded, local weak solution to* (1.2), (1.3), *and* (1.5). *Let $x_o \in \Omega$ be a point such that $u(x_o) > 0$ and $\rho > 0$ small enough to allow the inclusion*

$$Q_{\mathcal{M},\rho}(x_o) \subseteq \Omega, \quad \text{being } \mathcal{M} = \|u\|_{L^\infty(\Omega)}^{(2-p)/2} \rho^{\frac{p}{2}}. \tag{1.8}$$

Assume also that

$$\chi := p + (N - s)(p - 2) > 0. \tag{1.9}$$

Then there exist positive constants $K > 1$, $\bar{\delta}_o \in (0, 1)$ depending only on the data such that either

$$u(x_o) \le K\rho \tag{1.10}$$

or

$$u(x_o) \le K \inf_{Q_{\theta,\rho}(x_o)} u, \quad \text{with } \theta = \bar{\delta}_o u(x_o)^{\frac{2-p}{2}} \rho^{\frac{p}{2}}. \tag{1.11}$$

Condition (1.9) expresses the range of exponents available for the result to hold relatively to the weighted effect of singular and nondegenerate operators into play. When s decreases the range becomes tighter to the parabolic isotropic range for Harnack inequality to hold, i.e., $2N/(N + 1) < p < 2$. But when s increases the effect of regularization is stronger and this interval expands until it reaches $1 < p < 2$.

Theorems 1.1 and 1.2 are consequences of the following ones, which are worth of interest on their own.

We prove indeed the following shrinking property, which is typical of both singular parabolic equations ([10] Lemma 5.1 Chap. IV) and isotropic elliptic equations ([11] Prop 5.1 Chap X).

Theorem 1.3. *Let $\bar{x} \in \Omega$ and let u be a nonnegative, bounded, local weak solution to* (1.2), (1.3), *and* (1.5). *Suppose that for a point $\bar{x} \in \Omega$ and numbers $M, \rho > 0$, and $\nu \in (0, 1)$ it holds*

$$|[u \le M] \cap Q_{\theta,\rho}(\bar{x})| \le (1 - \nu)|Q_{\theta,\rho}(\bar{x})|, \quad \text{for } \theta = \rho^{\frac{p}{2}} (\delta M)^{\frac{2-p}{2}}, \tag{1.12}$$

and $Q_{2\theta,2\rho}(\bar{x}) \subset \Omega$, for a number $\delta = \delta(\nu) \in (0, 1)$. Then there exist constants $K > 1$ and $\delta_o \in (0, 1)$ depending only on the data and ν such that either

$$M \le K\rho \tag{1.13}$$

or for almost every $x \in Q_{\eta,2\rho}(\bar{x})$ we have

$$u(x) \geq \delta_o M/2, \quad \textit{where } \eta = (2\rho)^{\frac{p}{2}}(\delta_o M)^{\frac{2-p}{2}}. \tag{1.14}$$

The aforementioned theorem provides also an important expansion of positivity along the singular variables, conceived and modeled in a similar fashion than in ([13], Theorem 2.3).

Another fundamental tool for our analysis of local regularity is the following integral estimate, which can be seen as an Harnack estimate within the $L^1 - L^\infty$ topology, and is typical of singular parabolic equations (see for instance [10], Prop. 4.1 Chap VII).

Theorem 1.4. *Let u be a nonnegative, bounded, local weak solution to* (1.2), (1.3), *and* (1.5). *Fix a point* $\bar{x} \in \Omega$ *and numbers* $\theta, \rho > 0$ *such that* $Q_{8\theta,8\rho}(\bar{x}) \subset \Omega$. *Then there exists a positive constant* γ *depending only on the data such that either*

$$\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}} \leq \rho \tag{1.15}$$

or

$$\fint\fint_{Q_{\theta,\rho}(\bar{x})} u\,dx \leq \gamma \left\{ \inf_{B_{\frac{\theta}{2}}(\bar{x}')} \left(\fint_{B_{2\rho}(\bar{x}'')} u(\cdot, x'')dx'' \right)^{\frac{p}{X}} + \left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}} \right\}. \tag{1.16}$$

If additionally property (1.9) *holds, then either we have* (1.15) *or*

$$\sup_{Q_{\frac{\theta}{2},\frac{\rho}{2}}(\bar{x})} u \leq \gamma \left\{ \left(\frac{\rho^p}{\theta^2}\right)^{\frac{N-s}{X}} \inf_{B_{\frac{\theta}{2}}(\bar{x}')} \left(\fint_{B_{2\rho}(\bar{x}'')} u(\cdot, x'')dx'' \right)^{\frac{p}{X}} + \left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}} \right\}. \tag{1.17}$$

1.3 Novelty and significance

Considering fully anisotropic equations as (1.1), a standard statement of regularity requires a bound on the sparseness of the powers p_is. Indeed, in general, weak solutions can be unbounded, as proved in [17,26]. We refer to the surveys [27,28] for an exhaustive treatment of the subject and references. The problem of regularity for anisotropic operators behaving like (1.6) with measurable and bounded coefficients remains a major challenge after more than 50 years. Recently, some progress has been made in the parabolic prototype case, as for instance in [3,31] about Lipschitz continuity, [5] about intrinsic Harnack estimates [15], and in the singular case for Barenblatt-type solutions. Moreover, as we will see, various parabolic techniques have been applied, but in no circumstance Harnack estimates have been found when more than one spatial dimension was considered. This is due to the fact that usual parabolic techniques rely on the particular structure of a first derivative in time and are not suitable to manage stronger anisotropies. With the present work, limited to the case $p_i = 2$ for $i = 1, \dots, s$ and $p_i = p < 2$ for $i \in s+1, \dots, N$ we are able to prove a purely elliptic pointwise Harnack estimate when the operator acts on the first s variables. Furthermore, we have now a way to understand how these estimates degenerate when s varies; describing, roughly speaking, when the operator is closer to the p-Laplacian or to an uniformly elliptic operator (see the discussion after Theorem 1.2).

In the present work, we are interested in bounded solutions, therefore, leaving the problem of boundedness to the already rich literature. Our aim is to manage the anisotropic behavior of the operator interpreting its action in correspondence with a suitably adapted version of the technique developed by Chen and DiBenedetto (see the original paper [6] or the books [10,30]) in order to restore the homogeneity of the parabolic p-Laplacian. Indeed, because of the double derivative, equation (1.1) has a wilder heterogeneity of the operator than the parabolic p-Laplacian, and the intrinsic geometry will be set up according to the order and power of derivatives resulting in the dimensional analysis of the equation.

An interesting attempt in this direction has already been done by some of the authors in [23], in the case of only one nondegenerate variable (see also [25]). There an expansion of positivity is provided by applying an idea from [12], shaped on a proper exponential change of variables. Nevertheless, the change of variables in consideration is a purely parabolic tool, so that it does not allow the authors to go through more than one nondegenerate variable. The present work is conceived to fill this gap and to spread new light on the link between classical logarithmic estimates and anisotropic operators.

The two fundamental tools that we derive in our work are Theorems 1.3 and 1.4. Theorem 1.4 consists in a $L^1 - L^\infty$ Harnack inequality, which is independent of the other two theorems, although its proof relies as well on logarithmic estimates (2.13). The name *$L^1 - L^\infty$-inequality* refers to the fact that it is possible to control the supremum of the function through some L^1- integral norm of the function itself. This precious inequality can be used in turn to derive in straight way the Hölder continuity of solutions (see for instance [7] for a simple proof in the parabolic setting).

On the other hand, Theorem 1.3 provides both a shrinking property and an expansion of positivity. The special feature of Theorem 1.3 called *shrinking* property consists in the fact that from a whatever upper bound to the relative measure of some super-level of the solution it is possible to recover a pointwise estimate of positivity. Its proof is a proper consequence of logarithmic estimates (2.13) and a suitable choice of test functions (see functions f in Step 1 of the proof of Lemma 3.2). The *expansion of positivity* property refers to the possibility to expand along the space (in singular variables) the lower bound yet gained. The proof of this property is therefore linked to the measure theoretical approach of this shrinking property, and it is an adaptation of an idea of [13]. Here to end the proof of Theorem 1.3 we use in a crucial way the shrinking property to reach a critical mass and use Lemma 2.4.

Finally, in order to prove Theorem 1.2 we use an argument originally conceived by Krylov and Safonov in [21] to reach a certain controlled bound on the solution in terms of the solution itself, and then use repeatedly Theorem 1.4 to achieve an upper bound on the measure of some super-level set of the solution. Nonetheless, the argument of Krylov and Safonov gave us this information around an unknown point. Therefore, we apply Theorem 1.3 to expand the positivity until the desired neighborhood of the initial point and get the job done. The idea is an adaptation of the techniques originally developed in [13] to the case of anisotropic elliptic equations (1.2) and (1.3).

1.4 Structure of the article

In Section 2, we recall major functional tools and use them to derive fundamental properties of solutions as energy estimates, logarithmic estimates, and some integral estimates. Then in Section 3 we prove Theorem 1.3, in Section 4 we prove the Hölder continuity of solutions while in Section 5 we prove the $L^1 - L^\infty$ Harnack estimate (1.17). Section 6 is devoted to the proof of Theorem 1.2. Technicalities and standard material are given in Section A, to leave space along the previous text to what is really new.

Notations:
- If Ω is a measurable subset of $\mathbb{R}^N$, we denote by $|\Omega|$ its Lebesgue measure. We will write $\Omega \subset\subset \mathbb{R}^N$ when Ω is an open bounded set.
- For $r > 0$, $\bar{x} = (\bar{x}', \bar{x}'') \in \mathbb{R}^s \times \mathbb{R}^{N-s}$, we denote by $B_r(\bar{x})$ the ball of radius r and center $\bar{x}$; the standard polydisc is denoted by $Q_{\theta,\rho} = B_\theta(\bar{x}') \times B_\rho(\bar{x}'') \subset \mathbb{R}^N$. Furthermore, by $w_s = |B_1(0')|$ and $w_{N-s} = |B_1(0'')|$ we denote the measures of the respective unit balls.
- The symbol $\forall_{\text{ae}}$ stands for *–for almost every–*.
- For a measurable function u, by $\inf u$ and $\sup u$ we understand the essential infimum and supremum, respectively; when $u : \Omega \to \mathbb{R}$ and $a \in \mathbb{R}$, we omit the domain when considering sub/super level sets, letting $[u \lesseqgtr a] = \{x \in E : u(x) \lesseqgtr a\}$; if u is defined on some open set $\Omega \subset \mathbb{R}^N$, we let $\partial_i u = \frac{\partial}{\partial x_i} u$ denote the distributional derivatives.
- For numbers $B, C > 0$ we write $C \wedge B = \max\{B, C\}$.

- We make the usual convention that a constant $\gamma > 0$ depending only on the data, i.e., $\gamma = \gamma(N, 2, p, C_1, C_2, C)$, may vary from line to line along calculations.

2 Preliminaries

In this section, we collect the basic tools that will be used along the overall theory. For the sake of readability, simpler and well-known proofs are postponed to the Appendix (Section A), while most relevant passages that bring to light our method are detailed and highlighted.

2.1 Functional and standard tools

We recall the embedding $W_o^{1,\mathbf{p}}(\Omega) \hookrightarrow W_o^{1,\bar{p}^*}(\Omega)$ proved by Troisi in [29].

Lemma 2.1. *Let $\Omega \subset\subset \mathbb{R}^N$ and consider a function $u \in W_o^{1,\mathbf{p}}(\Omega)$, $p_i > 1$ for each $i \in \{1, \dots, N\}$. Assume $\bar{p} < N$ and let*

$$\bar{p}^* = \frac{N\bar{p}}{N - \bar{p}}. \tag{2.1}$$

Then there exists a positive constant $\gamma(N, \bar{p})$ such that

$$\|u\|^N_{L^{\bar{p}^*}(\Omega)} \leq \gamma \prod_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{2.2}$$

It is worth pointing out that without vanishing initial datum this embedding fails in general (see [22,18] for counter-examples). A simple calculation reveals that condition $\bar{p} < N$ is always in force in our case in study. Next lemma introduces a well-known weighted Poincaré inequality (see for instance Prop. 2.1 in [10]), which will be useful when estimating the logarithmic function.

Lemma 2.2. *Let B_ρ be a ball of radius $\rho > 0$ about the origin, and let $\varphi \in C(B_\rho)$ satisfy $0 \leq \varphi(x) \leq 1$ for each $x \in B_\rho$ together with the condition that the level sets $[\varphi > k] \cap B_\rho$ are convex for each $k \in (0, 1)$. Let $g \in W^{1,p}(B_\rho)$, and assume that the set*

$$\mathcal{E} = [g = 0] \cap [\varphi = 1]$$

has positive measure. Then, there exists a constant $\gamma > 0$ depending only upon N, p such that

$$\int_{B_\rho} \varphi |g|^p \mathrm{d}x \leq \gamma \rho^p \left(\frac{|B_\rho|}{|\mathcal{E}|} \right)^p \int_{B_\rho} \varphi |Dg|^p \mathrm{d}x. \tag{2.3}$$

2.2 Properties of solutions to (1.2) and (1.3)

The following classical *energy estimates* can be proved by a standard choice of test functions.

Lemma 2.3. *Let u be a bounded local weak solution to* (1.2) *with structure conditions* (1.3). *Then there exists a positive constant γ such that for any polydiscs $Q_{\theta,\rho}(\bar{x}) \subseteq \Omega$, any $k \in \mathbb{R}$, and any $\zeta \in C_o^\infty(Q_{\theta,\rho}(\bar{x}))$ such that $0 \leq \zeta \leq 1$ it holds*

$$\sum_{i=1}^{s}\int\int_{Q_{\theta,\rho}(\bar{x})} |\partial_i(u-k)_\pm|^2\zeta^2 \mathrm{d}x + \sum_{i=s+1}^{N}\int\int_{Q_{\theta,\rho}(\bar{x})} |\partial_i(u-k)_\pm|^p\zeta^2 \mathrm{d}x \\ \leq \gamma\int\int_{Q_{\theta,\rho}(\bar{x})}\left\{|(u-k)_\pm|^2\sum_{i=1}^{s}|\partial_i\zeta|^2 + |(u-k)_\pm|^p\sum_{i=s+1}^{N}|\partial_i\zeta|^p + C^p\chi_{[(u-k)_\pm>0]}\right\}\mathrm{d}x. \tag{2.4}$$

See Section A for the classical proof. Next lemma is a sort of measure theoretical maximum principle. It asserts that if a certain sub-level set of the solution reaches a critical mass, then the solution is above a multiple of the level on half sub-level set. We agree to refer to it as usual in the literature by the epithet *Critical Mass* lemma (*De Giorgi-type* lemma is used equivalently). We state it just for sub-level sets, a similar statement being true for super-level sets.

Lemma 2.4. *Let $\bar{x}\in\Omega$ and $\theta,\rho>0$ such that $Q_{4\theta,4\rho}(\bar{x})\subseteq\Omega$. Let μ^+,μ^-,ω be nonnegative numbers such that*

$$\mu^+ \geq \sup_{Q_{\theta,\rho}(\bar{x})} u, \quad \mu^- \leq \inf_{Q_{\theta,\rho}(\bar{x})} u, \quad \omega \geq \mu^+ - \mu^-.$$

Now, let u be a bounded function satisfying the energy estimates (2.4) *and fix $a,\xi\in(0,1)$. Then there exists a number $\nu\in(0,1)$ whose dependence from the data is specified by* (2.9) *and such that if*

$$|[u\leq\mu^-+\xi\omega]\cap Q_{\theta,\rho}(\bar{x})| \leq \nu|Q_{\theta,\rho}|, \tag{2.5}$$

then either $\xi\omega\leq\rho$ or

$$u\geq\mu^-+a\xi\omega, \quad \forall_{\text{ae}} x\in Q_{\theta/2,\rho/2}(\bar{x}). \tag{2.6}$$

Proof. We suppose without loss of generality that $\bar{x}=0$. For $j=0,1,2,\dots$ let us set

$$\begin{cases} k_j = \mu^- + a\xi\omega + \dfrac{(1-a)\xi\omega}{2^j}, \\ \rho_j = \dfrac{\rho}{2}+\dfrac{\rho}{2^{j+1}}, \quad \theta_j = \dfrac{\theta}{2}+\dfrac{\theta}{2^{j+1}}, \end{cases} \qquad A_j = Q_{\theta_j,\rho_j}\cap[u<k_j], \tag{2.7}$$

and let $\zeta_j\in C_o^\infty(Q_j)$ be a cutoff function between Q_j and Q_{j+1} such that $\zeta_j\equiv 1$ on Q_{j+1}, $0\leq\zeta_j\leq 1$, and therefore satisfying

$$|\partial_i\zeta_j|\leq\frac{2^{j+2}}{\theta} \quad \forall i\in\{1,\dots,s\}, \quad \text{and} \quad |\partial_i\zeta_j|\leq\frac{2^{j+2}}{\rho} \quad \forall i\in\{s+1,\dots,N\}.$$

Then, combining a precise use of the Hölder inequality and Troisi's embedding (2.2) to the energy estimates (2.4) leads us to the estimate

$$\begin{aligned} \left(\frac{(1-a)\xi\omega}{2^{j+1}}\right)^{\bar{p}}|A_{j+1}| &\leq \int\int_{Q_j}(u-k)_-^{\bar{p}}\zeta_j^{\bar{p}}\mathrm{d}x \\ &\leq \left(\int\int_{Q_j}[(u-k)_-\zeta_j]^{\frac{N\bar{p}}{N-\bar{p}}}\right)^{\frac{N-\bar{p}}{N}}|A_j|^{\frac{\bar{p}}{N}} \\ &\leq \left[\prod_{i=1}^{N}\left(\int\int_{Q_j}|\partial_i(u-k)_-|^{p_i}\mathrm{d}x\right)^{\frac{1}{p_i}}\right]^{\frac{\bar{p}}{N}}|A_j|^{\frac{\bar{p}}{N}} \\ &\leq \left[\int\int_{A_j}\left\{\sum_{i=1}^{s}|\partial_i(u-k)_\pm|^2\zeta^2 + \sum_{i=s+1}^{N}|\partial_i(u-k)_\pm|^p\zeta^2 + C^p\right\}\mathrm{d}x\right]|A_j|^{\frac{\bar{p}}{N}} \\ &\leq \gamma 2^{2j}\left\{\frac{(\xi\omega)^p}{\rho^p}\left[1+\frac{(\xi\omega)^{2-p}\rho^p}{\theta^2}+\left(\frac{C\rho}{\xi\omega}\right)^p\right]\right\}|A_j|^{1+\frac{\bar{p}}{N}}. \end{aligned} \tag{2.8}$$

By assumption $\xi\omega > \rho$ the third term on the right-hand side is smaller than 1. If we define $Y_j = |A_j|/|Q_j|$, we divide (2.8) by $|Q_{j+1}|$ and we observe that $|Q_j| \le \gamma 2^j |Q_{j+1}| \approx (\theta^s \rho^{N-s})$, and the previous estimate can be written as

$$Y_{j+1} \le \gamma \frac{2^{(2+p)j}(1-a)^{-\bar{p}}(\xi\omega)^{p-\bar{p}}}{\rho^p}\left[1 + \frac{(\xi\omega)^{2-p}\rho^p}{\theta^2}\right](\theta^s\rho^{N-s})^{\frac{\bar{p}}{N}} Y_j^{1+\frac{\bar{p}}{N}}$$
$$\le \gamma 2^{(2+p)j}(1-a)^{-\bar{p}}\left(\frac{\theta}{\rho^{p/2}(\xi\omega)^{\frac{2-p}{2}}}\right)^{\frac{s\bar{p}}{N}}\left[1 + \frac{(\xi\omega)^{2-p}\rho^p}{\theta^2}\right] Y_j^{1+\frac{\bar{p}}{N}},$$

by simple manipulation on various exponents. We evoke Lemma A.1 to declare that if

$$Y_o = \frac{|[u \le \mu^- + \xi\omega] \cap Q_{\theta,\rho}|}{|Q_{\theta,\rho}|} \le \gamma^{\frac{-N}{\bar{p}}}(1-a)^N\left(\frac{\theta}{\rho^{p/2}(\xi\omega)^{\frac{2-p}{2}}}\right)^{-s}\left[1 + \frac{(\xi\omega)^{2-p}\rho^p}{\theta^2}\right]^{-\frac{N}{\bar{p}}} =: \nu\ , \tag{2.9}$$

then $Y_j \to 0$ for $j \to \infty$ and the proof is concluded by specifying $Y_\infty = \lim_{j\to\infty} Y_j = 0$. □

Remark 2.1. We observe that within geometry (1.7) the choice $\theta = \rho^{\frac{p}{2}}(\xi\omega)^{\frac{2-p}{2}}$ sets ν free from any other dependence than the initial data.

The following lemma estimates the essential supremum of solutions by quantitative integral averages of the solution itself. Its proof is similar to the one of ([23], Prop. 8) and it is postponed to the Appendix.

Lemma 2.5. *Le u be a locally bounded local weak solution to* (1.2) *and* (1.3). *Let* $1 \le l \le 2$ *and*

$$N(\bar{p}-2) + l\bar{p} > 0. \tag{2.10}$$

Then there exist constants $\gamma, C > 1$ depending only on the data, such that for all polydiscs $Q_{2\theta,2\rho} \subset \Omega$ we have either

$$\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}} \le C\rho \quad \textit{or} \tag{2.11}$$

$$\sup_{Q_{\theta/2,\rho/2}} u \le \gamma\left(\frac{\rho^p}{\theta^2}\right)^{\left(\frac{N-s}{p}\right)\left(\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}\right)}\left(⨍⨍_{Q_{\theta,\rho}} u_+^l \mathrm{d}x\right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \gamma\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{2.12}$$

Note that (2.10) with $l = 1$ corresponds to (1.9). Finally, we give detailed description of the main analytical tool of the present work, the following *logarithmic estimates*.

Lemma 2.6. *Let u be a bounded weak solution to* (1.2), (1.3), *and* (1.5). *Then for any $Q_{2\theta,2\rho}(\bar{x}) \subset \Omega$, any $k \in \mathbb{R}$, and any function $f \in C^1(\mathbb{R};\mathbb{R}_+)$ with $f' > 0$, there exists a constant $\gamma > 1$ such that the following estimate holds for each $0 < t < \theta$, and each $\zeta \in W_o^{1,p}(B_{2\rho}(\bar{x}''))$,*

$$\int\int_{Q_{t,\rho}(\bar{x})} f'((u-k)_\pm)(t^2 - |x'-\bar{x}'|^2)\zeta^p(x'')\left\{\sum_{i=1}^{s}|\partial_i(u-k)_\pm|^2 + \sum_{i=s+1}^{N}|\partial_i(u-k)_\pm|^p\right\}\mathrm{d}x$$
$$\le \gamma t^{s+1}\frac{d}{t}\left(|B_t(\bar{x}')|^{-1}\int\int_{Q_{t,\rho}(\bar{x})}\left[\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\right]\zeta^p(x'')\mathrm{d}x\right) \tag{2.13}$$
$$+ \gamma\int\int_{Q_{t,\rho}(\bar{x})}(t^2 - |x'-\bar{x}'|^2)\left\{\sum_{i=s+1}^{N}\frac{[f((u-k)_\pm)]^p}{[f'((u-k)_\pm)]^{p-1}}(|\partial_i\zeta|^p + \zeta^p) + f'((u-k)_\pm)\right\}\mathrm{d}x.$$

Proof. We test equation (1.5) with a nonnegative function

$$\psi(x) = f(\pm(u(x) - k)_\pm)\phi(x')\zeta^p(x'') \in W_o^{1,[2,p]}(Q_{\theta,\rho}),$$

being $\phi \in W_o^{1,2}(B_{2\theta}(\bar{x}'))$ a test function, and we use the structure conditions (1.3) to obtain

$$\begin{aligned}
&\int\int f'((u-k)_\pm)\phi\zeta^p\left\{\sum_{i=1}^{s}|\partial_i(u-k)_\pm|^2 + C_1\sum_{i=s+1}^{N}|\partial_i(u-k)_\pm|^p\right\}\mathrm{d}x\\
&\le \int\int\left\{\sum_{i=s+1}^{N} Cf'((u-k)_\pm)|\partial_i(u-k_\pm)|\phi\zeta^p + \sum_{i=1}^{s}(-\partial_i\phi)f((u-k)_\pm)(\partial_i(u-k)_\pm)\zeta^p\right.\\
&\quad\left. + [C_2|\partial_i(u-k)_\pm|^{p-1} + C]f((u-k)_\pm)\phi\zeta^{p-1}|\partial_i\zeta|\right\}\mathrm{d}x,
\end{aligned}$$

the integrals being taken over the polydisc $Q_{\theta,\rho}(\bar{x})$. We use repeatedly Young's inequality to obtain, for the first term on the right

$$Cf'|\partial_i(u-k_\pm)|\phi\zeta^p \le f'\phi\left(\varepsilon|\partial_i(u-k)_\pm|^p\zeta^p + \gamma(\varepsilon)C^{\frac{p}{p-1}}\zeta^p\right),$$

while the third term on the right is estimated with

$$|\partial_i(u-k)_\pm|^{p-1}f\phi\zeta^{p-1}\partial_i\zeta(f')^{\frac{p-1}{p}}(f')^{\frac{1-p}{p}} \le \phi(\varepsilon f'|\partial_i(u-k)_\pm|^p\zeta^p + C(\varepsilon)|\partial\zeta|^pf^p(f')^{1-p}).$$

Similarly, fourth term is estimated by

$$Cf\phi\zeta^{p-1}|\partial_i\zeta|(f')^{\frac{p-1}{p}}(f')^{\frac{1-p}{p}} \le \phi\left(\varepsilon f^p(f')^{1-p}|\partial_i\zeta|^p + \gamma(\varepsilon)C^{\frac{p}{p-1}}f'\zeta^p\right).$$

Gathering all the pieces together and choosing accordingly ε small enough to reabsorb on the left the energy terms, we obtain

$$\begin{aligned}
&\int\int f'((u-k)_\pm)\phi\zeta^p\left[\sum_{i=1}^{s}|\partial_i(u-k)_\pm|^2 + \sum_{i=s+1}^{N}|\partial_i(u-k)_\pm|^p\right]\mathrm{d}x\\
&\le \gamma\int\int\left[f'((u-k)_\pm)\phi\zeta^p - \sum_{i=1}^{s}(\partial_i\phi)(\partial_i(u-k)_\pm)f((u-k)_\pm)\zeta^p\right]\mathrm{d}x\\
&\quad + \gamma\int\int\left[\sum_{i=s+1}^{N}\frac{f^p((u-k)_\pm)}{(f'(u-k)_\pm)^{p-1}}\phi(|\partial_i\zeta|^p + |\zeta^p|) + f'((u-k)_\pm)\phi\zeta^p\right]\mathrm{d}x.
\end{aligned}$$

Finally, choose $0 \le \zeta(x'') \le 1$ and $\phi(x') = (t^2 - |x' - \bar{x}'|^2)_+$ and estimate the second term on the right by using Green-Ostrogradsky's formula with

$$\begin{aligned}
&\int\int_{Q_{t,\rho}(\bar{x})} f((u-k)_\pm)\zeta^p(x'')\sum_{i=1}^{s}(\partial_i(u-k)_\pm)2(x_i' - \bar{x}_i')\mathrm{d}x\\
&= 2\int\int_{Q_{t,\rho}(\bar{x})}\sum_{i=1}^{s}\left[\partial_i\left(\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\cdot(x_i' - \bar{x}_i')\right) - 2s\left(\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\right)\right]\zeta^p\mathrm{d}x\\
&= 2t\int_{\partial B_t(\bar{x}')}\int_{B_\rho(\bar{x}'')}\left(\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\right)\zeta^p\mathrm{d}x - 2s\int\int_{Q_{t,\rho}(\bar{x})}\left(\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\right)\Bigg]\zeta^p\mathrm{d}x\\
&= 2t^{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{-s}\int\int_{Q_{t,\rho}(\bar{x})}\left(\int_0^{(u-k)_\pm} f(\tau)\mathrm{d}\tau\right)\zeta^p(x'')\mathrm{d}x\right],
\end{aligned} \tag{2.14}$$

and the proof is completed. □

3 Proof of Theorem 1.3

We start by proving two main lemmas, whose combination will provide an easy proof of Theorem 1.3. The first one turns a measure estimate given on the intrinsic polydisc into a measure estimate on each $(N-s)$-dimensional slice.

Lemma 3.1. *Let u be a bounded weak solution to* (1.2), *with structure conditions* (1.3) *and* (1.5). *Assume that for some $M, \rho > 0$ and some $\alpha \in (0,1)$ the following estimate holds*

$$|[u(\cdot) \le M] \cap Q_{\theta,\rho}(\bar{x})| \le (1-\alpha)|Q_{\theta,\rho}|, \quad \theta = \rho^{\frac{p}{2}}(\delta M)^{\frac{2-p}{2}}, \tag{3.1}$$

for some $\delta < \alpha^4(1+\alpha)^s/(1+\alpha^2)$. Then there exist numbers $s_1 > 1$, $\delta_1 \in (0,1)$ depending only on the data and α such that

$$M \le \rho \quad \textit{or} \tag{3.2}$$

$$\left|\left[u(y',\cdot) \le 2^{-s_1}M\right] \cap B_\rho(\bar{x}'')\right| \le (1-\alpha^4/2)|B_\rho(\bar{x}'')|, \quad \forall_{\mathrm{ae}}\, y' \in B_{(1-\delta_1)\theta}(\bar{x}'). \tag{3.3}$$

Proof. Let $\sigma, \delta_1 \in (0,1)$ be chosen later, let $\zeta(x'') \in C_o^\infty(B_\rho(\bar{x}''))$ be such that

$$\begin{cases} 0 \le \zeta \le 1, & \\ \zeta(x'') \equiv 1, & x'' \in B_{(1-\sigma)\rho}(\bar{x}''), \end{cases} \quad \& \ |\partial_i \zeta| \le \frac{\gamma}{\sigma\rho}, \quad \forall i = s+1,\dots,N.$$

Let $y' \in B_{(1-\delta_1)\theta}(\bar{x}')$ be a Lebesgue point for the function

$$\int_{B_\rho(\bar{x}'')} (u(\cdot,x)-M)_-^2 \zeta^p(x'')\,\mathrm{d}x,$$

let us call $z = (y', \bar{x}'')$ and use just the right-hand side of inequality (2.13), with $f(u) = u$, $f' \equiv 1$, $k = M$, to obtain

$$0 \le \frac{\mathrm{d}}{\mathrm{d}t}\left[t^{-s} \int\int_{Q_{t,\rho}(z)} \frac{(u-M)_-^2}{2}\zeta^p(x'')\,\mathrm{d}x\right] + \frac{\gamma t^{-s-1}}{(\sigma\rho)^p}\int\int_{Q_{t,\rho}(z)} t^2\left\{\sum_{i=s+1}^{N}(u-M)_-^p + (\sigma\rho)^p\right\}\mathrm{d}x.$$

Now we integrate this inequality on $t \in (0, \delta_1\theta)$ and estimate the various terms. The first term can be evaluated by

$$\left[t^{-s}\int\int_{Q_{t,\rho}(z)} \frac{(u-M)_-^2}{2}\zeta^p(x'')\,\mathrm{d}x\right]_0^{\delta_1\theta} = \frac{1}{(\delta_1\theta)^s}\int\int_{Q_{\delta_1\theta,\rho}(z)} \frac{(u-M)_-^2}{2}\zeta^p(x'')\,\mathrm{d}x - \int_{B_\rho(\bar{x}'')} \frac{(u(y',\cdot)-M)_-^2}{2}\zeta^p(x'')\,\mathrm{d}x,$$

using that y' is a Lebesgue point. Second term is estimated with

$$\gamma\int_0^{\delta_1\theta}\left(\frac{t^{-s+1}}{(\sigma\rho)^p}\int\int_{Q_{t,\rho}(z)} (u-M)_-^p\,\mathrm{d}x\right)\mathrm{d}t \le \frac{\gamma M^p}{(\sigma\rho)^p}(\delta_1\theta)^2|B_\rho(\bar{x}'')|,$$

and the third term similarly. Gathering all together we obtain

$$\int_{B_{(1-\sigma)\rho}(\bar{x}'')} (M-u(y',x''))_+^2\,\mathrm{d}x \le \frac{1}{(\delta_1\theta)^s}\int\int_{Q_{\delta_1\theta,\rho}(z)} (M-u)_+^2\,\mathrm{d}x + \gamma(\delta_1\theta)^2|B_\rho(\bar{x}'')|\left\{\left(\frac{M}{\sigma\rho}\right)^p + 1\right\}.$$

We observe that $B_{\delta_1}(y') \subset B_\theta(\bar{x}')$ and that by imposing the natural intrinsic geometry the term in parenthesis can be ruled. Indeed, we let $\theta = \rho^{p/2}(\delta N)^{(2-p)/2}$ and compute

$$
\begin{aligned}
&M^2\Big(1-\frac{1}{2^{s_1}}\Big)^2\,\Big|\Big[u(y',\cdot)\le\frac{M}{2^{s_1}}\Big]\cap B_{(1-\sigma)\rho}(\bar{x}'')\Big|\\
&\le \int\limits_{B_{(1-\sigma)\rho}(\bar{x}'')}(M-u(y',x''))_+^2\mathrm{d}x\\
&\le \frac{M^2}{(\delta_1\theta)^s}|[u\le M]\cap Q_{\theta,\rho}(\bar{x})|+\gamma M^2|B_\rho(\bar{x}'')|\bigg\{\frac{(\delta_1\theta)^2M^{p-2}}{(\sigma\rho)^p}+\frac{(\delta_1\theta)^2}{M^2}\bigg\}\\
&\le M^2|B_\rho(\bar{x}'')|\bigg\{\frac{1}{(\delta_1)^s}(1-\alpha)+\gamma\Big(\frac{\delta^{2-p}\delta_1}{\sigma^p}\Big)\bigg\},
\end{aligned}
$$

using $M\ge\rho$ and hypothesis (3.1). We estimate from below in the whole $B_\rho(\bar{x}'')$ by

$$
|\big[u(y',\cdot)\le M2^{-s_1}\big]\cap B_\rho(\bar{x}'')|\le|\big[u(y',\cdot)\le M2^{-s_1}\big]\cap B_{(1-\sigma)\rho}(\bar{x}'')|+(N-s)\sigma|B_\rho(\bar{x}'')|.
$$

Combining this remark with previous calculations we have the inequality

$$
\begin{aligned}
|\big[u(y',\cdot)\le M2^{-s_1}\big]\cap B_\rho(\bar{x}'')|&\le\big(1-2^{-s_1}\big)^{-2}|B_\rho(\bar{x}'')|\bigg\{\frac{1}{(\delta_1)^s}(1-\alpha)+\gamma\Big(\frac{\delta^{2-p}\delta_1}{\sigma^p}\Big)\bigg\}+(N-s)\sigma|B_\rho(\bar{x}'')|\\
&=|B_\rho(\bar{x}'')|\bigg\{\frac{(1-\alpha)}{\delta_1^s(1-2^{-s_1})^2}+\frac{\gamma\delta^{2-p}\delta_1}{\sigma^p(1-2^{-s_1})^2}+(N-s)\sigma\bigg\}.
\end{aligned}
$$

To conclude the proof we choose δ_1, s_1, σ, δ, from the conditions

$$
\big(1-2^{-s_1}\big)^2=1+\alpha^2,\quad \delta_1^{-s_1}=1+\alpha,\quad (N-s)\sigma=\alpha^4/4,\quad \delta^{2-p}=\frac{\alpha^4}{4}\Big(\frac{\gamma\delta_1}{\sigma^p(1-2^{-s_1})^2}\Big)^{-1}. \qquad\square
$$

The following lemma is what is called in the literature a *shrinking* lemma. Indeed, from a given relative measure information on a level set, it allows us to shrink as much as we need the relative measure on a lower-level set. Even more interesting, it provides also an expansion of positivity along singular variables.

Lemma 3.2. *Let $\bar{x}\in\Omega$ and u be a bounded weak solution to* (1.2), (1.3), *and* (1.5). *Let us suppose that for some $M,\rho>0$ and some $\beta\in(0,1)$ the following estimate holds for almost every $y'\in B_\theta(\bar{x}')$:*

$$
|[u(y',\cdot)\le M]\cap B_\rho(\bar{x}'')|\le(1-\alpha)|B_\rho|,\quad \textit{being }\ \theta=\rho^{\frac{p}{2}}(\delta M)^{\frac{2-p}{2}}, \tag{3.4}
$$

and $Q_{2\theta,4\rho}(\bar{x})\subset\Omega$. Then for every $\nu\in(0,1)$ there exist numbers $K>1$ and $\delta_o\in(0,1)$ depending only on the data and α, ν, n such that either

$$
M\le K\rho\quad \textit{or} \tag{3.5}
$$

$$
|[u(x',\cdot)\le\delta_oM]\cap B_{2\rho}(\bar{x}'')|\le\nu|B_{2\rho}(\bar{x}'')|,\quad \forall_{ae}x'\in B_\theta(\bar{x}'). \tag{3.6}
$$

Proof. We divide the proof into three steps.

Step 1. Normalization and logarithmic estimate.

Let us introduce the change of variables $\Phi: Q_{2\theta,4\rho}(\bar{x})\to Q_{1,1}(0)$ given by

$$
x'\to\frac{x'-\bar{x}'}{2\theta},\quad x''\to\frac{x''-\bar{x}''}{4\rho},\quad v=\frac{u}{M}.
$$

The new function v satisfies the following equation in $Q_{1,1}$:

$$
\sum_{i=1}^{s}\partial_{ii}v+\sum_{i=s+1}^{N}\partial_i(\tilde{A}_i(x,v,\nabla v))=0, \tag{3.7}
$$

with structure conditions

$$\begin{cases} \sum_{i=s+1}^{N} \tilde{A}_i(x, v, \nabla v)\partial_i v \geq \tilde{C}_1 \sum_{i=s+1}^{N} |\partial_i v|^p - \tilde{C}(\rho/M)^p, \\ |A_i(x, v, \nabla v)| \leq \tilde{C}_2 |\partial_i v|^{p-1} + \tilde{C}(\rho/M)^{p-1}. \end{cases} \tag{3.8}$$

By transformation Φ inequality (3.4) turns into

$$|[v(x', \cdot) > 1] \cap B_{1/4}(0'')| > \alpha |B_{1/4}(0'')|, \quad \forall_{\mathrm{ae}}\, x' \in B_{1/2}(0'). \tag{3.9}$$

The expansion of positivity relies on the following simple fact. The aforementioned inequality implies the measure estimate for x' in $B_{1/2}(0')$

$$|[v(x', \cdot) > 1] \cap B_1(0'')| > (w_{N,s} 4^{(N-s)})^{-1} \alpha |B_1(0'')| =: \tilde{\alpha} |B_1(0'')|. \tag{3.10}$$

Let now $\zeta(x'') \in C_o^{\infty}(B_1(0''))$ be a convex cutoff function between balls B_1 and $B_{1/2}$, i.e.,

$$0 \leq \zeta \leq 1, \quad \zeta|_{B_{1/2}(0'')} \equiv 1, \quad |\partial_i \zeta| \leq 2 \quad \forall i \in \{s+1, \ldots, N\}.$$

Let us fix numbers $j^* \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ and for $j = 1, 2, \ldots, j^*$ we define

$$f((v - \varepsilon^j)_-) = [\varepsilon^j (1 + \varepsilon) - (v - \varepsilon^j)_-]^{1-p}.$$

Then, inequality (2.13) reads, for $t \in (0, 1/2)$,

$$\begin{aligned} &\gamma^{-1} \sum_{i=s+1}^{N} \int\int_{Q_{t,1}} \frac{|\partial_i (v - \varepsilon^j)_-|^p}{[\varepsilon^j (1 + \varepsilon) - (v - \varepsilon^j)_-]^p} (t^2 - |x'|^2) \zeta^p(x'') \mathrm{d}x \\ &\leq t^{s+1} \frac{\mathrm{d}}{\mathrm{d}t} \left[t^{-s} \int\int_{Q_{t,1}} \left(\int_0^{(v-\varepsilon^j)_-} f(\tau) \mathrm{d}\tau \right) \zeta(x'')^p \mathrm{d}x \right] \\ &\quad + \int\int_{Q_{t,1}} \left\{ \sum_{i=s+1}^{N} (t^2 - |x'|^2)(|\partial_i \zeta(x'')|^p + \zeta(x'')^p) + \tilde{C} \left(\frac{\rho}{M\varepsilon^{j^*}} \right)^p (t^2 - |x'|^2) \right\} \mathrm{d}x. \end{aligned} \tag{3.11}$$

The last two terms on the right of (3.11) can be reduced, by assuming $M\varepsilon^{j^*} > \rho$, to

$$\sum_{i=s+1}^{N} t^{-s-1} |Q_{t,1}| t^2 + \tilde{C} \left(\frac{\rho}{M\varepsilon^{j^*}} \right)^p t^{-s-1} |Q_{t,1}| t^2 \leq \gamma t.$$

We observe that first integrands on the left of (3.11) are the directional derivatives of

$$g = \left[\ln \left(\frac{(1 + \varepsilon)\varepsilon^j}{(1 + \varepsilon)\varepsilon^j - (v - \varepsilon^j)_-} \right) \right]^p,$$

and, as $g \in W^{1,p}(B_1(0''))$ vanishes in $[v > e^j] \cap B_1|$, we can apply the weighted Poincaré inequality (2.3), using (3.10) to estimate the term $|\mathcal{E}| = |[v > \varepsilon^j] \cap B_1| \geq |[v > 1] \cap B_1|$.

A precise analysis of this last simple fact reveals its correspondence with (3.9) and (3.10) in terms of expansion of positivity. Putting all the pieces of the puzzle into (3.11) we arrive finally to the following logarithmic estimate, valid for each $j = 1, \ldots, j^*$, and $0 < t < \theta$,

$$\begin{aligned} &t^{-s-1} \int\int_{Q_{t,1}} \ln^p \left(\frac{(1 + \varepsilon)\varepsilon^j}{(1 + \varepsilon)\varepsilon^j - (v - \varepsilon^j)_-} \right) (t^2 - |x'|^2) \zeta^p(x'') \mathrm{d}x \\ &\leq \gamma \frac{\mathrm{d}}{\mathrm{d}t} \left(\int\int_{Q_{t,1}} t^{-s} \int_0^{(v-\varepsilon^j)_-} \frac{\mathrm{d}\tau}{[(1 + \varepsilon)\varepsilon^j - \tau]^{p-1}} \zeta^p(x'') \mathrm{d}x \right) + \gamma t. \end{aligned} \tag{3.12}$$

Step 2. First alternative.

Let us define $A_j(t) = [v < \varepsilon^j] \cap Q_{t,1}(0)$ and let

$$y_j = \sup_{0<t<1/2} Y_j(t), \quad \text{where } Y_j(t) = t^{-s} \int \int_{A_j(t)} \zeta^p(x'') \mathrm{d}x. \tag{3.13}$$

Now we show that if $M \geq K\rho$, then there exists a number $\xi = \xi(v) \in (0,1)$ such that $y_{j+1} \leq \max\{v, (1-\xi)y_j\}$ for each $j = 1, \dots, j^*$. This with a standard iteration procedure will end the proof. So we proceed by assuming $y_{j+1} > v$, and by continuity of the integral we can choose $t_o \in (0, 1/2)$ such that $Y_{j+1}(t_o) = y_{j+1}$ and divide the argument into two alternatives. Let

$$\Psi(t) := t^{-s} \int \int_{Q_{t,1}} \left(\int_0^{(v-\varepsilon^j)_-} \frac{\mathrm{d}\tau}{[(1+\varepsilon)\varepsilon^j - \tau]^{p-1}} \right) \zeta^p(x'') \mathrm{d}x, \tag{3.14}$$

and suppose $\Psi'(t_o) \leq 0$. Then inequality (3.12) implies that for each fixed $\sigma \in (0,1)$

$$\begin{aligned} & t_o^{-s-1} t_o^2 [1 - (1-\sigma)^2] \ln^p \left(\frac{1+\varepsilon}{2\varepsilon} \right) \int \int_{A_{j+1}(1-\sigma)t_o} \zeta^p(x'') \mathrm{d}x \\ & \leq t_o^{-s} \int \int_{A_j(1-\sigma)t_o} \ln^p \left(\frac{(1+\varepsilon)\varepsilon^j}{(1+\varepsilon)\varepsilon^j - (v-\varepsilon^j)_-} \right) \zeta^p(x'')(t_o^2 - |x'|^2) \mathrm{d}x \leq \gamma t_o, \end{aligned} \tag{3.15}$$

giving the estimate

$$t_o^{-s} \int \int_{A_{j+1}(1-\sigma)t_o} \zeta^p(x'') \mathrm{d}x \leq \gamma \sigma^{-2} \ln^{-p} \left(\frac{1+\varepsilon}{2\varepsilon} \right). \tag{3.16}$$

Now we determine σ, ε small enough to obtain $y_{j+1} \leq v$. Indeed, by $|Q_{t,1}| \leq \gamma t^{-s} |B_1(0'')|$ we see that the following estimate holds

$$\begin{aligned} y_{j+1} = Y_{j+1}(t_o) &= t_o^{-s} \int \int_{A_{j+1}(t_o)} \zeta^p(x'') \mathrm{d}x \\ &\leq t_o^{-s} \int \int_{Q_{(1-\sigma)t_o,1}} \chi_{[v<\varepsilon^{j+1}]} \zeta^p(x'') \mathrm{d}x + t_o^{-s} |Q_{t_o,1} / Q_{(1-\sigma)t_o,1}| \\ &\leq \gamma \left\{ \sigma^{-2} \ln^{-p} \left(\frac{1+\varepsilon}{2\varepsilon} \right) + \sigma(N-s)|B_1(0'')| \right\} \leq v, \end{aligned} \tag{3.17}$$

for the choices

$$\sigma = \frac{v}{2(N-s)\gamma}, \quad \varepsilon = (e^{\frac{2\gamma}{\sigma^2 v p}} - 1)^{-1}. \tag{3.18}$$

Step 3. Second alternative.

Let us suppose now that $\Psi'(t_o) > 0$ and that there exists $t_* = \inf\{t \in (t_o, 1/2) | \quad \Psi'(t) \leq 0\}$, so that by definition Ψ is monotone increasing before t_* and we have

$$\begin{aligned} & t_o^{-s} \int \int_{Q_{t_o,1}} \left(\int_0^{(v-\varepsilon^j)_-} \frac{\mathrm{d}\tau}{[(1+\varepsilon)\varepsilon^j - \tau]^{p-1}} \right) \zeta^p(x'') \mathrm{d}x \\ & \leq t_*^{-s} \int \int_{Q_{t_*,1}} \left(\int_0^{(v-\varepsilon^j)_-} \frac{\mathrm{d}\tau}{[(1+\varepsilon)\varepsilon^j - \tau]^{p-1}} \right) \zeta^p(x'') \mathrm{d}x. \end{aligned} \tag{3.19}$$

For the time t_* similar estimates to (3.12), (3.15), and (3.17) hold and we obtain similarly that

$$t_*^{-s}\int\int_{A_j(t_*)} \zeta^p(x'')\chi_{[(\varepsilon^j-v)>\varepsilon^j\tau]}\mathrm{d}x \le v/4 + \gamma v^{-2}\ln^{-p}\left(\frac{1+\varepsilon}{1+\varepsilon-\tau}\right) \le v/2. \tag{3.20}$$

In this case, the value of ε has been already chosen, so that the last inequality is valid, provided we restrict to levels τ such that

$$\gamma v^{-2}\ln^{-p}\left(\frac{1+\varepsilon}{1+\varepsilon-\tau}\right) \le v/4 \quad \Leftrightarrow \quad \tau \ge (1+\varepsilon)[1-e^{-h(v)}] =: \bar{\tau}, \tag{3.21}$$

for a function $h(v) = o(v^{3/p})$. Finally, we use (3.20) and (3.19) together with Fubini's theorem and a change of variables to estimate

$$\begin{aligned}
&t_*^{-s}\int\int_{A_j(t_*)}\left(\int_0^{(v-\varepsilon^j)_-}\frac{\mathrm{d}\tau}{[(1+\varepsilon)\varepsilon^j-\tau]^{p-1}}\right)\zeta^p(x'')\mathrm{d}x\\
&= \varepsilon^{j(2-p)}\int_0^1\frac{1}{[1+\varepsilon-\tau]^{p-1}}\left\{t_*^{-s}\int\int_{A_j(t_*)}\zeta^p(x'')\chi_{[\varepsilon^j-v\ge\varepsilon^j\tau]}\mathrm{d}x\right\}\mathrm{d}\tau\\
&\le \varepsilon^{j(2-p)}\left[y_j\int_0^{\bar{\tau}}\frac{\mathrm{d}\tau}{[1+\varepsilon-\tau]^{p-1}} + \frac{v}{2}\int_{\bar{\tau}}^1\frac{\mathrm{d}\tau}{[1+\varepsilon-\tau]^{p-1}}\right]\\
&\le \varepsilon^{j(2-p)}y_j\left[\int_0^{\bar{\tau}}\frac{\mathrm{d}\tau}{[1+\varepsilon-\tau]^{p-1}} + \frac{1}{2}\int_{\bar{\tau}}^1\frac{\mathrm{d}\tau}{[1+\varepsilon-\tau]^{p-1}}\right],
\end{aligned} \tag{3.22}$$

where we used $y_j > Y_{j+1}(t_*)$ in the first inequality and $y_j > y_{j+1} > v$ in the last one. We estimate from below (3.19) with a simple calculation, as

$$\int_0^{(v-\varepsilon^j)_-}\frac{\mathrm{d}\tau}{[(1+\varepsilon)\varepsilon^j-\tau]^{p-1}} \ge \varepsilon^{j(2-p)}(1-\varepsilon^{2-p})\int_0^1\frac{\mathrm{d}\tau}{[\varepsilon+\tau]^{p-1}}. \tag{3.23}$$

So we combine (3.19), (3.22), and (3.23) together to obtain

$$\begin{aligned}
&y_{j+1}\varepsilon^{j(2-p)}(1-\varepsilon^{2-p})\int_0^1\frac{\mathrm{d}\tau}{[\varepsilon+\tau]^{p-1}}\\
&\le \varepsilon^{j(2-p)}y_j\left(\int_0^1\frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}} - \frac{1}{2}\int_0^{1-\bar{\tau}}\frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}}\right)\\
&\le \varepsilon^{j(2-p)}y_j\left(\int_0^1\frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}} - \frac{(1-\bar{\tau})}{2}\int_0^1\frac{\mathrm{d}\tau}{(\varepsilon+\tau(1-\bar{\tau}))^{p-1}}\right)\\
&\le \varepsilon^{j(2-p)}y_j(1-(1-\bar{\tau})/2)\int_0^1\frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}}.
\end{aligned} \tag{3.24}$$

Finally, this implies

$$y_{j+1} \le \left(\frac{1-(1-\bar{\tau})/2}{1-\varepsilon^{2-p}}\right)y_j =: (1-\xi)y_j, \quad \text{with } \xi = 1-\left[\frac{1-(1-\bar{\tau})/2}{1-\varepsilon^{2-p}}\right] < 1,$$

redefining $\varepsilon = \min\{1/2, \varepsilon\}$ if needed. We prove now that if t_* does not exist, then the iteration inequality above is still satisfied. Indeed, in case no such t_* exists then $\Psi'(t) > 0$ for all $t \in [t_o, 1/2]$ and therefore $\Psi(t_o) \le \Psi(1/2)$. Moreover, by simple calculations analogous to (3.23) and (3.24) we recover the estimates

$$\Psi(t_o) \geq \varepsilon^{j(2-p)}(1-\varepsilon^{2-p})\left(\int_0^1 \frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}}\right) y_{j+1}, \tag{3.25}$$

$$\Psi(1/2) \leq 2^s \varepsilon^{j(2-p)} \int_0^1 \frac{\mathrm{d}\tau}{(\varepsilon+\tau)^{p-1}} \int\int_{Q_{1/2,1}} \chi_{[v\leq\varepsilon^j]}(x)\zeta^p(x'')\mathrm{d}x. \tag{3.26}$$

So, by combining $\Psi(t_o) \leq \Psi(1/2)$ with (3.25) and (3.26) we obtain the inequality

$$y_{j+1} \leq \frac{2^s}{1-\varepsilon^{2-p}} \int\int_{Q_{1/2,1}} \chi_{[v\leq\varepsilon^j]}\zeta^p(x'')\mathrm{d}x \leq \frac{2^s}{1-\varepsilon^{2-p}} \int\int_{Q_{1/2,1}} \chi_{[v\leq\varepsilon]}\zeta^p(x'')\mathrm{d}x. \tag{3.27}$$

If we test equation (1.5) with $\psi = [\varepsilon(1+\varepsilon)-(v-\varepsilon)_-]^{1-p}\zeta^p(x'')(1-|x'|^2)_+^2$ and use Young's inequality we can derive, similarly to (3.12), the estimate

$$\begin{aligned}
&\int\int_{Q_{1/2,1}} \ln^p\left(\frac{1+\varepsilon}{\varepsilon(1+\varepsilon)-(v-\varepsilon)_-}\right)\zeta^p(x'')\,\mathrm{d}x \\
&\leq \gamma \sum_{i=s+1}^N \int\int_{Q_{1,1}} \left(\frac{|\partial_i(v-\varepsilon)_-|}{[\varepsilon(1+\varepsilon)-(v-\varepsilon)_-]}\right)^p \zeta^p(x'')(1-|x'|^2)\mathrm{d}x \\
&\leq \gamma \int\int_{Q_{1,1}} \{[\varepsilon(1+\varepsilon)-(v-\varepsilon)_-]^{2-p}\zeta^p(x'')+(1-|x'|^2)\}\mathrm{d}x + \tilde{C}\left(\frac{\rho}{M\varepsilon^{j^*}}\right)^p |Q_{1,1}| \leq \gamma.
\end{aligned}$$

From this, by (3.26) we obtain

$$\int\int_{Q_{1/2,1}} \chi_{[v\leq\varepsilon]}\zeta^p(x'')\mathrm{d}x \leq \gamma \ln^{-p}\left(\frac{1+\varepsilon}{2\varepsilon}\right),$$

and therefore estimating (3.27) from above we conclude that $y_{j+1} \leq \nu$ by choosing ε small enough. Finally, for the sake of readability we just remark that in case Ψ is not regular enough it is possible to perform the same argument above by substituting Ψ' with its right Dini derivative, as in ([6], Section 7).

Conclusion.

Both the alternatives imply the estimate

$$y_{j+1} \leq \max\{\nu, (1-\xi)y_j\}, \quad \forall j = 1, \dots, j^*, \quad \xi = \xi(\nu) \in (0,1).$$

Iterating this inequality we arrive at

$$y_{j^*} \leq \max\left\{\nu, (1-\xi)^{j^*-1}y_1\right\}$$

and since $y_1 \leq 1$, we choose j^* such that $(1-\xi)^{j^*-1} \leq \nu$ to obtain for each $x' \in (0,1/2)$ the estimate

$$|\left[v(x',\cdot) < \varepsilon^{j^*}\right] \cap B_1(0'')| \leq y_{j^*} \leq \nu |B_1(0'')|. \tag{3.28}$$

The inverse transformation Φ^{-1} turns the obtained estimate into (3.6) with $\delta_o = \varepsilon^{j^*}$, therefore, finishing the proof of Lemma 3.2. □

3.1 Conclusion of the proof of Theorem 1.3

Let $\bar{x} \in \Omega$, $\rho > 0$ and $\alpha \in (0,1)$. We suppose that for $\delta(\alpha) \in (0,1)$ and $M > 0$ we have the information

$$Q_{2\theta,4\rho}(\bar{x}) \subset \Omega, \quad \text{and} \quad |[u \leq M] \cap Q_{\theta,\rho}(\bar{x})| \leq (1-\beta(\alpha))|Q_{\theta,\rho}|, \quad \text{being } \theta = \rho^{\frac{p}{2}}(\delta M)^{\frac{2-p}{p}}.$$

By Lemma 3.1 there exist numbers $s_1, \delta_1 > 0$ depending only on the data and α such that either $M \le \rho$ or

$$|\left[u(y',\cdot) \le 2^{-s_1}M\right] \cap B_\rho(\bar{x}'')| \le (1-\alpha^4/2)|B_\rho(\bar{x}'')|, \quad \forall_{\mathrm{ae}} y' \in B_{(1-\delta_1)\theta}(\bar{x}').$$

For ν as in (2.9), we apply Lemma 3.2 with

$$\bar{M} = 2^{-s_1}M, \quad \beta = \beta\alpha, \quad \bar{\delta} = (1-\delta_1)^{\frac{2}{p-2}}, \quad \bar{\theta} = \rho^{\frac{p}{2}}(\bar{\delta}M)^{\frac{2-p}{2}} = (1-\delta_1)\theta,$$

so that there exist numbers $K > 1$, $\delta_o \in (0,1)$ depending on the data and α, ν such that either $M \le K\rho$ or

$$|[u(x',\cdot) \le \delta_o M] \cap B_{4\rho}(\bar{x}'')| \le \nu |B_{4\rho}(\bar{x}'')|, \quad \forall_{ae}\ x' \in B_{\bar{\theta}}(\bar{x}').$$

To recover the correct intrinsic geometry (see Remark 2.1), we cut the slice-wise information on polydisc $Q_{\bar{\theta},4\rho}(\bar{x})$ to an information on a polydisc which is smaller along the nondegenerate variables, by

$$\theta_o^2 = (4\rho)^p(\bar{\delta}_o M)^{2-p}, \quad \bar{\delta}_o = 4^{\frac{-p}{2-p}}\delta_o,$$

so that $\theta_o \le \bar{\theta}$, increasing j^* in (3.28) in case of need. Finally, we apply the *Critical Mass* Lemma 2.4 to end the proof.

4 Hölder continuity. Proof of Theorem 1.1

We begin with the accommodation of degeneracy. Let $x_0 \in \Omega$ be an arbitrary point, $M = \sup_\Omega |u|$, and $\rho > 0$ such that

$$Q_\rho[M](x_0) := Q_{\rho^{\frac{p}{2}}(2M)^{\frac{2-p}{2}},\rho}(x_0) \subset \Omega.$$

Set

$$\mu_+ := \sup_\Omega u, \quad \mu_- := \inf_\Omega u \quad \omega = \mu_+ - \mu_-.$$

Now for $\alpha = 1/2$ we fix a number δ as defined in Theorem 1.3, let $\theta = \rho^{\frac{p}{2}}(\delta\omega)^{\frac{2-p}{2}}$, and consider the following two alternatives:

$$|[u \le \mu_- + \omega/2] \cap Q_{\theta,\rho}(x_0)| \le \frac{1}{2}|Q_{\theta,\rho}| \tag{4.1}$$

or

$$|[u \ge \mu_+ - \omega/2] \cap Q_{\theta,\rho}(x_0)| \le \frac{1}{2}|Q_{\theta,\rho}|. \tag{4.2}$$

From this, in both cases we may apply Theorem 1.3: in case of measure estimate (4.1) we apply it to the function $v^+ = (u - \mu_-)$, while in case of measure estimate (4.2) $v^- = (\mu_+ - u)$. Both these functions are solutions to an equation similar to (1.2) and (1.3), and therefore the aforementioned theorem can be applied, implying a reduction of oscillation

$$\operatorname{osc}_{Q_{\eta,\rho/4}} u \le \left(1 - \frac{\delta_o}{4}\right)\omega := \delta_* \omega, \quad \eta^2 = (\rho/4)^p(\delta\omega)^{2-p}.$$

Once we have this kind of controlled reduction of oscillation the whole procedure can be iterated in nested shrinking polydiscs and the rest of the proof is standard (see for instance Theorem 3.1 in [11], Chap. X).

5 $L^1 - L^\infty$ estimates. Proof of Theorem 1.4

Let us fix a point $\bar{x} \in \Omega$ and numbers $\rho, \theta > 0$ such that $Q_{8\theta,8\rho}(\bar{x}) \subset \Omega$. Let $y' \in B_{\theta/2}(\bar{x}')$ be a Lebesgue point for the function

$$y \to \int_{B_r(\bar{x}'')} u(y, x'')\zeta(x'')\mathrm{d}x''.$$

For $\sigma \in (0, 1)$, we consider a generic radius r such that $\rho \le (1-\sigma)r \le 2\rho$. Let $\zeta(x'') \in C_o^\infty(B_r(\bar{x}))$, $0 \le \zeta \le 1$, $\zeta \equiv 1$ in $B_{(1-\sigma)r}(\bar{x}'')$ be a cutoff function relative to the last $N-s$ variables between $B_{(1-\sigma)r}$ and B_r satisfying

$$|\partial_i\zeta| \le \frac{\gamma}{\sigma r}, \quad \forall i \in \{s+1, \dots, N\}.$$

We divide the proof into three steps. For ease of notation, let us call $Q_{t,r} := Q_{t,r}(y', \bar{x}'')$ and

$$\eta = \left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}}.$$

Step 1. An integral inequality

We test the equation (1.2) and (1.3) with $\phi(x) = (t^2 - |x'-y'|^2)_+\zeta^p(x'')$ for $0 < t < 2\theta$ and use Green's formula (2.14) with $f(u) \equiv 1$ to obtain

$$\begin{aligned} t^{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(|B_t(0')|^{-1}\int\int_{Q_{t,r}} u\zeta^p\mathrm{d}x\right) &= \sum_{i=1}^{s}\int\int_{Q_{t,r}} (\partial_i u)(x_i' - y_i')\zeta^p\mathrm{d}x \\ &\le \frac{p}{2}\sum_{s+1}^{N}\int\int_{Q_{t,r}} (t^2 - |x'-y'|^2)_+\, A_i(x, \nabla u)\zeta^{p-1}\partial_i\zeta\,\mathrm{d}x. \end{aligned} \tag{5.1}$$

Now we estimate (5.1) by using structure conditions (1.3) to obtain

$$\begin{aligned} t^{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-s}\int\int_{Q_{t,r}} u\zeta^p\mathrm{d}x\right) &\le \left(\int\int_{Q_{t,r}} (u+\eta)^{\beta(p-1)}\mathrm{d}x\right)^{\frac{1}{p}} \\ &\times \frac{\gamma(\omega_s)}{\sigma\rho}\sum_{i=s+1}^{N}\left(\int\int_{Q_{t,r}} (u+\eta)^{-\beta}|\partial_i u|^p(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p\mathrm{d}x\right)^{\frac{p-1}{p}} + \frac{\gamma t^2}{\sigma\rho}|Q_{t,r}|, \end{aligned} \tag{5.2}$$

using the Hölder inequality and multiplying and dividing for $(u+\eta)^{\beta(p-1)/p}$, $\beta > 0$.

Let us estimate the second integral term in (5.2). Let $\zeta(x'')$ be as before and let us test the equation (1.2) and (1.3) with

$$\psi(x) = (u(x) + \eta)^{1-\beta}(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p(x'') \in W_o^{1,\mathbf{p}}(Q_{t,r}).$$

Using structure conditions (1.3) and Young's inequality, we obtain

$$\begin{aligned} I &:= \gamma^{-1}\int\int_{Q_{t,r}} (u+\eta)^{-\beta}(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}}\left\{\sum_{i=1}^{s}|\partial_i u|^2 + \sum_{i=s+1}^{N}|\partial_i u|^p\right\}\mathrm{d}x \\ &\le \int\int_{Q_{t,r}} (u+\eta)^{-\beta}(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p\mathrm{d}x \\ &+ \sum_{i=1}^{s}\int\int_{Q_{t,r}} (\partial_i u)(u+\eta)^{1-\beta}(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}-1}(x_i' - y_i')\zeta^p\mathrm{d}x \\ &+ \sum_{i=s+1}^{N}\frac{1}{\sigma\rho}\int\int_{Q_{t,r}} (|\partial_i u|^{p-1} + 1)(u+\eta)^{1-\beta}(t^2 - |x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^{p-1}\mathrm{d}x. \end{aligned} \tag{5.3}$$

Applying repeatedly Young's inequality and reabsorbing on the left the terms involving energy estimates we obtain

$$
\begin{aligned}
I &\le \int\int_{Q_{t,r}} (u+\eta)^{2-\beta}(t^2-|x'-y'|^2)_+^{\frac{p}{p-1}-2}|x'-y'|^2\zeta^p \mathrm{d}x \\
&\quad + \frac{1}{(\sigma\rho)^p}\int\int_{Q_{t,r}} (u+\eta)^{p-\beta}(t^2-|x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p \mathrm{d}x \\
&\quad + \int\int_{Q_{t,r}} (u+\eta)^{-\beta}(t^2-|x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p \mathrm{d}x \\
&= I_1 + I_2 + \gamma\eta^{-\beta}t^{\frac{2p}{p-1}}|Q_{t,r}|.
\end{aligned} \tag{5.4}
$$

We estimate separately the various terms. For the first one, we have

$$
I_1 = \int\int_{Q_{t,r}} (u+\eta)^{2-\beta}(t^2-|x'-y'|^2)_+^{\frac{p}{p-1}-2}|x'-y'|^2\zeta^p \mathrm{d}x \le \gamma t^{\frac{2}{p-1}}\int\int_{Q_{t,r}} (u+\eta)^{2-\beta}\mathrm{d}x.
$$

Next we use $1 < p < 2$ to split $(u+\eta)^{p-\beta} = (u+\eta)^{p-2}(u+\eta)^{2-\beta} \le \eta^{p-2}(u+\eta)^{2-\beta}$ to obtain

$$
I_2 \le \frac{\gamma}{(\sigma\rho)^p}\eta^{p-2}t^{2\frac{p}{p-1}}\int\int_{Q_{t,r}} (u+\eta)^{2-\beta}\mathrm{d}x \le \frac{\gamma}{\sigma^p}t^{\frac{2}{p-1}}\int_{Q_{t,r}} (u+\eta)^{2-\beta}\mathrm{d}x.
$$

Inequalities above about I_1, I_2 and (5.4) lead us to the formula:

$$
\begin{aligned}
&\sum_{i=s+1}^{N}\int\int_{Q_{t,r}} |\partial_i u|^p(u+\eta)^{-\beta}(t^2-|x'-y'|^2)_+^{\frac{p}{p-1}}\zeta^p \mathrm{d}x \\
&\le \gamma\sigma^{-p}t^{\frac{2}{p-1}}\int\int_{Q_{t,r}} (u+\eta)^{2-\beta}\mathrm{d}x + \gamma\sigma^{-p}\eta^{-\beta}t^{\frac{2p}{p-1}}|Q_{t,r}|.
\end{aligned} \tag{5.5}
$$

We substitute (5.5) in (5.2), summing $(N-s)$ times the same quantity, to obtain

$$
\begin{aligned}
t^{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-s}\int\int_{Q_{t,r}} u\zeta^p \mathrm{d}x\right) &\le \frac{\gamma t^2}{\sigma\rho}|Q_{t,r}| + \frac{\gamma}{\sigma^p\rho}\left(t^{\frac{2}{p-1}}\int\int_{Q_{t,r}} (u+\eta)^{2-\beta}\mathrm{d}x + \eta^{-\beta}t^{\frac{2p}{p-1}}|Q_{t,r}|\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int\int_{Q_{t,r}} (u+\eta)^{\beta(p-1)}\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.6}
$$

Now we evaluate by Jensen's inequality separately for $\alpha = 2-\beta, \beta(p-1)$, the terms

$$
\int\int_{Q_{t,r}} (u+\eta)^{\alpha}\mathrm{d}x \le |Q_{t,r}|\left(⨍⨍_{Q_{t,r}} (u+\eta)\mathrm{d}x\right)^{\alpha}.
$$

Let

$$
\mathcal{A} := \left(⨍⨍_{Q_{t,r}} (u+\eta)\mathrm{d}x\right) = \left(⨍⨍_{Q_{t,r}} u\mathrm{d}x\right) + \eta,
$$

and therefore we estimate from the above inequality (5.6) by

$$t^{s+1}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-s}\int\int_{Q_{t,r}} u\zeta^p \mathrm{d}x\right) \le \frac{\gamma}{\rho\sigma^p}|Q_{t,r}|\left\{t^{\frac{2}{p}}\mathcal{A}^{2\left(\frac{p-1}{p}\right)} + t^2\eta^{-\beta(\frac{p-1}{p})}\mathcal{A}^{\beta(\frac{p-1}{p})} + t^2\sigma^{p-1}\right\}.$$

Now we divide left- and right-hand side of this inequality for $t^{s+1}\rho^{N-s}$, we take the supremum on times $0 < t < 2\theta$ on the right, and we integrate between $0 \le \tau \le t$, to obtain

$$\begin{aligned}\rho^{s-N}&\left[t^{-s}\int\int_{Q_{t,r}} u\zeta \mathrm{d}x - \lim_{\tau\downarrow 0} ⨍_{B_t(y')}\left(\int_{B_r(\bar{x}'')} u\zeta \mathrm{d}x''\right)\mathrm{d}x'\right]\\ &\le \frac{\gamma t}{\rho\sigma^p}\left\{\theta^{\frac{2}{p}-1}\sup_{0<t<2\theta}\mathcal{A}^{2(\frac{p-1}{p})} + \theta\eta^{-\beta(\frac{p-1}{p})}\sup_{0<t<2\theta}\mathcal{A}^{\beta(\frac{p-1}{p})} + \theta\sigma^{p-1}\right\}.\end{aligned}$$

Finally, we use that y' is a Lebesgue point and that $t < 2\theta$ to have the estimate

$$\begin{aligned}⨍⨍_{Q_{t,r}} u\zeta \mathrm{d}x \le& \frac{1}{|B_r|}\int_{B_r(\bar{x}'')} u(y',\cdot)\mathrm{d}x''\\ &+ \frac{\gamma}{\rho\sigma^p}\left\{\theta^{\frac{2}{p}}\sup_{0<t<2\theta}\mathcal{A}^{2(\frac{p-1}{p})} + \theta^2\eta^{-\beta(\frac{p-1}{p})}\sup_{0<t<2\theta}\mathcal{A}^{\beta(\frac{p-1}{p})} + \theta^2\sigma^{p-1}\right\}.\end{aligned} \tag{5.7}$$

Step 2. Integral inequality (5.7) *implies a nonlinear iteration*

We consider $\varepsilon \in (0,1)$ and use Young's inequality to (5.7) to obtain for the first term

$$\begin{aligned}&\frac{\gamma\theta^{\frac{2}{p}}}{\rho\sigma^p}\left[\left(\sup_{0<t<2\theta}⨍⨍_{Q_{t,r}} u\mathrm{d}x\right)^{\frac{2(p-1)}{p}} + \eta^{\frac{2(p-1)}{p}}\right]\\ &\le \varepsilon\sup_{0<t<2\theta}⨍⨍_{Q_{t,r}} u\mathrm{d}x + \gamma(\varepsilon)\left(\frac{\gamma\theta^{\frac{2}{p}}}{\rho\sigma^p}\right)^{\frac{p}{2-p}} + \frac{\gamma\theta^{\frac{2}{p}}}{\rho\sigma^p}\eta^{\frac{2(p-1)}{p}}.\end{aligned}$$

Then we require a condition $\beta < \frac{p}{(p-1)}$ to obtain for the second term

$$\begin{aligned}&\frac{\gamma\theta^2}{\rho\sigma^p}\eta^{-\beta(\frac{p-1}{p})}\left[\sup_{0<t<2\theta}\left(⨍⨍_{Q_{t,r}} u\mathrm{d}x\right)^{\beta(\frac{p-1}{p})} + \eta^{\beta(\frac{p-1}{p})}\right]\\ &\le \varepsilon\sup_{0<t<2\theta}⨍⨍_{Q_{t,r}} u\mathrm{d}x + \gamma(\varepsilon)\left(\frac{\theta^2\eta^{-\beta(\frac{p-1}{p})}}{\rho\sigma^p}\right)^{\frac{p}{p-\beta(p-1)}} + \frac{\gamma\theta^2}{\rho\sigma^p}.\end{aligned}$$

Finally, with these specifications, formula (5.7) is majorized by

$$\begin{aligned}⨍⨍_{Q_{t,(1-\sigma)r}} u\mathrm{d}x \le& \rho^{s-N}\int_{B_{2\rho}(\bar{x}'')} u(y',\cdot)\mathrm{d}x'' + \varepsilon\sup_{0<t<2\theta}⨍⨍_{Q_{t,r}} u\mathrm{d}x\\ &+ \gamma\varepsilon^{-\tilde{\gamma}}\sigma^{-\tilde{\gamma}}\eta\left\{1 + \eta^{-1}\left(\frac{\theta^2\eta^{-\beta(\frac{p-1}{p})}}{\rho}\right)^{\frac{p}{p-\beta(p-1)}} + \eta^{-1}\frac{\theta^2}{\rho}\right\},\end{aligned} \tag{5.8}$$

being $\tilde{\gamma} > 0$ a constant depending only on the data. The term in parenthesis is smaller than one if we contradict condition (1.15), and the right-hand side of (5.8) is estimated from above with

$$⨍⨍_{Q_{t,(1-\sigma)r}} u\mathrm{d}x \le \rho^{s-N}\int_{B_r(\bar{x}'')} u(y',\cdot)\mathrm{d}x'' + \varepsilon\sup_{0<t<2\theta}⨍⨍_{Q_{t,r}} u\mathrm{d}x + \gamma\varepsilon^{-\tilde{\gamma}}\sigma^{-\tilde{\gamma}}\eta. \tag{5.9}$$

Since $t \in (0, 2\theta)$ is an arbitrary number we choose the time $\bar{t}$ that achieves the supremum on the left of previous formula, and we observe that the right-hand side of (5.9) does not depend on t. This implies

$$\sup_{0<t<2\theta} \fint_{Q_{t,(1-\sigma)r}} \fint u\mathrm{d}x \le \rho^{s-N} \int_{B_{2\rho}(\bar{x}'')} u(y', \cdot)\mathrm{d}x'' + \varepsilon \sup_{0<t<2\theta} \fint_{Q_{t,r}} \fint u\mathrm{d}x + \gamma\varepsilon^{-\tilde{\gamma}}\sigma^{-\tilde{\gamma}}\eta.$$

We consider the increasing sequence $\rho_n = (1-\sigma_n)r = \rho(\sum_{i=1}^{n} 2^{-i}) \to 2\rho$ and the expanding polydisc $Q_n = \{Q_{t,\rho_n}\} : Q_{t,\rho} \to Q_{t,2\rho}$. By generality of the choice of r, previous formula implies the recurrence

$$\mathcal{S}_n := \sup_{0<t<2\theta} \fint_{Q_n} \fint u\mathrm{d}x \le \varepsilon\mathcal{S}_{n+1} + \left(\rho^{s-N} \int_{B_{2\rho}(\bar{x}'')} u(y', \cdot)\mathrm{d}x''\gamma\varepsilon^{-\tilde{\gamma}}\eta\right) b^n, \tag{5.10}$$

with $b = 2^{\tilde{\gamma}}$. Therefore, with an iteration as in (A6) we arrive to the conclusion

$$\sup_{0<t<2\theta} \fint_{Q_{t,\rho}} \fint u \,\mathrm{d}x \le \rho^{s-N} \int_{B_{2\rho}(\bar{x}'')} u(y', \cdot)\mathrm{d}x'' + \gamma\eta. \tag{5.11}$$

Step 3. Estimating the full L^∞ Norm from above.

We consider formula (2.12) with $l = 1$, which is

$$\sup_{Q_{\theta/2}, \rho/2} u \le \gamma\left(\frac{\rho^p}{\theta^2}\right)^{\frac{(N-s)}{X}} \left(\fint_{Q_{\theta,\rho}} \fint u_+\mathrm{d}x\right)^{\frac{p}{X}} + \gamma\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}}. \tag{5.12}$$

Let us insert in the integral term on the right of (5.12) our previously obtained formula (5.11) to obtain

$$\begin{aligned} \sup_{Q_{\theta/2}, \rho/2} u &\le \gamma\left(\frac{\rho^p}{\theta^2}\right)^{\frac{(N-s)}{X}} \left(\rho^{s-N} \int_{B_{2\rho}(\bar{x}'')} u(y', \cdot) + \gamma\eta\right)^{\frac{p}{X}} + \gamma\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}} \\ &\le \gamma\left(\frac{\rho^p}{\theta^2}\right)^{\frac{(N-s)}{X}} \left(\rho^{s-N} \int_{B_{2\rho}(\bar{x}'')} u(y', \cdot)\right)^{\frac{p}{X}} + \gamma\left(\frac{\theta^2}{\rho^p}\right)^{\frac{1}{2-p}}. \end{aligned} \tag{5.13}$$

The proof is complete.

6 Harnack inequality. Proof of Theorem 1.2

Here we prove the Harnack estimate (1.11), by use of Theorems 1.3 and 1.4. Without loss of generality we assume that $x_0 = 0$ and denote $u(x_0) = u_0$ to ease notation. First, we begin with an estimate reminiscent of [21] aiming to a bound from above and below in terms of the radius itself (estimate (6.2)).

Step 1. A Krylov-Safonov argument.

For a parameter $\lambda \in (0, 1)$ we consider the equation

$$\sup_{x'' \in B_\lambda\rho(0'')} u(0', x'') = u_0(1-\lambda)^{-\beta}. \tag{6.1}$$

Let λ_0 be the maximal root of equation (6.1) and by continuity let us fix a point $\bar{x}''$ by

$$u(0', \bar{x}'') = \max_{x'' \in B_{\lambda_0}\rho(0'')} u(0', x'') = u_0(1-\lambda_0)^{-\beta} =: M,$$

being $\bar{x}''$ a point in $B_{\lambda_0\rho}(0'')$ (Figure 1). Now let us define $\lambda_1 \in (0, 1)$ by

$$(1-\lambda_1)^{-\beta} = 4(1-\lambda_0)^{-\beta}, \quad \text{i.e.,} \quad \lambda_1 = 1 - 4^{-1/\beta}(1-\lambda_0), \quad \lambda_1 > \lambda_0,$$

and set also

$$2r := (\lambda_1 - \lambda_0)\rho = (1 - 4^{-1/\beta})(1-\lambda_0)\rho.$$

Then by definition of λ_0 it holds both

$$\sup_{x'' \in B_{\lambda_1 \rho}(0'')} u(0', x'') \le u_0(1-\lambda_1)^{-\beta} \quad \text{and} \quad B_{2r}(\bar{x}'') \subset B_{\lambda_1\rho}(0'').$$

This construction is shown in Figure 1. Henceforth, we arrive to the estimate

$$M = u(0', \bar{x}'') \le \sup_{x'' \in B_{2r}(\bar{x}'')} u(0', x'') \le u_0(1-\lambda_1)^{-\beta} = 4u_0(1-\lambda_0)^{-\beta} = 4M. \tag{6.2}$$

Now we use the bound obtained joint to the $L^1 - L^\infty$ estimate (1.17) to reach an estimate of the measure of sub-level sets of u, in order to apply Theorem 1.3.

Step 2. Estimation of the slice-wise measure of sub-levels of u.

Furthermore, we assume $u_0 \ge K\rho$, where K is the number of Theorem 1.3. Let us construct the polydisc

$$Q_{\eta,2r}(0', \bar{x}''), \quad \text{with } \eta = (2r)^{\frac{p}{2}} M^{\frac{2-p}{2}}, \quad \text{being } Q_{\eta,2r}(0, \bar{x}'') \subset Q_{M,\rho}(0) \subset \Omega.$$

Last inclusions are due to the fact that $r < \rho$, $M \le 2\|u\|_{L^\infty(\Omega)}$ and hypothesis (1.8). Hence, we apply Theorem 1.4 in $Q_{\eta,2r}(0, \bar{x}'')$ and use (6.2) to obtain the inequality

$$\max_{Q_{\frac{\eta}{2},\frac{r}{2}}(0',\bar{x}'')} u \le \gamma \left\{ \left(\frac{r^p}{\eta^2}\right)^{\frac{N-s}{\chi}} \left[\inf_{x' \in B_\eta(0')} \fint_{B_{2r}(\bar{x}'')} u(x', x'')\mathrm{d}x'' \right]^{\frac{p}{\chi}} + \left(\frac{\eta^2}{r^p}\right)^{\frac{1}{2-p}} \right\} \le \gamma M. \tag{6.3}$$

Next, we use again Theorem 1.4 for the polydisc $Q_{\delta\eta,r/4}(0, \bar{x}'')$ for a $\delta > 0$ to be determined later. By choice of $\bar{x}''$ we obtain the estimate

$$M = u(0', \bar{x}'') \le \gamma \left\{ \left(\frac{M^{p-2}}{\delta^2}\right)^{\frac{(N-s)}{\chi}} \left[\inf_{x' \in B_{\delta\eta}(0')} \fint_{B_{r/2}(\bar{x}'')} u(x', x'')\mathrm{d}x'' \right]^{\frac{p}{\chi}} + \delta^{\frac{2}{2-p}} M \right\}, \tag{6.4}$$

and we estimate the integral term by splitting the integral on the level $\tilde{\varepsilon}$ as

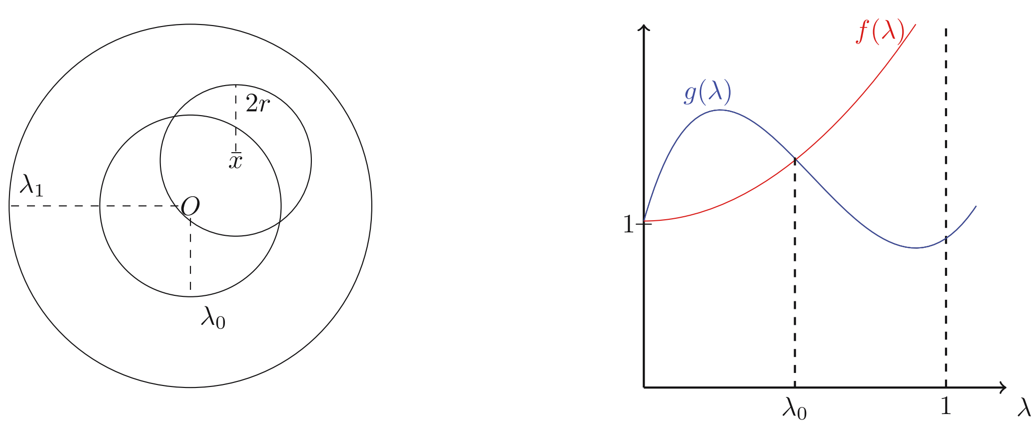

Figure 1: (Left) The geometric construction of $B_{2r}(\bar{x})$, with $\rho = 1$ and $u_0 = 1$. (Right) The idea of definition of λ_0 through equation (6.1). Function f (in red) represents the member on the right-hand side of equation (6.1) while function g (in blue) represents the left-hand side member.

$$\inf_{x'\in B_{\delta\eta}(0')} r^{s-N}\left(\int_{B_{r/2}(\bar{x}'')\cap[u\geq\tilde{\varepsilon}M]} u(x',x'')\mathrm{d}x'' + \tilde{\varepsilon}M|B_{r/2}(\bar{x}'')\cap[u(x',\cdot)<\tilde{\varepsilon}M]|\right).$$

First integral term is estimated by (6.3) and making the choice $\gamma\delta^{\frac{2}{2-p}} = 1/4$ we have from (6.4) that

$$\frac{M}{2} \leq \gamma\delta^{\frac{2(s-N)}{\chi}} M\left(\tilde{\varepsilon} + \frac{\inf_{x'\in B_{\delta\eta}(0')}|[u(x',\cdot)\geq\tilde{\varepsilon}M]\cap B_{r/2}(\bar{x}'')|}{|B_{r/2}(\bar{x}'')|}\right)^{\frac{p}{\chi}} + M/4. \tag{6.5}$$

Henceforth, by choosing also $\tilde{\varepsilon} \leq \frac{1}{2}\left(\frac{\delta^{2(N-s)}}{(4\gamma)^{\chi}}\right)^{1/p}$ we obtain the measure estimate

$$\frac{|u(x',\cdot)\geq\tilde{\varepsilon}M]\cap B_{r/2}(\bar{x}'')|}{|B_{r/2}(\bar{x}'')|} \geq \frac{1}{2}\left(\frac{\delta^{2(N-s)}}{(4\gamma)^{\chi}}\right)^{1/p} =: \alpha,$$

for each $x' \in B_{\delta\eta}(0')$. This gives us the inequality

$$|[u(x',\cdot)\leq\tilde{\varepsilon}M]\cap B_{r/2}(\bar{x}'')| \leq (1-\alpha)|B_{r/2}(\bar{x}'')|, \quad \forall_{ae}\ x'\in B_{\delta\eta}(0'). \tag{6.6}$$

If we reduce furthermore $\tilde{\varepsilon} < \min\{\tilde{\varepsilon}, (\delta 2^{p/2})^{\frac{2}{2-p}}\}$ and use that $1 < p < 2$, then we have

$$\theta := (r/2)^{\frac{p}{2}}(\tilde{\varepsilon}M)^{\frac{2-p}{2}} \leq \delta\eta = \delta r^{\frac{p}{2}} M^{\frac{2-p}{2}},$$

just in order to apply Theorem 1.3 in the polydisc $Q_{\theta,r/2}(0',\bar{x}'')$.

Finally, we expand the positivity applying iteratively Theorem 1.3 to u in appropriate neighborhoods of $(0',\bar{x}'')$, in order to expand positivity until we reach a neighborhood of the origin. A lower bound which is free from any dependence on u itself can be achieved by choosing β appropriately.

Step 3. Expansion of positivity and choice of β.

We consider the measure estimate (6.6): either holds (1.10) or

$$u(x) > \delta_o\tilde{\varepsilon}M/2, \quad \text{in } Q_{\eta,r}(0',\bar{x}''), \quad \text{being} \quad \eta = (r/2)^{\frac{p}{2}}(\delta_o\tilde{\varepsilon}M)^{\frac{2-p}{2}}.$$

This implies the measure estimate

$$|[u(x',\cdot)\leq\delta_o\tilde{\varepsilon}M/2]\cap B_r(\bar{x}'')| \leq |B_r|/2. \tag{6.7}$$

We can apply again Theorem 1.3. This time and next ones being $\alpha = 1/2$ fixed, there exists a number δ_* depending only on the data and such that for almost every $x \in Q_{\eta_*,2r}(0',\bar{x}'')$ we have

$$u(x) > \delta_* M^*/2, \quad \eta_*^2 = (2r)^p(\delta_* M^*)^{2-p} \quad \text{being } M^* = \delta_o\tilde{\varepsilon}M/2. \tag{6.8}$$

Now the procedure can be iterated a number $n \in \mathbb{N}$ of times such that $n \geq \log_2(4/(1-\lambda_0))$ in order to have

$$u(x) > (\delta_*/2)^n M^*, \quad \forall_{ae} x \in Q_{\eta_*(n),2^n r}(0',\bar{x}''), \quad \text{being } \eta_*^2(n) = (2r)^p(\delta_*^n M^*)^{2-p}.$$

We observe that in previous calculation of $\eta_*(n)$ the powers of 2 cancel each other out.

Since $\bar{x}'' \in B_{\lambda_0\rho}(0'')$, then $B_\rho(0'') \subset B_{2\rho}(\bar{x}'')$. If we assume $\beta > 2$ the choice of $\bar{n}$ above implies $2^{\bar{n}}r \geq 2\rho$ so that

$$u(x) > (\delta_*/2)^n M^*, \quad \forall_{ae} x \in Q_{\eta_*(n),\rho}(0), \quad \text{where} \tag{6.9}$$

$$\eta_*(n) = (2r)^{\frac{p}{2}}(\delta_*^n M^*)^{\frac{2-p}{2}} = (2r)^{\frac{p}{2}}(\delta_*^n\delta_o\tilde{\varepsilon}M)^{\frac{2-p}{2}} = (1-\lambda_0)^{p/2}\rho^{p/2}\{\delta_*^n\delta_o\tilde{\varepsilon}[u_0(1-\lambda_o)^{-\beta}2^{-1}]\}^{\frac{2-p}{2}} \geq \bar{\delta}_o\rho^{p/2}u_0^{\frac{2-p}{2}},$$

by properly choosing $\beta \geq p/(2-p)$ and redefining the constants. Observe that equation (6.9) is exactly (1.11). To end the proof, we will choose β big enough to free the lower bound

$$u(x) > (\delta_*/2)^n M^*,$$

by any dependence of the solution itself other than u_0. Indeed, decreasing ρ in case of need, let $n \in \mathbb{N}$ be a number big enough that

$$n \geq \bar{n}, \quad 1 \leq 2^n \left(\frac{r}{\rho} \right) \leq 2, \quad \Rightarrow \quad (1 - \lambda_0)^{-1} > 2^{n-2}(1 - 4^{-1/\beta}).$$

Then we have

$$(\delta_*/2)^n M^* = (\delta_*/2)^n [\delta_0 \tilde{\varepsilon} u_0 (1 - \lambda_0)^{-\beta}] \geq (\delta_* 2^{\beta-1})^n 2^{-2\beta} (1 - 4^{-1/\beta})^\beta \delta_0 \tilde{\varepsilon} u_0.$$

Decreasing δ_* in case of need, we choose finally $\beta > p/(2-p)$ so big that

$$\delta_* 2^{\beta-1} = 1, \quad K := 2^{-2\beta}(1 - 4^{-1/\beta})^\beta \delta_0 \tilde{\varepsilon}.$$

and the claim follows.

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A Appendix

In this section, we enclose all the details that have been postponed for reader's convenience.

A.1 Proof of the energy estimates (2.4)

We suppose without loss of generality that $\bar{x} = 0$. Let us test equation (1.4) with $\phi = \pm(u-k)_\pm\zeta^2 \in W_o^{1,[2,p]}(\Omega)$ for $\zeta \in C_o^\infty(\Omega)$ to obtain

$$\begin{aligned} 0 = \int\int_{Q_{\theta,\rho}} \Big\{ &\sum_{i=1}^{s} (\partial_i u)[(\pm\partial_i(u-k)_\pm)\zeta^2 + 2(u-k)_\pm(\partial_i\zeta)\zeta] \\ &+ \sum_{i=s+1}^{N} A_i(x,\nabla u)[(\pm\partial_i(u-k)_\pm)\zeta^2 + 2(u-k)_\pm(\partial_i\zeta)\zeta]\Big\} \mathrm{d}x. \end{aligned} \tag{A1}$$

We divide the terms in squared parenthesis and use (1.3) to obtain

$$\begin{aligned} &\sum_{i=1}^{s}\int\int_{Q_{\theta,\rho}} |\partial_i(u-k)_\pm|^2\mathrm{d}x + C_1\sum_{i=s+1}^{N}\int\int_{Q_{\theta,\rho}} |\partial_i(u-k)_\pm|^p\mathrm{d}x \\ &\le (N-s)C\int\int_{Q_{\theta,\rho}} \chi_{[u\gtrless k]}(x)\mathrm{d}x + \sum_{i=1}^{s}\int\int_{Q_{\theta,\rho}} \varepsilon|\partial_i u|^2\zeta^2 + C(\varepsilon)|(u-k)_\pm|^2|\partial_i\zeta|^2\mathrm{d}x \\ &\quad + 2\sum_{i=s+1}^{N}\int\int_{Q_{\theta,\rho}} [C_2|\partial_i(u-k)_\pm|^{p-1} + C][(u-k)_\pm(2\zeta)\partial_i\zeta]\mathrm{d}x, \end{aligned} \tag{A2}$$

where we have used Young's inequality $ab \le \varepsilon a^2 + C(\varepsilon)b^2$ on the term $(\partial_i u)[2(u-k)_\pm(\partial_i\zeta)\zeta]$ of the first line of (A1).

Two other similar applications of Young's inequality $ab \le \varepsilon a^p + C(\varepsilon)b^{\frac{p}{p-1}}$ to the last term on the right of (A2) reveal that this is smaller than the quantity

$$2\sum_{i=s+1}^{N}\int\int_{Q_{\theta,\rho}} [\varepsilon|\partial_i(u-k)_\pm|^p + 2C(\varepsilon)|(u-k)_\pm|^p|\partial_i\zeta|^p + \varepsilon(2\zeta)^p\chi_{\{(u-k)_\pm\ge 0\}}(x)]\mathrm{d}x.$$

Combining all these estimates and choosing $\varepsilon > 0$ to be a constant small enough concludes the proof.

A.2 Iteration lemmata

Here we recall two basic lemmas, extremely useful for the iteration techniques employed in our analysis and whose proofs can be found in ([10], Chap. I Sec.IV).

Lemma A.1. ([10] Chap. I, Sec. IV) *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying the recursive inequalities*

$$Y_{n+1} \le Cb^n Y_n^{1+\alpha}, \tag{A3}$$

where $C, b > 1$ and $\alpha \in (0,1)$ are given numbers. Then we have the logical implication

$$Y_o \le C^{\frac{-1}{\alpha}} b^{\frac{-1}{\alpha^2}} \quad \Rightarrow \quad \lim_{n\to\infty} Y_n = 0.$$

The following lemma is useful for reverse recursive inequalities of Section A.3.

Lemma A.2. ([10] Chap. I, Sec. IV) *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of equibounded positive numbers which satisfies the recursive inequality*

$$Y_n \le Cb^n Y_{n+1}^{1-\alpha}, \quad \alpha \in (0,1), \quad C,\ b > 1. \tag{A4}$$

Then

$$Y_0 \le \left(\frac{2C}{b^{1-1/\alpha}}\right)^{\frac{1}{\alpha}}. \tag{A5}$$

Remark A.1. If we just have a sequence of equibounded positive numbers $\{Y_n\}$ such that

$$Y_n \le \varepsilon Y_{n+1} + Cb^n, \quad C, b > 1, \quad \varepsilon \in (0,1), \tag{A6}$$

then by a simple iteration, setting $(\varepsilon b) = 1/2$ and letting $n \to \infty$ give (A5) with $\alpha = 1$. See for instance ([10], Lemma 4.3 page 13).

A.3 Proof of Lemma 2.5

Proof. We are going to perform a cross-iteration. Let $\{\sigma_j\}_{j\in\mathbb{N}} \subset (0,1)$ be the increasing sequence $\sigma_j = \sum_{i=1}^{j} 2^{-(i+1)}$. Let also $n \in \mathbb{N}$ be an index to define the decreasing sets

$$\begin{cases} Q_n = Q_{\theta_n,\rho_n}, \quad \tilde{Q}_n = Q_{\tilde{\rho}_n,\tilde{\theta}_n}, \\ Q_0 = Q_{\sigma_{j+1}\theta,\sigma_{j+1}\rho}, \quad Q_\infty = Q_{\sigma_j\theta,\sigma_j\rho}, \end{cases} \qquad \text{for} \begin{cases} \rho_n = \sigma_j\rho + \dfrac{(\sigma_{j+1}-\sigma_j)\rho}{2^n}, \quad \tilde{\rho}_n = \dfrac{\rho_n + \rho_{n+1}}{2}, \\ \theta_n = \sigma_j\theta + \dfrac{(\sigma_{j+1}-\sigma_j)\theta}{2^n}, \quad \tilde{\theta}_n = \dfrac{\theta_n + \theta_{n+1}}{2}. \end{cases}$$

Furthermore, contradicting (2.11), let $k \ge (\theta^2/\rho^p)^{\frac{1}{2-p}} \ge \rho$ be a number to be defined *a posteriori* and let us define the increasing sequence of levels and numbers

$$k_n = k - \frac{k}{2^n}, \quad M_n = \sup_{Q_n} u,$$

such that

$$M_0 =: M_{j+1} := \sup_{Q_{\sigma_{j+1}\theta,\sigma_{j+1}\rho}} u, \quad M_\infty =: M_j := \sup_{Q_{\sigma_j\theta,\sigma_j\rho}} u.$$

Step 1. First iteration (*valid without condition* (1.9))
We perform in the first place an iteration for $n \to \infty$ on shrinking polydiscs.
To this aim, we introduce cutoff functions ζ_n vanishing on ∂Q_n and equal to one in $\tilde{Q}_n$ that obey to

$$\begin{cases} |\partial_i\zeta_n| \le (\rho_n - \rho_{n+1})^{-1} = (2^{n+1}2^{j+1})/\rho, & \forall i = 1, \dots, s, \\ |\partial_i\zeta_n| \le (\theta n - \theta n + 1)^{-1} = (2^{n+1}2^{j+1})/\theta, & \forall i = s+1, \dots, N. \end{cases}$$

With these stipulations, the energy estimates (2.4) are

$$\begin{aligned} I_n &:= \int\int_{\tilde{Q}_n} \sum_{i=1}^{s} |\partial_i(u-k_{n+1})_+|^2\zeta_n^2 + \sum_{i=s+1}^{N} |\partial_i(u-k_{n+1})_+|^p\zeta_n^2 dx \\ &\le \gamma \int\int_{Q_n} \left\{ \frac{2^{2n}2^{2j}}{\theta^2}|(u-k_{n+1})_+|^2 + \frac{2^{np}2^{jp}}{\rho^p}|(u-k_{n+1})_+|^p \right\} dx + |A_n|, \end{aligned} \tag{A7}$$

being $A_n = Q_n \cap [u > k_{n+1}]$. Now for any $s > 0$ we observe that

$$|A_n| \le \frac{2^{s(n+1)}}{k^s} \iint\limits_{Q_n} (u-k_n)_+^s \mathrm{d}x.$$

This fact with $s = 2$ together with the Hölder inequality gives

$$\iint\limits_{Q_n} (u-k_{n+1})_+^p \mathrm{d}x \le \Bigg(\iint\limits_{Q_n} (u-k_{n+1})_+^2 \mathrm{d}x \Bigg)^{\frac{p}{2}} |A_n|^{1-\frac{p}{2}} \le \gamma \frac{2^{(2-p)n}}{k^{2-p}} \iint\limits_{Q_n} (u-k_n)_+^2 \mathrm{d}x.$$

Putting this into the aforementioned energy estimates leads us to

$$\begin{aligned} I_n &:= \iint\limits_{\tilde{Q}_n} \Bigg\{ \sum_{i=1}^{s} |\partial_i (u-k_{n+1})_+|^2 + \sum_{i=s+1}^{N} |\partial_i (u-k_{n+1})_+|^p \Bigg\} \mathrm{d}x \\ &\le \frac{\gamma 2^{2n} 2^{2j}}{\theta^2} \Bigg(1 + k^{p-2} \frac{\theta^2}{\rho^p} + \frac{\theta^2}{k^2} \Bigg) \iint\limits_{Q_n} |(u-k_n)_+|^2 \mathrm{d}x \\ &\le \frac{\gamma 2^{2n} 2^{2j}}{\theta^2} \iint\limits_{Q_n} |(u-k_n)_+|^2 \mathrm{d}x, \end{aligned} \tag{A8}$$

because $k \ge (\theta^2/\rho^p)^{\frac{1}{2-p}} \ge \rho$. Now an application of Troisi's Lemma 2.1 and (A8) give us the following inequality:

$$\begin{aligned} \iint\limits_{Q_{n+1}} (u-k_{n+1})_+^2 \mathrm{d}x &\le \iint\limits_{\tilde{Q}_n} (u-k_{n+1})_+^2 \tilde{\zeta}^2 \mathrm{d}x \\ &\le \Bigg(\iint\limits_{Q_n} [(u-k_{n+1})_+ \tilde{\zeta}]^{\frac{N\bar{p}}{N-\bar{p}}} \Bigg)^{2\left(\frac{N-\bar{p}}{N\bar{p}}\right)} |A_n|^{1-\frac{2}{\bar{p}}+\frac{2}{N}} \\ &\le \gamma \Bigg(\prod_{i=1}^{N} \|\partial_i (u-k_{n+1})_+\|_{L^{p_i}(Q_n)} \Bigg)^{\frac{2}{N}} |A_n|^{1-\frac{2}{\bar{p}}+\frac{2}{N}} \\ &\le \gamma \Bigg(I_n^{\sum_{i=1}^{N} \frac{1}{p_i}} \Bigg)^{\frac{2}{N}} |A_n|^{1-\frac{2}{\bar{p}}+\frac{2}{N}} \\ &\le \Bigg[\frac{\gamma 2^{2n} 2^{2j}}{\theta^2} \Bigg]^{\frac{2}{\bar{p}}} \Bigg(\frac{2^{2(n+1)}}{k^2} \Bigg)^{1-\frac{2}{\bar{p}}+\frac{2}{N}} \Bigg(\iint\limits_{Q_n} |(u-k_n)_+|^2 \mathrm{d}x \Bigg)^{1+\frac{2}{N}}. \end{aligned} \tag{A9}$$

Hence by setting

$$Y_n = \int\limits_{Q_n} (u-k_n)_+^2 \mathrm{d}x,$$

we arrive at the inequality

$$Y_{n+1} \le \gamma 2^{n(1+\frac{2}{N})} 2^{\frac{4j}{\bar{p}}} \theta^{-\frac{4}{\bar{p}}} k^{-2(\frac{N(\bar{p}-2)+2\bar{p}}{N\bar{p}})} Y_n^{1+\frac{2}{N}},$$

which converges by Lemma A.1 if

$$Y_0 \le \Bigg(\gamma 2^{\frac{4j}{\bar{p}}} \theta^{-\frac{4}{\bar{p}}} k^{-2\left(\frac{N(\bar{p}-2)+2\bar{p}}{N\bar{p}}\right)} \Bigg)^{-\frac{N}{2}} 2^{\frac{N+2}{N}\left(-\frac{N^2}{4}\right)} = \gamma 2^{-\frac{2Nj}{\bar{p}}} \theta^{\frac{2N}{\bar{p}}} k^{\frac{N(\bar{p}-2)+2\bar{p}}{\bar{p}}}.$$

This condition can be obtained by imposing

$$k = \gamma 2^{\tilde{\gamma} j} \theta^{\frac{-2N}{N(\bar{p}-2)+2\bar{p}}} \left(\int \int_{Q_{\sigma_{j+1}\theta, \sigma_{j+1}\rho}} u_+^2 dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+2\bar{p}}} \wedge \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{A10}$$

Therefore, we proceed by this choice of k and we develop the iteration to end up with

$$\sup_{Q_{\sigma_j\theta,\sigma_j\rho}} u \leq \gamma 2^{\tilde{\gamma} j} \theta^{\frac{-2N}{N(\bar{p}-2)+2\bar{p}}} \left(\int \int_{Q_{\sigma_{j+1}\theta, \sigma_{j+1}\rho}} u_+^2 dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+2\bar{p}}} + \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{A11}$$

Step 2. Second iteration (*condition* (1.9) *enters*)

At this stage, we would like to obtain an estimate with whatever power $1 \leq l \leq 2$ in the integral on the right, so that we collect $\sup_{Q_{\theta,\rho}} u = M$ to obtain

$$\sup_{Q_{\sigma_j\theta,\sigma_j\rho}} u \leq \gamma M_{j+1}^{\frac{(2-l)\bar{p}}{N(\bar{p}-2)+2\bar{p}}} 2^{\tilde{\gamma} j} \theta^{\frac{-2N}{N(\bar{p}-2)+2\bar{p}}} \left(\int \int_{Q_{\sigma_{j+1}\theta, \sigma_{j+1}\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+2\bar{p}}} + \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}.$$

Finally, we use Young's inequality with $p = \frac{N(\bar{p}-2)+2\bar{p}}{(2-l)\bar{p}}$ and $p' = (1-1/p)^{-1} = \frac{N(\bar{p}-2)+2\bar{p}}{N(\bar{p}-2)+l\bar{p}}$ to obtain the inequality

$$\sup_{Q_{\sigma_j\theta,\sigma_j\rho}} u \leq \varepsilon M_{j+1} + \gamma \varepsilon^{-\gamma} 2^{\tilde{\gamma} j} \theta^{\frac{-2N}{N(\bar{p}-2)+l\bar{p}}} \left(\int \int_{Q_{\sigma_{j+1}\theta, \sigma_{j+1}\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{A12}$$

Now we perform a second iteration on $\sigma_j = \sum_{i=1}^{j} 2^{-(i+1)}$. Clearly, $\sigma_0 = 1/2$ and $\sigma_\infty = 1$ and the polydiscs Q_{σ_j} increase up to $Q_{\theta,\rho}$. With these stipulations previous formula (A12) can be written as

$$Y_j \leq \varepsilon Y_{j+1} + \gamma 2^{\tilde{\gamma} j} \left\{ \varepsilon^{-\gamma} \theta^{\frac{-2N}{N(\bar{p}-2)+l\bar{p}}} \left(\int \int_{Q_{\sigma_{j+1}\theta, \sigma_{j+1}\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}} \right\} = \varepsilon Y_{j+1} + b^j I, \quad b = 2^{\tilde{\gamma}}.$$

We iterate as in (A6) and we obtain the inequality

$$\sup_{Q_{\theta/2,\rho/2}} u \leq \gamma \theta^{\frac{-2N}{N(\bar{p}-2)+l\bar{p}}} \left(\int \int_{Q_{\theta,\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{A13}$$

If we set $\bar{p}(N-s)/p = A$, then

$$\frac{(N-s)\bar{p}}{p} = A = -\left(\frac{s\bar{p} - 2N}{2} \right),$$

and inequality (A13) writes

$$\begin{aligned} \sup_{Q_{\theta/2,\rho/2}} u &\leq \gamma \theta^{\frac{-2N}{N(\bar{p}-2)+l\bar{p}}} (\theta^s \rho^{N-s})^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} \left(⨍ ⨍_{Q_{\theta,\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \gamma \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}} \\ &\leq \gamma \left(\frac{\rho^p}{\theta^2} \right)^{\frac{A}{N(\bar{p}-2)+\bar{p}l}} \left(⨍ ⨍_{Q_{\theta,\rho}} u_+^l dx \right)^{\frac{\bar{p}}{N(\bar{p}-2)+l\bar{p}}} + \gamma \left(\frac{\theta^2}{\rho^p} \right)^{\frac{1}{2-p}}. \end{aligned} \tag{A14}$$

□